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Small noise spectral analysis for a bistable system in large dimensions

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Introduction

The purpose of this thesis is to derive the sharp asymptotic of the convergence rate to equilibrium for a bistable dynamical system in the regime of large dimensions and small noise. We will investigate the following system of stochastic ordinary differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}X_i = X_i - X_i^3 + \frac{\mu}{4sin(\frac{\pi}{N})^2}(X_{i+1} - 2X_i + X_{i-1}) + \sqrt{\frac{2N}{\beta}}\xi_i, \tag{1}$$

where $\xi = (\xi_i)_{i=1}^N$ represents $\mathbb{R}^N$-valued white noise and we assume periodic boundary conditions on the random vector $X = (X_i)_{i=1}^N$, i.e. $X_0 = X_N$. The white noise $\xi(t)$ can be seen as the formal derivative of a $\mathbb{R}^N$-valued Wiener process $t \mapsto W(t)$. Because the paths of a Wiener process are almost surely not differentiable, the equation (1) is only formal. However, the standard approach to make it rigorous is to rewrite it as an integral equation in the following form

$$X_i(t) = x_i + \int_0^t X_i(s) - X_i^3(s) + \frac{\mu}{4sin(\frac{\pi}{N})^2}\big(X_{i+1}(s) - 2X_i(s) + X_{i-1}(s)\big)\ \mathrm{d}s + \sqrt{\frac{2N}{\beta}}W_t,$$

for almost every ω in the underlying probability space Ω, with initial condition $X_i(0) = x_i$. Observe that $t \mapsto X(t)$ is a stochastic process, i.e. $X(t)$ is a $\mathbb{R}^N$-valued random variable, and that we have suppressed the ω dependence of $X_i(t)$ and W_t in the equation for a better readability. Further we denote by $X^z(t)$ the solution of equation (1) with initial condition $X^z(0) = z \in \mathbb{R}^N$.

System (1) can also be seen as a discretisation of the stochastic Allen-Cahn equation in one space dimension on the interval $(0, \frac{2\pi}{\sqrt{\mu}})$, which can formally be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t,x) = \Phi(t,x) - \Phi^3(t,x) + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\Phi(t,x) + \sqrt{\frac{2}{\beta}}\xi(t,x), \tag{2}$$

where $\xi(t,x)$ represents white noise in time and space. This by [1] and others well studied equation, is also known as the stochastic Chaffee–Infante equation. For example, this model arises in the context of the ϕ_1^4 model in stochastic quantization. We can relate the stochastic ordinary differential equation (1) with the stochastic partial differential equation (2) by the approximation $X_i(t) \approx \Phi(t, \frac{2\pi}{\sqrt{\mu}}\frac{i}{N})$.

We will discuss the equation (1) in the limit of vanishing noise, i.e. we are interested in the limit $\beta \to \infty$. With a suitable potential $V : \mathbb{R}^N \to \mathbb{R}$ we can write this equation more compactly as $\frac{\mathrm{d}}{\mathrm{d}t}X = -\nabla V(X) + \sqrt{\frac{2N}{\beta}}\xi$. This stochastic differential equation describes the trajectories of N coupled particles in the velocity field $-\nabla V$, which are additionally exposed to a stochastic perturbation in the form of white noise. Looking at the energy landscape of V, we see that there are exactly three critical points, the points $I_- := (-1, .., -1)$ and $I_+ := (1, .., 1)$ which are minima, and the third critical point $(0,..,0)$, which is a saddle point. Hence I_-, I_+ are attracting orbits for the deterministic equation $\frac{\mathrm{d}}{\mathrm{d}t}X = -\nabla V(X)$, whereas $(0,..,0)$ is an unstable equilibrium point. Therefore we have a good heuristic understanding of the long time behaviour of solutions of the deterministic equation. Depending on where the solution starts, it will either converge to $(-1,..,-1)$ or $(1,..,1)$, as long as it does not start in the unstable point $(0,..,0)$. As soon as we add the stochastic perturbation, we obtain another interesting long time behaviour. At first the system will still approach one of the stable orbits $(-1,..,-1)$ or $(1,..,1)$, but for even larger times, we will see that so called metastable transitions occur, see [2], and the system will converge as a probability distribution to the equilibrium state $\mathrm{d}\mu_\beta = \frac{1}{Z_\beta}e^{-\beta V}\mathrm{d}x$, independently of the starting point. In the limit of vanishing noise, these probability measures μ_β again converge to the distribution $\frac{1}{2}(\delta_{I_+} + \delta_{I_-})$. This tells us, that even for arbitrary small noise, it is always possible for the particles to tunnel through the potential barrier from one zone of attraction into the other. We see that the points I_- and I_+ are stable for moderate time scales, but unstable for larger times. To quantify this new time scale where stochastic tunnel effects appear, one defines the stopping time $\tau_{\beta,N}$ as the first time the particle enters the other zone of attraction. The Eyring-Kramers formula for the finite dimensional case, for example see [3, 4], tells us in the limit of vanishing noise

$$E[\tau_{\beta,N}] = c \cdot e^{\frac{\beta}{4}} \cdot \big(1 + o_\beta(1)\big)$$

with a suitable positive constant c which can be computed explicitly.

Consider the diffusion operator L defined by the closure of $-\frac{N}{\beta}\Delta(\phi) + \nabla V \cdot \nabla\phi$ in the Hilbert space $L^2(\mu_\beta)$, which is associated to our stochastic differential equation in the sense that the evolution describing Markovian semi group

$$Q_t(f)(z) := E[f \circ X^z(t)]$$

is generated by $-L$. Note that L is selfadjoint and the constant function 1 is the eigenfunction to the eigenvalue 0. Let us denote with $\lambda_{\beta,N}$ the first non zero eigenvalue of L. By a standard spectral theoretic argument, we see that the convergence rate

$$\|E[f \circ X^z(t)] - \int f \;\mathrm{d}\mu_\beta\|^2_{L^2(\mu_\beta)} \leq \|f\|^2_{L^2(\mu_\beta)} e^{-\lambda_{\beta,N} t}$$

is governed by the spectral gap $\lambda_{\beta,N}$. In this sense, one can identify the convergence rate to the equilibrium state μ_β of the dynamical system (1) with the first non zero eigenvalue. Observe that the tunnel time $\tau_{\beta,N}$ is related to $\lambda_{\beta,N}$ as well by the following equation

$$E[\tau_{\beta,N}] = \frac{2}{\lambda_{\beta,N}} \cdot \big(1 + o_\beta(1)\big), \tag{3}$$

see [5, 6], which tells us that it is asymptotically equivalent to find the expected exit time or the first non zero eigenvalue.

It is known that the Eyring-Kramers formula for the expected exit times holds as well for the (infinite dimensional) Allen-Cahn equation (2), for example see [7, 8, 9]. In the context of finite discretization, this means that the asymptotic result

$$E[\tau_{\beta,N}] = c \cdot e^{\frac{\beta}{4}} \cdot \big(1 + o_{\beta,N}(1)\big)$$

holds with an error term $o_{\beta,N}(1)$ converging to 0 in the limit $\beta \to \infty$ uniformly in N, compare notation 4.4. In the relation $E[\tau_{\beta,N}] = \frac{2}{\lambda_{\beta,N}} \cdot \big(1 + o_\beta(1)\big)$ we have no insight about the N dependence of the error term $o_\beta(1)$, therefore it is not straightforward to see, that the central relation

$$\lambda_{\beta,N} = \frac{2}{c} \cdot e^{-\frac{\beta}{4}} \cdot \big(1 + o_{\beta,N}(1)\big) \tag{4}$$

with an uniformly in N controlled error term $o_{\beta,N}(1)$ holds as well. Because of the Eyring-Kramers formula, we know that this error term gets small in β for every discretisation size N, while it is shown in [10] that one can find useful N independent estimates on $\lambda_{\beta,N}$. It is content of this thesis, to prove that the error term gets small for $\beta \to \infty$ uniformly in N.

Performing a so called ground-state transformation, one can see that the operator L is up to a multiplicative constant unitary equivalent to a Schrödinger operator $H = -\Delta + \Phi$ with a suitable potential Φ, see Corollary 2.7. In the framework of Schrödinger operators, we would talk about the semi classical limit $h := \frac{1}{\beta} \to 0$ instead of vanishing noise, where h corresponds to Planck's constant, see [4]. In this setting, one also encounters tunnel effects, which have been investigated in various papers, such as [11] or [12]. In analogy to equation (3), it is described in [12], that the tunnel times in quantum mechanics are associated to the low lying spectrum of the operator H. Again in [12], this low lying spectrum is analysed. To be precise, the asymptotic exponential convergence rate of the spectral gap gets identified, using the theory of large deviations. Observe that we want, similar to the Eyring-Kramers formula, the sharp asymptotic of the spectral gap, and not only the exponential convergence rate. Never the less, using the Schrödinger operator representation H of our diffusion operator L can be a very useful tool, especially while working with a fixed dimension N. The problem is, that the natural bounds on H have a bad N dependence, which is why we will introduce different useful transformations in Definition 2.5.

This thesis is structured in the following way, in Section 1 we define and analyse the energy landscape of our potential V. Especially we want to find the critical points, see Lemma 1.10, and investigate convexity properties of V, which is content of Lemma 1.9. This will be useful, because we have a good understanding of the spectral gap of an operator associated to a convex potential.

In section 2 we discuss the fundamental techniques, which we require for the main proof. In particular, this includes the IMS localisation formula, which allows us to connect local results with global ones, see Corollary 2.19. Therefore we can restrict our investigations on test functions which have a support contained in some subsets $\Omega \subset \mathbb{R}^N$. This investigations should yield us a local version of the spectral gap, which we use in a combination of Corollary 2.19 and Lemma 2.15, a version of the Max–min principle, giving us insight about the spectrum. Furthermore we discuss logarithmic Sobolev inequalities in this section. For instance, they give us the right understanding of the spectral gap associated to a convex potential, especially see 2.12.

Even though we are interested in the first non zero eigenvalue, we have to do most of the work, to control the second non zero eigenvalue, which is carried out in Section 3. We will do this, by covering our space with subsets $\Omega_1, .., \Omega_6$, where we can find the right estimates for each Ω_j. The precise definition of this sets is given in Definition 3.1. Roughly speaking, Ω_1 and Ω_2 are neighbourhoods of the attracting orbits I_-, I_+ and the proof of the local spectral gap is carried out in Section 3.1. The saddle point of V is contained in Ω_3, while the infliction points are contained in Ω_5, Ω_6. The investigations of these regions are carried out in Section 3.2, respectively in Section 3.4. Finally, the region Ω_4 contains all points, which are away from the constants $(x, .., x)$. Here the proof of the local spectral gap is carried out in Section 3.3, and relies heavily on the fact, that the potential V is convex, if we restrict it to points which are orthogonal to the constants.

In section 4 we will be able to prove the asymptotic of the spectral gap, using the results of section 3. It is shown in [10], that the uniform bound on the second eigenvalue implies the desired asymptotic of the spectral gap. We will also carry out this proof, but only for the case of Dirichlet boundary conditions around one of the attracting orbits, see Definition 4.1. This operator again corresponds to a stochastic differential equation, where particles get absorbed at the boundary. If we denote with $\lambda^D_{\beta,N}$ the first non zero eigenvalue corresponding to Dirichlet boundary conditions, we will prove

$$\lambda^D_{\beta,N} = \frac{1}{2}\lambda_{\beta,N} \cdot \big(1 + o_{\beta,N}(1)\big) = \frac{1}{c} \cdot e^{-\frac{\beta}{4}} \cdot \big(1 + o_{\beta,N}(1)\big),$$

in analogy to formula (4). Observe that for this eigenvalue we have the relation

$$E[\tau_{\beta,N}] = \frac{1}{\lambda^D_{\beta,N}} \cdot \big(1 + o_\beta(1)\big)$$

instead of $E[\tau_{\beta,N}] = \frac{2}{\lambda_{\beta,N}} \cdot \big(1 + o_\beta(1)\big)$. The factor 2 only appears, because we are looking at the non generic case of a double well with equal well depths. This motivates, why $\lambda^D_{\beta,B}$ is more naturally connected to $E[\tau_{\beta,N}]$ and why we want to investigate the operator with Dirichlet boundary conditions as well.

1 Preliminaries

In this section we will define the double-well potential V_N and analyse its basic properties. In general we will follow the approach of [10], where the basic analysis of V_N is carried out in chapter 2.1.

1.1 Notation

Definition 1.1. *Given a $\mu > 1$, we define for every $N \geq 2$ a bilinear form on $\mathbb{R}^N$ (with v_i being the i-th component modulo N of v) by the formula*

$$K_N(v,w) := \frac{\mu}{4sin^2(\frac{\pi}{N})} \sum_{i=1}^{N} (v_i - v_{i-1})(w_i - w_{i-1}).$$

Furthermore let $V_N : \mathbb{R}^N \to \mathbb{R}$ be the potential

$$V_N(x) := \frac{1}{4N} \sum_{j=1}^{N} x_j^4 - \frac{1}{2N} \sum_{k=1}^{N} x_k^2 + \frac{1}{2N} K_N(x,x) + \frac{1}{4}.$$

With a slight abuse of notation, we will also write K_N for the linear mapping given by the formula

$$\big(K_N \cdot x, y\big) = K_N(x,y), \text{ for all } y \in \mathbb{R}^N,$$

where $\big(x, y\big) := \sum_{k=1}^{N} x_k \cdot y_k$ denotes the standard scalar product on $\mathbb{R}^N$.

Because of the special structure of our potential V_N, it will be convenient to introduce the following notation.

Notation 1.2. *For $x \in \mathbb{R}^N$ we denote the standard norm on $\mathbb{R}^N$*

$$|x|^2 := \sum_{k=1}^{N} x_k^2$$

and the mean

$$\bar{x} := \frac{1}{N} \sum_{k=1}^{N} x_k.$$

Furthermore let $W_N \subset \mathbb{R}^N$ be the set of all $y \in \mathbb{R}^N$ s.t. $\bar{y} = 0$. We can then, for every $x \in \mathbb{R}^N$, perform the following orthogonal decomposition

$$x = (\bar{x}, .., \bar{x}) + y(x),$$

with $y(x) \in W_N$.

Definition 1.3. *For $N \in \mathbb{N}_{\geq 2}$ and $z \in \mathbb{R}$ we define the function $F_{z,N} : W_N \to \mathbb{R}$*

$$\begin{aligned} F_{z,N}(y) &:= V_N((z,..,z)+y) - \frac{1}{4}z^4 + \frac{1}{2}z^2 - \frac{1}{4} \\ &= \frac{1}{4N}\sum_{k=1}^{N} y_k^4 + z\frac{1}{N}\sum_{k=1}^{N} y_k^3 + \frac{3}{2}z^2\frac{1}{N}\sum_{k=1}^{N} y_k^2 - \frac{1}{2N}\sum_{k=1}^{N} y_k^2 + \frac{1}{2N}\langle K_N \cdot y, y\rangle . \end{aligned}$$

We basically look at the function V_N restricted to the subset $(z,..,z) + W_N$, where we subtract terms which are constant in y.

1.2 Analysis of $F_{z,N}$

First we show a convexity result for $F_{z,N}$.

Lemma 1.4. *For all $y \in W_N$ we have*

$$\langle K_N \cdot y, y\rangle \geq \mu |y|^2.$$

Proof. For this proof, we look at K_N as a bilinear form defined on $\mathbb{C}^N$. Then the elements

$$v_k := \left(e^{i2\pi kj}\right)_{j=1}^{N},$$

with $k = 0,..,N-1$ form an orthogonal basis of Eigenvectors, where v_k correspond to the eigenvalue $\lambda_k := \mu\frac{sin(k\frac{\pi}{N})^2}{sin(\frac{\pi}{N})^2}$. Because we know $\lambda_k \geq \mu$ for all $1 \leq k < N$ we obtain for all $y \in W_N = \{v_0\}^{\perp}$

$$\langle K_N \cdot y, y\rangle \geq \mu |y|^2.$$

□

Corollary 1.5. *There exists a positive constant c, s.t. for all $N \in \mathbb{N}_{\geq 2}$, $z \in \mathbb{R}$ and $y \in W_N$*

$$HessF_{z,N}|_y \geq \frac{c}{N}.$$

Proof. For all $v \in W_N$ we define $x := (z,..,z) + y$. Using Lemma 1.4 we obtain

$$\begin{aligned} \mathrm{Hess}F_{z,N}|_y(v,v) = \mathrm{Hess}V_N|_x(v,v) &= \frac{1}{N}\sum_{k=1}^{N} 3x_k^2 v_k^2 - \frac{1}{N}\sum_{k=1}^{N} v_k^2 + \frac{1}{N}\langle K_N \cdot y, y\rangle \\ &\geq -\frac{1}{N}|v|^2 + \frac{\mu}{N}|v|^2 =: \frac{c}{N}|v|^2. \end{aligned}$$

□

For the proof of Corollary 1.5, it was necessary to assume $\mu > 1$. In the last part of this subsection we show some useful estimates of $F_{z,N}$.

Lemma 1.6. *We can control $F_{z,N}$ by the following two estimates. First of all we have for all $z \in \mathbb{R}$ and $y \in W_N$*

$$F_{z,N}(y) \geq \frac{1}{2N} \langle (K_N - I) \cdot y, y \rangle + \frac{z^2}{2N} \sum_{k=1}^{N} y_k^2.$$

Second, there exists a positive constant q s.t. for all $y \in W$, for all $z \in \mathbb{R}$ and all $x \in \mathbb{R}^N$ with $y(x) = y$ and $|\bar{x}| \leq 1$

$$F_{z,N}(y) \leq \frac{1}{2N} \sum_{k=1}^{N} x_k^4 + \frac{q \cdot (z^2+1)}{2N} \sum_{k=1}^{N} x_k^2 + \frac{1}{2N} \langle K_N \cdot x, x \rangle .$$

Proof. We compute for the first estimate

$$\begin{aligned} F_{z,N}(y) &= \frac{1}{2N} \langle (K_N - I) \cdot y, y \rangle + \frac{1}{4N} \sum_{k=1}^{N} \left(y_k^4 + 4zy_k^3 + 6z^2 y_k^2 \right) \\ &= \frac{1}{2N} \langle (K_N - I) \cdot y, y \rangle + \frac{1}{4N} \sum_{k=1}^{N} y_k^2 (y_k + 2z)^2 + \frac{1}{2N} z^2 \sum_{k=1}^{N} y_k^2 \\ &\geq \frac{1}{2N} \langle (K_N - I) \cdot y, y \rangle + \frac{1}{2N} z^2 \sum_{k=1}^{N} y_k^2. \end{aligned}$$

For the second estimate we define $b := q \cdot \left(z^2 + 1\right)$. Then this problem is equivalent to

$$\sum_{k=1}^{N} \frac{1}{4} y_k^4 + z y_k^3 + (\frac{3}{2} z^2 - 1) y_k^2 \leq \sum_{k=1}^{N} \frac{1}{2} y_k^4 + 2\bar{x} y_k^3 + 3(\bar{x})^2 y_k^2 + \frac{1}{2} (\bar{x})^4 + \frac{b}{2} y_k^2 + \frac{b}{2} (\bar{x})^2.$$

This especially means that we are done, if we can show that the function

$$f(y) := \frac{1}{4} \cdot y^2 + 2(\bar{x} - z) \cdot y + \left(3(\bar{x})^2 - \frac{3}{2} z^2 + 1 + b\right)$$

is positive. We see that the minimum point of f is at $y = 2z - \bar{x}$, with the value

$$f(2z - \bar{x}) = \frac{1}{4}(-2\bar{x} + 2z)^2 + (\bar{x} - z)(-2\bar{x} + 2z) + 1 - \frac{3}{2} z^2 + 3(\bar{x})^2 + q \cdot \left(z^2 + 1\right).$$

Since $\bar{x}$ is bounded we can take the constant q big enough, s.t. the function is always positive.

□

1.3 Analysis of V_N

First of all, we compute some derivatives of our potential V_N

$$\nabla V_N(x) = \frac{1}{N}\big(K_N \cdot x - x + (x_k^3)_k\big),$$
$$\mathrm{Hess}V_N(x) = \frac{1}{N}\big(K_N - I + (3x_k^2 \cdot \delta_{k,l})_{k,l}\big),$$
$$\Delta V_N(x) = \frac{\mu}{2sin^2(\frac{\pi}{N})} - 1 + \frac{3}{N}\sum_i x_i^2.$$

It will be our next goal, to show that V_N is uniformly convex on a "big enough" neighbourhood around the points $(-1, .., -1)$ and $(1, .., 1)$. As we will see later, these two points are the local minima of V_N.

For the next two auxiliary results, let us denote $\delta(v) := \max_{l,k} |v_l - v_k|$ for all $v \in \mathbb{R}^N$.

Lemma 1.7. *There exist constants $b > 0$ and $C > 0$, s.t. for all $v \in \mathbb{R}^N$ with $|v| = 1$ and $\delta(v) \geq \frac{C}{\sqrt{N}}$*

$$\langle (K_N - I) \cdot v, v\rangle \geq 1.$$

Proof. With the definition $C := \frac{32\pi^2}{\mu}$, we obtain using Lemma 5.3

$$\langle (K_N - I) \cdot v, v\rangle = \langle K_N \cdot v, v\rangle - 1 \geq N\frac{\mu}{16\pi^2}\delta(v)^2 - 1$$
$$\geq N\frac{\mu \cdot \frac{C}{N}}{16\pi^2} - 1 = 1.$$

□

Lemma 1.8. *There exists a constant D s.t. for all v with $|v| = 1$ and $\delta(v) < \frac{C}{\sqrt{N}}$*

$$|v_k| \leq \frac{D}{\sqrt{N}},$$

for all $k \in \{1..N\}$.

Proof. First of all, we claim that there exists a j s.t. $|v_j| \leq \frac{1}{\sqrt{N}}$. To prove this, let us assume $|v_j| > \frac{1}{\sqrt{N}}$ for all $j \in \{1, .., N\}$. From this we could conclude

$$|v|^2 = \sum_{j=1}^N |v_j|^2 > \sum_{j=1}^N \frac{1}{N} = 1,$$

a contradiction. Using the definition of $\delta(v)$ we obtain for all k

$$|v_k| \leq \delta(v) + |v_j| \leq \frac{C}{\sqrt{N}} + \frac{1}{\sqrt{N}}.$$

□

We define our neighbourhood $\Omega_{z_0,\gamma}$ as the set of all $x \in \mathbb{R}^N$ s.t. $\bar{x} > z_0$ and $|y(x)| < \gamma\sqrt{N} \cdot |\bar{x}|$ with a suitable choice of γ and z_0. We are now able to prove the following result.

Lemma 1.9. *For all $z_0 > \sqrt{\frac{1}{3}}$, there exist positive constants γ and c, s.t. for all $x \in \Omega_{z_0,\gamma}$*

$$\mathit{Hess}V_N|_x \geq \frac{c}{N}.$$

Proof. Let $v \in \mathbb{R}^N$ be an arbitrary vector with $|v| = 1$. First of all we consider the case $\delta(v) \geq C$, where we chose C as in Lemma 1.7. From this result we obtain

$$\begin{aligned}\mathrm{Hess}V_N|_x(v,v) &= \frac{3}{N}\sum_{k=1}^{N} 3x_k^2 v_k^2 + \frac{1}{N}\langle (K_N - I)\cdot v, v\rangle \geq \frac{1}{N}\langle (K_N - I)\cdot v, v\rangle \\ &\geq \frac{1}{N}\langle (K_N - I)\cdot v, v\rangle \geq \frac{1}{N}.\end{aligned}$$

In the other case that $\delta(v) \leq C$, we know from the Lemma 1.8, that there exists a D s.t. for all $k \in \{1, .., N\}$

$$|v_k| \leq \frac{D}{\sqrt{N}}.$$

With the decomposition $x = y(x) + (\bar{x}, .., \bar{x})$ we obtain, using $|y(x)| < \gamma\sqrt{N}\bar{x}$ and $|\bar{x}| > z_0$

$$\begin{aligned}\mathrm{Hess}V_N|_x(v,v) &\geq \frac{3}{N}\sum_{k=1}^{N} x_k^2 v_k^2 - \frac{1}{N}\sum_{k=1}^{N} v_k^2 = \frac{3}{N}\sum_{k=1}^{N} y(x)_k^2 v_k^2 + \frac{3}{N}\sum_{k=1}^{N} 2y(x)\bar{x}v_k^2 + \frac{3}{N}\sum_{k=1}^{N}\bar{x}^2 v_k^2 - \frac{1}{N} \\ &\geq \frac{3}{N}\left(\bar{x}^2\sum_{k=1}^{N} v_k^2 - 2|\bar{x}|\sum_{k=1}^{N}|y_k(x)|v_k^2\right) - \frac{1}{N} \geq \frac{3}{N}\left(\bar{x}^2 - 2\bar{x}D^2\frac{1}{N}\sum_{k=1}^{N}|y_k(x)|\right) - \frac{1}{N} \\ &\geq \frac{3}{N}\left(\bar{x}^2 - 2|\bar{x}|D^2\sqrt{\frac{1}{N}\sum_{k=1}^{N}|y_k(x)|^2}\right) - \frac{1}{N} \geq \frac{3}{N}\left(\bar{x}^2 - 2\bar{x}^2\gamma D^2\right) - \frac{1}{N} \\ &\geq \frac{3z_0^2(1 - 2\gamma D^2) - 1}{N}.\end{aligned}$$

Note that because of $z_0^2 > \frac{1}{3}$ the numerator in the last expression is positive for γ small enough. □

Lemma 1.10. *The potential V_N has exactly three critical points at $(-1,..,-1),(0,..,0)$ and $(1,..,1)$, where $(-1,..,-1)$ and $(1,..,1)$ are minima and $(0,..,0)$ is a saddle point where Hess$(V)((0,..,0))$ has exactly one negative eigenvalue.*

Proof. Computing ∇V_N shows that the three points are indeed critical points.
According to Corollary 1.5, the function $F_{z,N}$ is convex, with a minimum at $y = 0$. This means $\nabla V(y + (z,..,z)) = 0$ is only possible for $y = 0$. It is easy to show that z has to be one of the values $\{-1, 0, 1\}$. Because $\text{Hess}V_N|_{(1,..,1)} = \frac{1}{N}\big(K_N + 2 \cdot I\big)$ is convex, $(1,..,1)$ has to be a minimum. A symmetry argument shows that this also holds true for the point $(-1,..,-1)$. We still have to prove that $(0,..,0)$ is a saddle point. Because exactly one Eigenvalue of $\text{Hess}V_N|_{(0,..,0)} = \frac{1}{N}\big(K_N - I\big)$ is negative, we obtain that $(0,..,0)$ is a saddle point.

□

Without the assumption $\mu > 1$, we would lose the property that the potential has only three critical values. At the end of this subsection, we show a property which will be useful later on.

Lemma 1.11. *There exists a constant C s.t. for all $x \in \mathbb{R}^N$*

$$W(x) := \frac{1}{4}|\nabla \beta V_N(x)|^2 - \frac{1}{2}\Delta \beta V_N(x) \geq C,$$

where C might be negative and depend on N and β. Furthermore we have

$$\lim_{x\to\infty} W(x) = \infty.$$

Proof. Let us recall $\nabla V_N(x) = \frac{1}{N}\big(K_N \cdot x - x + (x_k^3)_k\big)$ and define $s := \frac{\beta}{N}(x_1^3 - x_1, .., x_N^3 - x_N)$. We write

$$\begin{aligned} W(x) &= |s|^2 + \frac{1}{N}|K_N \cdot x|^2 + \frac{1}{N} 2\left\langle K_N \cdot x, s\right\rangle - \Delta\beta V_N(x) \geq |s|^2 - c_0|x| \cdot |s| - c_1|x|^2 - c_2 \\ &\geq \frac{1}{2}|s|^2 - (c_1 + \frac{c_0^2}{2})|x|^2 - c_2 = \sum_{k=1}^{N}(c_3 x_k^6 + c_4 x_k^4 + c_5 x_k^2 + c_6), \end{aligned}$$

with suitable (positive) constants $c_0, .., c_6$. Because c_3 is positive, the mapping $y \mapsto c_3 y^6 + c_4 y^4 + c_5 y^2 + c_6$ is bounded from below, which yields us the first result. Because the mapping $y \mapsto c_3 y^6 + c_4 y^4 + c_5 y^2 + c_6 - y^2$ is bounded from below as well, we obtain for a suitable constant C'

$$W(x) - |x|^2 \geq N \cdot C',$$

and therefore $W(x)$ tends to infinity for $x \to \infty$. □

2 Auxiliary results

In this section we gather some definitions and results, which will be useful later on.

2.1 Dirichlet Form

Notation 2.1. *For a measurable function $V : \mathbb{R}^N \to \mathbb{R}$ we denote by μ_V the measure*

$$\mu_V(A) := \int_A e^{-V} \mathrm{d}x$$

Furthermore we will write $C_0^\infty(\Omega)$ for the set of all C^∞ functions $\phi : \Omega \to \mathbb{R}$ which have compact support and we will call the bilinear form

$$\mathcal{E}_V(\phi, \psi) := \int \nabla\phi \cdot \nabla\psi \, \mathrm{d}\mu_V,$$

defined on $C_0^\infty(\Omega)$ functions, the Dirichlet form associated to the potential V.

In the following we define a diffusion operator, which will be our main object.

Definition 2.2. *Let $V : \mathbb{R}^N \to \mathbb{R}$ be a C^2 function. We define L_V as the Friedrichs extension (see [13]) in the space $L^2(\mu_V)$ of the symmetric and positive operator*

$$L'_V : \begin{cases} C_0^\infty(\mathbb{R}^N) \to L^2(\mu_V), \\ \phi \mapsto -\Delta\phi + \nabla V \cdot \nabla\phi. \end{cases}$$

Lemma 2.3. *For all functions ϕ, ψ out of the set $C_0^\infty(\mathbb{R}^N)$ we obtain*

$$\mathcal{E}_V(\phi, \psi) = \langle L'_V \phi, \psi \rangle_{L^2(\mu_V)}.$$

Remark 2.4. *That the operator L'_V is symmetric and positive follows from Lemma 2.3. As described in [13], this is sufficient for the existence of the Friedrichs extension. In the following we are interested in the low lying spectrum of the operator*

$$L_{\beta,N} := \frac{N}{\beta} L_{\beta V_N},$$

where V_N is given by Definition 1.1. It is clear that $L_{\beta,N}$ is a positive selfadjoint operator.

2.2 Unitary Transformations

Definition 2.5. *Given a C^2 function $U : \mathbb{R}^N \to \mathbb{R}$, which we will call the reference potential, we define a unitary transformation*

$$T_{V,U} := \begin{cases} L^2(\mu_V) \to L^2(\mu_U), \\ \phi \mapsto exp\big(-\frac{1}{2}(V-U)\big)\cdot\phi. \end{cases}$$

We then define the transformed operators

$$L_V^U := T_{V,U}\cdot L_V \cdot T_{U,V} : dom(L_V^U) \to L^2(\mu_U),$$

where $dom(L_V^U)$ is the set of all $\phi \in L^2(\mu_U)$ s.t. $T_{U,V}\cdot\phi \in dom(L_V)$.

Lemma 2.6. *We obtain for $\phi \in C_0^\infty(\mathbb{R}^N)$,*

$$L_V^U\phi = L_U\phi + \Big(\frac{1}{2}L_U(V_N - U) + \frac{1}{4}|\nabla V_N - \nabla U|^2\Big)\phi.$$

Moreover, the set $C_0^\infty(\mathbb{R}^N)$ is a core for the operator L_V^U, if and only if it is a core for the operator L_V.

Proof. Let us denote $W := \frac{1}{2}(V-U)$ and compute

$$\begin{aligned} L_V^U(\phi) &= e^{-W}\cdot L_V(e^W\phi) = e^{-W}\cdot\Big(-\Delta(e^W\phi) + \nabla V\cdot\nabla(e^W\phi)\Big) \\ &= e^{-W}\cdot\Big(-\Delta W\phi e^W - 2\nabla W\cdot\nabla\phi e^W - \Delta\phi e^W + \nabla V\cdot\nabla\phi e^W + \nabla V\cdot\nabla W\phi e^W\Big) \\ &= -\Delta\phi + \big(\nabla V - 2\nabla W\big)\cdot\nabla\phi + \big(-\Delta W + \nabla V\cdot\nabla W\big)\phi \\ &= -\Delta\phi + \nabla U\cdot\nabla\phi + \big(-\Delta W + \nabla V\cdot\nabla W\big)\phi \\ &= L_V(\phi) + L_V(W)\cdot\phi. \end{aligned}$$

□

Corollary 2.7. *The special cases $U := V$ and $U := 0$ yield for $\phi \in C_0^\infty(\mathbb{R}^N)$*

$$\begin{aligned} L_V^V\phi &= L_V\phi, \\ L_V^0\phi &= -\Delta\phi + \Big(\frac{1}{4}|\nabla V|^2 - \frac{1}{2}\Delta V\Big)\phi. \end{aligned}$$

Remark 2.8. *The operator $H := L_V^0$ has the structure of a Schrödinger operator with the potential $\frac{1}{4}|\nabla V|^2 - \frac{1}{2}\Delta V$. The operator H can be seen as the restriction of the Witten Laplacian on the level of functions, which can more generally be defined on a Riemannian manifold, for example see ([4], [14]).*

Lemma 2.9. *The operator $L'_{\beta V_N}$ from Definition 2.2 is essentially selfajoint i.e. $L_{\beta V_N}$ is the unique selfadjoint extension of $L'_{\beta V_N}$ and the set of all smooth and compactly supported functions $C_0^\infty(\mathbb{R}^N)$ is a core for $L_{\beta V_N}$. We also know that the spectrum of $L_{\beta V_N}$ is discrete.*

Proof. We look at the unitary equivalent operator L_V^0. An operator of the type $-\Delta + W$ with a C^0 potential is, according to [13] Theorem 9.15, essentially selfadjoint, if W is semi bounded from below. If we define $W := \left(\frac{1}{4}|\nabla \beta V_N|^2 - \frac{1}{2}\Delta\beta V_N\right)$ we obtain $L^0_{\beta V_N} = -\Delta + W$. Furthermore, Lemma 1.11 tells us that $W \geq C$ for some constant C, and therefore we know that the set $C_0^\infty(\mathbb{R}^N)$ is a core for L_V^0 and hence a core for L_V, see Lemma 2.6. Lemma 1.11 also tells us, that W tends to infinity for $x \to \infty$. Therefore we know that the spectrum of our Schrödinger operator $-\Delta + W$ is discrete, see [13].

□

2.3 Logarithmic Sobolev Inequality

We will use the following results on multiple occasions. I want to refer to [15] and [16] for a detailed analysis of this topic. We will use slightly different definitions, which will be more convenient, since we do not want to cover the topic as general as in [15] or [16].

Definition 2.10. *We say that a C^2 potential V with $\int e^{-V}\mathrm{d}x < \infty$ satisfies the logarithmic Sobolev inequality with constant $\rho > 0$, if for all $\phi \in dom\left(\sqrt{L_V}\right)$, see Definition 2.2, with $\frac{1}{\int e^{-V}\mathrm{d}x}\int \phi^2 e^{-V}\mathrm{d}x = 1$*

$$\int \phi^2 \log(\phi^2)\ e^{-V}\mathrm{d}x \leq \rho \|\sqrt{L_V}\phi\|^2_{L^2(\mu_V)}.$$

For $\phi \in C_0^\infty(\mathbb{R}^N)$ this inequality becomes

$$\int \phi^2 \log(\phi^2)\ e^{-V}\mathrm{d}x \leq \rho \int |\nabla\phi|^2\ e^{-V}\mathrm{d}x.$$

Lemma 2.11 (Poincaré inequality)**.** *If the C^2 potential V satisfies a logarithmic Sobolev inequality with constant $\rho >$, then it satisfies a Poincaré inequality with the same constant. Satisfying such a Poincaré inequality means that for all $\phi \in dom\left(\sqrt{L_V}\right)$ with $\int \phi\ \mathrm{d}\mu = 0$*

$$\int \phi^2\ e^{-V}\mathrm{d}x \leq \rho \|\sqrt{L_V}\phi\|^2_{L^2(\mu_V)}.$$

For $\phi \in C_0^\infty(\mathbb{R}^N)$ this inequality becomes

$$\int \phi^2\ e^{-V}\mathrm{d}x \leq \rho \int |\nabla\phi|^2\ e^{-V}\mathrm{d}x.$$

Proof. See [15] Proposition 3.1.8.

□

One of the most useful criteria to show a logarithmic Sobolev inequality is the following result.

Lemma 2.12 (Bakry and Émery)**.** *Let $V : \mathbb{R}^N \to \mathbb{R}$ be a C^2 potential with $\mathrm{Hess}V|_x \geq m \cdot I > 0$ for all $x \in \mathbb{R}^N$. Then V satisfies a logarithmic Sobolev inequality with constant $\rho := \frac{1}{m}$.*

Proof. See [15] Theorem 3.1.29.

□

Lemma 2.13 (NGS Bound)**.** *Let V be a C^2 potential satisfying a logarithmic Sobolev inequality with constant ρ and let $\Omega \subset \mathbb{R}^N$ be a measurable set. Then we obtain for every continuous function G with $\int e^{-\frac{\rho}{2}G}\ e^{-V}\mathrm{d}x < \infty$ and all $\phi \in C_0^\infty(\mathbb{R}^N)$ with a support contained in Ω*

$$\int |\nabla\phi|^2\ e^{-V}\mathrm{d}x + \int G\phi^2\ e^{-V}\mathrm{d}x \geq C_\Omega \int \phi^2\ e^{-V}\mathrm{d}x,$$

with $C_\Omega := \frac{-2}{\rho}\log\left(\frac{\int_\Omega e^{-\frac{\rho}{2}G}\ e^{-V}\mathrm{d}x}{\int_{\mathbb{R}^N}\ e^{-V}\mathrm{d}x}\right)$.

Proof. See [16] Theorem 7.

□

Putting together 2.12 and 2.13, we obtain the following corollary.

Corollary 2.14. *Let $V : \mathbb{R}^N \to \mathbb{R}$ be a C^2 potential with $\mathrm{Hess}V|_x \geq m \cdot I > 0$ for all $x \in \mathbb{R}^N$ and let $\Omega \subset \mathbb{R}^N$ be a measurable set. Then we obtain for every continuous function G with $\int e^{-\frac{m}{2}G}\ e^{-V}\mathrm{d}x < \infty$ and all $\phi \in C_0^\infty(\mathbb{R}^N)$ with a support contained in Ω*

$$\int |\nabla\phi|^2\ e^{-V}\mathrm{d}x + \int G\phi^2\ e^{-V}\mathrm{d}x \geq C_\Omega \int \phi^2\ e^{-V}\mathrm{d}x,$$

with $C_\Omega := -2m\log\left(\frac{\int_\Omega e^{-\frac{1}{2m}G}\ e^{-V}\mathrm{d}x}{\int_{\mathbb{R}^N}\ e^{-V}\mathrm{d}x}\right)$.

2.4 Estimate on the Spectrum

We want to show, that the spectrum of the operator $L_{\beta,N}$ defined in Remark 2.4 contains at most two values lower then some positive β and N independend constant C. To do so, we need some kind of Max–min principle, for an introduction to this principle, also see [13].

Lemma 2.15 (Max–min principle)**.** *Let A be a selfadjoint operator. Assume that there exist constants $C, c_1, .., c_n$ and Hilbert space elements $v_1, .., v_n$ such that for all x in the domain of A (or all x out of a core of A)*

$$\langle Ax, x\rangle \geq C\|x\|^2 - \sum_{j=1}^{n} c_j \langle v_j, x\rangle^2 .$$

Then

$$|\sigma(A) \cap (-\infty, C)| \leq n.$$

This especially means that the part of the spectrum smaller than C is discrete and can only contain up to n different values. If A has discrete spectrum we obtain

$$\lambda_{n+1} \geq C,$$

where λ_j denotes the j-th eigenvalue of A.

Proof. Assume that there are $n+1$ different values $\lambda_1, .., \lambda_{n+1} \in \sigma(A)$ with $\lambda_j < C$. Let us choose an $\epsilon > 0$ small enough, such that $(\lambda_i - \epsilon, \lambda_i + \epsilon) \cap (\lambda_j - \epsilon, \lambda_j + \epsilon) = \emptyset$ for $i \neq j$ and $\lambda_i + \epsilon < C$. Then, pick $w_i \neq 0$ with $w_i \in ran\big(E((\lambda_i - \epsilon, \lambda_i + \epsilon))\big)$, where E denotes the spectral measure of A. Because of

$$dim\big(span(w_1, .., w_{n+1})\big) = n + 1 > n \geq codim\big(span(v_1, .., v_n)^{\perp}\big),$$

there exists an element $x \neq 0$ with $x \in span(w_1, .., w_{n+1}) \cap span(v_1, .., v_n)^{\perp}$. Because of $x \in span(w_1, .., w_{n+1}) \subset ran\big(E((-\infty, C))\big)$ and $x \neq 0$ we obtain

$$\langle Ax, x\rangle < C\|x\|^2.$$

On the other hand, $x \in span(v_1, .., v_n)^{\perp}$ together with the assumptions of this result implies

$$\langle Ax, x\rangle \geq C\|x\|^2 - \sum_{j=1}^{n} c_j \langle v_j, x\rangle^2 = C\|x\|^2.$$

This is a contradiction, hence there can only be up to n different values in $\sigma(A) \cap (-\infty, C)$. □

Our goal is to show the existence of a constant $C > 0$, such that

$$|\sigma(L_{\beta,N}) \cap [0, C)| \leq 2,$$

for all β big enough and arbitrary $N \geq 2$. For this we will need the following localisation results.

The next results will allow us to connect local results with global results. This technique can for example be found in [17]. First of all, we will introduce some notation.

Notation 2.16. *Let $\xi_1, ..\xi_n$ be a family of smooth functions with $\sum_{k=1}^n \xi_k^2 = 1$. Then we denote for a smooth ϕ the decomposition $\phi_1 := \xi_1 \cdot \phi, .., \phi_n := \xi_n \cdot \phi$.*

Lemma 2.17 (IMS localisation formula)**.** *For all smooth $\phi : \mathbb{R}^N \to \mathbb{R}$*

$$\sum_{k=1}^n \phi_k^2 = \phi^2,$$

$$\sum_{k=1}^n |\nabla \phi_k|^2 = |\nabla \phi|^2 + \sum_{k=1}^n |\nabla \xi_k|^2 \phi^2.$$

Proof. The first claim is obvious. For the second one, observe that we obtain $\sum_{k=1}^n \xi_k \nabla \xi_k = 0$ by differentiation of the equation $\sum_{k=1}^n \xi_k^2 = 1$. Now we can compute

$$\begin{aligned}\sum_{k=1}^n |\nabla \phi_k|^2 &= \sum_{k=1}^n |\nabla \xi_k \phi + \xi_k \nabla \phi|^2 = \sum_{k=1}^n |\nabla \xi_k \phi|^2 + \sum_{k=1}^n |\xi_k \nabla \phi|^2 + 2 \sum_{k=1}^n (\nabla \xi_k \phi) \cdot (\xi_k \nabla \phi) \\ &= \phi^2 \sum_{k=1}^n |\nabla \xi_k|^2 + |\nabla \phi|^2 \sum_{k=1}^n \xi_k^2 + 2\phi \nabla \phi \cdot \sum_{k=1}^n \xi_k \nabla \xi_k \\ &= \phi^2 \sum_{k=1}^n |\nabla \xi_k|^2 + |\nabla \phi|^2,\end{aligned}$$

where we made use of $\sum_{k=1}^n \xi_k^2 = 1$ again.

□

Corollary 2.18. *Let L_V be the operator from Definition 2.2. We have $\|\phi\|_{L^2(\mu_V)}^2 = \sum_{k=1}^n \|\phi_k\|_{L^2(\mu_V)}^2$ and*

$$|\langle L_V \phi, \phi \rangle_{L^2(\mu_V)} - \sum_{k=1}^n \langle L_V \phi_k, \phi_k \rangle_{L^2(\mu_V)}| \leq \| \sum_{k=1}^n |\nabla \xi_k|^2 \|_\infty \cdot \|\phi\|_{L^2(\mu_V)}^2.$$

Based on the IMS localisation formula, we are able to verify the correctness of one of our most important tools, which will allow us to work locally.

Corollary 2.19. *Let $\mathbb{R}^N = \Omega_1 \cup .. \cup \Omega_n$ be a cover of the whole space, $\xi_1, ..\xi_n$ a family of smooth functions with $\sum_{k=1}^n \xi_k^2 = 1$ and $\operatorname{supp} \xi_k \subset \Omega_k$, and let us assume that we have a family of functions $\psi_{k,1}, .., \psi_{k,m(k)} \in L^2(\mu_V)$ for all $k \in \{1, .., n\}$, s.t. for all smooth $\phi \in C_0^\infty(\mathbb{R}^N)$ with $\operatorname{supp} \phi \subset \Omega_k$*

$$\langle L_V \phi, \phi \rangle_{L^2(\mu_V)} \geq C \|\phi\|_{L^2(\mu_V)}^2 - \sum_{j=1}^{m(k)} | \int \phi \psi_{k,j} \, \mathrm{d}\mu|^2.$$

Under these assumptions, the part of the spectrum $\sigma(L)$ smaller then $C' := C - \| \sum_{k=1}^{n} |\nabla \xi_k|^2 \|_\infty$ can only contain up to $m := \sum_{k=1}^{n} m(k)$ different eigenvalues.

Proof. We define for all $k \in \{1, .., n\}$ and $j \in \{1, .., m(k)\}$ the functions $\theta_{k,j} := \xi_k \ \psi_{k,j}$, and denote the cut-off of a function ϕ with $\phi_k := \phi \xi_k$. Observe that *supp*$(\phi_k) \subset \Omega_k$. Corollary 2.18 yields us

$$
\begin{aligned}
\langle L_V \phi, \phi \rangle_{L^2(\mu_V)} - \| \sum_{k=1}^{n} |\nabla \xi_k|^2 \|_\infty \cdot \|\phi\|^2_{L^2(\mu_V)} &\geq \sum_{k=1}^{n} \langle L_V \phi_k, \phi_k \rangle_{L^2(\mu_V)} \\
&\geq \sum_{k=1}^{n} \left(C \|\phi_k\|^2_{L^2(\mu_V)} - \sum_{j=1}^{m(k)} | \int \phi_k \ \psi_{k,j} \ \mathrm{d}\mu|^2 \right) \\
&= C \|\phi\|^2_{L^2(\mu_V)} - \sum_{k=1}^{n} \sum_{j=1}^{m(k)} | \int \phi \xi_k \ \psi_{k,j} \ \mathrm{d}\mu|^2.
\end{aligned}
$$

Therefore we obtain for all $\phi \in C_0^\infty(\mathbb{R}^N)$

$$
\langle L_V \phi, \phi \rangle_{L^2(\mu_V)} \geq C' \|\phi\|^2_{L^2(\mu_V)} - \sum_{k=1}^{n} \sum_{j=1}^{m(k)} | \int \phi \ \theta_{k,j} \ \mathrm{d}\mu|^2,
$$

and applying Lemma 2.15 immediately yields us the desired statement. □

3 Estimate on the third Eigenvalue

In the proof that the third eigenvalue is bigger then some β and N independend constant C, we follow the strategy of [10], which contains a combination of the IMS localisation formula Lemma 2.17, Corollary 2.19 and the version of the max-min principle in Lemma 2.15. In a first step, we will cover $\mathbb{R}^N$ with regions $\Omega_1, .., \Omega_m$. We will then prove the condition for applying the max-min principle in each region Ω_k separately. The IMS localisation formula will then yield us the desired statement for the whole $\mathbb{R}^N$.

The success of our strategy is based on finding a good partition $\Omega_1, .., \Omega_n$ of the whole $\mathbb{R}^N$, s.t. $\langle L_{\beta,N}\phi, \phi\rangle \geq C|\phi\|^2$ or $\langle L_{\beta,N}\phi, \phi\rangle \geq C|\phi\|^2 - \langle\psi_j, \phi\rangle^2$ for all ϕ with support in some subset Ω_j. Corollary 2.19 yields

$$|\sigma(L_{\beta,N}) \cap [0, C)| \leq 2,$$

for a constant $C > 0$ and all β, N big enough. Observe that $|\sigma(L_{\beta,N}) \cap [0, C)| \leq 2$ can also be expressed as $\lambda_3^{\beta,N} \geq C$, where $\lambda_3^{\beta,N}$ denotes the third eigenvalue of $L_{\beta,N}$.

Recall the Notation 1.2 of $\bar{x}$ and $y(x)$, which we will use in the following definition.

Definition 3.1. *Let ρ, r be positive constants. Let $R(t \cdot e + y) := -t \cdot e + y$ be the reflection orthogonal to e, with $e := \frac{1}{\sqrt{N}}(1, .., 1)$. We decompose $\mathbb{R}^N = \Omega_1 \cup .. \cup \Omega_6$ in the following way*

We define the region Ω_1 as the half open cylinder containing the minima $(1, .., 1)$

$$\Omega_{1,N} := \{x \in \mathbb{R}^N : \frac{1}{\sqrt{3}} + \rho < \bar{x} < \infty,\ |y(x)| < 2\sqrt{N}r\},$$

and symmetrical $\Omega_{2,N} := R_e \cdot \Omega_{1,N}$, which contains the minima $(-1, .., -1)$. Furthermore we define Ω_3 as the cylinder

$$\Omega_{3,N} := \{x \in \mathbb{R}^N : -\frac{1}{\sqrt{3}} + \rho < \bar{x} < \frac{1}{\sqrt{3}} - \rho,\ |y(x)| < 2\sqrt{N}r\},$$

which contains the saddle point of V_N. If we define the on both sides open cylinder $Z := \{x \in \mathbb{R}^N : |y(x)| < 2\sqrt{N}r\}$, we observe that the complement $\mathbb{R}^N \setminus Z$ is covered by

$$\Omega_{4,N} := \{x \in \mathbb{R}^N : |y(x)| > \sqrt{N}r\}.$$

With the Definition

$$\Omega_{5,N} := \{x \in \mathbb{R}^N : \frac{1}{\sqrt{3}} - 2\rho < \bar{x} < \frac{1}{\sqrt{3}} + 2\rho,\ |y(x)| < 2\sqrt{N}r\}$$

and $\Omega_{6,N} := R_e \cdot \Omega_{5,N}$ we can cover the whole cylinder Z with the sets $Z = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_5 \cup \Omega_6$.

In the subsections 3.1, 3.2, 3.3 and 3.4 we will prove the following two results, which give us locally the desired estimate on the Dirichlet form, which is crucial for applying the Max-min principle.

Notation 3.2. *For convenience, we will denote the scalar product*

$$\langle \phi, \psi \rangle := \langle \phi, \psi \rangle_{L^2(\mu_{\beta V_N})} := \int \phi \cdot \psi \; e^{-\beta V_N} \mathrm{d}x$$

and the norm $\|\phi\|^2 := \langle \phi, \phi \rangle$.

Lemma 3.3. *We can choose* $r, \rho > 0$ *small enough, s.t. there exists a constant* C *and functions* ψ_l, ψ_r *with*

1. $\int_{\Omega_{1,N}} |\nabla \phi_1|^2 \; e^{-\beta V_N} \; \mathrm{d}x \geq \frac{\beta}{N} C \|\phi_1\|^2 - |\int \phi_1 \; \psi_r \; e^{-\beta V_N} \mathrm{d}x|^2$,

2. $\int_{\Omega_{2,N}} |\nabla \phi_2|^2 \; e^{-\beta V_N} \; \mathrm{d}x \geq \frac{\beta}{N} C \|\phi_2\|^2 - |\int \phi_2 \; \psi_l \; e^{-\beta V_N} \mathrm{d}x|^2$,

3. $\int_{\Omega_{k,N}} |\nabla \phi_k|^2 \; e^{-\beta V_N} \; \mathrm{d}x \geq \frac{\beta}{N} C \|\phi_k\|^2$ *for* $k = 3, .., 6$,

for all β *big enough, arbitrary* $N \in \mathbb{N}_{\geq 2}$ *and all functions* ϕ_k *with* $supp(\phi_k) \subset \Omega_{k,N}$.

Proof. Let γ be as in Lemma 1.9. If we choose $r < \gamma\rho$, we know that $\mathrm{Hess} V_N \geq \frac{c}{N}$ on the set $\Omega_{1,N}$ respectively on the set $\Omega_{2,N}$ and therefore apply the results of Subsection 3.1 with $\Omega := \Omega_{1,N}$ respectively $\Omega := \Omega_{2,N}$. This covers the proof for the first two statements. For $k = 3$ one finds, under the assumption $(\frac{1}{\sqrt{3}} - \rho)^2 + 4r^2 < \frac{1}{3}$, the proof in Subsection 3.2 and for $k = 4$ in 3.3. If we take ρ small enough, one finds the proof for $k = 4, 5$ in Subsection 3.4.

□

Lemma 3.4. *There exist* C^∞ *functions* $\xi_{1,N}, .., \xi_{6,N}$ *with the properties* $supp(\xi_{k,N}) \subset \Omega_{k,N}$, $\sum_{k=1}^6 \xi_{k,N}^2 = 1$ *and*

$$\sum_{k=1}^6 \| |\nabla \xi_{k,N}|^2 \|_\infty \leq \frac{K}{N},$$

with a N *independent constant* K.

Proof. From the way we have chosen our regions Ω_k it is clear that we can find functions $\xi_{k,N}$ with the properties $supp(\xi_{k,N}) \subset \Omega_{k,N}$ and $\sum_{k=1}^6 \xi_{k,N}^2 = 1$, s.t. we can write for all $N \in \mathbb{N}$ the functions $\xi_{k,N}$ as $x \mapsto h_k(\bar{x}, \frac{|y|^2}{N})$, where h_k is a C^∞ function which has a compactly supported gradient ∇h_k. The construction is carried out in Corollary 5.7. We are going to estimate the norm of the gradient of $\xi_{k,N} : x \mapsto h_k(\bar{x}, \frac{|y|^2}{N})$. Because of the compactly supported gradient of h_k, we can assume for a suitable constant d that $|\bar{x}| \leq d$ and $\frac{|y|^2}{N} \leq d$. We obtain

$$|\nabla \xi_{k,N}(x)|^2 = \sum_{j=1}^{N} (\partial_1 h_k \cdot \frac{1}{N} + \partial_2 h_k \cdot \frac{2y_j}{N})^2 \leq \frac{1}{N^2} |\nabla h_k|^2 \cdot \sum_{j=1}^{N} (1 + 4y_j^2)$$
$$\leq \frac{1}{N^2} |\nabla h_k|^2 \cdot (N + 4|y|^2) \leq \frac{1}{N} \||\nabla h|^2\|_\infty \cdot (1 + 4d).$$

This immediately yields us the existence of a N independent constant K s.t.

$$\sum_{k=1}^{6} \||\nabla \xi_{k,N}|^2\|_\infty \leq \frac{K}{N}.$$

□

Corollary 3.5. *There exists a constant D s.t. β big enough and arbitrary $N \in \mathbb{N}_{\geq 2}$*

$$|\sigma(L_{\beta,N}) \cap [0, D)| \leq 2.$$

Proof. Because of Lemma 3.3, our decompositions $\Omega_{1,N}, .., \Omega_{6,N}$ with the functions $\xi_{1,N}, .., \xi_{4,N}$ meet the requirements of Corollary 2.19, with the constant $\frac{\beta}{N} C$, hence we obtain

$$|\sigma(L_{\beta V_N}) \cap (-\infty, C'_{\beta,N})| \leq 2,$$

with $C' = \frac{\beta}{N} C - \sum_{k=1}^{6} \|\nabla \xi_k\|_\infty^2$. Lemma 3.4 tells us for β big enough and $D := \frac{C}{2}$

$$C_{\beta,N} \geq \frac{\beta}{N} C - \frac{K}{N} \geq \frac{\beta}{N} D.$$

Using the Definition $L_{\beta,N} := \frac{N}{\beta} L_{\beta V_N}$ yields us the desired estimate.

□

3.1 Localisation around the Minima $(\Omega_{1,N}, \Omega_{2,N})$

For a convex potential F with Hess$F \geq c$ we know because of Corollary 2.12, that for all ϕ with $\int \phi \, e^{-F} \mathrm{d}x = 0$

$$\int |\nabla \phi|^2 \, e^{-F} \mathrm{d}x \geq c \int \phi^2 \, e^{-F} \mathrm{d}x.$$

Using some kind of Max-min argument, this basically yields us the desired estimate for the region Ω_1, and therefore also for Ω_2. There is still a small technical difficulty, because our potential is not convex on the whole $\mathbb{R}^N$. The details are carried out in the rest of this subsection.

Assumption 3.6. *Let Ω be a convex set, $V : \mathbb{R}^N \to \mathbb{R}$ a C^2 function and let us assume Hess$V|_x \geq C \cdot I$ for all $x \in \overline{\Omega}$.*

Lemma 3.7. *Let us additionally assume that Ω is bounded. Then for all C^∞ functions ϕ which have a support contained in Ω and satisfy $\int_\Omega \phi \, e^{-V} \mathrm{d}x = 0$, we have*

$$\langle L_V \phi, \phi \rangle \geq C \|\phi\|^2.$$

If we define $\psi := \sqrt{\frac{1}{\int_\Omega \, e^{-V} \mathrm{d}x}} \mathbb{1}_\Omega$, we obtain

$$\langle L_V \phi, \phi \rangle \geq C \|\phi\|^2 - |\int \phi \; \psi e^{-V} \mathrm{d}x|^2,$$

now for all smooth ϕ with supp$\phi \subset \Omega$.

Proof. First of all, we can find an extension $F : \mathbb{R}^N \to \mathbb{R}$ of V which still satisfies Hess$F \geq C \cdot I$, for more details see Lemma 5.4. According to Lemma 2.12 the convex function F satisfies a logarithmic Sobolev inequality with constant $\frac{1}{C}$. Consequently Lemma 2.11 tells us

$$\int |\nabla \phi|^2 \, e^{-F} \mathrm{d}x \geq C \|\phi\|^2,$$

for all C^∞ functions with $\int_{\mathbb{R}^N} \phi \, e^{-V} \mathrm{d}x = 0$. If we additionally assume *supp*$(\phi) \subset \Omega$ we are able to exchange F with V_N in the inequality and in the condition $\int_{\mathbb{R}^N} \phi \, e^{-V} \mathrm{d}x = 0$, the domain of the integration $\mathbb{R}^N$ with the set Ω. Using Lemma 2.3 yields us the first statement.

For the second statement, define $c^2 := \int_{\mathbb{R}^N} \, e^{-F} \mathrm{d}x$, $\tilde{\psi} := \frac{1}{c} \mathbb{1}_\Omega$ and $\tilde{\phi} := \phi - \int \phi \; \tilde{\psi} e^{-V} \mathrm{d}x \cdot \frac{1}{c} \mathbb{1}_{\mathbb{R}^N}$. We can now compute, again using Lemma 2.3 and the results we already have

$$\langle L_V \phi, \phi \rangle = \int |\nabla \phi|^2 \, e^{-V} \mathrm{d}x = \int |\nabla \tilde{\phi}|^2 \, e^{-F} \mathrm{d}x \geq C \|\tilde{\phi}\|^2$$
$$= C \|\phi\|^2 - |\int \phi \; \tilde{\psi} e^{-V} \mathrm{d}x|^2 \geq C \|\phi\|^2 - |\int \phi \; \psi e^{-V} \mathrm{d}x|^2.$$

□

We can now prove an analogous result, without the restriction, that our set Ω needs to be bounded.

Corollary 3.8. *We define* $\psi := \sqrt{\frac{1}{\int_\Omega \ e^{-V}\mathrm{d}x}}\mathbb{1}_\Omega$ *and obtain*

$$\langle L_V\phi, \phi\rangle \geq C\|\phi\|^2 - |\int \phi \ \psi e^{-V}\mathrm{d}x|^2,$$

for all smooth ϕ *with compact support contained in* Ω*.*

Proof. Let ϕ be as in the assumption of this result. For every $R > 0$ s.t. the support of ϕ is bounded by the radius R, we define the convex and pre compact set $\Omega_R := \Omega \cap B_R(0)$, which satisfies the assumptions 3.6 and additionally contains the support of ϕ. We obtain

$$\begin{aligned}\langle L_V\phi, \phi\rangle &= \int_{\Omega_R} |\nabla\phi|^2 \ e^{-V}\mathrm{d}x \geq C\|\phi\|^2 - \frac{1}{\int_{\Omega\cap B_R(0)} \ e^{-V}\mathrm{d}x}|\int_\Omega \phi \ e^{-V}\mathrm{d}x|^2 \\ &\underset{R\to\infty}{\to} C\|\phi\|^2 - \frac{1}{\int_\Omega \ e^{-V}\mathrm{d}x}|\int_\Omega \phi \ e^{-V}\mathrm{d}x|^2 = C\|\phi\|^2 - |\int \phi \ \psi e^{-V}\mathrm{d}x|^2.\end{aligned}$$

□

3.2 Localisation around the Saddle Point ($\Omega_{3,N}$)

To prove the estimate for this region, we will perform a ground state transformation on the operator $L_{\beta,N}$, but only in one direction. Again we follow the techniques of [10]. Let us fix a $\delta > 0$ and let Ω be the set of all $x \in \mathbb{R}^N$ s.t. $|x|^2 \leq N(\frac{1}{3} - \delta)$. In this section, we will consider smooth functions ψ with $supp(\psi) \subset \Omega$.

Remark 3.9. *For $(\frac{1}{\sqrt{3}} - \rho)^2 + 4r^2 < \frac{1}{3}$ the set $\Omega_{3,N}$ is contained in Ω, for some $\delta > 0$.*

Lemma 3.10. *There exists a δ dependent constant $C > 0$ s.t. for all $x \in \Omega$ and $N \in \mathbb{N}_{\geq 2}$*

$$HessV_N|_x(e,e) \leq \frac{-C}{N},$$

where $e := \frac{1}{\sqrt{N}}(1,..,1)$.

Proof. Because of $\mathrm{Hess}V_N(x) = \frac{1}{N}\big(K_N - I + (3x_k^2 \cdot \delta_{k,l})_{k,l}\big)$ and $K_N \cdot e = 0$ we obtain

$$\mathrm{Hess}V_N|_x(e,e) = \frac{1}{N^2}\sum_k^N (3x_k^2 - 1) \leq \frac{1}{N^2}(N(1-3\delta) - \sum_k^N 1) = \frac{-3\delta}{N}.$$

□

Lemma 3.11. *The following estimate holds true for smooth functions ψ with $supp(\psi) \subset \Omega$, β big enough and arbitrary $N \in \mathbb{N}_{\geq 2}$*

$$\langle L_{\beta,N}\psi, \psi\rangle \geq \frac{C}{2}\|\psi\|^2.$$

Proof. Let us define $\phi := \psi \exp(-\frac{\beta}{2}V_N)$. We then obtain

$$\begin{aligned}
\langle L_{\beta,N}\psi, \psi\rangle &= \frac{N}{\beta}\int |\nabla\psi|^2 \exp(-\beta V_N)\mathrm{d}x \geq \frac{N}{\beta}\int |\nabla\psi \cdot e|^2 \exp(-\beta V_N)\mathrm{d}x \\
&= \frac{N}{\beta}\int |\nabla\big(\exp(\frac{\beta}{2}V_N)\phi\big) \cdot e \exp(-\frac{\beta}{2}V_N)|^2 \mathrm{d}x \\
&= \frac{N}{\beta}\int |\nabla\phi \cdot e + \phi\frac{\beta}{2}\nabla V_N \cdot e|^2 \mathrm{d}x \\
&= \frac{N}{\beta}\int |\nabla\phi \cdot e|^2 + 2(\nabla\phi \cdot e)\phi\frac{\beta}{2}(\nabla V_N \cdot e) + |\phi\frac{\beta}{2}\nabla V_N \cdot e|^2 \mathrm{d}x \\
&\geq \frac{N}{\beta}\int \phi(\nabla\phi \cdot e)\beta(\nabla V_N \cdot e)\mathrm{d}x = \frac{N}{\beta}\int \nabla\phi^2 \cdot \big(e\frac{\beta}{2}(\nabla V_N \cdot e)\big)\mathrm{d}x \\
&= -\frac{N}{\beta}\int \frac{\beta}{2}\phi^2 \nabla \cdot \big(e(\nabla V_N \cdot e)\big)\mathrm{d}x = -N\int \frac{1}{2}\phi^2 \mathrm{Hess}V_N|_x(e,e)\mathrm{d}x \\
&\geq N\frac{1}{2}\frac{C}{N}\int \phi^2 \mathrm{d}x = \frac{C}{2}\int |\psi \exp(-\frac{\beta}{2}V_N)|^2 \mathrm{d}x = \frac{C}{2}\|\psi\|^2.
\end{aligned}$$

□

3.3 Localisation away from the z-Axis ($\Omega_{4,N}$)

For the region $\Omega_{4,N} = \{x \in \mathbb{R}^N : |y(x)|^2 \geq N \cdot r^2\}$ we will use the fact, that V_N is uniformly convex, if restricted V_N to the set $(z, .., z) + W_N$. The following estimate on the Dirichlet form which "only acts on W_N"

$$\frac{N}{\beta} \int_{(z,..,z)+W_N} |\nabla\psi|^2 \, e^{-\beta V_N} \mathrm{d}y \geq C \int_{(z,..,z)+W_N} |\psi|^2 \, e^{-\beta V_N} \mathrm{d}y, \tag{5}$$

will be our main tool. Because the restriction of V_N is convex, we can obtain this estimate, by applying the NGS Bound 2.13 with perturbation $G := 0$.

Let us fix a constant $R > 0$. We will prove the following Lemma at the end of this subsection.

Lemma 3.12. *Let Γ be the set of all $y \in W$ s.t. $|y| \geq \sqrt{N}R$. Then there exists a R dependent, but z independent constant C, s.t. for arbitrary $N \in \mathbb{N}_{\geq 2}$, β big enough, $\psi : W \to \mathbb{R}$ smooth with supp$(\psi) \subset \Gamma$ and $z \in \mathbb{R}$*

$$\frac{N}{\beta} \int_W |\nabla\psi|^2 \, e^{-\beta F_{z,N}(y)} \mathrm{d}y \geq C \int_W |\psi|^2 \, e^{-\beta F_{z,N}(y)} \mathrm{d}y,$$

where $F_{z,N}$ is defined in Definition 1.3.

As a consequence of this result, we obtain the following corollary.

Corollary 3.13. *Let Ω be the set of all $x \in \mathbb{R}^N$ s.t. $|y(x)| \geq \sqrt{N}R$. Then there exists a R dependent constant C, s.t. for arbitrary $N \in \mathbb{N}_{\geq 2}$, β big enough and for all $\phi : \mathbb{R}^N \to \mathbb{R}$ smooth with supp$(\psi) \subset \Omega$, the following estimate holds true*

$$\langle L_{\beta,N}\phi, \phi\rangle \geq C\|\phi\|^2.$$

Proof. With $e := \frac{1}{\sqrt{N}}(1, .., 1)$ we compute

$$\begin{aligned}
\frac{\beta}{N} \langle L_{\beta,N}\phi, \phi\rangle &= \int_{\mathbb{R}^N} |\nabla\phi|^2 \, e^{-\beta V_N(x)} \mathrm{d}x \\
&= \int_{\mathbb{R}} \int_W |\nabla_y \phi\big(t \cdot e + y\big) + \nabla_e \phi\big(t \cdot e + y\big)|^2 \, e^{-\beta V_N\left(t \cdot e + y\right)} \mathrm{d}y \, \mathrm{d}t \\
&\geq \int_{\mathbb{R}} \int_W |\nabla_y \phi\big(t \cdot e + y\big)|^2 \, e^{-\beta V_N\left(t \cdot e + y\right)} \mathrm{d}y \, \mathrm{d}t.
\end{aligned}$$

Because the support of the function $\psi_t(y) := \phi\big(t \cdot e + y\big)$ is contained in the set Γ, we can apply Lemma 3.12

$$\begin{aligned}
\langle L_{\beta,N}\phi,\phi\rangle &\geq \int_{\mathbb{R}} \frac{N}{\beta}\int_W |\nabla\psi_t|^2\, e^{-\beta V_t}\mathrm{d}y\; e^{\frac{\beta}{4}\left(\frac{t}{\sqrt{N}}\right)^4-\frac{\beta}{2}\left(\frac{t}{\sqrt{N}}\right)^2+\frac{\beta}{4}}\mathrm{d}t \\
&\geq \int_{\mathbb{R}} C\|\psi_t\|_t^2\, e^{\frac{\beta}{4}\left(\frac{t}{\sqrt{N}}\right)^4-\frac{\beta}{2}\left(\frac{t}{\sqrt{N}}\right)^2+\frac{\beta}{4}}\mathrm{d}t \\
&= C\int_{\mathbb{R}}\int_W |\psi_t|^2\, e^{-\beta F_{\frac{t}{\sqrt{N}},N}(y)}\mathrm{d}y\; e^{\frac{\beta}{4}\left(\frac{t}{\sqrt{N}}\right)^4-\frac{\beta}{2}\left(\frac{t}{\sqrt{N}}\right)^2+\frac{\beta}{4}}\mathrm{d}t \\
&= C\int_{\mathbb{R}}\int_W |\phi(t\cdot e+y)|^2\, e^{-\beta\left(F_{\frac{t}{\sqrt{N}},N}(y)-\frac{1}{4}\left(\frac{t}{\sqrt{N}}\right)^4+\frac{\beta}{2}\left(\frac{t}{\sqrt{N}}\right)^2-\frac{1}{4}\right)}\mathrm{d}y\;\mathrm{d}t \\
&= C\int_{\mathbb{R}}\int_W |\phi(t\cdot e+y)|^2\, e^{-\beta V_N(t\cdot e+y)}\mathrm{d}y\;\mathrm{d}t = C\int_{\mathbb{R}^N}\phi^2\, e^{-\beta V_N}\mathrm{d}x.
\end{aligned}$$

□

In the rest of this section, we will prove Lemma 3.12. The key tool for our investigations is the following estimate.

Remark 3.14. *Because of Lemma 1.6 we know that there exist a positive constant q s.t. for all $N\in\mathbb{N}_{\geq 2}$, $y\in W_N$, $z\in\mathbb{R}$, t with $|t|\leq\sqrt{N}$ and $x:=t\cdot e+y$*

$$F_{z,N}(y)\leq \frac{1}{2N}\sum_{k=1}^N x_k^4+\frac{q\cdot(z^2+1)}{2N}\sum_{k=1}^N x_k^2+\frac{1}{2N}\langle K_N\cdot x,x\rangle\,.$$

Lemma 3.15. *There exists a constant β_0 s.t. for all $\beta\geq\beta_0$, $N\in\mathbb{N}_{\geq 2}$ and all $z\in\mathbb{R}$*

$$\int e^{-\beta F_{z,N}(y)}\;\mathrm{d}y\geq\frac{1}{\sqrt{\beta}}\left(\frac{N2\pi}{\beta}\right)^{\frac{N-1}{2}}\frac{1}{\det\!\left(K_N+q\left(z^2+1\right)\cdot I\right)}.$$

Proof. With the definition $b:=q\cdot\left(z^2+1\right)\geq q=:b_0$, Remark 3.14 tells us

$$e^{-\beta F_{z,N}(y)}\geq e^{-\frac{\beta}{N}\left(\frac{1}{2}\sum_{k=1}^N(t\cdot e+y)_k^4+\frac{b}{2}\sum_{k=1}^N(t\cdot e+y)_k^2+\frac{1}{2}\langle K_N\cdot y,y\rangle\right)}$$

holds for all $y\in W$ and t with $|t|\leq\sqrt{N}$. If we integrate this inequality and consider that the left side is independent of t we obtain using Lemma 5.12 (see also [10])

$$\begin{aligned}
2\sqrt{N}\int e^{-\beta F_{z,N}(y)}\;\mathrm{d}y &\geq \int_{-\sqrt{N}}^{\sqrt{N}}\int_W e^{-\frac{\beta}{N}\left(\frac{1}{2}\sum_{k=1}^N(t\cdot e+y)_k^4+\frac{b}{2}\sum_{k=1}^N(t\cdot e+y)_k^2+\frac{1}{2}\langle K_N\cdot y,y\rangle\right)}\mathrm{d}y\;\mathrm{d}t \\
&= \int_{\left\{x:\left|\frac{1}{N}\sum_{k=1}^N x_k\right|\leq 1\right\}} e^{-\frac{\beta}{N}\left(\frac{1}{2}\sum_{k=1}^N x_k^4+\frac{b}{2}\sum_{k=1}^N x_k^2+\frac{1}{2}\langle K_N\cdot x,x\rangle\right)}\;\mathrm{d}x \\
&= \left(\frac{N2\pi}{\beta}\right)^{\frac{N}{2}}\frac{1}{\det(K_N+b\cdot I)}\cdot(1+o_{\beta,N}(1)).
\end{aligned}$$

We conclude

$$\int e^{-\beta F_{z,N}(y)}\ \mathrm{d}y \geq \frac{1}{2\sqrt{N}}\Big(\frac{N2\pi}{\beta}\Big)^{\frac{1}{2}}\Big(\frac{N2\pi}{\beta}\Big)^{\frac{N-1}{2}}\frac{1}{\det(K_N+b\cdot I)}\cdot(1+o_{\beta,N}(1))$$
$$\geq \frac{1}{\sqrt{\beta}}\Big(\frac{N2\pi}{\beta}\Big)^{\frac{N-1}{2}}\frac{1}{\det(K_N+b\cdot I)},$$

for β.

□

Proof of Lemma 3.12. Let us fix a smooth function $\psi : W \to \mathbb{R}$ with $supp(\psi) \subset \Gamma$. Because of

$$\mathrm{Hess}F_{z,N}|_y \geq \frac{\mu-1}{N}\cdot I$$

we obtain, using the Log-Sobolev constant of the measure corresponding to $F_{z,N}$

$$\frac{N}{\beta}\int_W |\nabla\psi|^2\ e^{-\beta F_{z,N}(y)}\mathrm{d}y \geq -\frac{\mu-1}{2}\log\Big(\frac{\int_\Gamma e^{-\beta F_{z,N}(y)}\ \mathrm{d}y}{\int_W e^{-\beta F_{z,N}(y)}\ \mathrm{d}y}\Big)\|\psi\|_z^2.$$

For all $|y| \geq \sqrt{N}R$

$$\begin{aligned}
F_{z,N}(y) &= \frac{1}{2N}\langle (K_N-I)\cdot y, y\rangle + \frac{1}{4N}\sum_{k=1}^N\big(y_k^4+4zy_k^3+6z^2y_k^2\big)\\
&\geq \frac{1}{2N}\langle (K_N-I)\cdot y, y\rangle + \frac{1}{4N}\sum_{k=1}^N y_k^2(y_k+2z)^2 + \frac{1}{2N}z^2\sum_{k=1}^N y_k^2\\
&= \frac{1}{2N}\langle (K_N-\frac{1+\mu}{2}I)\cdot y, y\rangle + \frac{z^2+\mu-1}{2N}\sum_{k=1}^N y_k^2 + \frac{1}{4N}\sum_{k=1}^N y_k^2(y_k+2z)^2\\
&\geq \frac{1}{2N}\langle (K_N-\frac{1+\mu}{2}I)\cdot y, y\rangle + \frac{z^2+\mu-1}{2}R^2.
\end{aligned}$$

Then

$$\begin{aligned}
\int_\Gamma e^{-\beta F_{z,N}(y)}\ \mathrm{d}y &\leq e^{-\beta\frac{z^2+\mu-1}{2}R^2}\int_W e^{-\frac{\beta}{2N}\langle (K_N-\frac{1+\mu}{2}I)\rangle}\ \mathrm{d}y\\
&= e^{-\beta\frac{z^2+\mu-1}{2}R^2}\Big(\frac{2N\pi}{\beta}\Big)^{\frac{N-1}{2}}\frac{1}{\det(K_N|_W-\frac{1+\mu}{2}I_W)},
\end{aligned}$$

and together with the Lemmatas 5.8, 5.12 and 3.15

$$\begin{aligned}
\frac{\int_\Gamma e^{-\beta F_{z,N}(y)}\ \mathrm{d}y}{\int_W e^{-\beta F_{z,N}(y)}\ \mathrm{d}y} &\leq e^{-\beta\frac{z^2+\mu-1}{2}R^2}\sqrt{\beta}\frac{det\big(K_N+q\big(z^2+1\big)\cdot I\big)}{\det(K_N|_W-\frac{1+\mu}{2}I_W)}\\
&\leq e^{-\beta\frac{z^2+\mu-1}{2}R^2}\sqrt{\beta}e^{\lambda q\left(z^2+1\right)}G.
\end{aligned}$$

We conclude

$$\frac{N}{\beta}\int_W |\nabla\psi|^2 \ \exp(-\beta F_{z,N}(y))\mathrm{d}y$$
$$\geq \Big(\beta\frac{\mu-1}{2}\cdot\frac{z^2+\mu-1}{2}R^2 - \lambda q\big(z^2+1\big) - \log\big(G\sqrt{\beta}\big)\cdot\frac{\mu-1}{2}\Big)\|\psi\|_z^2 \geq C\|\psi\|_z^2,$$

for a suitable constant $C > 0$ and β big enough. □

3.4 Localisation around the Infliction Points ($\Omega_{5,N}, \Omega_{6,N}$)

The success of this subsection is based on finding a good reference potential, see Definition 2.2 and Lemma 2.6, s.t. we can find a suitable estimate for the unitary transformed operator $L'_{\beta,N}$. It will be the NGS Bound 2.13 which will yield us this estimate. Let us recall $\Omega_5 := \{x \in \mathbb{R}^N : \frac{1}{\sqrt{3}} - 2\rho < \bar{x} < \frac{1}{\sqrt{3}} + 2\rho,\ |y(x)| < 2\sqrt{N}r\}$.

Our goal will be to prove the following statement

Lemma 3.16. *For ρ small enough, there exists a constant $C > 0$, s.t. for all ϕ with supp(ϕ) $\subset \Omega_5$*

$$\frac{N}{\beta}\int |\nabla\phi|^2\ e^{-\beta V_N}\mathrm{d}x \geq C\|\phi\|^2,$$

for β big enough and arbitrary $N \in \mathbb{N}_{\geq 2}$.

Definition 3.17. *We define our reference potential, see Subsection 2.2, as*

$$U_N : \begin{cases} \mathbb{R}^N \to \mathbb{R} \\ x \mapsto V_N(x) + \frac{\lambda}{2}\bar{x}^2, \end{cases}$$

where we choose $1 < \lambda < \mu$. This yields us Hess$U_N \geq \frac{\lambda-1}{N} \cdot I$.

Remark 3.18. *From 2.6 we know that the unitary transformed operator $L'_{\beta,N} := T \cdot L_{\beta,N} \cdot T^{-1}$ with $T := T^{\beta V_N}_{\beta U_N}$ can be written as*

$$L'_{\beta,N}\phi = \frac{N}{\beta}L_{\beta U_N}\phi + \frac{N}{\beta}\Big(\frac{1}{2}L_{\beta U_N}(\beta V_N - \beta U_N) + \frac{1}{4}|\beta\nabla V_N - \beta\nabla U_N|^2\Big)\phi.$$

With the definition $W_N(x) := V_N(x) - U_N(x) = -\frac{\lambda}{2}\bar{x}^2$ and

$$\begin{aligned} G :&= \frac{1}{2}L_{\beta U_N}(\beta V_N - \beta U_N) + \frac{1}{4}|\beta\nabla V_N - \beta\nabla U_N|^2 \\ &= \frac{\beta^2}{4}\Big(|\nabla W_N|^2 + 2\nabla U_N \cdot \nabla W_N\Big) - \frac{\beta}{2}\Delta W_N \end{aligned}$$

we obtain

$$\langle L_{\beta,N}\phi, \phi\rangle = \frac{N}{\beta}\Big(\int |\nabla\phi|^2\ e^{-\beta U_N}\mathrm{d}x + \int G\phi^2\ e^{-\beta U_N}\mathrm{d}x\Big).$$

for all $\phi \in C_0^\infty(\mathbb{R}^N)$.

The potential βU_N satisfies a logarithmic Sobolev inequality with constant $\frac{\beta}{N} \cdot c := \frac{\beta}{N} \cdot \frac{1}{2(\lambda-1)}$, see Definition 2.10. Then the NGS Bound 2.13 tells us

$$\langle L_{\beta,N}\phi, \phi\rangle \geq -\frac{1}{c} \log\Big(\frac{\int_{B_{r,\delta}(z_0)} e^{-\frac{N}{\beta}cG} \; e^{-\beta U_N} \mathrm{d}x}{\int_{\mathbb{R}^N} \; e^{-\beta U_N} \mathrm{d}x}\Big) \|\psi\|^2.$$

If cG was positive enough, i.e. $cG \geq \frac{\beta}{N} C$, it would be easy to see by a monotonicity argument that we would be done. In general, we are not able to prove this, but we can show that adding a suitable quartic term will yield the positivity. This will be content of the next result.

Lemma 3.19. *Let c be an arbitrary positive constant. For all $1 < \lambda < \frac{4}{3}$ there exist positive constants C, ρ and r s.t. for all $x \in \Omega_5$*

$$c \cdot G + \frac{\beta^2}{4N^2} \sum_{k=1}^{N} x_k^4 \geq \frac{\beta^2}{N} C,$$

where G is defined as in Remark 3.18.

Proof. Let x be an element of Ω_5, i.e. $\frac{1}{\sqrt{3}} - 2\rho < \bar{x} < \frac{1}{\sqrt{3}} + 2\rho$ and $|y(x)|^2 \leq 4N \cdot r^2$. Because of $\Delta W \leq 0$, we are done if we can show

$$F := c\big(|N\nabla W_\lambda|^2 + 2N\nabla U_\lambda \cdot N\nabla W_\lambda\big) + \sum_{k=1}^{N} x_k^4 \geq N \cdot C.$$

Using

$$\begin{aligned} N\nabla W_\lambda &= -\lambda(\bar{x}, .., \bar{x}), \\ N\nabla U_\lambda &= (x_1^3, .., x_N^3) + (K_N - I) \cdot y(x) + (\lambda - 1)(\bar{x}, .., \bar{x}), \end{aligned}$$

we obtain

$$F = N \cdot c(2\lambda - \lambda^2)\bar{x}^2 - 2c\lambda\bar{x} \sum_{k=1}^{N} x_k^3 + \sum_{k=1}^{N} x_k^4.$$

Applying the orthogonal decomposition $x = y + (\bar{x}, .., \bar{x})$, where we suppress the x dependence of $y := y(x)$, we get

$$\begin{aligned} \sum_{k=1}^{N} x_k^4 &= \sum_{k=1}^{N} y_k^4 + 4\bar{x} \sum_{k=1}^{N} y_k^3 + 6\bar{x}^2 \sum_{k=1}^{N} y_k^2 + N \cdot \bar{x}^4, \\ \sum_{k=1}^{N} x_k^3 &= \sum_{k=1}^{N} y_k^3 + 3\bar{x} \sum_{k=1}^{N} y_k^2 + N \cdot \bar{x}^3, \end{aligned}$$

which leads to

$$F = N \cdot c\lambda\bar{x}^2\big(2-\lambda-2\bar{x}^2\big) + \sum_{k=1}^{N} y_k^4 + (4\bar{x}-2c\lambda\bar{x})\sum_{k=1}^{N} y_k^3 + (6\bar{x}^2-6c\lambda\bar{x}^2)\sum_{k=1}^{N} y_k^2 + N\cdot\bar{x}^4.$$

Using $1<\lambda<\frac{4}{3}$, we obtain for the function $d(x) := 2-\lambda-2\bar{x}^2$

$$d(\frac{1}{\sqrt{3}}) = \frac{4}{3}-\lambda > 0.$$

Because d is continuous, there exists a $0<\rho<\frac{1}{\sqrt{3}}$ and a $c>0$ s.t. for all $\frac{1}{\sqrt{3}}-2\rho < z < \frac{1}{\sqrt{3}}+2\rho$

$$d(z) > C'.$$

From this we can deduce

$$F \geq N\cdot c\lambda\Big(\sqrt{\frac{1}{3}}-2\rho\Big)^2 C' + \sum_{k=1}^{N} y_k^2\Big(y_k^2 + (4\bar{x}-2c\lambda z)y_k + (6\bar{x}^2-6c\lambda\bar{x}^2)\Big).$$

Let us look at the quadratic function

$$p(y) := y^2 + (4\bar{x}-2c\lambda\bar{x})y + (6\bar{x}^2-6c\lambda\bar{x}^2).$$

This function takes its minimum at $y_* = c\lambda\bar{x}-2\bar{x}$ with the value

$$p(y_*) = (c\lambda\bar{x}-2\bar{x})^2 + (-2c\lambda\bar{x}+4\bar{x})(c\lambda\bar{x}-2\bar{x}) + 6\bar{x}^2 - 6\bar{x}^2 c\lambda.$$

The expression above is defined on the compact set $\bar{x}\in[\frac{1}{\sqrt{3}}-2\rho, \frac{1}{\sqrt{3}}+2\rho]$ and is continuous, hence has a minimum C''. This leads to

$$F \geq N\cdot c\lambda\Big(\sqrt{\frac{1}{3}}-2\rho\Big)^2 C' - C''\sum_{k=1}^{N} y_k^2 \geq N\cdot\Big(c\lambda\Big(\sqrt{\frac{1}{3}}-2\rho\Big)^2 C' - 4C''r^2\Big) =: N\cdot C.$$

Observe that C is positive, if r is small enough.

□

Remark 3.20. *We denote the adjusted potential*

$$U_N'(x) := U_N(x) - \frac{1}{4N}\sum_{k=1}^{N} x^4.$$

For $c := \frac{1}{2(\mu-1)}$ we then obtain, using Corollary 2.14,

$$\frac{N}{\beta}\int_{\mathbb{R}^N} |\nabla\psi|^2\ e^{-\beta U_N}\mathrm{d}x + \frac{N}{\beta}\int_{\mathbb{R}^N} G\cdot|\psi|^2\ e^{-\beta U_N}\mathrm{d}x$$
$$\geq -\frac{1}{c}\log\Big(\frac{\int_{B_{r,\delta}(z_0)} e^{-\frac{N}{\beta}\left(cG+\frac{\beta^2}{N^2}\sum_{k=1}^{N}\frac{1}{4}x_k^4\right)}\ e^{-\beta U_N'}\mathrm{d}x}{\int_{\mathbb{R}^N}\ e^{-\beta U_N}\mathrm{d}x}\Big)\|\psi\|^2.$$

The key step is, that we decompose the measure $e^{-\beta U_N}\mathrm{d}x = e^{-\frac{\beta}{N}\sum_{k=1}^N x_k^4}\cdot e^{-\beta U_N'}\mathrm{d}x$ *and add the quadric part to the pertubation* G*, i.e.*

$$\int_{B_{r,\delta}(z_0)} e^{-\frac{N}{\beta}cG}\ e^{-\beta U_N}\mathrm{d}x = \int_{B_{r,\delta}(z_0)} e^{-\frac{N}{\beta}\left(cG+\frac{\beta^2}{N^2}\sum_{k=1}^N \frac{1}{4}x_k^4\right)}\ e^{-\beta U_N'}\mathrm{d}x.$$

Proof of 3.16. We define $c := \frac{1}{2(\mu-1)}$. We show, that there exists a positive constant C s.t. we obtain for all β big enough, arbitrary $N \geq 2$ and all smooth functions $\psi : \mathbb{R}^N \to \mathbb{R}$ with $supp(\psi) \subset \Omega_5$

$$\langle L_{\beta,N}\phi, \phi\rangle \geq C\|\phi\|^2.$$

Because $L_{\beta,N}$ and $L_{\beta,N}'$ are unitary equivalent, it is enough to prove

$$\langle L_{\beta,N}'\psi, \psi\rangle \geq C\|\psi\|^2.$$

Combining Lemma 3.19 and Remark 3.20, yields us the existence of a positive constant C s.t.

$$\frac{\langle L_{\beta,N}'\psi, \psi\rangle}{\|\psi\|^2} \geq -\frac{1}{c}\log\left(\frac{\int_{B_{r,\delta}(z_0)} e^{-\beta C}\ e^{-\beta U_N'}\mathrm{d}x}{\int_{\mathbb{R}^N}\ e^{-\beta U_N}\mathrm{d}x}\right) \geq \beta\frac{C}{c} - \frac{1}{c}\log\left(\frac{\int_{\mathbb{R}^N}\ e^{-\beta U_N'}\mathrm{d}x}{\int_{\mathbb{R}^N}\ e^{-\beta U_N}\mathrm{d}x}\right).$$

Because of Corollary 5.11 we know that we have in the regime of large β

$$\frac{\int_{\mathbb{R}^N}\ e^{-\beta U_N'}\mathrm{d}x}{\int_{\mathbb{R}^N}\ e^{-\beta U_N}\mathrm{d}x} \leq 2.$$

Furthermore, for β big enough we obtain

$$\frac{\langle L_{\beta,N}'\psi, \psi\rangle}{\|\psi\|^2} \geq \beta\frac{C}{c} - \frac{\log(2)}{c} \geq C.$$

□

4 Generalisation on Dirichlet Boundaries

4.1 The second Eigenvalue

In the case of Dirichlet boundary conditions, we can investigate analogously to section 3 the asymptotic behaviour of the low lying spectrum. Note that if we assume Dirichlet boundary conditions, 0 is not an eigenvalue any more. As a consequence we will analyse the second eigenvalue instead of the third one. In the second part of this section, we will additionally analyse the asymptotic behaviour of the first eigenvalue.

Definition 4.1. *Given a $R < 1$ we define $\Omega_{R,N} \subset \mathbb{R}^N$ as the set of all $x \in \mathbb{R}^N$ s.t. $|x - I_+| > \sqrt{N}R$, with $I_+ := (1, .., 1)$. In other words*

$$\Omega_{R,N} := \mathbb{R}^N \setminus B_{\sqrt{N}R}(I_+).$$

Further we denote the set of test functions $\mathfrak{D}_R$, as the set of all C^∞ functions $\phi : \Omega_{R,N} \to \mathbb{R}$ with compact support.

We can now define for a $V : \Omega_{R,N} \to \mathbb{R}$, which can be extended to a C^2 function on an open set $V \supset \overline{\Omega_{R,N}}$, in analogy to Definition 2.2, $A_{R,V}$ as the Friedrichs extension in the space $L^2(\mu_V)$ of the symmetric and positive operator

$$A'_{R,V} : \begin{cases} \mathfrak{D}_R \to L^2(\mu_V), \\ \phi \to -\Delta\phi + \nabla V \cdot \nabla\phi. \end{cases}$$

Furthermore we define $A_{\beta,N} := \frac{N}{\beta} A_{R,\beta V_N|_{\Omega_{R,N}}}$, where we suppress the R dependence in our notation. We have in analogy to Lemma 2.3 for all $\phi \in \mathfrak{D}_R$

$$\langle A_{R,V}\phi, \phi\rangle_{L^2(\mu_V)} = \int |\nabla\phi|^2 \, e^{-V} \mathrm{d}x.$$

In Corollary 3.5 we have shown $|\sigma(L_{\beta,N}) \cap [0, D)| \leq 2$. For the spectrum of $A_{\beta,N}$ we are able to prove the following stronger result.

Theorem 4.2. *There exists a constant $C > 0$ s.t. for all β big enough*

$$\sigma(A_{\beta,N}) \cap [0, D) = \{\lambda_0\},$$

with $\lambda_0 > 0$.

Remark 4.3. *Considering that $A_{\beta,N}$ is a positive operator, tells us that λ_0 cannot be negative.*

To prove Theorem 4.2, we repeat the proof we did for $L_{\beta,N}$. The only difference is, that we can prove an even stronger result for functions $\phi \in \mathfrak{D}_R$, which are localised in Ω_1, see Definition 3.1 for the definition of the decomposition $\Omega_1, .., \Omega_6$. Observe that Ω_1 and Ω_2 can not be treated similarly, because we only cut out the ball $B_{\sqrt{N}R}$ in the region Ω_1.

Notation 4.4. *In the following, we will write*

$$f(\beta, N) = o_{\beta,N}(1)$$

for a function f, which has the property

$$\lim_{\beta\to\infty} \sup_{N\in\mathbb{N}} |f(\beta, N)| = 0,$$

i.e. $f(\beta, N)$ converges to 0 uniformly in N.

Lemma 4.5. *There exists a constant C s.t. for all $\phi \in \mathfrak{D}_R$ with supp$(\phi) \subset \Omega_1$*

$$\frac{N}{\beta}\int |\nabla\phi|^2 \; e^{-\beta V_N}\mathrm{d}x \geq C\|\phi\|^2,$$

for all β big enough and arbitrary $N \in \mathbb{N}_{\geq 2}$.

Proof. Let F be a convex extension, as described in Lemma 5.4. Using Corollary 2.14 on the potential $V := \beta F$, subset $\Omega := \Omega_1$ and perturbation $G := 0$, yields us

$$\frac{N}{\beta}\int |\nabla\phi|^2 \; e^{-\beta F}\mathrm{d}x \geq C_{\beta,N}\|\phi\|^2,$$

with $C_{\beta,N} := -2c\log\left(\frac{\int_{\Omega_1\cap\Omega_{R,N}} \; e^{-\beta F}\mathrm{d}x}{\int_{\mathbb{R}^N} \; e^{-\beta F}\mathrm{d}x}\right)$, where c is given by Lemma 1.9. First of all, we estimate, using Lemma 5.13 and Corollary 5.9

$$\begin{aligned}\int_{\mathbb{R}^N} \; e^{-\beta F}\mathrm{d}x &\geq \int_{\Omega_1} \; e^{-\beta F}\mathrm{d}x = \int_{\Omega_1} \; e^{-\beta V_N}\mathrm{d}x \\ &= \int_{\Omega_1} \; e^{-\frac{\beta}{2N}\langle (K_N+2I)\cdot(x-I_+),(x-I_+)\rangle}\mathrm{d}x + o_{\beta,N}(1)\cdot\int \; e^{-\frac{\beta}{2N}\langle (K_N+2I)\cdot x,x\rangle}\mathrm{d}x \\ &= \int \; e^{-\frac{\beta}{2N}\langle (K_N+2I)\cdot x,x\rangle}\mathrm{d}x\cdot\big(1+o_{\beta,N}(1)\big).\end{aligned}$$

Furthermore, let $\delta > 0$ be small enough, s.t. Lemma 5.13 holds. Then we obtain

$$\begin{aligned}\int_{\Omega_1\cap\Omega_{R,N}} \; e^{-\beta V_N}\mathrm{d}x &= \int_{\Omega_1\cap\Omega_{R,N}} \; e^{-\beta V_N - \frac{\beta\delta}{N}|x|^2 + \frac{\beta\delta}{N}|x|^2}\mathrm{d}x \\ &\leq e^{-\beta R^2}\int \; e^{-\beta(V_N - \frac{\delta}{N}|x|^2)}\mathrm{d}x \\ &= e^{-\beta\delta R^2}\int \; e^{-\frac{\beta}{2N}\langle (K_N+2I)\cdot x,x\rangle}\mathrm{d}x\cdot\big(1+o_{\beta,N}(1)\big),\end{aligned}$$

which yields us the desired statement. □

Proof of Theorem 4.2. We already know that there exists a constant C and a function ψ_l (observe that there is no ψ_r compared to Lemma 3.3) for the potential V_N, s.t. for all β big enough, arbitrary $N \in \mathbb{N}_{\geq 2}$ and all functions ϕ_k with $supp(\phi_k) \subset \Omega_k$ and $\phi_k \in \mathfrak{D}_R$

1. $\int_{\Omega_1} |\nabla \phi_1|^2 \, e^{-\beta V_N} \, \mathrm{d}x \geq \frac{\beta}{N} C \|\phi_1\|^2$,

2. $\int_{\Omega_2} |\nabla \phi_2|^2 \, e^{-\beta V_N} \, \mathrm{d}x \geq \frac{\beta}{N} C \|\phi_2\|^2 - |\int \phi_2 \, \psi_l \, e^{-\beta V_N} \mathrm{d}x|^2$,

3. $\int_{\Omega_k} |\nabla \phi_k|^2 \, e^{-\beta V_N} \, \mathrm{d}x \geq \frac{\beta}{N} C \|\phi_k\|^2$ for $k \in \{3, .., 6\}$,

The first statement is proven in Lemma 4.5. The last two statements hold true for the same reasons as in Lemma 3.3. Because the product of a $\mathfrak{D}_R$ function with one of the $\xi_{1,N}, .., \xi_{6,N}$ functions stays a $\mathfrak{D}_R$ function, an analogue argument as in Corollary 2.19 yields us for $\phi \in \mathfrak{D}_R$

$$\frac{N}{\beta} \int |\nabla \phi|^2 \, e^{-\beta V_N} \mathrm{d}x \geq C \|\phi\|^2 - |\int \phi \, \psi_l \, e^{-\beta V_N} \mathrm{d}x|^2.$$

Let us define a norm on $\mathfrak{D}_R$

$$\|\phi\|^2_{\mathfrak{D}_r} := \int (\phi^2 + |\nabla \phi|^2) \, e^{-\beta V_N} \mathrm{d}x$$

and denote $X \subset L^2(\mu_{\beta V_N|\Omega_{R,N}})$ as the completion of $\left(\mathfrak{D}_R, \|.\|^2_{\mathfrak{D}_R}\right)$. Furthermore let us write $\mathcal{E} : X \to \mathbb{R}$ for the extension of the quadratic form $\phi \mapsto \int |\nabla \phi|^2 \, e^{-\beta V_N} \mathrm{d}x$. Because our operator $A_{\beta,N}$ is defined as the Friedrichs extension with respect to the quadratic form $\phi \mapsto \int |\nabla \phi|^2 \, e^{-\beta V_N} \mathrm{d}x$, see Definition 4.1, we obtain $dom(A_{\beta,N}) \subset X$ and for all $\phi \in dom(A_{\beta,N})$

$$\mathcal{E}(\phi, \phi) = \langle A_{\beta,N} \phi, \phi \rangle \, .$$

Let us take an arbitrary $\phi \in dom(A_{\beta,N})$. From the construction of X follows, that there exists a sequence $\phi_n \in \mathfrak{D}_R$ s.t. $\phi_n \to \phi$ in the norm $\|.\|^2_{\mathfrak{D}_R}$. Because $\mathcal{E}$ as well as the $L^2(\mu_{\beta V_N|\Omega_{R,N}})$ scalar product are continuous with respect to $\|.\|^2_{\mathfrak{D}_R}$, we obtain

$$\begin{aligned} \langle A_{\beta,N} \phi, \phi \rangle = \mathcal{E}(\phi, \phi) = \lim_n \mathcal{E}(\phi_n, \phi_n) \geq \limsup C \|\phi_n\|^2 - |\int \phi_n \, \psi_l \, e^{-\beta V_N} \mathrm{d}x|^2 \\ = C \|\phi\|^2 - |\int \phi \, \psi_l \, e^{-\beta V_N} \mathrm{d}x|^2. \end{aligned}$$

Applying Lemma 2.15 yields us the desired statement. □

4.2 The first Eigenvalue

To prove the sharp asymptotic of the the first eigenvalue, we need to find good test functions, which should be close to the real eigenvector. The control of the second eigenvalue, which we have already established, is crucial for the proof that our constructed test functions are "close" to the real first eigenvector. In [10] this strategy, i.e. using the control of the spectral gap between the first and second non zero eigenvalue to obtain the asymptotic behaviour of the first non zero eigenvalue, is carried out in the case of no boundaries. We will adapt this procedure in the case of Dirichlet boundaries.

Definition 4.6. *Let $\chi : \mathbb{R} \to \mathbb{R}$ be a C^∞ function with the properties $|\chi| \leq 1$, $\chi|_{[b,\infty)} = 0$ and $\chi|_{(-\infty,a]} = 1$, with $0 < a < b < 1$. Then we define our test functions $\psi_{\beta,N} : \mathbb{R}^N \to \mathbb{R}$ the following way*

$$\psi_{\beta,N}(x) := f_\beta(\bar{x}) := \sqrt{\frac{\beta}{2\pi}} \int_b^{\bar{x}} \chi(t) \cdot e^{-\frac{\beta t^2}{2}} \, \mathrm{d}t,$$

where $\bar{x}$ is defined as in 1.2. Observe that $\psi_{\beta,N} \in \mathfrak{D}_R$ for b small enough.

Lemma 4.7. *We have $\lim_{\beta\to\infty} f_\beta(\frac{y}{\beta} - 1) = 1$ and $\lim_{\beta\to\infty} f_\beta(\frac{y}{\beta} + 1) = 0$. Moreover $|f_\beta(z)| \leq 1$ for all $z \in \mathbb{R}$.*

The next results give us the control over crucial quantities of $\psi_{\beta,N}$.

Lemma 4.8. *We obtain the asymptotic result*

$$\int |\psi_{\beta,N}|^2 \, e^{-\beta V_N} \mathrm{d}x = \frac{\sqrt{\frac{2\pi N}{\beta}}^N}{\sqrt{\det\big(K_N + 2 \cdot I\big)}} \cdot \big(1 + o_{\beta,N}(1)\big).$$

Proof. Using the Laplace asymptotic of Lemma 5.13 allows us to substitute $\int |\psi_{\beta,N}|^2 \, e^{-\beta V_N} \mathrm{d}x$ with

$$\int_{\bar{x}>0} |\psi_{\beta,N}|^2 \, e^{-\frac{\beta}{2N}(K_N+2\cdot I)\left[x-(1,..,1)\right]} \mathrm{d}x + \int_{\bar{x}<0} |\psi_{\beta,N}|^2 \, e^{-\frac{\beta}{N}(K_N+2\cdot I)\left[x+(1,..,1)\right]} \mathrm{d}x.$$

If we denote by $\lambda_0, .., \lambda_{N-1}$ the eigenvalues of $K_N + 2 \cdot I$. We can compute using transformations and Lemma 4.7

$$\begin{aligned}
&\int_{\bar{x}\leq 0} |\psi_{\beta,N}|^2 \; e^{-\frac{\beta}{N}(K_N+2\cdot I)\left[x+(1,..,1)\right]}\mathrm{d}x \\
&\quad= \int_{\xi_0\leq 0} |f_\beta(\frac{\xi_0}{\sqrt{N}})|^2 \; e^{-\frac{\beta\lambda_0(\xi_0-1)^2}{2N}}\mathrm{d}\xi_0 \int_{\mathbb{R}^{N-1}} e^{-\frac{\beta}{2N}(\lambda_1\xi_1^2+..+\lambda_{N-1}\xi_{N-1}^2)}\mathrm{d}\xi_1..\mathrm{d}\xi_{N-1} \\
&\quad= \int_{\frac{y}{\beta}-1\leq 0} |f_\beta(\frac{y}{\beta}-1)|^2 \; e^{-\frac{\lambda_0 y^2}{2}}\mathrm{d}y\sqrt{\frac{N}{\beta}} \int_{\mathbb{R}^{N-1}} e^{-\frac{\beta}{2N}(\lambda_1\xi_1^2+..+\lambda_{N-1}\xi_{N-1}^2)}\mathrm{d}\xi_1..\mathrm{d}\xi_{N-1} \\
&\quad= \big(1+o_{\beta,N}(1)\big)\cdot \int_{\mathbb{R}} e^{-\frac{\beta\lambda_0(\xi_0-1)^2}{2N}}\mathrm{d}\xi_0 \int_{\mathbb{R}^{N-1}} e^{-\frac{\beta}{2N}(\lambda_1\xi_1^2+..+\lambda_{N-1}\xi_{N-1}^2)}\mathrm{d}\xi_1..\mathrm{d}\xi_{N-1} \\
&\quad=\big(1+o_{\beta,N}(1)\big)\cdot \frac{\sqrt{\frac{2\pi N}{\beta}}^N}{\det\big(K_N+2\cdot I\big)}.
\end{aligned}$$

Analogously we obtain

$$\int_{\bar{x}\geq 0} |\psi_{\beta,N}|^2 \; e^{-\frac{\beta}{N}(K_N+2\cdot I)\left[x-(1,..,1)\right]}\mathrm{d}x = o_{\beta,N}(1)\cdot \frac{\sqrt{\frac{2\pi N}{\beta}}^N}{\det\big(K_N+2\cdot I\big)}.$$

□

Lemma 4.9. *We obtain the asymptotic result*

$$\int |\nabla\psi_{\beta,N}|^2 \; e^{-\beta V_N}\mathrm{d}x = e^{-\frac{\beta}{4}}\frac{\beta}{2\pi N}\cdot\frac{\sqrt{\frac{2\pi N}{\beta}}^N}{\sqrt{\det\big(\tilde{Q}_N\big)}}\cdot\big(1+o_{\beta,N}(1)\big),$$

with $\tilde{Q}_N[x] := \langle (K_N - I)\cdot y(x), y(x)\rangle + N\bar{x}^2$.

Proof. If we define $F(x) := V_N(x) + \bar{x}^2$ we see that $\tilde{Q}_N$ describes the quadratic part of F, to be precise

$$F(x) = \frac{1}{4N}\sum_{k=1}^{N} x_k^4 + \frac{1}{2N}\tilde{Q}_N + \frac{1}{4}.$$

If we use again a suitable Laplace asymptotic, we obtain

$$\begin{aligned}\int |\nabla\psi_{\beta,N}|^2\ e^{-\beta V_N}\mathrm{d}x &= \int \frac{\beta}{2\pi N}|\chi(\bar{x})|^2 e^{-\beta\bar{x}^2}\ e^{-\beta V_N}\mathrm{d}x = \frac{\beta}{2\pi N}\int |\chi(\bar{x})|^2\ e^{-\beta F}\mathrm{d}x\\ &= e^{-\frac{\beta}{4}}\frac{\beta}{2\pi N}\int |\chi(\bar{x})|^2\ e^{-\frac{\beta}{2N}\tilde{Q}_N[x]}\mathrm{d}x\cdot\big(1+o_{\beta,N}(1)\big)\\ &= e^{-\frac{\beta}{4}}\frac{\beta}{2\pi N}\int e^{-\frac{\beta}{2N}\tilde{Q}_N[x]}\mathrm{d}x\cdot\big(1+o_{\beta,N}(1)\big)\\ &= e^{-\frac{\beta}{4}}\frac{\beta}{2\pi N}\cdot\frac{\sqrt{\frac{2\pi N}{\beta}}^N}{\det\big(\tilde{Q}_N\big)}\cdot\big(1+o_{\beta,N}(1)\big).\end{aligned}$$

□

Corollary 4.10. *Let $\lambda_{\beta,N}$ be the first eigenvalue of the diffusion operator $A_{\beta,N}$ with Dirichlet boundary conditions, see Definition 4.1, then*

$$\lambda_{\beta,N} \le \frac{e^{-\frac{\beta}{4}}}{2\pi}\sqrt{\frac{\det\big(K_N+2\cdot I\big)}{\det\big(\tilde{Q}_N\big)}}\cdot\big(1+o_{\beta,N}(1)\big).$$

Proof. Using the test function $\psi_{\beta,N}$ yields us

$$\lambda_{\beta,N} \le \frac{\langle A_{\beta,N}\psi_{\beta,N},\psi_{\beta,N}\rangle}{\|\psi_{\beta,N}\|^2} = \frac{N}{\beta}\frac{\int |\nabla\psi_{\beta,N}|^2\ e^{-\beta V_N}\mathrm{d}x}{\int |\psi_{\beta,N}|^2\ e^{-\beta V_N}\mathrm{d}x} = \frac{e^{-\frac{\beta}{4}}}{2\pi}\sqrt{\frac{\det\big(K_N+2\cdot I\big)}{\det\big(\tilde{Q}_N\big)}}\cdot\big(1+o_{\beta,N}(1)\big).$$

□

Remark 4.11. *The N asymptotic of $\sqrt{\frac{\det(K_N+2\cdot I)}{\det(\tilde{Q}_N)}}$ can be computed explicitly, see e.g. [10]. The essential part for us is, that there exists a constant $g>0$, s.t.*

$$\lim_{N\to\infty}\sqrt{\frac{\det\big(K_N+2\cdot I\big)}{\det\big(\tilde{Q}_N\big)}} = g.$$

Lemma 4.12. *There exists a constant q, s.t. for β big enough and arbitrary $N \geq 2$*

$$\int |A_{\beta,N}\psi_{\beta,N}|^2 \; e^{-\beta V_N} \mathrm{d}x \leq \frac{q}{\beta^2} e^{-\frac{\beta}{4}} \cdot \frac{\sqrt{\frac{2\pi N}{\beta}}^N}{\sqrt{\det(\tilde{Q}_N)}}.$$

Proof. We compute, using $\sum_{k=1}^N (K_N \cdot x)_k = 0$ and $\psi_{\beta,N} \in \mathfrak{D}_r$

$$A_{\beta,N}\psi_{\beta,N} = \frac{1}{\beta}\sqrt{\frac{\beta}{2\pi}} \cdot \Big(-\chi'(\bar{x}) + \frac{1}{N}\sum_{k=1}^N \beta\; \chi(\bar{x})\; x_k^3 \Big) e^{-\frac{\beta}{2}\bar{x}^2}.$$

Because of $supp(\chi') \subset [a,b]$ we obtain $|\chi'(\bar{x})| \leq S$ for some constant S and furthermore we claim

$$|\chi'(\bar{x})| \leq \frac{S}{a^3}\sum_{k=1}^N x_k^3.$$

In the case $\bar{x} < a$, we are done because of $\chi'(\bar{x}) = 0$. For $\bar{x} \geq a$ we can compute

$$a \leq \frac{1}{N}\sum_{k=1}^N |x_k| \leq \Big(\frac{1}{N}\sum_{k=1}^N x_k^3\Big)^{\frac{1}{3}},$$

which implies $|\chi'(\bar{x})| \leq S = \frac{S}{a^3}a^3 \leq \frac{S}{a^3}\frac{1}{N}\sum_{k=1}^N x_k^3$. We can now estimate

$$|A_{\beta,N}\psi_{\beta,N}| \leq e^{-\frac{\beta}{2}\bar{x}^2}\frac{1}{\beta}\sqrt{\frac{\beta}{2\pi}} \cdot \big(\frac{S}{a^3} + \beta\big)\frac{1}{N}\sum_{k=1}^N |x_k|^3 \leq e^{-\frac{\beta}{2}\bar{x}^2}\sqrt{\frac{\beta}{2\pi}}2 \cdot \frac{1}{N}\sum_{k=1}^N |x_k|^3,$$

for β big enough. Using $(\frac{1}{N}\sum_{k=1}^N |x_k|^3)^2 \leq \frac{1}{N}\sum_{k=1}^N x_k^6$ and the decomposition $V_N(x) + \bar{x}^2 = \frac{1}{4N}\sum_{k=1}^N x_k^4 + \frac{1}{2N}\tilde{Q}_N + \frac{1}{4} \geq \frac{1}{2N}\tilde{Q}_N + \frac{1}{4}$ from Lemma 4.9, we obtain

$$\int |A_{\beta,N}\psi_{\beta,N}|^2 \; e^{-\beta V_N}\mathrm{d}x \leq \int \frac{2\beta}{\pi}\frac{1}{N}\sum_{k=1}^N x_k^6 e^{-\beta\bar{x}^2} \; e^{-\beta V_N}\mathrm{d}x \leq e^{-\frac{\beta}{4}}\frac{2\beta}{\pi}\int \frac{1}{N}\sum_{k=1}^N x_k^6 \; e^{-\frac{\beta}{2N}\tilde{Q}_N}\mathrm{d}x.$$

Analogously to Lemma 5.12, we can compute the sixth momenta explicitly, and conclude that there exists a constant P s.t.

$$\int \frac{1}{N}\sum_{k=1}^N x_k^6 \; e^{-\frac{\beta}{2N}\tilde{Q}_N}\mathrm{d}x \leq P \cdot \frac{1}{\beta^3}\frac{\sqrt{\frac{2\pi N}{\beta}}^N}{\sqrt{\det(\tilde{Q}_N)}}.$$

In total we obtain

$$\int |A_{\beta,N}\psi_{\beta,N}|^2 \; e^{-\beta V_N}\mathrm{d}x \leq \frac{2}{\pi}P\frac{1}{\beta^2}e^{-\frac{\beta}{4}} \cdot \frac{\sqrt{\frac{2\pi N}{\beta}}^N}{\sqrt{\det(\tilde{Q}_N)}}.$$

□

The next corollary follows from Lemma 4.9 and Lemma 4.12.

Corollary 4.13. *We conclude*

$$\langle A_{\beta,N}\psi_{\beta,N}, A_{\beta,N}\psi_{\beta,N}\rangle = o_{\beta,N}(1)\,\langle A_{\beta,N}\psi_{\beta,N}, \psi_{\beta,N}\rangle\,.$$

4.2.1 Some Spectral Analysis

Assumption 4.14. *Let $\sigma(L) = \{\lambda\}\dot\cup[C,+\infty)\cap\sigma(L)$, $P := E([C,+\infty))$ and $P_\lambda := E(\{\lambda\})$ where E denotes the spectral measure of L. Furthermore, we have an $\epsilon > 0$ and an element $u \in dom(L)$, s.t. $\|u\| = 1$ and $\langle Lu, Lu\rangle \leq \epsilon\,\langle Lu, u\rangle$.*

Lemma 4.15. *Under those assumptions*

$$\langle LPu, Pu\rangle \leq \frac{\frac{\epsilon}{C}}{1-\frac{\epsilon}{C}}\lambda.$$

Proof. Using the spectral calculus we obtain (observe that $t1_{\sigma(L)\cap[C,+\infty)} \leq \frac{t^2}{C}1_{\sigma(L)}$)

$$\langle LPu, Pu\rangle = \int_{\sigma(L)\cap[C,+\infty)} t\;\mathrm{d}E_{u,u} \leq \frac{1}{C}\int_{\sigma(L)} t^2\;\mathrm{d}E_{u,u} = \frac{1}{C}\left\langle L^2u, u\right\rangle = \frac{1}{C}\left\langle Lu, Lu\right\rangle.$$

If we take the equation $\langle Lu, u\rangle = \langle LPu, Pu\rangle + \langle LP_\lambda u, P_\lambda u\rangle$ into account, which is true because of

$$\langle Lu, u\rangle = \int_{\sigma(L)} t\;\mathrm{d}E_{u,u} = \int_{\{0,\lambda\}} t\;\mathrm{d}E_{u,u} + \int_{\sigma(L)\cap[C,+\infty)} t\;\mathrm{d}E_{u,u} = \langle LPu, Pu\rangle + \langle LP_\lambda u, P_\lambda u\rangle\,,$$

we can deduce

$$\langle LPu, Pu\rangle \leq \frac{\epsilon}{C}\langle Lu, u\rangle = \frac{\epsilon}{C}(\langle LPu, Pu\rangle + \langle LP_\lambda u, P_\lambda u\rangle) = \frac{\epsilon}{C}(\langle LPu, Pu\rangle + \lambda\,\langle P_\lambda u, P_\lambda u\rangle).$$

Furthermore we obtain

$$\langle LPu, Pu\rangle \leq \frac{\epsilon}{C}\langle LPu, Pu\rangle + \frac{\epsilon}{C}\lambda\,\langle P_\lambda u, P_\lambda u\rangle\,.$$

Because of $\|u\| = 1$ we have $\langle P_\lambda u, P_\lambda u\rangle \leq 1$ which yields us the desired inequality.

□

Lemma 4.16. *Under the assumptions 4.14 we obtain*

$$|\langle Lu, u\rangle - \lambda| \leq \frac{2\epsilon}{C-\epsilon} \cdot \lambda.$$

Proof. Let us consider the case $\lambda = 0$. Then 4.15 yields us $x = 0$ and therefore $Pu \in ker(L)$. This is only possible, if $Pu = 0$. In this case u is the eigenvector to the eigenvalue $\lambda = 0$ and we are done.

In the case $\lambda \neq 0$ we define

$$z := \langle Lu, u\rangle,$$
$$x := \langle LPu, Pu\rangle,$$
$$r := \langle P_\lambda u, P_\lambda u\rangle.$$

We already know from Lemma 4.15

$$|\frac{x}{\lambda}| \leq \frac{\frac{\epsilon}{C}}{1-\frac{\epsilon}{C}}.$$

If we consider the equation $z = x + \lambda r$, we can deduce

$$|\frac{z-\lambda}{\lambda}| = |\frac{z-\lambda r + (\lambda r - \lambda)}{\lambda}| \leq |\frac{x}{\lambda}| + |r-1|.$$

Because of $\lambda \leq C$, we know

$$1 - r = \langle Pu, Pu\rangle \leq \frac{1}{C}\langle LPu, Pu\rangle \leq \frac{1}{C}\frac{\frac{\epsilon}{C}}{1-\frac{\epsilon}{C}}\lambda \leq \frac{\frac{\epsilon}{C}}{1-\frac{\epsilon}{C}}.$$

□

4.2.2 Asymptotic of $\lambda_{\beta,N}$

Theorem 4.17. *We obtain for the first eigenvalue $\lambda_{\beta,N}$ of $A_{\beta,N}$*

$$\lambda_{\beta,N} = \frac{e^{-\frac{\beta}{4}}}{2\pi}\sqrt{\frac{\det\left(K_N + 2\cdot I\right)}{\det\left(\tilde{Q}_N\right)}} \cdot \left(1 + o_{\beta,N}(1)\right).$$

Proof. Because of Theorem 4.2 we know that we have for β big enough

$$\sigma(A_{\beta,N}) \cap (-\infty, D) = \{\lambda_0\}.$$

Using the Lemmatas 4.8 and 4.12, we obtain for $u := \frac{\psi_{\beta,N}}{\|\psi_{\beta,N}\|}$ and arbitrary $N \in \mathbb{N}_{\geq 2}$

$$\langle Lu, u\rangle = \frac{e^{-\frac{\beta}{4}}}{2\pi}\sqrt{\frac{\det\left(K_N + 2\cdot I\right)}{\det\left(\tilde{Q}_N\right)}}\cdot\left(1 + o_{\beta,N}(1)\right),$$
$$\langle Lu, Lu\rangle \leq \frac{q}{2\pi\beta^2}.$$

We can now apply Lemma 4.16 for the operator $L := A_{\beta,N}$ and the element $u := u_{\beta,N}$, which yields us

$$\left|\frac{\langle A_{\beta,N}u_{\beta,N}, u_{\beta,N}\rangle}{\lambda_{\beta,N}} - 1\right| \leq \frac{\frac{q}{\pi\beta^2}}{D - \frac{q}{2\pi\beta^2}} \underset{\beta\to\infty}{\to} 0,$$

uniformly in N. From this we conclude

$$\lambda_{\beta,N} = \langle A_{\beta,N}u_{\beta,N}, u_{\beta,N}\rangle\cdot\left(1 + o_{\beta,N}(1)\right) = \frac{e^{-\frac{\beta}{4}}}{2\pi}\sqrt{\frac{\det\left(K_N + 2\cdot I\right)}{\det\left(\tilde{Q}_N\right)}}\cdot\left(1 + o_{\beta,N}(1)\right).$$

□

5 Appendix

5.1 Convexity of V_N

Lemma 5.1. *For $\delta, b > 0$ we define $V_{n,\delta,b} := \{w \in \mathbb{R}^n : w_1 = b, w_n - w_1 = \delta\}$ and a mapping*

$$E := \begin{cases} V_{n,\delta,b} \to \mathbb{R} \\ w \mapsto \sum_{k=1}^{n-1}(w_{k+1} - w_k)^2. \end{cases}$$

Then $w^* = \left(b + (j-1)\frac{\delta}{n-1}\right)_{j=1}^n$ minimizes E with $E(w^*) = \frac{\delta^2}{n-1}$.

Proof. Because of $\lim_{r\to\infty} E = \infty$ there exist a minimum w^* of E. Then for all $u \in \mathbb{R}^N$ with $u(1) = u(n)$

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} E(w^* + t \cdot u) = 2\sum_{k=1}^{n-1}(w^*_{k+1} - w^*_k)(u_{k+1} - u_k).$$

For all $1 < j < n$ we define u s.t. $u_k = 1$ for $1 < k \leq j$ and $u_k = 0$ for $k = 0$ and $k > j$. Testing with this vector u yields

$$0 = (w^*_2 - w^*_1) - (w^*_{j+1} - w^*_j).$$

From this follows $w^*_j = b + (j-1)(w^*_2 - w^*_1)$. Because of $\delta = w^*_n - w^*_1 = (n-1)(w^*_2 - w^*_1)$ this is only possible, if $w^* = \left(b + (j-1)\frac{\delta}{n-1}\right)_{j=1}^n$.

□

Definition 5.2. *For all $v \in \mathbb{R}^N$, we define*

$$\delta(v) := \max_{l,k} |v_l - v_k|.$$

Lemma 5.3. *For all $v \in \mathbb{R}^N$ and $N \in \mathbb{N}_{\geq 2}$*

$$\langle K_N \cdot v, v\rangle \geq N\frac{\mu \cdot \delta(v)^2}{16\pi^2}.$$

Proof. W.l.o.g. we can assume $v_n - v_0 = \delta(v)$ with $1 < n \leq N$.
We obtain (with $v_{N+1} := v_1$)

$$\langle K_N \cdot v, v\rangle = \frac{\mu}{4\sin(\frac{\pi}{N})^2}\sum_{k=1}^{N}(v_{k+1} - v_k)^2 \geq \frac{\mu}{4\sin(\frac{\pi}{N})^2}\sum_{k=1}^{n-1}(v_{k+1} - v_k)^2.$$

Because of $(v_j)_{j=1}^n \in V_{n,\delta(v),v_0}$ Lemma 5.1 tells us

$$\frac{\mu}{4\sin(\frac{\pi}{N})^2}\sum_{k=1}^{n-1}(v_{k+1}-v_k)^2 \geq \frac{\mu}{4\sin(\frac{\pi}{N})^2}\frac{\delta(v)^2}{n-1} \geq \frac{\mu}{4\sin(\frac{\pi}{N})^2}\frac{\delta(v)^2}{N} \geq N\frac{\mu\cdot\delta(v)^2}{16\pi^2}.$$

□

The next result is a corollary of the extension Theorem from [18].

Lemma 5.4. *Let $V:\Omega\to\mathbb{R}^N$ be a C^2 function with $HessV \geq C\cdot I$, where Ω is a convex and bounded subset of $\mathbb{R}^N$. Then the function V can be extended to a strictly convex C^2 function $F:\mathbb{R}^N\to\mathbb{R}$ with $HessF \geq C\cdot I$. By an extension, we mean that*

$$F|_\Omega = V.$$

Proof. The C^2 function

$$f:\begin{cases}\Omega\to\mathbb{R},\\ x\to V_N(x)-\frac{C}{2}|x|^2,\end{cases}$$

is strictly convex. Because f is also defined on a compact and convex domain, we can apply the extension Theorem from [18], which yields us that there exists a convex extension G of f. It is elementary to show that $F(x) := G(x)+\frac{C}{2}|x|^2$ satisfies the desired properties. □

5.2 Construction of $\xi_1,..,\xi_6$

Lemma 5.5. *For all $a<b$ we can find a function $f_{a,b}\in C^\infty(\mathbb{R})$ with $f_{a,b}|_{(-\infty,a]}=0$, $f_{a,b}|_{[b,\infty)}=1$ and $0\leq f_{a,b}(x)\leq 1$, s.t. $\sqrt{1-f_{a,b}^2}$ is a $C^\infty(\mathbb{R})$ function as well.*

Proof. We prove the statement for the prototypical case $a=0$ and $b=1$. To do so, we define the C^∞ function g as $g(x):=e^{-\frac{1}{x}}$ for $x>0$ and $g(x):=0$ for $x\leq 0$. We claim that $f(x):=\frac{f(x)}{f(x)+f(1-x)}$ has the desired properties, especially that $\sqrt{1-f^2}$ is again a C^∞ function. We compute

$$\begin{aligned}1-f^2(x) &= \frac{(g(x)+g(-x))^2-g(x)^2}{(g(x)+g(-x))^2} = \frac{2g(x)g(1-x)+g(1-x)^2}{(g(x)+g(1-x))^2}\\ &= g(1-x)\cdot\frac{2g(x)+g(1-x)}{(g(x)+g(1-x))^2}.\end{aligned}$$

and obtain

$$\sqrt{1-f^2(x)} = \sqrt{g(1-x)}\cdot\sqrt{2g(x)+g(1-x)}\cdot\sqrt{\frac{1}{(g(x)+g(1-x))^2}}.$$

Because $2g(x)+g(1-x)$ and $g(x)+g(1-x)$ are strictly positive, the functions $\sqrt{2g(x)+g(1-x)}$ and $\sqrt{\frac{1}{(g(x)+g(1-x))^2}}$ are C^∞. Furthermore

$$\sqrt{g(1-x)} = e^{-\frac{1}{1-x}^{\frac{1}{2}}} = e^{-\frac{1}{2-2x}} = g(2-2x)$$

is a C^∞ function, hence $\sqrt{1-f^2} \in C^\infty$.

□

Lemma 5.6. *Let $(a_0, b_0), .., (a_n, b_n)$ be a series of open intervals with $a_n = -\infty$, $b_0 = \infty$ and $a_j < b_{j+1} < a_{j-1}$. Then there exists a series of C^∞ functions $h_0, .., h_n$ with $\sum_{j=0}^n h_j^2 = 1$ and $supp(h_j) \subset [a_j, b_j]$. We can additionally assume that $supp(h'_j)$ is a bounded set.*

Proof. We define $h_0 := f_{a_0,b_1}$, see Lemma 5.5, $h_n := \sqrt{1 - f^2_{a_{n-1},b_n}}$ and for $0 < j < n$

$$h_j := f_{a_j,b_{j+1}} \cdot \sqrt{1 - f^2_{a_{j-1},b_j}}.$$

With this definition we obtain for $0 < j < n$, that $supp(h_j) = (-\infty, b_j] \cap [a_j, \infty) = [a_j, b_j]$ and $supp(h'_j) \subset [a_j, b_{j+1}] \cap [a_{j-1}, b_j]$. We also see that h_0 and h_n have the right support properties. We claim for all $k > 0$

$$\sum_{j=0}^{k-1} h_j^2 = f^2_{a_{k-1},b_k}.$$

For $k = 1$ this is obvious. Let us assume that this property holds for $k < n$, then

$$\sum_{j=0}^{k} h_j^2 = \sum_{j=0}^{k-1} h_j^2 + h_k^2 = f^2_{a_{k-1},b_k} + f^2_{a_k,b_{k+1}} \cdot (1 - f^2_{a_{k-1},b_k}).$$

For $x \leq b_{k+1}$ we obtain, using $x \leq b_{k+1} < a_{k-1}$, that $f_{a_{k-1},b_k}(x) = 0$ and hence

$$f_{a_{k-1},b_k}(x)^2 + f_{a_k,b_{k+1}}(x)^2 \cdot (1 - f_{a_{k-1},b_k}(x)^2) = f_{a_k,b_{k+1}}(x)^2.$$

In the other case that $x > b_{k+1}$, we have $f_{a_k,b_{k+1}}(x) = 1$ and consequently

$$\begin{aligned} f_{a_{k-1},b_k}(x)^2 + f_{a_k,b_{k+1}}(x)^2 \cdot (1 - f_{a_{k-1},b_k}(x)^2) &= f_{a_{k-1},b_k}(x)^2 + (1 - f_{a_{k-1},b_k}(x)^2) \\ &= 1 = f_{a_k,b_{k+1}}(x). \end{aligned}$$

Using this property for $k = n$ yields us

$$\sum_{j=0}^{n} h_j^2 = \sum_{j=0}^{n-1} h_j^2 + h_n^2 = f^2_{a_{n-1},b_n} + (1 - f^2_{a_{n-1},b_n}) = 1.$$

□

Corollary 5.7. *There exist C^∞ functions $\xi_{1,N},..,\xi_{6,N}$ with the properties supp$(\xi_{k,N}) \subset \Omega_{k,N}$, $\sum_{k=1}^{6} \xi_{k,N}^2 = 1$ and*

$$\xi_{k,N}(x) = h_k(\bar{x}, \frac{|y(x)|^2}{N}),$$

with suitable N independent functions h_k.

Proof. Let q_0, q_1, q_2, q_3, q_4 be as described in Lemma 5.6 with respect to the series of intervals $(\frac{1}{\sqrt{3}} + \rho, \infty), (\frac{1}{\sqrt{3}} - 2\rho, \frac{1}{\sqrt{3}} + 2\rho), (-\frac{1}{\sqrt{3}} + \rho, \frac{1}{\sqrt{3}} - \rho), (-\frac{1}{\sqrt{3}} - 2\rho, -\frac{1}{\sqrt{3}} + 2\rho)$, $(-\infty, -\frac{1}{\sqrt{3}} - \rho)$ and let w_0, w_1 be as described in Lemma 5.6 with respect to the series of intervals $(r, \infty), (-\infty, 2r)$. We can now define suitable functions h_k in the following way

$$\begin{aligned}
h_1(z, R) &:= q_0(z) \cdot w_1(R), \\
h_2(z, R) &:= q_4(z) \cdot w_1(R), \\
h_3(z, R) &:= q_2(z) \cdot w_1(R), \\
h_4(z, R) &:= w_2(R), \\
h_5(z, R) &:= q_1(z) \cdot w_1(R), \\
h_6(z, R) &:= q_3(z) \cdot w_1(R).
\end{aligned}$$

It is easy to see that $\xi_{k,N}(x) := h_k(\bar{x}, \frac{|y(x)|^2}{N})$ has the right support properties. Finally we compute

$$\sum_{k=1}^{6} h_k(z, R)^2 = w_2(R)^2 + \sum_{k=1, k \neq 4}^{6} h_k(z, R)^2 = w_2(R)^2 + w_1(R)^2 \sum_{j=0}^{4} q_j(z)^2 = w_2(R)^2 + w_1(R)^2 = 1.$$

□

5.3 Asymptotic Computations

Similar computations can be found in more detail in [10].

Lemma 5.8. *Let b_0 be a positive constant. Then there exist constants G, λ s.t. for all $b \geq b_0$ and $N \geq 2$*

$$\det(K_N + b \cdot I) \leq G e^{\lambda b} \cdot \det\left(K_N|_W - \frac{1+\mu}{2} \cdot I_W\right).$$

Proof. First of all, observe that with $d := b + \frac{1+\mu}{2}$

$$\frac{\det(K_N + b\cdot I)}{\det\left(K_N|_W - \frac{1+\mu}{2}\cdot I_W\right)} = b\cdot\det\left(\frac{K_N|_W + b\cdot I_W}{K_N|_W - \frac{1+\mu}{2}\cdot I_W}\right) = b\cdot\det\left(I_W + \frac{d}{K_N|_W - \frac{1+\mu}{2}\cdot I_W}\right)$$

$$= b\cdot\Pi_{k=1}^{N}\left(1+\frac{d}{\mu\frac{sin(k\frac{\pi}{N})^2}{sin(\frac{\pi}{N})^2}-\frac{\mu+1}{2}}\right) = b\cdot(1+\frac{2d}{\mu+1})^2\cdot\Pi_{k=2}^{N-1}\left(1+\frac{d}{\mu\frac{sin(k\frac{\pi}{N})^2}{sin(\frac{\pi}{N})^2}-\frac{\mu+1}{2}}\right)$$

$$\leq b\cdot(1+\frac{2d}{\mu+1})^2\cdot\Pi_{k=2}^{[\frac{N}{2}]}\left(1+\frac{d}{\mu\frac{sin(k\frac{\pi}{N})^2}{sin(\frac{\pi}{N})^2}-\frac{\mu+1}{2}}\right)^2.$$

Because of $N\geq 2$ we have $\frac{sin(k\frac{\pi}{N})^2}{sin(\frac{\pi}{N})^2}\geq\frac{1}{4}k^2$ for all $k\in\{1,..,[\frac{N}{2}]\}$. We conclude, using $\ln(1+x)\leq x$,

$$\frac{\det(K_N + b\cdot I)}{\det\left(K_N|_W - \frac{1+\mu}{2}\cdot I_W\right)} \leq b\cdot(1+\frac{2d}{\mu+1})^2\cdot\Pi_{k=2}^{[\frac{N}{2}]}\left(1+\frac{d}{\frac{\mu}{4}k^2-\frac{\mu+1}{2}}\right)^2$$

$$= b\cdot(1+\frac{2d}{\mu+1})^2\cdot e^{2\sum_{k=2}^{[\frac{N}{2}]}\log\left(1+\frac{d}{\frac{\mu}{4}k^2-\frac{\mu+1}{2}}\right)}$$

$$\leq b\cdot(1+\frac{2d}{\mu+1})^2\cdot e^{2d\sum_{k=2}^{\infty}\frac{1}{\frac{\mu}{2}k^2-\frac{\mu+1}{2}}}$$

$$= b\cdot(1+\frac{2b+1+\mu}{\mu+1})^2\cdot e^{(1+\mu)\sum_{k=2}^{\infty}\frac{1}{\frac{\mu}{4}k^2-\frac{\mu+1}{2}}}\cdot e^{b\cdot 2\sum_{k=1}^{\infty}\frac{1}{\frac{\mu}{4}k^2-\frac{\mu+1}{2}}}.$$

Observe that $T := \sum_{k=2}^{\infty}\frac{1}{\frac{\mu}{4}k^2-\frac{\mu+1}{2}}$ is finite and that $b\cdot(1+\frac{2b+1+\mu}{\mu+1})^2)\leq g\cdot e^b$ for a suitable constant g. We obtain

$$\frac{\det(K_N + b\cdot I)}{\det\left(K_N|_W - \frac{1+\mu}{2}\cdot I_W\right)} \leq g\cdot e^b\cdot e^{(1+\mu)\cdot T}\cdot e^{b\cdot 2T} = g\cdot e^{(1+\mu)\cdot T}\cdot e^{b\cdot\left(1+2T\right)} =: G\cdot e^{b\cdot\lambda}.$$

□

Corollary 5.9. *For arbitrary small $\rho > 0$ we obtain*

$$\int_{B_{\sqrt{N}\rho}(0)} e^{-\frac{\beta}{2N}\langle(K_N+2I)\cdot x,x\rangle}\mathrm{d}x = \int_{\mathbb{R}^N} e^{-\frac{\beta}{2N}\langle(K_N+2I)\cdot x,x\rangle}\mathrm{d}x\cdot\left(1+o_{\beta,N}(1)\right),$$

where $B_{\sqrt{N}\rho}(0)$ the ball around 0 with radius $\sqrt{N}\rho$.

Proof. We compute

$$\int_{\mathbb{R}^N\setminus B_{\sqrt{N}\rho}(0)} e^{-\frac{\beta}{2N}\langle(K_N+2I)\cdot x,x\rangle}\mathrm{d}x = \int_{\mathbb{R}^N\setminus B_{\sqrt{N}\rho}(0)} e^{-\frac{\beta}{2N}\langle(K_N+I)\cdot x,x\rangle-\frac{\beta|x|^2}{2N}}\mathrm{d}x$$

$$\leq e^{-\beta\rho^2}\int_{\mathbb{R}^N} e^{-\frac{\beta}{2N}\langle(K_N+I)\cdot x,x\rangle}\mathrm{d}x = e^{-\beta\rho^2}\int_{\mathbb{R}^N} e^{-\frac{\beta}{2N}\langle(K_N+2I)\cdot x,x\rangle}\mathrm{d}x\cdot\sqrt{\frac{\det\left(K_N+2I\right)}{\det\left(K_N+I\right)}}$$

$$\leq \int_{\mathbb{R}^N} e^{-\frac{\beta}{2N}\langle(K_N+2I)\cdot x,x\rangle}\mathrm{d}x\cdot T\cdot e^{-\beta\rho^2},$$

for some positive constant T. We have used Lemma 5.8 in the last step, to estimate

$$\det\big(K_N + 2I\big) \le Ge^{\lambda 2}\det\big(K_N|_W - \frac{1+\mu}{2}\cdot I_W\big) \\ \le Ge^{\lambda 2}\det\big(K_N|_W + I_W\big) = Ge^{\lambda 2}\det\big(K_N + I\big).$$

□

Lemma 5.10. *We have*

$$\int \frac{\beta}{N}\sum_{k=1}^{N} x_k^4 \;\; e^{-\frac{\beta}{2N}\big(\langle K_N\cdot x,x\rangle + a|x|^2 + b\cdot N\bar{x}^2\big)}\,\mathrm{d}x = o_{\beta,N}(1)\cdot\int e^{-\frac{\beta}{2N}\big(\langle K_N\cdot x,x\rangle + a|x|^2 + b\cdot N\bar{x}^2\big)}\,\mathrm{d}x$$

and

$$\int \frac{\beta}{N}\sum_{k=1}^{N} |x_k^3| \;\; e^{-\frac{\beta}{2N}\big(\langle K_N\cdot x,x\rangle + a|x|^2 + b\cdot N\bar{x}^2\big)}\,\mathrm{d}x = o_{\beta,N}(1)\cdot\int e^{-\frac{\beta}{2N}\big(\langle K_N\cdot x,x\rangle + a|x|^2 + b\cdot N\bar{x}^2\big)}\,\mathrm{d}x,$$

under the assumptions $a+b>0$ and $a>-\mu$.

Proof. The key element of this proof is our ability to explicitly compute fourth and sixth momenta of a Gaussian distribution

$$\int x_k^4 \; e^{-\frac{1}{2}\langle Q^{-1}x,x\rangle}\mathrm{d}x = \int \; e^{-\frac{1}{2}\langle Qx,x\rangle}\mathrm{d}x\cdot\big(3Q_{k,k}^2\big),$$
$$\int x_k^6 \; e^{-\frac{1}{2}\langle Q^{-1}x,x\rangle}\mathrm{d}x = \int \; e^{-\frac{1}{2}\langle Qx,x\rangle}\mathrm{d}x\cdot\big(15Q_{k,k}^3\big).$$

We define C as the inverse operator of $x\mapsto \frac{1}{2}\big(\langle K_N\cdot x,x\rangle + a|x|^2 + b\cdot N\bar{x}^2\big)$ and claim $C_{k,k}\le\frac{D}{N}$ for a suitable constant D and arbitrary $N\in\mathbb{N}_{\ge 2}$. If we perform a Fourier transformation, for example see the proof of 1.4, and denote with λ_l^C the eigenvalues of C, we obtain

$$0\le C_{k,k} = e_k\cdot(C\cdot e_k) = \frac{1}{N}\sum_{l=1}^{k} e^{i\frac{2\pi}{N}k\;l}\cdot\lambda_l^C \le \frac{1}{N}\sum_{l=1}^{k}\lambda_l^C.$$

The trace $\sum_{l=1}^{k}\lambda_l^C$ is uniformly bounded in N, which proves our claim. As a consequence we can estimate

$$\int \frac{\beta}{N}\sum_{k=1}^{N} x_k^4 \;\; e^{-\frac{\beta}{2N}\big(\langle K_N\cdot x,x\rangle + a|x|^2 + b\cdot N\bar{x}^2\big)}\,\mathrm{d}x = \int e^{-\frac{\beta}{2N}\big(\langle K_N\cdot x,x\rangle + a|x|^2 + b\cdot N\bar{x}^2\big)}\,\mathrm{d}x\cdot\frac{3\beta}{N}\sum_{k=1}^{N}\big(\frac{N}{\beta}\cdot C_{k,k}\big)^2 \\ \le \int e^{-\frac{\beta}{2N}\big(\langle K_N\cdot x,x\rangle + a|x|^2 + b\cdot N\bar{x}^2\big)}\,\mathrm{d}x\cdot\frac{3}{\beta}D^2.$$

For the second statement, we have to be careful because we take the absolute value of the x_k^3. We will overcome this inconvenience, by using the following estimate

$$|y| \leq \epsilon + \frac{1}{\epsilon} y^2,$$

for an arbitrary ϵ. We obtain

$$\int \frac{\beta}{N} \sum_{k=1}^{N} |x_k^3| \ \ e^{-\frac{\beta}{2N}\left(\langle K_N \cdot x, x\rangle + a|x|^2 + b \cdot N\bar{x}^2\right)} \ \mathrm{d}x \leq \int \frac{\beta}{N} \sum_{k=1}^{N} \left(\epsilon + \frac{1}{\epsilon} x_k^6\right) \ e^{-\frac{\beta}{2N}\left(\langle K_N \cdot x, x\rangle + a|x|^2 + b \cdot N\bar{x}^2\right)} \ \mathrm{d}x$$
$$= \beta \int e^{-\frac{\beta}{2N}\left(\langle K_N \cdot x, x\rangle + a|x|^2 + b \cdot N\bar{x}^2\right)} \ \mathrm{d}x \cdot \left(\epsilon + \frac{15}{\epsilon N} \sum_{k=1}^{N} \left(\frac{N}{\beta} \cdot C_{k,k}\right)^3\right)$$
$$\leq \beta \int e^{-\frac{\beta}{2N}\left(\langle K_N \cdot x, x\rangle + a|x|^2 + b \cdot N\bar{x}^2\right)} \ \mathrm{d}x \cdot \left(\epsilon + \frac{1}{\epsilon} \cdot \frac{15}{\beta^3} D^3\right).$$

With the choice $\epsilon_\beta := \sqrt{\frac{15D^3}{\beta^3}}$ we get

$$\int \frac{\beta}{N} \sum_{k=1}^{N} |x_k^3| \ \ e^{-\frac{\beta}{2N}\left(\langle K_N \cdot x, x\rangle + a|x|^2 + b \cdot N\bar{x}^2\right)} \ \mathrm{d}x \leq \int e^{-\frac{\beta}{2N}\left(\langle K_N \cdot x, x\rangle + a|x|^2 + b \cdot N\bar{x}^2\right)} \ \mathrm{d}x \ 2\beta \cdot \epsilon_\beta.$$

Because of $\lim_{\beta \to \infty} 2\beta \cdot \epsilon_\beta = 0$, we have the second statement.

□

Corollary 5.11. *Let U_N be as in Definition 3.17 and U_N' as in Remark 3.20, then for all β big enough and arbitrary $N \geq 2$*

$$\int_{\mathbb{R}^N} \ e^{-\beta U_N} \mathrm{d}x \geq \frac{1}{2} \int_{\mathbb{R}^N} \ e^{-\beta U_N'} \mathrm{d}x.$$

Proof. We apply Lemma 5.10 with $a := -1$ and $b := \lambda$ and obtain using $|e^{-a} - e^{-b}| \leq |a - b| \cdot e^{-\min(a,b)}$

$$\int_{\mathbb{R}^N} \ e^{-\beta U_N'} \mathrm{d}x - \int_{\mathbb{R}^N} \ e^{-\beta U_N} \mathrm{d}x = \int e^{-\frac{\beta}{2N}\left(\langle K_N \cdot x, x\rangle - |x|^2 + \lambda \cdot N\bar{x}^2\right)} - e^{-\frac{\beta}{2N}\left(\frac{1}{2}\sum_{k=1}^N x_k^4 + \langle K_N \cdot x, x\rangle - |x|^2 + \lambda \cdot N\bar{x}^2\right)} \ \mathrm{d}x$$
$$\leq \frac{\beta}{4} \int \frac{1}{N} \sum_{k=1}^{N} x_k^4 \ e^{-\frac{\beta}{2N}\left(\langle K_N \cdot x, x\rangle - |x|^2 + \lambda \cdot N\bar{x}^2\right)} \ \mathrm{d}x < \frac{1}{2} \cdot \int e^{-\frac{\beta}{2N}\left(\langle K_N \cdot x, x\rangle - |x|^2 + \lambda \cdot N\bar{x}^2\right)} \mathrm{d}x,$$

for β big enough. Hence

$$\int_{\mathbb{R}^N} \ e^{-\beta U_N} \mathrm{d}x = \int_{\mathbb{R}^N} \ e^{-\beta U_N'} \mathrm{d}x - \int e^{-\frac{\beta}{2N}\left(\langle K_N \cdot x, x\rangle - |x|^2 + \lambda \cdot N\bar{x}^2\right)} - e^{-\frac{\beta}{2N}\left(\frac{1}{2}\sum_{k=1}^N x_k^4 + \langle K_N \cdot x, x\rangle - |x|^2 + \lambda \cdot N\bar{x}^2\right)} \ \mathrm{d}x$$
$$> \int_{\mathbb{R}^N} \ e^{-\beta U_N'} \mathrm{d}x - \frac{1}{2} \int_{\mathbb{R}^N} \ e^{-\beta U_N'} \mathrm{d}x = \frac{1}{2} \int_{\mathbb{R}^N} \ e^{-\beta U_N'} \mathrm{d}x.$$

□

Lemma 5.12. *Let b_0, c be positive constants and let us define*

$$D := \{x : |\frac{1}{N}\sum_{k=1}^{N} x_k| \leq c\}.$$

We claim that there exist for all $\epsilon > 0$ a constant β_0 s.t. for all $b \geq b_0$, $\beta > \beta_0$ and arbitrary $N \in \mathbb{N}_{\geq 2}$

$$\int_D e^{-\frac{\beta}{N}\left(\frac{1}{2}\sum_{k=1}^N x_k^4 + \frac{b}{2}\sum_{k=1}^N x_k^2 + \frac{1}{2}\langle K_N \cdot x, x\rangle\right)} \,\mathrm{d}x > (1-\epsilon)\cdot\left(\frac{N2\pi}{\beta}\right)^{\frac{N}{2}} \frac{1}{\det(K_N + b\cdot I)}.$$

Proof. First of all, we compute

$$\begin{aligned}
&\int_{D^c} e^{-\frac{\beta}{N}\left(\frac{b}{2}\sum_{k=1}^N x_k^2 + \frac{1}{2}\langle K_N \cdot x, x\rangle\right)} \,\mathrm{d}x = \int_{|t|>c\sqrt{N}} e^{-\frac{\beta}{N}\frac{b}{2}t^2}\mathrm{d}t \int_W e^{-\frac{\beta}{N}\left(\frac{1}{2}\langle (K_N+b)\cdot y, y\rangle\right)} \,\mathrm{d}y \\
&\leq 2\int_{t>c\sqrt{N}} \frac{t}{c\sqrt{N}} e^{-\frac{\beta}{N}\frac{b}{2}t^2}\mathrm{d}t \int_W e^{-\frac{\beta}{N}\left(\frac{1}{2}\langle (K_N+b)\cdot y, y\rangle\right)} \,\mathrm{d}y \\
&= -2\frac{N}{\beta c\sqrt{N} b} e^{-\frac{\beta}{N}\frac{b}{2}t^2}\Big|_{t=c\sqrt{N}}^{\infty} \int_W e^{-\frac{\beta}{N}\left(\frac{1}{2}\langle (K_N+b)\cdot y, y\rangle\right)} \,\mathrm{d}y \leq 2\frac{\sqrt{N}}{\beta b}\frac{1}{c}\int_W e^{-\frac{\beta}{N}\left(\frac{1}{2}\langle (K_N+b)\cdot y, y\rangle\right)} \,\mathrm{d}y \\
&= 2\frac{1}{\sqrt{2\pi\beta b}}\frac{1}{c}\int e^{-\frac{\beta}{N}\frac{b}{2}t^2}\mathrm{d}t \int_W e^{-\frac{\beta}{N}\left(\frac{1}{2}\langle (K_N+b)\cdot y, y\rangle\right)} \,\mathrm{d}y \\
&\leq \sqrt{\frac{2}{\pi\beta b_0}}\frac{1}{c}\int e^{-\frac{\beta}{N}\left(\frac{b}{2}\sum_{k=1}^N x_k^2 + \frac{1}{2}\langle K_N \cdot x, x\rangle\right)} \,\mathrm{d}x \\
&< \frac{\epsilon}{2}\int e^{-\frac{\beta}{N}\left(\frac{b}{2}\sum_{k=1}^N x_k^2 + \frac{1}{2}\langle K_N \cdot x, x\rangle\right)} \,\mathrm{d}x,
\end{aligned}$$

for β big enough. This especially yields

$$\begin{aligned}
\int_{D^c} e^{-\frac{\beta}{N}\left(\frac{1}{2}\sum_{k=1}^N x_k^4 + \frac{b}{2}\sum_{k=1}^N x_k^2 + \frac{1}{2}\langle K_N \cdot x, x\rangle\right)} \,\mathrm{d}x &\leq \int_{D^c} e^{-\frac{\beta}{N}\left(\frac{b}{2}\sum_{k=1}^N x_k^2 + \frac{1}{2}\langle K_N \cdot x, x\rangle\right)} \,\mathrm{d}x \\
&< \frac{\epsilon}{2}\int e^{-\frac{\beta}{N}\left(\frac{b}{2}\sum_{k=1}^N x_k^2 + \frac{1}{2}\langle K_N \cdot x, x\rangle\right)} \,\mathrm{d}x.
\end{aligned}$$

Furthermore, using $|e^{-a} - e^{-b}| \leq |a-b|\cdot e^{-\min(a,b)}$,

$$\begin{aligned}
&\int e^{-\frac{\beta}{N}\left(\frac{b}{2}\sum_{k=1}^N x_k^2 + \frac{1}{2}\langle K_N \cdot x, x\rangle\right)} - e^{-\frac{\beta}{N}\left(\frac{1}{2}\sum_{k=1}^N x_k^4 + \frac{b}{2}\sum_{k=1}^N x_k^2 + \frac{1}{2}\langle K_N \cdot x, x\rangle\right)} \,\mathrm{d}x \\
&\leq \frac{\beta}{2}\int \frac{1}{N}\sum_{k=1}^N x_k^4 \, e^{-\frac{\beta}{N}\left(\frac{b}{2}\sum_{k=1}^N x_k^2 + \frac{1}{2}\langle K_N \cdot x, x\rangle\right)} \,\mathrm{d}x < \frac{\epsilon}{2}\cdot\int e^{-\frac{\beta}{N}\left(\frac{b}{2}\sum_{k=1}^N x_k^2 + \frac{1}{2}\langle K_N \cdot x, x\rangle\right)} \,\mathrm{d}x
\end{aligned}$$

for β. Observe that we have used Lemma 5.10 in the last inequality. Combining our estimates yields

$$\begin{aligned}
&\int_D e^{-\frac{\beta}{N}\left(\frac{1}{2}\sum_{k=1}^N x_k^4+\frac{b}{2}\sum_{k=1}^N x_k^2+\frac{1}{2}\langle K_N\cdot x,x\rangle\right)}\,\mathrm{d}x \geq \\
&\int e^{-\frac{\beta}{N}\left(\frac{b}{2}\sum_{k=1}^N x_k^2+\frac{1}{2}\langle K_N\cdot x,x\rangle\right)}\,\mathrm{d}x - \int_{D^c} e^{-\frac{\beta}{N}\left(\frac{1}{2}\sum_{k=1}^N x_k^4+\frac{b}{2}\sum_{k=1}^N x_k^2+\frac{1}{2}\langle K_N\cdot x,x\rangle\right)}\,\mathrm{d}x \\
&-\Big(\int e^{-\frac{\beta}{N}\left(\frac{b}{2}\sum_{k=1}^N x_k^2+\frac{1}{2}\langle K_N\cdot x,x\rangle\right)} - e^{-\frac{\beta}{N}\left(\frac{1}{2}\sum_{k=1}^N x_k^4+\frac{b}{2}\sum_{k=1}^N x_k^2+\frac{1}{2}\langle K_N\cdot x,x\rangle\right)}\,\mathrm{d}x\Big) \\
&\quad > \int e^{-\frac{\beta}{N}\left(\frac{b}{2}\sum_{k=1}^N x_k^2+\frac{1}{2}\langle K_N\cdot x,x\rangle\right)}\,\mathrm{d}x\cdot\left(1-\frac{\epsilon}{2}-\frac{\epsilon}{2}\right) \\
&\quad = (1-\epsilon)\cdot\left(\frac{N2\pi}{\beta}\right)^{\frac{N}{2}}\frac{1}{\det(K_N+b\cdot I)}.
\end{aligned}$$

□

Lemma 5.13 (Laplace asymptotic for V_N)**.** *Let $f_{\beta,N}:\mathbb{R}^N\to\mathbb{R}$ be a family of uniformly bounded functions and $V_N^\delta(x):=V_N(x)-\frac{1}{2N}\delta|x|^2$. Then we obtain for δ small enough*

$$\int f_{\beta,N}\ e^{-\beta V_N^\delta}\mathrm{d}x = \int_{\bar{x}\geq 0} f_{\beta,N}\ e^{-\frac{\beta}{2N}Q_N^r[x]}\mathrm{d}x + \int_{\bar{x}\leq 0} f_{\beta,N}\ e^{-\frac{\beta}{2N}Q_N^l[x]}\mathrm{d}x + o_{\beta,N}(1)\cdot\frac{\left(\frac{N2\pi}{\beta}\right)^{\frac{N}{2}}}{\det(K_N+2\cdot I)},$$

where we denote

$$\begin{aligned}
Q_N^r[x] &:= \langle (K_N+2\cdot I-\delta\cdot I)\cdot(x-(1,..,1)), x-(1,..,1)\rangle\,,\\
Q_N^l[x] &:= \langle (K_N+2\cdot I-\delta\cdot I)\cdot(x+(1,..,1)), x+(1,..,1)\rangle\,,
\end{aligned}$$

and suppress the δ dependence in the notation.

Proof. Let K be a bound for the family of functions. We then obtain

$$\begin{aligned}
&\Big|\int_{\bar{x}\geq 0} f_{\beta,N}\ e^{-\frac{\beta}{2N}Q_N^r[x]}\mathrm{d}x + \int_{\bar{x}\leq 0} f_{\beta,N}\ e^{-\frac{\beta}{2N}Q_N^l[x]}\mathrm{d}x - \int f_{\beta,N}\ e^{-\beta V_N^\delta}\mathrm{d}x\Big| \\
&\qquad \leq K\cdot\int_{\bar{x}\geq 0}|e^{-\beta V_N^\delta}-e^{-\frac{\beta}{2N}Q_N^r[x]}|\ \mathrm{d}x + K\cdot\int_{\bar{x}\leq 0}|e^{-\beta V_N^\delta}-e^{-\frac{\beta}{2N}Q_N^l[x]}|\ \mathrm{d}x.
\end{aligned}$$

Using $|e^{-a}-e^{-b}|\leq|a-b|\cdot e^{-\min(a,b)}$ and the following inequality, which follows from Lemma 3.6. in [10] for δ small enough,

$$V_N^\delta(x+(1,..,1)) \geq \frac{1}{2N}\langle (K_N-I)\cdot x,x\rangle + \frac{5}{8}\bar{x}^2 =: Q_N^0[x] \text{ for all } x \text{ with } \bar{x}\geq -1,$$

we obtain together with Lemma 5.10

$$
\begin{aligned}
\int_{\bar{x}\geq 0} |e^{-\beta V_N^\delta} - e^{-\frac{\beta}{2N} Q_N^r[x]}| \,\mathrm{d}x &\leq \int_{\bar{x}\geq 0} |V_N(x) - Q_N^r[x]| \; e^{-\frac{\beta}{2N} Q_N^0[x-(1,..,1)]} \mathrm{d}x \\
&\leq \int_{\mathbb{R}^N} \frac{\beta}{N} |\sum_{k=1}^N \frac{1}{4} x_k^4 + \sum_{k=1}^N x_k^3| \; e^{-\frac{\beta}{2N} Q_N^0[x]} \mathrm{d}x \leq o_{\beta,N}(1) \cdot \int \; e^{-\frac{\beta}{2N} Q_N^0[x]} \mathrm{d}x \\
&= o_{\beta,N}(1) \cdot \int \; e^{-\frac{\beta}{2N} Q_N^0[x]} \mathrm{d}x = o_{\beta,N}(1) \cdot \frac{\left(\frac{N 2\pi}{\beta}\right)^{\frac{N}{2}}}{\det(Q_N^0)} = o_{\beta,N}(1) \cdot \frac{\left(\frac{N 2\pi}{\beta}\right)^{\frac{N}{2}}}{\det(K_N + 2 \cdot I)}.
\end{aligned}
$$

Note that

$$
\det(K_N + 2 \cdot I) = C \cdot \det(Q_N^0).
$$

follows from Lemma 5.8, for a big enough constant C and arbitrary $N \in \mathbb{N}_{\geq 2}$. In the same way we can estimate the integral

$$
\int_{\bar{x}\leq 0} |e^{-\beta V_N^\delta} - e^{-\frac{\beta}{2N} Q_N^l[x]}| \,\mathrm{d}x.
$$

□

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