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Calculating the probability of a mid-price increase based on a stochastic model for order book dynamics

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Preface

Abstract

Algorithmic trading has increased massively over the last decade. Today, most of the traders are not sitting in trading pits, while buying or selling stocks by shouting and hand signaling. Different from the past, the transactions are mainly executed electronically by computer algorithms. Based on these procedures, using various information like volatility or liquidity, trading signals are sent.
The purpose of this thesis is to figure out if it is possible to predict market movements due to a stochastic model for the dynamics of an order book. However, in hardly any paper dealing with this sort of models the assumptions and computed results are accurately compared to those achieved from the order book data. Therefore, in this thesis, a stochastic model is illustrated, investigated and implemented by using parameters estimated by the data of 30 stocks of the Istanbul Stock Exchange. Afterwards, based on this model, the probability of a mid-price increase conditional on the current order book status is computed and compared to the outcomes received from the stock data.
The obtained results reflect the dead-on prediction of the model. The probabilities calculated based on the model are very similar to those achieved from the limit order book. Regarding future analysis, it could be interesting to examine if a trading strategy, considering the calculated probabilities of this thesis, is profitable.

Keywords

limit order book, transition probabilities, probability distributions, Laplace transform, birth-and-death process, mid-price increase

Vorwort

Kurzfassung

Während des letzten Jahrzehntes ist der algorithmische Handel stark angestiegen. Heute gibt es kaum noch Händler, die sich auf dem Parkett treffen, um dort mit Zurufen und Handzeichen die gewünschten Finanzinstrumente zu kaufen oder zu verkaufen. Im Gegensatz zur Vergangenheit werden heutzutage die Transaktionen hauptsächlich elektronisch mit Hilfe von Computeralgorithmen ausgeführt. Diese Algorithmen werden mit verschiedensten Informationen wie Volatilität oder Liquidität gespeist und basierend auf diesen statistischen Maßen wird ein Handelssignal gesendet.
Das Ziel dieser Arbeit ist es, herauszufinden, ob es möglich ist, auf Basis eines stochastischen Modells für die Dynamiken eines Orderbuches Marktbewegung vorherzusagen. Allerdings werden in den meisten wissenschaftlichen Arbeiten, welche von dieser Art von Modellen handeln, weder die Voraussetzungen für die Modellanwendung geprüft noch die erhaltenen Resultate genau mit den Ergebnissen, die die Daten des Limit-Orderbuches liefern, verglichen. Daher wird in dieser Arbeit ein stochastisches Modell erklärt, untersucht und mit den Parametern, die mit Hilfe der Orderbuchdaten von 30 Aktien der Istanbuler Börse geschätzt werden, implementiert. Danach wird basierend auf diesem Modell die Wahrscheinlichkeit für einen Anstieg des mittleren Preises, bedingt auf den aktuellen Stand des Orderbuches, berechnet und mit den Resultaten, die man mit Hilfe der Daten des Limit-Orderbuches erhält, verglichen.
Die erhaltenen Resultate zeigen, dass das Modell präzise Vorhersagen liefert. Die Wahrscheinlichkeiten, die basierend auf dem Modell berechnet worden sind, sind sehr ähnlich zu jenen, welche durch die Orderbuchdaten erhalten wurden. Betreffend zukünftiger Analysen wäre es interessant, zu überprüfen, ob eine Handelsstrategie, welche die berechneten Wahrscheinlichkeiten dieser Arbeit heranzieht, gewinnbringend ist.

Schlagwörter

Limit-Orderbuch, Übergangswahrscheinlichkeiten, Wahrscheinlichkeitsverteilungen, Laplacetransformation, Geburts- und Todesprozess, Anstieg des mittleren Preises

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CHAPTER 1

Introduction

1.1 Algorithmic trading

Algorithmic trading is defined as bidding automatically, meaning that computer platforms are used to submit and manage orders by applying algorithms. There are various algorithms used by different types of market participants. Some hedge funds and brokers/dealers supply the algorithm with liquidity or volatility information and trade by automatically generated signals. Other algorithms derive signals from the order book data if certain patterns are detected [HJM11].

1.1.1 High frequency trading

High frequency trading is one of the major recent innovations in financial markets. It is a subgroup of algorithmic trading and is defined by having short holding periods (meaning anything between a few seconds and a few hours) and trading frequently. The speed of these trading processes is incredible, meaning that the execution of the order occurs immediately after the arrival of information, faster than humans can even register the initial information. High frequency traders compete by having the most powerful computers, connections and programs and pay premiums to the exchanges for the privilege of locating their computers as close as possible to the trading venue [BW11].

In the first place, trading time measured in milliseconds is possible due to electronic trading systems instituted by important exchanges such as NYSE, NASDAQ, London Stock Exchange etc.. These systems aggregate all outstanding limit orders in the so called *limit order book* (LOB), which is visible for traders. Regarding the availability of "Level II" data, [1] dynamical models can be formulated, estimated and tested. Recently, a continuous-time Markov model, in which all required param-

[1]"Level II" market data show the orders which are currently pending for the market. They are also known as the depth of the market, because they display the number of contracts which are available at all prices - not limited to information about the bid and ask price.

eters are estimated from "Level II" order book data, has been proposed by [CST10].

The model in [CST10] treats the LOB as a queuing system, where the incoming orders and cancellations arrive in unit size, modeled by using independent Poisson processes. This queuing system can be described by a specified Markov process, called birth-and-death Markov chain model, where the states are representing the number of shares at a given price and the transition rates are represented by birth (the entry of a new limit order) and death rates (removal of a limit order by cancellation or matching with a new market order).
The technical problem of evaluating the Laplace transforms of the relevant first-passage times, which are needed to calculate the probability of a mid-price up movement, is overcome by a technique introduced in [HK12]. Hereby, the limit of truncated state spaces is analyzed. The entire approach is closely related to the methods proposed in [CST10, HK12].

1.2 Research question

This work's purpose is to examine whether the full access to the order book, meaning that the book is not limited to the best five or ten quotes, can be used to be faster and trade more profitably than other market participants. Therefore, this study tries to identify the kind of information which is necessary for predicting market movements.

Motivated by [CST10], various measures for predicting market movements can be computed. This thesis will concentrate on the probability of a mid-price increase at its next move conditional on the current order book status. The model of [CST10] will be explained, examined carefully and tested based on the data provided by the Istanbul Stock Exchange (ISE). Thus, the model will be illustrated with parameters estimated from the order book "Level II" data of 30 stocks, which determine the ISE-30 index.

1.3 Thesis outline

The structure of this thesis is organized as followed. In Chapter 2, the data and trading structure in the market are described. An overview of all mathematical basics needed for subsequent calculations is given. As a foretaste, the above mentioned Markov chain model is reviewed and transition probabilities for a linear birth-and-death process are calculated. In Chapter 3, the model parameters are estimated, the accuracy of the estimation is presented and the computation of the conditional probability of a mid-price increase is described. Chapter 4 outlines the numerical calculation and the results. Finally, in Chapter 5, the conclusion is presented.

CHAPTER 2

Material and Methods

This chapter contains a description of the order book data, the usage, some preparations [VZ13] and a section about mathematical foundations, which are used throughout this thesis. The last part supplies the reader with the main concept of the birth-and-death Markov chain model (birth-and-death process).

2.1 Data description and preparation

All data information is obtained by several papers of Marcela Valenzuela and Ilknur Zer [FRVZ14, VZ13].
The used data set consists of order and trade books of 30 stocks over a period of two months (06-01-2008 until 07-31-2008), which determine the ISE-30 index. For this sample period, the index corresponds to almost 75% of all data available in the order book.

The ISE is a fully computerized as well as a fully centralized stock exchange, meaning that all trades of the ISE-listed stocks have to be executed in ISE via electronic order submissions. Hence, the above described data fully captures the order flow.
All brokers have full access to the order book. Prior to the submission of an order, they can see the quantity available at different prices. Additional information - like the price and the quantity which is received for each order - is given. A trade occurs if the price of an order on the buy side fits one on the sell side.

One important fact is that, in 2008 ISE differed from other exchanges in some points. Therefore, four main issues need to be pointed out. First, until 2011, ISE was a non-anonymous open market, meaning that every trader was able to observe all submitted and traded orders. Therefore, no hidden orders existed. Before submitting an order, the market participants knew about the available quantity of all quoted prices. Second, walking through the order book was not possible, which means that the unexecuted proportion of the market order was converted to a limit order. Thirdly, each order was valid for a particular session (before midday or afternoon) or for the whole day. Fourth and last point, order modifications were allowed but the orders could not be canceled.

Table 2.1.1: Unprocessed order data. A detailed description of the column names is given in Appendix A Table A.0.1.

Date	OrderID	Ticker	OT	Quant.	Price	TIF	Time	CT	KTR[a]
01.07.08	107200800181191	AKBNK	1	3000	3.75	0	153023	1	1000
01.07.08	107200800181194	AKBNK	5	400	4.01	1	153035	2	0

[a] If the order is a kill the rest (KTR) type, it is matched with the available volume and the rest is canceled.

Table 2.1.2: Unprocessed trade data. A detailed description of the column names is given in Appendix A Table A.0.1.

Date	Ticker	Time	Quant.	Price	CT	BuyerID	SellerID
01.07.08	AKBNK	94031	11	5.40	0	107200800181191	107200800173428
01.07.08	AKBNK	94500	989	5.40	1	107200800180222	107200800181189

The reconstruction of the order book was done by [FRVZ14]. For one given price, the volume is calculated as the cumulative volume of all orders at exactly this price. The order book consists of the best 10 quotes of each side. This was considered as appropriate because the first 10 levels contain about 90% of all submitted orders.

Table 2.1.3: Reconstructed order book. A detailed description of the column names is given in Appendix A Table A.0.1.

Bid	Ask	S	VB1	VA1	AT	B2*	A2*	VB2*	VA2*
3.88	4.06	0.18	34326	27850	5S	3.98	4.08	182252	78585
3.88	4.04	0.16	34326	24426	2S	3.96	4.06	257804	43398

*This information is available for the best 10 quotes.

The files stock08061.mat, stock08062.mat, stock08063.mat, stock08071.mat, stock08072.mat and stock08073.mat served as basis for most of the calculations done in this thesis. Each of these files is a 1×10 structure (10 stocks in each file, two months) in MATLAB®2011b [MAT11]. Again, each structure entry consists of the stock number (for example AKBNK), a table including all submitted orders and one containing all trades according to this stock number, and the tick size[1]. The order table comprise almost the same information as given in Table 2.1.1 and 2.1.3 and the trade table is similar to Table 2.1.2.

[1] The tick size is the minimum price movement of a trading instrument. It is the minimum increment in which prices can change [Inv].

The files LOB0806min15.mat and LOB0807min15.mat were also prepared by [FRVZ14]. Each of these files is a 21×30 structure, determined by containing the LOB for each of the 30 stocks by the end of each interval $\tau = 1, 2, .., 21$, where the first snapshot ($\tau = 1$) includes all waiting orders submitted until 10:00, $\tau = 2$ corresponds to the time interval [9:30,10:15] and the last column entry of the structure includes all waiting orders, which are valid for both sessions and submitted until 17:00 for all available trading days. Each structure entry contains the ticker of the stock and the tick size too.
The LOB for its own consists of 23 columns and several rows. The number of rows is variable depending on the flow of incoming orders. As shown in Table 2.1.4 and Table 2.1.5, in the LOB two different bid and ask values with their respective volumes are given. The first two represent the bid and ask values after the order arrived in the trading system (immediate updated values), whereas the second two are the values observed when the trader submitted the order.

Table 2.1.4: LOB of AKBNK-10:00, which contains all waiting orders submitted until 10:00 - first part, tick size= 0.05. A detailed description of the column names is given in Appendix A Table A.0.1.

Date	OT	Quant.	Price	TIF	Time	CT	KTR	Bid	Ask	S
02.06.08	1	50000	5.25	0	34201	1	0	5.35	5.55	0.2
02.06.08	4	25000	5.45	1	34201	1	0	5.35	5.45	0.1
02.06.08	1	5000	5.35	1	34201	1	0	5.35	5.45	0.1
02.06.08	1	20000	5.20	0	34201	1	0	5.35	5.45	0.1

Table 2.1.5: LOB of AKBNK-10:00, which contains all waiting orders submitted until 10:00 - second part, tick size = 0.05 . The given bid and ask prices are dissimilar to the ones in the first part of the LOB (cf. Table 2.1.4). In this table they are representing the prices which traders observe when they submit the order. Therefore, the order of the trader is not taken into account. The same is essential for the volumes. A detailed description of the column names is given in Appendix A Table A.0.1.

VB1	VA1	AT	Trade Price	Bid$_T$	Ask$_T$	S	VB1$_T$	VA1$_T$	dist.$^{bid/ask}$	dist.^{trade}
194989	97438	3	5.40	5.35	5.55	0.2	194989	97438	2	3
194989	25000	4	5.40	5.35	5.55	0.2	194989	97438	-2	1
199989	25000	9	5.40	5.35	5.45	0.1	194989	97438	0	1
199989	25000	3	5.40	5.35	5.45	0.1	199989	97438	3	4

The differences between the stock files and the files LOB0806min15.mat and LOB0807min15.mat are the following.

- The stock files consist of all orders and trades in separate tables, whereas the latter only contain the waiting orders, meaning that the order and trade table were matched to construct the LOB. Thus, it is possible to obtain snapshots of the LOB at any given time and enables to observe the same information as a trader, for example the volume of orders waiting to be executed at a certain price range.

- The stock files only contain the bid and ask values noticed when the order is submitted.
- The files LOB0806min15.mat and LOB0807min15.mat are subdivided into smaller structures and contain information for every τ.

Therefore, the usage of the LOB files is appropriate for computing the probability of a mid-price increase, whereas the stock files are used for describing the incoming order flow.

2.1.1 Estimation of the limit order arrival rate

Regarding subsequent calculations, it is necessary to estimate the limit order arrival rate. This is explained by the following steps [CST10, FRVZ14].

1. Usage of stock08061.mat, stock08062.mat, stock08063.mat, stock08071.mat, stock08072.mat and stock08073.mat.
2. Removing market orders and rows with missing values.
3. Calculating the price distance of each limit order relative to the opposite best quote. For every order i, the price distance δ is defined as

$$\delta_{i,t}^{\text{buy}} = \frac{(p_t^A - p_{i,t}^{\text{buy}})}{\text{tick}_t}, t \in T \tag{2.1.1}$$

   and

$$\delta_{i,t}^{\text{sell}} = \frac{(p_{i,t}^{\text{sell}} - p_t^B)}{\text{tick}_t}, t \in T, \tag{2.1.2}$$

   where p_t^B and p_t^A are representing the bid and ask price at time point t, $p_{i,t}^{\text{buy}}/p_{i,t}^{\text{sell}}$ is the price of the ith limit buy/sell order and T represents the daily trading time.
4. Calculating the buy side arrival rate $\hat{\lambda}_{t,\theta,s}^{\text{buy}}(\delta)$ which is the total number of limit orders arrived at a given distance $\delta = 0, 1, 2, 3.., \delta_c$ for every stock S and trading day θ, where δ_c is the maximum distance which is taken into account.
5. Next, for a given day θ and price distance δ the stock-averaged arrival rate, meaning the arrival rate function averaged across 30 stocks is calculated as

$$\hat{\lambda}_{t,\theta}^{\text{buy}}(\delta) = \frac{1}{S}\sum_{s=1}^{S} \hat{\lambda}_{t,\theta,s}^{\text{buy}}(\delta)\,, \tag{2.1.3}$$

   where S is the total number of stocks.

6. In the next step, the limit order arrival rate function averaged across 450 trading minutes and 39 days is calculated as

$$\hat{\lambda}_t^{\text{buy}}(\delta) = \frac{1}{D}\sum_{\theta=1}^{D} \hat{\lambda}_{t,\theta}^{\text{buy}}(\delta)\,, \tag{2.1.4}$$

and finally

$$\hat{\lambda}^{\text{buy}}(\delta) = \frac{1}{T}\sum_{t=1}^{T} \hat{\lambda}_t^{\text{buy}}(\delta)\,, \tag{2.1.5}$$

where D represents the number of trading days and T stands for the daily trading time given in minutes.

Steps 4 to 6 were also done for the sell side.

2.1.2 Estimation of the market order arrival rate

To estimate the arrival rate of market orders, the following steps are executed.

1. Usage of stock08061.mat, stock08062.mat, stock08063.mat, stock08071.mat, stock08072.mat and stock08073.mat.
2. Separating limit and market orders after removing rows with missing values.
3. Calculating the total number of market orders arrived during the daily trading time T.
4. Computing the stock-averaged arrival rate, meaning that for a given $t \in T$ and day θ the averaged arrival rate across 30 stocks is calculated as

$$\tilde{\mu}_{t,\theta} = \frac{1}{S}\sum_{s=1}^{S} \tilde{\mu}_{t,\theta,s}\,, \tag{2.1.6}$$

where S is the total number of stocks.

5. Next, the market order arrival rate averaged across 450 trading minutes and 39 days is calculated as follows

$$\tilde{\mu}_t = \frac{1}{D}\sum_{\theta=1}^{D} \tilde{\mu}_{t,\theta} \tag{2.1.7}$$

and finally

$$\tilde{\mu} = \frac{1}{T}\sum_{t=1}^{T} \tilde{\mu}_t\,, \tag{2.1.8}$$

where D represents the number of trading days and T stands for the daily trading time in minutes.

6. As in [CST10], the average size of a limit order S_l was chosen as unit size. Thus, the final version of the market order arrival rate is given by

$$\hat{\mu} = \tilde{\mu}\frac{S_m}{S_l}\,, \tag{2.1.9}$$

where S_m denotes the average size of a market order.

2.1.3 Calculation of the interarrival time distribution

The distribution of the duration between every two successive orders at the same distance δ is calculated. This is done by applying the following steps.

1. Usage of stock08061.mat, stock08062.mat, stock08063.mat, stock08071.mat, stock08072.mat and stock08073.mat.

2. Removing rows with missing values.

3. Differentiating between buy and sell orders.

4. Splitting the data based on different distances δ and days, and calculating the duration s between every two successive orders.

5. Next, the data are assembled and the empirical probability, given by

$$\log\left(\frac{\text{number of durations} > s}{\text{number of durations}}\right)\,, \tag{2.1.10}$$

is calculated for $s = 0, 1, 2, \ldots, 500$.

2.1.4 Realized frequencies of a mid-price increase

To enable a comparison between the model based calculation of the probability of a mid-price increase and the outcome received from the LOB data, the following steps of calculation are done to obtain the realized frequencies of a mid-price increase.

1. Usage of LOB0806min15.mat and LOB0807min15.mat.

2. Removing rows with missing values.

3. Calculating the ratios of bid and ask shares, the spread p_S, signifying the distance between bid and ask price divided by the tick size, and the mean of the bid and the ask price, which is represented by the mid-price p_M. To actualize the calculations, the bid and ask prices described in Table 2.1.5 are used. The same procedure applies to their corresponding shares.

4. For each of these ratios and a given spread, the probability of a mid-price increase at its next move is computed as $\frac{C_{\text{increase}}}{C_{\text{change}}}$, where C_{increase} counts the events $p_{M_{\text{old}}} < p_{M_{\text{new}}}$ if the mid-price changes for the first time and C_{change} totals the occurrences of $p_{M_{\text{old}}} \neq p_{M_{\text{new}}}$ after a mid-price change.

All calculations were done using MATLAB®2011b.

2.2 Foundations

The aim of this section is to provide the reader with the mathematical foundations, which are used at a later stage. First, a short introduction about the least squares method, the Laplace transform, continued fractions and completely monotone functions is given. Next, the trapezoidal rule, the Euler summation and the Poisson summation formula are explained. Additional information regarding convolutions is given, followed by a description of some important distributions and processes. The last part supplies the reader with the main concept about Markov processes.

2.2.1 Least squares method

The term least squares describes a frequently used approach for solving overdetermined or inexactly specified systems of equations in an approximate sense, this means instead of solving the equations exactly, the sum of the squares of the residuals is minimized [Mata].

In this thesis, the MATLAB®2011b function `lsqcurvefit` was used as least squares method. This function solves non-linear curve-fitting (data-fitting) problems in least squares sense, which means that it finds coefficients x that solve the problem

$$\| F(x, xdata) - ydata \|_2^2 = \min_x \sum_i (F(x, xdata_i) - ydata_i)^2 \tag{2.2.1}$$

given input data $xdata$ and the observed output $ydata$, where $xdata$ and $ydata$ are matrices or vectors. $F(x, xdata)$ is a matrix-valued or vector-valued function of the same size as $ydata$. For more information see [Matb].

2.2.2 Laplace transform

According to [Kal13], a general definition of the Laplace transform is given.

Definition 2.2.1 (Laplace transform). *Let $f : [0, \infty) \to \mathbb{C}$ be measurable. If $t \mapsto e^{-st} f(t) \notin L^1[0, \infty)$ for all real $s \in \mathbb{R}$, then $\sigma(f) := \infty$. Otherwise, let*

$\sigma(f) \in [-\infty, \infty)$ be the infimum of all $s \in \mathbb{R}$ such that $t \mapsto e^{-st} f(t) \in L^1[0, \infty)$. The Laplace transform of f is defined by

$$\hat{f} : \begin{cases} (\sigma(f), \infty) \to \mathbb{C} \\ s \mapsto \int_0^\infty f(t) e^{-st} dt \,. \end{cases} \tag{2.2.2}$$

For subsequent analysis, the Laplace transform of probability density functions is needed.

Definition 2.2.2. *Given a probability density function $f : [0, \infty) \to \mathbb{R}$ of some random variable X, its Laplace transform is defined by*

$$\hat{f}(s) = \int_0^\infty e^{-st} f(t) dt \,. \tag{2.2.3}$$

An important property in association with the Laplace transform is that if X and Y are two independent random variables with densities $f_X, f_Y : [0, \infty) \to \mathbb{R}$ and their well-defined Laplace transforms $\hat{f}_X, \hat{f}_Y$ then

$$\hat{f}_{X+Y}(s) = \mathbb{E}[e^{-s(X+Y)}] = \mathbb{E}[e^{-sX}]\mathbb{E}[e^{-sY}] = \hat{f}_X(s)\hat{f}_Y(s) \,. \tag{2.2.4}$$

Moreover, for the Laplace transform of a distribution function $F(t)$ the following holds

$$\hat{F}(s) = \int_0^\infty e^{-st} F(t) dt \,. \tag{2.2.5}$$

For $s \neq 0$ partial integration yields

$$\hat{F}(s) = -\frac{e^{-st}}{s} F(t) \Big|_0^\infty + \int_0^\infty \frac{e^{-st}}{s} f(t) dt \,. \tag{2.2.6}$$

Finally, using Definition 2.2.2 gives

$$\hat{F}(s) = \frac{1}{s} \int_0^\infty e^{-st} f(t) dt = \frac{1}{s} \hat{f}(s) \,. \tag{2.2.7}$$

Next, it is necessary to introduce the inverse Laplace transform [ACW99, Dav01, Dyk14].

Theorem 2.2.1 (Inverse Laplace transform)**.** *Let $\hat{f}$ be the Laplace transform of some f defined as in Definition 2.2.2. Then, the function value $f(t)$ can be recovered by*

$$f(t) = \frac{1}{2\pi \mathrm{i}} \int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} e^{st} \hat{f}(s) ds, \; t > 0 \,. \tag{2.2.8}$$

Here a is real number and has to be chosen greater than the real parts of all singularities of $\hat{f}$.

If f is a bounded continuous probability density function, $a = 0$ in (2.2.8) is sufficient [AW92].

According to [Kal13], some important properties of the Laplace transform are introduced.

Proposition 2.2.1. *For some functions f, g as in Definition 2.2.1 the following holds.*

- *If f is real valued, then,*

$$\hat{f}(\overline{z}) = \overline{\hat{f}(z)}\,. \tag{2.2.9}$$

Thus, for the real part $\mathrm{Re}(\hat{f}(\overline{z})) = \mathrm{Re}(\hat{f}(z))$ and for the imaginary part $\mathrm{Im}(\hat{f}(\overline{z})) = -\mathrm{Im}(\hat{f}(z))$. In particular, $\hat{f}$ is real valued too.

- *Moreover, $\widehat{f * g} = \hat{f}\hat{g}$.*

2.2.3 Continued fractions and completely monotone functions

Next, it is useful to introduce so-called continued fractions. All achievements in this section are based on [AW99].

Definition 2.2.3 (Continued fractions)**.** *The continued fraction $\{w_n : n \geq 1\}$ associated with a sequence $\{a_n : n \geq 1\}$ of partial numerators and a sequence $\{b_n : n \geq 1\}$ of partial denominators, where a_n and $b_n \in \mathbb{C}$ for all n is given by*

$$w_n = t_1 \circ t_2 \circ \cdots \circ t_n(0),\ n \geq 1\,. \tag{2.2.10}$$

Here $\circ$ denotes the composition operator and

$$t_k(u) = \frac{a_k}{b_k + u},\ k \geq 1\,. \tag{2.2.11}$$

Thus, w_n is the n-fold composition of the mappings $t_k(u)$ in (2.2.11).

If $\lim_{n\to\infty} w_n = w$ for some $w \in \mathbb{C}$, then the continued fraction is said to be convergent and the limit w is called the value of the continued fraction. In this case, one writes

$$w \equiv \Phi_{n=1}^{\infty} \frac{a_n}{b_n} \text{ or } w = \frac{a_1}{b_1+}\frac{a_2}{b_2+}\cdots\,, \tag{2.2.12}$$

where the last notation is a compact expression for

$$w = \frac{a_1}{b_1 + \frac{a_2}{b_2 + \cdots}}\,. \tag{2.2.13}$$

One special continued fraction is called *S fraction*. Regarding this case, w can be written as

$$w(s) = \frac{1}{1+}\frac{a_2 s}{1+}\frac{a_3 s}{1+}\cdots\,, \tag{2.2.14}$$

where $a_k > 0$ for all k.

According to [AW99], these special continued fractions have an important attribute, which is needed for subsequent calculations.

Remark 2.2.1. *Probability applications of continued fractions are tempting if there is a completely monotone probability density function. In this case, the associated Laplace transform can be represented by S fractions, which have desirable convergence properties.*

Definition 2.2.4 (Completely monotone functions). *A function f on $[0, \infty)$ is completely monotone if it is continuous on $[0, \infty)$, infinitely differentiable on $(0, \infty)$ and has derivations of all orders that alternate in sign, meaning that*

$$(-1)^n f^{(n)}(t) \geq 0 \text{ for all } t \geq 0 \text{ and } n \geq 0\,. \tag{2.2.15}$$

2.2.4 The trapezoidal rule

According to [AP11], one important numerical method which approximates the value of an integral $\phi := \int_a^b f dx$ with continuous integrand $f : [a, b] \to \mathbb{R}$ is the trapezoidal rule.

The interval $[a, b]$ is divided into $n \in \mathbb{N}$ sub-intervals of equal length

$$h = \frac{b-a}{n}\,, \tag{2.2.16}$$

where h is called the step size.
Next, define $n + 1$ evaluation points

$$x_0 = a,\ x_1 = a + h,\ x_2 = a + 2h,\ \ldots\ , x_n = a + nh = b\,. \tag{2.2.17}$$

Now, the values of f can be evaluated at these points

$$y_0 = f(x_0),\ y_1 = f(x_1),\ y_2 = f(x_2),\ \ldots\ , y_n = f(x_n)\,. \tag{2.2.18}$$

The integral is approximated by using n trapezoids formed by using straight line segments between (x_{i-1}, y_{i-1}) and (x_i, y_i) for $1 \leq i \leq n$. Figure 2.2.1 visualizes this approximation method.

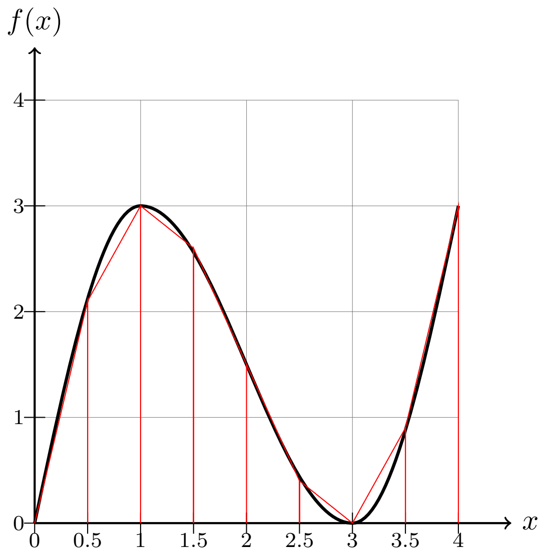

Figure 2.2.1: Visualization of the trapezoidal rule with parameters $a = 0, b = 4, h = 0.5$ and $n = 8$.

The area of the first trapezoid is obtained by adding the area of a rectangle to a triangle. Thus, one gets

$$A = y_0 h + \frac{1}{2}(y_1 - y_0)h = \frac{(y_0 + y_1)h}{2} . \tag{2.2.19}$$

The following approximation is obtained by adding the area of n trapezoids

$$\int_a^b f(x)dx \approx \frac{(y_0 + y_1)h}{2} + \frac{(y_1 + y_2)h}{2} + \ldots + \frac{(y_{n-1} + y_n)h}{2} , \tag{2.2.20}$$

which simplifies to

$$\begin{aligned} \int_a^b f(x)dx &\approx \frac{h}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n) \\ &= h\left(\frac{y_0 + y_n}{2} + \sum_{k=1}^{n-1} y_k\right) . \end{aligned} \tag{2.2.21}$$

2.2.5 Poisson summation formula

According to [BZ97, Kal13], a short introduction on the Fourier transform and the Poisson summation formula is given.

If $f \in L^1(\mathbb{R})$ and $s \in \mathbb{R}$, then $|f(t)e^{-2\pi ist}| = |f(t)|$ and thus, $t \mapsto f(t)e^{-2\pi ist}$ belongs to $L^1(\mathbb{R})$.

Definition 2.2.5 (Fourier transform). *For $f \in L^1(\mathbb{R})$, the Fourier transform F : $\mathbb{R} \to \mathbb{C}$ is defined by*

$$F(s) = \int_{-\infty}^{\infty} f(t) e^{-2\pi \mathrm{i} st} dt \ , s \in \mathbb{R} \, . \tag{2.2.22}$$

Definition 2.2.6 (Poisson summation formula). *For some sufficiently well-behaved functions $f \in L^1(\mathbb{R})$, the Poisson summation formula is given by*

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{k \in \mathbb{Z}} F(k) \, , \tag{2.2.23}$$

where F is the Fourier transform of f.

Next, some properties of the Fourier transform are introduced.

Proposition 2.2.2. *For some f as in Definition 2.2.5 and $r, z \in \mathbb{R}, r \neq 0$, the following holds.*

- *By $f_z(t) = f(z+t)$, $t \in \mathbb{R}$, follows $f_z \in L^1(\mathbb{R})$ and $F_z(s) = e^{2\pi \mathrm{i} sz} F(s)$.*
- *By $g(t) = |r| f(rt)$, $t \in \mathbb{R}$, follows $g \in L^1(\mathbb{R})$ and $G(s) = F\left(\frac{1}{r} s\right)$.*

The usage of Proposition 2.2.2 asserts the following version of the Poisson summation formula.

$$\sum_{n \in \mathbb{Z}} f(t + nP) = \sum_{k \in \mathbb{Z}} \frac{1}{P} F\left(\frac{k}{P}\right) e^{\mathrm{i} \frac{2\pi k}{P} t} \, , \tag{2.2.24}$$

where $P > 0$ represents the period.

2.2.6 Euler summation

Following [AW92], Euler summation is used to accelerate the convergence of alternating series. It is defined in terms of finite differences. In case of alternating series, Euler summation is described as the average of the last m partial sums weighted by a binomial probability with parameters m and $p = \frac{1}{2}$. The Euler sum applied to m terms after initial n terms is interpreted as

$$E(t, m, n) = \sum_{k=0}^{m} \binom{m}{k} 2^{-m} s_{n+k}(t) \, , \tag{2.2.25}$$

where s_n is the nth partial sum given by

$$s_n(t) = \sum_{k=0}^{n} (-1)^k a_k(t), \tag{2.2.26}$$

where $a_k(t)$ is the kth summand of the approximated infinite series.

2.2.7 Convolutions

The achievements in this section are based on [Kal13, Kus11].

Definition 2.2.7 (Convolution). *The convolution of two functions $f, g \in L^1(\mathbb{R}^d)$ given by $f * g : \mathbb{R}^d \to \mathbb{C}$ is defined by*

$$f * g(x) = \int_{\mathbb{R}^d} f(x-y)g(y)dy, \ x \in \mathbb{R}^d . \tag{2.2.27}$$

One important fact regarding convolutions is the following.

Theorem 2.2.2. *Let X_1 and X_2 be two independent random variables with continuous density functions f_{X_1} and f_{X_2} respectively. Then the sum $Z = X_1 + X_2$ is a random variable with density function f_Z, where f_Z is the convolution of f_{X_1} and f_{X_2}, meaning that*

$$\begin{aligned} f_Z(s) = f_{X_1+X_2}(s) &= \int_{-\infty}^{\infty} f_{X_1}(s-x_2)f_{X_2}(x_2)dx_2 \\ &= \int_{-\infty}^{\infty} f_{X_2}(s-x_1)f_{X_1}(x_1)dx_1 . \end{aligned} \tag{2.2.28}$$

Proof. Let $Z = X_1 + X_2$. Then

$$\begin{aligned} \mathbb{P}(X_1 + X_2 \leq z) &= \int \int_{x_1+x_2 \leq z} f_{X_1}(x_1)f_{X_2}(x_2)dx_1dx_2 \\ &= \int_{-\infty}^{\infty} \Big(\int_{-\infty}^{z-x_2} f_{X_1}(x_1)dx_1 \Big) f_{X_2}(x_2)dx_2 \\ &= \int_{-\infty}^{\infty} \Big(\int_{-\infty}^{z} f_{X_1}(s-x_2)ds \Big) f_{X_2}(x_2)dx_2 \\ &\overset{\text{Fubini}}{=} \int_{-\infty}^{z} \underbrace{\Big(\int_{-\infty}^{\infty} f_{X_1}(s-x_2)f_{X_2}(x_2)dx_2 \Big)}_{f_Z(s)} ds . \end{aligned} \tag{2.2.29}$$

■

2.2.8 Discrete and continuous probability distributions

This subsection is mainly based on [Gur11]. Two distributions, starting with the discrete one, are introduced.

Definition 2.2.8 (Poisson distribution). *A $\mathbb{N}$-valued random variable X is said to have a Poisson distribution with parameter $\lambda > 0$ if*

$$\mathbb{P}(X = x) = \frac{\lambda^x e^{-\lambda}}{x!} \textit{ for all } x \in \mathbb{N} . \tag{2.2.30}$$

One of the most important continuous distributions for continuous-time Markov chains is the exponential distribution since it is commonly used to model waiting times between occurrences.

Definition 2.2.9 (Exponential distribution). *A continuous random variable X is said to have an exponential distribution with parameter $\lambda > 0$ if the probability density function is given by*

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0\,. \end{cases} \tag{2.2.31}$$

One relevant fact is that the minimum of independent exponentially distributed random variables is again exponentially distributed.

Theorem 2.2.3. *Let $X_1, X_2, \ldots, X_n$ be independent exponentially distributed random variables with parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then the minimum*

$$\min\{X_1, X_2, \ldots, X_n\} \tag{2.2.32}$$

is again exponentially distributed with parameter $\lambda = \lambda_1 + \lambda_2 + \ldots + \lambda_n$.

Proof. The random variable X_i has the distribution function $F_{X_i}(x) = \mathbb{P}(X_i \leq x) = 1 - e^{-\lambda_i x}$, for all $i = 1, \ldots, n$. Let now $Y = \min\{X_1, X_2, \ldots, X_n\}$, then the complementary distribution function is given by

$$\begin{aligned} \mathbb{P}(Y > x) &= \mathbb{P}(X_1 > x, X_2 > x, \ldots, X_n > x) \\ &\overset{X_i \text{ are ind.}}{=} \mathbb{P}(X_1 > x)\mathbb{P}(X_2 > x) \cdots \mathbb{P}(X_n > x) \\ &= e^{-\lambda_1 x} e^{-\lambda_2 x} \cdots e^{-\lambda_n x} \\ &= e^{-(\lambda_1 + \lambda_2 + \ldots + \lambda_n)x} \\ &= e^{-\lambda x}\,, \end{aligned} \tag{2.2.33}$$

where $\lambda = \lambda_1 + \lambda_2 + \ldots + \lambda_n$.

■

2.2.9 Markov process

All achievements in this subsection are mainly based on the researches of [Con11, Kle11, Rhe13, Ros83].

Definition 2.2.10 (Stochastic process). *A family of random variables $(X_t)_{t \in T}$, $X_t : \Omega \to \mathbb{R}^d$, $t \in T$ is called a $\mathbb{R}^d$-valued stochastic process.*

If T is countable, then $(X_t)_{t \in T}$ is called a *discrete-time process*. Whereas, if T is a sub-interval of $[0, \infty)$, then the process is called *time-continuous*.

Definition 2.2.11 (Markov chain). *A stochastic process $(X_n)_{n\in\mathbb{N}}$ is called a discrete-time Markov chain if for all times $n \in \mathbb{N}$ and states $(i_0, i_1, .., i_n) \in S$*

$$\mathbb{P}(X_n = i_n | X_0 = i_0, .., X_{n-1} = i_{n-1}) = \mathbb{P}(X_n = i_n | X_{n-1} = i_{n-1}) \quad (2.2.34)$$

holds, where $S = \mathbb{Z}$ is the set of values of X and called state space.

In other words the future depends only on the present state without consulting past states.

The equation in Definition 2.2.11 is called *Markov property* and

$$\mathbb{P}(X_n = j | X_{n-1} = i), \; i, j \in S = \mathbb{Z} \quad (2.2.35)$$

is said to be the *transition probability* from state i to state j, denoted by $p_{i,j}(n-1, n)$.

A discrete-time Markov chain is called time-homogeneous if $p_{i,j}(n-1, n)$ does not depend on n. In other words if

$$\mathbb{P}(X_n = j | X_{n-1} = i) = \mathbb{P}(X_{n+m} = j | X_{n+m-1} = i) \quad (2.2.36)$$

for $m \in \mathbb{N}$ holds.

Counting process

A counting process $\{N_t | t \geq 0\}$ is a stochastic process such that N_t represents the number of events occurred in the interval $[0, t)$ with initial value $N_0 = 0$. Obviously, N_t is non-negative and integer-valued for all $t \geq 0$.

Poisson process

The Poisson process is one of the most important models used in the queuing theory. It is a specific counting process with Poisson distributed increments [Ros83].

Definition 2.2.12 (Poisson-Process). *A stochastic process with $c\grave{a}dl\grave{a}g$[2] paths on a probability space $[\Omega, \mathfrak{A}, \mathbb{P}]$ is called (homogeneous) Poisson-Process $P_{\lambda,t}$ with intensity $\lambda > 0$ and $t \in [0, \infty)$ if the following holds:*

- $P_{\lambda,0} = 0, \; \mathbb{P} - a.s.$
- $P_{\lambda,t} - P_{\lambda,s} \sim \mathcal{P}_{\lambda,t-s}$ *for all $s < t$, where $\mathcal{P}_{\lambda,t-s}$ stands for the Poisson distribution with parameter $\lambda \cdot (t - s)$.*
- *If $t_1 < t_2 < .. < t_n, n \in \mathbb{N}$, then $P_{\lambda,t_i} - P_{\lambda,t_{i-1}}$ is stochastically independent for every $i, \; 2 \leq i \leq n$.*

[2]right-continuous and the left-hand limit exists

Remark 2.2.2 (Properties of the Poisson process). *A few properties of the Poisson process are given*

1. *A homogeneous Poisson process is a Markov process.*
2. *The interarrival times, these are the times between any two increments, are exponentially distributed with parameter* λ*.*
3. *If* $P_{\lambda,t}$ *is a Poisson process, then* $\hat{P}_{\lambda,t} = P_{\lambda,t+s} - P_{\lambda,s}$ *is a Poisson process for all* $0 < s < t$*, this means that the increments of a homogeneous Poisson process are stationary.*
4. *For every* λ *and* t*,* $\mathbb{E}(P_{\lambda,t}) = \mathbb{V}(P_{\lambda,t}) = \lambda \cdot t$ *holds.*

Continuous-time Markov chains and the Kolmogorov forward and backward equations

A continuous-time Markov process with discrete state space is called continuous-time Markov chain.

Definition 2.2.13. *Let* $(X_t)_{t\geq 0}$ *be a continuous-time Markov chain, then formula (2.2.34) is given by*

$$\mathbb{P}(X_t = j | X_{t_1} = i_1, X_{t_2} = i_2, .., X_{t_n} = i_n) = \mathbb{P}(X_t = j | X_{t_n} = i_n)\,, \quad (2.2.37)$$

where $n > 1$*,* $0 < t_1 < t_2 < \ldots < t_n < t$ *and* $(i_1, i_2, \ldots, i_n, j) \in S$*.*

Define

$$\begin{aligned} p_{i,j}(s, t+s) &= \mathbb{P}(X_{t+s} = j | X_s = i) \quad \text{and} \\ p_{i,j}(0, t) &= \mathbb{P}(X_t = j | X_0 = i) = p_{i,j}(t)\,. \end{aligned} \quad (2.2.38)$$

The probability of moving from state i at time u to state j at t is given by

$$p_{i,j}(u, t) = \sum_k p_{i,k}(u, s) \cdot p_{k,j}(s, t), \quad u < s < t\,. \quad (2.2.39)$$

Definition 2.2.14 (Landau symbol). *The Landau symbol* $o(t)$ *is a function such that*

$$\lim_{t \searrow 0} \frac{o(t)}{t} = 0\,. \quad (2.2.40)$$

Furthermore, the transition rates are defined as follows.

Definition 2.2.15.

$$q_{i,j}(t) = \lim_{h \searrow 0} \frac{p_{i,j}(t, t+h)}{h}\,. \quad (2.2.41)$$

Both, $1 - p_{i,i}(t, t + dt) = q_i(t)dt + o(dt)$ and $p_{i,j}(t, t + dt) = q_{i,j}(t)dt + o(dt)$, are used to get the Kolmogorov forward and backward equations[3].

Kolmogorov forward equation with initial condition $p_{i,j}(u, u) = \delta_{i,j}$:

$$\frac{\partial}{\partial t} p_{i,j}(u, t) = \sum_{g, g \neq j} p_{i,g}(u, t) q_{g,j}(t) - p_{i,j}(u, t) q_j(t), \;\; u < t \,. \tag{2.2.42}$$

Kolmogorov backward equation with end condition $p_{i,j}(t, t) = \delta_{i,j}$:

$$\frac{\partial}{\partial u} p_{i,j}(u, t) = q_i(u) p_{i,j}(u, t) - \sum_{g, g \neq i} q_{i,g}(u) p_{g,j}(u, t), \;\; u < t \,. \tag{2.2.43}$$

2.3 Extension of the continuous-time Markov chain: The birth-and-death process

This section provides the reader with an important example of a continuous-time Markov chain. This process is called birth-and-death process. In the following, the birth-and-death process is explained and afterwards transition probabilities regarding this process are calculated.
Most of the achievements are based on the researches of [Con11, Cra12, Mit07, Rhe13].

2.3.1 Introduction

The birth-and-death process is a stochastic process with the property that the net change across an infinitesimal time interval dt is either -1 (death), 0 or 1 (birth) and the state i signifies the current size of population. When a birth occurs, the process goes from state i to $i + 1$ and in the case of a death from i to $i - 1$. Besides, it is assumed that the birth and death events are independent of each other.

Such a process can be explained as follows. After the process enters state i, it holds there for some random length of time, exponentially distributed with parameter $(\lambda_i + \mu_i)$. When leaving i, the process goes to state $i+1$ with probability

$$\frac{\lambda_i}{\lambda_i + \mu_i} \tag{2.3.1}$$

or otherwise to state $i - 1$ with probability

$$\frac{\mu_i}{\lambda_i + \mu_i} \,. \tag{2.3.2}$$

[3] Instead of $\mu_{i,j}(t)$ as in [Rhe13], $q_{i,j}(t)$ is used.

Assume that $T_B(i)$ and $T_D(i)$ are two exponentially distributed random variables with parameters λ_i and μ_i respectively describing the holding time in state i. $T_B(i)$ can be seen as the time until a birth and $T_D(i)$ as the time until a death occurs. A transition from state i to state $i+1$ is made if $T_B(i) < T_D(i)$ holds. This event occurs with probability

$$\begin{aligned}\mathbb{P}(T_B(i) < T_D(i)) &= \mathbb{P}(T_B(i) - T_D(i) < 0) \\ &= \int_0^\infty \int_0^y \lambda_i e^{-\lambda_i x} \mu_i e^{-\mu_i y} dx dy \\ &= \int_0^\infty e^{-\mu_i y} \mu_i dy - \int_0^\infty e^{-(\lambda_i + \mu_i) y} \mu_i dy \\ &= 1 - \frac{\mu_i}{\lambda_i + \mu_i} = \frac{\lambda_i}{\lambda_i + \mu_i} \,.\end{aligned} \tag{2.3.3}$$

Figure 2.3.1 illustrates the process described above.

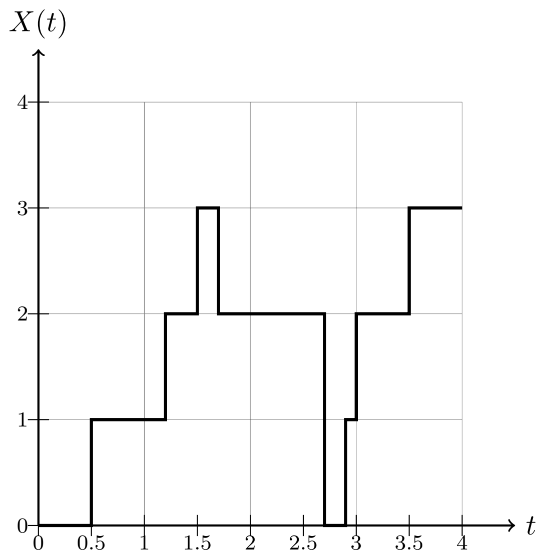

Figure 2.3.1: Counting process interpretations for a birth-and-death process, where the up movements represent the births and the down movements signify the deaths. The holding times are illustrated by the horizontal lines.

The time when the process first crosses the threshold $i+1$ in case of a birth or $i-1$ in the event of a death is called first-passage time.

Definition 2.3.1 (First-passage time). *Given a stochastic process $(X_t)_{t\geq 0}$ with state space $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$. If $X_s = i$, the first-passage time to some point $j \neq i \in \mathbb{Z}^+$ is specified with*

$$\tau_{i,j} = \inf\{t : X_t = j\}, \; s < t \,. \tag{2.3.4}$$

2.3.2 Transition probabilities

Given a state i, the transition probabilities under the assumption $\mu_0 = \lambda_{-1} = 0$ are given by

$$p_{i,j}(t, t+dt) = \begin{cases} \lambda_i(t)dt + o(dt) & \text{if } j = i+1 \\ \mu_i(t)dt + o(dt) & \text{if } j = i-1 \\ 1 - (\lambda_i(t) + \mu_i(t))dt + o(dt) & \text{if } j = i \\ o(dt) & \text{otherwise}. \end{cases}$$

Therefore, the transition rates are defined by

$$q_{i,j}(t) = \begin{cases} \lambda_i(t) & \text{if } j = i+1 \\ \mu_i(t) & \text{if } j = i-1 \\ 1 - (\lambda_i(t) + \mu_i(t)) & \text{if } j = i \\ 0 & \text{otherwise}. \end{cases}$$

Now, the above defined probabilities and the Kolmogorov forward equation are used to derive the differential equation for the probability $p_{i,j}(s,t)$ with initial condition $p_{i,j}(s,s) = 0$ if $i \neq j$ and $p_{i,j}(s,s) = 1$ for $i = j$

$$\frac{\partial p_{i,j}(s,t)}{\partial t} = -p_{i,j}(s,t)(\lambda_j(t) + \mu_j(t)) + p_{i,j+1}(s,t)\mu_{j+1}(t) + p_{i,j-1}(s,t)\lambda_{j-1}(t)\,. \tag{2.3.5}$$

For visualization, Figure 2.3.2 illustrates a drawing of the Kolmogorov forward equation.

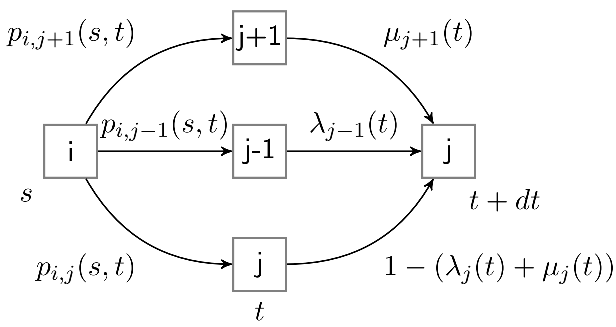

Figure 2.3.2: Kolmogorov forward equation for $p_{i,j}(s,t)$.

In a more general way, using the transition matrix Q,

$$Q = \begin{pmatrix} -\lambda_0 & -\lambda_0 & 0 & 0 & \dots \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & 0 & \dots \\ 0 & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{2.3.6}$$

(2.3.5) can be written as $\frac{\partial P(s,t)}{\partial t} = QP(s,t)$ with initial condition $P(s,s) = I$. Q is also called *generator matrix*.

2.3.3 A simple linear birth-and-death process

For a simple linear birth-and-death process $(X_t)_{t\geq 0}$, meaning that $\lambda_k = k\lambda$ and $\mu_k = k\mu$, it is possible to solve the transition probabilities explicitly by finding a probability generating function solution to the Kolmogorov forward equation [Cra12].

According to [Bai64, Cra12, Mor79], the calculation of the solution is illustrated. Therefore, let

$$G_a(s,t) = \sum_{k=0}^{\infty} s^k p_{a,k}(t)\,, \tag{2.3.7}$$

where $G_a(s,t)$ denotes the probability generating function of $(X_t)_{t\geq 0}$. Putting $i = a$, $j = k$ and $p_{a,k}(0,t) = p_{a,k}(t)$, multiplying both sides with s^k and summing over k in (2.3.5) gives

$$\begin{aligned}
\frac{\partial G_a(s,t)}{\partial t} &= \sum_{k=0}^{\infty} s^k \frac{dp_{a,k}(t)}{dt} \overset{(2.3.5)}{=} \sum_{k=0}^{\infty} s^k(-(\lambda_k(t) + \mu_k(t))p_{a,k}(t) \\
&\qquad + \mu_{k+1}(t)p_{a,k+1}(t) + \lambda_{k-1}(t)p_{a,k-1}(t) \\
&\overset{\lambda_k=\lambda k, \mu_k=\mu k}{=} -\sum_{k=0}^{\infty} s^k(\lambda k + \mu k)p_{a,k}(t) + \sum_{k=0}^{\infty} s^k \mu(k+1)p_{a,k+1}(t) \\
&\qquad + \sum_{k=1}^{\infty} s^k \lambda(k-1)p_{a,k-1}(t) \\
&= -(\lambda+\mu)s\sum_{k=0}^{\infty} ks^{k-1}p_{a,k}(t) + \mu\sum_{k=0}^{\infty}(k+1)s^k p_{a,k+1}(t) \\
&\qquad + \lambda s^2 \sum_{k=1}^{\infty}(k-1)s^{k-2}p_{a,k-1}(t) \\
&= -(\lambda+\mu)s\sum_{k=1}^{\infty} ks^{k-1}p_{a,k}(t) + \mu\sum_{k=1}^{\infty} ks^{k-1}p_{a,k}(t) \\
&\qquad + \lambda s^2 \sum_{k=1}^{\infty} ks^{k-1}p_{a,k}(t)\,.
\end{aligned} \tag{2.3.8}$$

Note that

$$\frac{\partial G_a(s,t)}{\partial s} = \frac{\partial}{\partial s}\sum_{k=0}^{\infty} p_{a,k}(t)s^k = \sum_{k=1}^{\infty} kp_{a,k}(t)s^{k-1} \tag{2.3.9}$$

and thus,

$$\frac{\partial G_a(s,t)}{\partial t} = (\lambda s - \mu)(s-1)\frac{\partial G_a(s,t)}{\partial s} \tag{2.3.10}$$

with the initial condition[4] $G_a(s,0) = s^a$.

Before continuing with the calculation of the transition probabilities, a short excursion on Lagrange's method for solving partial-differential equations is done [Eng80].

Excursion on Lagrange's equation

A linear partial-differential equation of first order, involving a dependent variable z and two independent variables x and y, of the form

$$P(x,y,z)z_x + Q(x,y,z)z_y = R(x,y,z)\,, \tag{2.3.11}$$

where P, Q and R are functions of x, y and z, is called Lagrange's linear equation. Using this, a solution can be found by the following steps:

1. Write the auxiliary equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.
2. Find any two independent solutions of the auxiliary equations and let the two solutions be $u(x,y,z) =$ constant and $v(x,y,z) =$ constant.
3. The general solution of Lagrange's equation is $u = \phi(v)$ or $\phi(u,v) = 0$, where ϕ is a differentiable arbitrary function.

Rearranging formula (2.3.10) gives

$$\frac{\partial G_a(s,t)}{\partial t} - (\lambda s - \mu)(s-1)\frac{\partial G_a(s,t)}{\partial s} = 0\,. \tag{2.3.12}$$

Therefore, it can be solved by Lagrange's method for partial-differential equations and the following auxiliary equations can be obtained

$$\frac{dt}{P} = \frac{ds}{Q} = \frac{dG_a}{R} \rightarrow \frac{dt}{1} = -\frac{ds}{(\lambda s - \mu)(s-1)} = \frac{dG_a}{0}\,. \tag{2.3.13}$$

For the first equation, separation of variables yields

$$\int -\frac{1}{(\lambda s - \mu)(s-1)}ds = \int 1\,dt\,. \tag{2.3.14}$$

Integration of both sides results in

$$\frac{\log(\lambda s - \mu) - \log(s-1)}{\lambda - \mu} = t + c\,, \tag{2.3.15}$$

[4] This follows from $p_{a,k}(0) = 0,$ for all $k \neq a$.

where finally mathematical conversion yields

$$\log\left(\frac{\lambda s-\mu}{s-1}\right)=t(\lambda-\mu)+c\,. \tag{2.3.16}$$

Thus, the first solution is given by

$$v(t,s,G_a)=\frac{\lambda s-\mu}{s-1}e^{-t(\lambda-\mu)}\,. \tag{2.3.17}$$

The second one results from

$$\int 0\,dt=\int 1\,dG_a \tag{2.3.18}$$

and is constituted by

$$u(t,s,G_a)=G_a\,. \tag{2.3.19}$$

Hence, the general solution of Lagrange's equation is

$$G_a(s,t)=\phi\left(\frac{\lambda s-\mu}{s-1}e^{-t(\lambda-\mu)}\right)\,. \tag{2.3.20}$$

Putting $t=0$ and assuming that $a=1$ in (2.3.20) as well as in (2.3.7) delivers

$$G_1(s,0)=\phi\left(\frac{\lambda s-\mu}{s-1}\right)=\sum_{k=1}^{\infty}p_{1,k}(0)s^k=s^1=s\,. \tag{2.3.21}$$

Setting

$$x=\frac{\lambda s-\mu}{s-1}\,, \tag{2.3.22}$$

gives

$$s=\frac{x-\mu}{x-\lambda} \tag{2.3.23}$$

and thus,

$$\phi(x)=\frac{x-\mu}{x-\lambda}\,. \tag{2.3.24}$$

Therefore,

$$G_1(s,t)=\frac{(\lambda s-\mu)e^{-t(\lambda-\mu)}-\mu(s-1)}{(\lambda s-\mu)e^{-t(\lambda-\mu)}-\lambda(s-1)}\,. \tag{2.3.25}$$

In general, by the initial condition $G_a(s,0) = s^a$ one gets

$$G_a(s,t) = \left(\frac{(\lambda s - \mu)e^{-t(\lambda-\mu)} - \mu(s-1)}{(\lambda s - \mu)e^{-t(\lambda-\mu)} - \lambda(s-1)} \right)^a . \tag{2.3.26}$$

According to [Git10], the transition probabilities $p_{a,b}(t)$ are received by finding the bth coefficient of $G_a(s,t)$, that is

$$p_{a,b}(t) = [s^b]G_a(s,t) . \tag{2.3.27}$$

Therefore, (2.3.26) is first converted to obtain a fraction of the form

$$\left(\frac{As + B}{Cs + D} \right)^a , \tag{2.3.28}$$

where

$$\begin{aligned} A = (\lambda e^{-t(\lambda-\mu)} - \mu),\ B = (\mu - \mu e^{-t(\lambda-\mu)}), \\ C = (\lambda e^{-t(\lambda-\mu)} - \lambda) \text{ and } D = (\lambda - \mu e^{-t(\lambda-\mu)}) . \end{aligned} \tag{2.3.29}$$

Next, the binomial theorem yields

$$\begin{aligned} \left(\frac{As + B}{Cs + D} \right)^a &= (As + B)^a (Cs + D)^{-a} \\ &= \sum_{j \geq 0} \sum_{m \geq 0} \binom{a}{j} \binom{-a}{m} s^j A^j s^m C^m B^{a-j} D^{-a-m} \\ &\overset{l=m+j}{=} \sum_{j \geq 0} \sum_{l \geq j} \binom{a}{j} \binom{-a}{l-j} s^j s^{l-j} A^j C^{l-j} B^{a-j} D^{-a-l+j} . \end{aligned} \tag{2.3.30}$$

By deducing the bth coefficient, one obtains

$$\begin{aligned} [s^b]G_a(s,t) &= [s^b] \sum_{j \geq 0} \sum_{l \geq j} \binom{a}{j} \binom{-a}{l-j} s^l A^j C^{l-j} B^{a-j} D^{-a-l+j} \\ &= \sum_{j \geq 0} \sum_{j \leq l} [s^b] s^l \binom{a}{j} \binom{-a}{l-j} A^j C^{l-j} B^{a-j} D^{-a-l+j} \\ &\overset{0 \text{ for } b \neq l}{=} \sum_{j \geq 0} \sum_{j \leq l, l=b} [s^b] s^l \binom{a}{j} \binom{-a}{l-j} A^j C^{l-j} B^{a-j} D^{-a-l+j} , \end{aligned} \tag{2.3.31}$$

where $[s^b]$ is a linear and continuous operator relative to the formal topology [Git10].

For $j > b$ the sum is empty and therefore, one gets

$$\begin{aligned} [s^b]G_a(s,t) &= \sum_{0\le j\le b} [s^b]s^b \binom{a}{j}\binom{-a}{b-j} A^j C^{b-j} B^{a-j} D^{-a-b+j} \\ &= \sum_{0\le j\le b} \binom{a}{j}\binom{-a}{b-j} A^j C^{b-j} B^{a-j} D^{-a-b+j} \,. \end{aligned} \tag{2.3.32}$$

Since $a \in \mathbb{N}$ implies $\binom{a}{j} = 0$ for all $j > a$, the following is obtained

$$\begin{aligned} p_{a,b}(t) = [s^b]G_a(s,t) &= \sum_{j=0}^{\min\{a,b\}} \binom{a}{j}\binom{-a}{b-j} A^j C^{b-j} B^{a-j} D^{-a-b+j} \\ &= \sum_{j=0}^{\min\{a,b\}} \binom{a}{j}\binom{-a}{b-j} (\lambda e^{-t(\lambda-\mu)} - \mu)^j \\ &\quad (\lambda e^{-t(\lambda-\mu)} - \lambda)^{b-j} (\mu - \mu e^{-t(\lambda-\mu)})^{a-j} (\lambda - \mu e^{-t(\lambda-\mu)})^{-a-b+j} \,. \end{aligned} \tag{2.3.33}$$

Finally, one uses the relations

$$\begin{aligned} &\binom{-a}{b-j} = (-1)^{b-j}\binom{a+b-j-1}{b-j} \text{ and} \\ &\binom{a+b-j-1}{b-j} = \binom{a+b-j-1}{a-1} \end{aligned} \tag{2.3.34}$$

to see

$$\begin{aligned} p_{a,b}(t) = &\sum_{j=0}^{min\{a,b\}} \binom{a}{j}\binom{a+b-j-1}{a-1} (-1)^{b-j} (\lambda e^{-t(\lambda-\mu)} - \mu)^j \\ &(\lambda e^{-t(\lambda-\mu)} - \lambda)^{b-j} (\mu - \mu e^{-t(\lambda-\mu)})^{a-j} (\lambda - \mu e^{-t(\lambda-\mu)})^{-a-b+j} \,. \end{aligned} \tag{2.3.35}$$

Usually, it is more difficult to solve general birth-and-death processes. A useful method for finding the transition probabilities $p_{a,b}(t)$ is to apply the Laplace transform to both sides of the forward equation (2.3.5). This has the effect of turning the infinite system of differential equations into a recurrence relation [Cra12].

Hereafter, this thesis will focus on a birth-and-death process introduced by [CST10].

CHAPTER 3

The birth-and-death Markov chain model

This chapter contains achievements which are mainly based on the researches of [AW92, AW95, CST10, FRVZ14, HK12]. First, the reader is provided with a short description of the LOB and the corresponding dynamics. Next, some preparation regarding the computation of the probability of a mid-price increase such as parameter estimations and model implementations are done. Finally, the probability of a mid-price increase at its next move is calculated, which in the end is compared to the frequencies achieved from the ISE-30 data.

3.1 Limit order books

A *limit order* represents the maximum or minimum price at which an investor is willing to buy or sell. It is posted to an electronic trading system. The quantity of all outstanding limit orders can be summarized by counting the quantities at each price level. The price which exceeds all other buy prices is called the bid price, whereas the lowest price for which there is an outstanding limit sell order is called the ask price. The unexecuted limit orders are waiting for execution in the LOB.

If a new order appears, two possible scenarios can occur: The order is matched with the best available price in the LOB (*market order*) or adds to the book and sits there until it is executed against a market order or it is canceled (*cancellation order*)[1]. Whenever an order on the opposite side of the book hits the quote, executions occur based on priority rules[2]. Therefore, the price dynamics are the result of the interplay between the order book and the order flow [FRVZ14].

[1] Note that there were no cancellation orders at ISE until 2010 to ensure the liquidity in the market. In ISE, the system cancels all waiting orders by the end of the day.

[2] Price priority means that the limit orders with better prices will be executed before the worse ones and time priority implies a first-in-first-out principle.

3.1.1 Model framework and notation

Before introducing the birth-and-death Markov chain model, some preparations have to be done. This section is mainly based on [CST10, HK12].

Remark 3.1.1. *Limit orders are placed on a finite price grid, $\Pi = \{1, 2, .., n\}$, representing prices measured in multiples of a tick, where n is chosen large enough such that the probability of placing an order further away is highly unlikely.*

Definition 3.1.1. *The state of the LOB is described by two $\mathbb{Z}^+$ continuous-time processes A and B with*

$$(A(t))_{t\geq 0} = (A_1(t), A_2(t), ..., A_n(t))_{t\geq 0} \tag{3.1.1}$$

representing the sell process and

$$(B(t))_{t\geq 0} = (B_1(t), B_2(t), ..., B_n(t))_{t\geq 0} \tag{3.1.2}$$

representing the buy process, where $A_p(t)$ is the number of outstanding limit sell orders for a price p, $p \in \Pi$ at time $t \geq 0$, the same holds for $B_p(t)$.

Remark 3.1.2. *Note that $A_p(t) \wedge B_p(t) = 0,$ for all $p \in \Pi$ and $t \geq 0$, meaning that buy and sell orders at the same price are not possible, because otherwise a trade would take place.*

Definition 3.1.2 (Ask-, Bid-price). *The ask price $p_A(t)$ at time t is defined by*

$$p_A(t) = \inf\{p \in \Pi | A_p(t) > 0\} \wedge (n+1)\,. \tag{3.1.3}$$

Similarly, for the bid price $p_B(t)$

$$p_B(t) = \sup\{p \in \Pi | B_p(t) > 0\} \vee 0 \tag{3.1.4}$$

holds.

Remark 3.1.3. *If at time t there is not any ask order in the book, $p_A(t) = n+1$ and if no bid orders exist, $p_B(t) = 0$. Additionally, $p_B(t) < p_A(t)$ for all t holds.*

Definition 3.1.3 (Mid-price). *The mid-price $p_M(t)$ at time t is defined by*

$$p_M(t) = \frac{p_A(t) + p_B(t)}{2}\,. \tag{3.1.5}$$

Definition 3.1.4 (Bid-ask spread). *The bid-ask spread $p_S(t)$ at time t is defined by*

$$p_S(t) = p_A(t) - p_B(t)\,. \tag{3.1.6}$$

The number of outstanding orders at a certain distance from the bid/ask can be defined as follows.

Definition 3.1.5 (Outstanding volume). *At time $t \geq 0$, the number of outstanding buy orders placed at a distance δ from the ask is defined by*

$$Q_\delta^B(t) = \begin{cases} B_{p_A(t)-\delta}(t) & \text{if } 0 < \delta < p_A(t) \\ 0 & \text{if } p_A(t) \leq \delta \end{cases} \tag{3.1.7}$$

and

$$Q_\delta^A(t) = \begin{cases} A_{p_B(t)+\delta}(t) & \text{if } 0 < \delta \leq n - p_B(t) \\ 0 & \text{if } n - p_B(t) < \delta \end{cases} \tag{3.1.8}$$

represents the number of sell orders placed at a distance δ from the bid.

Remark 3.1.4. *At time $t \geq 0$, $(A(t), B(t))$ and $(p_A(t), p_B(t), Q^A(t), Q^B(t))$ contain the same information, however, the second representation highlights the depth of the order book relative to the opposite best quotes.*

3.1.2 Limit order book dynamics

An update of the LOB at some sell price p at time t is defined by

$$A_{p+1}(t) = A(t) \pm (0, \ldots, 1, \ldots, 0),\ A \in (\mathbb{Z}^+)^n, t \geq 0, p \in \Pi\,. \tag{3.1.9}$$

Equally, for buy prices p an update is given by

$$B_{p+1}(t) = B(t) \pm (0, \ldots, 1, \ldots, 0),\ B \in (\mathbb{Z}^+)^n, t \geq 0, p \in \Pi\,, \tag{3.1.10}$$

where the 1 in the vector is at the pth position. Hence, only orders of unit size[3] are assumed [CST10].
Next, the following assumptions hold[4]:

- At time t, a limit sell order at price level $p > p_B(t)$ increases the quantity at level p: $A(t) \to A_{p+1}(t)$.
- At time t, a limit buy order at price level $p < p_A(t)$ increases the quantity at level p: $B(t) \to B_{p+1}(t)$.
- At time t, a market sell order decreases the quantity at the bid price: $B(t) \to B_{p_B-1}(t)$
- At time t, a market buy order decreases the quantity at the ask price: $A(t) \to A_{p_A-1}(t)$.

Following [CST10], one assumes that these above mentioned events are modeled using independent Poisson processes, meaning that the interarrival times are exponentially distributed (cf. Definition 2.2.12, Remark 2.2.2).

[3] The average size of limit orders is taken as unit.
[4] Different from [CST10, HK12], there is no need to model dynamics for cancellation orders.

More precisely, for $\delta \geq 1$

- Limit buy orders arrive at distance δ from the opposite best quote at independent exponential times with rate $\lambda^{\text{buy}}(\delta)$ and limit sell orders with rate $\lambda^{\text{sell}}(\delta)$.

and for $\delta \leq 0$

- Market orders arrive at independent exponential times with rate μ.

Note that the described events are independent.

Given these assumptions, A and B are continuous-time Markov chains with state space $(\mathbb{Z}^+)^n$ and the following transition rates at time $t \geq 0$:

- $A(t) \to A_{p+1}(t)$ with rate $\lambda^{\text{sell}}(p - p_B(t))$ for $p > p_B(t)$.
- $B(t) \to B_{p+1}(t)$ with rate $\lambda^{\text{buy}}(p_A(t) - p)$ for $p < p_A(t)$.
- $A(t) \to A_{p_A-1}(t)$ with rate μ.
- $B(t) \to B_{p_B-1}(t)$ with rate μ.

Before assuming that the order arrivals are Poisson processes, it should be checked if this assumption is adequate for the data used in this thesis. Therefore, the interarrival times are tested for an exponential distribution (cf. Remark 2.2.2, second item).

If a random variable X is exponentially distributed with parameter $\lambda > 0$,

$$\mathbb{P}(X > s) = e^{-\lambda s} \Leftrightarrow log(\mathbb{P}(X > s)) = -\lambda s \tag{3.1.11}$$

holds (cf. Definition 2.2.9). This directly implies that the logarithm of the empirical probability is linear proportional to s.

For checking the assumption above, the calculation in Section 2.1.3 is used. Figure 3.1.1 shows a good outcome since the logarithm of the empirical probability has a linear tendency in respect to s. However, focusing on smaller s values and checking the probability density function instead of the distribution function again indicates acceptable results regarding the distribution assumptions (cf. Figure 3.1.2). These results are different from those observed in [ACD$^+$09].

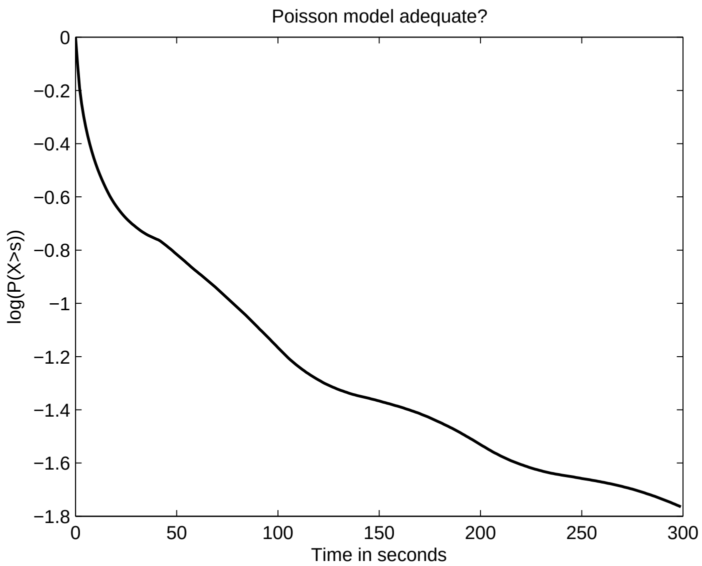

Figure 3.1.1: Distribution of the interarrival times, $s = 0, 1, \ldots, 300$, containing more than 80% of the data. Thus, this limit was considered as appropriate.

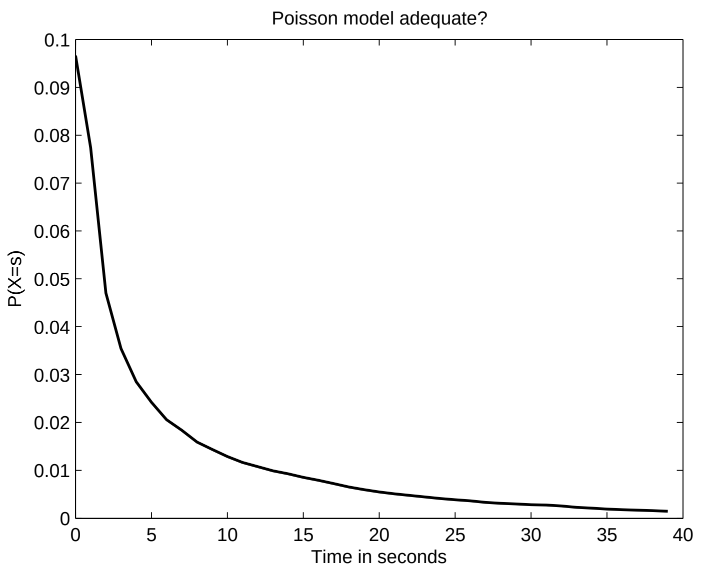

Figure 3.1.2: Probability density function of the interarrival times, $s = 0, 1, \ldots, 40$, containing more than 50% of the data.

Empirical estimates exhibit that the incoming order arrival rates depend on the distance to the bid/ask price, meaning that the incoming orders arrive more frequently the closer they are to the current bid/ask price [BMP02, CST10, FRVZ14]. This phenomenon will be analyzed in Section 3.2.

According to [HK12], a κ-truncated model is defined instead of using an infinite state space $\mathbb{Z}^+$ at each price level. This is because, in reality, the number of shares in an order book are bounded by a finite number κ.

Remark 3.1.5. *For the κ-truncated model, $\mathbb{Z}_\kappa = \{0, 1, \ldots, \kappa\}$ represents the number of buy or sell orders at a given price. Therefore, the continuous-time Markov chains, $A^{(\kappa)}$ and $B^{(\kappa)}$, are defined on the state space $(\mathbb{Z}_\kappa)^n$ with the same transition rates, $\lambda^{buy}(\delta), \lambda^{sell}(\delta)$ and μ as A and B, except for the following: If a transition would cause a transgression of κ, the state is reset to κ and the rate of incoming limit orders $\lambda^{buy}(\delta)/\lambda^{sell}(\delta)$ at state κ is reset to zero.*

3.2 Parameter estimation

In this model, orders of unit size are assumed (cf. Section 2.1.2, sixth step). The parameters $\lambda^{\text{buy}}(\delta), \lambda^{\text{sell}}(\delta)$ and μ have to be estimated from the order book data.

For calibrating these parameters, the calculations of the limit and market order arrival rates defined in Section 2.1.1 and Section 2.1.2 are used.

The arrival rate function of the limit buy orders can be estimated by

$$\hat{\lambda}^{\text{buy}}(\delta)\,, \tag{3.2.1}$$

where δ represents the distance between a limit buy order and the opposite best quote. For the sell side, the arrival rate function is calculated in an analogous way and is given by

$$\hat{\lambda}^{\text{sell}}(\delta)\,. \tag{3.2.2}$$

In [CST10], it is suggested that the order arrival rate of the buy side and the sell side is found to be very well fitted by a power law function

$$\hat{\lambda}^{\text{buy/sell}}(\delta) = \frac{k}{\delta^\alpha}\,, \tag{3.2.3}$$

where δ denotes the distance from a given price p to the opposite best quote.

Figure 3.2.1 indicates that a power law fit would provide good results for the ISE-30 data too, because the calculated values are similar to a power law function. Thus, the least squares method was used to fit the arrival rate function of the buy orders and the sell orders respectively (cf. Section 2.2.1):

$$\min_{\alpha,k} \sum_{\delta=1}^{30} (\hat{\lambda}^{\text{buy/sell}}(\delta) - \frac{k}{\delta^\alpha})\,. \tag{3.2.4}$$

In Figure 3.2.2 the fit of the arrival rate function of the buy orders is illustrated, whereas Figure 3.2.3 shows the fit for the sell orders. In conclusion, the power law function is a good fit for the buy and the sell side. Unfortunately, in both cases this fit underestimates the tail.

It is often assumed that the tails and not the whole function, as in formula (3.2.3), are given by a power law. This thesis will not concentrate on the tail. Instead, focusing on small δ-values is appropriate, because much more arrivals occur in this area. Thus, the arrival rate function should be fitted accurately for small δ-values.

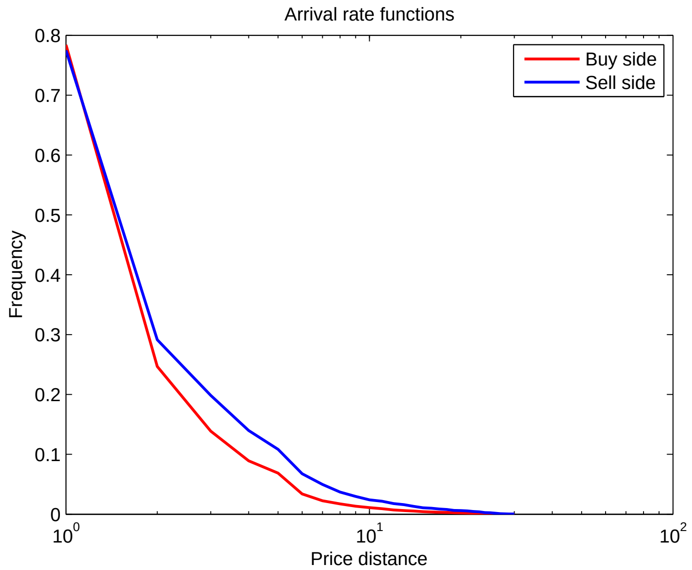

Figure 3.2.1: Arrival rate function of the buy and the sell side with $\delta_c = 30$.

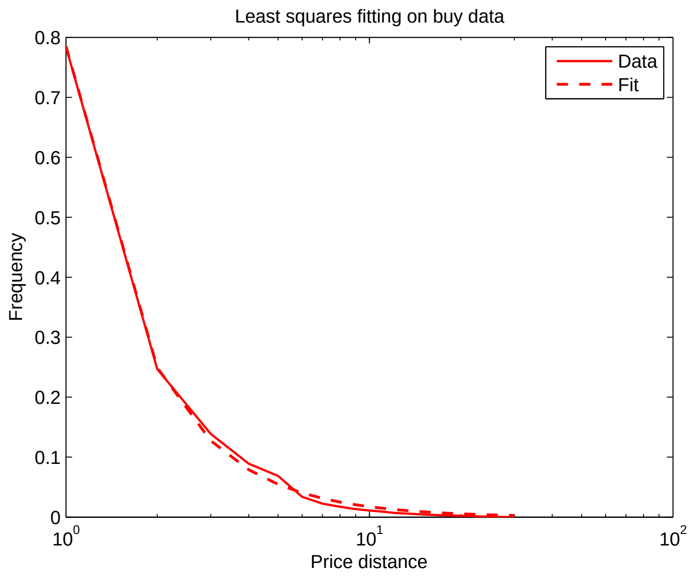

Figure 3.2.2: Arrival rate function of the buy side. The fit corresponds to a power law function with parameters $k = 0.785$ and $\alpha = 1.656$.

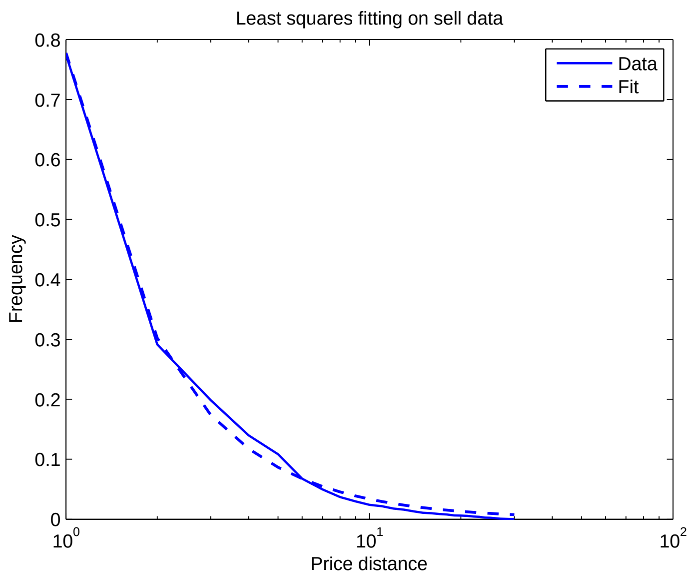

Figure 3.2.3: Arrival rate function of the sell side. The fit corresponds to a power law function with parameters $k = 0.777$ and $\alpha = 1.363$.

In general, the estimated parameters, k and α, are quite similar for the buy and the sell side, meaning that the arrival rate functions of both sides vary slightly. Therefore, the arrival rates, $\hat{\lambda}^{\text{buy}}(\delta)$ and $\hat{\lambda}^{\text{sell}}(\delta)$, of the limit buy and sell orders are not distinguished. Furthermore, $\hat{\lambda}(\delta)$ will stand for both functions $\hat{\lambda}^{\text{buy}}(\delta)$

and $\hat{\lambda}^{\text{sell}}(\delta)$. This finding goes along with the output in [CST10]. Thus, for the computation of the arrival rate of limit orders one does not distinguish between buy and sell orders and in the end the arrival rate function is calculated for all available orders.

Figure 3.2.4 illustrates the calculated arrival rate function for both sides, making no distinction between buy and sell orders. Again the fit corresponds to a power law function. Figure 3.2.5 reveals that the power law fit delivers accurate results.

The outcomes are dissimilar to those obtained in [FRVZ14]. This is because in [FRVZ14] the bid and ask prices described in Table 2.1.4 were used and the LOB probability density function was obtained by computing the percentage of total volume supplied/demanded at a given δ.

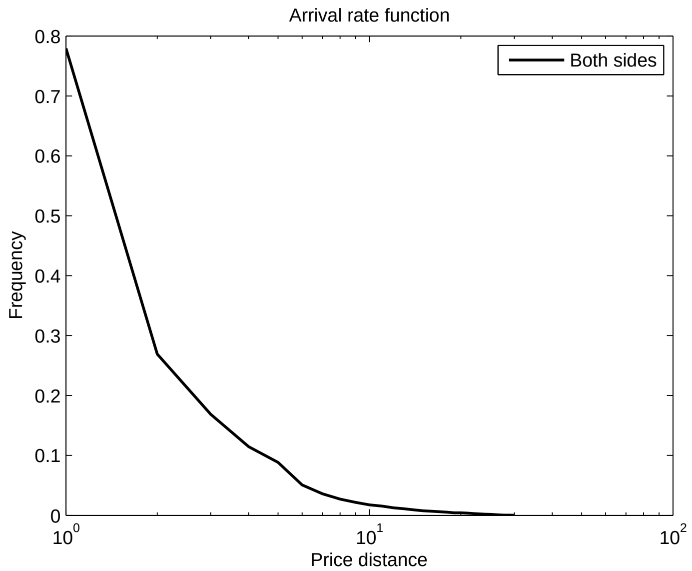

Figure 3.2.4: Arrival rate function of both sides with $\delta_c = 30$.

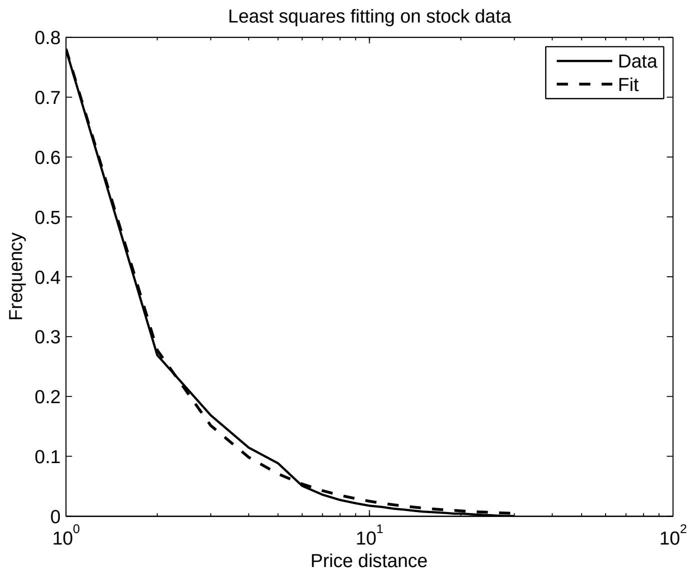

Figure 3.2.5: Arrival rate function of both sides. The fit corresponds to a power law function with parameters $k = 0.781$ and $\alpha = 1.493$.

The estimated parameter values for ISE-30 are given in Table 3.2.1. For the limit orders only the estimated arrival rates at distances $1 \leq \delta \leq 10$ from the opposite best quotes are listed.

Table 3.2.1: Estimated parameters for the ISE-30 data. The row $\hat{\lambda}(\delta)$ stands for the values of the valuation points of the arrival rate function and $\hat{\mu}$ signifies the arrival rate of the market orders. The parameters α and k are obtained from the power law fit.

δ	1	2	3	4	5	6	7	8	9	10
$\hat{\lambda}(\delta)$	0.780	0.269	0.169	0.115	0.089	0.051	0.036	0.027	0.022	0.018
$\hat{\mu}$	1.507									
α	1.493									
k	0.781									

3.3 First-passage times of a birth-and-death process

The aim of this section is to calculate the first-passage times of a birth-and-death process and derive the probability of a mid-price up movement at its next move afterwards. Almost all approaches in this section are taken from [AW92, AW99, CST10, HK12].

Consider a general $\mathbb{Z}^+$-valued birth-and-death process $(X_t)_{t\geq 0}$, where the value at t is called the state at time t. As described in Section 3.1.2, the birth-and-death process considered in this thesis has constant birth rates $\lambda = \lambda_j$

and constant death rates $\mu = \mu_j$ for all states j.

Let τ_b denote the random variable representing the first-passage time (cf. Definition 2.3.1) of X to zero, given that it started in state b. Recall that if the current state is $j \geq 2$, the process must reach $j-1$ first in order to reach state $j-2$. The first-passage time τ_b can be written as the sum of independent random variables

$$\tau_b = \tau_{b,b-1} + \tau_{b-1,b-2} + \ldots + \tau_{1,0}\,, \tag{3.3.1}$$

where $\tau_{j,j-1}$ represents the first-passage time of the birth-and-death process from state j to the neighboring state $j-1$ for all $j \in \{1, \ldots, b\}$ [AW99, HK12].

Let $\nu_j = \lambda + \mu = \nu$, then the density of the dwell time at state j is given by $\nu_j e^{-\nu_j t} = \nu e^{-\nu t}$. Furthermore, $\frac{\lambda_j}{\nu_j} = \frac{\lambda}{\nu}$ represents the probability of a transition to $j+1$, whereas $\frac{\mu_j}{\nu_j} = \frac{\mu}{\nu}$ indicates the transition probability to $j-1$ (cf. Section 2.3.1).

Moreover, if $f_{j,j-1}$ denotes the probability density function of $\tau_{j,j-1}$ one obtains the recursive formula

$$f_{j,j-1}(t) = \frac{\mu}{\nu}\nu e^{-\nu t} + \frac{\lambda}{\nu}\nu e^{-\nu t} * f_{j+1,j}(t) * f_{j,j-1}(t), j \geq 1\,, \tag{3.3.2}$$

where $*$ signifies convolution (cf. Definition 2.2.7). This formula is easy to understand because getting from state j to $j-1$ is possible either by death or by birth. In the latter case, the process has to go to state j again to finally reach state $j-1$.

Let $f_b = f_{b,0}$ be the probability density function of τ_b and $\hat{f}_b$ its Laplace transform as given by Definition 2.2.2. If $\hat{f}_{j,j-1}$ denotes the Laplace transform of $f_{j,j-1}$ for $j = 1, \ldots, b$, one gets

$$\hat{f}_b(s) = \prod_{j=1}^{b} \hat{f}_{j,j-1}(s) \tag{3.3.3}$$

from formula (2.2.4).
Thus, in order to calculate the Laplace transform $\hat{f}_b$ it is sufficient to compute the Laplace transforms $\hat{f}_{j,j-1}$ for $j = 1, \ldots, i$ as first-passage times to neighboring states are tractable because their probability density functions are always completely monotone (cf. Definition 2.2.4 and Remark 2.2.1) [AW92].

The idea is to derive the Laplace transforms of $f_{j,j-1}$ for $j = 1, \ldots, b$, multiplying these to get the Laplace transform of f_b and finally numerically inverting this resulting Laplace transform to derive the probability density function f_b.

The Laplace transform of the probability density function $f_{j,j-1}$ is given by

$$\hat{f}_{j,j-1}(s) = \mathbb{E}[e^{-s\tau_{j,j-1}}] = \int_0^\infty e^{-st} f_{j,j-1}(t)dt\,. \tag{3.3.4}$$

The following calculations have to be done to obtain the probability density function of τ_b.
Take the Laplace transform on both sides of (3.3.2), use

$$\int_0^\infty e^{-st}e^{-\nu t}dt = \frac{1}{\nu+s} \tag{3.3.5}$$

and the second property in Proposition 2.2.1 to get

$$\hat{f}_{j,j-1}(s) = \frac{\mu}{\nu+s} + \frac{\lambda}{\nu+s}\hat{f}_{j+1,j}(s)\hat{f}_{j,j-1}(s)\,. \tag{3.3.6}$$

Thus, rearranging and factoring out $\hat{f}_{j,j-1}(s)$ yields

$$\hat{f}_{j,j-1}(s) = \frac{\mu}{\nu+s-\lambda\hat{f}_{j+1,j}(s)}\,. \tag{3.3.7}$$

Iterating on (3.3.7) produces a continued fraction as introduced in Definition 2.2.3

$$\hat{f}_{j,j-1}(s) = -\frac{1}{\lambda}\Phi_{k=i}^{\infty}\frac{-\lambda\mu}{\nu+s}\,. \tag{3.3.8}$$

Applying (3.3.3), one obtains

$$\hat{f}_b(s) = \left(-\frac{1}{\lambda}\right)^b \left(\prod_{i=1}^{b}\Phi_{k=i}^{\infty}\frac{-\lambda\mu}{\nu+s}\right)\,. \tag{3.3.9}$$

The use of continued fractions is applied to compute Laplace transforms via infinite-series representations. It is often possible to calculate probability density functions and distribution functions efficiently by numerically inverting Laplace transforms. Unfortunately, explicit expressions for these transforms are often unavailable [AW99].

Laplace transforms of density functions of first-passage times to neighboring states can be illustrated by special continued fractions, called S fractions, which have desirable convergence properties [AW99]. Thus, at this point, it is important to refer to [HK12]. Therefore, a truncated birth-and-death process $X^{(\kappa)}$ on $\mathbb{Z}_\kappa$ with birth rate λ of size 1 and death rate μ of size 1 , which has been introduced previously, is considered.

The processes X and $X^{(\kappa)}$ are identical below state $\kappa-1$, thus, interchangeable if κ is chosen large enough. For the truncated process $X^{(\kappa)}$, $\tau_{j,j-1}^{(\kappa)}$ denotes the first-passage time from state j to state $j-1$ and $\tau_b^{(\kappa)}$ the first-passage time from state b to zero respectively.

Let $\hat{f}^{(\kappa)}_{j,j-1}$ be the Laplace transform of $f^{(\kappa)}_{j,j-1}$. Since the first-passage time from κ to $\kappa-1$ is exponentially distributed with rate μ,

$$\hat{f}^{(\kappa)}_{\kappa,\kappa-1}(s) = \frac{\mu}{\nu+s} \tag{3.3.10}$$

and because of Remark 3.1.5 this results in

$$\hat{f}^{(\kappa)}_{\kappa,\kappa-1}(s) = \frac{\mu}{\mu+s}\,. \tag{3.3.11}$$

For $\kappa-1$ one obtains

$$\hat{f}^{(\kappa)}_{\kappa-1,\kappa-2}(s) = \frac{\mu}{\nu+s-\lambda\hat{f}^{(\kappa)}_{\kappa,\kappa-1}(s)}\,. \tag{3.3.12}$$

Therefore, the following recursive formula for $\hat{f}^{(\kappa)}_{j,j-1}(s)$ can be written as

$$\hat{f}^{(\kappa)}_{j,j-1}(s) = \frac{\mu}{\nu+s-\lambda\hat{f}^{(\kappa)}_{j+1,j}(s)} \tag{3.3.13}$$

with $j = \kappa-1, \kappa-2, \ldots, 1$.

Next, following [HK12], it is proved that $\hat{f}^{(\kappa)}_{j,j-1}(s)$ converges as $\kappa \to \infty$. Therefore, let $F^{(\kappa)}_{j,j-1}$ be the distribution function of $\tau^{(\kappa)}_{j,j-1}$.

Lemma 3.3.1. *The following stochastic order relation as a function of κ holds*

$$\tau^{(\kappa)}_{j,j-1} \le \tau^{(\kappa+1)}_{j,j-1}, \ \textit{for } \kappa \ge j\,, \tag{3.3.14}$$

which is equivalent to

$$F^{(\kappa)}_{j,j-1}(t) \ge F^{(\kappa+1)}_{j,j-1}(t), \ \textit{for all } t \ge 0 \textit{ and } \kappa \ge j\,. \tag{3.3.15}$$

Sketch of Proof.
Note that $X^{(\kappa)}$ and $X^{(\kappa+1)}$ have the same probability of an upward move on $\{1,2,\ldots,\kappa-1\}$ and the same probability of a downward move on $\{1,2,\ldots,\kappa\}$. Thus, the probability of both processes to exceed state $\kappa-1$ for $s<t$ is the same. Conditional on exceeding state $\kappa-1$ in s, the process $X^{(\kappa)}$ has a greater probability to reach state $j-1$ at time t, because $X^{(\kappa+1)}$ may first reach state $\kappa+1$ before arriving at $j-1$.

■

Hence, $F^{(\kappa)}_{j,j-1}(t)$ decreases to a limit $F^{(\infty)}_{j,j-1}(t)$ as $\kappa\to\infty$, for $t \ge 0$.

Next,

$$F^{(\kappa)}_{j,j-1}(t) \geq F_{j,j-1}(t) \text{ for all } t \geq 0 \text{ and } \kappa \geq j\,, \tag{3.3.16}$$

where $F_{j,j-1}$ denotes the distribution function of $\tau_{j,j-1}$, the first-passage time from state j to $j-1$ for the unbounded process X. Therefore, in the limit

$$F^{(\infty)}_{j,j-1}(t) \geq F_{j,j-1}(t) \text{ for all } t \geq 0 \tag{3.3.17}$$

and thus, $F^{(\infty)}_{j,j-1}(t) \to 1$ for all $j \geq 1$ as $t \to \infty$. Hence, it is the distribution function of a random variable on $[0, \infty)$.

Since

$$\hat{f}^{(\kappa)}_{j,j-1}(s) = \int_0^\infty e^{-st} dF^{(\kappa)}_{j,j-1}(t) \tag{3.3.18}$$

and putting

$$\hat{f}_{j,j-1}(s) = \int_0^\infty e^{-st} dF^{(\infty)}_{j,j-1}(t)\,, \tag{3.3.19}$$

by Lemma 3.3.2, $\hat{f}^{(\kappa)}_{j,j-1}(s)$ converges to $\hat{f}_{j,j-1}(s)$ at $s \in \mathbb{C}$ with positive real part as $\kappa \to \infty$.

Lemma 3.3.2. *If probability distribution functions $F_n(x) \to F(x)$ as $n \to \infty$ for all $x \in \mathbb{R}$ at which F is continuous, $n \in \mathbb{N}$, then $\mathbb{E}_n[u] \to \mathbb{E}[u]$ for every bounded piecewise continuous function $u(x) : \mathbb{R} \to \mathbb{C}$, where $\mathbb{E}_n[u]$ and $\mathbb{E}[u]$ are expectations with respect to F_n and F.*

Proof. This is a consequence of the Portmanteau theorem, which gives a number of equal descriptions of weak convergence [Kus11][5].

■

Hence, $\tau_{j,j-1} = \tau^{(\infty)}_{j,j-1}$ will signify the limit of the first-passage times $\tau^{(\kappa)}_{j,j-1}$ as $\kappa \to \infty$ for all $j \geq 1$. These times will be indistinguishable from those of the original process X and thus, they will be approximated by recursion using a finite but large κ. Therefore, hereafter, κ will be omitted.

3.4 Probability of a mid-price increase

There are two possible events which cause a mid-price up-or-down movement. Either if the first move in the mid-price occurs at the first-passage time of the bid

[5] If $\mathbb{R}$ represents the metric space and F_n, F denote the distribution functions of the measures $\mathbb{P}_n$ and $\mathbb{P}$ respectively, then the probability measures $\mathbb{P}_n$ converge weakly to $\mathbb{P}$ if and only if $\lim_{n\to\infty} F_n(x) = F(x)$ for all points $x \in \mathbb{R}$ at which F is continuous.

or ask volume to zero or if the spread is greater than one and a limit order arrives between the actual bid and ask [CST10].

For the computation of the probability that the mid-price increases at its next move, the calculations of Section 3.3 and the estimated parameters in Section 3.2 (cf. Table 3.2.1) are used. Let T be the time of the first change in the mid-price,

$$T = \inf\{t \geq 0, p_M(t) \neq p_M(0)\}\,. \tag{3.4.1}$$

Assume that at time $t = 0$ are a orders at the ask price $p_A(0)$ and b orders at the bid price $p_B(0)$. Furthermore, the spread $p_S(0)$ is S ticks. Let τ_a be the first-passage time at price $p_A(0)$ from state a to zero and τ_b the first time when all b orders at price $p_B(0)$ disappear. Next, τ_S^δ is the first time a limit sell order arrives δ ticks away from the bid price and τ_B^δ is the first time a limit buy order arrives δ ticks away from the ask price. Both processes, A and B, described in Section 3.1.2 are constructed as the process X defined in Section 3.3.

Given these assumptions, the probability that the next change in the mid-price is an increase can be written as

$$\mathbb{P}(p_M(T) > p_M(0) | A_{p_A}(0) = a, B_{p_B}(0) = b, p_S(0) = S)\,, \tag{3.4.2}$$

where $S > 0$.

Furthermore, let

$$\Lambda_S = \sum_{\delta=1}^{S-1} \lambda(\delta)\,, \tag{3.4.3}$$

where $\lambda(\delta)$ is the limit order arrival rate at distance δ estimated in Section 3.2.

The Laplace transforms of f_a^S and f_b^S are given by $\hat{f}_a^S$ and $\hat{f}_b^S$ and can be computed as described in (3.3.3) and (3.3.10)-(3.3.13).

Next,

$$\tau_{aB} \quad = \tau_a \wedge \tau_B^1 \wedge \ldots \wedge \tau_B^{S-1} \tag{3.4.4}$$

and

$$\tau_{bA} \quad = \tau_b \wedge \tau_A^1 \wedge \ldots \wedge \tau_A^{S-1} \tag{3.4.5}$$

and let $f_{\tau_{aB}}$ and $f_{\tau_{bA}}$ be their corresponding probability density functions. Then τ_{aB} and τ_{bA} are independent since $\tau_a, \tau_B^1, \ldots, \tau_B^{S-1}$ and $\tau_b, \tau_A^1, \ldots, \tau_A^{S-1}$ are mutually independent.

Before calculating the conditional probability given in (3.4.2), the following Lemma is introduced.

Lemma 3.4.1. *Let Z be an exponentially distributed random variable with parameter Λ, σ a random variable with Laplace transform $\hat{f}_\sigma$. Moreover, Z and σ are independent. Then, the Laplace transform of $f_{\sigma\wedge Z}$ is given by*

$$\hat{f}_\sigma(\Lambda+s) + \frac{\Lambda}{\Lambda+s}(1-\hat{f}_\sigma(\Lambda+s)) \,. \tag{3.4.6}$$

Proof. First, the density $f_{\sigma\wedge Z}$ of the random variable $\sigma\wedge Z$ in terms of the density f_σ of the random variable σ is computed. Thus, one obtains for all $t \geq 0$

$$\begin{aligned} F_{\sigma\wedge Z}(t) &= \mathbb{P}(\sigma\wedge Z \leq t) \\ &= 1-\mathbb{P}(\sigma\wedge Z > t) = 1-\mathbb{P}(\sigma > t)\mathbb{P}(Z>t)\,. \end{aligned} \tag{3.4.7}$$

Since Z is exponentially distributed with rate Λ, one gets

$$F_{\sigma\wedge Z}(t) = 1-(1-F_\sigma(t))e^{-\Lambda t} = 1-e^{-\Lambda t} + F_\sigma(t)e^{-\Lambda t}\,. \tag{3.4.8}$$

The density $f_{\sigma\wedge Z}$ is obtained by differentiating both sides with respect to t

$$\begin{aligned} f_{\sigma\wedge Z}(t) = \frac{dF_{\sigma\wedge Z}(t)}{dt} &= \Lambda e^{-\Lambda t} + f_\sigma(t)e^{-\Lambda t} - \Lambda e^{-\Lambda t}F_\sigma(t) \\ &= f_\sigma(t)e^{-\Lambda t} + \Lambda(1-F_\sigma(t))e^{-\Lambda t} \end{aligned} \tag{3.4.9}$$

for all $t \geq 0$. Moreover, $f_{\sigma\wedge Z}$ is a probability density function and therefore, the Laplace transform (cf. Definition 2.2.2) of $f_{\sigma\wedge Z}$, which represents the probability density function of $\sigma\wedge Z$, is given by

$$\begin{aligned} \hat{f}_{\sigma\wedge Z}(s) &= \int_0^\infty e^{-st} f_{\sigma\wedge Z}(t)dt \\ &= \int_0^\infty e^{-st}(f_\sigma(t)e^{-\Lambda t} + \Lambda(1-F_\sigma(t))e^{-\Lambda t})dt \\ &= \int_0^\infty e^{-(\Lambda+s)t} f_\sigma(t)dt + \Lambda\int_0^\infty e^{-(\Lambda+s)t}dt - \Lambda\int_0^\infty e^{-(\Lambda+s)t}F_\sigma(t)dt \\ &= \hat{f}_\sigma(\Lambda+s) - \frac{e^{-(\Lambda+s)t}\Lambda}{\Lambda+s}\Big|_0^\infty \\ &\quad - \Lambda\left(-\frac{e^{-(\Lambda+s)t}F_\sigma(t)}{\Lambda+s}\Big|_0^\infty + \int_0^\infty \frac{e^{-(\Lambda+s)t}}{\Lambda+s}f_\sigma(t)dt\right) \\ &= \hat{f}_\sigma(\Lambda+s) + \frac{\Lambda}{\Lambda+s} - \frac{\Lambda}{\Lambda+s}(\hat{f}_\sigma(\Lambda+s)) \\ &= \hat{f}_\sigma(\Lambda+s) + \frac{\Lambda}{\Lambda+s}(1-\hat{f}_\sigma(\Lambda+s))\,. \end{aligned} \tag{3.4.10}$$

■

Theorem 3.4.1 (Probability of a mid-price increase)**.** *Let* $\hat{f}_k^S$ *for* $k = \{a, b\}$ *be given by*

$$\begin{aligned}\hat{f}_k^S(s) &\stackrel{(3.3.3)}{=} \prod_{j=1}^{k} \hat{f}_{j,j-1}^S(s) \\ &\stackrel{(3.3.7)}{=} \prod_{j=1}^{k} \frac{\mu}{\nu(S) + s - \lambda(S)\hat{f}_{j+1,j}^S(s)} .\end{aligned} \tag{3.4.11}$$

Then, (3.4.2) is given by

$$\mathbb{P}(\tau_{aB} < \tau_{bA}) = \int_0^\infty \int_0^1 y f_{\tau_{aB}}^S(sy) f_{\tau_{bA}}^S(y) ds dy, \tag{3.4.12}$$

where $f_{\tau_{aB}}^S(t)$ *and* $f_{\tau_{bA}}^S(t)$ *are derived from the inverse Laplace transforms of*

$$\hat{f}_{\tau_{aB}}^S(s) = \hat{f}_a^S(\Lambda_S + s) + \frac{\Lambda_S}{\Lambda_S + s}(1 - \hat{f}_a^S(\Lambda_S + s)) \tag{3.4.13}$$

and

$$\hat{f}_{\tau_{bA}}^S(s) = \hat{f}_b^S(\Lambda_S + s) + \frac{\Lambda_S}{\Lambda_S + s}(1 - \hat{f}_b^S(\Lambda_S + s)) \tag{3.4.14}$$

respectively.
When $S = 1$*, (3.4.12) reduces to*

$$\mathbb{P}(\tau_a < \tau_b) = \int_0^\infty \int_0^1 y f_a^1(sy) f_b^1(y) ds dy\,, \tag{3.4.15}$$

where $f_a^1(t)$ *and* $f_b^1(t)$ *are derived from the inverse Laplace transforms of* $\hat{f}_a^1(s)$ *and* $\hat{f}_b^1(s)$ *respectively.*

Proof. First, the case $S = 1$ is considered and afterwards the analysis is extended to the case $S > 1$.
$S = 1$: The mid-price changes for the first time if one of the two processes, B_{p_B} and A_{p_A}, reaches state 0 (cf. Definition 3.1.3). The mid-price increases if the first-passage time τ_a, this is the first-passage time of the shares at the ask from a to zero, is smaller than the first-passage time τ_b. That is because the bid price represents the maximum of all bid prices, whereas the ask price is the minimum of all sell prices. If the volume at the ask is zero, then the next best sell price, meaning the new minimum of all available sell prices, for which $p_{A_{\text{old}}} < p_{A_{\text{new}}}$ and therefore, $p_{M_{\text{old}}} < p_{M_{\text{new}}}$ holds, becomes the ask price. This process is valid in an analogous way for the buy side (maximum instead of minimum). Therefore,

$$\mathbb{P}(p_M(T) > p_M(0) | A_{p_A}(0) = a, B_{p_B}(0) = b, p_S(0) = S)$$

is given by $\mathbb{P}(\tau_a < \tau_b) = \mathbb{P}(\tau_a - \tau_b < 0)$, where τ_a and τ_b are independent first-passage times.
Due to the fact that τ_a and τ_b are independent and non-negative, this probability is given by

$$\mathbb{P}(\tau_a - \tau_b < 0) = \int_{-\infty}^{\infty} \int_{-\infty}^{y} \mathbb{1}_{s \geq 0} \mathbb{1}_{y \geq 0} f_a^1(s) f_b^1(y) ds dy \,. \tag{3.4.16}$$

Changing the integral boundaries in respect to the indicator functions yields

$$\mathbb{P}(\tau_a - \tau_b < 0) = \int_{0}^{\infty} \int_{0}^{y} f_a^1(s) f_b^1(y) ds dy \,. \tag{3.4.17}$$

Substituting sy for s gives

$$\mathbb{P}(\tau_a - \tau_b < 0) = \int_{0}^{\infty} \int_{0}^{1} y f_a^1(sy) f_b^1(y) ds dy \,. \tag{3.4.18}$$

$S > 1$: Now, the case $S > 1$ is considered. Let τ_A^δ and τ_B^δ for $\delta = 1, 2, \ldots, S-1$ be defined as above. The time of the first change in mid-price is now given by

$$T = \tau_a \wedge \tau_b \wedge \min\{\tau_A^1, \tau_B^1, \tau_A^2, \tau_B^2, \ldots, \tau_A^{S-1}, \tau_B^{S-1}\} \,. \tag{3.4.19}$$

Note that B_{p_B} and A_{p_A} are independent of the mutually independent arrival times $\tau_A^\delta, \tau_B^\delta$ for $\delta = 1, 2, \ldots, S-1$ and the arrival times are exponentially distributed with corresponding rate $\lambda(\delta)$ for $\delta = 1, 2, \ldots, S-1$.
The first change in the mid-price is an increase either if there is an arrival of a limit buy order within the bid-ask spread or if A_{p_A} hits zero before either there is an arrival of a limit ask order inside the spread or B_{p_B} hits zero. Therefore,

$$\mathbb{P}(p_M(T) > p_M(0) | A_{p_A}(0) = a, B_{p_B}(0) = b, p_S(0) = S)$$

can be written as

$$\begin{aligned} \mathbb{P}(\tau_a \wedge \tau_B^1 \wedge \ldots \wedge \tau_B^{S-1} < \tau_b \wedge \tau_A^1 \wedge \ldots \wedge \tau_A^{S-1}) \\ = \mathbb{P}(\tau_a \wedge \tau_B^\Sigma < \tau_b \wedge \tau_A^\Sigma), \end{aligned} \tag{3.4.20}$$

where τ_B^Σ and τ_A^Σ are independent exponential random variables , both with rate Λ_S since the arrival times are independent (cf. Theorem 2.2.3).
To compute the probability of a mid-price increase, first, the Laplace transform of the minimum $\tau_a \wedge \tau_B^\Sigma$ and $\tau_b \wedge \tau_A^\Sigma$ has to be calculated. Therefore, after substituting τ_B^Σ for Z and τ_B^Σ respectively Lemma 3.4.1 is used.
As in the first case, due to independence and non-negativity of τ_{aB} and τ_{bA}, the probability $\mathbb{P}(\tau_a \wedge \tau_B^\Sigma - \tau_b \wedge \tau_A^\Sigma < 0) = \mathbb{P}(\tau_{aB} < \tau_{bA})$ can be rewritten as

$$\mathbb{P}(\tau_{aB} < \tau_{bA}) = \int_{-\infty}^{\infty} \int_{-\infty}^{y} \mathbb{1}_{s \geq 0} \mathbb{1}_{y \geq 0} f_{\tau_{aB}}^S(s) f_{\tau_{bA}}^S(y) ds dy \,. \tag{3.4.21}$$

Changing the integral boundaries in respect to the indicator functions yields

$$\mathbb{P}(\tau_{aB} - \tau_{bA} < 0) = \int_0^\infty \int_0^y f^S_{\tau_{aB}}(s) f^S_{\tau_{bA}}(y) ds dy \,. \tag{3.4.22}$$

Substituting sy for s gives

$$\mathbb{P}(\tau_{aB} - \tau_{bA} < 0) = \int_0^\infty \int_0^1 y f^S_{\tau_{aB}}(sy) f^S_{\tau_{bA}}(y) ds dy \,, \tag{3.4.23}$$

which is the probability that the mid-price increases at its next move when there are a orders at the ask, b orders at the bid and the spread equals S ticks.

■

CHAPTER 4

Numerical computation and results

This chapter contains a detailed description of numerically inverting Laplace transforms and the computation procedure, followed by an illustrating report of the achieved results concerning the probabilities calculated on the estimated model parameters. Most of the realizations in this section are based on the researches of [AW92, AW95, AW06].

4.1 Numerical inversion of Laplace transforms

Before the probability of a mid-price increase is computed, an algorithm for numerically inverting Laplace transforms is given. In Theorem 3.4.1, the inverse Laplace transforms of $\hat{f}^S_{\tau_{aB}}, \hat{f}^S_{\tau_{bA}}$, $\hat{f}^1_a$ and $\hat{f}^1_b$ have to be calculated to get the desired probability. For that reason, numerical inversion is necessary.
Throughout this section, f is always representing a probability density function, therefore, it is real-valued. Moreover, f is assumed to be bounded and continuous.

Following [AW92], f can be computed by numerically inverting the Laplace transform $\hat{f}$ (cf. Theorem 2.2.1)

$$\begin{aligned} f(t) &= \frac{1}{2\pi\mathrm{i}}\int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} e^{ts}\hat{f}(s)ds \\ &\overset{s=a+\mathrm{i}u}{=} \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(a+\mathrm{i}u)t}\hat{f}(a+\mathrm{i}u)du\,. \end{aligned} \tag{4.1.1}$$

Since

$$e^{(a+\mathrm{i}u)t} = e^{at}(\cos(ut)+\mathrm{i}\sin(ut))\,, \tag{4.1.2}$$

one obtains

$$f(t) = \frac{e^{at}}{2\pi}\int_{-\infty}^{\infty}(\cos(ut)+\mathrm{i}\sin(ut))\hat{f}(a+\mathrm{i}u)du\,. \tag{4.1.3}$$

Using the fact that

$$\hat{f}(a + \mathrm{i}u) = \mathrm{Re}(\hat{f}(a + \mathrm{i}u)) + \mathrm{i}\ \mathrm{Im}(\hat{f}(a + \mathrm{i}u))\,, \tag{4.1.4}$$

yields

$$\begin{aligned} f(t) = \frac{e^{at}}{2\pi}\Big(&\int_{-\infty}^{\infty} \mathrm{Re}(\hat{f}(a+\mathrm{i}u))\cos(ut) - \mathrm{Im}(\hat{f}(a+\mathrm{i}u))\sin(ut)du \\ &+ \mathrm{i}\int_{-\infty}^{\infty} \mathrm{Im}(\hat{f}(a+\mathrm{i}u))\cos(ut) + \mathrm{Re}(\hat{f}(a+\mathrm{i}u))\sin(ut)du\Big)\,. \end{aligned} \tag{4.1.5}$$

Nevertheless, f is real-valued and thus, the imaginary part of (4.1.5) is zero. Therefore,

$$\begin{aligned} f(t) = \frac{e^{at}}{2\pi}\int_{-\infty}^{\infty} &\mathrm{Re}(\hat{f}(a+\mathrm{i}u))\cos(ut) \\ &- \mathrm{Im}(\hat{f}(a+\mathrm{i}u))\sin(ut)du\,. \end{aligned} \tag{4.1.6}$$

Since the integral in Theorem 2.2.1 is 0 for $t < 0$, one obtains

$$\begin{aligned} f(t) = \frac{e^{-at}}{2\pi}\Big(&\int_{-\infty}^{\infty} \mathrm{Re}(\hat{f}(a+\mathrm{i}u))\cos(-ut)du \\ &- \int_{-\infty}^{\infty} \mathrm{Im}(\hat{f}(a+\mathrm{i}u))\sin(-ut)du\Big) = 0\,. \end{aligned} \tag{4.1.7}$$

Moreover,

$$\sin(ut) = -\sin(-ut),\ \cos(ut) = \cos(-ut) \tag{4.1.8}$$

yields

$$\begin{aligned} &\int_{-\infty}^{\infty} \mathrm{Re}(\hat{f}(a+\mathrm{i}u))\cos(ut)du = \\ &-\int_{-\infty}^{\infty} \mathrm{Im}(\hat{f}(a+\mathrm{i}u))\sin(ut)du\,. \end{aligned} \tag{4.1.9}$$

Thus, for $t \geq 0$

$$f(t) = \frac{e^{at}}{\pi}\int_{-\infty}^{\infty} \mathrm{Re}(\hat{f}(a+\mathrm{i}u))\cos(ut)du\,. \tag{4.1.10}$$

And finally, since

$$\mathrm{Re}(\hat{f}(a+\mathrm{i}u)) = \mathrm{Re}(\hat{f}(a-\mathrm{i}u)) \tag{4.1.11}$$

(cf. Proposition 2.2.1) one gets

$$f(t) = \frac{2e^{at}}{\pi}\int_{0}^{\infty} \mathrm{Re}(\hat{f}(a+\mathrm{i}u))\cos(ut)du\,. \tag{4.1.12}$$

As described in [AW92], the trapezoidal rule given by the formula (2.2.21), which tends to work well for periodic functions because of errors canceling, is used to calculate the Laplace transform inversion formula (4.1.12) numerically. According to [AW92], formula (2.2.21) also applies for $a = -\infty$ or $b = \infty$ and thus, one obtains

$$\begin{aligned} f(t) &= \frac{2e^{at}}{\pi} \int_0^\infty \mathrm{Re}(\hat{f}(a + \mathrm{i}u)) \cos(ut) du \\ &\approx \frac{he^{at}}{\pi} \mathrm{Re}(\hat{f}(a)) + \frac{2he^{at}}{\pi} \sum_{k=1}^{\infty} \mathrm{Re}(\hat{f}(a + \mathrm{i}kh)) \cos(kht)\,. \end{aligned} \tag{4.1.13}$$

Producing nearly an alternating series by putting $h = \frac{\pi}{2t}$ and $a = \frac{A}{2t}$ for $t \neq 0$, delivers

$$f(t) \approx \frac{e^{\frac{A}{2}}}{2t} \mathrm{Re}(\hat{f}\Big(\frac{A}{2t}\Big)) + \frac{e^{\frac{A}{2}}}{t} \sum_{k=1}^{\infty} \cos\left(\frac{k\pi}{2}\right) \mathrm{Re}(\hat{f}\Big(\frac{A + k\pi\mathrm{i}}{2t}\Big))\,. \tag{4.1.14}$$

Using the fact that $\cos(\frac{k\pi}{2}) = 0$ for all $k \in 2\mathbb{N} - 1$, yields

$$f(t) \approx \frac{e^{\frac{A}{2}}}{2t} \mathrm{Re}(\hat{f}\Big(\frac{A}{2t}\Big)) + \frac{e^{\frac{A}{2}}}{t} \sum_{k=1}^{\infty} (-1)^k \mathrm{Re}(\hat{f}\Big(\frac{A + 2k\pi\mathrm{i}}{2t}\Big))\,. \tag{4.1.15}$$

This numerical inversion is applied to $f^S_{\tau_{aB}}$, $f^S_{\tau_{bA}}$, f^1_a and f^1_brespectively.

Proposition 4.1.1 (Discretization Error). *According to [AW95], the discretization error associated with (4.1.15) is given by*

$$e_d = e_d(f, t, A) = \sum_{k=1}^{\infty} e^{-kA} f((2k + 1)t)\,. \tag{4.1.16}$$

Furthermore, if $|f(t)| \leq C$ *for* $C \in [0, \infty)$ *the discretization error is bounded by*

$$|e_d| \leq \frac{Ce^{-A}}{1 - e^{-A}} \underbrace{\approx}_{\text{if } e^{-A} \text{ is small}} Ce^{-A}\,. \tag{4.1.17}$$

Since f *is a distribution function,*

$$|e_d| \leq \frac{e^{-A}}{1 - e^{-A}} \approx e^{-A} \tag{4.1.18}$$

holds.

Proof. The calculation of the discretization error is based on the usage of the Poisson summation formula defined in Section 2.2.5.
First, let $g(t) = e^{-bt} f(t)$ for $b > 0$ be defined over the entire real line by setting

$g(t) = 0$ for $t < 0$. Now, the idea is to replace this function by the periodic function

$$g_p(t) = \sum_{n\in\mathbb{Z}} g(t + \frac{2\pi n}{h}) \tag{4.1.19}$$

of period $P = \frac{2\pi}{h}$. This can be done due to the bounding condition of f and the dumping factor e^{-bt}. Next, the Poisson summation formula with $P = \frac{2\pi}{h}$ is applied, which requires that the periodic function $g_p \in L^1(\mathbb{R})$ has to be a sufficiently well-behaved function [BZ97].
Therefore, g_p can be presented by

$$g_p(t) = \sum_{n\in\mathbb{Z}} g(t + \frac{2\pi n}{h}) \overset{(2.2.24)}{=} \frac{h}{2\pi} \sum_{k\in\mathbb{Z}} G\Big(\frac{kh}{2\pi}\Big) e^{\mathrm{i}kht}\,, \tag{4.1.20}$$

where G is the Fourier transform of g.
Using Definition 2.2.5 gives

$$g_p(t) = \frac{h}{2\pi} \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} g(s) e^{-\mathrm{i}skh} ds\, e^{\mathrm{i}tkh}\,. \tag{4.1.21}$$

Next, $g(t) = 0$ for $t < 0$ leads to

$$g_p(t) = \frac{h}{2\pi} \sum_{k\in\mathbb{Z}} \int_{0}^{\infty} g(s) e^{-\mathrm{i}skh} ds\, e^{\mathrm{i}tkh}\,. \tag{4.1.22}$$

By the relation $g(t) = e^{-bt} f(t)$, one gets

$$\begin{aligned} g_p(t) &= \frac{h}{2\pi} \sum_{k\in\mathbb{Z}} \int_{0}^{\infty} f(s) e^{-bs} e^{-\mathrm{i}skh} ds\, e^{\mathrm{i}tkh} \\ &= \frac{h}{2\pi} \sum_{k\in\mathbb{Z}} \int_{0}^{\infty} f(s) e^{-s(b+\mathrm{i}kh)} ds\, e^{\mathrm{i}tkh}\,. \end{aligned} \tag{4.1.23}$$

Finally, using Definition 2.2.1 yields

$$g_p(t) = \frac{h}{2\pi} \sum_{k\in\mathbb{Z}} \hat{f}(b + \mathrm{i}kh) e^{\mathrm{i}tkh} \tag{4.1.24}$$

Now, the periodic function g_p has two different representations. Thus, one obtains

$$\begin{aligned} g_p(t) &= \sum_{k\in\mathbb{Z}} g\left(t + \frac{2\pi k}{h}\right) \\ &= \sum_{k\in\mathbb{Z}} f\left(t + \frac{2\pi k}{h}\right) e^{-b(t+\frac{2\pi k}{h})} \\ &= \frac{h}{2\pi} \sum_{k\in\mathbb{Z}} \hat{f}(b + \mathrm{i}kh) e^{\mathrm{i}kht}\,. \end{aligned} \tag{4.1.25}$$

Focusing on the single term $k = 0$ yields

$$\frac{h}{2\pi}\sum_{k\in\mathbb{Z}}\hat{f}(b+\mathrm{i}kh)e^{\mathrm{i}kht} = \sum_{k\in\mathbb{Z},k\neq 0} f\left(t+\frac{2\pi k}{h}\right)e^{-b(t+\frac{2\pi k}{h})} + e^{-bt}f(t)\,, \tag{4.1.26}$$

which leads to

$$e^{-bt}f(t) = \frac{h}{2\pi}\sum_{k\in\mathbb{Z}}\hat{f}(b+\mathrm{i}kh)e^{\mathrm{i}kht} - \sum_{k\in\mathbb{Z},k\neq 0} f\left(t+\frac{2\pi k}{h}\right)e^{-b(t+\frac{2\pi k}{h})}\,. \tag{4.1.27}$$

Multiplying both sides with e^{bt} supplies

$$f(t) = \frac{h}{2\pi}\sum_{k\in\mathbb{Z}}\hat{f}(b+\mathrm{i}kh)e^{(b+\mathrm{i}kh)t} - \sum_{k\in\mathbb{Z},k\neq 0} f\left(t+\frac{2\pi k}{h}\right)e^{-\frac{2\pi k}{h}b}\,. \tag{4.1.28}$$

Moreover, f is a density function of non-negative random variables and thus, for $h < \frac{2\pi}{t}$

$$\sum_{k\in\mathbb{Z},k\neq 0} f\left(t+\frac{2\pi k}{h}\right)e^{-\frac{2\pi k}{h}b} \tag{4.1.29}$$

can be written as

$$\sum_{k=1}^{\infty} f\left(t+\frac{2\pi k}{h}\right)e^{-\frac{2\pi k}{h}b}\,. \tag{4.1.30}$$

Furthermore, one obtains

$$\frac{h}{2\pi}\sum_{k\in\mathbb{Z}}\hat{f}(b+\mathrm{i}kh)e^{(b+\mathrm{i}kh)t} = \frac{h}{2\pi}e^{bt}\sum_{k\in\mathbb{Z}}\mathrm{Re}(\hat{f}(b+\mathrm{i}kh))\cos(kht) - \mathrm{Im}(\hat{f}(b+\mathrm{i}kh))\sin(kht)\,. \tag{4.1.31}$$

Setting $h = \frac{\pi}{t} < \frac{2\pi}{t}$, $b = \frac{A}{2t}$ and using the fact that $\sin(k\pi) = 0$ for all $k \in \mathbb{Z}$ gives

$$f(t) = \frac{e^{\frac{A}{2}}}{2t}\sum_{k\in\mathbb{Z}}\cos(k\pi)\,\mathrm{Re}(\hat{f}\left(\frac{A+2k\pi\mathrm{i}}{2t}\right)) - \sum_{k=1}^{\infty}e^{-kA}f((2k+1)t)\,. \tag{4.1.32}$$

Finally, $\cos(k\pi) = \cos(-k\pi) = (-1)^k$ and $\mathrm{Re}(\hat{f}\left(\frac{A+2k\pi \mathrm{i}}{2t}\right)) = \mathrm{Re}(\hat{f}\left(\frac{A-2k\pi \mathrm{i}}{2t}\right))$ for $k \neq 0$ (cf. Proposition 2.2.1) yields

$$
\begin{aligned}
f(t) = {} & \frac{e^{\frac{A}{2}}}{2t} \mathrm{Re}(\hat{f}\left(\frac{A}{2t}\right)) + \frac{e^{\frac{A}{2}}}{t} \sum_{k=1}^{\infty} (-1)^k \mathrm{Re}(\hat{f}\left(\frac{A+2k\pi \mathrm{i}}{2t}\right)) \\
& - \sum_{k=1}^{\infty} e^{-kA} f((2k+1)t) .
\end{aligned}
\tag{4.1.33}
$$

It is obvious that the first two terms of the right coincides with the trapezoidal-rule approximation in (4.1.15). Thus, the third term on the right-hand side is the discretization error associated with the trapezoidal rule [AW95].

The proof of the second statement follows by usage of the triangle inequality and the bounding condition regarding f. First,

$$
\begin{aligned}
|e_d| &= \Big| \sum_{k=1}^{\infty} e^{-kA} f((2k+1)t) \Big| \\
&\leq \sum_{k=1}^{\infty} |e^{-kA}| \underbrace{|f((2k+1)t)|}_{\leq C} \leq \sum_{k=1}^{\infty} |e^{-kA}| C .
\end{aligned}
\tag{4.1.34}
$$

For $A > 0$, one obtains $0 < e^{-A} < 1$ and hence,

$$
\sum_{k=1}^{\infty} |e^{-kA}| C \leq C \frac{e^{-kA}}{1 - e^{-kA}} . \tag{4.1.35}
$$

Since f is a distribution function, implying that $|f(t)| \leq 1$ for all t, the error is bounded by $\frac{e^{-A}}{1-e^{-A}} \approx e^{-A}$.

■

Considering (4.1.16) exhibits to choose A large enough to make the discretization error small, nevertheless, a large A can make the computation of (4.1.15) more difficult.

According to [AW95], Euler summation (cf. Section 2.2.6) is used to solve the remaining problem, that is to numerically calculate (4.1.15). This mode of calculation can be applied effectively since the infinite sum in (4.1.15) is an alternating series.
Moreover, Euler summation to m terms after initial n terms is a standard method to accelerate (4.1.15).

Let $s_n(t)$ be the approximation of $f_h(t)$ with the infinite series truncated to n terms, that is

$$
s_n(t) = \frac{e^{\frac{A}{2}}}{2t} \mathrm{Re}(\hat{f}\Big(\frac{A}{2t}\Big)) + \frac{e^{\frac{A}{2}}}{t} \sum_{k=1}^{n} (-1)^k a_k(t) , \tag{4.1.36}
$$

where

$$a_k(t) = \mathrm{Re}(\hat{f}\Big(\frac{A + 2k\pi \mathrm{i}}{2t}\Big)) \,. \tag{4.1.37}$$

Thus, the Euler sum is given by

$$E(m, n, t) = \sum_{k=0}^{m} \binom{m}{k} 2^{-m} s_{n+k}(t) \,. \tag{4.1.38}$$

This sum is the binomial average of the terms $s_n, s_{n+1}, \ldots, s_{n+m}$.
Next, for the estimation of the error associated with Euler summation the difference

$$E(m, n+1, t) - E(m, n, t) \tag{4.1.39}$$

is used. Following [AW95], this is a good error estimate if f is sufficiently smooth.

This numerical approximation of the inverse Laplace transform is implemented in MATLAB®2011b and is based on [AW06]. The inversion routine called *Euler algorithm* is used, which is an implementation of the Fourier-series method and the Euler summation [AW92, AW06].
Thus, implementing $\hat{f}^S_{\tau_{aB}}, \hat{f}^S_{\tau_{bA}}, \hat{f}^1_a$ and $\hat{f}^1_b$ and using the numerical approximation of the inverse Laplace transform already implemented is necessary to obtain the probability of a mid-price increase (cf. Theorem 3.4.1).

4.2 Numerical computation of the probability of a mid-price increase

In this subsection, a short description regarding the implementation of (3.4.12) and (3.4.15) is given. First, (3.4.13) and (3.4.14) are calculated with the use of the estimated parameters $\hat{\lambda}(\delta)$ and $\hat{\mu}$. Next, the numerical approximation of inverse Laplace transforms already implemented in MATLAB®2011b is used to receive $f^S_{\tau_{aB}}, f^S_{\tau_{bA}}, f^1_{\tau_a}$ and $f^1_{\tau_b}$. The function is called `euler_inversion` and returns an approximation of the inverse Laplace transform of some function f evaluated at certain values, which are again supplied by an implemented algorithm called `lgwt` [Matc, Matd]. The latter computes the Legendre-Gauss weights and nodes to solve definite integrals. Based on the evaluated points, the Gauss quadrature is used to calculate the double integral in (3.4.12) and (3.4.15).
To facilitate this calculation of an infinite integral, the following transformation is done to obtain finite integrals.

Recall the before received probabilities of an upward mid-price movement.
One obtains

$$\int_0^\infty \int_0^1 y f^S_{\tau_{aB}}(sy) f^S_{\tau_{bA}}(y) ds dy \tag{4.2.1}$$

for $S > 1$ and

$$\int_0^\infty \int_0^1 y f_a^1(sy) f_b^1(y) ds dy \tag{4.2.2}$$

for $S = 1$ (cf. formula (3.4.12) and formula (3.4.15)).
The following calculations are only presented for the first case, for the second case the steps are analog.
The arc tangent is used to receive a finite integral. Evaluating the arc tangent in $x = \infty$ gives $\frac{\pi}{2}$. Another essential evaluation point regarding the transformation is $x = 0$, which is the sole root of the function. Therefore, the infinite integral can be rewritten as

$$\begin{aligned}\mathbb{P}(\tau_{aB} < \tau_{bA}) = &\int_0^{\frac{\pi}{2}} \int_0^1 (\tan(y)^2 + 1) \tan(y) f_{\tau_{aB}}^S(s \tan(y)) \\ &\cdot f_{\tau_{bA}}^S(\tan(y)) ds dy\end{aligned} \tag{4.2.3}$$

Using the Gauss quadrature, this double integral can be rewritten as

$$\begin{aligned}&\int_0^{\frac{\pi}{2}} \int_0^1 (\tan(y)^2 + 1) \tan(y) f_{\tau_{aB}}^S(s \tan(y)) f_{\tau_{bA}}^S(\tan(y)) ds dy \\ &= \sum_{i=1}^{n1} \sum_{j=1}^{n2} w_i^{[0,\frac{\pi}{2}]} w_j^{[0,1]} (1 + \tan(m_i^{[0,\frac{\pi}{2}]})^2) \tan(m_i^{[0,\frac{\pi}{2}]}) \\ &\cdot f_{\tau_{aB}}^S(m_j^{[0,1]} \tan(m_i^{[0,\frac{\pi}{2}]})) f_{\tau_{bA}}^S(\tan(m_i^{[0,\frac{\pi}{2}]}))\,,\end{aligned} \tag{4.2.4}$$

where $n1$ and $n2$ are the numbers of evaluation points, w_i and w_j are called the weights and m_i and m_j represent their corresponding nods.
At this point, an implemented algorithm is used, which calculates the weights and nods for a given number of evaluation points and interval boundaries. These nods are the evaluation points used in the Euler algorithm. Finally, the obtained values are substituted into formula (4.2.4) to achieve the desired probabilities.

4.3 Numerical results

In this section the conditional probabilities of various events calculated based on the above introduced birth-and-death Markov chain model are compared to the frequencies of these events computed based on the LOB data (cf. Table 2.1.4 and Table 2.1.5).

Before these probabilities are matched, the results of the calculations regarding the approximating process $X^{(\kappa)}$ if the truncation state $\kappa \geq j$ increases are given. This is, to see how the process behaves if κ grows.
Following [HK12], first, the convergence of the Laplace transforms of the probability density functions of the first-passage times for $j = 1, 5, 8, 9, 10$, meaning the

first-passage times from state 10 to 9, 9 to 8, 8 to 7, 5 to 4 and 1 to 0, evaluated at $t = 10 + 5\mathrm{i}$ is illustrated. Table 4.3.1 indicates that the Laplace transform of $f^{(\kappa)}_{j,j-1}$ for the states $j = 1, 5, 8, 9, 10$ evaluated at $t = 10 + 5\mathrm{i}$ is identical for every state j if $\kappa \geq 12$. The obtained outcomes are dissimilar to those observed in [HK12]. They showed that the Laplace transforms for $j = 8, 9, 10$ evaluated at $t = 10 + 5\mathrm{i}$ tend to remain constant for $\kappa \geq 15$. Therefore, to ensure that the model used in this paper has been correctly calculated, formula (3.3.13) was changed to the formula obtained in [HK12] using their estimated parameters. Table 4.3.2 represents the outcomes using the formula of [HK12].

The reason for the differences regarding the convergence of the Laplace transforms is that in this thesis there was no need for calculating cancellation rates. Therefore, the death rate used in this model is given by the constant rate μ, whereas in [HK12] the death rate at state $j \geq 1$ is constituted by $\mu_j = \mu + \theta(i-1)j$, where $\theta(i-1)$ illustrates the cancellation rate for the ith level quote. Since all calculations in [HK12] were done for $S = 1$, the death rate is given by $\mu_j = \mu + \theta(0)j$.

Table 4.3.1: The Laplace transform of $f^{(\kappa)}_{j,j-1}$ for $j = 1, 5, 8, 9, 10$ evaluated at $t = 10 + 5\mathrm{i}$ in a truncated birth-and-death process $X^{(\kappa)}$ for $\kappa = 10, 11, \ldots, 20$ with parameters $\hat{\lambda}(\delta)$, $\hat{\mu}$ and $S = 1$.

κ	$j = 1$	$j = 5$	$j = 8$	$j = 9$	$j = 10$
10	$0.1055 - 0.0435\mathrm{i}$	$0.1055 - 0.0435\mathrm{i}$	$0.1055 - 0.0435\mathrm{i}$	$0.1055 - 0.0436\mathrm{i}$	$0.1102 - 0.0479\mathrm{i}$
11	$0.1055 - 0.0435\mathrm{i}$	$0.1055 - 0.0435\mathrm{i}$	$0.1055 - 0.0435\mathrm{i}$	$0.1055 - 0.0435\mathrm{i}$	$0.1055 - 0.0436\mathrm{i}$
12	$0.1055 - 0.0435\mathrm{i}$	$0.1055 - 0.0435\mathrm{i}$	$0.1055 - 0.0435\mathrm{i}$	$0.1055 - 0.0435\mathrm{i}$	$0.1055 - 0.0435\mathrm{i}$
13	$0.1055 - 0.0435\mathrm{i}$	$0.1055 - 0.0435\mathrm{i}$	$0.1055 - 0.0435\mathrm{i}$	$0.1055 - 0.0435\mathrm{i}$	$0.1055 - 0.0435\mathrm{i}$
14	$0.1055 - 0.0435\mathrm{i}$	$0.1055 - 0.0435\mathrm{i}$	$0.1055 - 0.0435\mathrm{i}$	$0.1055 - 0.0435\mathrm{i}$	$0.1055 - 0.0435\mathrm{i}$
15	$0.1055 - 0.0435\mathrm{i}$	$0.1055 - 0.0435\mathrm{i}$	$0.1055 - 0.0435\mathrm{i}$	$0.1055 - 0.0435\mathrm{i}$	$0.1055 - 0.0435\mathrm{i}$
16	$0.1055 - 0.0435\mathrm{i}$	$0.1055 - 0.0435\mathrm{i}$	$0.1055 - 0.0435\mathrm{i}$	$0.1055 - 0.0435\mathrm{i}$	$0.1055 - 0.0435\mathrm{i}$
17	$0.1055 - 0.0435\mathrm{i}$	$0.1055 - 0.0435\mathrm{i}$	$0.1055 - 0.0435\mathrm{i}$	$0.1055 - 0.0435\mathrm{i}$	$0.1055 - 0.0435\mathrm{i}$
18	$0.1055 - 0.0435\mathrm{i}$	$0.1055 - 0.0435\mathrm{i}$	$0.1055 - 0.0435\mathrm{i}$	$0.1055 - 0.0435\mathrm{i}$	$0.1055 - 0.0435\mathrm{i}$
19	$0.1055 - 0.0435\mathrm{i}$	$0.1055 - 0.0435\mathrm{i}$	$0.1055 - 0.0435\mathrm{i}$	$0.1055 - 0.0435\mathrm{i}$	$0.1055 - 0.0435\mathrm{i}$
20	$0.1055 - 0.0435\mathrm{i}$	$0.1055 - 0.0435\mathrm{i}$	$0.1055 - 0.0435\mathrm{i}$	$0.1055 - 0.0435\mathrm{i}$	$0.1055 - 0.0435\mathrm{i}$

Table 4.3.2: The Laplace transform of $f^{(\kappa)}_{j,j-1}$ for $j = 1, 5, 8, 9, 10$ evaluated at $t = 10 + 5\mathrm{i}$ in a truncated birth-and-death process $X^{(\kappa)}$ for $\kappa = 10, 11, \ldots, 20$ with parameters $\lambda^{(1)} = 7.46, \lambda^{(2)} = 0.80, \mu = 3.16$ and $\theta(0) = 0.71$.

κ	$j = 1$	$j = 5$	$j = 8$	$j = 9$	$j = 10$
10	0.1759 − 0.0463i	0.2794 − 0.0689i	0.3467 − 0.0833i	0.3832 − 0.0961i	0.4773 − 0.1178i
11	0.1759 − 0.0463i	0.2794 − 0.0689i	0.3454 − 0.0812i	0.3670 − 0.0865i	0.4029 − 0.0986i
12	0.1759 − 0.0463i	0.2794 − 0.0689i	0.3454 − 0.0808i	0.3655 − 0.0844i	0.3863 − 0.0894i
13	0.1759 − 0.0463i	0.2794 − 0.0689i	0.3454 − 0.0808i	0.3654 − 0.0840i	0.3847 − 0.0872i
14	0.1759 − 0.0463i	0.2794 − 0.0689i	0.3454 − 0.0808i	0.3654 − 0.0840i	0.3846 − 0.0868i
15	0.1759 − 0.0463i	0.2794 − 0.0689i	0.3454 − 0.0808i	0.3654 − 0.0840i	0.3846 − 0.0867i
16	0.1759 − 0.0463i	0.2794 − 0.0689i	0.3454 − 0.0808i	0.3654 − 0.0840i	0.3846 − 0.0867i
17	0.1759 − 0.0463i	0.2794 − 0.0689i	0.3454 − 0.0808i	0.3654 − 0.0840i	0.3846 − 0.0867i
18	0.1759 − 0.0463i	0.2794 − 0.0689i	0.3454 − 0.0808i	0.3654 − 0.0840i	0.3846 − 0.0867i
19	0.1759 − 0.0463i	0.2794 − 0.0689i	0.3454 − 0.0808i	0.3654 − 0.0840i	0.3846 − 0.0867i
20	0.1759 − 0.0463i	0.2794 − 0.0689i	0.3454 − 0.0808i	0.3654 − 0.0840i	0.3846 − 0.0867i

Next, the Laplace transform of the density function $f^{(\kappa)}_{j,j-1}$ evaluated at $t = 0.01, 0.02, \ldots, 10$ for $j = 4$ and $\kappa = 4, 5, \ldots, 15$ is given. Table 4.3.3 represents the calculated values and the Figures 4.3.1, 4.3.2 and 4.3.3 illustrate the Laplace transform of $f^{(\kappa)}_{j,j-1}$ for $j = 4$ and $\kappa = 4, 10$ and 15 respectively. The Laplace transforms are almost even, especially for $\kappa = 10$ and $\kappa = 15$. Regarding the model used in this paper, the Laplace transforms for other states j are the same for equal κ distances, meaning that $f^{(\kappa)}_{j,j-1} = f^{(\kappa-j+i)}_{i,i-1}$ for all states j and i. This is because formula (3.3.13) is independent of the state and therefore, the outcome is the same for equal κ distances. This is not true for the model used in [HK12] because the cancellation rate depends on the state.

Table 4.3.3: The Laplace transform of the probability density function $f^{(\kappa)}_{j,j-1}$ of the process $\tau_{j,j-1}$, meaning the first-passage time from j to $j-1$, for $j = 4$ evaluated at $t = 0.1, 1, 10$ with parameters $\hat{\lambda}(\delta)$, $\hat{\mu}$ and $S = 1$.

κ	$\hat{f}^{(\kappa)}_{j,j-1}(0.1)$	$\hat{f}^{(\kappa)}_{j,j-1}(1)$	$\hat{f}^{(\kappa)}_{j,j-1}(10)$
4	0.938	0.601	0.131
5	0.910	0.535	0.124
6	0.899	0.525	0.124
7	0.894	0.524	0.124
8	0.892	0.524	0.124
9	0.891	0.524	0.124
10	0.891	0.524	0.124
11	0.891	0.524	0.124
12	0.891	0.524	0.124
13	0.891	0.524	0.124
14	0.891	0.524	0.124
15	0.891	0.524	0.124

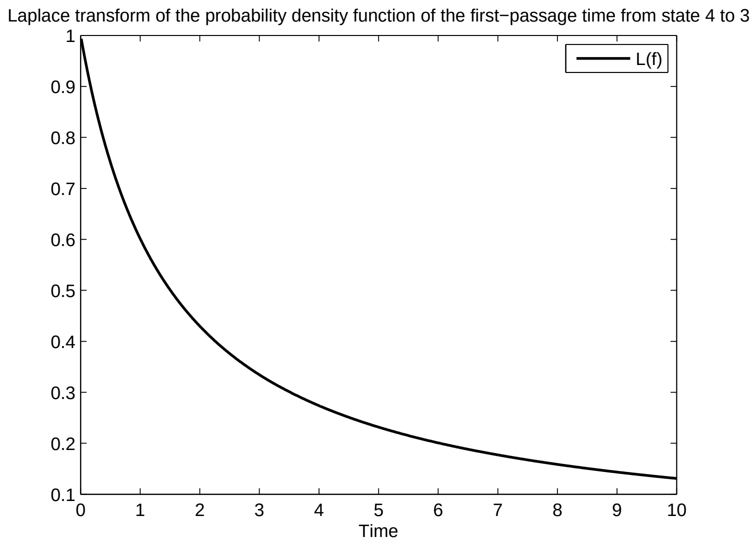

Figure 4.3.1: The Laplace transform of the probability density function $f^{(\kappa)}_{j,j-1}$ of the process $\tau_{j,j-1}$, meaning the first-passage time from j to $j-1$, for $j = 4$ evaluated at $t = 0.01, 0.02, \ldots, 10$ with parameters $\hat{\lambda}(\delta)$, $\hat{\mu}$, $S = 1$ and $\kappa = 4$.

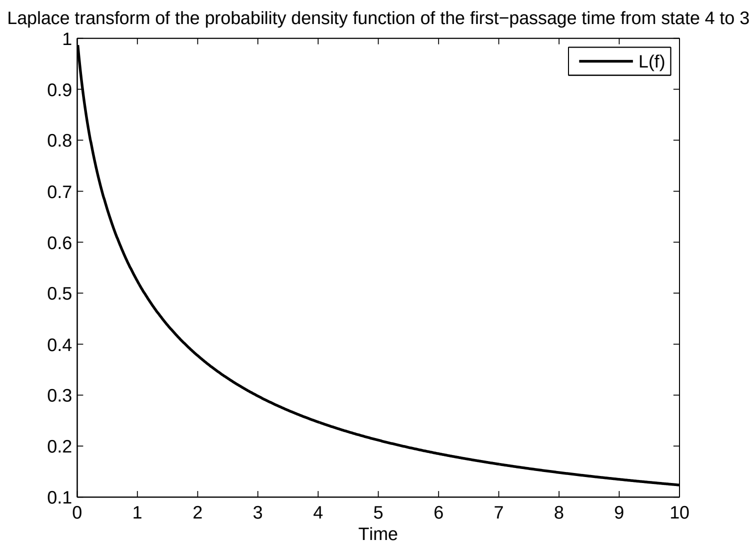

Figure 4.3.2: The Laplace transform of the probability density function $f^{(\kappa)}_{j,j-1}$ of the process $\tau_{j,j-1}$, meaning the first-passage time from j to $j-1$, for $j = 4$ evaluated at $t = 0.01, 0.02, \ldots, 10$ with parameters $\hat{\lambda}(\delta)$, $\hat{\mu}$, $S = 1$ and $\kappa = 10$.

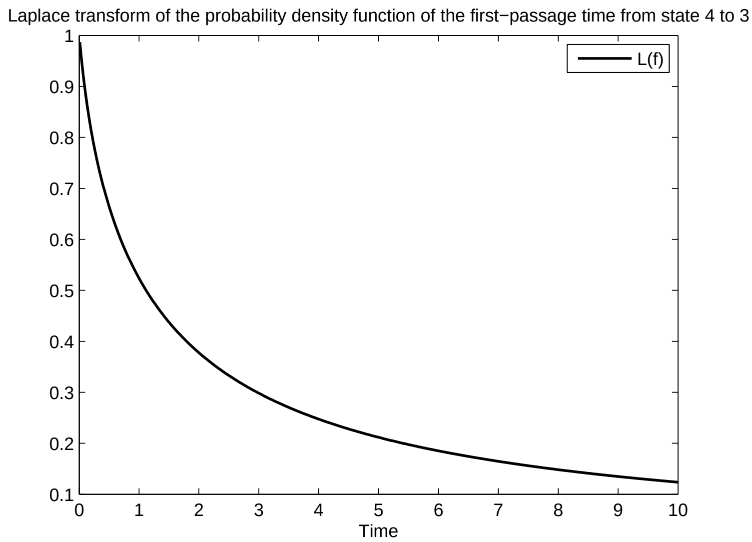

Figure 4.3.3: The Laplace transform of the probability density function $f_{j,j-1}^{(\kappa)}$ of the process $\tau_{j,j-1}$, meaning the first-passage time from j to $j-1$, for $j = 4$ evaluated at $t = 0.01, 0.02, \ldots, 10$ with parameters $\hat{\lambda}(\delta)$, $\hat{\mu}$, $S = 1$ and $\kappa = 15$.

Following the described calculation process, next, in Table 4.3.4 the Laplace transform of $f_b^{(\kappa)}$ evaluated at $t = 0.1, 1, 10$ for $b = 4$ and $\kappa = 4, 5, \ldots, 15$ is presented. The Figures 4.3.4, 4.3.5 and 4.3.6 show the Laplace transform $\hat{f}_b^{(\kappa)}$ for state $b = 4$ with truncation states $\kappa = 4, 10$ and 15 respectively. Again, it is obvious that the Laplace transforms are almost even, especially for $\kappa = 10$ and $\kappa = 15$.

Table 4.3.4: The Laplace transform $\hat{f}_b^{(\kappa)}$ of the process τ_b, meaning the first-passage time from b to zero, for $b = 4$ evaluated at $t = 0.1, 1, 10$ with parameters $\hat{\lambda}(\delta)$, $\hat{\mu}$ and $S = 1$.

κ	$\hat{f}_b^{(\kappa)}(0.1)$	$\hat{f}_b^{(\kappa)}(1)$	$\hat{f}_b^{(\kappa)}(10)$
4	0.686	0.089	$2.477 \cdot 10^{-4}$
5	0.652	0.077	$2.338 \cdot 10^{-4}$
6	0.638	0.075	$2.337 \cdot 10^{-4}$
7	0.633	0.075	$2.337 \cdot 10^{-4}$
8	0.630	0.075	$2.337 \cdot 10^{-4}$
9	0.630	0.075	$2.337 \cdot 10^{-4}$
10	0.629	0.075	$2.337 \cdot 10^{-4}$
11	0.629	0.075	$2.337 \cdot 10^{-4}$
12	0.629	0.075	$2.337 \cdot 10^{-4}$
13	0.629	0.075	$2.337 \cdot 10^{-4}$
14	0.629	0.075	$2.337 \cdot 10^{-4}$
15	0.629	0.075	$2.337 \cdot 10^{-4}$

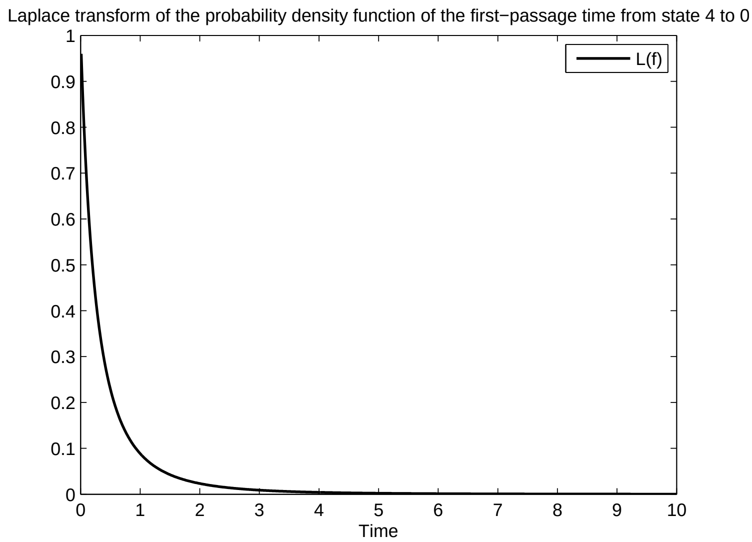

Figure 4.3.4: The Laplace transform $\hat{f}_b^{(\kappa)}$ of the first-passage time τ_b from state 4 to zero with parameters $\hat{\lambda}(\delta)$, $\hat{\mu}$, $S = 1$ and $\kappa = 4$.

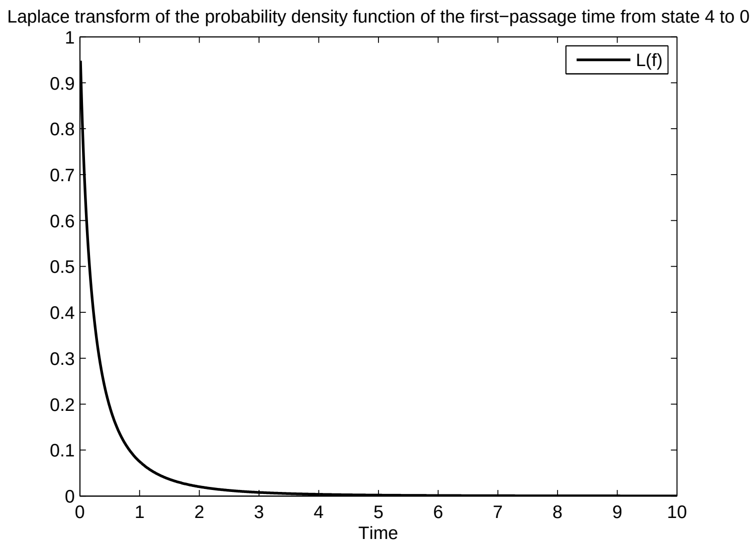

Figure 4.3.5: The Laplace transform $\hat{f}_b^{(\kappa)}$ of the first-passage time τ_b from state 4 to zero with parameters $\hat{\lambda}(\delta)$, $\hat{\mu}$, $S = 1$ and $\kappa = 10$.

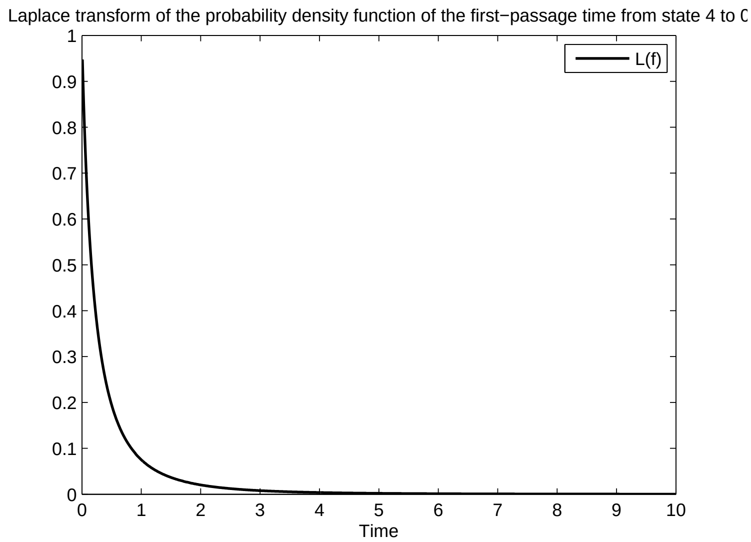

Figure 4.3.6: The Laplace transform $\hat{f}_b^{(\kappa)}$ of the first-passage time τ_b from state 4 to zero with parameters $\hat{\lambda}(\delta)$, $\hat{\mu}$, $S = 1$ and $\kappa = 15$.

Table 4.3.5 represents the density function $f_b^{(\kappa)}$ of the first-passage time from b to zero evaluated at $t = 0.1, 1, 10$. In Table 4.3.6 the distribution function $F_b^{(\kappa)}$ of the first-passage time from b to zero evaluated at the same points as $f_b^{(\kappa)}$ is shown. The values of the distribution function can be calculated as in formula (2.2.7). From both, tables and figures, it is evident that $\kappa = 10$ is an appropriate choice for further calculations because the obtained results tend to remain constant for larger κ values.

Table 4.3.5: The density function $f_b^{(\kappa)}$ of the process τ_b, meaning the first-passage time from b to zero, for $b = 4$ evaluated at $t = 0.1, 1, 10$ with parameters $\hat{\lambda}(\delta)$, $\hat{\mu}$ and $S = 1$.

κ	$f_b^{(\kappa)}(0.1)$	$f_b^{(\kappa)}(1)$	$f_b^{(\kappa)}(10)$
4	$6.992 \cdot 10^{-4}$	0.127	0.017
5	$6.860 \cdot 10^{-4}$	0.111	0.025
6	$6.860 \cdot 10^{-4}$	0.110	0.028
7	$6.860 \cdot 10^{-4}$	0.110	0.028
8	$6.860 \cdot 10^{-4}$	0.110	0.028
9	$6.860 \cdot 10^{-4}$	0.110	0.027
10	$6.860 \cdot 10^{-4}$	0.110	0.027
11	$6.860 \cdot 10^{-4}$	0.110	0.027
12	$6.860 \cdot 10^{-4}$	0.110	0.027
13	$6.860 \cdot 10^{-4}$	0.110	0.027
14	$6.860 \cdot 10^{-4}$	0.110	0.027
15	$6.860 \cdot 10^{-4}$	0.110	0.027

Table 4.3.6: The distribution function $F_b^{(\kappa)}$ of the process τ_b, meaning the first-passage time from b to zero, for $b = 4$ evaluated at $t = 0.1, 1, 10$ with parameters $\hat{\lambda}(\delta)$, $\hat{\mu}$ and $S = 1$.

κ	$F_b^{(\kappa)}(0.1)$	$F_b^{(\kappa)}(1)$	$F_b^{(\kappa)}(10)$
4	$1.823 \cdot 10^{-5}$	0.048	0.954
5	$1.795 \cdot 10^{-5}$	0.043	0.891
6	$1.795 \cdot 10^{-5}$	0.043	0.878
7	$1.795 \cdot 10^{-5}$	0.043	0.872
8	$1.795 \cdot 10^{-5}$	0.043	0.870
9	$1.795 \cdot 10^{-5}$	0.043	0.869
10	$1.795 \cdot 10^{-5}$	0.043	0.869
11	$1.795 \cdot 10^{-5}$	0.043	0.869
12	$1.795 \cdot 10^{-5}$	0.043	0.869
13	$1.795 \cdot 10^{-5}$	0.043	0.869
14	$1.795 \cdot 10^{-5}$	0.043	0.869
15	$1.795 \cdot 10^{-5}$	0.043	0.869

Figure 4.3.7 and Figure 4.3.8 display the graphs of $f_b^{(\kappa)}$ and $F_b^{(\kappa)}$ respectively when $b = 4$ and the truncation state $\kappa = 10$. The graphs are similar to those obtained in [HK12]. In contrast to [HK12], the calculations were done for the state $b = 4$. This value was chosen to be appropriate because in the next step the first-passage times from $b \in \{1, 2, 3, 4\}$ to zero are computed.

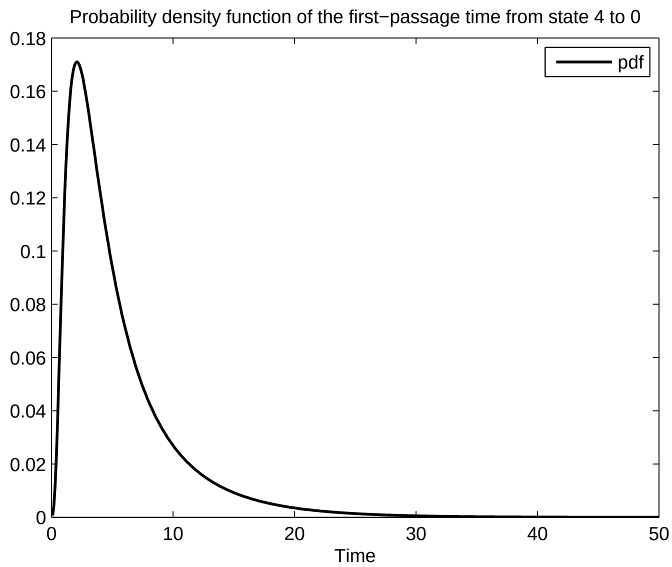

Figure 4.3.7: The density function $f_b^{(\kappa)}$ of the first-passage time τ_b from state 4 to zero with parameters $\hat{\lambda}(\delta)$, $\hat{\mu}$, $S = 1$ and $\kappa = 10$.

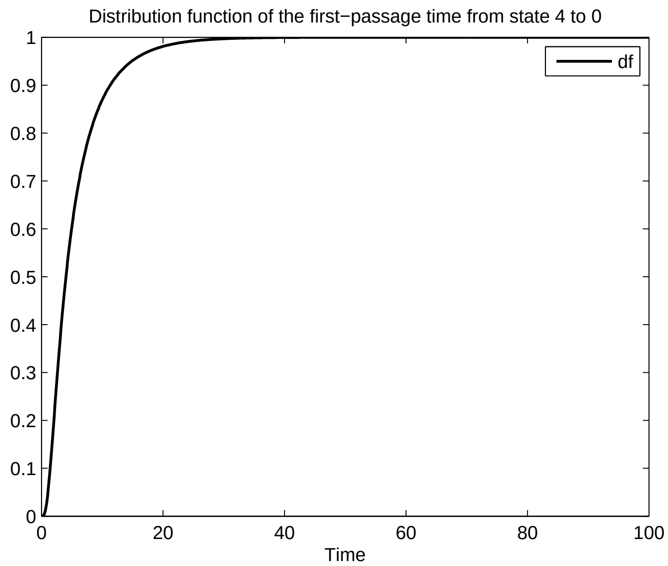

Figure 4.3.8: The distribution function $F_b^{(\kappa)}$ of the first-passage time τ_b from state 4 to zero with parameters $\hat{\lambda}(\delta)$, $\hat{\mu}$, $S = 1$ and $\kappa = 10$.

In Table 4.3.7 the results based on the birth-and-death Markov chain model for different shares a, b and $S = 1$ are given. The truncation bound was chosen to be 10. This table is compared to Table 4.3.8, which contains the calculated frequencies based on the LOB data.
The same calculations are done for $S = 2$. Therefore, Table 4.3.9 represents the model based results, whereas Table 4.3.10 shows the results achieved from the order book data.
Regarding the calculations based on the LOB data, it has to be mentioned that cases as $a = 1, b = 1$ or $a = 1, b = 2$ are not likely to find in an order book. Thus, it has been decided to calculate the ratios of a and b for $a \in \{1, 2, 3, 4\}$, $b \in \{1, 2, 3, 4\}$ and to compare these to the ratios of the bid and ask volumes (cf. Section 2.1.4).

A comparison of Table 4.3.7 and Table 4.3.8 and also of the Tables 4.3.9 and 4.3.10 indicates that the birth-and-death Markov chain model predicts the outcomes achieved from the LOB data quite good. All tables show that the larger the b values the higher is the probability of a mid-price increase, whereas the larger the a values the lower is the probability of a mid-price increase. Similar results are obtained in [CST10, HK12].

Table 4.3.7: Probability that the mid-price increases at its next move with parameters $\hat{\lambda}(\delta)$, $\hat{\mu}, \kappa = 10$ and $S = 1$, where the column labels indicate the number of initial shares at the ask and the row labels signify the number of initial shares at the bid.

	1	2	3	4
1	0.500	0.279	0.170	0.111
2	0.721	0.500	0.348	0.246
3	0.829	0.651	0.500	0.380
4	0.886	0.751	0.617	0.497

Table 4.3.8: Probability that the mid-price increases at its next move calculated based on the LOB data, where the column labels indicate the number of initial shares at the ask and the row labels signify the number of initial shares at the bid with parameters $S = 1$ and $\kappa = 10$.

	1	2	3	4
1	0.515	0.355	0.306	0.276
2	0.605	0.515	0.410	0.355
3	0.659	0.553	0.515	0.442
4	0.670	0.650	0.528	0.515

Table 4.3.9: Probability that the mid-price increases at its next move with parameters $\hat{\lambda}(\delta)$, $\hat{\mu}, \kappa = 10$ and $S = 2$, where the column labels indicate the number of initial shares at the ask and the row labels signify the number of initial shares at the bid.

	1	2	3	4
1	0.500	0.342	0.290	0.274
2	0.658	0.500	0.433	0.407
3	0.710	0.567	0.500	0.470
4	0.727	0.594	0.530	0.500

Table 4.3.10: Probability that the mid-price increases at its next move calculated based on the LOB data, where the column labels indicate the number of initial shares at the ask and the row labels signify the number of initial shares at the bid with parameters $S = 2$ and $\kappa = 10$.

	1	2	3	4
1	0.532	0.485	0.465	0.519
2	0.554	0.532	0.493	0.485
3	0.599	0.560	0.532	0.504
4	0.675	0.554	0.548	0.532

CHAPTER 5

Conclusion

A stochastic model which describes the LOB dynamics was introduced. The order arrivals were modeled using independent Poisson processes and the associated parameters were estimated from the order book. For the computation of a mid-price increase at its next move conditional on the current order book status, Laplace transform methods were used. Due to the fact that the probability density functions of first-passage times to neighboring states are completely monotone, the Laplace transforms of these times have desirable convergence properties. Thus, the infinite state space was replaced by a truncated one and therefore, it was possible to apply an approximation method of the Laplace transform of the first-passage time's density function to zero.

The results indicate that the model is working and predicts the outcome achieved from the LOB data quite well. At some points, for example when focusing on the outcomes of the Laplace transform, the results differ from those observed in similar studies. This is reflected by the non-existent cancellation orders.

The model can be extended in various ways to make it more realistic. A different approach could be to model the order arrivals by Brownian motions instead of governing the occurrences of market events, meaning limit and market orders, which are assumed to arrive in unit size, by independent Poisson processes. This would conserve the Markov property of the process and would achieve more flexibility regarding order arrival sizes. An important fact is that since 2010 orders at the ISE are allowed to be canceled. Therefore, this model is not actual and has to be reviewed, meaning that the parameters based on recent ISE data have to be re-estimated, to obtain a real-time model. Additionally, introducing a model which allows dependent arrival rates as well as arrival times of limit and market orders could be interesting.

APPENDIX A

Data

Remark A.0.1 (Aggressiveness Type). *According to [VZ13], orders are divided into five categories based on the limit price position.*

1. *Category* 1 *(large market order buy):* $Q_{buy} \geq Q_{ask}$ *and* $P_{buy} \geq P_{ask}$
2. *Category* 2 *(small market order buy):* $Q_{buy} < Q_{ask}$ *and* $P_{buy} \geq P_{ask}$
3. *Category* 3 *(buy limit order within the quote):* $P_{ask} > P_{buy} > P_{bid}$
4. *Category* 4 *(buy limit order at the quote order):* $P_{ask} > P_{buy} = P_{bid}$
5. *Category* 5 *(buy limit order away from the quotes):* $P_{ask} > P_{bid} > P_{buy}$

The sell side is constructed analogously (Category 6 *to* 10*).*

Table A.0.1: Description of the columns of the order book

OrderID	Identity number of the submitted order.
Order Type (OT)	Type of the order (if $1, 2, 3$ then the order is a buy order, otherwise $4, 5, 6, 7, 8, 9$ it is a sell order).
Ticker	Stock number.
Quantity (Quant.)	Number of shares to be bought or sold.
Price	Price of the order.
Time in force (TIF)	0 if the order is valid/active for one session, 1 if it is valid for the whole day.
Time	Time of the order submitted/traded.
Client Type (CT)	Whether the order is submitted by an individual or institutional client (2 if institutional).
KTR	Immediate or cancel orders (1000 if the order is of this type).
Bid	Best buy price, i.e highest available buy price.
Ask	Best sell price, i.e lowest available sell price.
Spread (S)	Difference between bid and ask.
dist.$^{\text{bid/ask}}$	Price distance measured in ticks to the best available price (bid or ask)[a].
dist.$^{\text{trade}}$	Price distance to the trade price.
Volume at bid (VB1)	Numbers of shares at the bid.
Volume at ask (VA1)	Numbers of shares at the ask.
Aggressiveness Type (AT)	See remark A.0.1.
B2	Second best buy price.
A2	Second best sell price.
VB2	Number of shares at B2.
VA2	Number of shares at A2.

[a] Depending on whether the order is a buy or a sell order.

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