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Hook-length formulas for trees: a general approach

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I dedicate this work to my family, who always supported me, all my friends, who stood by me and all the people, who guided me through the last years!

Abstract:

Starting with an expansion technique for binary trees obtained by Han [Han08a], several other authors, e.g. Chen, Yang, Kuba and Panholzer [KP13, CGG09, Yan08] have unified and extended this approach to various other tree families, such as k-ary trees, labelled trees and forests, weighted trees and increasing trees.

In this thesis, we take a look at the original hook-length expansion technique, explain how Han obtained it, give an explanation of how exactly hook-length formulas are derived and list examples of the most significant ones. Furthermore, we examine, how this formula was extended to fit other tree families and, on the basis of this results, show unifications and connections between certain tree families.

Keywords: Hook-length formulas, Binary trees, K-ary trees, Ordered trees, Labelled trees, Weigthed trees, Increasing trees.

Zusammenfassung:

Beginnend mit einer Expansionsmethode für binäre Bäume, abgeleitet von Han [Han08a], haben mehrere andere Autoren, z.B. Chen, Yang Kuba und Panholzer [KP13, CGG09, Yan08] diesen Ansatz vereinheitlicht und auf verschiedene andere Baumfamilien wie k-äre Bäume, markierte Bäume und Wälder, gewichtete Bäume und aufsteigend markierte Bäume ausgeweitet.

In dieser Arbeit werfen wir einen Blick auf die ursprüngliche Hakenlängenexpansionsmethode, erläutern, wie Han sie erhalten hat, erklären, wie genau Hakenlängenformeln abgeleitet werden, und listen Beispiele mit den wichtigsten Ergebnissen auf. Außerdem untersuchen wir, wie diese Formel auf andere Baumfamilien erweitert wurde, und zeigen basierend auf dieser Grundlage Vereinheitlichungen und Verbindungen zwischen bestimmten Baumfamilien auf.

Schlüsselwörter: Hakenlängenformeln, Binäre Bäume, K-äre Bäume, Geordnete Bäume, Markierte Bäume, Gewichtete Bäume, Aufsteigend markierte Bäume.

Statutory declaration

I hereby declare that I have written this diploma thesis independently and that I have not used any sources and aids other than the specified ones and that I have identified the ideas taken directly or indirectly from foreign sources as such. So far, I have not submitted the work to any other examination office in the same or similar form. It has not been published yet. I agree that the work will be checked for plagiarism with the help of a plagiarism detection service.

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Contents

1 Introduction

Given a rooted tree T, the hook-length of a vertex $v \in T$, denoted by $h_v := h(v)$, is the number of decendants of node v. Several identities involving the hook-length, so called hook-length formulas, have been discovered, essentially starting with the following formula, which Postinikov derived for binary trees $\mathfrak{B}(n)$ of size n, see [Pos05]:

$$\frac{n!}{2^n} \sum_{T \in \mathfrak{B}(n)} \prod_{v \in T} \left(1 + \frac{1}{h_v}\right) = (n+1)^{n-1}.$$

Following this, Han developed a versatile expansion technique, see [Han08a]. This technique can be used for deriving further hook-length formulas for binary trees by showing that for any given power series $F(z)$ with $F(0) = 0$, it's possible to determine the weight function $\rho(n)$, if

$$\sum_{n \geq 1} \left(\sum_{T \in \mathfrak{B}(n)} \prod_{v \in T} \rho(h_v) \right) z^n = F(z),$$

holds.
Han used his expansion technique to obtain further hook-length formulas, see [Han08b, Han08c]. For some of these formulas combinatorial proofs do exist, for others they are yet to be found.
Based on this technique Chen, Gao and Guo found several extensions for k-ary trees, ordered trees and forests and labelled trees and forests, see [CGG09]. By finding appropriate generating functions, they derived new hook formulas or replicated results from various authors.
By considering weighted tree families and due to the fact, that different simply generated tree models are obtained by using weighted ordered trees, where each node v in an ordered tree $T \in \mathfrak{O}$ gets a certain degree-weight factor depending on the out-degree of v, Kuba and Panholzer [KP13] proposed an expansion technique which further unifies recent results.

This thesis is organized as follows. In Chapter 2 we list the most basic terminology, define generating functions and list often used identities and the most important theorems. The third chapter is dedicated to the derivation and the proof of Han's expansion technique. We then use this technique and derive some hook-length formulas for binary trees before considering two special types of binary trees: Fibonacci trees

and complete binary trees.
Our goal in the fourth chapter is to extend the results from the third chapter to other trees. We look at k-ary trees, orderded trees and forests and labelled trees and forests. We also give serveral hook-length formulas for each of the considered tree families. The aim of Chapter 5 is to achieve a generalization of all previous results by expanding the expansion technique using weighted trees.
In Chapter 6 we take a closer look at increasingly labelled trees and extend our results to k-labelled increasing trees.

2 Basic definitions and terminology

In this section we list mathematical definitions and necessary identities, which we will use several times throughout the whole thesis.

2.1 Trees and weight-functions

Since we are dealing with various types of trees, forests and weight-functions in this thesis, we start with a review of the necessary terminology.

Definition 2.1. *A tree is an undirected graph in which any two vertices are connected by exactly one path. In other words, any acyclic connected graph is a tree.*
A rooted tree is a tree in which one vertex has been designated the root, see [Knu97, p. 285-399].

Definition 2.2. *A forest is a disjoint union of trees.*

We will define the special types of trees we examine at the beginning from their respective chapter.
Furthermore, we need a few basics about weight-functions, which we will further expand in Chapter 5.

Definition 2.3. *Given a rooted tree T, the hook-length of a vertex $v \in T$, denoted by $h_v := h(v)$, is the number of decendants of vertex v, where we use the convention that each vertex v is always considered to be a decendant of itself. A vertex w is a decendant of v, iff v lies in the unique path from the root of T to w.*

Since our calculations can get quite complicated from time to time, it's convenient to introduce the following:

Definition 2.4. *Given a weight function $\rho : \mathbb{N}^+ \to \mathbb{C}$ we can associate with any given tree T of some rooted tree family the hook-weight $w_{hook}(T)$ via*

$$w_{hook}(T) := \prod_{v \in T} \rho(h_v). \tag{2.1}$$

We call ρ the hook-function or hook-weight-function.

2.2 Generating functions

Due to the importance of generating functions in this thesis, we dedicate this section for a proper introduction of the key aspects.

Definition 2.5. *The ordinary generating function of a sequence* $(a_n)_{n\geq 0}$ *is*

$$A(z) = \sum_{n\geq 0} a_n z^n.$$

Therefore, the sought-after information is saved in the coefficients of the formal power-series $A(z)$.
Since we deal with combinatorial enumeration problems that involve labelled objects, it's sometimes more convenient to use the exponential generating function than ordinary generating functions.

Definition 2.6. *The exponential generating function* $A(z)$ *of a sequence* $(a_n)_{n\geq 0}$ *is defined as*

$$A(z) = \sum_{n=0}^{\infty} a_n \frac{z^n}{n!}.$$

To extract the coefficients we use the coefficient extraction operator.

Definition 2.7. *Let* $A(z) = \sum_{n\geq 0} a_n z^n$ *be a generating function. Then* $[z^n]$ *denotes the coefficient extraction operator and* $[z^n]\, A(z) := a_n$ *delivers the coefficient of* z^n *in the formal power series expansion of* $A(z)$*. Two formal power series* $A(z), B(z)$ *are equal, if and only if* $[z^n]A(z) = [z^n]B(z)$, $\forall n \in \mathbb{N}$,

Example 1. Let $A(z) = \sum_{n\geq 0} \frac{2^n}{n!} z^n$. Then $[z^n]A(z)$ delivers $\frac{2^n}{n!}$.

We can also define arithmetic operations on the set of formal power-series $R[[z]] := \{\sum_{n\geq 0} a_n z^n : a_n \in \mathbb{R}\}$ in the following way:

$$+ : \quad A(z) + B(z) = \sum_{n\geq 0} (a_n + b_n) z^n,$$

$$\cdot : \quad A(z) \cdot B(z) = \sum_{n\geq 0} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) z^n.$$

Now we need the definition of an inverse power-series.

Definition 2.8. *Let* $A(z) = \sum_{n\geq 0} a_n z^n$ *be the ordinary generating function of a sequence* a_n*. We call the generating function* $B(z) = \sum_{n\geq 0} b_n z^n$ *(multiplicatively) inverse to* $A(z)$ *if* $A(z) \cdot B(z) = 1$*.*

Theorem 2.9. *The generating function* $A(z) = \sum_{n\geq 0} a_n z^n$ *has an (multiplicative) inverse generating function if and only if* $a_0 \neq 0$*.*

Proof. A proof can be found in [Wil94, p. 31]. □

If one represents a sequence as a generating function, certain manipulations of the sequence correspond to special manipulations of the generating function. We list the most important ones:

Lemma 2.10. *Let* $A(z) = \sum_{n\geq 0} a_n z^n$ *be the ordinary generating function of a sequence* a_n *and* $k \in \mathbb{N}$*. Then it holds*

Generating function	*sequence*
$A'(z)$	$((n+1)a_{n+1})_{n\geq 0}$
$\frac{1}{2}(A(z) + A(-z))$	$a_0, 0, a_2, 0, a_4, \ldots$
$\frac{1}{2}(A(z) - A(-z))$	$0, a_1, 0, a_3, 0 \ldots$
$z^k A(z)$	$(a_{n-k})_{n\geq 0}$

Proof. Through simple calculation. □

We'll also often need the Cauchy product of power series:

Definition 2.11. *The Cauchy product of two power series is defined as follows:*

$$\left(\sum_{n\geq 0} a_n z^n\right) \cdot \left(\sum_{n\geq 0} b_n z^n\right) = \sum_{n\geq 0} \left(\sum_{\substack{n_1+n_2=n \\ n_1,n_2\geq 0}} a_{n_1} b_{n_2}\right) z^n \tag{2.2}$$

2.2.1 Important power series

Following results and identities concerning power series are quite useful and we'll benefit from them throughout the whole thesis:

- Taylor series, see Taylor's theorem (Theorem 2.12), of essential functions:

  - Expansion of e^z:

$$e^z = \sum_{n\geq 0} \frac{1}{n!} z^n. \tag{2.3}$$

– Mercator series:

$$\ln(1+z) = \sum_{n\geq 1} \frac{(-1)^{n+1}}{n} z^n. \tag{2.4}$$

– Another useful logarithm expansion:

$$\ln\left(\frac{1}{1-z}\right) = \sum_{n\geq 1} \frac{1}{n} z^n. \tag{2.5}$$

– Sum of the geometric series:
For $|z| < 1$ the n-th partial sum of $A(z) = \sum_{n\geq 0} z^n$ fulfills

$$\sum_{n=0}^{m-1} z^n = \frac{1-z^m}{1-z}.$$

As $m \to \infty$, the absolute value of z must fulfill $|z| < 1$ for the series to converge. The sum then becomes

$$\sum_{n\geq 0} z^n = \frac{1}{1-z}. \tag{2.6}$$

There exists a useful generalisation of this identity for $m \geq 1 \in \mathbb{N}$:

$$\frac{1}{(1-z)^m} = \sum_{n\geq 0} \binom{m+n-1}{m-1} z^n. \tag{2.7}$$

- The binomial theorem, see [Coo49].
The binomial theorem is used to expand binomial expressions $(a+b)^n$ raised to any given power by

$$(a+b)^n = \sum_{k\geq 0} \binom{n}{k} a^k b^{n-k}. \tag{2.8}$$

We in particular need one special case ($m \in \mathbb{N}$):

$$(1+z)^m = \sum_{n\geq 0} \binom{m}{n} z^n. \tag{2.9}$$

2.3 Main theorems

Following two theorems are of great importance for numerous proofs and derivations presented in this thesis.

Theorem 2.12 (Taylor's theorem)**.** *Let $k \geq 1$ be an integer and let the function $f : \mathbb{R} \to \mathbb{R}$ be k-times differentiable at the point $a \in \mathbb{R}$. Then there exists a function $h_k : \mathbb{R} \to \mathbb{R}$, such that*

$$f(x) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n + h_k(x)(x-a)^k,$$

with $\lim_{x \to a} h_k(x) = 0$.

Proof. A proof can be found in [For08]. □

Following this theorem, we can define the Taylor series:

Definition 2.13. *The Taylor series of a real or complex-valued function $f(x)$, that is infinitely differentiable at a real or complex number a, is the power series, which fulfills*

$$f(x) = \sum_{n \geq 0} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Theorem 2.14 (Lagrange Inversion formula)**.** *Let $\phi(u) = \sum_{n \geq 0} \phi_n u^n$ be a power series in $\mathbb{C}[[u]]$ with $\phi_0 \neq 0$. Then, the equation $y = z\phi(y)$ admits a unique solution in $\mathbb{C}[[z]]$, whose coefficients are given by*

$$[z^n]y(z) = \frac{1}{n}[u^{n-1}]\phi(u)^n, \quad n \geq 1.$$

Furthermore, let H be an arbitrary function. Then it holds

$$[z^n]H(y(z)) = \frac{1}{n}[u^{n-1}]\left(H'(u)\phi(u)^n\right), \quad n \geq 1, \tag{2.10}$$

which is known as the ***Lagrange-Bürmann formula****.*

Proof. A proof can be found in [FS09]. □

3 Han's expansion formula for binary trees

Han obtained a general way to calculate the weight function corresponding to the expansion of a series considering binary trees. Before we generalize his result for various other families, we give an explanation of his method.

Before we start to examine how Han derived his method, it is recommended to define binary trees recursively, due to easy handling and nice properties of the regarding power series.

Definition 3.1. *A binary tree T with size $|T| = n \geq 1$ is recursicely defined as:*

- *There is one specially designated vertex $v \in T$, which is called the root of T.*
- *The remaining vertices (excluding the root) are then displayed in an ordered pair (T_1, T_2), where T_1, T_2 are either binary trees or empty subtrees.*

A vertex is called leaf, if its two subtrees are both empty.

Let $\mathfrak{B}(n)$ denote the set of all binary trees of size n, so that

$$\mathfrak{B} := \bigcup_{n \geq 1} \mathfrak{B}(n),$$

is the set of all binary trees.
We will use this way often to define trees and furthermore splitting it into the root and his subtrees. Most theorems, lemmas and corollaries are proved by this recursive definition and considering all possibilities of building a tree.

In his paper, see [Han08a], Han uses the concept of the hook-length expansion to derive hook-length formulas for binary trees, defined as follows:

Definition 3.2. *The left-side of the following equation is called hook-length expansion:*

$$\sum_{T \in \mathfrak{B}} z^{|T|} \prod_{v \in T} \rho(h_v) = F(z), \tag{3.1}$$

where $F(z) \in K[[z]]$ is a formal power series with coefficients in K.

We will use the following equivalent form of (3.1) throughout the whole thesis:

$$F(z) = \sum_{n \geq 1} \left(\sum_{T \in \mathfrak{B}(n)} \prod_{v \in T} \rho(h_v) \right) z^n. \tag{3.2}$$

Its called expansion technique, because it starts with an expansion of the generating function linked to the generating function of the sum of the products of the weights of all trees with the same given amount of vertices.
Furthermore, Han defines $F(z) = f_1 z + f_2 z^2 + \cdots$, the generating function for binary trees under the weight-function ρ.
Now we can prove Han's Theorem:

Theorem 3.3. *Let $\mathfrak{B}$ be the family of binary trees associated with a weight function ρ and let $F(z)$ be the generating function of the total weights of trees of size $n \geq 1$:*

$$\sum_{n\geq 1}\left(\sum_{T\in\mathfrak{B}(n)}\prod_{v\in T}\rho(h_v)\right)z^n = F(z). \tag{3.3}$$

Then the weight-function ρ satisfies

$$\rho(n) = \frac{[z^n]F(z)}{[z^{n-1}](1+F(z))^2}, \quad n \geq 1. \tag{3.4}$$

Proof. To simplify proofs it is advantageous to define $\mathfrak{B}(0) = \{\epsilon\}$, i.e., the empty tree, although $\mathfrak{B}(0)$ is formally not contained in $\mathfrak{B}$. We define

$$[z^n]F(z) = f_n = \sum_{T\in\mathfrak{B}(n)}\prod_{v\in T}\rho(h_v).$$

Due to our recursive definition of binary trees, we can identify each binary tree with its root v and two subtrees T_1 and T_2. So every binary tree corresponds with a triplet (T_1, T_2, v), where $h_v = n$ and

$$T_1 \in \mathfrak{B}(n_1),\ T_2 \in \mathfrak{B}(n_2),\ n_1 + n_2 = n - 1,\ n_1, n_2 \geq 0.$$

Since n_1 and n_2 can be empty, according to our definition, but according to (3.2) $F(z)$ doesn't consider empty binary trees, we have to take a look at $1 + F(z)$ instead by adding $f_0 = 1$.

Considering, how a binary tree splits into its root, its subtrees and dealing with all possibilities, we can rearrange (3.3) and get:

$$
\begin{aligned}
[z^n]F(z) = &\sum_{T\in\mathfrak{B}(n)}\prod_{v\in T}\rho(h_v)\\
&= \rho(n)\sum_{\substack{n_1+n_2=n-1\\ n_1,n_2\geq 0}}\left(\sum_{T_1\in\mathfrak{B}(n_1),T_2\in\mathfrak{B}(n_2)}\prod_{u\in T_1,v\in T_2}\rho(h_u)\cdot\rho(h_v)\right)\\
&= \rho(n)\sum_{\substack{n_1+n_2=n-1\\ n_1,n_2\geq 0}}\left(\sum_{T_1\in\mathfrak{B}(n_1)}\prod_{v\in T_1}\rho(h_v)\right)\cdot\left(\sum_{T_2\in\mathfrak{B}(n_2)}\prod_{u\in T_2}\rho(h_u)\right)\\
&= \rho(n)\sum_{\substack{n_1+n_2=n-1\\ n_1,n_2\geq 0}} f_{n_1}\cdot f_{n_2}.
\end{aligned}
$$

Using (2.2) and simplifying with the $[z^n]$ operator leads to

$$
\begin{aligned}
&[z^n]F(z) = \rho(n)[z^{n-1}](1+F(z))^2\\
&\Rightarrow \rho(n) = \frac{[z^n]F(z)}{[z^{n-1}](1+F(z))^2}, \quad n\geq 1.
\end{aligned}
$$

□

Han's expansion technique enables us to calculate $F(z)$ for a given weight function ρ, or vica versa, to calculate ρ for a given $F(z)$ under the premise, that $\frac{[z^n]F(z)}{[z^{n-1}](1+F(z))^2}$ has a nice form.

If both ρ and $F(z)$ have compact forms, we call the resulting formula hook-length formula. The hook-length formula is then given by the identity of the coefficients:

$$
\sum_{T\in\mathfrak{B}(n)}\prod_{v\in T}\rho(h_v) = [z^n]F(z).
$$

3.1 Hook-length formulas for binary trees

In this section, we'll use (3.4) and several known power series to duplicate some results given in [Han08a, Pos05, Han08c, Han08b] and conclude hook-length formulas.

Corollary 3.4. *It holds*

$$1 + \sum_{n \geq 1} \left(\sum_{T \in \mathfrak{B}(n)} \prod_{v \in T} \frac{1}{h_v 2^{h_v - 1}} \right) z^n = e^z.$$

Proof. Han presented two proofs for this theorem.
We present one proof using (3.4), the other proof uses induction and the recursice definition of binary trees, see [Han08b].

With (2.3) we can obtain

$$(e^z)^2 = e^{2z} = \sum_{n \geq 0} \frac{(2z)^n}{n!} = \sum_{n \geq 0} \frac{2^n}{n!} z^n.$$

Therefore, by setting $F(z) = e^z - 1$, and with (3.4)

$$\rho(n) = \frac{[z^n]F(z)}{[z^{n-1}](1 + F(z))^2} = \frac{[z^n]e^z - 1}{[z^{n-1}]e^{2z}} = \frac{\frac{1}{n!}}{\frac{2^{n-1}}{(n-1)!}} = \frac{(n-1)!}{2^{n-1}n!}$$

$$\rho(n) = \frac{1}{n \cdot 2^{n-1}}, \quad n \geq 1,$$

yielding the hook-length formula

$$\sum_{T \in \mathfrak{B}(n)} \prod_{v \in T} \frac{1}{h_v 2^{h_v - 1}} = \frac{1}{n!},$$

and therefore

$$1 + \sum_{n \geq 1} \left(\sum_{T \in \mathfrak{B}(n)} \prod_{v \in T} \frac{1}{h_v 2^{h_v - 1}} \right) z^n = e^z.$$

□

Corollary 3.5. *It holds*

$$1+\sum_{n\geq 1}\left(\sum_{T\in\mathfrak{B}(n)}\prod_{v\in T}\frac{1}{h_v}\right)z^n=\frac{1}{1-z}.$$

Proof. We use (3.4) and $F(z)=\frac{1}{1-z}-1$. From (2.6) we know

$$\frac{1}{1-z}=\sum_{n\geq 0}z^n\Rightarrow\frac{1}{1-z}-1=\frac{z}{1-z}=\sum_{n\geq 1}z^n.$$

Since $F'(z)$ fulfills

$$F'(z)=\frac{1}{(1-z)^2}=(1+F(z))^2=\sum_{n\geq 1}nz^{n-1},\tag{3.5}$$

we, according to (3.4), get

$$\begin{aligned}\rho(n)&=\frac{[z^n]F(z)}{[z^{n-1}](1+F(z))^2}=\frac{[z^n]\frac{z}{1-z}}{[z^{n-1}]\frac{1}{(1-z)^2}}\\&=\frac{1}{n},\quad n\geq 1.\end{aligned}$$

Thus, we obtain the hook-length formula

$$\sum_{T\in\mathfrak{B}(n)}\prod_{v\in T}\frac{1}{h_v}=1.$$

and

$$1+\sum_{n\geq 1}\left(\sum_{T\in\mathfrak{B}(n)}\prod_{v\in T}\frac{1}{h_v}\right)z^n=\frac{1}{1-z}.$$

Additionally we present the following combinatorial proof:
It has been shown [GS05], that the number of ways $|\mathbb{L}_I(T)|$ to label the vertices of T with $\{1,2,\ldots,n\}$, so that every vertex has a higher value than all its descendants, is

$$|\mathbb{L}_I(T)|=\frac{n!}{\prod_{v\in T}h_v}.\tag{3.6}$$

These trees are called increasingly labelled binary trees, which we will further examine in Chapter 6, where we will prove a generalization of (3.6) as well. Furthermore it has been shown, that each increasingly labelled binary tree with n vertices is in bijection with a permutation of order n, see [Sta86, p.24].

$$\sum_{T \in \mathfrak{B}(n)} |\mathbb{L}_I(T)| = n!,$$

or

$$\sum_{T \in \mathfrak{B}(n)} n! \prod_{v \in T} \frac{1}{h_v} = n!.$$

Now we divide by $n!$ and transform the resulting equation into a generating function:

$$1 + \sum_{n \geq 1} \left(\sum_{T \in \mathfrak{B}(n)} \prod_{v \in T} \frac{1}{h_v} \right) z^n = \sum_{n \geq 0} z^n.$$

With (2.9) this equation is equal to

$$1 + \sum_{n \geq 1} \left(\sum_{T \in \mathfrak{B}(n)} \prod_{v \in T} \frac{1}{h_v} \right) z^n = \frac{1}{1-z}.$$

□

Corollary 3.6. *It holds*

$$1 + \sum_{n \geq 1} \left(\sum_{T \in \mathfrak{B}(n)} \prod_{v \in T} 1 \right) z^n = \frac{1 - \sqrt{1-4z}}{2z}. \tag{3.7}$$

Proof. Since the right side of (3.7) is the formal power series of the Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$, the corollary implies, that the number of binary trees with n vertices is equal to the n-th Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$, see [Sta99, p.220].
This fact alone would be sufficient to prove the corollary. Instead we'll use (3.4), define

$F(z) = \frac{1-\sqrt{1-4z}}{2z} - 1$ and obtain

$$\begin{aligned}(1+F(z))^2 &= \left(\frac{1-\sqrt{1-4z}}{2z}\right)^2 = \frac{1-2\cdot\sqrt{1-4z}+1-4z}{4z^2} \\ &= \frac{2\cdot(1-\sqrt{1-4z}-2z)}{4z^2} = \frac{1-\sqrt{1-4z}-2z}{2z^2} \\ &= \frac{1-\sqrt{1-4z}}{2z^2} - \frac{2z}{2z^2} = \frac{1-\sqrt{1-4z}}{2z}\cdot\frac{1}{z} - \frac{1}{z} \\ &= \left(\frac{1-\sqrt{1-4z}}{2z} - 1\right)\cdot\frac{1}{z} = F(z)\cdot\frac{1}{z}.\end{aligned}$$

We use this result and conclude for $n \geq 1$:

$$[z^{n-1}](1+F(z))^2 = [z^{n-1}]F(z)/z = [z^n]F(z). \tag{3.8}$$

So with (3.4) and (3.8), we get

$$\rho(n) = \frac{[z^n]F(z)}{[z^{n-1}](1+F(z))^2} = \frac{[z^n]F(z)}{[z^n]F(z)} = 1, \quad n \geq 1.$$

We obtain the hook-length formula

$$\sum_{T\in\mathfrak{B}(n)}\prod_{v\in T} 1 = \frac{1}{n+1}\binom{2n}{n},$$

which proves the corollary. □

In our introduction, we mentioned the basic formula from Postnikov and with use of the Lagrange Inversion formula (2.10) and Han's theorem (3.3), we can now prove it in a different way, than Postnikov did.

Corollary 3.7 (Postnikov)**.** *It holds*

$$1 + \sum_{n\geq 1}\left(\sum_{T\in\mathfrak{B}(n)}\prod_{v\in T}\left(1+\frac{1}{h_v}\right)\right)z^n = \sum_{n\geq 0}\frac{(n+1)^{n-1}2^n}{n!}z^n. \tag{3.9}$$

Proof. Let P(z) be a power series which fulfills

$$P(z) = e^{z\cdot P(z)}.$$

Now we substitute $W(z) = \ln(P(z))$, which implies $P(z) = e^{W(z)}$. So we can rearrange

$$P(z) = e^{z \cdot P(z)} \Rightarrow \ln(P(z)) = z \cdot P(z), \tag{3.10}$$

$$W(z) = z \cdot e^{W(z)}. \tag{3.11}$$

$W(z)$ is the exponential generating function of the numbers n^{n-1} enumerating rooted labelled trees of size n. Therefore $e^{W(z)}$ denotes the exponential generating function enumerating rooted labelled forests of size n. With the Lagrange Inversion formula (2.10) for $H(z) = H'(z) = e^z$ and $\phi(z) = e^z$, we get the following equation

$$\begin{aligned}
[z^n]e^{W(z)} &= \frac{1}{n}[z^{n-1}]e^z \cdot e^{nz} = \frac{1}{n}[z^{n-1}]e^{(n+1)z} \\
&= \frac{1}{n}[z^{n-1}]\sum_{k\geq 0}\frac{(n+1)^k}{k!}z^k \\
&= \frac{1}{n} \cdot \frac{(n+1)^{n-1}}{(n-1)!} \\
&= \frac{(n+1)^{n-1}}{n!}.
\end{aligned}$$

With these coefficients, we have the explicit expansion of $P(z)$:

$$P(z) = 1 + \sum_{n\geq 1}\frac{(n+1)^{n-1}}{n!}z^n,$$

which implies

$$P(2z) = 1 + \sum_{n\geq 1}\frac{(n+1)^{n-1}2^n}{n!}z^n. \tag{3.12}$$

The right-hand side of (3.9) is exactly $P(2z)$. To use (3.4) we further need $P(2z)^2$, which is just a slight modification of our previous calculations.

In this case we get $H(z) = (e^z)^2$, $H'(z) = 2 \cdot (e^z)^2$ and $\phi(z) = e^z$. Thus it follows

$$\begin{aligned}
[z^n](e^{W(z)})^2 &= \frac{1}{n}[z^{n-1}]2e^z e^z \cdot e^{n \cdot z} \\
&= \frac{2}{n}[z^{n-1}] \sum_{k \geq 0} \frac{(n+2)^k}{k!} z^k \\
&= \frac{2}{n} \cdot \frac{(n+2)^{n-1}}{(n-1)!} \\
&= \frac{2(n+2)^{n-1}}{n!}.
\end{aligned}$$

So by (3.4), (3.12) and defining $F(z) = P(2z) - 1$, it follows

$$\begin{aligned}
\rho(n) &= \frac{[z^n]F(z)}{[z^{n-1}](1+F(z))^2} = \frac{[z^n]P(2z) - 1}{[z^{n-1}]P(2z)^2} \\
&= \frac{(n-1)!(n+1)^{n-1} \cdot 2^n}{2 \cdot n!(n+1)^{n-2} 2^{n-1}} \\
&= \frac{(n+1)}{n} = 1 + \frac{1}{n}, \quad n \geq 1.
\end{aligned}$$

Thus, we obtain

$$\frac{n!}{2^n} \sum_{T \in \mathfrak{B}(n)} \prod_{v \in T} \left(1 + \frac{1}{h_v}\right) = (n+1)^{n-1},$$

and

$$1 + \sum_{n \geq 1} \left(\sum_{T \in \mathfrak{B}(n)} \prod_{v \in T} \left(1 + \frac{1}{h_v}\right)\right) z^n = \sum_{n \geq 0} \frac{(n+1)^{n-1} 2^n}{n!} z^n.$$

□

We can further generalize (3.9) to:

Corollary 3.8. *It holds*

$$1 + \sum_{n \geq 1} \left(\sum_{T \in \mathfrak{B}(n)} \prod_{v \in T} \frac{(m + h_v)^{h_v - 1}}{h_v (2m + h_v - 1)^{h_v - 2}} \right) z^n = \sum_{n \geq 0} m(n+m)^{n-1} \frac{(2z)^n}{n!}. \qquad (3.13)$$

Proof. Let P(z) again be a power series which fulfills

$$P(z) = e^{z \cdot P(z)}.$$

Now we substitute $W(z) = \ln(P(z))$, which implies $P(z) = e^{W(z)}$. Thus, we can rearrange

$$P(z) = e^{z \cdot P(z)} \Rightarrow \ln(P(z)) = z \cdot P(z),$$
$$W(z) = z \cdot e^{W(z)}.$$

With the Lagrange Inversion formula (2.10) for $H(z) = (e^z)^m$, $H'(z) = m(e^z)^m$ and $\phi(z) = e^z$ we get the following equation:

$$\begin{aligned}[z^n](e^{W(z)})^m &= \frac{1}{n}[z^{n-1}]me^{m \cdot z}e^{n \cdot z} = \frac{m}{n}[z^{n-1}]e^{(n+m)z} \\ &= \frac{m}{n}[z^{n-1}]\sum_{k \geq 0}\frac{(n+m)^k}{k!}z^k \\ &= \frac{m}{n} \cdot \frac{(n+m)^{n-1}}{(n-1)!} \\ &= m\frac{(n+m)^{n-1}}{n!}.\end{aligned}$$

Therefore, the explicit power series is

$$P(z)^m = 1 + \sum_{n \geq 1}\frac{m(n+m)^{n-1}}{n!}z^n. \tag{3.14}$$

The right-hand side of (3.13) is $P(2z)^m$ and with (3.4) and $F(z) = P(2z)^m - 1$ we get

$$\begin{aligned}\rho(n) &= \frac{[z^n]F(z)}{[z^{n-1}](1+F(z))^2} = \frac{[z^n]P(2z)^m - 1}{[z^{n-1}]P(2z)^{2m}} = \frac{(n-1)!m(n+m)^{n-1} \cdot 2^n}{2m \cdot n!(n+2m-1)^{n-2}2^{n-1}} \\ &= \frac{(n+m)^{n-1}}{n(2m+n-1)^{n-2}}, \quad n \geq 1,\end{aligned}$$

yielding

$$1+\sum_{n\geq 1}\left(\sum_{T\in\mathfrak{B}(n)}\prod_{v\in T}\frac{(m+h_v)^{h_v-1}}{h_v(2m+h_v-1)^{h_v-2}}\right)z^n=\sum_{n\geq 0}m(n+m)^{n-1}\frac{(2z)^n}{n!}.$$

Remark. *In [CY08] Chen and Yang present a combinatorial proof of Postnikov's hook-length formula.*

□

Corollary 3.9.

$$1+\sum_{n\geq 1}\left(\sum_{T\in\mathfrak{B}(n)}\prod_{v\in T}\frac{\prod_{i=1}^{m}(m-1+i)}{h_v\prod_{i=1}^{m-1}(h_v+m-1+i)}\right)z^n=\frac{1}{(1-z)^m}.$$

Proof. From (2.7)

$$\frac{1}{(1-z)^m}=\sum_{n\geq 0}\binom{m+n-1}{m-1}z^n \Rightarrow \frac{1}{(1-z)^m}-1=F(z)=\sum_{n\geq 1}\binom{m+n-1}{m-1}z^n,$$

and (3.4) we conclude, for $n\geq 1$,

$$\rho(n)=\frac{[z^n]F(z)}{[z^{n-1}](1+F(z))^2}=\frac{[z^n]\frac{1}{(1-z)^m}-1}{[z^{n-1}]\frac{1}{(1-z)^{2m}}}=\frac{\binom{m+n-1}{m-1}}{\binom{n+2m-2}{2m-1}}=\frac{\frac{(m+n-1)!}{n!(m-1)!}}{\frac{(2m+n-2)!}{(n-1)!(2m-1)!}}$$

$$=\frac{1}{n}\cdot\frac{m\cdot(m+1)\cdot(m+2)\cdots(2m-1)}{(m+n)\cdot(m+n+1)\cdot(m+n+2)\cdots(2m+n-2)},$$

which yields the hook-length formula

$$\sum_{T\in\mathfrak{B}(n)}\prod_{v\in T}\frac{\prod_{i=1}^{m}(m-1+i)}{h_v\prod_{i=1}^{m-1}(m+h_v-1+i)}=\binom{m+n-1}{m-1},$$

therefore proving our corollary. Han used the symmetric properties of the binomial coefficient $\binom{n}{k}=\binom{n}{n-k}$ to get the equivalent result:

$$1+\sum_{n\geq 1}\sum_{T\in\mathfrak{B}(n)}\prod_{v\in T}\frac{\prod_{i=1}^{h_v-1}(m+i)}{2h_v\prod_{i=1}^{h_v-2}(2m+i)}z^n=\frac{1}{(1-z)^m}.$$

□

The following corollary can be used to prove several other hook-length formulas, including some of our previous results by choosing parameters a and m accordingly.

Corollary 3.10.

$$\sum_{n\geq 1}\left(\sum_{T\in\mathfrak{B}(n)}\prod_{v\in T}\frac{\prod_{i=1}^{h_v-1}(m(a+1)+2ah_v-i(a-1))}{2h_v\prod_{i=1}^{h_v-2}(2m(a+1)+2a(h_v-1)-i(a-1))}\right)z^n$$
$$=\sum_{n\geq 1}\left(\frac{m(a+1)}{n!}\prod_{i=1}^{n-1}(m(a+1)+2an-i(a-1))\right)z^n. \tag{3.15}$$

Proof. Let $P(z)$ be the following power series

$$P(z)=1+(a-1)zP(z)^{\frac{2a}{(a-1)}},$$

and furthermore we substitute $B(z)=P(z)-1$, to use the Lagrange Inversion formula (2.14). We also need

$$(1+z)^m=\sum_{n\geq 0}\binom{m}{n}z^n,$$

obtained from (2.9). Now we have

$$B(z)=(a-1)z(B(z)+1)^{\frac{2a}{(a-1)}}.$$

With (2.14), $H(z)=(z+1)^m$, $H'(z)=m(z+1)^{m-1}$ and $\phi(z)=(a-1)(z+1)^{\frac{2a}{a-1}}$ we get the following

$$\begin{aligned}[z^n](B(z)+1)^m &= \frac{1}{n}[z^{n-1}]m(z+1)^{m-1}(a-1)^n(z+1)^{\frac{2an}{a-1}}\\ &= \frac{m(a-1)^n}{n}[z^{n-1}](z+1)^{m-1}(z+1)^{\frac{2an}{a-1}}\\ &= \frac{m(a-1)^n}{n}[z^{n-1}](z+1)^{m-1+\frac{2an}{a-1}}.\end{aligned}$$

With (2.9) we get

$$[z^n](B(z)+1)^m = \frac{m(a-1)^n}{n}[z^{n-1}]\sum_{i\geq 0}\binom{m-1+\frac{2an}{a-1}}{i}z^i$$

$$= \frac{m(a-1)^n}{n}[z^{n-1}]\sum_{i\geq 0}\frac{(m-1+\frac{2an}{a-1})!}{i!(m-1+\frac{2an}{a-1}-i)!}z^i$$

$$= \frac{m(a-1)^n}{n(n-1)!}\prod_{i=0}^{n-2}\left(m-1+\frac{2an}{a-1}-i\right).$$

After an index shift and excluding $\frac{1}{a-1}$ from the product, we get

$$[z^n](B(z)+1)^m = \frac{m(a-1)}{n!}\prod_{i=1}^{n-1}(m(a-1)+2an-i(a-1)).$$

Since we got the coefficient for every arbitrary exponent m, we use $\frac{m(a+1)}{a-1}$ as exponent instead to match the right-hand side of (3.15). In conclusion, we have

$$[z^n]P(z)^{m\frac{a+1}{a-1}} = \frac{m\frac{a+1}{a-1}(a-1)}{n!}\prod_{i=1}^{n-1}(m\frac{a+1}{a-1}(a-1)+2an-i(a-1))$$

$$= \frac{m(a+1)}{n!}\prod_{i=1}^{n-1}(m(a+1)+2an-i(a-1)).$$

By (3.4) and setting $F(z)=P(z)^{m\frac{a+1}{a-1}}-1$ we get

$$\rho(n) = \frac{[z^n]F(z)}{[z^{n-1}](1+F(z))^2} = \frac{[z^n]P(z)^{m\frac{a+1}{a-1}}-1}{[z^{n-1}]P(z)^{2m\frac{a+1}{a-1}}}$$

$$= \frac{\frac{m(a+1)}{n!}\prod_{i=1}^{n-1}(m(a+1)+2an-i(a-1))}{\frac{2m(a+1)}{(n-1)!}\prod_{i=1}^{n-2}(2m(a+1)+2a(n-1)-i(a-1))}$$

$$= \frac{\prod_{i=1}^{n-1}(m(a+1)+2an-i(a-1))}{2n\prod_{i=1}^{n-2}(2m(a+1)+2a(n-1)-i(a-1))},\quad n\geq 1,$$

yielding the hook-length formula

$$\sum_{T\in\mathfrak{B}(n)}\prod_{v\in T}\frac{\prod_{i=1}^{h_v-1}(m(a+1)+2ah_v-i(a-1))}{2h_v\prod_{i=1}^{h_v-2}(2m(a+1)+2a(h_v-1)-i(a-1))}$$
$$=\frac{m(a+1)}{n!}\prod_{i=1}^{n-1}(m(a+1)+2an-i(a-1)).$$

□

Corollary 3.11. *It holds*

$$1+\sum_{n\geq 1}\left(\sum_{T\in\mathfrak{B}(n)}\prod_{v\in T}\rho(h_v)\right)z^n=\frac{1+z}{1+z^3},$$

with, for $k\in\mathbb{N}$,

$$\rho(n)=\begin{cases}\frac{1}{k}, & \textit{if } n=3k-2,\\ 0, & \textit{if } n=3k-1,\\ \frac{-1}{k}, & \textit{if } n=3k.\end{cases}$$

Proof. We start with

$$F(z)=\frac{1+z}{1+z^3}-1$$
$$\Rightarrow (1+F(z))^2=\frac{1+2z+z^2}{(1+z^3)^2}$$
$$=\frac{1}{(1+z^3)^2}+\frac{2z}{(1+z^3)^2}+\frac{z^2}{(1+z^3)^2},$$

and now write the explicit form of every power series appearing by (2.7). We start with $(1+F(z))^2$:

$$\frac{1}{(1+z^3)^2}=\sum_{n\geq 0}(n+1)(-1)^n z^{3n},$$

$$\frac{2z}{(1+z^3)^2} = \sum_{n\geq 0} 2(n+1)(-1)^n z^{3n+1},$$

$$\frac{z^2}{(1+z^3)^2} = \sum_{n\geq 0} (n+1)(-1)^n z^{3n+2}.$$

Equivalently, we repeat this with $F(z) = \frac{1+z}{1+z^3} - 1 = \frac{1}{1+z^3} - 1 + \frac{z}{1+z^3}$:

$$\frac{1}{1+z^3} - 1 = \sum_{n\geq 1} (-1)^n z^{3n},$$

$$\frac{z}{1+z^3} = \sum_{n\geq 0} (n+1)(-1)^n z^{3n+1}.$$

Now we use (3.4) and distinction of cases and also see, that each power series contained in $(1+F(z))^2$ and $F(z)$ contributes to just one case. Thus we get

$$\rho(3n) = \frac{[z^{3n}]F(z)}{[z^{3n-1}](1+F(z))^2} = \frac{(-1)^n}{n(-1)^{n-1}} = \frac{-1}{n},$$

$$\rho(3n-1) = \frac{[z^{3n-1}]F(z)}{[z^{3n-2}](1+F(z))^2} = \frac{0}{n(-1)^n} = 0,$$

$$\rho(3n-2) = \frac{[z^{3n-2}]F(z)}{[z^{3n-3}](1+F(z))^2} = \frac{(-1)^{n-1}}{n(-1)^{n-1}} = \frac{1}{n},$$

which completes the proof. □

3.1.1 Hook-length formulas for Fibonacci trees

Definition 3.12. *A Fibonacci tree T is a binary tree such that the right subtree of each vertex v is either an empty tree or a binary tree with only one vertex.*

Let $\mathfrak{B}_F(n)$ denote the set of all Fibonacci trees of size n and

$$\mathfrak{B}_F = \bigcup_{n\geq 1} \mathfrak{B}_F(n),$$

the set of all Fibonacci trees.

Theorem 3.13. *Let $\mathfrak{B}_F$ be the family of Fibonacci trees associated with a weight function ρ and let $F(z)$ be the generating function of the total weights of Fibonacci trees of size n:*

$$F(z) = \sum_{n\geq 1}\left(\sum_{T\in\mathfrak{B}_F(n)}\prod_{v\in T}\rho(h_v)\right)z^n.$$

Then the hook-function ρ satisfies

$$\rho(n) = \begin{cases} \frac{[z^n]F(z)}{[z^{n-1}]F(z)+\rho(1)[z^{n-2}](1+F(z))}, & \text{if } n\geq 2,\\ [z^1]F(z), & \text{if } n=1. \end{cases} \tag{3.16}$$

Proof. Considering, how a Fibonacci tree of size $n\geq 3$ splits into a left subtree with either $n-1$ vertices, if the right subtree is empty or a left and a right subtree with $n-2$ and 1 vertices, respectively, we get

$$\begin{aligned}
[z^n]F(z) &= \sum_{T\in\mathfrak{B}_T(n)}\prod_{v\in T}\rho(h_v)\\
&= \rho(n)\sum_{T\in\mathfrak{B}_T(n-1)}\prod_{v\in T}\rho(h_v) + \rho(n)\rho(1)\sum_{T\in\mathfrak{B}_T(n-2)}\prod_{v\in T}\rho(h_v)\\
&= \rho(n)[z^{n-1}]F(z) + \rho(n)\rho(1)[z^{n-2}]F(z)\\
\Rightarrow \rho(n) &= \frac{[z^n]F(z)}{[z^{n-1}]F(z)+\rho(1)[z^{n-2}]F(z)}.
\end{aligned}$$

For $n=2$ these computations are also valid, but we need to add $f_0=1$, since $\mathfrak{B}_F(0)=\{\epsilon\}$, and obtain the following formula:

$$\rho(n) = \frac{[z^n]F(z)}{[z^{n-1}]F(z)+\rho(1)[z^{n-2}](1+F(z))}, \quad n\geq 2.$$

Finally, there exists only one Fibonacci tree with size 1, therefore

$$[z^1]F(z) = \sum_{T\in\mathfrak{B}_F(1)}\prod_{v\in T}\rho(h_v) = \rho(1).$$

□

Now we give an explanation, why these kind of trees are called Fibonacci trees.

Definition 3.14. *The Fibonacci numbers are the sequence of numbers* $(f_n)_{n\geq 0}$ *defined by the linear recurrence equation:*

$$f_n = f_{n-1} + f_{n-2}, \quad n \geq 2,$$

with $f_0 = f_1 = 1$.

Example 2. It's well known, see [Rio62], that

$$\tilde{F}(z) := \sum_{n\geq 0} f_n z^n = \frac{1}{1-z-z^2},$$

is the generating function of the Fibonacci numbers. We define $F(z) = \tilde{F}(z) - 1$. Since $\rho(1) = [z^1]F(z) = 1$ we use (3.16) and the definition of Fibonacci numbers to get

$$\rho(n) = \frac{[z^n]F(z)}{[z^{n-1}]F(z) + \rho(1)[z^{n-2}](1+F(z))} = \frac{f_n}{f_{n-1}+f_{n-2}} = 1, \quad n \geq 2.$$

Therefore

$$1 + \sum_{n\geq 1}\left(\sum_{T\in\mathfrak{B}_F(n)} \prod_{v\in T} 1\right) z^n = \frac{1}{1-z-z^2},$$

which includes, that the number of Fibonacci trees with n vertices is given by the n-th Fibonacci number.

Corollary 3.15. *It holds*

$$1 + \sum_{n\geq 1}\left(\sum_{T\in\mathfrak{B}_F(n)} \prod_{v\in T} \frac{((m+h_v-1)(m+h_v-2))}{(h_v(mh_v+h_v-2))}\right) z^n = \frac{1}{(1-z)^m}, \quad m \geq 1.$$

Proof. We know from (2.7), that

$$\frac{1}{(1-z)^m} = \sum_{n\geq 0}\binom{m+n-1}{m-1} z^n.$$

Let us define $F(z) = \frac{1}{(1-z)^m} - 1$. Thus, with (3.16) we get

$$\rho(1) = [z^1]F(z) = \binom{m}{1} = m,$$

and,

$$\begin{aligned}
\rho(n) &= \frac{[z^n]F(z)}{[z^{n-1}]F(z) + \rho(1)[z^{n-2}](1+F(z))} \\
&= \frac{\binom{m+n-1}{m-1}}{\binom{m+n-2}{m-1} + m\binom{m+n-3}{m-1}} \\
&= \frac{\frac{(m+n-1)!}{n!(m-1)!}}{\frac{(m+n-2)!}{(n-1)!(m-1)!} + m\frac{(m+n-3)!}{(n-2)!(m-1)!}} = \frac{\frac{(m+n-1)!}{n!}}{\frac{(m+n-2)!}{(n-1)!} + m\frac{(n-1)(m+n-3)!}{(n-1)!}} \\
&= \frac{\frac{(m+n-1)(m+n-2)}{n}}{\frac{(m+n-2)}{1} + m\frac{(n-1)}{1}} = \frac{(m+n-1)(m+n-2)}{n(m+n-2+mn-m)} \\
&= \frac{(m+n-1)(m+n-2)}{n(mn+n-2)}, \quad n \geq 2.
\end{aligned}$$

This leads to the stated result. □

3.1.2 Hook-length formulas for complete binary trees

Definition 3.16. *A complete binary tree T is a binary tree such that the two subtrees of each vertex v are either both empty or both non empty, with the possible exception when v is the latest vertex in the so-called inorder traversal. The inorder traversal is a certain way to visit every vertex in a binary tree exactly once, see [Knu97, p. 319]. In the latter case, the right subtree of v is empty when T possesses an even number of vertices.*

Let $\mathfrak{B}_C(n)$ denote the set of all complete binary trees with n vertices, such that

$$\mathfrak{B}_C = \bigcup_{n\geq 1} \mathfrak{B}_C(n),$$

is the set of all complete binary trees.

Theorem 3.17. *Let $\mathfrak{B}_C$ be the family of complete binary trees associated with a weight function ρ and let $F(z)$ be the generating function of the total weights of complete binary trees of size n:*

$$F(z) = \sum_{n\geq 1} \left(\sum_{T\in\mathfrak{B}_C(n)} \prod_{v\in T} \rho(h_v) \right) z^n.$$

Then the hook-function ρ satisfies

$$\rho(n) = \begin{cases} \frac{[z^n]F(z)}{[z^{n-1}]\frac{1}{2}(F(z)-F(-z))(1+F(z))}, & \text{if } n \geq 2, \\ [z^1]F(z), & \text{if } n = 1. \end{cases} \tag{3.17}$$

Proof. Considering, how a complete binary tree of size $n \geq 2$ splits into a root v, a left subtree with an odd number of vertices, and the remaining vertices in the right subtree, also

$$T_1 \in \mathfrak{B}_C(n_1),\ T_2 \in \mathfrak{B}_C(n_2), \quad n_i \geq 0, \quad \sum_{i=1}^{2} n_i = n-1,\ n_1 \text{ odd},$$

we get

$$\begin{aligned}
[z^n]F(z) &= \sum_{T\in\mathfrak{B}_C(n)} \prod_{v\in T} \rho(h_v) \\
&= \rho(n) \sum_{\substack{n_1+n_2=n-1 \\ n_1 \text{ odd} \\ n_1,n_2\geq 0}} \left(\sum_{T_1\in\mathfrak{B}_C(n_1),T_2\in\mathfrak{B}_C(n_2)} \left(\prod_{u\in T_1, v\in T_2} \rho(h_u)\cdot\rho(h_v) \right) \right) \\
&= \rho(n) \sum_{\substack{n_1+n_2=n-1 \\ n_1 \text{ odd} \\ n_1,n_2\geq 0}} \left(\sum_{T_1\in\mathfrak{B}_C(n_1)} \prod_{v\in T_1} \rho(h_v) \right) \cdot \left(\sum_{T_2\in\mathfrak{B}_C(n_2)} \prod_{u\in T_2} \rho(h_u) \right) \\
&= \rho(n) \sum_{\substack{n_1+n_2=n-1 \\ n_1 \text{ odd} \\ n_1,n_2\geq 0}} f_{n_1}\cdot f_{n_2} = \rho(n) \sum_{\substack{0\leq k\leq n-1 \\ k \text{ odd}}} f_k \cdot f_{n-1-k},
\end{aligned}$$

where for the last identity we again defined $f_0 := 1$. From (2.10) we know, that the generating function of a sequence, which contains just the odd coefficients is $\frac{1}{2}(F(z) - F(-z))$. Together with (2.2) and taking into account that we had to add $f_0 = 1$ we obtain

$$\rho(n) = \frac{[z^n]F(z)}{[z^{n-1}]\frac{1}{2}(F(z)-F(-z))(1+F(z))}, \quad n \geq 2.$$

Also, there exists only one complete binary tree of size 1, therefore
$[z^1]F(z) = \sum_{T\in\mathfrak{B}_C(1)} \prod_{v\in T} \rho(h_v) = \rho(1)$. □

Example 3. Define $F(z) = e^z - 1 = \sum_{n\geq 1} \frac{1}{n!} z^n$. Since $[z^n]F(z) = \frac{1}{n!}$, for $n \geq 1$, and

$$[z^{n-1}]\frac{1}{2}(F(z) - F(-z))(1 + F(z)) = [z^{n-1}]\frac{1}{2}(e^z - e^{-z})e^z$$

$$= [z^{n-1}]\frac{1}{2}(e^{2z} - 1) = \frac{2^{n-2}}{(n-1)!},$$

together with (3.17) it follows

$$\rho(n) = \frac{[z^n]F(z)}{[z^{n-1}]\frac{1}{2}(F(z) - F(-z))(1 + F(z))} = \frac{\frac{1}{n!}}{\frac{2^{n-2}}{(n-1)!}}$$

$$= \frac{1}{n2^{n-2}}, \quad n \geq 1.$$

Therefore, we get following hook-length formula

$$\sum_{T \in \mathfrak{B}_C(n)} \prod_{v \in T} \frac{1}{h_v 2^{h_v - 2}} = \frac{1}{n!},$$

yielding

$$1 + \sum_{n\geq 1} \left(\sum_{T \in \mathfrak{B}_C(n)} \prod_{v \in T} \frac{1}{h_v 2^{h_v - 2}} \right) z^n = e^z.$$

Corollary 3.18. *It holds*

$$1 + \sum_{n\geq 1} \left(\sum_{T \in \mathfrak{B}_C(n)} \prod_{v \in T} \rho(h_v) \right) z^n = \frac{1}{1 - z},$$

with

$$\rho(n) = \begin{cases} 1, & \text{if } n = 1, \\ \frac{1}{k}, & \text{if } n = 2k + 1, \\ \frac{1}{k}, & \text{if } n = 2k. \end{cases}$$

Proof. To use (3.17) with $F(z) = \frac{1}{1-z} - 1$ we first consider

$$\begin{aligned}\frac{1}{2}(F(z) - F(-z))(1 + F(z)) &= \frac{1}{2} \cdot \left(\frac{1}{(1-z)^2} - \frac{1}{1-z^2} \right) \\ &= \frac{1}{2} \cdot \frac{2z}{(1-z)^2(1+z)} = \frac{z+z^2}{(1-z^2)^2} \\ &= (z+z^2)\frac{1}{(1-z^2)^2}.\end{aligned}$$

With (2.9) we get

$$\begin{aligned}[z^{n-1}](z+z^2)\frac{1}{(1-z^2)^2} &= [z^{n-1}]\frac{z}{(1-z^2)^2} + \frac{z^2}{(1-z^2)^2} \\ &= [z^{n-1}]\sum_{n\geq 0}(n+1)z^{2n+1} + \sum_{n\geq 0}(n+1)z^{2(n+1)},\end{aligned}$$

which yields

$$[z^{2n}](z+z^2)\frac{1}{(1-z^2)^2} = n = [z^{2n-1}](z+z^2)\frac{1}{(1-z^2)^2}.$$

With (3.17), for $n \geq 1$, we get

$$\begin{aligned}\rho(2n) &= \frac{[z^{2n}]F(z)}{[z^{2n-1}]\frac{1}{2}(F(z) - F(-z))(1+F(z))} = \frac{1}{n}, \\ \rho(2n+1) &= \frac{[z^{2n+1}]F(z)}{[z^{2n}]\frac{1}{2}(F(z) - F(-z))(1+F(z))} = \frac{1}{n}, \\ \rho(1) &= [z^1]F(z) = 1.\end{aligned}$$

□

4 Hook-length formulas for further tree families

In this chapter, we follow [CGG09] and derive hook-length formulas for various families of trees and forests, namely k-ary trees and ordered trees/forests. We'll also examine certain generating functions to find weight functions ρ.

4.1 Hook-length formulas for k-ary trees

Definition 4.1. *A k-ary tree is an ordered rooted unlabelled tree, where each vertex has exactly k subtrees. We also allow a subtree to be empty.*

For our purpose we'll split a k-ary tree into its root and k subtrees, as we have already done with binary trees.

Definition 4.2. *Let $\mathfrak{T}_k(n)$ denote the set of k-ary trees with n vertices and let $\mathfrak{T}_k$ be defined as*

$$\mathfrak{T}_k := \bigcup_{n\geq 1} \mathfrak{T}_k(n).$$

$\mathfrak{T}_k$ is the family of all k-ary trees.

Note, that for $k = 2$ we get binary trees, so our previous results will be special cases for k-ary trees.

Definition 4.3. *The hook-length expansion for k-ary trees is defined as*

$$\sum_{n\geq 1} \left(\sum_{T\in\mathfrak{T}_k} \prod_{v\in T} \rho(h_v) \right) z^n = F(z). \tag{4.1}$$

Equivalent to binary trees the weight function ρ is calculated as follows.
We define $F(z) = f_1 z + f_2 z^2 + f_3 z^3 + \cdots$ to be the generating function for k-ary trees associated with the weight function ρ. Therefore, with each $T \in \mathfrak{T}_k(n)$, $n \geq 1$ we can associate a set $(T_1, T_2, ..., T_k, v)$, consisting of the k subtrees and the root vertex v.
By using the same technique as for binary trees, we can use this expansion to calculate $F(z)$ for a given ρ, or to calculate ρ for a given $F(z)$.

Theorem 4.4. *Let $\mathfrak{T}_k$ be the family of k-ary trees associated with a weight function ρ and let $F(z)$ be the generating function of the total weights of k-ary trees of size n:*

$$\sum_{n\geq 1}\left(\sum_{T\in\mathfrak{T}_k}\prod_{v\in T}\rho(h_v)\right)z^n = F(z). \tag{4.2}$$

Then the hook-function ρ satisfies

$$\rho(n) = \frac{[z^n]F(z)}{[z^{n-1}](1+F(z))^k}, \quad n\geq 1. \tag{4.3}$$

Proof. Recall (2.1): $w_{hook}(T) := \prod_{v\in T}\rho(h_v)$. Considering, how a tree splits into its root, its subtrees and dealing with all possibilities, we get

$$T_i \in \mathfrak{T}_k(n_i), \quad n_i \geq 0, \quad \sum_{i=1}^{k} n_i = n-1.$$

Since n_i can be empty according to our definition, but $F(z)$ isn't defined for empty k-ary trees, we again take a look at $1+F(z)$, to close this gap and add $f_0 = 1$.
We can rearrange (4.2) yielding

$$\begin{aligned}
[z^n]F(z) &= \sum_{T\in\mathfrak{T}_k(n)}\prod_{v\in T}\rho(h_v)\\
&= \rho(n)\sum_{\substack{n_1+n_2+\cdots+n_k=n-1\\ n_1,n_2,\ldots,n_k\geq 0}}\left(\sum_{T_1\in\mathfrak{T}_k(n_1),T_2\in\mathfrak{T}_k(n_2),\ldots,T_k\in\mathfrak{T}_k(n_k)}\left(\prod_{l=1}^{k} w_{hook}(T_l)\right)\right)\\
&= \rho(n)\sum_{\substack{n_1+n_2+\cdots+n_k=n-1\\ n_1,n_2,\ldots,n_k\geq 0}}\left(\sum_{T_1\in\mathfrak{T}_k(n_1)} w_{hook}(T_1)\right)\cdots\left(\sum_{T_k\in\mathfrak{T}_k(n_k)} w_{hook}(T_k)\right)\\
&= \rho(n)\sum_{\substack{n_1+n_2+\cdots+n_k=n-1\\ n_1,n_2,\ldots,n_k\geq 0}}\left(\prod_{l=1}^{k} f_{n_l}\right).
\end{aligned}$$

By using the $[z^n]$ operator and (2.2) k-times to compute

$$(1+F(z))^k = \sum_{n\geq 0}\left(\sum_{\substack{n_1+n_2+\cdots+n_k=n\\ n_1,n_2,\ldots,n_k\geq 0}}\left(\prod_{l=1}^{k} f_{n_l}\right)\right)z^n, \tag{4.4}$$

we obtain

$$[z^n]F(z) = \rho(n)[z^{n-1}](1+F(z))^k$$

$$\Rightarrow \rho(n) = \frac{[z^n]F(z)}{[z^{n-1}](1+F(z))^k}. \quad n \geq 1.$$

□

Example 4. Define $F(z) = e^z - 1$. Since $[z^n]F(z) = \frac{1}{n!}$, $n \geq 1$, and

$$[z^{n-1}](1+F(z))^k = [z^{n-1}]e^{kz} = \frac{k^{n-1}}{(n-1)!},$$

we use (4.3), thus

$$\rho(n) = \frac{[z^n]F(z)}{[z^{n-1}](1+F(z))^k} = \frac{\frac{1}{n!}}{\frac{k^{n-1}}{(n-1)!}} = \frac{1}{nk^{n-1}}.$$

Therefore, we obtain the hook-length formula

$$\sum_{T \in \mathfrak{T}_k(n)} \prod_{v \in T} \frac{1}{h_v k^{h_v - 1}} = \frac{1}{n!},$$

and therefore

$$1 + \sum_{n \geq 1} \left(\sum_{T \in \mathfrak{T}_k(n)} \prod_{v \in T} \frac{1}{h_v k^{h_v - 1}} \right) z^n = e^z.$$

Since the results for binary trees and k-ary trees are calculated the same way, we are going to just present one universal hook-length formula, which includes most previous results as special cases.

Corollary 4.5. *It holds*

$$\sum_{n \geq 1} \left(\sum_{T \in \mathfrak{T}_k(n)} \prod_{v \in T} \frac{\prod_{i=1}^{h_v - 1} (ma + k(a-1)h_v - i(a-k))}{kh_v \prod_{i=1}^{h_v - 2} (kma + k(a-1)(h_v - 1) - i(a-k))} \right) z^n$$
$$= \sum_{n \geq 1} \left(\frac{ma}{n!} \prod_{i=1}^{n-1} (ma + k(a-1)n - i(a-k)) \right) z^n. \tag{4.5}$$

Proof. We define

$$P(z) = (a-k)z(1+B(z))^{\frac{k(a-1)}{a-k}},$$

and

$$B(z) = (1-P(z))^{\frac{a}{a-k}}.$$

With (2.10) we get

$$\begin{aligned}
[z^n]((1-P(z))^{\frac{a}{a-k}})^m &= \frac{1}{n}[z^{n-1}]m\frac{a}{a-k}(1+z)^{m\frac{a}{a-k}-1}(a-k)^n(1+z)^{\frac{k(a-1)n}{a-k}} \\
&= \frac{ma(a-k)^{n-1}}{n}[z^{n-1}](1+z)^{\frac{ma}{a-k}-1+\frac{k(a-1)n}{a-k}} \\
&= \frac{ma(a-k)^{n-1}}{n}[z^{n-1}]\sum_{l\geq 0}\binom{\frac{ma}{a-k}-1+\frac{k(a-1)n}{a-k}}{l}z^l \\
&= \frac{ma(a-k)^{n-1}}{n}[z^{n-1}]\sum_{l\geq 0}\left(\frac{(\frac{ma}{a-k}-1+\frac{k(a-1)n}{a-k})!}{l!(\frac{ma}{a-k}-1+\frac{k(a-1)i}{a-k}-l)!}\right)z^l \\
&= \frac{ma(a-k)^{n-1}}{n!}\prod_{i=0}^{n-2}\left(\frac{ma}{a-k}-1+\frac{k(a-1)n}{a-k}-i\right).
\end{aligned}$$

Extracting $\frac{1}{a-k}$ $(n-1)$-times from the product simplifies it to

$$\begin{aligned}
[z^n]((1-P(z))^{\frac{a}{a-k}})^m &= \frac{ma(a-k)^{n-1}}{n!(a-k)^{n-1}}\prod_{i=0}^{n-2}(ma+k(a-1)n-(a-k)(i+1)) \\
&= \frac{ma}{n!}\prod_{i=1}^{n-1}(ma+k(a-1)n-i(a-k)).
\end{aligned}$$

Since $B(0) = (1-P(0))^{\frac{a}{a-k}} = 1$ we consider $F(z) = (1-P(z))^{\frac{ma}{a-k}} - 1$ to match the requirements of Theorem 4.4. We also need $[z^{n-1}](1+F(z))^k$, which is equal to $((1-P(z))^{\frac{a}{a-k}})^{mk}$, and because our calculation holds for every exponent $m \in \mathbb{N}$, we just need to substitute m for km and get

$$[z^{n-1}](1+F(z))^k = \frac{kma}{(n-1)!}\prod_{i=1}^{n-2}(kma+k(a-1)(n-1)-i(a-k)).$$

So, by (4.3),

$$\rho(n) = \frac{[z^n]F(z)}{[z^{n-1}](1+F(z))^k}$$

$$= \frac{\frac{ma}{n!}\prod_{i=1}^{n-1}(ma+k(a-1)n-i(a-k))}{\frac{kma}{(n-1)!}\prod_{i=1}^{n-2}(kma+k(a-1)(n-1)-i(a-k))}$$

$$= \frac{\prod_{i=1}^{n-1}(ma+k(a-1)n-i(a-k))}{kn\prod_{i=1}^{n-2}(kma+k(a-1)(n-1)-i(a-k))}, \quad n \geq 2.$$

Therefore

$$\sum_{n\geq 1}\left(\sum_{T\in\mathfrak{T}_k(n)}\prod_{v\in T}\frac{\prod_{i=1}^{h_v-1}(ma+k(a-1)h_v-i(a-k))}{kh_v\prod_{i=1}^{h_v-2}(kma+k(a-1)(h_v-1)-i(a-k))}\right)z^n$$

$$= \sum_{n\geq 1}\left(\frac{ma}{n!}\prod_{i=1}^{n-1}(ma+k(a-1)n-i(a-k))\right)z^n.$$

□

This formula is an unification and an extension to several known formulas. For example, by setting $k=2$ and substituting a for $a+1$, we obtain (3.15).
If we set $a=k$ and $m=1$, we obtain

$$\sum_{T\in\mathfrak{T}_k(n)}\prod_{v\in T}\frac{\prod_{i=1}^{h_v-1}k(1+(k-1)h_v)}{kh_v\prod_{i=1}^{h_v-2}k(k+(k-1)(h_v-1))} = \frac{k}{n!}\prod_{i=1}^{n-1}k(1+(k-1)n).$$

Since the enumeration index i does not occur, this simplifies to

$$\sum_{T\in\mathfrak{T}_k(n)}\prod_{v\in T}\frac{(k(1+(k-1)h_v))^{n-1}}{k^{h_v-1}h_v(k+(k-1)(h_v-1))^{h_v-2}} = \frac{k^n}{n!}(1+(k-1)n)^{n-1}$$

$$\iff \sum_{T\in\mathfrak{T}_k(n)}\prod_{v\in T}\frac{(1+(k-1)h_v)^{n-1}}{h_v(1+(k-1)(h_v))^{h_v-2}} = \frac{k^n}{n!}(1+(k-1)n)^{n-1}$$

$$\iff \sum_{T\in\mathfrak{T}_k(n)}\prod_{v\in T}\frac{(1+(k-1)h_v)}{h_v} = \frac{k^n}{n!}(1+(k-1)n)^{n-1}$$

$$\iff \sum_{T\in\mathfrak{T}_k(n)}\prod_{v\in T}\left(k-1+\frac{1}{h_v}\right) = \frac{k^n}{n!}(1+(k-1)n)^{n-1},$$

which is the extension of Postnikov's hook-length formula (3.9) to k-ary trees. We can also extend (3.13) to k-ary trees by setting $a = k$ to get

$$\sum_{T\in\mathfrak{T}_k(n)}\prod_{v\in T}\frac{\prod_{i=1}^{h_v-1}(km+k(k-1)h_v)}{kh_v\prod_{i=1}^{h_v-2}(mk^2+k(k-1)(h_v-1))} = \frac{km}{n!}\prod_{i=1}^{n-1}(km+k(k-1)n)$$

$$\iff \sum_{T\in\mathfrak{T}_k(n)}\prod_{v\in T}\frac{(k(m+(k-1)h_v))^{h_v-1}}{kh_v(k(mk+(k-1)(h_v-1)))^{h_v-2}} = \frac{k^n m}{n!}(m+(k-1)n)^{n-1}$$

$$\iff \frac{n!}{k^n}\sum_{T\in\mathfrak{T}_k(n)}\prod_{v\in T}\frac{(m+(k-1)h_v)^{h_v-1}}{h_v(mk+(k-1)(h_v-1))^{h_v-2}} = m(k(m+(k-1)n))^{n-1}$$

$$\iff n!\sum_{T\in\mathfrak{T}_k(n)}\prod_{v\in T}\frac{(m+(k-1)h_v)^{h_v-1}}{kh_v(mk+(k-1)(h_v-1))^{h_v-2}} = m(k(m+(k-1)n))^{n-1}.$$

4.2 Hook-length formulas for ordered trees and ordered forests

Trees are often used to implement databases, but to do so, we have to declare a certain ordering for the descendants of each vertex. This leads us to ordered trees.

Definition 4.6. *An ordered tree (or planted plane tree) is a rooted tree in which an ordering is specified for the children of each vertex.*

Remark. *An ordered binary tree is sometimes called a binary search tree.*

Let $\mathfrak{O}(n)$ denote the set of ordered trees with n vertices, so that

$$\mathfrak{O} := \bigcup_{n\geq 1} \mathfrak{O}(n),$$

is the set of all ordered trees.

Theorem 4.7. *Let $\mathfrak{O}$ be the family of ordered trees associated with a weight function ρ and let $F(z)$ be the generating function of the total weights of ordered trees of size n:*

$$\sum_{n\geq 1}\left(\sum_{T\in\mathfrak{O}(n)}\prod_{v\in T}\rho(h_v)\right)z^n = F(z).$$

Then the hook-function ρ satisfies

$$\rho(n) = \begin{cases} \frac{[z^n]F(z)}{[z^{n-1}]\frac{1}{1-F(z)}}, & \textit{if } n \geq 2, \\ [z^1]F(z), & \textit{if } n = 1. \end{cases} \tag{4.6}$$

Proof. Yet again, for every ordered tree T of size $n \geq 2$, we can split the tree in its root v and a sequence of subtrees.
This yields a $(j+1)$-tuple $(T_1, T_2, ..., T_j, v), (j \geq 1)$, where $h_v = n$, and

$$T_i \in \mathfrak{O}(n_i), \quad n_i \geq 1, \quad \sum_{i=1}^{j} n_i = n-1.$$

Let $f_n = [z^n]F(z)$. By definition and considering every possibility to build an ordered tree, it holds

$$[z^n]F(z) = \sum_{T\in\mathfrak{O}(n)} \prod_{v\in T} \rho(h_v)$$

$$= \rho(n) \sum_{j\geq 1} \left(\sum_{\substack{n_1+n_2+\cdots+n_j=n-1 \\ n_1,n_2,\ldots,n_j\geq 1}} \left(\sum_{T_1\in\mathfrak{O}(n_1),\ldots T_j\in\mathfrak{O}(n_j)} \left(\prod_{l=1}^{j} w_{hook}(T_l) \right) \right) \right)$$

$$151577 = \rho(n) \sum_{j\geq 1} \left(\sum_{\substack{n_1+n_2+\cdots+n_j=n-1 \\ n_1,n_2,\ldots,n_j\geq 1}} \left(\sum_{T_1\in\mathfrak{O}(n_1)} w_{hook}(T_1) \right) \cdots \left(\sum_{T_j\in\mathfrak{O}(n_j)} w_{hook}(T_j) \right) \right)$$

$$= \rho(n) \sum_{j\geq 1} \left(\sum_{\substack{n_1+n_2+\cdots+n_j=n-1 \\ n_1,n_2,\ldots,n_j\geq 1}} f_{n_1} f_{n_2} \cdots f_{n_j} \right).$$

Simplified with the $[z^n]$ operator and using (4.4) we get

$$[z^n]F(z) = \rho(n)[z^{n-1}]\frac{1}{1-F(z)}$$

$$\Rightarrow \rho(n) = \frac{[z^n]F(z)}{[z^{n-1}]\frac{1}{1-F(z)}}, \quad n \geq 2.$$

Finally, there exists only one ordered tree with size 1, therefore

$$[z^1]F(z) = \sum_{T\in\mathfrak{B}_F(1)} \prod_{v\in T} \rho(h_v) = \rho(1).$$

□

Corollary 4.8. *It holds*

$$\sum_{n\geq 1} \left(\sum_{T\in\mathfrak{O}(n)} \prod_{v\in T} \left(1 - \frac{1}{h_v}\right)^{h_v-1} \right) z^n = \sum_{n\geq 1} \frac{(n-1)^{n-1}}{n!} z^n. \tag{4.7}$$

Proof. In order to use (4.6) we need to calculate $\frac{1}{1-F(z)}$ first. The right-hand side of (4.7) is exactly (3.14) with $m = -1$,

$$P(z)^m = 1 + \sum_{n\geq 1} \frac{m(n+m)^{n-1}}{n!} z^n \Rightarrow P(z)^{-1} = 1 - \sum_{n\geq 1} \frac{(n-1)^{n-1}}{n!} z^n.$$

Therefore,

$$\left(1+\sum_{n\geq 1}\frac{(n+1)^{n-1}}{n!}z^n\right)^{-1} = 1-\sum_{n\geq 1}\frac{(n-1)^{n-1}}{n!}z^n.$$

Finally we switch the exponent -1 to the other side and get the wanted result

$$1+\sum_{n\geq 1}\frac{(n+1)^{n-1}}{n!}z^n = \left(1-\sum_{n\geq 1}\frac{(n-1)^{n-1}}{n!}z^n\right)^{-1}.$$

We just showed, that

$$\frac{1}{1-F(z)} = 1+\sum_{n\geq 1}\frac{(n+1)^{n-1}}{n!}\cdot z^n,$$

and by (4.6) we get

$$\begin{aligned}\rho(n) &= \frac{[z^n]F(z)}{[z^{n-1}]\frac{1}{1-F(z)}} = \frac{\frac{(n-1)^{n-1}}{n!}}{\frac{n^{n-2}}{(n-1)!}}\\ &= \frac{(n-1)^{n-1}}{n\cdot n^{n-2}} = \frac{(n-1)^{n-1}}{n^{n-1}} = \left(\frac{n-1}{n}\right)^{n-1}\\ &= \left(1-\frac{1}{n}\right)^{n-1}, \quad n\geq 1.\end{aligned}$$

Thus, we obtain the hook-length formula

$$\sum_{T\in\mathfrak{O}(n)}\prod_{v\in T}\left(1-\frac{1}{h_v}\right)^{h_v-1} = \frac{(n-1)^{n-1}}{n!},$$

and therefore

$$\sum_{n\geq 1}\left(\sum_{T\in\mathfrak{O}(n)}\prod_{v\in T}\left(1-\frac{1}{h_v}\right)^{h_v-1}\right)z^n = \sum_{n\geq 1}\frac{(n-1)^{n-1}}{n!}z^n.$$

□

Now we expand our previous results to ordered forests.

Definition 4.9. *An ordered forest is a disjoint union of ordered trees, in which an ordering is also specified.*
We denote the set of ordered forests with n vertices with $\mathfrak{O}_F(n)$.

Theorem 4.10. *Let $\mathfrak{O}_F$ be the family of ordered forests associated with a weight function ρ and let $F(z)$ be the generating function of the total weights of forests of size n:*

$$\sum_{n\geq 0}\left(\sum_{F\in\mathfrak{O}_F(n)}\prod_{v\in F}\rho(h_v)\right)z^n = F(z). \tag{4.8}$$

Then the hook-function ρ satisfies

$$\rho(n) = -\frac{[z^n]F(z)^{-1}}{[z^{n-1}]F(z)}, \quad n\geq 1. \tag{4.9}$$

Proof. We'll prove this theorem in Chapter 5. Another proof can be found in [CGG09]. □

Example 5. Let $F(z) = e^z$. It holds $[z^{n-1}]F(z) = \frac{1}{(n-1)!}$, for $n\geq 1$, and

$$[z^n]F(z)^{-1} = [z^n]e^{-z} = [z^n]\sum_{n\geq 0}\frac{(-1)^n}{n!}z^n$$
$$= \frac{(-1)^n}{n!}, \quad n\geq 1.$$

Via (4.9), we get

$$\rho(n) = -\frac{[z^n]F(z)^{-1}}{[z^{n-1}]F(z)} = -\frac{\frac{(-1)^n}{n!}}{\frac{1}{(n-1)!}} = \frac{(-1)^{n+1}}{n},$$

which yields

$$\sum_{T\in\mathfrak{O}_F(n)}\prod_{v\in T}\frac{(-1)^{h_v+1}}{h_v} = \frac{1}{n!}, \quad n\geq 1,$$

and

$$\sum_{n\geq 0}\left(\sum_{T\in\mathfrak{O}_F(n)}\prod_{v\in T}\frac{(-1)^{h_v+1}}{h_v}\right)z^n = e^z.$$

Theorem 4.11. *For ordered forests it holds*

$$\begin{aligned}&\sum_{n\geq 1}\left(\sum_{F\in\mathfrak{O}_F(n)}\prod_{v\in F}\frac{\prod_{i=1}^{h_v-1}((2h_v-m)a-(a+1)i)}{\prod_{i=1}^{h_v-2}((2h_v-2+m)a-(a+1)i)}\right)z^n\\&=\sum_{n\geq 1}\left(\frac{ma}{n!}\prod_{i=1}^{n-1}((2n+m)a-(a+1)i)\right)z^n.\end{aligned}\tag{4.10}$$

Proof. We define

$$G(z)=(a-1)z(1+G(z))^{\frac{2a}{a+1}}$$

and

$$F(z)=(1+G(z))^{\frac{ma}{a+1}}.$$

With Lagrange Inversion formula (2.10), $H(z)=(1+z)^{\frac{ma}{a+1}}$, $H'=\frac{ma}{a-1}(1+z)^{\frac{ma}{a+1}-1}$ and $\phi(z)=(a+1)\cdot(1-z)^{\frac{2a}{a+1}}$ we get

$$\begin{aligned}[z^n](1+G(z))^{\frac{ma}{a+1}}&=\frac{1}{n}[z^{n-1}]\frac{ma}{a+1}(1+z)^{\frac{ma}{a+1}-1}(a+1)^n(1+z)^{\frac{2an}{a+1}}\\&=\frac{1}{n}\cdot\frac{ma}{a+1}(a+1)^n[z^{n-1}](1+z)^{\frac{ma}{a+1}-1}(1+z)^{\frac{2an}{a+1}}\\&=\frac{ma}{n}\cdot(a+1)^{n-1}[z^{n-1}](1+z)^{\frac{a(m-1+2n)-1}{a+1}}.\end{aligned}$$

With (2.9),

$$\begin{aligned}[z^n](1+G(z))^{\frac{ma}{a+1}}&=\frac{ma}{n}\cdot(a+1)^{n-1}[z^{n-1}]\sum_{l\geq 0}\binom{\frac{a(m-1+2n)-1}{a+1}}{l}z^l\\&=\frac{ma}{n}\cdot(a+1)^{n-1}[z^{n-1}]\sum_{l\geq 0}\frac{\left(\frac{a(m-1+2n)-1}{a+1}\right)!}{l!\left(\frac{a(m-1+2n)-1}{a+1}-l\right)}z^l\\&=\frac{ma}{n!}\cdot(a+1)^{n-1}\prod_{i=0}^{n-2}\left(\frac{(2n+m-1)a-1}{a+1}-i\right)\end{aligned}$$

After simplifying and an index shift we get

$$[z^n](1+G(z))^{\frac{ma}{a+1}} = \frac{ma}{n!}\cdot\frac{(a+1)^{n-1}}{(a+1)^{n-1}}\prod_{i=0}^{n-2}((2n+m-1)a-1-i(a+1))$$

$$= \frac{ma}{n!}\prod_{i=1}^{n-1}((2n+m)a-a-1-(i-1)(a+1))$$

$$= \frac{ma}{n!}\prod_{i=1}^{n-1}((2n+m)a-i(a+1)).$$

Substituting m with $-m$ yields

$$[z^n](1+G(z))^{\frac{-ma}{a+1}} = [z^n]\left((1+G(z))^{\frac{ma}{a+1}}\right)^{-1}$$

$$= \frac{-ma}{n!}\prod_{i=1}^{n-1}((2n-m)a-i(a+1)), \quad n\geq 1.$$

Therefore, with $F(z)=(1+G(z))^{\frac{ma}{a+1}}$ and using (4.9), we obtain

$$\rho(n) = -\frac{[z^n]F(z)^{-1}}{[z^{n-1}]F(z)}$$

$$= -\frac{-\frac{ma}{n!}\prod_{i=1}^{n-1}((2n-m)a-i(a+1))}{\frac{ma}{n!}\prod_{i=1}^{n-2}((2n-2+m)a-i(a+1))}, \quad n\geq 1,$$

concluding to

$$\sum_{n\geq 1}\left(\sum_{F\in\mathfrak{O}_F(n)}\prod_{v\in F}\frac{\prod_{i=1}^{h_v-1}(2h_v-m)a-(a+1)i}{\prod_{i=1}^{h_v-2}(2h_v-2+m)a-(a+1)i}\right)z^n$$

$$=\sum_{n\geq 1}\left(\frac{ma}{n!}\prod_{i=1}^{n-1}((2n+m)a-(a+1)i)\right)z^n.$$

□

With (4.10) one can derive other hook-length formulas by adjusting a nd m accordingly.

4.3 Hook-length formulas for labelled trees and labelled forests

Now we give the expansion formula for labelled trees and forests and then derive hook formulas.

Definition 4.12. *A labelled tree of size n is a tree, where its vertices are labelled with distinct integers of the set $\{1, 2, ..., n\}$. We denote the family of labelled unordered trees with $\mathfrak{U}_L$ and the family of labelled unordered forests with $\mathfrak{U}_{LF}$.*

Since we look at unordered trees now, we have to use the multinomial coeffcients:

Definition 4.13. *The multinomial coefficient is defined as follows, where $k_i, n \in \mathbb{N}$ and $k_1 + k_2 + \cdots + k_m = n$:*

$$\binom{n}{k_1, k_2, \ldots, k_m} = \frac{n!}{k_1!k_2! \cdots k_m!}. \tag{4.11}$$

The multinomial coefficient has a direct combinatorial interpretation, as the number of ways of depositing n distinct objects into m distinct bins, with k_1 objects in the first bin, k_2 objects in the second bin, and so on.

Theorem 4.14. *Let $\mathfrak{U}_L$ be the family of labelled unordered trees associated with a weight function ρ and let $F(z)$ be the generating function of the total weights of labelled unordered trees of size n:*

$$\sum_{n\geq 1} \left(\sum_{T\in\mathfrak{U}_L(n)} \prod_{v\in T} \rho(h_v) \right) \frac{z^n}{n!} = F(z).$$

Then the hook-function ρ satisfies

$$\rho(n) = \frac{[z^n]F(z)}{[z^{n-1}]e^{F(z)}}, \quad n \geq 1. \tag{4.12}$$

Proof. We split the tree in its root v and a possibly empty sequence of subtrees. This yields a $(j+1)$-tuple $(T_1, T_2, ..., T_j, v), (j \geq 0)$, where $h_v = n$ and

$$T_i \in \mathfrak{U}_L(n_i), \quad n_i \geq 1, \quad \sum_{i=1}^{j} n_i = n - 1.$$

Let $f_n = [z^n]F(z) = \frac{1}{n!} \sum_{T\in\mathfrak{U}_L(n)} \prod_{v\in T} \rho(h_v)$. There are n ways to label the root vertex and the remaining $n-1$ labels are distributed to its subtrees. We use the multinomial coefficient $\binom{n-1}{n_1,n_2,\ldots,n_j}$ to count all possibilities of doing so. It holds

$$
\begin{aligned}
[z^n]F(z) &= \frac{1}{n!}\sum_{T\in\mathfrak{U}_L(n)}\prod_{v\in T}\rho(h_v) \\
&= \frac{\rho(n)}{(n-1)!}\sum_{j\geq 0}\frac{1}{j!}\sum_{\substack{n_1+n_2+\cdots+n_j=n-1\\ n_1,n_2,\ldots,n_j\geq 1}}\left(\sum_{T_1\in\mathfrak{U}_L(n_1),\ldots,\mathfrak{U}_L(n_j)}\left(\prod_{i=1}^{j} w_{hook}(T_i)\binom{n-1}{n_1,n_2,\ldots,n_j}\right)\right) \\
&= \frac{\rho(n)n}{n!}\sum_{j\geq 0}\frac{1}{j!}\sum_{\substack{n_1+n_2+\cdots+n_j=n-1\\ n_1,n_2,\ldots,n_j\geq 1}}\left(\sum_{T_1\in\mathfrak{U}_L(n_1),\ldots,\mathfrak{U}_L(n_j)}\left(\prod_{i=1}^{j} w_{hook}(T_i)\binom{n-1}{n_1,n_2,\ldots,n_j}\right)\right) \\
&= \frac{\rho(n)(n-1)!}{(n-1)!}\sum_{j\geq 0}\frac{1}{j!}\sum_{\substack{n_1+n_2+\cdots+n_j=n-1\\ n_1,n_2,\ldots,n_j\geq 1}}\left(\sum_{T_1\in\mathfrak{U}_L(n_1)} w_{hook}(T_1)\frac{1}{n_1!}\right)\cdots\left(\sum_{T_j\in\mathfrak{U}_L(n_j)} w_{hook}(T_j)\frac{1}{n_j!}\right) \\
&= \rho(n)\sum_{j\geq 0}\frac{1}{j!}\sum_{\substack{n_1+n_2+\cdots+n_j=n-1\\ n_1,n_2,\ldots,n_j\geq 1}} f_{n_1}f_{n_2}\cdots f_{n_j}.
\end{aligned}
$$

Simplified with the $[z^n]$ operator we get

$$
\begin{aligned}
[z^n]F(z) &= \rho(n)[z^{n-1}]e^{F(z)} \\
\Rightarrow \rho(n) &= \frac{[z^n]F(z)}{[z^{n-1}]e^{F(z)}} \quad n\geq 1.
\end{aligned}
$$

□

Example 6. Let $F(z) = \ln\left(\frac{1}{(1-z)^m}\right)$. With (2.5) we obtain

$$F(z) = \ln\left(\frac{1}{(1-z)^m}\right) = \mathrm{m}\cdot\ln\left(\frac{1}{1-z}\right) = \sum_{n\geq 1}\frac{m}{n}z^n.$$

Since $F(z) = \ln\left(\frac{1}{(1-z)^m}\right)$, this leads to

$$e^{F(z)} = \frac{1}{(1-z)^m}.$$

So, with (4.12) and (2.7) we obtain for $n \geq 1$

$$\rho(n) = \frac{[z^n]F(z)}{[z^{n-1}]e^{F(z)}} = \frac{\frac{m}{n}}{\binom{m+n-2}{m-1}},$$

concluding to

$$\sum_{T \in \mathfrak{U}_L(n)} \prod_{v \in T} \frac{\frac{m}{h_v}}{\binom{m+h_v-2}{m-1}} = \frac{m}{n},$$

and therefore

$$\sum_{n \geq 1} \left(\sum_{T \in \mathfrak{U}_L(n)} \prod_{v \in T} \frac{\frac{m}{h_v}}{\binom{m+h_v-2}{m-1}} \right) \frac{z^n}{n!} = \ln\left(\frac{1}{(1-z)^m} \right).$$

Now we expand (4.12) to labelled forests.

Theorem 4.15. *Let $\mathfrak{U}_{LF}$ be the family of labelled unordered forests associated with a weight function ρ and let $F(z)$ be the generating function of the total weights of labelled unordered forests of size n:*

$$\sum_{n \geq 0} \left(\sum_{T \in \mathfrak{U}_{LF}(n)} \prod_{v \in T} \rho(h_v) \right) \frac{z^n}{n!} = F(z).$$

Then the hook-function ρ satisfies

$$\rho(n) = \frac{[z^n]\ln(F(z))}{[z^{n-1}]F(z)}, \quad n \geq 1. \tag{4.13}$$

Proof. We take a look at the generating function $F(z)$ for labelled forests. Since a tree of size n can be seen as a root together with the forest of subtrees with $n-1$ vertices, the generating function for trees is $zF(z)$, see [GS05]. A forest is a set of trees, therefore the generating function for forests is $e^{zF(z)}$. $F(z)$ satisfies the functional equation

$$F(z) = e^{zF(z)}.$$

By this exponential formula, we obtain

$$\sum_{n \geq 0} \left(\sum_{T \in \mathfrak{U}_{LF}(n)} \prod_{v \in T} \rho(h_v) \right) \frac{z^n}{n!} = \exp\left\{ \sum_{n \geq 1} \left(\sum_{T \in \mathfrak{U}_L(n)} \prod_{v \in T} \rho(h_v \right) \frac{z^n}{n!} \right\}.$$

Taking logarithm yields

$$\ln(F(z)) = \sum_{n\geq 1} \left(\sum_{T\in\mathfrak{U}_L(n)} \prod_{v\in T} \rho(h_v) \right) \frac{z^n}{n!}.$$

With (4.12) we get

$$\begin{aligned}\rho(n) &= \frac{[z^n]F(z)}{[z^{n-1}]e^{F(z)}} = \frac{[z^n]\ln(F(z))}{[z^{n-1}]e^{\ln(F(z))=}} \\ &= \frac{[z^n]\ln(F(z))}{[z^{n-1}]F(z)}, \quad n \geq 1.\end{aligned}$$

□

Corollary 4.16. *It holds*

$$\begin{aligned}&\sum_{n\geq 1} \left(\sum_{F\in\mathfrak{U}_{LF}(n)} \prod_{v\in F} \frac{\prod_{k=1}^{h_v-1}(ah_v-(a-1)k)}{h_v\prod_{k=1}^{h_v-2}(a(h_v-1+m)-(a-1)k)} \right) \frac{z^n}{n!} \\ &\qquad = \sum_{n\geq 1} \left(ma \prod_{k=1}^{n-1}(a(n+m)-(a-1)k) \right) z^n.\end{aligned}$$

Proof. The proof is nearly equivalent to the proof of (4.10). We define

$$G(z) = (a-1)z(1+G(z))^{\frac{2a}{a+1}},$$

and

$$F(z) = (1+G(z))^{\frac{ma}{a+1}}.$$

With the Lagrange Inversion formula (2.10), we get

$$[z^n]F(z) = \frac{ma}{n!}\prod_{k=1}^{n-1}(a(n+m)-(a-1)k),$$

and

$$[z^n]\ln(F(z)) = \frac{1}{n}[z^{n-1}]\frac{ma}{a-1}(1+z)^{-1}(a-1)^n(1+z)^{\frac{an}{a-1}}$$

$$= \frac{ma}{n!}\prod_{k=1}^{n-1}(an-(a-1)k).$$

So, with (4.13)

$$\rho(n) = \frac{[z^n]\ln(F(z))}{[z^{n-1}]F(z)} = \frac{\frac{ma}{n!}\prod_{k=1}^{n-1}(an-(a-1)k)}{\frac{ma}{(n-1)!}\prod_{k=1}^{n-2}(a(n-1+m)-(a-1)k)}$$

$$= \frac{\prod_{k=1}^{n-2}(an-(a-1)k)}{n\prod_{k=1}^{n-2}(a(n-1+m)-(a-1)k)}.$$

□

The last theorem was proven in [CGG09] and is a unification of several known hook-length formulas for forests.

If we set $a = 1$ and $m = 1$, we obtain

$$\sum_{n\geq 1}\left(\sum_{F\in\mathfrak{U}_{LF}(n)}\prod_{v\in F}1\right)\frac{z^n}{n!} = \sum_{n\geq 1}(n+1)^{n-1}z^n,$$

which implies, that the number auf labelled forests with n vertices is $(n+1)^{n-1}$. This is due to Cayley's formula, see [Cay89]. It states, that for every positive integer n, the number of unordered trees with n labelled vertices is n^{n-2}.
Each labelled rooted forest can be turned into a labelled tree with one extra vertex, by adding a vertex with label $n+1$ and connecting it to all roots of the trees in the forest.

5 Hook-length formulas for weighted tree and forest families

In this chapter we'll follow the definitions and calculations given by [KP13] to derive further hook-length formulas for certain important and well-studied combinatorial objects called simply generated tree families. For this purpose, the objects considered are again ordered trees, recall:

Definition 5.1. *An ordered tree (or planted plane tree) is a rooted tree in which an ordering is specified for the children of each vertex.*

Definition 5.2. *Let $\mathfrak{O}(n)$ denote the set of ordered trees with n vertices and let $\mathfrak{O}$ be defined as*

$$\mathfrak{O} = \bigcup_{n \geq 1} \mathfrak{O}(n).$$

$\mathfrak{O}$ is the family of all ordered trees.

To extend Han's expansion technique to weighted tree families, we need the definition of a degree-weight.

Definition 5.3. *A sequence of complex numbers $(\phi_j)_{j \geq 0}$ with $\phi_0 \neq 0$ defines the multiplicative degree-weight ϕ_j of a vertex v with out-degree j (out-degree: number of children, denoted as deg(v)). Then, the degree-weight $w_{deg}(T)$ of any ordered tree T is given by the product of all degree-weight factors of the vertices v of T:*

$$w_{deg}(T) := \prod_{v \in T} \phi_{deg(v)}.$$

Definition 5.4. *The family $\mathfrak{F}$ of weighted ordered trees contains all ordered trees $T \in \mathfrak{O}$ paired with their degree-weights to form pairs $(T, w_{deg}(T))$.*

Before we can use the expansion technique for weighted ordered tree families, we have to define a weight to any given ordered tree.

Definition 5.5. *Let T be an ordered tree. Given a hook weight function $\rho : \mathbb{N} \to \mathbb{C}$ and a degree-weight sequence $(\phi_j)_{j \geq 0}$ we define*

$$w(T) := w_{deg}(T) \cdot w_{hook}(T),$$

and associate this weight to any given ordered tree T. Recall (2.1),

$$w_{hook}(T) := \prod_{v \in T} \rho(h_v).$$

Now we are ready to prove the corresponding expansion technique for weighted ordered trees to find a general approach for all kinds of tree families, including those we already examined in this thesis.

Remark. *In [Yan08], Yang extended Han's results to binomial families of trees, a special form of weighted ordered trees. Binomial trees include ordered trees, but they do not include, for example, complete binary trees, which we examined in Chapter 3.*

Theorem 5.6. *Given a family $\mathfrak{F}$ of weighted ordered trees associated to a degree-weight generating function $\phi(t) = \sum_{j\geq 0} \phi_j t^j$, let $F(z)$ be the generating function of the total weights of trees of size n:*

$$F(z) = \sum_{n\geq 1} \left(\sum_{T \in \mathfrak{O}(n)} w(T) \right) z^n = \sum_{n\geq 1} \left(\sum_{T \in \mathfrak{O}(n)} w_{deg}(T) \prod_{v \in T} \rho(h_v) \right) z^n.$$

Then the hook-function ρ satisfies

$$\rho(n) = \frac{[z^n]F(z)}{[z^{n-1}]\phi(F(z))}, \quad n \geq 1. \tag{5.1}$$

Proof. Once again we use the decomposition of a tree, following our previous results. Now we have to consider all possibilities to build a tree of size $|T| \geq 2$. So we split our tree into its root v and the subtrees T_i attached to the root:

$$T_i \in \mathfrak{O}(n_i), \quad n_i \geq 1, \quad \sum_{i=1}^{j} n_i = n - 1.$$

Note, that the root gives a degree-weight factor $\phi_{\deg(v)} = \phi_j$ assuming the root v has out-degree $j \geq 1$ and $\rho(h_v) = \rho(n)$. By definition we have

$$[z^n]F(z) = \sum_{T \in \mathfrak{O}(n)} w(T) = \sum_{T \in \mathfrak{O}(n)} w_{\deg}(T) \cdot w_{hook}(T) \tag{5.2}$$

and

$$w(T) = w_{\deg}(T) \cdot w_{hook}(T) = \prod_{v \in T} \rho(h_v) \cdot \phi_{\deg(v)} \tag{5.3}$$

$$= \phi_j \cdot \rho(n) \prod_{l=1}^{j} w_{\deg}(T_l) \cdot w_{hook}(T_l) = \phi_j \cdot \rho(n) \prod_{l=1}^{j} w(T_l). \tag{5.4}$$

Now we combine (5.4) and (5.2) to generate all possible trees:

$$[z^n]F(z) = \sum_{T \in \mathfrak{O}(n)} w(T) = \sum_{T \in \mathfrak{O}(n)} w_{\deg}(T) \cdot w_{hook}(T) =$$

$$= \rho(n) \sum_{j \geq 1} \phi_j \left(\sum_{\substack{n_1+n_2+\cdots+n_j=n-1 \\ n_1,n_2,\ldots,n_j \geq 1}} \left(\sum_{T_1 \in \mathfrak{O}(n_1),\ldots,T_j \in \mathfrak{O}(n_j)} \prod_{l=1}^{j} w(T_l) \right) \right)$$

$$= \rho(n) \sum_{j \geq 1} \phi_j \sum_{\substack{n_1+n_2+\cdots+n_j=n-1 \\ n_1,n_2,\ldots,n_j \geq 1}} \left(\sum_{T_1 \in \mathfrak{O}(n_1)} w(T_1) \right) \cdot \left(\sum_{T_2 \in \mathfrak{O}(n_2)} w(T_2) \right) \cdots \left(\sum_{T_l \in \mathfrak{O}(n_l)} w(T_l) \right)$$

$$= \rho(n) \sum_{j \geq 1} \phi_j \left(\sum_{\substack{n_1+n_2+\cdots+n_j=n-1 \\ n_1,n_2,\ldots,n_j \geq 1}} \left(\prod_{l=1}^{j} f_{n_l} \right) \right).$$

Simplified with the $[z^n]$ operator and using (4.4) and the definition of $\phi(t)$ we obtain

$$[z^n]F(z) = \rho(n)[z^{n-1}]\phi(F(z))$$

$$\Rightarrow \rho(n) = \frac{[z^n]F(z)}{[z^{n-1}]\phi(F(z))}, \quad n \geq 2.$$

The case $n = 1$ is easily proven by the definition $f_1 = \phi_0 \rho(1)$ and therefore

$$\rho(1) = \frac{[z^1]F(z)}{[z^0]\phi(F(z))} = \frac{f_1}{\phi_0},$$

showing, that (5.1) holds for all $n \geq 1$. □

Now, we can use this theorem to show some applications to important tree models, duplicate some of our results and derive hook-length formulas for yet to be examined tree families. But first we need to define an isomorphism between trees.

Definition 5.7. *Let T' be a rooted tree and T be a ordered tree. We write shape$(T') = T$, if there exists an isomorphism from T' to T that preserves the linear order of the children of the vertices.*

Various tree families are equivalent to families of weighted ordered trees with specific degree-weights. This situation occurs if, for a rooted tree familiy $\mathfrak{R}$, there exists a weighted ordered tree family $\mathfrak{F}$ with a degree-weight generating function $\phi(t)$, such that for each ordered tree $T \in \mathfrak{O}$ the following relation holds:

$$w_{\mathrm{deg}}(T) = \prod_{v \in T} \phi_{\mathrm{deg}(v)} = \sum_{T' \in \mathfrak{R}, \mathrm{shape}(T') = T} 1.$$

Since the hook-weight is also preserved by an isomorphism, with ρ defined as an arbitrary hook-function and $T \in \mathfrak{O}$ we further get

$$\begin{aligned} w(T) = w_{hook}(T) \cdot w_{\mathrm{deg}}(T) = w_{hook}(T) \cdot \sum_{T' \in \mathfrak{R}, \mathrm{shape}(T') = T} 1 \\ = \sum_{T' \in \mathfrak{R}, \mathrm{shape}(T') = T} w_{hook}(T) = \sum_{T' \in \mathfrak{R}, \mathrm{shape}(T') = T} w_{hook}(T'). \end{aligned}$$

By summerarizing over all ordered trees T with $|T| = n$, we get

$$\sum_{T \in \mathfrak{O}(n)} w(T) = \sum_{T \in \mathfrak{O}(n)} \sum_{T' \in \mathfrak{R}, \mathrm{shape}(T') = T} w_{hook}(T') = \sum_{T' \in \mathfrak{R}(n)} w_{hook}(T'),$$

which leads to following result:

$$F(z) = \sum_{n \geq 1} \left(\sum_{T \in \mathfrak{O}(n)} w(T) \right) z^n = \sum_{n \geq 1} \left(\sum_{T' \in \mathfrak{R}(n)} w_{hook}(T') \right) z^n. \tag{5.5}$$

Equation (5.5) means, that our result (5.1) for weighted tree families associated to a degree-weight generating function $\phi(t)$ is also true for the corresponding rooted tree family.

5.1 Confirmation of previous results

Now we take a look back und prove some of our previous results with weighted trees and certain degree-weight functions.
First we consider binary trees, so we have to find the equivalent weighted ordered tree model. To see, how to choose the coefficients of the degree-weight generating function,

we examine an arbitrary ordered tree $T \in \mathfrak{O}$ and count the number

$$w_{\deg}(T) = \sum_{T' \in \mathfrak{R}, \text{shape}(T')=T} 1,$$

of binary tress $T' \in \mathfrak{B}$ with $\text{shape}(T') = T$. Since a binary tree can only have at most 2 children we got $\phi_j = 0$ for $j \geq 3$. If $\deg(v) = 1$, the child of v, assuming that $\deg(v) \leq 2$ for all $v \in T$, could be attached as a left child or a right child, therefore we got $w_{\deg}(T) = 2^{|\{v \in T : \deg(v)=1\}|}$. If $\deg(v) = 0$ or $\deg(v) = 2$, there is only one possibility, so overall we got:
$\phi_0 = 1$, $\phi_1 = 2$, $\phi_2 = 1$ and as stated before $\phi_j = 0$, for $j \geq 3$.

In either case, we get $w_{\deg}(T) = 2^{|\{v \in T : \deg(v)=1\}|} = \prod_{v \in T} \phi_{\deg(v)}$ for ordered trees T with $\deg(v) \leq 2$, for all $v \in T$, and $w_{\deg}(T) = 0$ otherwise. Finally, let

$$F(z) = \sum_{n \geq 1} \left(\sum_{T \in \mathfrak{B}(n)} \prod_{v \in T} \rho(h_v) \right) z^n,$$

be the generating function of the total weights of binary trees. Then, according to (5.1) and matching with Han's result (3.4), we get

$$\begin{aligned} \rho(n) &= \frac{[z^n]F(z)}{[z^{n-1}]\phi(F(z))} = \frac{[z^n]F(z)}{[z^{n-1}](1 + 2F(z) + F(z)^2)} \\ &= \frac{[z^n]F(z)}{[z^{n-1}](1 + F(z))^2}. \end{aligned}$$

Using the same pattern, we can generalize this result for the family $\mathfrak{T}_k$ of k-ary trees. Every vertex has k positions, where up to k children can be attached, so overall $\phi_j = \binom{k}{j}$, for $j \geq 0$, with $\binom{k}{j} = 0$, for $j > k$. As shown in (2.9) it holds,

$$(1 + z)^k = \sum_{j \geq 0} \binom{k}{j} z^j.$$

So the generating function of $\binom{k}{j}$ is $(1 + t)^k$, yielding $\phi(t) = (1 + t)^k$. Thus by setting

$$F(z) = \sum_{n \geq 1} \left(\sum_{T \in \mathfrak{T}_k(n)} \prod_{v \in T} \rho(h_v) \right) z^n,$$

and applying (5.1), we get (4.3):

$$\rho(n) = \frac{[z^n]F(z)}{[z^{n-1}]\phi(F(z))} = \frac{[z^n]F(z)}{[z^{n-1}](1+F(z))^k}.$$

Since our approach is based on ordered trees, they are easily obtained by using the degree weights $\phi_j = 1$, for $j \geq 0$. From (2.6) we know:

$$\sum_{n\geq 0} 1 \cdot z^n = \frac{1}{1-z},$$

yielding, that the generating function for this sequence is $\phi(t) = \frac{1}{1-t}$.
Therefore, with

$$F(z) = \sum_{n\geq 1} \left(\sum_{T\in\mathfrak{O}(n)} \prod_{v\in T} \rho(h_v) \right) z^n,$$

and (5.1), we get (4.6):

$$\rho(n) = \frac{[z^n]F(z)}{[z^{n-1}]\phi(F(z))} = \frac{[z^n]F(z)}{[z^{n-1}]\frac{1}{1-F(z)}}, \quad n \geq 1.$$

5.2 Hook-length formulas for further tree families

In this chapter, we take a look at tree families, which are yet to be examined, and derive further hook-length formulas. We start with even trees.

5.2.1 Even trees

Definition 5.8. *An even tree is a rooted ordered tree, where each vertex has an even number of descendants.*

Due to this definition the degree weights for even trees are easily obtained by:

$$\phi_j = \begin{cases} 1, & \text{if } j \geq 2,\ j \text{ even}, \\ 0, & \text{if } j \text{ odd}. \end{cases}$$

The way to cancel out every odd coeffient in a generating function is to consider $F(z) = \frac{1}{2}(A(z) + A(-z))$, see (2.10). This, together with the generating function of the sequence $(1, 1, 1, \cdots)$, which is $\frac{1}{1-t}$, see (2.6), leads to:

$$\begin{aligned}\phi(t) &= \frac{1}{2}\left(\frac{1}{1-t} + \frac{1}{1+t}\right) = \frac{1}{2}\left(\frac{1+t+1-t}{1-t^2}\right) \\ &= \frac{1}{2}\left(\frac{2}{1-t^2}\right) = \frac{1}{1-t^2}.\end{aligned}$$

With (5.1) follows the weight-function for even trees:

$$\rho(n) = \frac{[z^n]F(z)}{[z^{n-1}]\phi(F(z))} = \frac{[z^n]F(z)}{[z^{n-1}]\frac{1}{1-F(z)^2}}, \quad n \geq 1. \tag{5.6}$$

5.2.2 Motzkin-trees

Definition 5.9. *A Motzkin-tree, also called unary-binary tree, is a rooted ordered tree, where each vertex has 0, 1 or 2 children.*

Let $\mathfrak{M}(n)$ denote the set of Motzkin-trees with n vertices.
As the name suggests, they are related to Motzkin-numbers and Motzkin-paths, which, just like Catalan-numbers, have diverse applications in geometry, combinatorics and number theory, see [DS77].
Since we already know the degree-weights of binary trees ($\phi_0 = 1$, $\phi_1 = 2$, $\phi_2 = 1$ and $\phi_j = 0$ for $j \geq 3$) and we know there are no attachment positions in Motzkin-trees, we get $\phi_0 = 1$, $\phi_1 = 1$, $\phi_2 = 1$ and $\phi_j = 0$, for $j \geq 3$, and thus the degree-weight generating function $\phi(t) = 1 + t + t^2$ and therefore:

$$\rho(n) = \frac{[z^n]F(z)}{[z^{n-1}]\phi(F(z))} = \frac{[z^n]F(z)}{[z^{n-1}](1 + F(z) + F(z)^2)}, \quad n \geq 1. \tag{5.7}$$

Example 7. For this example it's convenient to define $\mathrm{EN}(T)$ as the set of leaves of the rooted tree T. $\mathrm{EN}(T)$ consists of all vertices v satisfying $\deg(v) = 0$ and thus $h_v = 1$.
First we want to apply (5.7) on the function

$$F(z) = \frac{1}{2}(e^z - 1) = \frac{1}{2}\left(\sum_{n\geq 0}\frac{1}{n!}z^n - 1\right) = \frac{1}{2}\left(\sum_{n\geq 1}\frac{1}{n!}z^n\right).$$

We obtain

$$[z^n]F(z) = \frac{1}{2 \cdot n!}, \quad n \geq 1.$$

By simplifying the denominator of (5.7), we get

$$\begin{aligned}
[z^{n-1}]\phi(F(z)) &= [z^{n-1}](1+F(z)+F(z)^2)\\
&= [z^{n-1}]\left(1+\frac{1}{2}(e^z-1)+\frac{1}{4}\cdot e^{2z}-\frac{1}{2}\cdot e^z+\frac{1}{4}\right)\\
&= [z^{n-1}]\frac{3}{4}-\frac{1}{4}\cdot e^{2z} = [z^{n-1}]\frac{3}{4}-\frac{1}{4}\sum_{n\geq 0}\frac{2^n}{n!}z^n\\
&= [z^{n-1}]\frac{3}{4}-\sum_{n\geq 0}\frac{2^{n-2}}{n!}z^n\\
&= \begin{cases} \frac{2^{n-3}}{(n-1)!}, & \text{if } n\geq 2,\\ \frac{1}{2}, & \text{if } n=1. \end{cases}
\end{aligned}$$

Finally, with (5.1) and our results, we get

$$\begin{aligned}
\rho(n) = \frac{[z^n]F(z)}{[z^{n-1}]\phi(F(z))} &= \frac{\frac{1}{2\cdot n!}}{\frac{2^{n-3}}{(n-1)!}} = \frac{\frac{1}{n}}{\frac{2^{n-2}}{1}}\\
&= \frac{1}{n2^{n-2}}, \quad n\geq 2.
\end{aligned}$$

In conclusion we derive the hook-function

$$\rho(n) = \begin{cases} \frac{1}{n2^{n-2}}, & \text{if } n\geq 2,\\ 1, & \text{if } n=1, \end{cases}$$

and the hook-length formula

$$\sum_{T\in\mathfrak{M}(n)}\prod_{v\in T\setminus EN(T)}\frac{1}{h_v 2^{h_v-2}} = \frac{1}{2n!}.$$

Therefore, it follows

$$\sum_{n\geq 1}\left(\sum_{T\in\mathfrak{M}(n)}\prod_{v\in T\setminus EN(T)}\frac{1}{h_v 2^{h_v-2}}\right)z^n = \sum_{n\geq 1}\frac{1}{2n!}z^n.$$

Rearranging yields the equivalent hook-length formula

$$2n!\sum_{T\in\mathfrak{M}(n)}\prod_{v\in T\setminus EN(T)}\frac{1}{h_v 2^{h_v-2}} = 1,$$

and thus

$$\sum_{n\geq 1} 2n! \sum_{T\in\mathfrak{M}(n)} \prod_{v\in T\setminus EN(T)} \frac{1}{h_v 2^{h_v-2}} = \frac{z}{1-z}.$$

In Chapter 3 we mentioned the so-called tree-function $W(z)$, which satisfies the functional equation $W(z) = z\cdot e^{W(z)}$. In (3.14), we have proven

$$P(z)^m = 1 + \sum_{n\geq 1} \frac{m(n+m)^{n-1}}{n!} z^n,$$

with $P(z) = e^{W(z)}$. To get the explicit form of $W(z)^m$ we multiply $P(z)^m$ by z^m and get

$$\begin{aligned} W(z)^m = z^m \cdot P(z)^m &= z^m + z^m \cdot \sum_{n\geq 1} \frac{m(n+m)^{n-1}}{n!} z^n \\ &= z^m + \sum_{n\geq 1} \frac{m(n+m)^{n-1}}{n!} z^{n+m} = z^m + \sum_{n>m} \frac{mn^{n-m-1}}{(n-m)!} z^n \\ &= \sum_{n\geq m} \frac{mn^{n-m-1}}{(n-m)!} z^n. \end{aligned}$$

For the next corollary we need

$$\begin{aligned} W(z) &= \sum_{n\geq 1} \frac{n^{n-2}}{(n-1)!} z^n \\ &= \sum_{n\geq 1} \frac{n^{n-1}}{n!} z^n, \end{aligned} \tag{5.8}$$

and

$$W(z)^2 = \sum_{n\geq 2} \frac{2n^{n-3}}{(n-2)!} z^n. \tag{5.9}$$

Corollary 5.10. *It holds*

$$\sum_{n\geq 1} n! \sum_{T\in\mathfrak{M}(n)} \frac{1}{2^{|EN(T)|-1}} \prod_{v\in T\setminus EN(T)} \left(1+\frac{1}{h_v}\right) z^n = \sum_{n\geq 1} (n+1)^{n-1} z^n,$$

Proof. Recall $EN(T)$ denotes the set of leaves of the tree T.

We now use the same $\phi(t) = 1+t+t^2$, but the different function $F(z) = \frac{1}{2}\left(\frac{W(2z)}{2z} - 1\right)$. To apply (5.1), we need $[z^n]F(z)$ and $[z^{n-1}]\phi(F(z))$, which now we are able to obtain easily by using (5.8):

$$
\begin{aligned}
[z^n]F(z) &= [z^n]\frac{1}{2}\left(\frac{W(2z)}{2z} - 1\right) = [z^n]\frac{1}{2}\left(\frac{\sum_{n\geq 1} 2^n \frac{n^{n-1}}{n!} \cdot z^n}{2z} - 1\right) \\
&= [z^n]\frac{1}{2}\left(\sum_{n\geq 1} 2^{n-1}\frac{n^{n-1}}{n!} \cdot z^{n-1} - 1\right) = [z^n]\frac{1}{2}\left(\sum_{n\geq 2} 2^{n-1}\frac{n^{n-1}}{n!} \cdot z^{n-1}\right) \\
&= [z^n]\frac{1}{2}\left(\sum_{n\geq 1} 2^n\frac{(n+1)^n}{(n+1)!} \cdot z^n\right) = [z^n]\sum_{n\geq 1} 2^{n-1}\frac{(n+1)^n}{(n+1)!} \cdot z^n \\
&= \frac{2^{n-1}(n+1)^n}{(n+1)!}, \quad n \geq 1,
\end{aligned}
$$

and

$$
\begin{aligned}
[z^{n-1}]\phi(F(z)) &= [z^{n-1}]1 + F(z) + F(z)^2 \\
&= [z^{n-1}]1 + \frac{1}{2}\left(\frac{W(2z)}{2z} - 1\right) + \frac{1}{4}\left(\frac{W(2z)^2}{(2z)^2} - 2 \cdot \frac{W(2z)}{2z} + 1\right) \\
&= [z^{n-1}]1 - \frac{1}{2} + \frac{1}{4} + \frac{1}{2} \cdot \frac{W(2z)}{2z} - \frac{1}{2} \cdot \frac{W(2z)}{2z} + \frac{1}{4} \cdot \frac{W(2z)^2}{(2z)^2} \\
&= [z^{n-1}]\frac{3}{4} + \frac{1}{4} \cdot \frac{W(2z)^2}{(2z)^2}, \quad n \geq 1.
\end{aligned}
$$

For $n \geq 2$ and with (5.9), we calculate

$$
\begin{aligned}
[z^{n-1}]\frac{1}{4} \cdot \frac{W(2z)^2}{(2z)^2} &= [z^{n+1}]\frac{1}{16} \cdot W(2z)^2 = [z^{n+1}]\frac{1}{16}\sum_{n\geq 2}\frac{2^{n+1}n^{n-3}}{(n-2)!}z^n \\
&= \frac{1}{16} \cdot \frac{2^{n+2}(n+1)^{n-2}}{(n-1)!} \\
&= \frac{2^{n-2}(n+1)^{n-2}}{(n-1)!}.
\end{aligned}
$$

Finally, with (5.1) and our results we get

$$\rho(n) = \frac{[z^n]F(z)}{[z^{n-1}]\phi(F(z))} = \frac{\frac{2^{n-1}(n+1)^n}{(n+1)!}}{\frac{2^{n-2}(n+1)^{n-2}}{(n-1)!}} = \frac{2(n+1)}{n}$$
$$= 2\left(1+\frac{1}{n}\right).$$

In conclusion we derive the hook-function

$$\rho(n) = \begin{cases} 2\left(1+\frac{1}{n}\right), & \text{if } n \geq 2, \\ 1, & \text{if } n = 1, \end{cases}$$

and therefore

$$\sum_{n\geq 1}\left(\sum_{T\in\mathfrak{M}(n)} 1 \prod_{v\in T\setminus EN(T)} 2\left(1+\frac{1}{h_v}\right)\right) z^n = \sum_{n\geq 1} \frac{2^{n-1}(n+1)^n}{(n+1)!} z^n.$$

This implies the hook-length formula

$$\sum_{T\in\mathfrak{M}(n)} 1 \prod_{v\in T\setminus EN(T)} 2\left(1+\frac{1}{h_v}\right) = \frac{2^{n-1}(n+1)^n}{(n+1)!}$$

$$\iff \sum_{T\in\mathfrak{M}(n)} 2^{n-|EN(T)|} \prod_{v\in T\setminus EN(T)} \left(1+\frac{1}{h_v}\right) = \frac{2^{n-1}(n+1)^{n-1}}{n!}$$

$$\iff \frac{n!}{2^{n-1}} \sum_{T\in\mathfrak{M}(n)} 2^{n-|EN(T)|} \prod_{v\in T\setminus EN(T)} \left(1+\frac{1}{h_v}\right) = (n+1)^{n-1}$$

$$\iff n! \sum_{T\in\mathfrak{M}(n)} 2^{n-|EN(T)|-(n-1)} \prod_{v\in T\setminus EN(T)} \left(1+\frac{1}{h_v}\right) = (n+1)^{n-1}$$

$$\iff n! \sum_{T\in\mathfrak{M}(n)} \frac{1}{2^{|EN(T)|-1}} \prod_{v\in T\setminus EN(T)} \left(1+\frac{1}{h_v}\right) = (n+1)^{n-1}.$$

Altogether, we obtain

$$\sum_{n\geq 1}\left(n! \sum_{T\in\mathfrak{M}(n)} \frac{1}{2^{|EN(T)|-1}} \prod_{v\in T\setminus EN(T)} \left(1+\frac{1}{h_v}\right)\right) z^n = \sum_{n\geq 1} (n+1)^{n-1} z^n,$$

which is the corresponding formula to Postnikov's result (3.9) for Motzkin-trees. □

5.2.3 Labelled ordered trees

In this section, we describe various labelled tree models, using a variation of our approach for weighted ordered trees. For this purpose, we extend the definition given in Chapter 4.

Definition 5.11. *A labelled tree of size n is a tree, where its nodes are labelled with distinct integers of the set* $\{1, 2, ..., n\}$.
We denote the family of labelled ordered trees with $\mathfrak{O}_L$ *and* $\mathbb{L}(T)$ *the set of labellings of a given ordered tree* $T \in \mathfrak{O}$.

Apparently $|\mathbb{L}(T)| = n!$, for $T \in \mathfrak{O}(n)$. Following this, we consider a labelled ordered tree $T_L \in \mathfrak{O}_L$ as a pair $T_L = (T, L)$ with $T \in \mathfrak{O}$ and $L \in \mathbb{L}_T$ a labelling with distinct integers of $\{1, 2, ..., |T|\}$.
Now we have to associate each labelled rooted tree T' with a non-empty set of ordered labelled trees. To achieve this we use shapeset notation:

Definition 5.12. *For a labelled rooted tree family* $\mathfrak{R}_L$*, we associate to each tree* $T' \in \mathfrak{R}_L$ *a non empty set of ordered labelled trees by*

$$\textit{shapeset}(T') = \{T_L^{(1)}, T_L^{(2)}, ..., T_L^{(m)}\}, \quad m \geq 1, \quad \textit{with } T_L^{(i)} \in \mathfrak{O}_L, 1 \leq i \leq m,$$

where the trees $T', T_L^{(1)}, ..., T_L^{(m)}$*, are "label-isomorphic" meaning that for all* $1 \leq i \leq m$ *and labels* x, y *it holds:*

vertex labelled x *is a child of vertex labelled* y *in* T' $\iff$ *vertex labelled* x *is a child of vertex labelled* y *in* $T_L^{(i)}$.

Equivalent to our approach with unlabelled tree models we assume there exists a weighted ordered tree family $\mathfrak{F}$ with a degree-weight generating function $\phi(t)$, such that for each labelled ordered tree $T_L \in \mathfrak{O}_L$ the following relation holds:

$$w_{\deg}(T_L) = \sum_{T' \in \mathfrak{R}_L : T_L \in \text{ shapeset}(T')} \frac{1}{|\text{shapeset}(T')|}.$$

With ρ defined as an arbitrary hook function and $T_L \in \mathfrak{O}_L$ we further get

$$w(T_L) = w_{hook}(T_L) \cdot w_{\deg}(T_L) = w_{hook}(T_L) \cdot \sum_{T' \in \mathfrak{R}_L : T_L \in \text{shapeset}(T')} \frac{1}{|\text{shapeset}(T')|}$$

$$= \sum_{T' \in \mathfrak{R}_L : T_L \in \text{shapeset}(T')} \frac{w_{hook}(T_L)}{|\text{shapeset}(T')|}.$$

Following our isomorphic shapeset definition, we further conclude

$$w_{hook}(T') = w_{hook}(T_L^{(i)}), \quad 1 \leq i \leq m,$$

which leads to

$$w(T_L) = \sum_{T' \in \mathfrak{R}_L : T_L \in \text{shapeset}(T')} \frac{w_{hook}(T')}{|\text{shapeset}(T')|}.$$

Furthermore, we obtain (recall that $|\mathbb{L}(T)| = n!$):

$$\sum_{T \in \mathfrak{O}(n)} w(T) = \sum_{T \in \mathfrak{O}(n)} \frac{w(T)}{n!} \sum_{L \in \mathbb{L}(n)} 1 = \frac{1}{n!} \cdot \sum_{T_L \in \mathfrak{O}_L(n)} w(T_L)$$

$$= \frac{1}{n!} \cdot \sum_{T_L \in \mathfrak{O}_L(n)} \sum_{T' \in \mathfrak{R}_L : T_L \in \text{shapeset}(T')} \frac{w_{hook}(T')}{|\text{shapeset}(T')|}$$

$$= \frac{1}{n!} \cdot \sum_{T' \in \mathfrak{R}_L(n)} |\text{shapeset}(T')| \cdot \frac{w_{hook}(T')}{|\text{shapeset}(T')|}$$

$$= \sum_{T' \in \mathfrak{R}_L(n)} \frac{w_{hook}(T')}{n!},$$

which leads to

$$F(z) = \sum_{n \geq 1} \left(\sum_{T \in \mathfrak{O}(n)} w(T) \right) z^n = \sum_{n \geq 1} \left(\sum_{T' \in \mathfrak{R}_L(n)} w_{hook}(T') \right) \frac{z^n}{n!}.$$

This means, that our result (5.1) for weighted tree families associated to a degree-weight generating function $\phi(t)$ is also true for the corresponding labelled tree family $\mathfrak{R}_L$.

Example 8. We now examine the family $\mathfrak{C}$ of labelled cyclic trees.

Definition 5.13. *A labelled cyclic tree is a labelled tree where each vertex is either an* *end-vertex or its children are arranged via circular shifts such that the child with the smallest label is always the leftmost child.*

To use (5.1) we have to assign the right degree-weights to our weighted ordered tree family $\mathfrak{O}$. In case of an end-vertex we set $\phi_0 = 1$. Since $T, T' \in \mathfrak{C}$ are considered equal, if the order of children of each vertex $v' \in T'$ can be obtained by cyclic movements of the children of the corresponding vertex $v \in T$, and for j children, there are j different equal cyclic positions. Therefore we conclude $\phi_j = \frac{1}{j}$, for $j \geq 1$.
Further we need (2.6):

$$\ln\left(\frac{1}{1-z}\right) = \sum_{n\geq 1} \frac{1}{n} z^n.$$

Overall we get $\phi(t) = 1 + \ln\left(\frac{1}{1-t}\right)$, which allows us to use (5.1) by defining $F(z) = 1 - e^{-W(z)}$, with $W(z)$ defined in (5.8). In (3.14), we already derived

$$\begin{aligned}
P(z)^m &= (e^{W(z)})^m = 1 + \sum_{n\geq 1} \frac{m(n+m)^{n-1}}{n!} z^n \\
\Rightarrow e^{-W(z)} &= (e^{W(z)})^{-1} = 1 + \sum_{n\geq 1} \frac{-(n-1)^{n-1}}{n!} z^n \\
\Rightarrow 1 - e^{-W(z)} &= \sum_{n\geq 1} \frac{(n-1)^{n-1}}{n!} z^n \\
\Rightarrow [z^n]F(z) &= \frac{(n-1)^{n-1}}{n!}.
\end{aligned}$$

Now we just need

$$\begin{aligned}
[z^{n-1}]\phi(F(z)) &= [z^{n-1}]1 + \ln\left(\frac{1}{1-F(z)}\right) \\
&= [z^{n-1}]1 + \ln\left(\frac{1}{1-1+e^{-W(z)}}\right) = [z^{n-1}]1 + \ln\left(\frac{1}{e^{-W(z)}}\right) \\
&= [z^{n-1}]1 + \ln(e^{W(z)}) = [z^{n-1}]1 + W(z), \quad n \geq 1.
\end{aligned}$$

With (5.8) we obtain

$$[z^{n-1}]\phi(F(z)) = \frac{(n-1)^{n-2}}{(n-1)!}, \quad n \geq 2.$$

So, with (5.1) it follows

$$\rho(n) = \frac{[z^n]F(z)}{[z^{n-1}]\phi(F(z))} = \frac{\frac{(n-1)^{n-1}}{n!}}{\frac{(n-1)^{n-2}}{(n-1)!}}$$

$$= \frac{n-1}{n} = 1 - \frac{1}{n}, \quad n \geq 2,$$

yielding

$$\rho(n) = \begin{cases} 1 - \frac{1}{n}, & \text{if } n \geq 2, \\ 1, & \text{if } n = 1. \end{cases}$$

Thus, it follows

$$\sum_{T \in \mathfrak{C}(n)} \prod_{v \in T \setminus EN(T)} \left(1 - \frac{1}{h_v}\right) = (n-1)^{n-1},$$

and

$$\sum_{n \geq 1} \left(\sum_{T \in \mathfrak{C}(n)} \prod_{v \in T \setminus EN(T)} \left(1 - \frac{1}{h_v}\right) \right) z^n = \sum_{n \geq 1} (n-1)^{n-1} z^n.$$

5.2.4 Weighted ordered forests

To further generalise some of our previous results about ordered forests, we now introduce forests of weighted ordered trees. Then our already known formulas appear as particular instances.

Definition 5.14. *A given finite sequence $F = (T_1, ..., T_k)$ of ordered trees, where we allow the empty sequence $F = \epsilon$, is called ordered forest. We call the family of ordered forests $\mathfrak{O}_F$ and define their size $|F| = |T_1| + |T_2| + \cdots |T_k|$, with $|\epsilon| = 0$.*

Now we define a weight for each ordered forest.

Definition 5.15. *Let $F \in \mathfrak{O}_F$ be a given ordered forest. Then F has a weight $w(F)$ defined as:*

$$w(F) := \phi_k \prod_{l=1}^{k} w(T_l), \quad \text{if } F = (T_1, ..., T_k), \quad \text{and} \quad w(\epsilon) = \phi_0.$$

Recall that $(\phi_j)_{j \geq 0}$ is the degree-weight sequence associated to a family $\mathfrak{O}$ of weighted ordered trees.

Following to this definition, the family $\mathfrak{O}_F$ of weighted ordered forests consists of all ordered forests F together with their weights $w(F)$. The generating functions

$$F(z) := \sum_{n\geq 1} \sum_{T\in\mathfrak{O}(n)} w(T)z^n,$$

of the total weights of ordered trees of size n, and the generating function

$$G(z) := \sum_{n\geq 0} \sum_{F\in\mathfrak{O}_F(n)} w(F)z^n,$$

of the total weights of forests of size n are, due to our definition of the weight of a forest, simply connected by

$$G(z) = \phi(F(z)).$$

Theorem 5.16. *Given a family $\mathfrak{O}_F$ of weighted ordered forests associated to a degree-weight generating function $\phi(t) = \sum_{j\geq 0} \phi_j t^j$, with $\phi_0 \neq 0$ and $\phi_1 \neq 0$, let $G(z)$ be the generating function of the total weights of forests of size n,*

$$G(z) = \sum_{n\geq 0} \left(\sum_{F\in\mathfrak{O}_F(n)} w(F) \right) z^n. \tag{5.10}$$

Then the hook-weight function ρ satisfies

$$\rho(n) = \frac{[z^n]\tilde{\phi}^{[-1]}(G(z) - \phi_0)}{[z^{n-1}]G(z)}, \quad n \geq 1, \tag{5.11}$$

where $\tilde{\phi}(t) := \phi(t) - \phi_0$, and $\tilde{\phi}^{[-1]}(t)$ denotes the inverse function of $\tilde{\phi}(t)$, with respects to composition, i.e. $\tilde{\phi}(\tilde{\phi}^{[-1]}(t)) = \tilde{\phi}^{[-1]}(\tilde{\phi}(t)) = t$

Proof. Due to $\phi_0 \neq 0$ and Theorem 2.9, there exists a formal power series $\tilde{\phi}^{[-1]}(t) = \sum_{j\geq 1} \psi_j t^j$, which is inverse to $\tilde{\phi}(t)$.
From $G(z) = \phi(F(z))$ and our definition of $\tilde{\phi}(t)$ we get the relation $G(z) = \phi_0 + \tilde{\phi}(F(z))$, so

$$F(z) = \sum_{n\geq 1} \sum_{T\in\mathfrak{O}(n)} w(T)z^n = \tilde{\phi}^{[-1]}(G(z) - \phi_0).$$

Therefore we can use (5.1) to obtain

$$\rho(n) = \frac{[z^n]F(z)}{[z^{n-1}]\phi(F(z))} = \frac{[z^n]\tilde{\phi}^{[-1]}(G(z)-\phi_0)}{[z^{n-1}](\phi(\tilde{\phi}^{[-1]}(G(z)-\phi_0)))}$$

$$= \frac{[z^n]\tilde{\phi}^{[-1]}(G(z)-\phi_0)}{[z^{n-1}](\phi_0+\tilde{\phi}(\tilde{\phi}^{[-1]}(G(z)-\phi_0)))}$$

$$= \frac{[z^n]\tilde{\phi}^{[-1]}(G(z)-\phi_0)}{[z^{n-1}]G(z)}.$$

□

In Chapter 4, we obtained expansion formulas for different kinds of forests. We'll use (5.11) to confirm our results.
With the degree-weight generating function $\phi(t) = \frac{1}{1-t}$ for plane forests we are yet to prove the result (4.9):

$$\rho(n) = -\frac{[z^n](G(z))^{-1}}{[z^{n-1}]G(z)}.$$

To use (5.11) we need $\tilde{\phi}^{[-1]}(t)$. We know, that

$$\tilde{\phi}(t) = \phi(t) - \phi_0 = \frac{1}{1-t} - 1 = \frac{t}{1-t}$$

$$\Rightarrow \tilde{\phi}(t)(1-t) = t$$

$$\Rightarrow \tilde{\phi}(t) = t(1+\tilde{\phi}(t))$$

$$\Rightarrow t = \frac{\tilde{\phi}(t)}{1+\tilde{\phi}(t)}.$$

Therefore $\tilde{\phi}^{[-1]}(t) = \frac{t}{1+t}$. We conclude

$$\rho(n) = \frac{[z^n]\tilde{\phi}^{[-1]}(G(z)-1)}{[z^{n-1}]G(z)} = \frac{[z^n](1-\frac{1}{G(z)})}{[z^{n-1}]G(z)}$$

$$= -\frac{[z^n]G(z)^{-1}}{[z^{n-1}]G(z)}, \quad n \geq 1,$$

which proves (4.9).

For forests of labelled trees, we obtained result (4.13):

$$\rho(n) = \frac{[z^n] \ln(F(z))}{[z^{n-1}]F(z)}.$$

With the degree-weight function for forests of labelled trees $\phi(t) = e^t$, we obtain $\tilde{\phi}(t) = e^t - 1$, yielding

$$\tilde{\phi}(t) = e^t - 1 \Rightarrow \tilde{\phi}(t) + 1 = e^t,$$

$$\ln(\tilde{\phi}(t) + 1) = t.$$

Consequently $\tilde{\phi}^{[-1]}(t) = \ln(1 + t)$. Hence, with (5.11), it follows

$$\rho(n) = \frac{[z^n](\tilde{\phi}^{[-1]}(G(z) - 1))}{[z^{n-1}]G(z)} = \frac{[z^n] \ln(G(z))}{[z^{n-1}]G(z)}, \quad n \geq 1.$$

Finally we use (5.11) for a new family of forests called labelled cyclic forests $\mathfrak{C}_F$.

Definition 5.17. *A given finite sequence $F = (T_1, ..., T_k)$ of cyclic trees, where cyclic rearrangments of a particular sequence of trees are considered to be equal, is called labelled cyclic forest. We allow the empty sequence $F = \epsilon$.*

We call the family of labelled cyclic forests $\mathfrak{C}_F$ and define their size $|F| = |T_1| + |T_2| + \cdots + |T_k|$, with $|\epsilon| = 0$.
Recall, that for cyclic trees we found the degree-weight function $\phi(t) = 1 + \ln(\frac{1}{1-t})$ and with

$$\phi(t) = 1 + \ln(\frac{1}{1-t}) \Rightarrow e^{\phi(t)} = e^{1+\ln(\frac{1}{1-t})}$$

$$\Rightarrow e^{\phi(t)} = e \cdot e^{\ln(\frac{1}{1-t})} \Rightarrow e^{\phi(t)} = e \cdot \frac{1}{1-t} \Rightarrow 1 - t = \frac{e}{e^{\phi(t)}}$$

$$\Rightarrow t = 1 - e^{1-\phi(t)},$$

we get $\tilde{\phi}^{[-1]}(t) = 1 - e^{1-t}$. With (5.11), the tree-function $W(z) = \sum_{n\geq1} \frac{n^{n-1}}{(n)!} z^n$, and

$G(z) = W(z) + 1$, we obtain

$$\rho(n) = \frac{[z^n]\tilde{\phi}^{[-1]}(G(z)-1)}{[z^{n-1}]G(z)} = -\frac{[z^n]e^{1-G(z)}}{[z^{n-1}]G(z)}$$
$$= -\frac{[z^n]e^{-W(z)}}{[z^{n-1}]W(z)+1}.$$

In Chapter 3, we got result (3.14) for a power-series $P(z)$, which satisfies $P(z) = e^{W(z)}$, recall:

$$P(z)^m = \sum_{n\geq 0} \frac{m(n+m)^{n-1}}{n!} z^n.$$

For $m = -1$ we obtain $P(z)^{-1} = e^{-W(z)} = \sum_{n\geq 0} -\frac{(n-1)^{n-1}}{n!} z^n$. We get

$$\rho(n) = -\frac{[z^n]e^{-W(z)}}{[z^{n-1}]W(z)+1} = \frac{\frac{(n-1)^{n-1}}{n!}}{\frac{(n-1)^{n-2}}{(n-1)!}} = \frac{n-1}{n}$$
$$= 1 - \frac{1}{n}, \quad n \geq 2,$$

which yields the hook-length formula

$$\sum_{F\in\mathfrak{C}_F(n)} \prod_{v\in F\setminus EN(F)} \left(1 + \frac{1}{h_v}\right) = n^{n-1},$$

and thus

$$\sum_{n\geq 1} \left(\sum_{F\in\mathfrak{C}_F(n)} \prod_{v\in F\setminus EN(F)} \left(1 + \frac{1}{h_v}\right)\right) z^n = \sum_{n\geq 1} n^{n-1} z^n.$$

6 Hook-length formulas for increasing trees

In this chapter, we follow the calculations and results for increasing trees given by Kuba and Panholzer in [KP13, KP16].

Definition 6.1. *An increasing tree $\tilde{T}$ is a specific labelled tree with size n, where the nodes are uniquely labelled by integers from 1 to n, so that every sequence of labels in any path starting from the root is increasing.*

Definition 6.2. *Let $\tilde{\mathfrak{O}}(n)$ denote the set of ordered trees with n vertices and let $\tilde{\mathfrak{O}}$ be defined as*

$$\tilde{\mathfrak{O}} = \bigcup_{n\geq 1} \tilde{\mathfrak{O}}(n).$$

Analogous to the previous chapter we will define the family $\mathfrak{I}$ of weighted increasing trees as follows:

Definition 6.3. *The family $\mathfrak{I}$ consists of all ordered increasing trees $\tilde{T} \in \tilde{\mathfrak{O}}$ together with their degree-weights $w_{deg}(\tilde{T})$, where $w_{deg}(\tilde{T})$ is defined as:*

$$w_{deg}(\tilde{T}) := \prod_{v\in\tilde{T}} \phi_{deg(v)},$$

with a degree-weight sequence $(\phi_j)_{j\geq 0}, \phi_0 \neq 0$. The hook-weight $\tilde{w}_{hook}(\tilde{T})$, given for a hook-function $\tilde{\rho}$, and the weight $w(\tilde{T})$ of an increasing ordered tree is defined as:

$$\tilde{w}_{hook}(\tilde{T}) := \prod_{v\in\tilde{T}} \tilde{\rho}(h_v), \quad w(\tilde{T}) := w_{deg}(\tilde{T}) \cdot \tilde{w}_{hook}(\tilde{T}).$$

Definition 6.4. *Let $\phi(t) := \sum_{j\geq 0} \phi_j t^j$ denote the degree-weight generating function. Then the exponential generating function of the total degree weights is given by:*

$$T(z) := \sum_{n\geq 1} T_n \frac{z^n}{n!}, \qquad T_n := \sum_{\tilde{T}\in\tilde{\mathfrak{O}}(n)} w_{deg}(\tilde{T}).$$

Now we are able to prove the corresponding expansion technique for weighted increasing trees.

Theorem 6.5. *Given a family $\mathfrak{I}$ of weighted increasing trees associated to a degree-weight generating function $\phi(t)$, let $\tilde{F}(z)$ be the exponential generating function of the total weights of increasing trees of size n:*

$$\sum_{n\geq 1}\left(\sum_{\tilde{T}\in\tilde{\mathfrak{H}}(n)} w(\tilde{T})\right)\frac{z^n}{n!} = \tilde{F}(z). \tag{6.1}$$

Then the hook-function $\tilde{\rho}$ satisfies

$$\tilde{\rho}(n) = \frac{[z^{n-1}]\tilde{F}'(z)}{[z^{n-1}]\phi(\tilde{F}(z))} = \frac{n[z^n]\tilde{F}(z)}{[z^{n-1}]\phi(\tilde{F}(z))}, \quad n\geq 1. \tag{6.2}$$

Proof. We use $[z^n]\tilde{F}(z) = [z^n]\sum_{n\geq 1} f_n\frac{z^n}{n!} = f_n\frac{1}{n!}$, with $f_n = \sum_{\tilde{T}\in\tilde{\mathfrak{H}}(n)} w(\tilde{T})$ and the top-bottom decomposition of a tree into a root vertex v and the subtrees attached to it. We consider a tree $\tilde{T}$ with size $n\geq 2$ and assume that the root v has out-degree $j\geq 1$. Recall the root vertex gives a degree weight factor $\phi_{deg(v)} = \phi_j$ and a hook-weight $\tilde{\rho}(h_v) = \tilde{\rho}(n)$. We split our tree into its root v and the subtrees $\tilde{T}_i$ attached to the root:

$$\tilde{T}_i\in\tilde{\mathfrak{H}}(n_i), \quad n_i\geq 1, \quad \sum_{i=1}^{j} n_i = n-1.$$

We also need

$$w(\tilde{T}) = w_{deg}(\tilde{T})\cdot\tilde{w}_{hook}(\tilde{T}) = \phi_j\cdot\tilde{\rho}(n)\prod_{l=1}^{j} w_{deg}(\tilde{T}_l)\cdot\tilde{w}_{hook}(\tilde{T}_l)$$

$$= \phi_j\cdot\tilde{\rho}(n)\prod_{l=1}^{j} w(\tilde{T}_l),$$

and the multinomial coefficient (4.11), since after removing the root v the remaining $n-1$ vertices are distributed to the j subtrees and we need to consider all the ways to do so.
We start of with $[z^{n-1}]\tilde{F}'$ and get

$$[z^{n-1}]\tilde{F}' = [z^{n-1}]\sum_{n\geq 1}\sum_{\tilde{T}\in\tilde{\mathfrak{H}}(n)} w(\tilde{T})\cdot\frac{z^{n-1}}{(n-1)!}$$

$$= \sum_{\tilde{T}\in\tilde{\mathfrak{H}}(n)}\frac{w(\tilde{T})}{(n-1)!}.$$

By now considering all possible ways to build an increasing tree of size $n \geq 2$ we conclude

$$[z^{n-1}]\tilde{F}' = [z^{n-1}] \sum_{n \geq 1} \sum_{\tilde{T} \in \tilde{\mathfrak{H}}(n)} w(\tilde{T}) \cdot \frac{z^{n-1}}{(n-1)!} = \sum_{\tilde{T} \in \tilde{\mathfrak{H}}(n)} \frac{w(\tilde{T})}{(n-1)!}$$

$$= \frac{\tilde{\rho}(n)}{(n-1)!} \sum_{j \geq 1} \phi_j \left(\sum_{\substack{n_1 + \cdots + n_j = n-1 \\ n_1, n_2, \ldots, n_j \geq 1}} \left(\sum_{\tilde{T}_1 \in \tilde{\mathfrak{H}}(n_1), \ldots, \tilde{T}_j \in \tilde{\mathfrak{H}}(n_j)} \prod_{l=1}^{j} w(\tilde{T}_l) \right) \binom{n-1}{n_1, n_2, \ldots, n_j} \right)$$

$$= \tilde{\rho}(n) \sum_{j \geq 1} \phi_j \sum_{\substack{n_1 + n_2 + \cdots + n_j = n-1 \\ n_1, n_2, \ldots, n_j \geq 1}} \left(\sum_{\tilde{T}_1 \in \tilde{\mathfrak{H}}(n_1)} w(\tilde{T}_1) \right) \frac{1}{n_1!} \cdots \left(\sum_{\tilde{T}_j \in \tilde{\mathfrak{H}}(n_j)} w(\tilde{T}_l) \right) \frac{1}{n_j!}$$

$$= \tilde{\rho}(n) \sum_{j \geq 1} \phi_j \sum_{\substack{n_1 + \cdots + n_j = n-1 \\ n_1, n_2, \ldots, n_j \geq 1}} \prod_{l=1}^{j} \frac{f_{n_1}}{n_1!} \frac{f_{n_2}}{n_2!} \cdots \frac{f_{n_j}}{n_j!}.$$

Simplified with the $[z^n]$ operator and using (4.4) and the definition of $\phi(t) = \sum_{n \geq 0} \phi_n t^n$ we obtain

$$[z^{n-1}]\tilde{F}' = \tilde{\rho}(n)[z^{n-1}]\phi(\tilde{F}(z))$$

$$\Rightarrow \tilde{\rho}(n) = \frac{[z^{n-1}]\tilde{F}'}{[z^{n-1}]\phi(\tilde{F}(z))}, \quad n \geq 1,$$

which, due to (2.10), leads to

$$\tilde{\rho}(n) = \frac{[z^{n-1}]\tilde{F}'(n)}{[z^{n-1}]\phi(\tilde{F}(z))} = \frac{n[z^n]\tilde{F}(z)}{[z^{n-1}]\phi(\tilde{F}(z))}, \quad n \geq 2.$$

Now we check the inital case $n = 1$. Since $[z^1]\tilde{F}' = \sum_{\tilde{T} \in \tilde{\mathfrak{H}}(1)} \frac{w(\tilde{T})}{1!} = \phi_0 \cdot \tilde{\rho}(1)$ we get

$$\tilde{\rho}(1) = \frac{1[z^1]\tilde{F}(z)}{[z^0]\phi(\tilde{F}(z))} = \frac{\phi_0 \cdot \tilde{\rho}(1)}{\phi_0},$$

which proves the theorem for all $n \geq 1$. □

Alternatively, by starting with a family $\mathfrak{O}$ of weighted ordered trees associated to a degree-weight generating function $\phi(t)$ and a set $\mathbb{L}_I(T)$ of all increasing labellings with distinct integers from $\{1, 2, ..., |T|\}$ for each ordered tree $T \in \mathfrak{O}$, we can also define the family of weighted increasing trees as follows:

Definition 6.6. *The family $\mathfrak{I}$ of weighted increasing trees consists of pairs (T,L), where $T \in \mathfrak{O}$ is an ordered tree and $L \in \mathbb{L}_I$ an increasing labelling, together with their degree-weights $w_{deg(T)}$.*

We use this definition and the next lemma to prove (6.2) by using (5.1).

Lemma 6.7. *For any ordered tree $T \in \mathfrak{O}(n)$ the number of increasing labellings $|\mathbb{L}_I(T)|$ of T is equal to*

$$|\mathbb{L}_I(T)| = \frac{n!}{\prod_{v \in T} h_v}. \tag{6.3}$$

Proof. Since we take a look at some k-labelled increasing tree families later on, we prove (6.7) for k-labellings, see (6.10). The lemma is then a special case for $k = 1$. □

With this lemma we can now show the strong connection between our previous results for weighted ordered trees and hook-length formulas for weighted increasing trees.

Lemma 6.8. *If the hook-weight functions $\tilde{\rho}$ and ρ of the families $\mathfrak{I}$ and $\mathfrak{O}$, respectively, satisfy the relation*

$$\tilde{\rho}(n) = n\rho(n), \tag{6.4}$$

then the following relation between the total weights of weighted increasing trees $\tilde{T} \in \tilde{\mathfrak{O}}$ and weighted ordered trees $T \in \mathfrak{O}$ holds:

$$\sum_{n\geq 1} \sum_{\tilde{T} \in \tilde{\mathfrak{O}}} w_{deg}(\tilde{T}) \cdot \prod_{v \in \tilde{T}} \tilde{\rho}(h_v) \frac{z^n}{n!} = \sum_{n\geq 1} \sum_{T \in \mathfrak{O}} w_{deg}(T) \cdot \prod_{v \in T} \rho(h_v) z^n. \tag{6.5}$$

Proof. We prove this by equating coefficients, so it follows

$$\frac{1}{n!}\sum_{\tilde{T}\in\tilde{\mathfrak{D}}} w_{deg}(\tilde{T})\cdot\prod_{v\in\tilde{T}}\tilde{\rho}(h_v) = \frac{1}{n!}\sum_{T\in\mathfrak{D}}\sum_{L\in\mathbb{L}_I(T)} w_{deg}(T)\cdot\prod_{v\in T}\tilde{\rho}(h_v)$$

$$= \frac{1}{n!}\sum_{T\in\mathfrak{D}} w_{deg}(T)\cdot\prod_{v\in T}\tilde{\rho}(h_v)\cdot|\mathbb{L}_I(T)| = \frac{1}{n!}\sum_{T\in\mathfrak{D}} w_{deg}(T)\cdot\prod_{v\in T}\tilde{\rho}(h_v)\cdot\frac{n!}{\prod_{v\in T} h_v}$$

$$= \sum_{T\in\mathfrak{D}} w_{deg}(T)\cdot\prod_{v\in T}\tilde{\rho}(h_v)\cdot\frac{1}{\prod_{v\in T} h_v}.$$

Since

$$\tilde{\rho}(n) = n\rho(n) \iff \frac{\tilde{\rho}(n)}{n} = \rho(n),$$

we finally get

$$\sum_{T\in\mathfrak{D}} w_{deg}(T)\cdot\prod_{v\in T}\tilde{\rho}(h_v)\cdot\frac{1}{\prod_{v\in T} h_v} = \sum_{T\in\mathfrak{D}} w_{deg}(T)\cdot\prod_{v\in T}\rho(h_v).$$

□

So, assuming $\tilde{\rho}(n) = n\rho(n)$ holds, and defining

$$F(z) = \sum_{n\geq 1}\left(\sum_{T\in\mathfrak{D}(n)} w_{deg}(T)\prod_{v\in T}\rho(h_v)\right)z^n,$$

$$\tilde{F}(z) = \sum_{n\geq 1}\left(\sum_{\tilde{T}\in\tilde{\mathfrak{D}}(n)} w_{deg}(\tilde{T})\prod_{v\in\tilde{T}}\tilde{\rho}(h_v)\right)\frac{z^n}{n!},$$

with (5.1) we get $F(z) = \tilde{F}(z)$.

Example 9. Let $\mathfrak{I}_k$ denote the family of increasingly labelled k-ary trees. In Chapter 4 we derived the following hook-length formula:

$$n!\sum_{T\in\mathfrak{I}_k(n)}\prod_{v\in T}\frac{(m+(k-1)h_v)^{h_v-1}}{kh_v(mk+(k-1)(h_v-1))^{h_v-2}} = m(k(m+(k-1)n))^{n-1},$$

for k-ary trees. Now we use (6.4) and (6.5) to obtain a hook-length formula for increasingly labelled k-ary trees:

$$\sum_{T\in\mathfrak{I}_k(n)}\prod_{v\in T}\frac{h_v(m+(k-1)h_v)^{h_v-1}}{h_vk(mk+(k-1)(h_v-1))^{h_v-2}}=m(k(m+(k-1)n))^{n-1}$$

$$\iff\sum_{T\in\mathfrak{I}_k(n)}\prod_{v\in T}\frac{(m+(k-1)h_v)^{h_v-1}}{k(mk+(k-1)(h_v-1))^{h_v-2}}=m(k(m+(k-1)n))^{n-1},$$

yielding

$$\sum_{n\geq 1}\left(\sum_{T\in\mathfrak{I}_k(n)}\prod_{v\in T}\frac{(m+(k-1)h_v)^{h_v-1}}{k(mk+(k-1)(h_v-1))^{h_v-2}}\right)\frac{z^n}{n!}=\sum_{n\geq 1}m(k(m+(k-1)n))^{n-1}\frac{z^n}{n!}.$$

6.1 Bilabelled increasing trees

We take a look at bilabelled increasing trees first, before we expand the results to general k-labelled increasing trees.

Definition 6.9. *A bilabelled increasing tree T is a specific labelled tree of size n, where the nodes are uniquely labelled by sets of size 2 of integers from 1 to $2n$, so that each label of a child vertex is larger than all the labels of its parent vertex.*

Then the family $\mathfrak{I}_2$ of bilabelled increasing trees consists of all trees $T\in\mathfrak{O}$ together with their weights $w(T)$ and the set of increasing bilabellings $\mathbb{L}_I^2(T)$, so we can identify it with triplets $(T,w(T),L(T))$, where $L(T)\in\mathbb{L}_I^2(T)$.
Let $T_n:=\sum_{T\in\mathfrak{O}(n)}w(T)\cdot|\mathbb{L}_I^2(T)|$ be the number of bilabelled increasing trees with n vertices and $T(z)$ denote the exponential generating function $T(z):=\sum_{n\geq 1}T_n\frac{z^{2n}}{(2n)!}$ of the number of bilabelled increasing trees $T\in\mathfrak{I}_k(n)$ with n vertices.
In [KP16] it is shown, that this function $T(z)$ fulfills the following autonomous second order differential equation:

$$T''(z)=\phi(T(z)),\quad T(0)=0,\quad T'(0)=0. \tag{6.6}$$

Our next goal is to give an implicit representation of $T(z)$. Therefore we translate the second order differential equation into a first-order differential equation.

Lemma 6.10. *The exponential generating function $T(z)$ of bilabelled increasing trees with degree-weight generating function $\phi(t)$ satisfies the first oder differential equation:*

$$T'(z) = \sqrt{2\Phi(T(z))}, \quad T(0) = 0,$$

with $\Phi(z) = \int_0^z \phi(t)dt$. $T(z)$ is given implicitly via

$$\int_0^{T(z)} \frac{1}{\sqrt{2\Phi(T(x))}} dx = z.$$

Proof. We start by transforming (6.6) into an first-order differential equation by multiplying it with $T'(z)$.

$$T'(z)T''(z) = T'(z)\phi(T(z)).$$

Integrating both sides yields

$$\frac{(T'(z))^2}{2} = \Phi(T(z)) \Rightarrow T'(z) = \sqrt{2\Phi(T(z))}, \quad T(0) = 0.$$

To obtain the implicit representation we need separation of variables, see [EBCD12, p. 42-51]. It states that the solution of the differential equation

$$y'(x) = f(y(x))g(x), \quad y(x_0) = y_0,$$

is given by the solution y of following equation:

$$\int_{y_0}^{y(x)} \frac{ds}{f(s)} = \int_{x_0}^{x} g(s)ds.$$

In our case this leads to

$$\int_0^{T(z)} \frac{1}{\sqrt{2\Phi(T(x))}} dx = z + C.$$

Since $T(0) = 0$, the constant fulfills $C = 0$, completing the proof. □

Further we need the number of different increasing bilabellings of a tree.

Corollary 6.11. *The number $|\mathbb{L}_I^2(T)|$ of different increasing bilabellings of a tree T of size n is given by*

$$|\mathbb{L}_I^2(T)| = \frac{(2n)!}{\prod_{v\in T}\left((2h_v)(2h_v-1)\right)}. \tag{6.7}$$

Proof. By Lemma 6.14 for $k = 2$. □

Now we can prove the next hook-length formula:

Corollary 6.12. *It holds*

$$\sum_{n\geq 1}\left(\sum_{T\in\mathfrak{O}(n)}\prod_{v\in T}\frac{\phi_{deg(v)}}{2h_v(2h_v-1)}\right)z^{2n} = \sum_{n\geq 1}\frac{T_n}{(2n)!}z^{2n}. \tag{6.8}$$

Proof. With (6.7) and $T_n := \sum_{T\in\mathfrak{O}(n)} w(T)\cdot|\mathbb{L}_I^2(T)|$, we get that the total weight T_n of increasingly bilabelled trees of size n is given as follows:

$$T_n = \sum_{T\in\mathfrak{O}(n)}\frac{w_{deg}(T)(2n)!}{\prod_{v\in T}\left(2h_v(2h_v-1)\right)}.$$

Dividing by $(2n)!$ leads to

$$\sum_{T\in\mathfrak{O}(n)}\frac{w_{deg}(T)}{\prod_{v\in T}\left(2h_v(2h_v-1)\right)} = \frac{T_n}{(2n)!}$$

$$\iff \sum_{T\in\mathfrak{O}(n)}\prod_{v\in T}\frac{\phi_{deg(v)}}{2h_v(2h_v-1)} = \frac{T_n}{(2n)!},$$

concluding to

$$\sum_{n\geq 1}\left(\sum_{T\in\mathfrak{O}(n)}\prod_{v\in T}\frac{\phi_{deg(v)}}{2h_v(2h_v-1)}\right)z^{2n} = \sum_{n\geq 1}\frac{T_n}{(2n)!}z^{2n}.$$

□

Example 10. We consider the family of unordered bilabelled increasing trees with the degree-weight generating function $\phi(t) = e^t$. Now we use Lemma 6.10:

$$\Phi(x) = \int_0^x \phi(t)dt = \int_0^x e^t dt = e^x - 1$$

$$\Rightarrow \int_0^{T(z)} \frac{1}{\sqrt{2(e^x - 1)}} = z.$$

Substituting $u = \sqrt{e^x - 1} \Rightarrow u^2 = e^x - 1 \Rightarrow u^2 + 1 = e^x$ yields

$$\frac{du}{dx} = \frac{1}{2\sqrt{e^x - 1}} e^x = \frac{u^2 + 1}{2u} \Rightarrow dx = \frac{2u}{u^2 + 1}.$$

Thus

$$\frac{1}{\sqrt{2}} \int_0^{T(z)} \frac{1}{\sqrt{2(e^x - 1)}} = \frac{1}{\sqrt{2}} \int_0^{\sqrt{e^{T(z)}-1}} \frac{1}{u} \cdot \frac{2u}{u^2+1} du = \frac{2}{\sqrt{2}} \int_0^{\sqrt{e^{T(z)}-1}} \frac{1}{1+u^2} du$$

$$= \sqrt{2} \arctan(u)\Big|_0^{\sqrt{e^{T(z)}-1}} = \sqrt{2} \arctan\left(\sqrt{e^{T(z)} - 1}\right).$$

Therefore

$$\sqrt{2} \arctan\left(\sqrt{e^{T(z)} - 1}\right) = z \Rightarrow \arctan\left(\sqrt{e^{T(z)} - 1}\right) = \frac{z}{\sqrt{2}}$$

$$\Rightarrow \sqrt{e^{T(z)} - 1} = \tan\left(\frac{z}{\sqrt{2}}\right) \Rightarrow e^{T(z)} = \tan^2\left(\frac{z}{\sqrt{2}}\right) + 1$$

$$\Rightarrow T(z) = \ln\left(\tan^2\left(\frac{z}{\sqrt{2}}\right) + 1\right).$$

Extracting coefficients leads to the so-called reduced tangent numbers

$$T_n = (2n)![z^{2n}]T(z) = \tilde{E}_n.$$

Together with $\phi(t) = e^t \Rightarrow \phi_j = \frac{1}{j!}$ and (6.8) we obtain

$$\sum_{n\geq 1} \left(\sum_{T \in \mathfrak{D}(n)} \prod_{v \in T} \frac{1}{\deg(v)! 2h_v(2h_v - 1)} \right) z^{2n} = \sum_{n \geq 1} \frac{\tilde{E}_n}{(2n)!} z^{2n}.$$

6.2 k-labelled increasing trees

Analogous to bilabelled increasing trees we define:

Definition 6.13. *An k-labelled increasing tree T is a specific labelled tree of size n, where the nodes are uniquely labelled by sets of size k of integers from 1 to kn, so that each label of a child vertex is larger than all the labels of its parent vertex.*

Then the family $\mathfrak{I}_k$ of k-labelled increasing trees consists of all trees $T \in \mathfrak{O}$ together with their weights $w(T)$ and the set of increasing bilabellings $\mathbb{L}_I^k(T)$, so we can identify it with triplets $(T, w(T), L(T))$, where $L(T) \in \mathbb{L}_I^k(T)$.
The exponential generating function $T(z) = \sum_{n\geq 1} T_n \frac{z^{kn}}{(kn)!}$ fulfills

$$T^{(k)}(z) = \phi(T(z)), \quad T^{(l)}(z) = 0, \quad 0 \leq l \leq k-1. \tag{6.9}$$

Lemma 6.14. *For any tree $T \in \mathfrak{O}(n)$ the number of increasing k-labellings $|\mathbb{L}_I^k(T)|$ of T is equal to*

$$|\mathbb{L}_I^k(T)| = \frac{(kn)!}{\prod_{v\in T} (kh_v)^{\underline{k}}}, \tag{6.10}$$

where $n^{\underline{k}} = n(n-1)(n-2)\cdot(n-k+1)$.

Proof. We use induction on the size n of a tree.
For $n = 1$ there is obviouly exactly one increasing labelling and $h_v = 1$. Thus, $|\mathbb{L}_I^k(T)| = \frac{k!}{\prod_{v\in T} k^{\underline{k}}} = \frac{k!}{k!} = 1$.
Let the lemma hold for $n-1$ and consider a tree T of size $|T| = n$. We split the tree into its root v, which has an arbitrary out-degree j and its j subtrees. Due to T being an increasing tree, the root vertex v must have the labellings $\{1, 2, \ldots, k\}$.
The subtrees $T_1(n_1), T_2(n_2), \ldots, T_j(n_j)$ fulfill $1 \leq n_1, n_2, \ldots, n_j \leq n-1$. Its easy to see, that each subtree, after an order preserving relabelling, is itself an increasing k-labelled tree. Since the remaining $kn-k$ labels are distributed to the j subtrees, we use the multinomial coeffcients (4.11):

$$\binom{kn-k}{kn_1, kn_2, \ldots, kn_j} = \frac{(kn-k)!}{(kn_1)!(kn_2)!\ldots(kn_j)!}.$$

Together with the induction hypothesis we obtain:

$$
\begin{aligned}
|\mathbb{L}_I^k(T)| &= \binom{kn-k}{kn_1, kn_2, \dots, kn_j} \cdot |\mathbb{L}_I^k(T_1)| \cdot |\mathbb{L}_I^k(T_2)| \cdots |\mathbb{L}_I^k(T_j)| \\
&= \frac{(kn-k)!}{\prod_{l=1}^j (kn_l)!} \prod_{l=1}^j \frac{(kn_l)!}{\prod_{v\in T_l} (kh_v)^{\underline{k}}} \\
&= \frac{(kn-k)!}{\prod_{l=1}^j (kn_l)!} \frac{\prod_{l=1}^j (kn_l)!}{\prod_{l=1}^j \prod_{v\in T_l} (kh_v)^{\underline{k}}} \\
&= \frac{(kn-k)!}{\prod_{l=1}^j \prod_{v\in T_l} (kh_v)^{\underline{k}}}.
\end{aligned}
$$

After reattaching our root vertex, we have to consider its labels. Since we have an increasing tree, the root node must be labelled with the k smallest labels, which adds the weight $(kn)^{\underline{k}}$. This concludes to:

$$
\begin{aligned}
|\mathbb{L}_I^k(T)| &= \frac{(kn-k)!}{\prod_{l=1}^j \prod_{v\in T_l} (kh_v)^{\underline{k}}} = (kn-k)! \cdot \frac{(kn)^{\underline{k}}}{\prod_{v\in T} (kh_v)^{\underline{k}}} \\
&= \frac{(kn)!}{\prod_{v\in T} (kh_v)^{\underline{k}}}.
\end{aligned}
$$

□

With (6.10) we can obtain a generalization of (6.8).

Theorem 6.15. *Let $\mathfrak{I}_k$ be the family of k-labelled increasing trees. Then it holds*

$$
\sum_{n\geq 1} \left(\sum_{T\in\mathfrak{O}(n)} \prod_{v\in T} \frac{\phi_{deg(v)}}{(kh_v)^{\underline{k}}} \right) z^{kn} = \sum_{n\geq 1} \frac{T_n}{(kn)!} z^{kn}. \tag{6.11}
$$

Proof. With (6.10) and $T_n := \sum_{T\in\mathfrak{O}(n)} w(T) \cdot |\mathbb{L}_I^k(T)|$, we get that the total weight T_n of increasingly k-labelled trees of size n is given as follows:

$$
T_n = \sum_{T\in\mathfrak{O}(n)} \frac{w_{deg}(T)(kn)!}{\prod_{v\in T} (kh_v)^{\underline{k}}}.
$$

Dividing by $(kn)!$ leads to

$$\sum_{T\in\mathfrak{O}(n)} \frac{w_{deg}(T)}{\prod_{v\in T}(kh_v)^{\underline{k}}} = \frac{T_n}{(kn)!}$$

$$\iff \sum_{T\in\mathfrak{O}(n)} \prod_{v\in T} \frac{\phi_{deg(v)}}{(kh_v)^{\underline{k}}} = \frac{T_n}{(kn)!},$$

concluding to

$$\sum_{n\geq 1} \left(\sum_{T\in\mathfrak{O}(n)} \prod_{v\in T} \frac{\phi_{deg(v)}}{(kh_v)^{\underline{k}}} \right) z^{kn} = \sum_{n\geq 1} \frac{T_n}{(kn)!} z^{kn}.$$

□

To show an application of (6.11), we consider the family of unordered trilabelled increasing trees. Let $\phi(t) = e^t$ again be the degree-weight generating function. For the next hook-length formula we need the Blasius differential equation.

Definition 6.16. *The Blasius differential equation is the following third-order ordinary differential equation:*

$$y'''(z) + y''(z)y(z) = 0, \quad y(0) = 0, \quad y'(0) = 0, \quad \lim_{z\to\infty} y'(z) = 1, \tag{6.12}$$

and arises in the theory of fluid boundary layers, see [Bla08].

The solution of this differential equation satisfies

$$y(z) = \sum_{n\geq 0} (-1)^n \frac{p_n \zeta^{n+1}}{(3n+2)!} z^{3n+2}, \quad \zeta = y''(0) = 0.4695\ldots, \tag{6.13}$$

with $p_0 = 1$ and $(p_n)_{n\geq 1}$ a certain positive integer sequence. Now we can prove:

Corollary 6.17. *It holds*

$$\sum_{n\geq 1} \left(\sum_{T\in\mathfrak{O}(n)} \prod_{v\in T} \frac{1}{deg(v)!3h_v(3h_v-1)(3h_v-2)} \right) z^{3n} = \sum_{n\geq 1} \frac{p_{n-1}}{(3n)!} z^{3n},$$

where $(p_n)_{n\geq 0}$ denote the coefficients of the Blasius function $y(z)$.

Proof. Since $\phi(t) = e^t \Rightarrow \phi_j = \frac{1}{j!}$ and due to (6.11) by setting $k = 3$ we obtain

$$\sum_{n\geq 1}\left(\sum_{T\in\mathfrak{O}(n)}\prod_{v\in T}\frac{1}{deg(v)!3h_v(3h_v-1)(3h_v-2)}\right)z^{3n} = \sum_{n\geq 1}\frac{T_n}{(3n)!}z^{3n}.$$

Therefore, we just have to show $T_n = p_{n-1}$.
By (6.9), the exponential generating function $T(z)$ satisfies the third-order non-linear autonomous differential equation

$$T'''(z) = e^{T(z)}, \quad T^{(n)}(0) = 0, \quad n = 0, 1, 2.$$

Now we differentiate this equation und set $F(z) = T'(z)$:

$$T''''(z) = T'(z)e^{T(z)}$$

$$\Rightarrow F'''(z) = F''(z)F(z)$$

$$\Rightarrow F'''(z) - F''(z)F(z) = 0, \quad F(0) = F'(0) = 0, \quad F'' = 1.$$

Since $F(z) = \sum_{n\geq 1} T_n \frac{z^{3n-1}}{(3n-1)!}$, we obtain $F(-z) = \sum_{n\geq 1}(-1)^{n-1}T_n\frac{z^{3n-1}}{(3n-1)!}$. Comparing this to (6.13) and considering $F''(0) = 1$ one gets

$$F(z) = \frac{1}{\zeta^{\frac{1}{3}}}y\left(-\frac{z}{\zeta^{\frac{1}{3}}}\right) = \frac{1}{\zeta^{\frac{1}{3}}}\sum_{n\geq 0}\frac{p_n\zeta^{n+1}}{(3n+2)!}\frac{z^{3n+2}}{\zeta^n\zeta^{\frac{2}{3}}}$$

$$= \sum_{n\geq 0}p_n\frac{z^{3n+2}}{(3n+2)!} = \sum_{n\geq 1}T_n\frac{z^{3n-1}}{(3n-1)!}$$

which yields our wanted result: $T_n = p_{n-1}$. □

7 Summary

In this thesis we derived the following hook-length formulas:

- Binary trees:
  - $$\sum_{T\in\mathfrak{B}(n)}\prod_{v\in T}\frac{1}{h_v 2^{h_v-1}}=\frac{1}{n!}.$$
  - $$\sum_{T\in\mathfrak{B}(n)}\prod_{v\in T}\frac{1}{h_v}=1.$$
  - $$\sum_{T\in\mathfrak{B}(n)}\prod_{v\in T}1=\frac{1}{n+1}\binom{2n}{n}.$$
  - $$\frac{n!}{2^n}\sum_{T\in\mathfrak{B}(n)}\prod_{v\in T}\left(1+\frac{1}{h_v}\right)=(n+1)^{n-1}.$$
  - $$\sum_{T\in\mathfrak{B}(n)}\prod_{v\in T}\frac{(m+h_v)^{h_v-1}}{h_v(2m+h_v-1)^{h_v-2}}=m(n+m)^{n-1}\frac{2^n}{n!}.$$
  - $$\sum_{T\in\mathfrak{B}(n)}\prod_{v\in T}\frac{\prod_{i=1}^{h_v-1}(m+i)}{2h_v\prod_{i=1}^{h_v-2}(2m+i)}=\binom{m+n-1}{m-1}.$$
  - $$\sum_{T\in\mathfrak{B}(n)}\prod_{v\in T}\frac{\prod_{i=1}^{h_v-1}(m(a+1)+2ah_v-i(a-1))}{2h_v\prod_{i=1}^{h_v-2}(2m(a+1)+2(h_v-1)-i(a-1))}$$
    $$=\frac{m(a+1)}{n!}\prod_{i=1}^{n-1}(m(a+1)+2an-i(a-1)).$$

- Fibonacci trees (f_n denotes the n-th Fibonacci number):
  -
  $$\sum_{T\in\mathfrak{B}_F(n)}\prod_{v\in T}1=f_n.$$
  -
  $$\sum_{T\in\mathfrak{B}_F(n)}\prod_{v\in T}\frac{(m+h_v-1)(m+h_v-2)}{h_v(mh_v+h_v-2)}=\binom{m+n-1}{m-1}.$$
- Complete binary trees:
  -
  $$\sum_{T\in\mathfrak{B}(n)}\prod_{v\in T}\frac{1}{h_v2^{h_v-2}}=\frac{1}{n!}.$$
  -
  $$\sum_{T\in\mathfrak{B}(n)}\prod_{v\in T}\rho(h_v)=1,$$

  with

  $$\rho(n)=\begin{cases}1, & \text{if } n=1,\\ \frac{1}{k}, & \text{if } n=2k+1,\\ \frac{1}{k}, & \text{if } n=2k.\end{cases}$$
- k-ary trees
  -
  $$\sum_{T\in\mathfrak{T}_k(n)}\prod_{v\in T}\frac{1}{h_vk^{h_v-1}}=\frac{1}{n!}.$$
  -
  $$\sum_{T\in\mathfrak{T}_k(n)}\prod_{v\in T}\frac{\prod_{i=1}^{h_v-1}\left(ma+k(a-1)h_v-i(a-k)\right)}{kh_v\prod_{i=1}^{h_v-2}\left(kma+k(a-1)(h_v-1)-i(a-k)\right)}$$
  $$=\frac{ma}{n!}\prod_{i=1}^{n-1}(ma+k(a-1)n-i(a-k)).$$

–

$$\sum_{T\in\mathfrak{T}_k(n)}\prod_{v\in T}\left(k-1+\frac{1}{h_v}\right)=\frac{k^n}{n!}(1+(k-1)n)^{n-1}.$$

–

$$n!\sum_{T\in\mathfrak{T}_k(n)}\prod_{v\in T}\frac{(m+(k-1)h_v)^{h_v-1}}{kh_v(mk+(k-1)(h_v-1))^{h_v-2}}=m(k(m+(k-1)n))^{n-1}.$$

- Ordered trees and forests:

–

$$\sum_{T\in\mathfrak{O}(n)}\prod_{v\in T}\left(1-\frac{1}{h_v}\right)^{h_v-1}=\frac{(n-1)^{n-1}}{n!}.$$

–

$$\sum_{T\in\mathfrak{O}_F(n)}\prod_{v\in T}\frac{(-1)^{h_v+1}}{h_v}=\frac{1}{n!}.$$

–

$$\sum_{F\in\mathfrak{O}_F(n)}\prod_{v\in F}\frac{\prod_{i=1}^{h_v-1}((2h_v-m)a-(a+1)i)}{\prod_{i=1}^{h_v-2}((2h_v-2+m)a-(a+1)i)}$$
$$=\frac{ma}{n!}\prod_{i=1}^{n-1}((2n+m)a-(a+1)i).$$

- Labelled trees and forests:
  -
$$\sum_{T\in\mathfrak{U}_L(n)}\prod_{v\in T}\frac{\frac{m}{h_v}}{\binom{m+h_v-2}{m-1}}=\frac{m}{n}.$$
  -
$$\sum_{F\in\mathfrak{U}_{LF}(n)}\prod_{v\in F}\frac{\prod_{k=1}^{h_v-1}(ah_v-(a-1)k)}{h_v\prod_{k=1}^{h_v-2}(a(h_v-1+m)-(a-1)k)}$$
$$=ma\prod_{k=1}^{n-1}(a(n+m)-(a-1)k).$$
  -
$$\sum_{F\in\mathfrak{U}_{LF}(n)}\prod_{v\in F}1=(n+1)^{n-1}.$$
- Motzkin-trees:
  -
$$2n!\sum_{T\in\mathfrak{M}(n)}\prod_{v\in T\setminus EN(T)}\frac{1}{h_v2^{h_v-2}}=1.$$
  -
$$n!\sum_{T\in\mathfrak{M}(n)}\frac{1}{2^{|EN(T)|-1}}\prod_{v\in T\setminus EN(T)}\left(1+\frac{1}{h_v}\right)=(n+1)^{n-1}.$$
- Cyclic trees:
  -
$$\sum_{T\in\mathfrak{C}(n)}\prod_{v\in T\setminus EN(T)}\left(1-\frac{1}{h_v}\right)=(n-1)^{n-1}.$$

- Labelled cyclic forests:
  - $$\sum_{F\in\mathfrak{C}_F(n)}\prod_{v\in F\setminus EN(F)}\left(1+\frac{1}{h_v}\right)=n^{n-1}.$$
- Increasing trees
  - Increasingly labelled k-ary trees:

$$\sum_{T\in\mathfrak{I}_k(n)}\prod_{v\in T}\frac{(m+(k-1)h_v)^{h_v-1}}{k(mk+(k-1)(h_v-1))^{h_v-2}}=m(k(m+(k-1)n))^{n-1}.$$

  - Bilabelled increasing trees:

$$\sum_{T\in\mathfrak{O}(n)}\prod_{v\in T}\frac{\phi_{deg(v)}}{2h_v(2h_v-1)}=\frac{T_n}{(2n)!},$$

    where T_n denotes the number of bilabelled weighted increasing trees of size n.
  - Unordered trilabelled increasing trees:

$$\sum_{T\in\mathfrak{O}(n)}\prod_{v\in T}\frac{1}{deg(v)!3h_v(3h_v-1)(3h_v-2)}=\frac{p_{n-1}}{(3n)!},$$

    where $(p_n)_{n\geq 0}$ denote the coefficients of the Blasius function (6.12).

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