D I P L O M A R B E I T

Pricing Derivatives of American and Game Type in Incomplete Markets

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Introduction

During the last few years, various suggestions have been made how to price European-type contingent claims in incomplete markets. By contrast, there is only very little corresponding literature dealing with American options and in particular with Game options.

In the following I am describing two different pricing approaches for such contingent claims: utility maximization of terminal wealth and local utility maximization. The role of the arbitrage-free price in complete financial markets will now be played by the neutral derivative value. This is the unique price so that the speculators's optimal portfolio contains no contingent claims. In other words, this is the unique derivative price so that nobody can benefit from trading the contingent claim.

As we will see, American contingent claims can be treated as special cases of Game contingent claims. Moreover, all results lead to the well known simple pricing formulas for contingent claims in the special case of a complete financial market.

I am grateful to the supervisor of my work Walter Schachermayer, who has been able to give me useful advice. Special thanks go to Christoph Kühn for his clear explanations of the paper (Kallsen und Kuehn 2002) on which my 'Diplomarbeit' is based. Without his valuable help I would have needed much longer to understand central ideas of that paper.

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Chapter 1

Stochastic Calculus

This chapter is mainly based on (Jacod and Shiryaev 2003), (Kallsen 2002) and (Kallsen und Shiryaev 2002). In addition, we adopt the notation used therein throughout all other chapters.

1.1 The General Theory of Stochastic Processes

We assume that there is given a complete probability space $(\Omega, \mathcal{F}, P)$. In addition we are given a filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$. By a filtration we mean a family of sub-σ-algebras $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ of $\mathcal{F}$ that is increasing, i.e. $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ for $0 \leq s < t < \infty$.

Definition 1.1 *A filtered complete probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\geq 0})$ is said to satisfy the usual conditions if*

(i) $\mathcal{F}_0$ contains all the P-null sets of $\mathcal{F}$,

(ii) $\mathcal{F}_t = \mathcal{F}_{t+} := \cap_{u>t}\mathcal{F}_u$, for all $t \geq 0$; that is, the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ is right continuous.

We assume that the usual conditions are always satisfied.

Definition 1.2 *(i) A stochastic process X on $(\Omega, \mathcal{F}, P)$ is a collection of random variables $(X_t)_{t\in\mathbb{R}_+}$. The process X is said to be adapted if X_t is $\mathcal{F}_t$-measurable for every $t \in \mathbb{R}_+$.*

(ii) A process X is called càdlàg, for "continu à droite avec des limites à gauche" in French, if all its paths are right-continuous and admit left-hand limits.

Processes that are càdlàg are frequently called RCLL-processes, for right-continuous with left-hand limits processes. If X is càdlàg (or only left continuous) we define two other processes $X_- = (X_{t-})_{t\in\mathbb{R}_+}$ and $\Delta X = (\Delta X_t)_{t\in\mathbb{R}_+}$ by

$$\begin{cases} X_{0-} = X_0, \quad X_{t-} = \lim_{s\uparrow t} X_s \quad \text{for } t > 0 \\ \Delta X_t = X_t - X_{t-} \end{cases} \tag{1.1}$$

and hence $\Delta X_0 = 0$.

Remark 1.3 *This process is in generally not càdlàg, though it is adapted and for a.a.* ω, $t \longmapsto \triangle X_t = 0$ *except for at most countably many* t.

Proof. Assume that the set $J := \{t : \triangle X_t \neq 0\}$ contains more than countably many elements. Then there exists a $t_0 \in \mathbb{R}$ such that for every $\varepsilon > 0$ the set $(t_0 - \varepsilon, t_0 + \varepsilon) \cap J$ still contains more than countably many elements. But

$$(t_0 - \varepsilon, t_0 + \varepsilon) \cap J = \cup_{n \in \mathbb{N}} \{t \in (t_0 - \varepsilon, t_0 + \varepsilon) : |\Delta X_t| > \tfrac{1}{n}\}$$

which implies that there occur more than countably many jumps of size larger than $\frac{1}{n_0}$ in the interval $(t_0 - \varepsilon, t_0 + \varepsilon)$ for some $n_0 \in \mathbb{N}$. But this cannot be possible, since the process X is càdlàg and therefore admits right-hand and left-hand limits. ■

Definition 1.4 *(i) Two stochastic processes* X *and* Y *are called modifications if* $X_t = Y_t$ *a.s. for each* t, *i.e.*

$$P(X_t = Y_t) = 1 \quad \forall t \in \mathbb{R}_+$$

(ii) Two processes X *and* Y *are called indistinguishable if a.s., for all* t, $X_t = Y_t$, *i.e.*

$$P(X_t = Y_t; \forall t \in \mathbb{R}_+) = 1$$

Obviously two indistinguishable processes X and Y are also modifications.

Definition 1.5 *(i) A random set is a subset of* $\Omega \times \mathbb{R}_+$.

(ii) The predictable σ*-algebra is the* σ*-algebra* $\mathcal{P}$ *on* $\Omega \times \mathbb{R}_+$ *that is generated by all adapted, left-continuous processes, i.e. it is the smallest* σ*-algebra making all such processes measurable. A process or a random set that is* $\mathcal{P}$*-measurable is called predictable.*

(iii) The optional σ*-algebra is the* σ*-algebra* $\mathcal{O}$ *on* $\Omega \times \mathbb{R}_+$ *that is generated by all adapted, càdlàg processes. A process or a random set that is* $\mathcal{O}$*-measurable is called optional.*

(iv) A predictable (respectively optional) time is a mapping $T : \Omega \to \overline{\mathbb{R}}_+$ *such that the stochastic interval* $[0, T)$ *is predictable (respectively optional).*

(v) A càdlàg process X *is called quasi-left continuous if* $\triangle X_T = 0$ *a.s. on the set* $\{T < \infty\}$ *for every predictable time* T.

Theorem 1.6 *Every process* X *that is left-continuous and adapted is optional.*

Definition 1.7 *(i) A random variable* $T : (\Omega, \mathcal{F}) \to \mathbb{R}_+$ *is called a stopping time of the filtration* $(\mathcal{F}_t)_{t \geq 0}$, *if the event* $\{T \leq t\}$ *belongs to the* σ*-algebra* $\mathcal{F}_t$, *for every* $t \geq 0$.

(ii) If T *is a stopping time, we denote by* $\mathcal{F}_T$ *the collection of all sets* $A \in \mathcal{F}$ *such that* $A \cap \{T \leq t\} \in \mathcal{F}_t$ *for all* $t \in \mathbb{R}_+$.

In the following we shall consider exclusively real-valued processes $X = (X_t)_{t \in \mathbb{R}_+}$ on a probability space $(\Omega, \mathcal{F}, P)$, adapted to a given filtration $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ and such that $E(X_t) < \infty$ holds for every $t \geq 0$.

Definition 1.8 *An d-dimensional stochastic process X is said to be a submartingale (respectively, a supermartingale) if, for every $0 \le s \le t < \infty$, we have P-almost surely: $E(X_t|\mathcal{F}_s) \ge X_s$ (respectively, $E(X_t|\mathcal{F}_s) \le X_s$). We shall say that X is a martingale if it is both a submartingale and a supermartingale.*

Definition 1.9 *If $\mathcal{C}$ is a class of processes, we denote by $\mathcal{C}_{loc}$ the localized class, defined such: a process X belongs to $\mathcal{C}_{loc}$ if and only if there exists an increasing sequence of stopping times $(T_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} T_n = \infty$ a.s. and that $X_t^{T_n} := X_{t\wedge T_n}$ belongs to $\mathcal{C}$ for every $n \in \mathbb{N}$. The sequence $(T_n)_{n\in\mathbb{N}}$ is called a localizing sequence for X (relative to $\mathcal{C}$).*

Definition 1.10 *(i) We denote by $\mathcal{M}$ the class of all uniformly integrable martingales, i.e. the set of all martingales such that*

$$\lim_{\alpha\to\infty} \sup_{t\in\mathbb{R}_+} E\left(|X_t| \cdot \chi_{\{|X_t|>\alpha\}}\right) = 0$$

(ii) A stochastic process X is called a (continuous) local martingale if $X \in \mathcal{M}_{loc}$ (respectively, $X \in \mathcal{M}^c_{loc}$ if X is continuous).

(iii) $\mathcal{L}$ denotes the set of all local martingales M such that $M_0 = 0$.

(iv) We denote by $\mathcal{H}^2$ the set of all square-integrable martingales, that is of all martingales X such that $\sup_{t\in\mathbb{R}_+} E(X_t^2) < \infty$. By $\mathcal{H}^2_{loc}$ we denote the corresponding localized class called the class of locally square-integrable martingales.

Remark 1.11 *Every martingale is a local martingale due to the Optional Sampling Theorem (put $T_n = n$). However, the converse is generally not true even for local martingales which are uniformly integrable.*

Theorem 1.12 *Every local martingale that is a.s. lower bounded is a supermartingale.*

Proof. Let X be such a local martingale, i.e. there exists a $K \in \mathbb{R}$ such that $X_t \ge -K$ *a.s.* for all $t \in \mathbb{R}_+$ and a localizing sequence $(T_n)_{n\in\mathbb{N}}$. Then with Fatou's lemma it follows for $s > t$ (everything *a.s.*)

$$\begin{aligned} E(X_s|\mathcal{F}_t) &= E(\lim_{n\to\infty} X_{s\wedge T_n}|\mathcal{F}_t) = E(\underline{\lim}_{n\to\infty} X_{s\wedge T_n} + K|\mathcal{F}_t) - K \le \\ &\le \underline{\lim}_{n\to\infty} E((X_{s\wedge T_n} + K)(\chi_{\{T_n>t\}} + \chi_{\{T_n\le t\}})|\mathcal{F}_t) - K = \\ &= \underline{\lim}_{n\to\infty} \chi_{\{T_n\le t\}}(X_{s\wedge T_n} + K) + \\ &\quad + \underline{\lim}_{n\to\infty} \chi_{\{T_n>t\}} E(X_{s\wedge T_n} + K|\mathcal{F}_t) - K = \\ &= 0 + 1 \cdot (X_t + K) - K = X_t \end{aligned}$$

■

Lemma 1.13 $(\mathcal{M}_{loc})_{loc} = \mathcal{M}_{loc}$*, i.e. the class of local martingales is stable under localization.*

Since every martingale is a local martingale, it is a trivial conclusion that $\mathcal{M}_{loc}$ is also the localized class obtained from the class of martingales.

We say that a process X admits a terminal variable X_∞ if X_t converges *a.s.* to a limit X_∞ as $t \to \infty$. In such a case, the variable X_T is *a.s.* well defined for any stopping time $T : \Omega \to \overline{\mathbb{R}}_+$, with $X_T = X_\infty$ on $\{T = \infty\}$.

Lemma 1.14 *Let X be an adapted càdlàg process with terminal random variable X_∞. Then X is a uniformly integrable martingale if and only if for each stopping time $T : \Omega \to \overline{\mathbb{R}}_+$, the variable X_T is integrable and satisfies $E(X_T) = E(X_0)$.*

Proof.

- $\Longleftarrow$ First we note that X_∞ is integrable by hypothesis. Then if $t \in \mathbb{R}_+$ and $A \in \mathcal{F}_t$, we define the stopping time T by $T = t$ on A and $T = \infty$ on the complement A^c. We have $E(X_T) = E(X_t\chi_A) + E(X_\infty\chi_{A^c})$ and $E(X_\infty) = E(X_\infty\chi_A) + E(X_\infty\chi_{A^c})$. Our assumption implies that $E(X_T) = E(X_\infty)$, hence $E(X_t\chi_A) = E(X_\infty\chi_A)$ by difference. This being true for all $A \in \mathcal{F}_t$, it follows that $X_t = E(X_\infty|\mathcal{F}_t)$, i.e. that X is a martingale. We now have due to Jensen's Inequality

  $$|X_t| \cdot \chi_{\{|X_t|>\alpha\}} \leq E\left(|X_\infty| \cdot \chi_{\{|X_\infty|>\alpha\}}|\mathcal{F}_t\right), \tag{1.2}$$

  which yields

  $$\lim_{\alpha\to\infty} \sup_{t\in\mathbb{R}_+} E\left(|X_t| \cdot \chi_{\{|X_t|>\alpha\}}\right) \leq \lim_{\alpha\to\infty} E\left(|X_\infty| \cdot \chi_{\{|X_\infty|>\alpha\}}\right) = 0 \tag{1.3}$$

  and therefore the uniform integrability of X.

- $\Longrightarrow$ If X is a uniformly integrable martingale, then X_t converges $a.s.$ and in L^1 to a terminal variable X_∞, and $X_T = E(X_\infty|\mathcal{F}_T)$ for all stopping times T. This particularly implies that $E(X_T) = E(X_0) = E(X_\infty)$ for every stopping time T.

■

It follows that every martingale $(X_t)_{0\leq t\leq T}$ on a finite time horizon is a uniformly integrable martingale, since it admits a terminal variable $X_\infty = X_T$.

Definition 1.15 *A process X is of class (D) if the set of random variables $\{X_T : T$ finite-valued stopping time$\}$ is uniformly integrable.*

Proposition 1.16 *a) Each uniformly integrable martingale is a process of class (D).*

b) A local martingale is a uniformly integrable martingale if and only if it is a process of class (D).

Lemma 1.17 *Let X be a càdlàg process bounded from below and above by uniformly integrable martingales Y and Z, i.e. $Y \leq X \leq Z$. Then X is a process of class (D).*

Proof. Since $X = (X-Y)+Y$, we may assume that X is non-negative, because the sum of two uniformly integrable processes is again uniformly integrable. Then it follows that

$$E\left(|X_T| \cdot \chi_{\{|X_T|>\alpha\}}\right) \leq E\left(|Z_T| \cdot \chi_{\{|Z_T|>\alpha\}}\right) \leq E\left(|Z_\infty| \cdot \chi_{\{|Z_\infty|>\alpha\}}\right)$$

for every finite valued stopping time T due to Lemma 1.14. Since the right side of this equation does not depend on T it follows that we also have

$$\lim_{\alpha\to\infty} \sup_{T<\infty} E\left(|X_T| \cdot \chi_{\{|X_T|>\alpha\}}\right) \leq \lim_{\alpha\to\infty} E\left(|Z_\infty| \cdot \chi_{\{|Z_\infty|>\alpha\}}\right) = 0 \qquad (1.4)$$

which yields the assertion. ■

Before continuing with semimartingales and stochastic integration, we state a useful property about the nature of local martingales. First of all let us introduce an new definition.

Definition 1.18 *(i) Two local martingales M and N are called orthogonal if their product MN is a local martingale.*

(ii) A local martingale is called a purely discontinuous local martingale if $X_0 = 0$ and if it is orthogonal to all continuous local martingales.

However a purely discontinuous local martingale X is usually not the sum of its jumps. First of all, the series $\sum_{s\leq t} \Delta X_s$ usually diverges and even if it converges, its sum usually differs from X_t. For example $M_t = N_t - a(t)$ is a purely discontinuous local martingale if N is a Poisson process with (by definition continuous) intensity function $a(.)$. But it is indeed the case that $\sum_{s\leq t} \Delta M_s = N_t \neq M_t$. This sort of martingale is the prototype of all purely continuous local martingales and explains why in many places those are also called compensated sums of jumps.

Theorem 1.19 *Let $a > 0$. Any local martingale M admits a (non-unique) decomposition $M = M_0 + M' + M''$, where M' and M'' are local martingales with $M'_0 = M''_0 = 0$, M' has finite variation and $|\Delta M''| \leq a$ (which yields $M'' \in \mathcal{H}^2_{loc}$).*

Theorem 1.20 *Any local martingale M admits a unique (up to indistinguishability) decomposition*

$$M = M_0 + M^c + M^d$$

where $M^c_0 = M^d_0 = 0$, M^c is a continuous local martingale, and M^d is a purely discontinuous local martingale.

M^c is called the continuous part of M, and M^d its purely discontinuous part.

Theorem 1.21 *To each pair (M, N) of locally square integrable martingales one associates a predictable process $\langle M, N\rangle \in \mathcal{V}$, unique up to indistinguishability, such that $MN - \langle M, N\rangle$ is a local martingale. Moreover,*

$$\langle M, N\rangle = \tfrac{1}{4}(\langle M + N, M + N\rangle - \langle M - N, M - N\rangle).$$

The process $\langle M, N\rangle$ is called the predictable quadratic covariation, or the quadratic characteristic of the pair (M, N).

1.2 Semimartingales

Definition 1.22 *We denote by $\mathcal{V}^+$ (respectively $\mathcal{V}$) the set of all real-valued processes A that are càdlàg and adapted with $A_0 = 0$, and whose paths $t \longmapsto A_t(\omega)$ are non-decreasing (respectively are of finite variation over each finite interval $[0,t]$, for all $t \in \mathbb{R}_+$).*

Note that if $A \in \mathcal{V}^+$ it admits a terminal variable A_∞ that takes its values in $\overline{\mathbb{R}}_+$:

$$A_\infty = \lim_{t\to\infty} A_t.$$

We denote by $Var(A)$ the variation process of A, that is the process such that $Var(A)_t(\omega)$ is the total variation of the function $s \mapsto A_s(\omega)$ on the interval $[0,t]$. Of course, $Var(A) = A$ if $A \in \mathcal{V}^+$.

Definition 1.23 *(i) $\mathcal{A}^+$ is the set of all $A \in \mathcal{V}^+$ that are integrable: $E(A_\infty) < \infty$. $\mathcal{A}$ is the set of all $A \in \mathcal{V}$ that have integrable variation: $E(Var(A)_\infty) < \infty$.*

(ii) $\mathcal{A}^+_{loc}$ and $\mathcal{A}_{loc}$ are the localized classes constructed from $\mathcal{A}^+$ and $\mathcal{A}$. A process in $\mathcal{A}^+_{loc}$ (respectively $\mathcal{A}^+_{loc}$) is called a locally integrable adapted increasing process (respectively an adapted process with locally integrable variation).

Definition 1.24 *(i) A Semimartingale is a process X of the form $X = X_0 + M + A$ where X_0 is finite-valued and $\mathcal{F}_0$-measurable, where $M \in \mathcal{L}$, and where $A \in \mathcal{V}$. We denote by $\mathcal{S}$ the space of all semimartingales.*

(ii) A special semimartingale is a semimartingale X which admits a decomposition $X = X_0 + M + A$ as above, with A is predictable. We denote by $\mathcal{S}_p$ the set of all special martingales.

It is clear that $\mathcal{M}_{loc} \subset \mathcal{S}_p$ and that $\mathcal{V} \subset \mathcal{S}$. All semimartingales are càdlàg and adapted. The decomposition in 1.24 is unique if A is a predictable element of $\mathcal{V}$ and is therefore usually called the canonical decomposition of X.

Although it may not be quite apparent from the definition above, the space of semimartingales is a very pleasant space: it stays stable under a large variety of transformations: under stopping, under "absolutely continuous changes of probability measure", under "changes of filtration". Moreover, it is the largest possible class of processes with respect to which one may "reasonable integrate all locally bounded predictable processes.

The following two propositions furthermore characterize special semimartingales.

Proposition 1.25 *Let X be a semimartingale. There is equivalence between:*

a) X is a special semimartingale.

b) The process $Y_t = \sup_{s\le t} |X_s - X_0|$ belongs to $\mathcal{A}^+_{loc}$.

c) There exists a decomposition $X = X_0 + M + A$ where $A \in \mathcal{A}_{loc}$.

Proposition 1.26 *If a semimartingale X satisfies $|\triangle X| \leq a$, it is special and its canonical decomposition $X = X_0 + M + A$ satisfies $|\triangle A| \leq a$ and $|\triangle M| \leq 2a$ (in particular if X is continuous, then M and A are continuous).*

Proposition 1.27 *Let U, V be special semimartingales. If X is a semimartingale with $U \leq X \leq V$, then X is a special semimartingale as well.*

Proof. Since $X = (X - U) + U$, it suffices to consider the case $U = 0$. Let $B := \sum_{t \leq \cdot} \Delta X_t \chi_{\{|\Delta X_t| > 1\}}$ and $\tilde{X} := X - B$. By Proposition 1.26, $\tilde{X}$ is a special semimartingale. Moreover, since B is càdlàg it has pathwise only finitely many jumps on any finite interval. As V is a special semimartingale, we have with Proposition 1.25 that $\sup_{t \leq \cdot} |V_t - V_0| \in \mathcal{A}^+_{loc}$. Let $(T_n)_{n \in \mathbb{N}}$ be a sequence of stopping times with $T_n \to \infty$ P-almost surely such that $|\{t \leq T_n : |\Delta X_t| > 1\}| \leq n$ and $E(\sup_{t \leq T_n} |V_t - V_0|) < \infty$ and $V_0 \leq n$ on $\{T_n > 0\}$. From $|\Delta X| \leq \max(V, V_-) \leq V + V_-$ we can now conclude $Var(B)_{T_n} = \sum_{t \leq T_n} |\Delta X_t| \chi_{\{|\Delta X_t| > 1\}} \leq 2n \sup_{t \leq T_n} V_t$ for the variation process of B, which implies that $E(Var(B)_{T_n}) < \infty$ for any $n \in \mathbb{N}$. Therefore, $B \in \mathcal{A}^+_{loc}$ and hence it is a special semimartingale due to Proposition 1.25. ■

1.2.1 Stochastic Integration

Let again $A \in \mathcal{V}$. For each $\omega \in \Omega$, the path: $t \longmapsto A_t(\omega)$ is the distribution function of a signed measure (a positive measure if A is increasing) on $\mathbb{R}_+$, that is finite on each interval $[0, t]$, and that is finite on $\mathbb{R}_+$ if and only if $Var(A)_\infty(\omega) < \infty$. We denote this measure by $dA_t(\omega)$.

Theorem 1.28 *Let $A \in \mathcal{V}$ and H be an predictable process. Then $t \mapsto H_t(\omega)$ is Borel-measurable and hence we can define the integral process, denoted by $H \cdot A$ or by $\int_0^{\cdot} H_s dA_s$, as follows:*

$$H \cdot A_t(\omega) = \begin{cases} \int_0^t H_s(\omega) dA_s(\omega) & \textit{if } \int_0^t |H_s(\omega)| \, d[Var(A)]_s(\omega) < \infty \\ +\infty & \textit{otherwise.} \end{cases} \tag{1.5}$$

The problem is to define an integral process $H \cdot X$ when X does not belong to $\mathcal{V}$, but is only a semimartingale: hence the paths $X_{\cdot}(\omega)$ do not define a measure $dX_s(\omega)$ on $\mathbb{R}_+$ (for instance if X is Brownian motion, then almost all paths $t \mapsto X_t(\omega)$ have infinite variation over each finite interval).

When H is simple enough, it is very easy. More precisely, we denote by $\mathcal{E}$ the set of all processes of the form

$$\begin{cases} \text{either} & H = Y\chi_{\{0\}}, \ Y \text{ is bounded } \mathcal{F}_0\text{-measurable} \\ \text{or} & H = Y\chi_{(r,s]}, \ r < s, \ Y \text{ is bounded } \mathcal{F}_r\text{-measurable.} \end{cases} \tag{1.6}$$

For such an H, the integral process $H \cdot X_t = \int_0^t H_s dX_s = \int_{(0,t]} H_s dX_s$ has only one "natural" definition, namely:

$$H \cdot X_t = \begin{cases} 0 & \text{if } H = Y\chi_{\{0\}} \\ Y(X_{s \wedge t} - X_{r \wedge t}) & \text{if } H = Y\chi_{(r,s]}. \end{cases} \tag{1.7}$$

The following theorem gives the answer to the problem stated above and lists some useful properties of the stochastic integral.

Theorem 1.29 *Let X be a semimartingale. The map $H \mapsto H \cdot X$ defined on $\mathcal{E}$ by (1.7) has an extension, still denoted by $H \mapsto H \cdot X$ (and is called stochastic integral of H with respect to X), to the space of all locally bounded predictable processes H, with the following properties*

a) *(i) $H \cdot X$ is a semimartingale, in particular it is càdlàg and adapted.*
(ii) If X is a local martingale, then so is $H \cdot X$.
(iii) If $X \in \mathcal{V}$ then $H \cdot X \in \mathcal{V}$ and $H \cdot X$ coincides with the process defined in (1.5), which is sometime called the Stieltjes integral process.

b) *$H \mapsto H \cdot X$ is linear, i.e. $(aH + K) \cdot X$ and $aH \cdot X + K \cdot X$ are indistinguishable.*

c) *$\Delta(H \cdot X) = H \cdot \Delta X$.*

d) *$X^T = X_0 + \chi_{[0,T]} \cdot X$ and $(H \cdot X)^T = (H\chi_{[0,T]}) \cdot X$ for all stopping times T.*

e) *If a sequence $(H^n)_{n\in\mathbb{N}}$ of predictable stochastic processes converges pointwise to a limit H, and if $|H^n| \leq K$ where K is a locally bounded predictable process, then $(H^n \cdot X)_t \to (H \cdot X)_t$ in measure for all $t \in \mathbb{R}_+$.*

f) *The extension is unique in the sense, that if $H \mapsto \alpha(H)$ is another extension with the same properties, then $\alpha(H)$ and $H \cdot X$ are indistinguishable.*

Remark 1.30 *One could define the integral process by (1.5) for all H of the form (1.6) but without the measurability condition on Y, that is for simple processes that are not predictable. But the extension is essentially possible for predictable processes only. Similarly, formula (1.7) makes sense for every process X, semimartingale or not. But the extension is possible only when X is a semimartingale. As mentioned before, it is a fundamental result by Bichteler, Dellacherie and Mokobodzki, which explains why the space of semimartingales is so important.*

We could still enlarge the class of integrands to some non locally bounded processes. That is rather difficult for a semimartingale; but when $X \in \mathcal{H}^2_{loc}$ it is simple enough. We associate to all $X \in \mathcal{H}^2_{loc}$ the following classes of processes:

Definition 1.31 *We denote by $L^2(X)$ (respectively $L^2_{loc}(X)$) the set of all predictable processes H such that the process $H^2 \cdot \langle X, X\rangle$ is integrable (respectively locally integrable).*

Note that all locally bounded predictable processes belong to $L^2_{loc}(X)$, because we know that $\langle X, X\rangle \in \mathcal{A}^+_{loc}$.

We now want to define stochastic integrals for d-dimensional semimartingales X. The situation is rather simple when X has components in $\mathcal{V}$. In this case, we can find an increasing optional process F and an optional $\mathbb{R}^d$-valued process $a = (a^i)_{1\leq i\leq d}$ such that

$$X^i = a^i \cdot F \tag{1.8}$$

and further F and a may be chosen predictable if X is so. Then we set:

Definition 1.32 *$L^0(X)$ is the set of all d-dimensional predictable processes H such that the increasing process*

$$\left|\sum_i H^i a^i\right| \cdot F$$

is finite-valued. We then put

$$H \cdot X = \left(\sum_i H^i a^i\right) \cdot F. \tag{1.9}$$

Remark 1.33 *Here, neither the set $L^0(X)$ nor the integral process $H \cdot X$ depend on the choice of the pair (a, F) satisfying (1.8). Moreover, this integral definition coincides with the one from Theorem 1.29 on the set $\mathcal{V} \cap \mathcal{H}^2_{loc}$.*

Note, that even in this simple case we may have $H \in L^0(X)$ without each component H^i belonging to $L^0(H^i)$.

We are now ready for the general case: X is an arbitrary d-dimensional semimartingale, i.e. every component X^i is a semimartingale. Using Theorem 1.19 we can write it in the form

$$X = X_0 + M + A \quad M^i \in \mathcal{H}^2_{loc}, \quad A^i \in \mathcal{V}. \tag{1.10}$$

Definition 1.34 *We say that a d-dimensional predictable process H is integrable w.r.t. X, if there exists a decomposition 1.10 such that $H \in L^2_{loc}(M) \cap L^0(A)$, and in this case we define the integral process by*

$$H^T \cdot X = H \cdot M + H \cdot A. \tag{1.11}$$

We denote by $L(X)$ the set of all (predictable) integrable processes H.

Remark 1.35 *The stochastic integral process $H^T \cdot X$ above does not depend on the decomposition 1.10 due to Remark 1.33. Also, any locally bounded predictable process H belongs to $L(X)$, and in this case $H^T \cdot X$ can also be defined as $\sum_{i=1}^{d} H^i \cdot X^i$, i.e. as the natural extension of the stochastic integral introduced in Theorem 1.29. For a detailed description see e.g. (Jacod and Shiryaev 2003).*

1.2.2 Itô's formula

Definition 1.36 *The quadratic covariation of two semimartingales X and Y (the quadratic variation of X, when $Y = X$) is the following process:*

$$[X, Y] = XY - X_0 Y_0 - X_- \cdot Y - Y_- \cdot X \tag{1.12}$$

(it is defined uniquely, up to indistinguishability).

Proposition 1.37 *Let X be a semimartingale. There is a unique (up to indistinguishability) continuous local martingale X^c with $X^c_0 = 0$, such that any decomposition $X = X_0 + M + A$ in sense of Definition 1.24 meets $M^c = X^c$ (up to indistinguishability again).*

X^c is called the continuous martingale part of X.

Now we turn to Itô's formula. In the following $D_i f$ and $D_{ij} f$ denote the partial derivatives $\partial f/\partial x^i$ and $\partial^2 f/\partial x^i \partial x^j$.

Theorem 1.38 *Let $X = (X^1, \ldots, X^d)$ be a d-dimensional semimartingale, and $f \in C^2(\mathbb{R}^d)$. Then $f(X)$ is a semimartingale and we have:*

$$\begin{aligned} f(X_t) &= f(X_0) + \sum_{1\le i\le d} D_i f(X_-) \cdot X^i + \frac{1}{2} \sum_{1\le i,j\le d} D_{ij} f(X_-) \cdot \qquad (1.13) \\ &\quad \cdot \langle X^{i,c}, X^{j,c} \rangle + \sum_{0\le s\le t} \left(f(X_s) - f(X_{s-}) - \sum_{1\le i\le d} D_i f(X_{s-}) \Delta X_s^i \right). \end{aligned}$$

Of course, this formula implicitly means that all terms are well-defined. In particular the last two terms are processes with finite variation (the first one continuous, the second one "purely discontinuous").

One can even use time dependent functions f by applying Itô's formula to $\widetilde{X}_t := (X_t^1, \ldots, X_t^d, t)$.

1.3 Random Measures

Theorem 1.39 *Let $A \in \mathcal{A}_{loc}^+$. There is a process, called the compensator of A and denoted by A^p, which is unique up to indistinguishability, and which is characterized by being a predictable process in $\mathcal{A}_{loc}^+$ meeting any of the following equivalent statements:*

a) $A - A^p$ is a local martingale.

b) $E(A_T^p) = E(A_T)$ for all stopping times T.

Sometimes, A^p is called "predictable compensator" of A, or also "dual predictable projection" of A.

Let us now consider a fundamental example: point processes and the Poisson process. By definition, an adapted point process is a process $N \in \mathcal{V}^+$ that takes values in $\mathbb{N}$, and whose jumps are equal to 1 (i.e. the jump process ΔN takes only the values 0 and 1). If N_t is the number of "events" occurring in the interval $(0, t]$, this assumption means that two or more events cannot occur exactly at the same time. We can associate the following sequence of stopping times to the point process N:

$$T_n = \inf\{t : N_t = n\}. \qquad (1.14)$$

Note that $T_0 = 0$, that $T_n < T_{n+1}$ on the set $\{T_n < \infty\}$, and that $\lim_{n\to\infty} T_n = \infty$, since $N \in \mathcal{V}^+$. Conversely, the sequence (T_n) completely characterizes the process N, since we have

$$N_t = \sum_{n\ge 1} \chi_{[T_n,\infty)}. \qquad (1.15)$$

Finally we note that any adapted point process is locally bounded, because $N_{T_n} \le n$.

Definition 1.40 *(i) An extended Poisson process on $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\ge 0})$ is an adapted point process N such that* *a)* $E(N_t) < \infty$ *for each* $t \in \mathbb{R}_+$,

b) $N_t - N_s$ *is independent of the* σ*-algebra* $\mathcal{F}_s$ *for all* $0 \leq s < t$.

(ii) The function $a(t) = E(N_t)$ *is called the intensity of* N*. If this function is continuous, we say that* N *is a Poisson process. If this function is* $a(t) = t$*, we say that* N *is a standard Poisson process.*

Proposition 1.41 *Let* N *be an be an extended Poisson process on* $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\geq 0})$ *with intensity* $a(.)$*. Then the compensator of* N *is* $N_t^p = a(t)$.

Proof. The definition of an extended Poisson process immediately yields $E(N_t - N_s|\mathcal{F}_s) = a(t) - a(s)$ for $s \leq t$. Hence $X_t = N_t - a(t)$ is a martingale and $A_t = a(t)$ is a predictable (because deterministic) process in $\mathcal{A}_{loc}^+$, which yields the assertion. ∎

Definition 1.42 *A random measure on* $\mathbb{R}_+ \times \mathbb{R}^d$ *is a family* $\mu = (\mu(\omega; dt, dx) : \omega \in \Omega)$ *of nonnegative measures on* $(\mathbb{R}_+ \times \mathbb{R}^d, \mathcal{B}_+ \otimes \mathcal{B}^d)$ *satisfying* $\mu(\omega; \{0\} \times \mathbb{R}^d) = 0$ *identically.*

We put $\widetilde{\Omega} = \Omega \times \mathbb{R}_+ \times \mathbb{R}^d$, with the σ-fields $\widetilde{\mathcal{O}} = \mathcal{O} \otimes \mathcal{B}^d$ and $\widetilde{\mathcal{P}} = \mathcal{P} \otimes \mathcal{B}^d$. A function W on $\widetilde{\Omega}$ that is $\widetilde{\mathcal{O}}$-measurable (respectively $\widetilde{\mathcal{P}}$-measurable) is called an optional (respectively a predictable) function.

Let μ be a random measure and W an optional function on $\widetilde{\Omega}$. Since $(t, x) \mapsto W(\omega, t, x)$ is $\mathcal{B}_+ \otimes \mathcal{B}^d$-measurable for each $\omega \in \Omega$, we can define the integral process $W * \mu$ by

$$W * \mu_t(\omega) = \\ = \begin{cases} \int\limits_{[0,t]\times\mathbb{R}^d} W(\omega, s, x)\mu(\omega; ds, dx) & \text{if } \int\limits_{[0,t]\times\mathbb{R}^d} |W(\omega, s, x)|\, \mu(\omega; ds, dx) < \infty \\ +\infty & \text{otherwise.} \end{cases}$$

Example 1.43 *Let* $W(s, x)ds$ *denote the infinitesimal amount of dividend paid by a stock at time* s *if the stock price is* x*. If* μ *is the probability measure associated to the stock price development, then* $W * \mu_t$ *denotes the amount of dividend paid by the stock up to time* t*. Moreover, if* W *depends on* ω*, then dividend payments are even path-dependent. Note that this is still a trivial example, since the restriction of* μ *onto* (ω, s)*, namely* $\mu(\omega, s, dx)$ *is always the Dirac measure at some point* $x \in \mathbb{R}^d$.

Definition 1.44 *(i) A random measure* μ *is called optional (respectively predictable) if the process* $W * \mu$ *is optional (respectively predictable) for every optional (respectively predictable) function* W.

(ii) An optional measure μ *is called integrable if the random variable* $\mathbf{1} * \mu_\infty = \mu(., \mathbb{R}^+ \times \mathbb{R}^d)$ *is integrable (or equivalently, if* $\mathbf{1} * \mu \in \mathcal{A}^+$*).*

(iii) An optional random measure μ *is called* $\widetilde{\mathcal{P}}$*-*σ*-finite if there exists a strictly positive predictable function* V *on* $\widetilde{\Omega}$ *such that the random variable* $V * \mu_\infty$ *is integrable (or equivalently,* $V * \mu \in \mathcal{A}^+$*).*

The following result is a generalization of Theorem (1.39).

Theorem 1.45 *Let μ be an optional $\widetilde{\mathcal{P}}$-σ-finite random measure. There exists a random measure, called the compensator of μ and denoted by μ^p, which is unique up to a P-null set, and which is characterized as being a predictable random measure satisfying one of the following equivalent properties:*

a) $E(W * \mu^p_\infty) = E(W * \mu_\infty)$ for every nonnegative $\widetilde{\mathcal{P}}$-measurable function W on $\widetilde{\Omega}$.

b) For every $\widetilde{\mathcal{P}}$-measurable function W on $\widetilde{\Omega}$ such that $|W| * \mu \in \mathcal{A}^+_{loc}$, then $|W| * \mu^p$ belongs to $\mathcal{A}^+_{loc}$, and $W * \mu^p$ is the compensator of the process $W * \mu$ (or equivalently, $W * \mu - W * \mu^p$ is a local martingale).

Sometimes μ^p is also called "predictable compensator", or "dual predictable projection", of μ.

Proposition 1.46 *Let X be an adapted càdlàg $\mathbb{R}^d$-valued process. Then*

$$\mu^X(\omega, dt, dx) = \sum_s \chi_{\{\Delta X_s(\omega)\neq 0\}} \varepsilon_{(s,\Delta X_s(\omega))}(dt, dx)$$

defines an integer-valued random measure on $\mathbb{R}_+ \times \mathbb{R}^d$, where ε_a denotes the Dirac measure at point a.

Such an integer-valued random measure may be considered as the "counting measure" associated to a random point process in $\mathbb{R}_+ \times \mathbb{R}^d$, whose jumps are characterized by the pairs $(T_n, \Delta X_{T_n})(\omega)$, where T_n is defined recursively via $T_0 = 0$ and $T_{n+1} = \inf\{t > T_n : \Delta X_t \neq 0\}$.

We now want to construct a stochastic integral with respect to a "compensated" integer-valued random measure. For this we associate to any measurable function W on $\widetilde{\Omega}$ the process

$$\widehat{W}_t(\omega) = \begin{cases} \int_{\mathbb{R}^d} W(\omega, t, x)\upsilon(\omega; \{t\} \times dx) & \text{if } \int_{\mathbb{R}^d} |W(\omega, t, x)|\,\upsilon(\omega; \{t\} \times dx) < \infty \\ +\infty & \text{otherwise} \end{cases}$$

where $\upsilon = \upsilon^X = \mu^p$ is an appropriate version of the compensator of the random measure μ.

Definition 1.47 *(i) We denote by $G_{loc}(\mu)$ the set of all $\widetilde{\mathcal{P}}$-measurable real-valued functions W on $\widetilde{\Omega}$ such that the process $\widetilde{W}_t(\omega) = W(\omega, t, \Delta X_t(\omega))$ $\chi_{\{\Delta X_s(\omega)\neq 0\}}(\omega, t) - \widehat{W}_t(\omega)$ satisfies $[\sum_{s\leq\cdot} (\widetilde{W}_s)^2]^{1/2} \in \mathcal{A}^+_{loc}$.*

(ii) If $W \in G_{loc}(\mu)$ we call stochastic integral of W with respect to $\mu - \upsilon$ and we denote by $W * (\mu - \upsilon)$ any purely discontinuous local martingale such that ΔX and $\widetilde{W}$ are indistinguishable.

Remark 1.48 *This definition actually makes sense - for details see (Jacod and Shiryaev 2003) - and the mapping $W \mapsto W*(\mu-\upsilon)$ is linear up to indistinguishability. Moreover, if W is a predictable function on $\widetilde{\Omega}$, such that $|W| * \mu \in \mathcal{A}^+_{loc}$ (or equivalently $|W| * \upsilon \in \mathcal{A}^+_{loc}$ according to Theorem 1.45) then $W \in G_{loc}(\mu)$ and*

$$W * (\mu - \upsilon) = W * \mu - W * \upsilon.$$

1.4 Exponential Semimartingales

Now we consider the equation

$$Y = 1 + Y_- \cdot X \quad \text{(or equivalently: } dY = Y_- dX \text{ and } Y_0 = 1) \tag{1.16}$$

where X is a given semimartingale, and Y is an unknown càdlàg process. By analogy with the ordinary differential equation $\frac{dy}{dx} = y$, we will call the solution Y the exponential of X. There exits an useful characterization of the solutions from equation (1.16) above.

Theorem 1.49 *Let X be a semimartingale. Then equation (1.16) has one and only one (up to indistinguishability) càdlàg adapted solution. This solution is a semimartingale, is denoted by $\mathcal{E}(X)$, and is given by*

$$\mathcal{E}(X)_t = \exp(X_t - X_0 - \tfrac{1}{2}\langle X^c, X^c\rangle_t) \prod_{s \le t} (1 + \Delta X_s) e^{-\Delta X_s} \tag{1.17}$$

where the (possibly) infinite product is absolutely convergent. Furthermore,

a) If X has finite variation, then so has $\mathcal{E}(X)$.

b) If X is a local martingale, then so is $\mathcal{E}(X)$.

c) Let $T := \inf(t : \Delta X_t = -1)$. Then $\mathcal{E}(X) \neq 0$ on the interval $[0, T)$, and $\mathcal{E}(X)_- \neq 0$ on the interval $[0, T]$, and $\mathcal{E}(X) = 0$ on the interval $[T, \infty)$.

Example 1.50 *Geometric Brownian motion H is the exponential of standard Brownian motion B. According to Theorem 1.49, H is given by*

$$dH_t = H_t dB_t$$

or explicitly by

$$H_t = \mathcal{E}(B)_t = \exp(B_t - \tfrac{1}{2}\langle B, B\rangle_t) = e^{B_t - \frac{1}{2}t}. \tag{1.18}$$

The mapping $X \mapsto \mathcal{E}(X)$ can be inverted. In analogy to real calculus, we call its converse $\mathcal{L}(X)$ the stochastic logarithm of X. Moreover, for any real-valued semimartingale X with $X_0 = 0$ and $\Delta X > -1$, we call $\widetilde{X} := \log(\mathcal{E}(X))$ the logarithmic transform of X.

Definition 1.51 *Let X be a real-valued semimartingale. X is called exponentially special if $\exp(X - X_0)$ is a special semimartingale.*

Definition 1.52 *Let X be a real-valued semimartingale. A predictable process $V \in \mathcal{V}$ is called exponential compensator of X if $\exp(X - X_0 - V) \in \mathcal{M}_{loc}$, i.e. if it is a local martingale.*

Put differently, we decompose $\exp(X - X_0) = MU$ where $M \in \mathcal{M}_{loc}$ and $U = \exp(V)$ is a positive predictable process of finite variation. It can be shown that such a decomposition exists if and only if $\exp(X - X_0)$ is a special semimartingale:

Lemma 1.53 *A real-valued semimartingale X has an exponential compensator if and only if it is exponentially special. In this case, the exponential compensator is up to indistinguishability unique.*

Definition 1.54 *(i) Let $\varphi \in L^1(X)$ (see Definition 5.1) such that $\varphi^T \cdot X$ is exponentially special. The Laplace cumulant process $\tilde{K}^X(\varphi)$ of X in φ is defined as the compensator of the special semimartingale $(\varphi^T \cdot X)\tilde{} := \mathcal{L}(\exp(\varphi^T \cdot X))$. For $\varphi = 1$ we write $\tilde{K}^X := \tilde{K}^X(1)$.*

(ii) The modified Laplace cumulant process $K^X(\varphi)$ of X in φ is the logarithmic transform of $\tilde{K}^X(\varphi)$, i.e. $K^X(\varphi) := \log(\mathcal{E}(\tilde{K}^X(\varphi)))$. For $\varphi = 1$ we write $K^X := K^X(1)$.

Theorem 1.55 *Let $\varphi \in L^1(X)$ such that $\varphi^T \cdot X$ is exponentially special. Then $K^X(\varphi)$ is the exponential compensator of $\varphi^T \cdot X$. More specifically,*

$$\begin{aligned} Z \quad : &= \exp(\varphi^T \cdot X - K^X(\varphi)) \qquad (1.19)\\ &= \frac{\exp(\varphi^T \cdot X)}{\mathcal{E}(\tilde{K}^X(\varphi))} \\ &= \mathcal{E}\left(\varphi^T \cdot X^c + \frac{e^{\varphi^T x} - 1}{1 + \hat{W}(\varphi)} * (\mu^X - \nu)\right) \in \mathcal{M}_{loc}, \end{aligned}$$

where $\hat{W}(\varphi)_t := \int (e^{\varphi^T x} - 1)\nu(\{t\} \times dx)$.

The proof of this theorem can be found (Kallsen und Shiryaev 2002).

1.5 Characteristics of Semimartingales

The notion of "characteristics" of a semimartingale is designed to replace (or rather, to extend) the three terms: drift, variance of the Gaussian part and Lévy measure, that characterize the distribution of a process with independent increments, also called Lévy processes.

In the following, we consider a d-dimensional semimartingale $X = (X^1, \ldots, X^d)$ and we write $X \in \mathcal{S}^d$.

Definition 1.56 *A transition kernel $F(a, db)$ of a measurable space $(A, \mathcal{A})$ into another measurable space $(B, \mathcal{B})$ is a family $(F(a, .) : a \in A)$ of positive measures on $(B, \mathcal{B})$, such that $F(., C)$ is $\mathcal{A}$-measurable for each $C \in \mathcal{B}$.*

Definition 1.57 *We call $\mathcal{C}_t^d$ (for truncation function) the class of all functions $h : \mathbb{R}^d \to \mathbb{R}^d$ which are bounded, with compact support, and satisfy $h(x) = x$ in a neighborhood of 0.*

Let $h \in \mathcal{C}_t^d$. Then $\triangle X_s - h(\triangle X_s) \neq 0$ only if $|\triangle X_s| > b$ for some $b > 0$ and the following formulae

$$\begin{aligned} \check{X}(h)_t \quad : &= \sum_{s \le t} [\triangle X_s - h(\triangle X_s)] \qquad (1.20)\\ X(h) \quad : &= X - \check{X}(h) \end{aligned}$$

define a d-dimensional process $\check{X}(h)$ in $\mathcal{V}^d$ (i.e. its components are in $\mathcal{V}$) and a d-dimensional semimartingale $X(h)$. Moreover $\triangle X(h) = h(\triangle X)$, which is bounded and hence by Lemma (1.28) $X(h)$ is a special semimartingale (i.e. its components are in $\mathcal{S}_p$) and we consider its canonical decomposition

$$X(h) = X_0 + M(h) + B(h), \quad M(h) \in \mathcal{L}^d, \quad B(h) \text{ predictable in } \mathcal{V}^d. \tag{1.21}$$

Definition 1.58 *Let $h \in \mathcal{C}_t^d$ be fixed. We call characteristics of X the triplet (B, C, υ) consisting in:*

(i) $B = (B^i)_{1 \le i \le d}$ is a predictable process in $\mathcal{V}^d$, namely the process $B = B(h)$ appearing in (1.21).

(ii) $C = (C^{ij})_{1 \le i,j \le d}$ is a continuous process in $\mathcal{V}^{d \times d}$, namely

$$C^{ij} = \left\langle X^{i,c}, X^{j,c} \right\rangle \tag{1.22}$$

where X^c is the continuous martingale part of X.

(iii) υ is a predictable random measure on $\mathbb{R}_+ \times \mathbb{R}^d$, namely the compensator of the random measure μ^X associated to the jumps of X.

Remark 1.59 *We see that C and υ do not depend on the choice of the function h, while $B = B(h)$ does. From Definition 1.58, the characteristics are unique up to a P-null set (because the decomposition (1.21), as well as X^c and the bracket $\left\langle X^{i,c}, X^{j,c} \right\rangle$ and the compensator of μ^X themselves are unique up to a null set only). Nevertheless this allows for a good version of the characteristics.*

Proposition 1.60 *One can find a version of the characteristics (B, C, υ) of X which is of the form*

$$\begin{aligned} B^i &= b^i \cdot A \\ C^{ij} &= c^{ij} \cdot A \\ \upsilon(\omega; dt, dx) &= dA_t(\omega) F_{\omega,t}(dx) \end{aligned} \tag{1.23}$$

where:

a) A is a predictable process in $\mathcal{A}_{loc}^+$ which may be chosen continuous if and only if X is quasi-left-continuous,

b) $b = (b^i)_{1 \le i \le d}$ is a d-dimensional predictable process,

c) $c = (c^{ij})_{1 \le i,j \le d}$ is a predictable process with values in the set of all symmetric nonnegative $d \times d$ matrices,

d) $F_{\omega,t}(dx)$ is a transition kernel from $(\Omega \times \mathbb{R}_+, \mathcal{P})$ into $(\mathbb{R}^d, \mathcal{B}^d)$ which satisfies

$$\begin{gathered} F_{\omega,t}(\{0\}) = 0, \quad \int F_{\omega,t}(dx)(|x|^2 + 1) \le 1 \\ \triangle A_t(\omega) > 0 \implies b_t(\omega) = \textstyle\int F_{\omega,t}(dx) h(x) \\ \triangle A_t(\omega) F_{\omega,t}(\mathbb{R}^d) \le 1 \end{gathered} \tag{1.24}$$

It also follows from c) and (1.24) above that this "good" version of (B, C, υ) satisfies identically

$$s \leq t \implies (C_t^{ij} - C_s^{ij})_{1\leq i,j\leq d} \text{ is a symmetric nonnegative matrix;}$$
$$((|x|^2 + 1) * \upsilon \in \mathcal{A}_{loc};\tag{1.25}$$
$$\triangle B_t = \int h(x)\upsilon(\{t\} \times dx).$$

We usually call (b, c, F, A) differential characteristics of X.

Proposition 1.61 *Let $h, h' \in \mathcal{C}_t^d$. Then, up to indistinguishability,*

$$B(h) - B(h') = (h - h') * \nu \tag{1.26}$$

Remark 1.62 *This in particular implies for the trivial truncation function $h'(x) = x$, that $B(h) = B+(h(x)-x)*\nu$ or equivalently $b(h) = b+(h(x)-x)\cdot F$.*

Proposition 1.63 *Let X be a special semimartingale with characteristics $(B(h), C, \nu)$ relative to $h \in \mathcal{C}_t^d$. Then the canonical decomposition $X = X_0 + N + A$ satisfies:*

$$\begin{aligned} A = &\quad B(h) + (x - h(x)) * \nu \\ \Delta A = &\quad \int x\nu(\{t\} \times dx). \end{aligned}\tag{1.27}$$

In the following we set for each stopping time T

$$P_T = \text{restriction of } P \text{ to } \mathcal{F}_T.$$

Definition 1.64 *Let P and P' denote two measures:*

(i) We say that P' is locally absolutely continuous with respect to P, and we write $P' \overset{loc}{\ll} P$, if $P'_t \ll P_t$ for all $t \in \mathbb{R}$.

(ii) Let μ be an random measure on $\mathbb{R}_+ \times \mathbb{R}^d$. The M_μ^P is the positive measure on $(\widetilde{\Omega}, \mathcal{F} \otimes \mathcal{R}_+ \otimes \mathcal{B}^d)$ defined by $M_\mu^P(W) = E(W * \mu_\infty)$ for all measurable nonnegative functions W.

(iii) Assume that the restriction of the measure M_μ^P to $(\widetilde{\Omega}, \widetilde{\mathcal{P}})$ is σ-finite. Then for every nonnegative measurable function W the "conditional expectation" $W' = M_\mu^P(W|\widetilde{\mathcal{P}})$ is the M_μ^P-a.e. unique $\widetilde{\mathcal{P}}$-measurable function such that

$$M_\mu^P(WU) = M_\mu^P(W'U) \quad \text{for all nonnegative } \widetilde{\mathcal{P}}\text{-measurable } U.$$

Proposition 1.65 *Assume that $P' \overset{loc}{\ll} P$ and let Z be the density process. Let M' be a càdlàg adapted process. Then $M'Z$ is a P-martingale if and only if M' is a P'-martingale.*

Proof. Let $A \in \mathcal{F}_t$. Then $E_{P'}(\chi_A M'_t) = E_P(\chi_A Z_t M'_t)$. Therefore $E_{P'}(M'_t - M'_s|\mathcal{F}_s) = 0$ for $s \leq t$ if and only if $E_P(Z_t M_t - Z_s M_s|\mathcal{F}_s) = 0$ and the equivalence follows. ■

Theorem 1.66 (Girsanov) *Assume that $P' \overset{loc}{\ll} P$, and let X be a d-dimensional semimartingale with characteristics (B, C, ν) relative to a given truncation function h. Then there exists a $\widetilde{\mathcal{P}}$-measurable nonnegative function Y and a predictable process $\beta = (\beta^i)_{1 \le i \le d}$ satisfying*

$$|h(x)(Y-1)| * \nu_t < \infty \quad P'\text{-a.s. for } t \in \mathbb{R}_+ \tag{1.28}$$

$$\left| \sum_{1 \le j \le d} c^{ij} \beta^j \right| \cdot A_t < \infty \text{ and } \left(\sum_{1 \le j,k \le d} \beta^j c^{jk} \beta^k \right) \cdot A_t < \infty \quad P'\text{-a.s. for } t \in \mathbb{R}_+ \tag{1.29}$$

and such that a version of the characteristics of X relative to P' are

$$\begin{aligned} B'^i = {} & B^i + \left(\sum_{1 \le j \le d} c^{ij} \beta^j \right) \cdot A + h^i(x)(Y-1) * \nu \\ C' = {} & C \\ \nu' = {} & Y \cdot \nu. \end{aligned} \tag{1.30}$$

Moreover, Y and β meet all the above conditions, if and only if

$$\begin{aligned} Y Z_- = {} & M^P_{\mu^X}(X|\widetilde{\mathcal{P}}) \\ \langle Z^c, X^{i,c} \rangle = {} & \left(\sum_{1 \le j \le d} c^{ij} \beta^j \right) \cdot A_t, \end{aligned} \tag{1.31}$$

(up to a P-null set, of course), where Z is the density process, Z^c is its continuous martingale part relative to P, and $\langle Z^c, X^{i,c} \rangle$ is the bracket relative to P.

Lemma 1.67 *Let X be a semimartingale with characteristics (B, C, ν) relative to some truncation function $h \in \mathcal{C}_t^d$ and differential characteristics (b, c, F, A). Furthermore, let $H \in \mathcal{L}^1(X)$ in sense of Definition 5.1. Then then the characteristics $(\widetilde{B}, \widetilde{C}, \widetilde{\nu})$ of $H^T \cdot X$ relative to some truncation function $h_1 \in \mathcal{C}_t^1$ are of the form*

$$\widetilde{B} = \widetilde{b} \cdot A, \quad \widetilde{C} = \widetilde{c} \cdot A, \quad \widetilde{\nu} = A \cdot \widetilde{F}, \tag{1.32}$$

where

$$\begin{aligned} \widetilde{b}_t = {} & H_t^T b_t + \int (h_1(H_t^T x) - H_t^T h(x)) F_t(dx), \\ \widetilde{c}_t = {} & H_t^T c_t H_t, \\ \widetilde{F}_t(G) = {} & \int \chi_G(H_t^T x) F_t(dx) \text{ for any } G \in \mathcal{B} \text{ with } 0 \notin G. \end{aligned} \tag{1.33}$$

Example 1.68 *An important class of semimartingales are the Lévy processes. These are d-dimensional semimartingales with differential characteristics (b, c, F, t), with $b \in \mathbb{R}^d$ and positive definite $c \in \mathbb{R}^{d \times d}$. Therefore, they are usually characterized by the so called Lévy-Chintschin triplet (b, c, F) only. Lévy processes have independent stationary increments, i.e. for such a process X the distribution of $X_t - X_s$ $(0 \le s \le t)$ only depends on the difference $t - s$ and is independent of the σ-field $\mathcal{F}_s$. Many well known processes are indeed Lévy processes, e.g.*

(i) linear functions, $(b, c, F) = (b, 0, 0)$

(ii) standard d-dimensional Brownian motion, $(b, c, F) = (0, I_d, 0)$

(iii) the standard Poisson process, $(b, c, F) = (0, 0, \varepsilon_1)$.

1.6 σ-Localization

The concept of σ-localization is a generalization of localization in the general theory of stochastic processes. For any semimartingale X and any predictable set $D \subseteq \Omega \times \mathbb{R}_+$, we write $X^D := X_0\chi_D(0) + \chi_D \cdot X$, where $\chi_D(0)(\omega) := \chi_D((\omega, 0))$ for $\omega \in \Omega$. In particular, we have $X^{[0,T]} = X^T$ for any stopping time T (see Theorem 1.29).

Definition 1.69 *For any class $\mathcal{C}$ of semimartingales we define the σ-localized class $\mathcal{C}_\sigma$ as follows: A process X belongs to $\mathcal{C}_\sigma$ if and only if there exists an increasing sequence $(D_n)_{n\in\mathbb{N}}$ of predictable sets such that $D_n \uparrow \Omega \times \mathbb{R}_+$ up to a nullset and $X^{D_n} \in \mathcal{C}$ for any $n \in \mathbb{N}$.*

This is obviously a generalization of the classical localization procedure, because if $(\tau_n)_{n\in\mathbb{N}}$ is a localizing sequence, then $D_n := [0, \tau_n]$ is the corresponding σ-localizing sequence. We therefore have $\mathcal{C} \subseteq \mathcal{C}_{loc} \subseteq \mathcal{C}_\sigma$ for every class $\mathcal{C}$ of semimartingales.

Definition 1.70 *The class of σ-martingales (respectively σ-supermartingales and σ-submartingales) is the class of processes obtained via σ-localization from the class of martingales (respectively supermartingales and submartingales).*

The following two results are due to (Goll and Kallsen 2001):

Lemma 1.71 *Let X be a semimartingale in $\mathbb{R}^d$ with differential characteristics (b, c, F, A). Fix $i \in \{1, \dots, d\}$. Then X^i is a σ-supermartingale if and only if $\int \left|x^i - h^i(x)\right| F(dx) < \infty$ and*

$$b^i + \int (x^i - h^i(x))F(dx) \leq 0 \tag{1.34}$$

$(P \otimes A)$-almost everywhere. If we replace ≤ 0 with $= 0$ or ≥ 0, we obtain corresponding statements for σ-martingales and σ-submartingales, respectively.

Remark 1.72 *Thus, a negative drift for semimartingale X does not automatically mean that it is a already a supermartingale or local supermartingale, but only a σ-martingale. Nevertheless, the following statement gives a sufficient condition when this is indeed the case.*

Proposition 1.73 *Let X be a non-negative σ-supermartingale with $E(X_0) < \infty$. Then X is a supermartingale.*

Definition 1.74 *For any real-valued semimartingale X we define*

$$\begin{aligned} \|X\|_{\mathcal{H}^1} := \quad & \inf\{E(|X_0| + Var(A)_\infty + \sqrt{[M, M]_\infty}): \\ & X = X_0 + M + A \text{ with } M \in M_{loc},\, A \in \mathcal{V}\}, \end{aligned} \tag{1.35}$$

where $Var(A)$ denotes the variation process of A. By $\mathcal{H}^1$ we denote the set of all real-valued semimartingales with $\|X\|_{\mathcal{H}^1} < \infty$.

Proposition 1.75 *Let L, X, U be real-valued semimartingales with $L \leq X \leq U$ and such that $\chi_{\{L_- < X_-\}} \cdot X$ is a σ-submartingale and $\chi_{\{X_- < U_-\}} \cdot X$ is a σ-supermartingale. Then*

$$\|X\|_{\mathcal{H}^1} \leq c\,(\|L\|_{\mathcal{H}^1} + \|U\|_{\mathcal{H}^1}) \tag{1.36}$$

for some $c \in \mathbb{R}_+$ which is independent of L, X, U.

The proof of this statement can be found in (**?**)

Proposition 1.76 *Let X be an adapted real-valued process and $(T_n)_{n\in\mathbb{N}}$ an increasing sequence of stopping times such that X^{T_n} is a semimartingale for any $n \in \mathbb{N}$. If we have $sup_{n\in\mathbb{N}} \left\|X^{T_n}\right\|_{\mathcal{H}^1} < \infty$, then X^{T_∞} is a semimartingale, where $T_\infty := \sup_{n\in\mathbb{N}} T_n$.*

Proof. Since $sup_{n\in\mathbb{N}} \left\|X^{T_n}\right\|_{\mathcal{H}^1} = \lim_{n\to\infty} \left\|X^{T_n}\right\|_{\mathcal{H}^1} < \infty$ we see that $(X^{T_n})_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{H}^1$. Due to completeness - see (Dellacherie and Meyer 1982), $VII.98$ - there is a limit in $\mathcal{H}^1$ which coincides with X on the set $[0, T_n]$ for all $n \in \mathbb{N}$ and therefore also on the set $[0, T_\infty)$. ■

Chapter 2

Brownian Motion - A Special Case

The following chapter is mainly based on (Øksendal 1998) and (Karatzas and Shreve 1998). Here we will restrict ourself to the special case of stochastic integration with respect to Brownian motion. In view of the preceding chapter, this may seem redundant, but before turning our focus to pricing derivatives in incomplete markets, it is mandatory to understand basic principles of complete markets first. Without this knowledge it is likely to loose track of the distinctive features making up incomplete financial markets.

2.1 Stochastic Differential Equations

Definition 2.1 *Let $\mathcal{K} = \mathcal{K}(S,T)$ be the class of functions*

$$f_t(\omega) : [0,\infty) \times \Omega \to \mathbb{R}$$

such that

(i) $(\omega, t) \to f_t(\omega)$ is $\mathcal{F} \times \mathcal{B}$-measurable, where $\mathcal{B}$ denotes the Borel σ - algebra on $[0,\infty)$.

(ii) $f_t(\omega)$ is $\mathcal{F}_t$-adapted.

(iii) $E\left(\int\limits_S^T f_t(\omega)^2 dt\right) < \infty$.

Definition 2.2 *A function $\phi \in \mathcal{K}$ is called elementary if it has the form*

$$\phi_t(\omega) = \sum_j e_j(\omega) \cdot \chi_{[t_j, t_{j+1})}(t) \tag{2.1}$$

where $S = t_0 < t_1 < \ldots < t_m = T$ for some $0 \leq S < T$. Since $\phi \in \mathcal{K}$ the functions e_j have to be $\mathcal{F}_{t_j}$-measurable.

For such functions we naturally define

$$\int_S^T \phi_t(\omega) dB_t(\omega) = \sum_j e_j(\omega) \left[B_{t_{j+1}} - B_{t_j}\right](\omega). \tag{2.2}$$

Lemma 2.3 (Simple version of the Itô isometry) *If an elementary function ϕ_t is bounded then*

$$E\left(\int_S^T \phi_t(\omega) dB_t(\omega)\right)^2 = E\left(\int_S^T \phi_t(\omega)^2 dt\right). \tag{2.3}$$

Definition 2.4 (The Itô integral) *Let $f \in \mathcal{K}$. Then the Itô integral of f (from S to T) is defined by*

$$\int_S^T f_t(\omega) dB_t(\omega) = \lim_{n\to\infty} \int_S^T \phi_n(\omega, t) dB_t(\omega) \quad \textit{(limit in } L^2(P)) \tag{2.4}$$

where $\{\phi_n\}_{n\in\mathbb{N}}$ is a sequence of elementary functions such that

$$E\left(\int_S^T (f_t(\omega) - \phi_n(\omega, t))^2 dt\right) \to 0 \quad \textit{as } n \to \infty. \tag{2.5}$$

Remark 2.5 *Such a sequence $\{\phi_n\}_{n\in\mathbb{N}}$ satisfying (2.5) actually exists. Moreover, by (2.3) the limit in (2.4) exists and does not depend on the actual choice of $\{\phi_n\}_{n\in\mathbb{N}}$, as long as (2.5) holds. For details see e.g. (Øksendal 1998).*

Remark 2.6 *Since Brownian motion is a very special semimartingale, it is not surprising that we can define a meaningful stochastic integral for a much wider class of integrands, i.e. we can extend the class of usable processes form locally bounded predictable integrands to the class $\mathcal{K}$.*

From (2.3) and (2.4) we get the following important

Corollary 2.7 (The Itô isometry)

$$E\left(\int_S^T f_t(\omega) dB_t(\omega)\right)^2 = E\left(\int_S^T f_t(\omega)^2 dt\right) \quad \textit{for all } f \in \mathcal{K}(S,T). \tag{2.6}$$

Definition 2.8 *Let $B = (B^1, B^2, \dots, B^m)^T$ be m-dimensional Brownian motion. Then $\mathcal{K}^{n\times m}(S,T)$ denotes the set of $n \times m$ matrices $k = [k_t^{ij}(\omega)]$ where each entry $k_t^{ij}(\omega) \in \mathcal{K}(S,T)$. If $k \in \mathcal{K}^{n\times m}(S,T)$ we define, using matrix notation*

$$\int_S^T k dB = \int_S^T \begin{pmatrix} k^{11} & \cdots & k^{1m} \\ \vdots & & \vdots \\ k^{n1} & \cdots & k^{nm} \end{pmatrix} \begin{pmatrix} dB^1 \\ \vdots \\ dB^m \end{pmatrix}$$

to be the $n \times 1$ matrix (column vector) whose i'th component is the following sum of 1-dimensional Itô integrals:

$$\sum_{j=1}^{m} \int_S^T k_s^{ij}(\omega) dB_s^j(\omega).$$

If $n = 1$ we write $\mathcal{K}^m(S,T)$ and we also put

$$\mathcal{K}^{n\times m} = \mathcal{K}^{n\times m}(0,\infty) = \bigcap_{T>0} \mathcal{K}^{n\times m}(0,T).$$

Definition 2.9 *$\mathcal{K}'(S,T)$ denotes the class of processes $f_t(\omega) \in \mathbb{R}$ satisfying (i) and (ii) of Definition 2.1 and the weaker condition*

(iii)' $P\left(\int_S^T f_s(\omega)^2 ds\right) < \infty = 1.$

We obviously have that $\mathcal{K}' \supseteq \mathcal{K}$.

Definition 2.10 (Extension of the Itô integral) *We may now define*

$$\int_S^T f_t(\omega) dB_t(\omega) = \lim_{n\to\infty} \int_S^T \phi_n(\omega,t) dB_t(\omega) \quad \textit{(limit in probability)},\ f \in \mathcal{K}'(S,T)$$

and $\mathcal{K}'^{m\times n}(S,T)$, $\mathcal{K}'^m(S,T)$, $\mathcal{K}'^{m\times n}$ similarly to $\mathcal{K}^{m\times n}(S,T)$, $\mathcal{K}^m(S,T)$, $\mathcal{K}^{m\times n}$, respectively.

Remark 2.11 *If $f \in \mathcal{K}$ then the Itô integral is a martingale with respect to the filtration $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$. However if $f \in \mathcal{K}'$ then in general it is only a local martingale. See e.g. (Karatzas and Shreve 1991).*

Definition 2.12 (1-dimensional Itô process) *Let B_t be 1-dimensional Brownian motion on $(\Omega, \mathcal{F}, P)$. A (1-dimensional) Itô process (stochastic integral) is a stochastic process X_t on $(\Omega, \mathcal{F}, P)$ of the form*

$$X_t = X^0 + \int_0^t b_s(\omega) ds + \int_0^t \sigma_s(\omega) dB_s, \tag{2.7}$$

where $\sigma \in \mathcal{K}'$, such that

$$P\left(\int_0^t \sigma_s(\omega)^2 ds < \infty \textit{ for all } t \geq 0\right) = 1, \tag{2.8}$$

and b is $\mathcal{F}_t$-adapted and such that

$$P\left(\int_0^t |b_s(\omega)|\, ds < \infty \textit{ for all } t \geq 0\right) = 1. \tag{2.9}$$

If X_t is an Itô process of the form (2.7), equation (2.7) is usually written in the shorter differential form

$$dX_t = bdt + \sigma dB_t.$$

It turns out that these Itô processes as sums of a dB_s-and a ds-integral represent a family of processes that is stable under smooth maps. The following is the simplified version of Theorem 1.38 in the setting of Brownian motion:

Theorem 2.13 (The 1-dimensional Itô formula) *Let X_t be an Itô process given by*

$$dX_t = bdt + \sigma dB_t.$$

Let $g(t,x) \in C^2([0,\infty) \times \mathbb{R})$ (i.e. g is twice continuously differentiable). Then

$$Y_t := g(t, X_t)$$

is again an Itô process, and

$$dY_t = \frac{\partial g}{\partial t}(t, X_t)dt + \frac{\partial g}{\partial x}(t, X_t)dX_t + \tfrac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, X_t) \cdot (dX_t)^2, \tag{2.10}$$

where $(dX_t)^2 = (dX_t) \cdot (dX_t)$ is computed according to the rules

$$dt \cdot dt = dt \cdot dB_t = dB_t \cdot dt = 0, \quad dB_t \cdot dB_t = dt. \tag{2.11}$$

Let $B_t(\omega) = \big(B_t^1(\omega), \ldots, B_t^m(\omega)\big)^T$ denote m-dimensional Brownian motion. If each of the processes $b_t^i(\omega)$ and $\sigma_t^{ij}(\omega)$ satisfies the conditions for 1-dimensional Itô processes ($1 \leq i \leq n$, $1 \leq j \leq m$). Then we can write in matrix notation

$$dX_t = bdt + \sigma dB_t \tag{2.12}$$

where

$$X_t = \begin{pmatrix} X_t^1 \\ \vdots \\ X_t^n \end{pmatrix}, \quad b = \begin{pmatrix} b^1 \\ \vdots \\ b^n \end{pmatrix}, \quad \sigma = \begin{pmatrix} \sigma^{11} & \cdots & \sigma^{1m} \\ \vdots & & \vdots \\ \sigma^{n1} & \cdots & \sigma^{nm} \end{pmatrix}.$$

Such a process X is called an n-dimensional Itô process.

Theorem 2.14 (The general Itô formula) *Let*

$$dX_t = bdt + \sigma dB_t$$

be an n-dimensional Itô process as above. Let $g(t,x) = \big(g^1(t,x), \ldots, g^p(t,x)\big)$ be a C^2 map from $[0,\infty) \times \mathbb{R}^n$ into $\mathbb{R}^p$. Then the process

$$Y_t(\omega) = g(t, X_t)$$

is again an Itô process, whose k-th component number, Y^k is given by

$$dY^k = \frac{\partial g^k}{\partial t}(t, X)dt + \sum_i \frac{\partial g^k}{\partial x^i}(t, X)dX^i + \tfrac{1}{2}\sum_{i,j} \frac{\partial^2 g^k}{\partial x^i \partial x^j}(t, X)dX^i dX^j$$

where $dB^i \cdot dB^j = \delta_{ij} dt$, $dB^i \cdot dt = dt \cdot dB^i = dt \cdot dt = 0$.

Theorem 2.15 (Itô representation theorem) *Let $F \in L^2(\Omega, \mathcal{F}_T, P)$. Then there exists a unique stochastic process $f_t(\omega) \in \mathcal{K}^m(0,T)$ such that*

$$F(\omega) = E(F) + \int_0^T f_t(\omega) dB_t. \tag{2.13}$$

Theorem 2.16 (Integration by parts) *Suppose $f_s(\omega) = f_s$ only depends on s and that f is continuous and of bounded variation in $[0,t]$. Then*

$$\int_0^t f_s dB_s = f_t B_t - \int_0^t B_s df_s.$$

Theorem 2.17 (Existence and eniqueness theorem for SDEs) *Let $T > 0$ and $b : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n \times m}$ be measurable functions satisfying*

$$|b(t,x)| + |\sigma(t,x)| \leq C(1 + |x|); \quad x \in \mathbb{R}^n,\ t \in [0,T] \tag{2.14}$$

for some constant C, (where $|\sigma|^2 = \sum \left|\sigma^{ij}\right|^2$) and such that

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq D\,|x - y|\,; \quad x, y \in \mathbb{R}^n,\ t \in [0,T] \tag{2.15}$$

for some constant D. Let Z be a random variable which is independent of the σ-algebra $\mathcal{F}_\infty$ generated by $\{B_s : s \geq 0\}$ and such that

$$E\left(|Z|^2\right) < \infty.$$

Then the stochastic differential equation

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad 0 \leq t \leq T,\ X_0 = Z \tag{2.16}$$

has a unique continuous solution $X_t(\omega)$ with the property that

$$X_t(\omega) \text{ is adapted to the filtration } \mathcal{F}_t^Z \text{ generated by } Z \text{ and } \{B_s : s \leq t\} \tag{2.17}$$

and

$$E\left(\int_0^T |X_t|^2 \, dt\right) < \infty \tag{2.18}$$

Theorem 2.18 (Girsanov theorem) *Let Y be a d-dimensional Itô process of the form*

$$dY_t = bdt + dB_t; \quad t \leq T,\ Y_0 = 0.$$

where $T \leq \infty$ is a given constant and B is n-dimensional Brownian motion. Put

$$Z_t = \exp\left(-\int_0^t b_s(\omega) dB_s - \tfrac{1}{2}\int_0^t b_s^2(\omega) ds\right); \quad t \leq T. \tag{2.19}$$

Assume that $b_s(\omega)$ satisfies Novikov's condition

$$E\left(\exp\left(\frac{1}{2}\int_0^T b_s^2(\omega)ds\right)\right) < \infty \tag{2.20}$$

where $E = E_P$ is the expectation w.r.t. P. Define the measure Q on $(\Omega, \mathcal{F}_T)$ by

$$dQ(\omega) = Z_T(\omega)dP(\omega) \tag{2.21}$$

Then Y is an n-dimensional Brownian motion w.r.t. the probability law Q, for $t \leq T$.

2.2 Market, Portfolio and Arbitrage

First we give the mathematical definitions of some fundamental finance concepts in the setting of Brownian motion. These concepts can be applied with at most minor modifications to more general semimartingale models, which are examined in the subsequent chapters.

Definition 2.19 *(i) A market is an $\mathcal{F}_t$-adapted $(n+1)$-dimensional Itô process $X_t = (X_t^0, X_t^1, \ldots, X_t^n)$; $0 \leq t \leq T$ which we will assume to have the form*

$$dX_t^0 = \rho_t(\omega)X_t^0 dt; \quad X_0^0 = 1 \tag{2.22}$$

and

$$\begin{aligned} dX_t^i &= \mu_t^i(\omega)dt + \sum_{j=1}^m \sigma_t^{ij}(\omega)dB_t^j \\ &= \mu_t^i(\omega) + \sigma_t^i(\omega)dB_t; \quad X_0^i = x^i, \end{aligned} \tag{2.23}$$

where σ^i is row number i of the $n \times m$ matrix (σ^{ij}); $1 \leq i \leq n \in \mathbb{N}$.

(ii) The market $\{X_t\}_{t\in[0,T]}$ is called normalized if $X_t^0 \equiv 1$.

(iii) A portfolio in the market $\{X_t\}_{t\in[0,T]}$ is an $(n+1)$-dimensional (ω,t)-measurable and $\mathcal{F}_t$-adapted stochastic process

$$\varphi_t(\omega) = (\varphi_t^0(\omega), \varphi_t^1(\omega), \ldots, \varphi_t^n(\omega)); \quad 0 \leq t \leq T. \tag{2.24}$$

(iv) The value at time t of a portfolio φ is defined by

$$V_t(\omega) = V_t^\varphi(\omega) = \varphi_t^T X_t = \sum_{i=0}^n \varphi_t^i X_t^i. \tag{2.25}$$

(v) The portfolio φ is called self-financing if

$$\int_0^T \left\{\left|\varphi_s^0 \rho_s X_s^0 + \sum_{i=1}^n \varphi_s^i \mu_s^i\right| + \sum_{j=1}^m \left(\sum_{i=1}^n \varphi_s^i \sigma_s^{ij}\right)^2\right\} ds < \infty \quad a.s. \tag{2.26}$$

and

$$dV_t = \varphi_t^T dX_t \tag{2.27}$$

i.e.

$$V_t = V_0 + \int_0^t \varphi_s^T dX_s \quad \textit{for } t \in [0,T]. \tag{2.28}$$

Comments to the Definition above.

(i) We think of $X_t^i = X_t^i(\omega)$ as the price of security/asset number i at time t. The assets number $1,\ldots,n$ are called risky because of the presence of their diffusion terms. They can for example represent stock investments. The asset number 0 is called safe because of the absence of a diffusion term (although $\rho_t(\omega)$ may depend on ω). This asset can for example represent a bank investment. For simplicity we will assume that $\rho_t(\omega)$ is bounded, although it is usually enough to to assume that

$$\int_0^T |\rho_t|\, dt < \infty \ a.s. \tag{2.29}$$

(ii) Note that we can always make the market normalized by defining

$$\overline{X}_i = (X_t^0)^{-1} X_t^i \quad \text{for } 1 \le i \le n. \tag{2.30}$$

The market

$$\overline{X}_t = (1, \overline{X}_t^1, \ldots, \overline{X}_t^n)$$

is called the normalization of $\{X_t\}_{t\in[0,T]}$. Thus normalization corresponds to regarding the price X_t^0 of the safe investment as the unit of price (the numeraire) and computing the other prices in terms of this unit. Since

$$X_t^0 = \exp\left(\int_0^t \rho_s(\omega) ds\right)$$

we have

$$\xi_t := X_t^0 = \exp\left(-\int_0^t \rho_s(\omega) ds\right) > 0 \quad \text{for all } t \in [0,T] \tag{2.31}$$

and

$$d\overline{X}_t^i = d(\xi_t X_t^i) = \xi_t[(\mu^i - \rho X^i)dt + \sigma^i dB_t] \quad \text{for } 1 \le i \le n \tag{2.32}$$

or

$$d\overline{X}_t = \xi_t[dX_t - \rho_t X_t dt]. \tag{2.33}$$

(iii) The components $\varphi_t^0(\omega), \ldots, \varphi_t^n(\omega)$ represent the number of units of the security number $0,\ldots,n$, respectively, which an investor holds at time t.

(iv) This is simply the total value of all investments held at time t.

(v) Note that condition (2.26) is required to make (2.28) well-defined. See Definition of $\mathcal{K}'^{m\times n}$. The requirement (2.27) stems from the corresponding discrete time model: If investments φ_{t_k} are made at discrete time $t = t_k$, then the increase in the wealth $\Delta V_{t_k} = V_{t_{k+1}} - V_{t_k}$ should be given by

$$\Delta V_{t_k} = \varphi_{t_k}^T \Delta X_{t_k} \tag{2.34}$$

where $\Delta X_{t_{k+1}} = X_{t_{k+1}} - X_{t_k}$ is the change in prices, provided that no money is brought in or taken out from the system i.e provided the portfolio is self-financing. If we consider our continuous time model as a limit of the discrete time case as $\Delta t_k = t_{k+1} - t_k$ goes to 0, then (2.27) (with the Itô interpretation of the integral) follows from (2.34).
Note that if φ is self-financing for X and

$$\overline{V}_t^{\varphi} = \varphi_t^T \overline{X}_t = \xi_t V_t^{\varphi} \tag{2.35}$$

is the value process of the normalized market (also called discounted value process), then by Itô's formula and (2.33) we have

$$\begin{aligned} d\overline{V}_t^{\varphi} &= \xi_t dV_t^{\varphi} + V_t^{\varphi} d\xi_t \\ &= \xi_t \varphi_t^T dX_t - \rho_t \xi_t V_t^{\varphi} dt \\ &= \xi_t \varphi_t^T [dX_t - \rho_t X_t dt] \\ &= \varphi_t d\overline{X}_t. \end{aligned} \tag{2.36}$$

Hence φ is also self-financing for the normalized market.

Remark 2.20 *Note that by combining (2.25) and (2.27) we get*

$$\varphi_t^0 X_t^0 + \sum_{i=1}^n \varphi_t^i X_t^i = V_0 + \int_0^t \varphi_s^0 dX_s^0 + \sum_{i=1}^n \int_0^t \varphi_s^i dX_s^i.$$

Hence if we denote by

$$Y_t^0 = \varphi_t^0 X_t^0$$

the money invested in the riskless asset at time, then

$$dY_t^0 = \rho_t Y_t^0 dt + dA_t, \tag{2.37}$$

where

$$A_t = \sum_{i=1}^n \left(\int_0^t \varphi_s^i dX_s^i - \varphi_t^i X_t^i \right). \tag{2.38}$$

$A_t - A_0$ can be interpreted as the total amount of money transferred up to time t form the risky assets $1, \ldots, n$ to the riskless asset 0. Via differentiation we can check that (2.37) has the solution

$$\xi_t Y_t^0 = \varphi_0^0 + \int_0^t \xi_s dA_s$$

or

$$\varphi_t^0 = \varphi_0^0 + \int_0^t \xi_s dA_s. \tag{2.39}$$

Using integration by parts we may rewrite this as

$$\varphi_t^0 = \varphi_0^0 + \xi_t A_t - A_0 - \int_0^t A_s d\xi_s$$

or

$$\varphi_t^0 = V_0 + \xi_t A_t + \int_0^t \rho_s A_s \xi_s ds. \tag{2.40}$$

In particular, if $\rho = 0$ *this gives*

$$\varphi_t^0 = V_0 + A_t. \tag{2.41}$$

Therefore, if $\varphi_t^1 \dots, \varphi_t^n$ *are chosen, we can always make the portfolio* $\varphi_t = (\varphi_t^0, \varphi_t^1, \dots, \varphi_t^n)$ *self-financing by choosing* φ_t^0 *according to (2.40).*

Example 2.21 *Let* $\varphi_t = (\varphi^0, \dots, \varphi^n)$ *be a constant portfolio. Then* φ *is self-financing.*

Proof. Consider that $\varphi^1, \dots, \varphi^n$ are given and that we want to make the portfolio self-financing. From (2.38) we get that

$$A_t = \sum_{i=1}^n \varphi^i \left(\int_0^t dX_s^i - X_t^i \right) = - \sum_{i=1}^n \varphi^i X_0^i$$

is constant. To make the portfolio self-financing we choose φ_t^0 according to (2.39) which gives $\varphi_t^0 \equiv \varphi_0^0$, since $dA \equiv 0$. ■

We now make the following fundamental definition:

Definition 2.22 *A portfolio* φ_t *which satisfies (2.26) and which is self-financing is called admissible if the corresponding value process* V_t^φ *is* (t, ω) *a.s. lower bounded, i.e. there exists* $K = K(\varphi) < \infty$ *such that*

$$V_t^\varphi(\omega) \geq -K \quad \text{for a.a. } (\omega, t) \in \Omega \times [0, T]. \tag{2.42}$$

The restriction (2.42) reflects a natural condition in real life finance: There must be a limit how much debt the creditors can tolerate. Note that if V_t^φ is (ω, t) *a.s.* lower bounded, then the same holds true for $\overline{V}_t^\varphi$ and vice versa, since we have assumed that $\rho_t(\omega)$ is bounded.

Definition 2.23 *An admissible portfolio* φ_t *is called an arbitrage in the market* $\{X_t\}_{t \in [0,T]}$ *if the corresponding value process* V_t^φ *satisfies*

$$V_T^\varphi \geq 0 \quad \text{a.s. and} \quad P(V_T^\varphi > 0) > 0.$$

In other words, φ_t is an arbitrage if it gives an increase in the value from time $t = 0$ to time $t = T$ *a.s.*, and a strictly positive increase with positive probability. So φ_t may generate a profit without any risk of loosing money.

Intuitively, the existence of an arbitrage is a sign of lack of equilibrium in the market: No real market equilibrium can exist in the long run if there are arbitrages there. Therefore it is important to be able to determine if a given market allows arbitrage or not. Not surprisingly, this question turns out to be closely related to what conditions we pose on the portfolios that should be allowed to use. We have defined our admissible portfolios in Definition 2.22 above, where condition (2.42) was motivated from a modelling point of view. One could also obtain a mathematically sensible theory with other conditions instead, for example L^2-conditions which imply that

$$E(V_t^2) < \infty \quad \text{for all } t \in [0, T]. \tag{2.43}$$

In any case, some additional conditions are required on the self-financing portfolios: If we only require the portfolio to be self-financing (and satisfying (2.26)) we can generate virtually any final value V_T, as illustrated by the following striking result, which is due to Dudley (1997):

Theorem 2.24 *Let F be an $\mathcal{F}_T$-measurable random variable and let B be m-dimensional Brownian motion. Then there exists a $\phi \in \mathcal{K}'^m$ such that*

$$F(\omega) = \int_0^T \phi_t(\omega)^T dB_t. \tag{2.44}$$

F could for example be any given constant. This clearly contradicts the real life situation in finance, so a realistic model must put stronger restrictions than (2.26) on the portfolios allowed.

How can we decide if a given market $\{X_t\}_{t\in[0,T]}$ allows arbitrage or not? First we establish an auxiliary result:

Lemma 2.25 *The price process $\{X_t\}_{t\in[0,T]}$ has an arbitrage if and only if the normalized price process $\{\overline{X}_t\}_{t\in[0,T]}$ has an arbitrage*

Proof. We first note that since we have assumed that $\rho_t(\omega)$ is bounded that

$$0 < \xi_T < \infty$$

Because of $\overline{V}_t^\varphi = \xi_t V_t^\varphi$ the following holds true for every admissible portfolio φ_t

$$P(V_T^\varphi \geq 0) = P(\overline{V}_T^\varphi \geq 0) \tag{2.45}$$

and

$$P(V_T^\varphi > 0) = P(\overline{V}_T^\varphi > 0) \tag{2.46}$$

Therefore, if φ_t is an arbitrage in the market $\{X_t\}_{t\in[0,T]}$ it is also an arbitrage in the market $\{\overline{X}_t\}_{t\in[0,T]}$ and vice versa. ■

The following simple result is the basis for many further investigations in no-arbitrage theory:

Lemma 2.26 *Suppose there exists a measure Q on $\mathcal{F}_T$ such that $P \sim Q$ and such that the normalized price process $\{\overline{X}_t\}_{t\in[0,T]}$ is a local martingale w.r.t. Q. Then the market $\{X_t\}_{t\in[0,T]}$ has no arbitrage.*

Proof. Suppose φ_t is an arbitrage for $\{\overline{X}_t\}_{t\in[0,T]}$. Let $\overline{V}_t^{\varphi}$ be the corresponding value process for the normalized market with $\overline{V}_0^{\varphi} = 0$. Then $\overline{V}_t^{\varphi}$ is a lower bounded local martingale w.r.t. Q. Therefore $\overline{V}_t^{\varphi}$ is a supermartingale w.r.t. Q, by Theorem 1.12. Hence

$$E_Q(\overline{V}_T^{\varphi}) \leq \overline{V}_0^{\varphi} = 0. \tag{2.47}$$

But since $\overline{V}_T^{\varphi} \geq 0$ P-almost surely we have $\overline{V}_T^{\varphi} \geq 0$ Q-almost surely (because $Q \ll P$) and since $P(\overline{V}_T^{\varphi} > 0) > 0$ we have $Q(\overline{V}_T^{\varphi} > 0) > 0$ (because $P \ll Q$). This implies that

$$E_Q(\overline{V}_T^{\varphi}) > 0,$$

which contradicts (2.47). Hence arbitrage opportunities do not exist for the normalized price process $\{\overline{X}_t\}_{t\in[0,T]}$. Using Lemma 2.25 it follows that $\{X_t\}_{t\in[0,T]}$ has no arbitrage. ■

Thus Lemma 2.26 states that if there exists an equivalent local martingale measure then the market has no arbitrage. In fact, then the market also satisfies the stronger condition "no free lunch with vanishing risk" (NFLVR). Conversely, if the market satisfies the NFLVR condition, then there exists an equivalent martingale measure. See (Delbaen and Schachermayer 1995). In this chapter we will settle with a weaker result, which nevertheless is very useful for many applications:

Theorem 2.27 *a) Suppose there exists a process $u_t(\omega) \in \mathcal{K}^m(0,T)$ such that, with $\widehat{X}_t(\omega) = \big(X_t^1(\omega), \ldots, X_t^n(\omega)\big)$,*

$$\sigma_t(\omega)u_t(\omega) = \mu_t(\omega) - \rho_t(\omega)\widehat{X}_t(\omega) \quad \text{for a.a. } (\omega, t) \tag{2.48}$$

and such that

$$E\left(\exp\left(\frac{1}{2}\int_0^T u_t^2(\omega)dt\right)\right) < \infty. \tag{2.49}$$

Then the market $\{X_t\}_{t\in[0,T]}$ has no arbitrage.

b) Conversely, if the market $\{X_t\}_{t\in[0,T]}$ has no arbitrage, then there exists an $\mathcal{F}_t^{(m)}$-adapted, (ω,t)-measurable process $u_t(\omega)$ such that

$$\sigma_t(\omega)u_t(\omega) = \mu_t(\omega) - \rho_t(\omega)\widehat{X}_t(\omega) \quad \text{for a.a. } (\omega, t).$$

Proof.

a) We may assume that $\{X_t\}_{t\in[0,T]}$ is normalized, i.e. $\rho = 0$ (see Lemma 2.25). Define the measure $Q = Q_u$ on $\mathcal{F}_T$ by

$$dQ = \exp\left(-\int_0^T u_t(\omega)dB_t - \frac{1}{2}\int_0^T u_t^2(\omega)dt\right)dP(\omega). \tag{2.50}$$

Then $Q \sim P$ and by Girsanov Theorem 2.18 the process

$$\widetilde{B}_t := \int_0^t u_s(\omega)ds + B_t \tag{2.51}$$

is a Q-Brownian motion. In terms of $\widetilde{B}_t$ we have

$$dX_t^i = \mu^i dt + \sigma^i dB_t = \mu^i dt + \sigma^i (d\widetilde{B}_t - u_t(\omega)dt) = \sigma^i d\widetilde{B}_t; \quad 1 \le i \le n.$$

Hence X is a local Q-martingale and the conclusion follows from Lemma 2.26.

b) Conversely, assume that the market has no arbitrage and is normalized. For $t \in [0,T]$, $\omega \in \Omega$ let

$$\begin{aligned} F_t &= \{\omega; \text{ the equation (2.48) has no solution }\} \\ &= \left\{ \begin{array}{l} \omega;\ \mu_t(\omega) \text{ does not belong to the} \\ \text{linear span of the columns of } \sigma_t(\omega) \end{array} \right\} \\ &= \left\{\omega;\ \exists v = v_t(\omega) \text{ with } \sigma_t^T(\omega)v_t(\omega) = 0 \text{ and } v_t(\omega)\cdot\mu_t(\omega) \neq 0\right\}. \end{aligned}$$

Define

$$\varphi_t^i(\omega) = \begin{cases} sign(v_t(\omega)\cdot\mu_t(\omega))v_i & \text{for } \omega \in F_t \\ 0 & \text{for } \omega \notin F_t \end{cases}$$

for $1 \le i \le n$ and $\varphi_t^0(\omega)$ according to (2.41). Since $\sigma_t(\omega)$, $\mu_t(\omega)$ are $\mathcal{F}_t$-adapted and (ω,t)-measurable, it follows that we can choose $\varphi_t(\omega)$ to be $\mathcal{F}_t$-adapted and (ω,t)-measurable also. Moreover, $\varphi_t(\omega)$ is self-financing and it generates the following gain in the value function

$$\begin{aligned} V_t^\varphi(\omega) - V_0^\varphi &= \int_0^t \sum_{i=1}^n \varphi_s^i(\omega)dX_s^i \\ &= \int_0^t \chi_{F_s}(\omega)\,|v_s(\omega)\cdot\mu_s(\omega)|\,ds + \int_0^t \sum_{j=1}^m \left(\sum_{i=1}^n \varphi_s^i(\omega)^T \sigma_s^{ij}(\omega)\right) dB_s^j \\ &= \int_0^t \chi_{F_s}(\omega)\,|v_s(\omega)\cdot\mu_s(\omega)|\,ds \\ &+ \int_0^t sign(v_s(\omega)\cdot\mu_s(\omega))\chi_{F_s}(\omega)\left(\sigma_s^T(\omega)v_s(\omega)\right)^T dB_s^j \\ &= \int_0^t \chi_{F_s}(\omega)\,|v_s(\omega)\cdot\mu_s(\omega)|\,ds \ge 0 \quad \text{for all } t \in [0,T] \end{aligned}$$

Since the market has no arbitrage we must have that

$$\chi_{F_s}(\omega) = 0 \quad \text{for } a.a\ (\omega,t)$$

i.e. that (2.48) has a solution for $a.a.$ (ω,t).

■

Example 2.28 *(i) Consider the price process X given by*

$$dX_t^0 = 0, \quad dX_t^1 = 2dt + dB_t^1, \quad dX_t^2 = -dt + dB_t^1 + dB_t^2.$$

In this case we have

$$\mu = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

and the system $\sigma u = \mu$ has the unique solution

$$u = \begin{pmatrix} u^1 \\ u^2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}.$$

From Theorem 2.27a) we conclude that $\{X_t\}_{t\in[0,T]}$ has no arbitrage.

(ii) Next, consider the price process Y_t given by

$$\begin{aligned} dY_t^0 &= 0, \quad dY_t^1 = 2dt + dB_t^1 + dB_t^2, \\ dY_t^2 &= -dt - dB_t^1 - dB_t^2. \end{aligned}$$

Here the system of equations $\sigma u = \mu$ gets the form

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} u^1 \\ u^2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

which has no solutions. So the market has an arbitrage, according to Theorem 2.27b). Indeed, if we choose

$$\varphi_t = (\varphi^0, 1, 1)$$

we get

$$\begin{aligned} V_T^\varphi &= V_0^\varphi + \int_0^T 2dt + dB_t^1 + dB_t^2 - dt - dB_t^1 - dB_t^2 \\ &= V_0^\varphi + T. \end{aligned}$$

In particular, if we choose φ^0 constant such that $V_0^\varphi = \varphi^0 Y_0^0 + Y_0^1 + Y_0^2 = 0$, then φ is self-financing according to Example 2.21 and therefore an arbitrage.

2.3 Attainability and Completeness

We start this section by stating without a proof the following useful result, which is a special case of Proposition 17.1 in (Yor 1997).

Lemma 2.29 *Suppose a process $u_t(\omega) \in \mathcal{K}^m(0,T)$ satisfies the condition*

$$E\left(\exp\left(\frac{1}{2}\int_0^T u_s^2(\omega)ds\right)\right) < \infty. \tag{2.52}$$

Define the measure $Q = Q_u$ *on* $\mathcal{F}_T$ *by*

$$dQ(\omega) = Z_0(t)dP(\omega) \tag{2.53}$$

with

$$Z_0(t) := \exp\left(-\int_0^T u_t(\omega)dB_t - \frac{1}{2}\int_0^T u_t^2(\omega)dt\right) \tag{2.54}$$

Then

$$\widetilde{B}_t := \int_0^t u_s(\omega)ds + dB_t \tag{2.55}$$

is $\mathcal{F}_T$*-Brownian motion (and hence a* $\mathcal{F}_T$*-martingale) w.r.t.* Q *and every* $F \in L^2(\Omega, \mathcal{F}_T, Q)$ *has a unique representation*

$$F(\omega) = E_Q(F) + \int_0^T \phi_t(\omega)^T d\widetilde{B}_t, \tag{2.56}$$

where $\phi_t(\omega)$ *is an* $\mathcal{F}_T$*-adapted,* (ω, t)*-measurable* $\mathbb{R}^n$*-valued process such that*

$$E_Q\left(\int_0^T \phi_t^2(\omega)dt\right) < \infty. \tag{2.57}$$

Remark 2.30 *Note that the filtration* $\{\widetilde{\mathcal{F}}_t\}_{t\in[0,T]}$ *generated by* $\{\widetilde{B}_t\}_{t\in[0,T]}$ *is contained in* $\{\mathcal{F}_t\}$ *by (2.55), but not necessarily equal to* $\{\mathcal{F}_t\}$*. Therefore the representation (2.56) is not a consequence of the Itô representation theorem (2.15) or the Dudley theorem (Theorem 2.24), which in this setting would require* F *to be* $\widetilde{\mathcal{F}}_T$*-measurable.*

Remark 2.31 *Condition (2.52) is the so called Novikov condition. It is sufficient for* $\{Z_0(t)\}$ *to be a martingale. In particular, if* u *is bounded in* t *and* ω*, then* $\{Z_0(t)\}$ *is a martingale.*

Next we make the following simple, but useful observation:

Lemma 2.32 *Let* $\overline{X}_t = \xi_t X_t$ *the normalized prices process as in (2.30)-(2.33). Suppose* φ_t *is an admissible portfolio for the market* $\{X_t\}_{t\in[0,T]}$ *with value process*

$$V_t^\varphi = \varphi_t^T X_t. \tag{2.58}$$

Then φ_t *is also an admissible portfolio for the normalized market* $\{\overline{X}_t\}_{t\in[0,T]}$ *with value process*

$$\overline{V}_t^\varphi := \varphi_t^T \overline{X}_t = \xi_t V_t^\varphi \tag{2.59}$$

and vice versa.

Proof. Note, that $\overline{V}^\varphi$ is lower bounded if and only V_t^φ is lower bounded (since ρ_t is bounded). Consider first the market consisting of the price process X. Let φ_t be an admissible portfolio for this market with value process V_t^φ. Then by (2.36) we have

$$d\overline{V}_t^\varphi = \varphi_t^T d\overline{X}_t. \tag{2.60}$$

Hence φ_t is self-financing and therefore also admissible for $\{\overline{X}_t\}_{t\in[0,T]}$. This argument goes both ways, so the lemma is proofed. ■

Lemma 2.33 *Suppose there exists an m-dimensional process $u_t(\omega) \in \mathcal{K}^m(0,T)$ such that, with $\widehat{X}_t(\omega) = \big(X_t^1(\omega), \ldots, X_t^n(\omega)\big)$,*

$$\sigma_t(\omega)u_t(\omega) = \mu_t(\omega) - \rho_t(\omega)\widehat{X}_t(\omega) \quad \text{for a.a. } (\omega, t) \tag{2.61}$$

and

$$E\left(\exp\left(\tfrac{1}{2}\int_0^t u_s^2(\omega)ds\right)\right) < \infty. \tag{2.62}$$

Define the measure $Q = Q_u$ and the process $\widetilde{B}$ as in (2.53), (2.55), respectively. Then $\widetilde{B}$ is a Brownian motion w.r.t. Q and in terms of $\widetilde{B}$ we have the following representation of the normalized market $\overline{X}_t = \xi_t X_t$:

$$\begin{aligned} d\overline{X}_t^0 &= 0 & (2.63)\\ d\overline{X}_t^i &= \xi_t \sigma_t^i d\widetilde{B}_t; \quad 1 \le i \le n. & (2.64) \end{aligned}$$

In particular, if $\int_0^T E_Q(\xi_t^2 \sigma_t^i)dt < \infty$, then $\overline{X}_i$ is a Q-martingale for $i = 1, \ldots, n$ and Q is an so called equivalent martingale measure. In any case the normalized value process $\overline{V}_t^\varphi$ of an admissible portfolio φ is a local Q-martingale given by

$$d\overline{V}_t^\varphi = \xi_t \sum_{i=1}^n \varphi_t^i \sigma^i d\widetilde{B}_t. \tag{2.65}$$

Proof. The first statement follows from the Girsanov theorem. To prove the representation (2.64) we compute

$$\begin{aligned} d\overline{X}_t^i &= d(\xi_t X_t^i) = \xi_t dX_t^i + X_t^i d\xi_t \\ &= \xi_t[(\mu_t^i - \rho_t X_t^i)dt + \sigma_t^i dB_t] \\ &= \xi_t[(\mu_t^i - \rho_t X_t^i)dt + \sigma_t^i(d\widetilde{B}_t - u_t^i dt)] \\ &= \xi_t \sigma^i d\widetilde{B}_t. \end{aligned}$$

In particular, if $\int_0^T E_Q(\xi_t^2 \sigma_t^i (\sigma_t^i)^T)dt < \infty$, then $\overline{X}_t^i$ is a martingale w.r.t. Q because then $\xi_t \sigma_t^i \in \mathcal{K}(0,T)$ and using Remark 2.11. Finally, the representation (2.65) follows from (2.60) and (2.64). ■

Condition 2.34 ***From now on we assume that there exists a process*** $u_t(\omega) \in \mathcal{K}(0,T)$ ***satisfying (2.61) and (2.62) and we let*** Q ***and*** $\widetilde{B}$ ***be as in (2.53), (2.55).*** *Furthermore we assume the normalized value process $\overline{V}_t^\varphi$ to be indeed a martingale. Terminal payoff structures that require portfolio process such that this is not the case (i.e. the value process is a supermartingale) are according to (Karatzas and Shreve 1991) not well understood and may also be excluded because they are undesirable.*

Definition 2.35 *(i) A (European) contingent T-claim (or just a T-claim or claim) is a lower bounded $\mathcal{F}_T^{(m)}$-measurable random variable $F(\omega)$.*

(ii) *We say that a claim $F(\omega)$ is attainable in the market $\{X_t\}_{t\in[0,T]}$ if there exists an admissible portfolio φ_t and a real number z such that*

$$F(\omega) = V_z^{\varphi}(T) := z + \int_0^T \varphi_t dX_t \quad a.s.$$

If such a φ_t exists, we call it a replicating or hedging portfolio for F.

(iii) *The market $\{X_t\}_{t\in[0,T]}$ is called complete if every bounded T-claim is attainable.*

In other words, a claim $F(\omega)$ is attainable if there exists a real number z such that if we start with z as our initial fortune we can find an admissible portfolio φ_t which generates a value $V_z^{\varphi}(T)$ at time T which $a.s.$ equals F.

Remark 2.36 (i) *The boundedness condition in Definition 2.35 is technically convenient, but other, related definitions are also possible. Note that if the market is complete in the sense of c), then it often follows that many unbounded claims are attainable as well, see Proposition 2.40.*

(ii) *Note that due to our condition (2.34) the normalized value process $\overline{V}_t^{\varphi}$ is a martingale and not just a local martingale w.r.t. Q. Moreover, this together with the Itô representation theorem (2.15) ensures that the replicating portfolio is unique which is generally not the case.*

Which claims are attainable? Which markets are complete? These are questions of fundamental importance for many further investigations. The following result already give some partial answers.

Theorem 2.37 *The market $\{X_t\}_{t\in[0,T]}$ is complete if and only if $\sigma_t(\omega)$ has a left inverse $\Lambda_t(\omega)$ for a.a. (ω, t), i.e. there exits an $\mathcal{F}_t^{(m)}$-adapted matrix valued process $\Lambda_t(\omega) \in \mathbb{R}^{m\times n}$ such that*

$$\Lambda_t(\omega)\sigma_t(\omega) = I_m \quad \textit{for a.a. } (\omega, t). \tag{2.66}$$

Remark 2.38 *Note that the property (2.66) is equivalent to the property*

$$\textit{rank } \sigma_t(\omega) = m \quad \textit{for a.a. } (\omega, t). \tag{2.67}$$

Proof of Theorem 2.37.

$\Longleftarrow$ Assume that (2.66) hold. Let Q be as in (2.53), (2.55). Let F be a bounded T-claim. We want to prove that there exists an admissible portfolio $\varphi_t = (\varphi_t^1, \ldots, \varphi_t^n)$ and a real number z such that if we put

$$V_z^{\varphi}(t) = z + \int_0^t \varphi_s^T dX_s \quad \text{for } 0 \le t \le T$$

then $V_z^{\varphi}(t)$ is a Q-martingale and

$$V_z^{\varphi}(T) = F(\omega) \quad a.s.$$

By (2.65) this is equivalent to

$$\xi_T F(\omega) = \overline{V}_T^{\varphi} = z + \int_0^T \xi_t \sum_{i=1}^n \varphi_t^i \sigma_t^i t d\widetilde{B}_t.$$

By Lemma 2.29 we have a unique representation

$$\xi_T F(\omega) = E_Q(\xi_T F) + \int_0^T \phi_t^T d\widetilde{B}_t = E_Q(\xi_T F) + \int_0^T \sum_{j=1}^m \phi_t^j d\widetilde{B}_t^j$$

for some $\phi_t(\omega) = (\phi_t^1(\omega), \ldots, \phi_t^m(\omega)) \in \mathbb{R}^m$. Hence we put

$$z = E_Q(\xi_T F)$$

and we choose $\widehat{\varphi}_t = (\varphi_t^1, \ldots, \varphi_t^n)$ such that

$$\int_0^T \xi_t \sum_{i=1}^n \varphi_t^i \sigma_t^i = \phi_t^j \quad \text{for } 1 \le j \le m$$

i.e. such that

$$\xi_t \widehat{\varphi}_t^T \sigma_t = \phi_t.$$

By (2.66) this equation in $\widehat{\varphi}_t$ has the solution

$$\widehat{\varphi}_t = X_t^0 \phi_t \Lambda_t.$$

By choosing $\widehat{\varphi}^0$ according to (2.40) the portfolio becomes self-financing. Moreover, since

$$\xi_t V_z^{\varphi}(t) = z + \int_0^t \widehat{\varphi}_s^T d\overline{X}_s = z + \int_0^t \phi_s^T d\widetilde{B}_s,$$

we get the useful formula

$$\xi_t V_z^{\varphi}(t) = E_Q(\xi_T V_z^{\varphi}(T)|\mathcal{F}_t) = E_Q(\xi_T F|\mathcal{F}_t). \tag{2.68}$$

In particular, $V_z^{\widehat{\varphi}}(t)$ is lower bounded and thus $\widehat{\varphi}$ an admissible strategy. Hence the market $\{X_t\}_{t\in[0,T]}$ is complete.

$\Longrightarrow$ Conversely, assume that $\{X_t\}_{t\in[0,T]}$ is complete. Then $\{\overline{X}_t\}_{t\in[0,T]}$ is complete, so we may assume that $\rho = 0$. The calculation in part a) shows that the value process $V_z^{\varphi}(t)$ generated by an admissible portfolio $\varphi_t = (\varphi_t^1, \ldots, \varphi_t^n)$ is

$$V_z^{\varphi}(t) = z + \int \varphi^T \sigma d\widetilde{B}. \tag{2.69}$$

Since $\{X_t\}_{t\in[0,T]}$ is complete we can hedge any bounded T-claim. Choose an $\mathcal{F}_t^{(m)}$-adapted process $\phi_t(\omega) \in \mathbb{R}^m$ such that $E(\int_0^T \phi_t^2(\omega)dt) < \infty$ and define $F(\omega) := \int_0^T \phi_t(\omega) d\widetilde{B}_t$. Then $E_Q(F^2) < \infty$ by Itô isometry and so we can find a sequence of bounded T-claims $F_k(\omega)$ such that

$$F_k \to F \quad \text{in } L^2(Q) \quad \text{and} \quad E_Q(F_k) = 0.$$

By completeness there exists for all $k \in \mathbb{N}$ an admissible portfolio $\varphi_{(k)} = (\varphi^0_{(k)}, \dots, \varphi^n_{(k)})$ such that $V^{\varphi_{(k)}} = \int_0^t \varphi^T_{(k)} \sigma d\widetilde{B}$ is a Q-martingale and

$$F_k(\omega) = V_T^{\varphi_{(k)}} = \int_0^T \varphi^T_{(k)} \sigma d\widetilde{B}.$$

Then by Itô isometry the sequence $\{\varphi_{(k)}\sigma\}_{k \in \mathbb{N}}$ is a Cauchy sequence in $L^2(\lambda \times Q)$, where λ denotes the Lebesgue measure on $[0,T]$. Hence there exists $\psi_t(\omega) = (\psi^1_t(\omega), \dots, \psi^m_t(\omega)) \in L^2(\lambda \times Q)$ such that

$$\varphi^T_{(k)} \sigma \to \psi \quad \text{in } L^2(\lambda \times Q).$$

But then

$$\int_0^t \psi^T d\widetilde{B} = \lim_{k \to \infty} \int_0^t \varphi^T_{(k)} \sigma d\widetilde{B} = \lim_{k \to \infty} E(F_k | \widetilde{\mathcal{F}}_t) = E(F | \widetilde{\mathcal{F}}_t) = \int_0^t \phi^T d\widetilde{B}$$

$a.s.$ for all $t \in [0,T]$, where $\widetilde{\mathcal{F}}_t$ is the σ-algebra generated by $\{\widetilde{B}_s : s \leq t\}$. Hence by uniqueness we have $\phi_t(\omega) = \psi_t(\omega)$ for $a.a.$ (ω, t). By taking a subsequence we obtain that for $a.a.$ (ω, t) there exists a sequence $x_{(k)}(\omega, t) = (x^1_{(k)}(\omega, t), \dots, x^m_{(k)}(\omega, t)) \in \mathbb{R}^m$ such that

$$\lim_{k \to \infty} x_{(k)}(\omega, t) \sigma_t(\omega) \to \phi(\omega, t).$$

This implies that $\phi_t(\omega)$ belongs to the linear span of the rows $\{\sigma^i_t(\omega)\}_{1 \leq i \leq n}$ of $\sigma_t(\omega)$. Since $\phi \in L^2(\lambda \times Q)$ was arbitrary, we conclude that the linear span of $\{\sigma^i_t(\omega)\}_{1 \leq i \leq n}$ is the whole $\mathbb{R}^m$ for $a.a.$ (ω, t). So $rank(\omega, t) = m$ and there exists $\Lambda_t(\omega) \in \mathbb{R}^{m \times n}$ such that

$$\Lambda_t(\omega) \sigma_t(\omega) = I_m.$$

■

Corollary 2.39 *a) If $n = m$ then the market is complete if and only if $\sigma_t(\omega)$ is invertible for a.a. (ω, t).*

b) If the market is complete, then

$$rank\ \sigma_t(\omega) = m \quad \text{for a.a. } (\omega, t).$$

In particular, $n \geq m$. Moreover, the process $u_t(\omega)$ satisfying (2.61) is unique.

Proof.

a) This is a direct consequence of Theorem 2.37, since the existence of a left inverse implies invertibility when $m = n$.

b) The existence of a left inverse of an $n \times m$ matrix is only possible if the rank is equal to m, which again implies that $n \geq m$. Moreover, the only solution $u_t(\omega)$ of (2.61) is given by

$$u_t(\omega) = \Lambda_t(\omega)\left(\mu_t(\omega) - \rho_t(\omega)\widehat{X}_t(\omega)\right).$$

■

The following proposition generalizes the setting of a complete market to unbounded T-claims.

Proposition 2.40 *Suppose $\{X_t\}_{t\in[0,T]}$ is a complete normalized market and that (2.61) and (2.62) hold. Then any lower bounded claim F such that $E_Q(F) < \infty$ is attainable.*

Proof. We use the same arguments as in the proof of Theorem 2.37. We choose bounded T-claims F_k, such that

$$F_k \to F \quad \text{in } L^2(Q) \quad \text{and} \quad E_Q(F_k) = E_Q(F).$$

By completeness there exit admissible portfolios $\varphi_{(k)} = (\varphi^0_{(k)}, \ldots, \varphi^n_{(k)})$ and constants $V_k(0)$ such that

$$F_k(\omega) = V_k(0) + \int_0^T \varphi^T_{(k)}(s)dX_s = V_k(0) + \int_0^T \varphi^T_{(k)}(s)\sigma_s d\widetilde{B}_s.$$

It follows that $V_k(0) = E_Q(F_k) \to E_Q(F)$ as $k \to \infty$. By Itô isometry the sequence $\{\varphi^T_{(k)}\sigma\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\lambda \times Q)$. Hence there exists $\psi_t(\omega) = (\psi^1_t(\omega), \ldots, \psi^m_t(\omega)) \in L^2(\lambda \times Q)$ such that

$$\varphi^T_{(k)}\sigma \to \psi \quad \text{in } L^2(\lambda \times Q).$$

Define $\theta := (\psi\sigma^{-1})^T$. We now have $a.s.$

$$\begin{aligned} E_Q(F) + \int_0^T \theta^T_s dX_s &= E_Q(F) + \int_0^T \psi^T d\widetilde{B}_s \\ = \lim_{k\to\infty} V_k(0) + \lim_{k\to\infty} \int_0^T \varphi^T_{(k)}\sigma d\widetilde{B} &= \lim_{k\to\infty} F_k = F(\omega), \end{aligned} \tag{2.70}$$

by Itô isometry, which implies that θ is indeed a replicating portfolio for F. The admissibility now follows easily: Since F is a lower bounded claim, it follows that in normalized market $\{X_t\}_{t\in[0,T]}$ the left-hand side of Equation 2.70 cannot fall below this lower boundary value due to no-arbitrage arguments. We consequently have bounded losses for the portfolio θ, which is therefore admissible. ■

Example 2.41 *Define* $X_t^0 \equiv 1$ *and*

$$\begin{pmatrix} dX_t^1 \\ dX_t^2 \\ dX_t^3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} dt + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} dB_t^1 \\ dB_t^2 \end{pmatrix}.$$

Then $\rho = 0$ *and the equation (2.61) gets the form*

$$\sigma u = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

which has the unique solution $u_1 = 1$, $u_2 = 2$. *Since* u *is constant, it is clear that (2.61) and (2.62) hold. It is immediate that rank* $\sigma = 2$, *so (2.67) holds and the market is complete by Theorem 2.37. Since*

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2,$$

we see that in this case

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

is a left inverse of σ.

Example 2.42 *Let* $X_t^0 = 1$ *and*

$$dX_t^1 = 2dt + dB_t^1 + dB_t^2.$$

Then $\mu = 2$, $\sigma = (1,1) \in \mathbb{R}^{1\times 2}$, *so* $n = 1 \leq 2 = m$. *Hence this market cannot be complete, by Corollary 2.39. So there exist bounded* T*-claims which cannot be hedged. We now want to find such a* T*-claim. Let* $\varphi_t = (\varphi_t^0, \varphi_t^1)$ *be an admissible portfolio. The the corresponding value process* $V_z^\varphi(t)$ *is given by (see (2.69))*

$$V_z^\varphi(t) = z + \int_0^t \varphi_s^1 (d\widetilde{B}_s^1 + d\widetilde{B}_s^2).$$

So if φ *hedges a* T*-claim* $F(\omega)$ *we have*

$$F(\omega) = z + \int_0^t \varphi^1(s)(d\widetilde{B}_s^1 + d\widetilde{B}_s^2). \tag{2.71}$$

Choose $F(\omega) = g(\widetilde{B}_T^1)$, *where* $g : \mathbb{R} \to \mathbb{R}$ *is bounded. Then by Itô representation theorem applied to the 2-dimensional Brownian motion* $\widetilde{B}_t = (\widetilde{B}_t^1, \widetilde{B}_t^2)$ *there is a unique* $\phi_t(\omega) = (\phi_t^1(\omega), \phi_t^2(\omega))$ *such that*

$$g(\widetilde{B}_T^1) = E_Q\left(g(\widetilde{B}_T^1)\right) + \int_0^T \phi_s^1 d\widetilde{B}_s^1 + \phi_s^2 d\widetilde{B}_s^2$$

and by the Itô representation theorem applied to $\widetilde{B}_t^1$*, we must have* $\phi^2 = 0$*, i.e.*

$$g(\widetilde{B}_T^1) = E_Q\left(g(\widetilde{B}_T^1)\right) + \int_0^T \phi_s^1 d\widetilde{B}_s^1$$

Comparing this with (2.71) we see that no such φ^1 *exists. So* $F(\omega) = g(\widetilde{B}_T^1)$ *cannot be hedged.*

Remark 2.43 *There is a striking characterization of completeness in terms of equivalent martingale measures, due to (Harrison and Pliska 1983) and (Jacod 1979):*

> *A market* $\{X_t\}_{t\in[0,T]}$ *is complete if and only if there is one and only one equivalent martingale measure for the normalized market* $\{\overline{X}_t\}_{t\in[0,T]}$*.*

Chapter 3

American and Game Options

The following chapter is again based on (Øksendal 1998) and (Karatzas and Shreve 1998).

3.1 American Contingent Claims

European options differ to American options in the way that in the latter case the buyer of the option is free to choose any exercise time τ before or at the given expiration time T and not only the expiration date T. The guaranteed payoff of an American option may depend on both τ and ω. The exercise time τ may be stochastic (depend on ω), but only in such a way that the decision to exercise before or at time t only depends on the history up to time t. More precisely, we require that for all t we have

$$\{\omega : \tau(\omega) \leq t\} \in \mathcal{F}_t.$$

In other words, τ must be an $\mathcal{F}_t$-stopping time.

Definition 3.1 *An American contingent T-claim is an $\mathcal{F}_t$-adapted, (ω, t)- measurable and a.s. lower bounded continuous stochastic process $F_t = F_t(\omega)$; $t \in [0, T]$, $\omega \in \Omega$. An American options on such a claim $F_t(\omega)$ gives the owner of the option the right to choose any stopping time $\tau(\omega) \leq T$ as exercise time for the option, resulting in a payment $F_{\tau(\omega)}(\omega)$ to the owner.*

Let $F_t = F_t(\omega)$ be an American contingent claim. Suppose we were offered a guarantee to be paid the amount $F_{\tau(\omega)}(\omega)$ at the (stopping) time $\tau(\omega) \leq T$ that we are free to choose. How much would we be willing to pay for such a guarantee?

One could argue: If I - the buyer - pay the price y for this guarantee, then I will have an initial fortune (debt) $-y$ in my investment strategy. With this initial fortune $-y$ it must be possible to find a stopping time $\tau \leq T$ and an admissible portfolio φ such that

$$V^{\varphi,-y}_{\tau(\omega)}(\omega) + F_{\tau(\omega)}(\omega) \geq 0 \quad a.s.$$

Thus the maximal price $p = p_A(F)$ the buyer is willing to pay is

(Buyer's price of the American contingent claim F) (3.1)

$$p_A(F) = \sup\{y;\ \text{There exists a stopping time } \tau \le T \text{ and an admissible portfolio } \varphi \text{ such that } V^{\varphi,-y}_{\tau(\omega)}(\omega) := -y + \int_0^{\tau(\omega)} \varphi_s dX_s \ge -F_{\tau(\omega)}(\omega)\ a.s.\}$$

On the other hand, the seller could argue as follows: If I - the seller - receive the price z for such a guarantee, then with this initial fortune z it must be possible to find an admissible portfolio φ which generates a value process which at any time is not less the amount promised to pay to the buyer:

$$V_t^{\varphi,z}(\omega) \ge F_t(\omega) \quad a.s. \text{ for all } t \in [0,T].$$

Thus the minimal price $q = q_A(F)$ the seller is willing to accept is

(Sellers's price of the American contingent claim F) (3.2)

$$q_A(F) = \inf\{z;\ \text{There exists an admissible portfolio } \varphi \text{ such that for all } t \in [0,T] \text{ we have } V_t^{\varphi,z}(\omega) := z + \int_0^t \varphi_s dX_s \ge F_t(\omega)\ a.s.\}$$

We can now prove a fundamental result for the pricing of American options.

Theorem 3.2 *a) Let Q be any equivalent martingale measure and let $F_t = F_t(\omega)$; $t \in [0,T]$ be an American contingent T-claim such that*

$$\sup_{\tau \le T} E_Q(\xi_\tau F_\tau) < \infty \tag{3.3}$$

Then

$$p_A(F) \le \sup_{\tau \le T} E_Q(\xi_\tau F_\tau) \le q_A(F) \le \infty. \tag{3.4}$$

b) Suppose, in addition to the conditions in a), that (2.61), (2.62) hold and let Q be as in (2.53). If the market $\{X_t\}_{t\in[0,T]}$ is complete, then we have

$$p_A(F) = \sup_{\tau \le T} E_Q(\xi_\tau F_\tau) = q_A(F) \le \infty. \tag{3.5}$$

Furthermore, there is a stopping time τ^* attaining this supremum and there is a hedging portfolio φ^* such that

$$F_{\tau^*} = p_A(F) + \int_0^{\tau^*} \varphi_s^* dX_s \tag{3.6}$$

Proof.

a) Suppose $y \in \mathbb{R}$ and there exists a stopping time $\tau \leq T$ and an admissible portfolio φ such that

$$-V_\tau^{\varphi,-y} = -y + \int_0^\tau \varphi_s dX_s \geq -F_\tau \quad a.s.$$

Then we have with (2.65) for the normalized market

$$\overline{V}_\tau^{\varphi,-y} = -y + \int_0^\tau \sum_{i=1}^n \varphi_s^i \xi_s \sigma_s^i d\widetilde{B}_s \geq -\xi_\tau F_\tau \quad a.s.$$

Since $\int_0^\tau \sum_{i=1}^n \varphi_s^i \xi_s \sigma_s^i d\widetilde{B}_s$ is a lower bounded local Q-martingale, it is a supermartingale according to Theorem (1.12). Taking expectations with respect to Q we get

$$y \leq E_Q(\xi_\tau F_\tau) \leq \sup_{\tau \leq T} E_Q(\xi_\tau F_\tau).$$

Since this holds for all such y we conclude that

$$p_A(F) \leq \sup_{\tau \leq T} E_Q(\xi_\tau F_\tau). \tag{3.7}$$

Similarly, suppose $z \in \mathbb{R}$ and there exists an admissible portfolio φ such that

$$V_t^{\varphi,z}(\omega) = z + \int_0^t \varphi_s dX_s \geq F_t \quad a.s. \text{ for all } t \in [0,T].$$

Then, as above, if $\tau \leq T$ is a stopping time we get

$$z + \int_0^\tau \sum_{i=1}^n \varphi_s^i \xi_s \sigma_s^i d\widetilde{B}_s \geq \xi_\tau F_\tau \quad a.s.$$

Again, taking expectations with respect to Q and then supremum over $\tau \leq T$ we get

$$z \geq \sup_{\tau \leq T} E_Q(\xi_\tau F_\tau).$$

Since this holds for all such z, we get

$$q_A(F) \geq \sup_{\tau \leq T} E_Q(\xi_\tau F_\tau). \tag{3.8}$$

b) The proof of the second part of the theorem requires additional knowledge about optimal stopping problems and existence of optimal stopping times. For a detailed description see (Karatzas 1988) or (Karatzas and Shreve 1998).

■

3.2 Game Contingent Claims

Definition 3.3 *A Game Contingent Claim (GCC) in discrete time is a contract between investors A and B consisting of a maturity date $N < \infty$, of selection of a cancellation time $\sigma \in \mathcal{T}(0, N)$ by A, of selection of an exercise time $\tau \in \mathcal{T}(0, N)$ by B and of $\mathcal{F}_n$-adapted payoff processes $\infty > U_n \geq L_n \geq 0$, so that A pledges to pay to B at time $\sigma \wedge \tau = \min(\sigma, \tau)$ the sum*

$$R(\sigma, \tau) := U_\sigma \chi_{\sigma < \tau} + L_\tau \chi_{\tau \leq \sigma}.$$

Definition 3.4 *A Game Contingent Claim (GCC) in continuous time is a contract between investors A and B consisting of a maturity $T < \infty$, of selection of a cancellation time $\sigma \in \mathcal{T}(0, T)$ by A, of selection of an exercise time $\tau \in \mathcal{T}(0, T)$ by B and of $\mathcal{F}_t$-adapted càdlàg payoff processes $\infty > U_n \geq L_n \geq 0$, so that A pledges to pay to B at time $\sigma \wedge \tau = \min(\sigma, \tau)$ the sum*

$$R(\sigma, \tau) := U_\sigma \chi_{\sigma < \tau} + L_\tau \chi_{\tau \leq \sigma}.$$

These are contracts which enable both their buyer and seller to stop them at any time. For example, then the buyer can exercise the right to buy (call option) or sell (put option) a specified security for certain agreed price. If the contract is terminated by the seller he must pay a certain penalty to the buyer. Due to the cancellation right of the seller, Game options can be sold cheaper (or at most for the same price) than usual American options. Their introduction could help sellers of American options to reduce their risk and diversify financial markets.

In (Kifer 2000) it is already shown how to price Game options in complete financial markets, in discrete as well in continuous time. We omit the pricing formula here, since it is a special case of Theorem 6.3 as we will see later on.

Chapter 4

Incomplete Markets

4.1 Reasons for Incompleteness

4.1.1 Trading Constraints

In many cases otherwise complete financial markets are incomplete due to portfolio constraints. If often occurs in such markets that a given contingent claim cannot be hedged perfectly, no matter how large the initial wealth of the would-be hedging agent. Trading constraints are usually given by a nonempty, closed, convex set K in which the portfolio vector $\varphi = (\varphi^1, \ldots, \varphi^n)$ is constrained to take values.

Example 4.1 *Let us consider the following possible constraint sets K on portfolio processes.*

(i) Unconstrained case: $K = \mathbb{R}^n$.

(ii) Prohibition of short-selling: $K = [0, \infty)^n$.

(iii) Some securities are not available for trading: $K = \mathbb{R}^s \times \{0\}^{n-s}$.

(iv) Constraints on short-selling: $K = [-\alpha, \infty)^n$.

(v) K is a nonempty, closed, convex cone in $\mathbb{R}^n$.

(vi) Rectangular constraints: $K = I_1 \times \ldots \times I_n$ with $I_r = [\alpha_r, \beta_r]$, $-\infty \leq \alpha \leq 0 \leq \beta \leq \infty$ and with the understanding that I_r is open on the right (respectively, left) if $\beta_r = \infty$ (respectively $\alpha_r = \infty$).

4.1.2 Discontinuous Stock Returns

With the generalization from Brownian motion to general Lévy processes there arises the conceptual problem that financial models become incomplete, i.e. not every claim can be replicated. This has the consequence that simple no-arbitrage arguments alone are not sufficient to determine unique derivative prices. Consider a financial market consisting of two tradable securities: one risky asset modeled by a Lévy process and one riskless bond. Then, besides Brownian motion and Lévy processes with constant jump size, i.e. the sum of a multiple of a Poisson process and a linear drift, are the only examples where the market is complete - see e.g. (Cox and Rossr 1976). This illustrates that complete markets are very special cases and incomplete markets is what we should generally expect.

4.2 Possible Pricing Approaches

In complete markets, arbitrage arguments suffice to derive unique prices for contingent claims. During the last years, various suggestions have been made how to price contingent claims in incomplete markets.

4.2.1 Lower- and Superhedging

Suppose you have sold e.g. an European contingent claims and you want to hedge yourself by trading only in the underlyings. In frictionless complete markets you simply buy the duplicating portfolio in order to completely offset the risk. The incomplete markets the situation is less obvious.

If you want to be as safe as in the complete case you should invest in a so called superhedging portfolio. This a portfolio that with sufficient initial wealth leads to a final payoff that dominates almost surely the payoff of the contingent claim. Given a contingent claim, the upper hedging price h_{up} of the claim is defined to be the smallest initial capital that permits construction of a superhedging portfolio.

Conversely, the buyer of the contingent claim wishes to manage his debt so that the payoff of the contingent claim at the final time is sufficient to cover his debt. Therefore, the lower hedging price h_{low} is defined to be the largest sum the buyer can pay for the contingent claim and still have the payoff from the contingent claim cover his debt almost surely at the final time.

Simple arbitrage arguments show that $h_{low} \leq h_{up}$ and that the price of the contingent claim cannot lie outside the interval $[h_{low}, h_{up}]$, but are incapable of determining a single price inside the interval, unless this interval contains only one point.

In many cases, e.g. in some complete markets with particular trading constraints, this approach already yields unsatisfactory results. For example an upper hedging price of $+\infty$ or the trivial hedging portfolio that simply buys and holds the underlying stock. See e.g. (Karatzas and Shreve 1998), Examples 5.7.3.

4.2.2 Utility-based Indifference Pricing

Utility-based indifference pricing is a concept which has been applied explicitly to American options. Here, one takes the perspective of a particular market participant and fixes the number of shares of the claim (say, 1 for an option buyer or -1 for an options seller). The indifference premium is a price such that the optimal expected utility among all portfolios containing the specified number of options coincides with the optimal expected utility among all portfolios without the option. Put differently, the investor is indifferent to including the options into the portfolio. Taking the perspective of the option buyer, it turns out that for American options the indifference price is indeed the supremum of the indifference prices of the implied European options. Surprisingly, this is not true for the option seller: Unless exponential utility is chosen, it may happen that a reasonable indifference premium for an American option exceeds the indifference price of all implied European claims. See (Kuehn 2002).

4.2.3 Neutral Derivative Pricing

Neutral prices occur if traders maximize their expected utility and derivative supply and demand are balanced. More precisely, a derivative price process is called neutral if an optimal portfolio does not need to contain the contingent claim.

Both utility-based indifference pricing and neutral pricing rely on expected utility maximization and indifference to trading the option. However, indifference pricing takes and asymmetric point of view. Moreover, it depends decisively on the number of claims under consideration. As far as options are concerned, intermediate trades are not allowed. Therefore, this approach is particularly well suited for over-the-counter trades: Suppose that the buyer wants to purchase a specific contingent claim. Then he has to pay the seller at least his indifference price in order to prompt him to enter the contract.

The concept of neutral pricing, on the other hand, takes a symmetric point of view. It assumes that options are traded in arbitrary positive and negative amounts. Neutral prices are the unique prices such that neither the buyer nor the seller takes advantage from trading the claim.

Chapter 5

Utility Maximization

The derivative pricing approach in the subsequent chapter relies an assumptions concerning investors who maximize their expected utility. We will discuss two kinds of portfolio optimization problems in this chapter, based on the classical utility of terminal wealth and on local utility.

The mathematical framework for this is frictionless market model as follows: Fix a terminal time $T \in \mathbb{R}_+$ and a filtered probability space that satisfies the usual conditions. The price processes of traded securities $1, \dots, m$ are expressed in terms of a numeraire security 0. Put differently, these securities are modelled by their discounted price process $\widehat{S} := (\widehat{S}^1, \dots, \widehat{S}^m)$. We assume that $\widehat{S}$ is a $\mathbb{R}^m$-valued special semimartingale.

In the following chapters, trading strategies are modelled by $\mathbb{R}^m$-valued, predictable stochastic processes $\varphi = (\varphi^1, \dots, \varphi^m)$, where φ^i_t denotes the number of share of security i in the investor's portfolio at time t.

Definition 5.1 *We denote by $\mathcal{L}^1(S)$ the set of all trading strategies φ satisfying*

$$\int_0^T \left(|\varphi_t^T b_t| + \varphi_t^T c_t \varphi_t + \int \left((\varphi_t^T x)^2 \wedge |\varphi_t^T x| \right) F_t(dx) \right) dA_t \in L^1(\Omega, \mathcal{F}, P) \quad (5.1)$$

or equivalently

$$E\left(\left(|\varphi_t^T b_t| + \varphi_t^T c_t \varphi_t + \int \left((\varphi_t^T x)^2 \wedge |\varphi_t^T x| \right) F_t(dx) \right) \cdot A_T \right) < \infty. \quad (5.2)$$

5.1 Utility Functions

Investors are assumed to maximize some kind of utility. In this section we note some properties of utility functions we consider. The following properties are common to all such functions in the subsequent.

Definition 5.2 *A utility function is a concave, nondecreasing, upper semicontinuous function $U : \mathbb{R} \to [-\infty, \infty)$ satisfying*

(i) The half-line $dom(U) := \{x \in \mathbb{R} : U(x) > -\infty\}$ is a nonempty set of the form $[\overline{x}, \infty)$ or $(\overline{x}, \infty)$ with $\overline{x} \in \mathbb{R} \cup \{-\infty\}$.

(ii) U' is continuous, positive, and strictly decreasing on the interior of dom(U), and

$$U'(\infty) := \lim_{x\to\infty} U'(x) = 0.$$

Here are some common utility functions. Take $p \in (-\infty, 1)\backslash\{0\}$ and set

$$U^{(p)}(x) := \begin{cases} x^p/p & x > 0, \\ \lim_{\xi\downarrow 0} \xi^p/p & x = 0, \\ -\infty & x < 0. \end{cases} \tag{5.3}$$

For $p = 0$, set

$$U^{(0)}(x) := \begin{cases} \ln(x) & x > 0, \\ -\infty & x \leq 0. \end{cases} \tag{5.4}$$

5.2 Utility from Terminal Wealth

Definition 5.3 *A strategy $\varphi \in L(S)$ belongs to the set Φ of all admissible strategies if its discounted wealth process $V(\varphi) := \varepsilon + \varphi^T \cdot S$ is nonnegative, i.e. there are no debts allowed.*

Trading constraints are expressed in terms of subsets of the set of all trading strategies. More specifically, we consider a process Γ whose values are convex cones in $\mathbb{R}^m$. The constrained set of trading strategies $\Phi(\Gamma)$ is the subset of all admissible strategies φ which satisfy $(\varphi^1, \ldots, \varphi^m)_t \in \Gamma_t$ pointwise on $\Omega \times \Gamma_t$. Important examples for Γ_t can be found in Subsection 4.1.1.

The investor's preferences are modelled by a strictly concave utility function $U : \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\}$ which is continuously differentiable on $(0, \infty)$ and satisfies $\lim_{x\to\infty} U(x) = \infty$, $\lim_{x\to\infty} U'(x) = 0$, and $\limsup_{x\to\infty} \frac{xU'(x)}{U(x)} < 1$ (i.e. it is of reasonable asymptotic elasticity in the sense of (Kramkov and Schachermayer 1999), Definition 2.2). The aim is to make the best out of the money in the following sense:

Definition 5.4 *We say that $\varphi \in \Phi(\Gamma)$ is an optimal strategy for terminal wealth under the constraints Γ if it maximizes $\tilde{\varphi} \mapsto E(U(V_T(\tilde{\varphi})))$ over all $\tilde{\varphi} \in \Phi(\Gamma)$. By convention, we set $E(U(V_T(\tilde{\varphi}))) := -\infty$ if $E(\min(U(V_T(\tilde{\varphi})), 0)) = -\infty$.*

Optimal portfolios are characterized by the following result.

Lemma 5.5 *Let $\varphi \in \Phi(\Gamma)$ with finite expected utility. Then we have equivalence between*

a) φ is optimal for terminal wealth under the constrains Γ.

b) $U'(V_T(\varphi))((\psi - \varphi)^T \cdot S_T)$ is integrable and has non-positive expectation for any $\psi \in \Phi(\Gamma)$ with $E(U(V_T(\psi))) > -\infty$.

Proof.

b)$\Longrightarrow$a): Let $\psi \in \Phi(\Gamma)$ with $E(U(V_T(\psi))) > -\infty$. Since U is concave, we have

$$\begin{aligned} E(U(\varepsilon + \psi^T \cdot S_T)) &\leq E(U(\varepsilon + \varphi^T \cdot S_T)) \\ &\quad + E(U'(\varepsilon + \varphi^T \cdot S_T)((\psi - \varphi)^T \cdot S_T)) \\ &\leq E(U(\varepsilon + \varphi^T \cdot S_T)), \end{aligned}$$

which yields the assertion.

a)$\Longrightarrow$b): Let $\psi \in \Phi(\Gamma)$ with $E(U(V_T(\psi))) > -\infty$. Define $\widetilde{\psi} := \varphi + \frac{1}{2}(\psi - \varphi)$ and $\psi^{(\lambda)} := \varphi + \lambda(\psi - \varphi)$ for $\lambda \in [0,1]$ (in particular $\widetilde{\psi} = \psi^{(\frac{1}{2})}$). Since $\Phi(\Gamma)$ is convex and U is concave, we have that $\widetilde{\psi} \in \Phi(\Gamma)$ and $E(U(V_T(\widetilde{\psi}))) > -\infty$. From

$$\begin{aligned} -\infty &< E(U(\varepsilon + \psi^T \cdot S_T)) \\ &\leq E(U(\varepsilon + \varphi^T \cdot S_T)) + E(U'(\varepsilon + \varphi^T \cdot S_T)((\psi - \varphi)^T \cdot S_T)) \end{aligned}$$

and $E(U(V_T(\varphi))) < \infty$ it follows that $E((U'(\varepsilon + \varphi^T \cdot S_T)((\psi - \varphi)^T \cdot S_T))^-) < \infty$. Similarly,

$$\begin{aligned} -\infty &< E(U(\varepsilon + \psi^T \cdot S_T)) \\ &\leq E(U(\varepsilon + \widetilde{\psi}^T \cdot S_T)) + \tfrac{1}{2}E(U'(\varepsilon + \widetilde{\psi}^T \cdot S_T)((\psi - \varphi)^T \cdot S_T)) \end{aligned}$$

implies that $E((U'(\varepsilon + \widetilde{\psi}^T \cdot S_T)((\psi - \varphi)^T \cdot S_T))^-) < \infty$.
Let $\lambda \in (0, \frac{1}{2}]$. By optimality of φ, we have

$$0 \geq E(U(\varepsilon + (\psi^{(\lambda)})^T \cdot S_T)) - E(U(\varepsilon + \varphi^T \cdot S_T)), \tag{5.5}$$

which equals $\lambda E(\xi^{(\lambda)}((\psi - \varphi)^T \cdot S_T))$ for some random variable $\xi^{(\lambda)}$ with values in $[U'(\varepsilon + \varphi^T \cdot S_T), U'(\varepsilon + \widetilde{\psi}^T \cdot S_T)]$ or $[U'(\varepsilon + \widetilde{\psi}^T \cdot S_T), U'(\varepsilon + \varphi^T \cdot S_T)]$, respectively. Note that $(\xi^{(\lambda)}((\psi - \varphi)^T \cdot S_T))^- \leq U'(\varepsilon + \varphi^T \cdot S_T)((\psi - \varphi)^T \cdot S_T))^- + U'(\varepsilon + \widetilde{\psi}^T \cdot S_T)((\psi - \varphi)^T \cdot S_T))^- \in L^1(\Omega, \mathcal{F}, P)$.
Since $(\psi^{(\lambda)})^T \cdot S_T \to \varphi^T \cdot S_T$ for $\lambda \to 0$, we have because of continuity of U' that the length of the above intervals tends to zero as $\lambda \to 0$. We therefore have that $\xi^{(\lambda)} \to U'(\varepsilon + \varphi^T \cdot S_T)$ *a.s.* for $\lambda \to 0$. Fatou's lemma finally yields

$$\begin{aligned} E(U'(\varepsilon + \varphi^T \cdot S_T)((\psi - \varphi)^T \cdot S_T)) &= E(\underline{\lim}_{\lambda \to 0} \xi^{(\lambda)}((\psi - \varphi)^T \cdot S_T)) \\ &\leq \underline{\lim}_{\lambda \to 0} E(\xi^{(\lambda)}((\psi - \varphi)^T \cdot S_T)). \end{aligned}$$

From (5.5) it follows that $E(U'(V_T(\varphi))((\psi - \varphi)^T \cdot S_T)) \leq 0$ as claimed.

■

5.3 Local Utility

The concept of maximizing expected local utility in this section is related to maximization of expected utility from consumption but, contrary to this common approach, the discounted financial gains are consumed immediately. It means that we optimize the expected utility of the gains over infinitesimal time intervals. This is a local concept which is motivated by stochastic limit theorems.

The local utility maximization approach has several advantages. Firstly, it is much easier to determine optimal strategies than in the classical utility maximization framework. Secondly, optimal strategies will be more robust against long term model misspecification since they depend only on the local behavior of the security prices. Thirdly, there is no dependence on a terminal date T.

5.3.1 Discrete Time

Our general mathematical framework for a frictionless market model with a finite number of traded securities is as follows. We work with a filtered complete probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\geq 0})$ - that satisfies the usual conditions - and fix a terminal time $T \in \mathbb{R}_+$. Securities $0, \dots, m$ are modelled by their respective price processes $S^0, \dots S^m$. Security 0 plays the special role as a numeraire by which all other assets are discounted. From now on we consider only the discounted price process $\widehat{S} := \frac{1}{S_0}(S^0, \dots, S^m) = (1, \frac{S^1}{S^0}, \dots, \frac{S^m}{S^0})$. We assume that it is a $\mathbb{R}^{m+1}$-valued special semimartingale. Trading strategies are modelled by $\mathbb{R}^{m+1}$-valued, predictable stochastic processes $\varphi = (\varphi^0, \dots, \varphi^m)$, where φ^i_t denotes the number of security i in your portfolio at time t. If the stochastic integral exists (e.g. if φ is a locally bounded and predictable), we can define the real-valued discounted gain process $V(\varphi)$ by

$$V_t(\varphi) := \int_0^t \varphi_s d\widehat{S}_s. \tag{5.6}$$

Trading constraints are given in form of a set $\Gamma := \{\psi \in \mathbb{R} : g^j(\psi) \leq 0$ for $j = 1, \dots, p$ and $g^j(\psi) = 0$ for $j = p+1, \dots, q\}$, where $g^1, \dots, g^p : \mathbb{R}^m \to \mathbb{R}$ are differentiable, convex, mappings and $g^{p+1}, \dots, g^q : \mathbb{R}^m \to \mathbb{R}$ are affine mappings. By $\mathcal{M} = \mathcal{M}(\Gamma)$ we denote the set of all trading strategies φ such that $(\varphi^1_t(\omega), \dots, \varphi^m_t(\omega)) \in \Gamma$ for any $(\omega, t) \in \Omega \times [0, T]$. For typical choices see Subsection 4.1.1. From now on we assume that the subset $\{\psi \in \mathbb{R}^m : g^j(\psi) < 0$ or $j = 1, \dots, p$ and $g^j(\psi) = 0$ for $j = p+1, \dots, q\}$ of Γ is non-empty. Moreover we write $g^j(\psi) := g^j(\psi^1, \dots, \psi^m)$ for $\psi = (\psi^0, \psi^1, \dots \psi^m) \in \mathbb{R}^{m+1}$.

In this subsection we restrict ourselves to a discrete time market, i.e., we assume that $\widehat{S}$ is piecewise constant on the open interval between integer times. In this case $V_t(\varphi) = \sum_{s=1}^{t} \varphi_s \Delta \widehat{S}_s$ (where $\Delta \widehat{S}_s := \widehat{S}_s - \widehat{S}_{s-1}$).

As an investor, you may want to choose your trading strategy in some optimal way. Our notion of optimality is based on maximization of expected utility of discounted one-period gains.

Definition 5.6 *(i) We call a function $U : \mathbb{R} \to \mathbb{R}$ utility function if*

(a) U is twice continuously differentiable.

(b) The derivatives U', U'' are bounded and $\lim_{x\to\infty} U'(x) = 0$.

(c) $U(0) = 0$, $U'(0) = 1$.

(d) $U'(x) > 0$ for any $x \in \mathbb{R}$.

(e) $U''(x) < 0$ for any $x \in \mathbb{R}$.

(ii) By $\mathcal{L}^1(\widehat{S})$ we denote the set of all trading strategies φ with $E(\sum_{t=1}^{T} \left|\varphi_t^T \Delta \widehat{S}_t\right|) < \infty$.

(iii) We call a strategy $\varphi \in \mathcal{M} \cap \mathcal{L}^1(\widehat{S})$ U-optimal for $\mathcal{M}$ if

$$E\left(\sum_{t=1}^{T} U(\Delta V_t(\varphi))\right) \geq E\left(\sum_{t=1}^{T} U(\Delta V_t(\tilde{\varphi}))\right)$$

for any $\tilde{\varphi} \in \mathcal{M} \cap \mathcal{L}^1(\widehat{S})$ (where $\Delta V_t(\varphi) := V_t(\varphi) - V_{t-1}(\varphi)$).

Remark 5.7 *(i) Properties (d) and (e) (monotonicity and concavity) are common intuitive postulates for utility functions. Assumptions (a) and (b) are made for mathematical ease and to allow results that are applicable to a wide class of security price processes (i.e. under relatively mild moment conditions). (c) is just a convenient normalization that does not affect the generality of the approach.*

(ii) A typical example for such a utility function is $U_\kappa : \mathbb{R} \to \mathbb{R}$, $x \mapsto \frac{1}{\kappa}(1+\kappa x - \sqrt{1+\kappa^2 x^2})$, where the parameter $\kappa = -U''(0) \in (0, \infty)$ can be interpreted as the investor's risk aversion. For small κ the mapping U_κ is close to linear around the origin, whereas for $\kappa \to \infty$ it converges pointwise to the function $x \mapsto 2\min(x, 0)$. A large value of κ is advisable for a hedger who is not primarily interested in profits but above all wants to avoid losses. Since the mappings U_κ are of a simple analytic form, we call them standard utility functions.

(iii) The conditions $\varphi \in \mathcal{L}^1(\widehat{S})$ and in Definition 5.6(i) ensure that $\sum_{t=1}^{T} U(\Delta V_t(\varphi))$ is integrable. More precisely,

$$\begin{aligned} E\left(\left|\sum_{t=1}^{T} U(\Delta V_t(\varphi))\right|\right) &\leq \sum_{t=1}^{T} E\left(|U(\Delta V_t(\varphi))|\right) \leq \\ &\leq T \cdot c \cdot E(\Delta V_t(\varphi)) < \infty \end{aligned}$$

for some constant c since U' is bounded.

(iv) Since $\varphi_t \mapsto E(U(\Delta V_t(\varphi))) = E(U(\varphi_t \cdot \Delta\widehat{S}_t))$ can be maximized independently for $t = 1, \ldots, T$, the following equivalence holds. $\varphi \in \mathcal{M} \cap \mathcal{L}^1(\widehat{S})$ U-optimal for $\mathcal{M}$ if and only if

$$E\Big(U(\varphi_t \cdot \Delta\widehat{S}_t))\Big) \geq E\Big(U(\tilde{\varphi}_t \cdot \Delta\widehat{S}_t)\Big) \tag{5.7}$$

for any $t \in \{1, \ldots, T\}$ and any $\tilde{\varphi} \in \mathcal{M} \cap \mathcal{L}^1(\widehat{S})$. This shows that U-optimality effectively means maximization of the expected utility of one period gains. Moreover, we see that the choice of the terminal date T does not effect the optimality of a portfolio (as long as T is remote).

5.3.2 Continuous Time

In this section we turn to continuous-time processes. The general mathematical framework is as in the preceding subsection. We assume that $\widehat{S}$ is a $\mathbb{R}^{m+1}$-valued special semimartingale with truncation function h and semimartingale characteristics (B, C, υ) respectively differential characteristics (b, c, F, A). By Proposition 1.60 and Proposition 1.61 one can write (B, C, υ) in the form

$$B_t + (x - h(x)) * \nu_t = \int_0^t b_s dA_s, \quad C_t = \int_0^t c_s dA_s, \quad \nu = A \otimes F, \tag{5.8}$$

since $\widehat{S}$ is special. We furthermore assume that A is an right-continuous increasing function from $\mathbb{R}_+ \to \mathbb{R}^m$ and therefore deterministic.

We denote by $\mathcal{L}^1(\widehat{S})$ the space of all trading strategies φ satisfying (5.1) according to Definition 5.1.

We want to extend the approach from the previous section to such markets. However, the notion of optimality in Definition 5.7 is based on one-period gains which do not have a natural counterpart in continuous time. One may consider U-optimality in spirit of Condition 5.7 relative to some time grid with fixed small mesh size Δt, but the resulting optimal strategy will usually depend on the chosen time interval Δt, which is unsatisfactory. Nevertheless, the situation is not hopeless. For small Δt, the following limit theorem holds.

Theorem 5.8 *Let $(\Sigma_n)_{n\in\mathbb{N}}$ be a sequence of discrete sets $\Sigma_n = \{t_0^n, \ldots, t_{m_n}^n\}$ with $0 = t_0^n < t_1^n < \ldots < t_{m_n}^n = T$. Assume that $\|\Sigma_n\| := \sup\{t_i^n - t_{i-1}^n : i \in \{1, \ldots, m_n\}\} \to 0$ for $n \to \infty$. Let φ be any trading strategy in $\mathcal{L}^1(\widehat{S})$. Then we have*

$$E\left(\sum_{i=1}^{m_n} U(V_{t_i^n}(\varphi) - V_{t_{i-1}^n}(\varphi))\right) \to E\left(\int_0^T \gamma_t(\varphi_t)dA_t\right) \quad \text{for } n \to \infty, \tag{5.9}$$

where $\gamma_t(.)$ is defined as follows.

Definition 5.9 *For any $\psi \in \mathbb{R}^{m+1}$, $t \in \mathbb{R}^+$ we call the random variable*

$$\gamma_t(\psi) := \psi^T b_t + \frac{U''(0)}{2}\psi^T c_t \psi + \int (U(\psi^T x) - \psi^T x)F_t(dx) \tag{5.10}$$

local utility of ψ in t.

The rather lengthy and technical proof of the theorem above can be found in (Kallsen 1999).

Remark 5.10 *(i) Interpretation: The first term in Equation 5.10 is linear and naturally reflects the drift parts of the stock prices. The second term represents the disadvantage for the investor due to the concavity of the utility function U and finally the last term corresponds to the jumps of the stock prices.*

(ii) If $\widehat{S}$ is as in the preceding subsection, the convergence in Theorem 5.8 is even an equality for any n large enough.

The previous theorem inspires the following definition.

Definition 5.11 *We call a strategy $\varphi \in \mathcal{M}\cap\mathcal{L}^1(\widehat{S})$ U-optimal for $\mathcal{M}$ if*

$$E\left(\int_0^T \gamma_t(\varphi_t)dA_t\right) \geq E\left(\int_0^T \gamma_t(\tilde{\varphi}_t)dA_t\right) \tag{5.11}$$

for any $\tilde{\varphi} \in \mathcal{M}\cap\mathcal{L}^1(\widehat{S})$.

The condition $\varphi \in \mathcal{L}^1(\widehat{S})$ ensures that $E(\int_0^T \gamma_t(\varphi_t)dA_t)$ and also $V(\varphi)$ actually exist. One may argue that U-optimal strategies may not be practically feasible in some cases since we have hardly restricted the set of portfolios under consideration. The following lemma from (Kallsen 1999) indicates that any U-optimal strategy con usually be approximated in a suitable way by very simple portfolios.

Lemma 5.12 *Suppose that*

$$\int_0^T \left(|b_t| + |c_t| + \int |x|^2 \wedge |x| \, F_t(dx) \right) dA_t \in L^1(\Omega, \mathcal{F}, P). \tag{5.12}$$

Then for any $\varphi \in \mathcal{M} \cap \mathcal{L}^1(\widehat{S})$ *there exits a sequence of strategies* $(\varphi^{(n)})_{n \in \mathbb{N}}$ *in* $\mathcal{M} \cap \mathcal{L}^1(\widehat{S})$ *such that we have*

a) $\varphi^{(n)} = \sum_{i=1}^{m_n} \xi_i^{(n)} \chi_{(T_{i-1}^{(n)}, T_i^{(n)}]}$ *for some* $m_n \in \mathbb{N}$*, stopping times* $0 \leq T_0^{(n)} \leq \ldots \leq T_{m_n}^{(n)} \leq T$ *and bounded,* $\mathcal{F}_{T_{i-1}^{(n)}}$*-measurable,* $\mathbb{R}^{m+1}$*-valued random variables* $\xi_i^{(n)}$ *for* $i = 1, \ldots, m_n$*,*

b) $\lim_{n \to \infty} \varphi^{(n)} \to \varphi$ $(P \otimes A)$*-almost surely on* $\Omega \times [0, T]$*,*

c) $\lim_{n \to \infty} E\left(\int_0^T \gamma_t(\varphi_t^{(n)}) dA_t \right) = E\left(\int_0^T \gamma_t(\varphi_t) dA_t \right)$.

Remark 5.13 *Clearly,* $\varphi \in \mathcal{M} \cap \mathcal{L}^1(\widehat{S})$ *is* U*-optimal for* $\mathcal{M}$ *if and only if, for any* $\tilde{\varphi} \in \mathcal{M} \cap \mathcal{L}^1(\widehat{S})$*, we have* $\gamma_t(\varphi_t) \geq \gamma_t(\tilde{\varphi}_t)$ $(P \otimes A)$*-almost surely on* $\Omega \times [0, T]$*.*

The following proposition additionally characterizes the uniqueness of U-optimal strategies.

Proposition 5.14 *A strategy* $\varphi \in \mathcal{M} \cap \mathcal{L}^1(\widehat{S})$ *is* U*-optimal for* $\mathcal{M}$ *if and only if, outside some* $(P \otimes A)$*-null set* N*, we have* $\gamma_t(\varphi_t) \geq \gamma_t(\psi)$ *for any* $\psi \in \mathbb{R} \times M$*.*

Proof.

$\Longrightarrow$: This direction is obvious due to the monotonicity of the integral in (5.11) with respect to γ_t.

$\Longleftarrow$: Fix $\psi \in \mathbb{R} \times M$. For any $n \in \mathbb{N}$ let $E_n \subseteq \Omega \times [0, T]$ denote the predictable set of points (ω, t) where

$$(\left|\psi^T b_t\right| + \psi^T c_t \psi + \int (\psi^T x)^2 \wedge \left|\psi^T x\right|) F_t(dx)(\omega) \leq n. \tag{5.13}$$

Note that $\psi \chi_{E_n} + \varphi \chi_{E_n^c} \in \mathcal{M} \cap \mathcal{L}^1(\widehat{S})$ for any $n \in \mathbb{N}$. From $E_n \uparrow \Omega \times [0, T]$ for $n \to \infty$ and Remark 5.13 it follows that there exists a $(P \otimes A)$-null set N^ψ such that $\gamma_t(\varphi_t) \geq \gamma_t(\psi)$ outside N^ψ. Fix a countable dense subset D of $\mathbb{R} \times M$ and let $N := \cup_{\psi \in D} N^\psi$. Since the mapping $\psi \mapsto \gamma_t(\psi)(\omega)$ is continuous for $(P \otimes A)$-almost (ω, t), the assertion follows.

■

Lemma 5.15 *Fix $(\omega, t) \in \Omega \times [0,T]$ and define the mapping $h : \mathbb{R}^{m+1} \to \mathbb{R}$ by $h(\psi) := -\gamma_t(\psi)$ for $\psi \in \mathcal{M} \cap \mathcal{L}^1(\widehat{S})$. Then h is a convex mapping.*

Proof. To see this we examine the terms of $\gamma_t(\psi)$ in Equation 5.10 separately: The first term and the second part of the integral are linear with respect to ψ. The second term is concave because $U'' < 0$ and the first part of the integral is again concave because U is concave. Therefore, $\psi \mapsto \gamma_t(\psi)$ is a concave mapping and therefore h convex. ■

As suspected, the computation of optimal strategies is relatively easy. They can be determined using only the local behavior of $\widehat{S}$.

Theorem 5.16 *A strategy $\varphi \in \mathcal{M} \cap \mathcal{L}^1(\widehat{S})$ is U-optimal of $\mathcal{M}$ if and only if the following condition holds: For $(P \otimes A)$-almost all $(\omega, t) \in \Omega \times [0,T]$ there exist $\lambda_1, \ldots, \lambda_q \in \mathbb{R}$ with $\lambda_j \geq 0$ and $\lambda_j g^j(\varphi_t) = 0$ for $j = 1, \ldots p$ such that*

$$b_t^i + U''(0) c_t^{i\cdot} \varphi_t + \int x^i (U'(\varphi_t^T x) - 1) F_t(dx) \tag{5.14}$$

$$- \sum_{j=1}^{q} \lambda_j D_i g^j(\varphi_t) = 0 \quad \textit{for } i = 0, \ldots, m$$

where $D_i f$ denotes the i-th partial derivative of a function $f : \mathbb{R}^{m+1} \to \mathbb{R}$.

Proof. According to e.g. (Billingsley 1979), the function h defined in Lemma 5.15 is differentiable with partial derivatives

$$D_i h(\psi) = -b_t^i - U(0)'' c_t^{i\cdot} \psi - \int x^i (U'(\psi^T x) - 1) F_t(dx).$$

Let (P) denote the convex optimization problem corresponding to $h : \mathbb{R}^{m+1} \to \mathbb{R}$ with the constraints $g^j(\psi) \leq 0$ for $j = 1, \ldots, p$ and $g^j(\psi) = 0$ for $j = p+1, \ldots, q$ in the sense of Definition A.5.

$\Longrightarrow$: Outside the $(P \otimes A)$-null set from Proposition 5.14, we have that $\varphi_t(\omega)$ is an optimal solution to the corresponding convex program (P). As known from basic results about convex optimization (see Theorem A.6), there exists a Kuhn-Tucker vector $(\lambda_1, \ldots, \lambda_q)$ which yields the assertion because of Theorem A.7 and Theorem A.8.

$\Longleftarrow$: As known from basic results about convex optimization (see Theorem A.8), $\varphi_t(\omega)$ is an optimal solution to (P). Hence $\gamma_t(\varphi_t)(\omega) \geq \gamma_t(\psi)(\omega)$ for any $\psi \in \mathbb{R} \times M$ and Proposition 5.14 completes the proof.

■

Remark 5.17 *As pointed out in (Kallsen 1999), a careful inspection of the proof of Proposition 5.14 and Theorem 5.16 reveals that all statements remain true in the slightly more general setting of $A \in \mathcal{A}_{loc}^+$. Note that the local utility depends on the chosen process A. However, the definition of U-optimality and the other statements in this section to not depend on the particular choice of A.*

Definition 5.18 *In the following, the family of trading strategies under consideration is the set Φ' of all predictable $\mathbb{R}^m$-valued processes $\varphi = (\varphi^1, \ldots, \varphi^m)$ satisfying the integrability condition $\varphi \in \mathcal{L}^1(S)$. Similarly to above, we denote by $\Phi'(\Gamma)$ the set of all trading strategies in Φ' meeting the cone constraints Γ.*

Assumption 5.19 *In order to avoid technical proofs in the setting of local utility, we assume that der exists polyhedral cones $K_1, \ldots, K_n \subseteq \mathbb{R}^m$ and predictable sets $D_1, \ldots, D_n$ such that*

$$\Gamma_t(\omega) = \cap_{\{i \in \{1,\ldots,n\}:(\omega,t) \in D_i\}} K_i \text{ for } (\omega, t) \in \Omega \times [0, T]. \tag{5.15}$$

Remark 5.20 *Using Theorem A.3, this particularly implies that for every pair $(\omega, t) \in \Omega \times [0, T]$ there exits a finite collection of vectors $a_1, \ldots, \alpha_m$ such that*

$$\Gamma_t(\omega) = \left\{ x \in \mathbb{R}^m : a_i^T x \leq 0 \text{ for } i = 1, \ldots, m \right\}, \tag{5.16}$$

i.e. the functions g^j forming the constraints are of the form $g^j(x) = a_j^T x$ for $j = 1, \ldots, p$ and $Dg^j \equiv a_j$ for $j = 1, \ldots, p$.

In constrained markets, locally optimal portfolios can again be determined by pointwise solution of equations in $\mathbb{R}^m$:

Theorem 5.21 *A trading strategy $\varphi \in \Phi'(\Gamma)$ is locally optimal under the constraints Γ if and only if*

$$b_t^i + U''(0) c_t^{i \cdot} \varphi_t + \int x^i (U'(\varphi_t^T x) - 1) F_t(dx) \in \Gamma_t^\circ \tag{5.17}$$

$(P \otimes A)$-almost everywhere, where $\Gamma_t^\circ := \{y \in \mathbb{R}^m : x^T y \leq 0 \text{ for any } x \in \Gamma_t\}$ denotes the polar cone of Γ_t.

Proof. In view of Farkas' Lemma A.4 and Remark 5.20, the Theorem follows from Theorem 5.16. Strictly speaking, Theorem 5.16 considers a narrower set-up where A and Γ are deterministic. As it is pointed out in Remark 5.17, the statements remain valid for $A \in \mathcal{A}_{loc}^+$. Moreover, a careful inspection of the Proposition 5.14 and Theorem 5.16 reveals that these results hold for random constraints of the above type as well. ■

Chapter 6

Neutral Pricing

In this section we turn to derivative pricing. More exactly, we propose a way to extend a market model for the underlyings to a model for both underlyings and derivatives. In a sense, the approach mimics the reasoning in complete models, but under stronger preferences.

In complete models there exist unique arbitrage-free derivative values. The assertion that real market prices have to coincide with these values can be easily justified. It suffices to assume the existence of traders who exploit favorable market conditions once they detect them. The existence of such derivative speculators explains why the market price cannot deviate too strongly from the right value: If it did, the huge demand for (respectively supply of) the mispriced security would push its price immediately closer to the rational value. The only assumption on the preferences of the speculators is that they do not reject riskless profits - which most people may agree on. The elegance of this approach comes at a price. It only works in complete models, or more exactly, for attainable claims.

We want to extend this reasoning on incomplete markets by imposing stronger assumptions on the preferences of derivative speculators. We suppose that they trade by maximizing a certain kind of utility. The role of the arbitrage-free price will now be played by the neutral derivative value. This is the unique price such that the speculators's optimal portfolio contains no contingent claim. Similarly as in the complete case we argue that the speculators' presence should prevent the market price from deviating too strongly from the neutral value.

6.1 Terminal Wealth

6.1.1 The Neutral Pricing Measure

Optimal strategies for terminal wealth are often characterized in terms of EMM's. They plays a key role in many papers that apply martingale or duality methods to utility maximization. Suppose that φ is an optimal strategy for terminal wealth without constraints (i.e. for $\Gamma = \mathbb{R}^d$). If the probability space is finite, then $\frac{U'(V_T(\varphi))}{E(U'(V_T(\varphi)))}$ is the density of some equivalent martingale measure (EMM), see e.g. (Kallsen 2001). In addition, this measure solves some dual minimization problem, according to (Schachermayer 2001), Theorem 2.3. In general markets, the density process of P^* is replaced with a supermartingale which may not be the density process of a probability measure, let alone an EMM, see (Kramkov and Schachermayer 1999), Examples 5.1. Nevertheless, in many models of practical importance the dual measure P^* exists and it is at least a σ-martingale measure, i.e. $\widehat{S}^1, \dots \widehat{S}^m$ are σ-martingales relative to P^*. Since it plays a key role in the neutral pricing approach we call P^* neutral pricing measure for terminal wealth.

Definition 6.1 *Suppose that φ is an optimal strategy for terminal wealth without constraints and, moreover, has finite expected utility. If $\frac{U'(V_T(\varphi))}{E(U'(V_T(\varphi)))}$ is the density of some σ-martingale measure P^*, we call P^* dual measure or neutral pricing measure for terminal wealth.*

6.1.2 The Pricing Formula

We are now ready to turn to the valuation of game contingent claims. The general setting is as in the previous section. We distinguish two kinds of securities: underlyings $1, \dots, m$ and derivatives $m+1, \dots, m+n$. From now on we assume that the sock prices $S^1, \dots, S^m$ are already discounted and that the derivatives are game contingent claims with discounted exercise process L^i and discounted cancellation process U^i, where L^i and U^i are semimartingales with $L^i < U^i$ as well as $L^i_- < U^i_-$ on $[0, T)$ (this has a purely technical reason) and $L^i_T = U^i_T$ for $i = m+1, \dots, m+n$. European and American options are treated as special cases of game contingent claims as it is explained in Remark 6.5 below. We call semimartingales $S^{m+1}, \dots, S^{m+n}$ derivative price processes if $L^i \leq S^i \leq U^i$ for $i = m+1, \dots, m+n$. As noted above, we are interested in derivative price processes that have a neutral effect on the market in the sense that they do not cause any supply of or demand for contingent claims by derivative speculators.

Speculators may not be able to hold arbitrary amounts of game contingent claims because these contracts can be cancelled. If the market price approaches the upper cancellation value U^i, it may happen that all options vanish from the market because they are terminated by the sellers. Thus a long position in the option is no longer feasible. Conversely, all derivative contracts may be exercised by the claim holders if the market price coincides with the exercise value L^i. This terminates short positions in the claim. However, as long as the derivative price stays above the exercise value, nobody will exercise the option because selling it on the market yields higher reward. Similarly, there is no danger that the seller of a GCC cancels the contract as long as the cancellation value exceeds the market price. Summing up, the derivative speculators are facing trading constraints Γ given by

$$\Gamma_t = \{x \in \mathbb{R}^{m+n}\text{: For } i = m+1, \dots, m+n \text{ we have } x^i \geq 0 \text{ if } S^i_{t-} = L^i_{t-} \text{ and } x^i \leq 0 \text{ if } S^i_{t-} = U^i_{t-}\}. \tag{6.1}$$

We start by assuming that derivative speculators are identical investors trying to maximize expected utility from terminal wealth. Moreover, we suppose that the neutral pricing measure for terminal wealth P^* in the sense of Definition 6.1 exists for the underlyings's market $S^1, \dots, S^m$. At this stage, the only information on the derivatives is their discounted terminal payoffs $R^{m+1}, \dots, R^{m+n}$ at time T, which are supposed to be $\mathcal{F}_T$-measurable random variables. As explained above, our goal is to determine neutral price processes in the sense of the following

Definition 6.2 *We call derivative price processes $S^{m+1}, \dots, S^{m+n}$ neutral for terminal wealth if there exists a strategy $\overline{\varphi}$ in the extended market $S^1, \dots, S^{m+n}$ which is optimal for terminal wealth under the constraints Γ an satisfies $\overline{\varphi}^{m+1} = \dots = \overline{\varphi}^{m+n} = 0$.*

The following main result of this section treats existence and uniqueness of neutral derivative price processes. Moreover, it shows that they are recovered as the value of a Dynkin game relative to the neutral pricing measure P^*.

Theorem 6.3 *Suppose that $\sup_{t\in[0,T]} \left|L_t^i\right|$ and $\sup_{t\in[0,T]} \left|U_t^i\right|$ are P^*-integrable for $i = m+1, \dots, m+n$. Then there exist neutral derivative price processes. These are given by*

$$\begin{aligned} S_t^i &= essinf_{\tau^U \in \mathcal{T}(t,T)}\ esssup_{\tau^L \in \mathcal{T}(t,T)} E_{P^*}(R^i(\tau^L, \tau^U)|\mathcal{F}_t) \qquad (6.2)\\ &= esssup_{\tau^L \in \mathcal{T}(t,T)}\ essinf_{\tau^U \in \mathcal{T}(t,T)} E_{P^*}(R^i(\tau^L, \tau^U)|\mathcal{F}_t) \end{aligned}$$

for $t \in [0,T]$, $i = m+1, \dots, m+n$, where $\mathcal{T}(t,T)$ denotes the set of $[t,T]$-valued stopping times and

$$R^i(\tau^L, \tau^U) := \begin{cases} L^i_{\tau^L} & \text{if } \tau^L \leq \tau^U \\ U^i_{\tau^U} & \text{otherwise.} \end{cases} \qquad (6.3)$$

If there exists an $x \in \mathbb{R}$ and an admissible strategy $\widehat{\varphi}$ such that

$$V_t^{\widehat{\varphi},x} = x + (\widehat{\varphi}^T \cdot S)_t \geq U_s^i \quad \text{for } t \in [0,T],\ i = m+1, \dots m+n \qquad (6.4)$$

and $V_t^{\widehat{\varphi},x}$ is a P^*-martingale, then the neutral derivative price processes defined in (6.2) are unique. In any case, the extended market $S^1, \dots, S^{m+n}$ satisfies condition NFLVR in the sense of the definition below.

Definition 6.4 *We say that the market $S = (S^1, \dots, S^{m+n})$ satisfies the condition no free lunch with vanishing risk (NFLVR) if 0 is the only non-negative element of the $L^\infty(\Omega, \mathcal{F}, P)$-closure of the set $C := \{f \in L^\infty(\Omega, \mathcal{F}, P) : f \leq \psi^T S_T$ for some $\psi \in \Phi(\Gamma)\}$.*

PROOF OF THEOREM 6.3: For clarity we split the proof into three parts: existence, uniqueness and the NFLVR condition.

EXISTENCE:

Step 1: By Theorem B.5 there exist right-continuous adapted processes S^{m+1}, $\dots, S^{m+n}$ satisfying Equation 6.2 if $L^{m+1}, \dots, L^{m+n}$ and $U^{m+1}, \dots U^{m+n}$ are bounded. As (Kifer 2000) points out, these results of (Lepeltier et Maingueneau 1984) hold also true if L^i, U^i satisfy the above integrability conditions. Fix $i \in \{m+1, m+n\}$. Define stopping times

$$T_1^k := \inf\left\{t \in \mathbb{R}_+ : S_t^i \geq U_t^i - \tfrac{1}{k}\right\} \quad \text{for } k \in \mathbb{N},\ T_1 := \sup_{k\in\mathbb{N}} T_1^k. \qquad (6.5)$$

By Theorem B.5 $(S^i)^{T_1^k}$ is a P^*-supermartingale for any $k \in \mathbb{N}$, which is càdlàg due to (Dellacherie and Meyer 1982), Theorem $VI.3$, . Obviously, $(S^i)^{T_1^k}$ converges for $k \to \infty$ P^*-almost surely to

$$R := U_{T_1} \chi_{\cup_{k\in\mathbb{N}}\{T_1^k = T_1\}} + U_{T_1-} \chi_{\cap_{k\in\mathbb{N}}\{T_1^k < T_1\}}. \qquad (6.6)$$

Define an adapted right-continuous process $\overline{S}^i$ by

$$\overline{S}^i_t := \begin{cases} S^i_t & \text{if } t < T_1 \text{ or } t = 0 \\ U_{T_1-} & \text{if } 0 \neq t \geq T_1 \text{ and } T_1^k < T_1 \text{ for any } k \in \mathbb{N} \\ U_{T_1} & \text{if } 0 \neq t \geq T_1 \text{ and } T_1^k = T_1 \text{ for some } k \in \mathbb{N} \end{cases}, \tag{6.7}$$

i.e.

$$\overline{S}^i_t = \sum_{k\in\mathbb{N}} (S^i)^{T_1^k} \chi_{(T_1^{k-1}, T_1^k]} + R\chi_{(\cup_{k\in\mathbb{N}}[0,T_1^k])^c} \tag{6.8}$$

with the convention $(T_1^{-1}, T_1^0] := [T_1^0]$.

Let $s, t \in [0, T]$ with $s \leq t$. If $s \in (\cup_{k\in\mathbb{N}}[0, T_1^k])^c$, then $\overline{S}^i_s = R = \overline{S}^i_t$ and hence $E_{P^*}(\overline{S}^i_t|\mathcal{F}_s) = \overline{S}^i_s$. Now let $s \in (T_1^{k-1}, T_1^k]$ for some $k \in \mathbb{N}$. Then

$$\overline{S}^i_s = S^i_s = (S^i)^{T_1^l}_s \geq E_{P^*}((S^i)^{T_1^l}_t|\mathcal{F}_s) = E_{P^*}((\overline{S}^i)_{T_1^l\wedge t}|\mathcal{F}_s) \quad \text{for } l \geq k. \tag{6.9}$$

Moreover, dominated convergence yields that $\overline{S}^i_{T_1^l\wedge t} \to \overline{S}^i_t$ in L^1 for $l \to \infty$, since we already have $(\overline{S}^i)_{T_1^l\wedge t} \to \overline{S}^i_t$ pointwise. Consequently, we also have $E_{P^*}((\overline{S}^i)_{T_1^l\wedge t}|\mathcal{F}_s) \to E_{P^*}(\overline{S}^i_t|\mathcal{F}_s)$ in L^1 for $l \to \infty$ and therefore $a.s$, because

$$\begin{aligned} &E_{P^*}\left(\left|E_{P^*}((\overline{S}^i)_{T_1^l\wedge t}|\mathcal{F}_s) - E_{P^*}(\overline{S}^i_t|\mathcal{F}_s)\right|\right) \\ &\leq E_{P^*}\left(E_{P^*}\left(\left|\overline{S}^i_{T_1^l\wedge t} - \overline{S}^i_t\right| |\mathcal{F}_s\right)\right) \\ &= E_{P^*}\left(\left|\overline{S}^i_{T_1^l\wedge t} - \overline{S}^i_t\right|\right) \to 0. \end{aligned} \tag{6.10}$$

Hence $\overline{S}^i_s \geq E_{P^*}(\overline{S}^i_t|\mathcal{F}_s)$. Altogether, it follows that $\overline{S}^i$ is a P^*-supermartingale. Hence, $(S^i)^{T_1}$ is a semimartingale.

For $l \in \mathbb{N}\backslash\{0, 1\}$ define $T_l := \sup_{k\in\mathbb{N}} T_l^k$ where

$$\begin{aligned} T_l^k &:= \inf\{t \geq T_{l-1} : S^i_t \leq L^i_t + \tfrac{1}{k}\} \quad \text{for } l = 2, 4, 6, \ldots \\ T_l^k &:= \inf\{t \geq T_{l-1} : S^i_t \geq U^i_t - \tfrac{1}{k}\} \quad \text{for } l = 3, 5, 7, \ldots \end{aligned} \tag{6.11}$$

Similarly to above, one shows by induction that $(S^i)^{T_l}$ is a semimartingale for any $l \in \mathbb{N}$. To proof this, we use that $(S^i)^{T_{l-1}}$ and $(S^i)^{T_l}$ coincide on $[0, T_{l-1})$ and that $(S^i)^{T_l}$ is a semimartingale on $[T_{l-1}, T]$ imitating the steps above.
Step 2: We keep the notation form the previous step. Fix $l \in \mathbb{N}$. For $t_0 \in [0, T]$ and $k \in \mathbb{N}$ define stopping times

$$\tau_{t_0,k} := \inf\left\{t \geq t_0 : (S^i)^{T_l}_t \leq (L^i)^{T_l}_t + \tfrac{1}{k}\right\} \wedge T. \tag{6.12}$$

From Theorem B.5 with $X' = -U^{T_l}$, $X = L^{T_l}$ and $\underline{X} = S^{T_l}$ it follows that $\chi_{(t_0,\tau_{t_0,k}]} \cdot (S^i)^{T_l}$ is a P^*-submartingale for any $t_0 \in [0, T]$, $k \in \mathbb{N}$. In particular, we have from Lemma 1.67 and Lemma 1.71

$$b^* + \int (x - h(x))F * (dx) \geq 0 \tag{6.13}$$

$(P \otimes A)$-almost everywhere on $(t_0, \tau_{t_0,k}]$, where (b^*, c^*, F^*, A^*) denote the P^*-differential characteristics of the semimartingale $(S^i)^{T_l}$. Since

$$\left\{(L^i)^{T_l}_- < (S^i)^{T_l}_-\right\} \cap (0, T] = \cup_{t_0\in\mathbb{Q}\cap[0,T]} \cup_{k\in\mathbb{N}} (t_0, \tau_{t_0,k}], \tag{6.14}$$

it follows that Equation 6.13 holds $(P \otimes A)$-almost everywhere on $\{(L^i)^{T_l}_- < (S^i)^{T_l}_-\}$. Therefore, $\chi_{\left\{(L^i)^{T_l}_- < (S^i)^{T_l}_-\right\}} \cdot (S^i)^{T_l}$ is a P^*-σ-submartingale according to Lemma 1.71 and Lemma 1.67.
Analogously, it follows that $\chi_{\left\{(S^i)^{T_l}_- < (U^i)^{T_l}_-\right\}} \cdot (S^i)^{T_l}$ is a P^*-σ-supermartingale, and hence $\chi_{\left\{(L^i)^{T_l}_- < (S^i)^{T_l}_- < (U^i)^{T_l}_-\right\}} \cdot (S^i)^{T_l}$ is a P^*-σ-martingale.
Step 3: We keep keep the notation from the previous steps. Let $T_\infty := \lim_{l\to\infty} T_l$. It is obvious that $T_\infty \le T$, because this holds true for all stopping times T_l. Since L^i, U^i are P^*-special semimartingales with integrable L^i_0, U^i_0 they are locally in class $\mathcal{H}^1$ in the sense of Definition 1.74 and relative to P^* (see (Dellacherie and Meyer 1982), $VII.99$). Denote by $(\sigma_k)_{k\in\mathbb{N}}$ a corresponding localizing sequence. Fix $k \in \mathbb{N}$. By Proposition 1.75, applied to $(L^i)^{T_l\wedge\sigma_k}, (S^i)^{T_l\wedge\sigma_k}$ and $(U^i)^{T_l\wedge\sigma_k}$, it follows that

$$\begin{aligned}\sup_{l\in\mathbb{N}} \left\|(S^i)^{T_l\wedge\sigma_k}\right\|_{\mathcal{H}^1} &= \lim_{l\to\infty} \left\|(S^i)^{T_l\wedge\sigma_k}\right\|_{\mathcal{H}^1} \\ &\le \left\|(S^i)^{\sigma_k}\right\|_{\mathcal{H}^1} \\ &\le c\left(\left\|(L^i)^{\sigma_k}\right\|_{\mathcal{H}^1} + \left\|(U^i)^{\sigma_k}\right\|_{\mathcal{H}^1}\right) < \infty,\end{aligned} \tag{6.15}$$

which in turn implies that $(S^i)^{T_\infty\wedge\sigma_k}$ is a semimartingale due to Proposition 1.76 ($(S^i)^{T_n}$ is a semimartingale and because semimartingales are stable under stopping). Therefore, $(S^i)^{T_\infty}$ is a local semimartingale and hence a semimartingale. In particular, it has left-hand limits at T_∞. Now assume that $T_\infty(\omega) < T$ for some $\omega \in \Omega$. Since $L^i_{t-} < U^i_{t-}$ for $t < T$, this implies $L^i_{T_\infty(\omega)-}(\omega) < U^i_{T_\infty(\omega)-}(\omega)$ and therefore due to the definition of the stopping times T_l, that the left-hand limit of $(S^i)^{T_\infty}$ at T_∞ does not exist for ω. This is a contradiction and thus it is only possible that $T_\infty = T$. Consequently, $S^i = (S^i)^T = (S^i)^{T_\infty}$ is a semimartingale.
Step 4: Let Z denote the density process of P^* and φ an optimal strategy for terminal wealth in the market $S^1, \ldots, S^m$. We want to show that the $\mathbb{R}^{m+n}$-valued process $\overline{\varphi} := (\varphi, 0) \in \Phi(\Gamma)$ is an optimal strategy for terminal wealth under the constraints Γ, now referring to the extended market $S := (S^1, \ldots, S^{m+n})$. Since $ZE(U'(V_T(\varphi)))$ coincides with the optimal solution $\widehat{Y}(y)$ to the dual problem in (Kramkov and Schachermayer 1999), Theorem 2.2, we have that

$$V_t(\varphi)\widehat{Y}(y) = (\varphi^T \cdot (S^1, \ldots, S^m))ZE(U'(V_T(\varphi))) \tag{6.16}$$

is a (uniformly integrable) martingale and therefore also $(\varphi^T \cdot (S^1, \ldots, S^m))Z$. This implies that $\overline{\varphi}^T \cdot S = \varphi^T \cdot (S^1, \ldots, S^m)$ is a P^*-martingale due to Proposition 1.65.

Consider a trading strategy $\psi \in \Phi(\Gamma)$ in the extended market. Denote by (b^*, c^*, F^*, A) the P^*-characteristics of S. The same argument as in Step 2 shows that

$$b^{*,i} + \int (x^i - h^i(x))F^*(dx) \ge 0 \tag{6.17}$$

$(P \otimes A)$-almost everywhere on $\{L^i_- < S^i_-\}$ and ≤ 0 on $\{S^i_- < U^i_-\}$ for $i = m+1, \ldots, m+n$. Since $S^i, \ldots, S^m$ are P^*-σ-martingales, we have that

$$b^{*,i} + \int (x^i - h^i(x))F^*(dx) = 0 \quad \text{for } i = 1, \ldots m. \tag{6.18}$$

From the form of constraints Γ is follows that $\psi^i \leq 0$ for $\{S^i_- = U^i_-\} \subseteq \{L^i_- < S^i_-\}$ and that $\psi^i \geq 0$ for $\{L^i_- = S^i_-\} \subseteq \{S^i_- < U^i_-\}$. It follows with $\{L^i_- < S^i_- < U^i_-\} = \{L^i_- < S^i_-\} \cap \{S^i_- < U^i_-\}$ that

$$\psi^i \left(b^{*,i} + \int (x^i - h^i(x)) F^*(dx) \right) \leq 0 \quad \text{for } i = m+1, \dots m+n, \tag{6.19}$$

which yields that

$$\psi^T \left(b^* + \int (x - h(x)) F^*(dx) \right) \leq 0 \tag{6.20}$$

$(P \otimes A)$-almost everywhere. In view of Proposition 1.73 and Lemma 1.71, this implies that $\psi^T \cdot S$ is a P^*-σ-supermartingale. By Proposition 1.73 this process and hence also $(\psi - \overline{\varphi})^T \cdot S$ is even a P^*-supermartingale. In particular, we have

$$E\left(U'(V_T(\overline{\varphi}))((\psi - \overline{\varphi})^T \cdot S)\right) = E\left(U'(V_T(\overline{\varphi}))\right) E_{P^*}\left((\psi - \overline{\varphi})^T \cdot S\right) \leq 0. \tag{6.21}$$

Due to Lemma 5.5, $\overline{\varphi}$ is an optimal strategy for terminal wealth under the constraints Γ. Hence, $S^{m+1}, \dots, S^{m+n}$ are neutral price processes for terminal wealth.

UNIQUENESS: Assume that $\widetilde{S}^{m+1}, \dots, \widetilde{S}^{m+n}$ are neutral derivative price processes corresponding to some optimal strategy $\widetilde{\varphi} = (\widetilde{\varphi}^1, \dots, \widetilde{\varphi}^m, 0, \dots, 0)$ in the extended market $\widetilde{S} := (S^1, \dots, S^m, \widetilde{S}^{m+1}, \dots, \widetilde{S}^{m+n})$. Since $\widetilde{\varphi}$ does not contain any derivative, we have that $(\widetilde{\varphi}^1, \dots, \widetilde{\varphi}^m)$ is an optimal strategy for the small market $S^1, \dots, S^m$ with the same expected utility. Similarly, the expected utility of φ in the small market and of $\overline{\varphi} = (\varphi, 0)$ in the extended market $\widetilde{S}$ tally. Since φ is optimal in the small market $S^1, \dots, S^m$, it follows that $\overline{\varphi} \in \Phi'(\Gamma)$ is optimal in the extended market $\widetilde{S}$ under the constraints Γ. Hence we may w.l.o.g. assume that $\widetilde{\varphi} = \overline{\varphi}$.

Fix $i \in \{m+1, \dots, m+n\}$. Firstly, we show that $\chi_D \cdot \widetilde{S}^i$ is a P^*-σ-submartingale for any predictable subset D of $\{L^i_- < \widetilde{S}^i_-\}$. We define stopping times

$$T_k := \inf\{t \geq 0 : \left| \left(\chi_D \cdot \widetilde{S}^i\right)_t \right| > k \text{ or } V^{\widehat{\varphi},x}_t > k\}.$$

Due to condition 6.4 we always have $0 \leq \widetilde{S}^i \leq U^i \leq V^{\widehat{\varphi},x}$ and therefore

$$\left| \Delta(\chi_D \cdot \widetilde{S}^i)_{T_k} \right| = \left| \left(\chi_D \cdot \Delta\widetilde{S}^i\right)_{T_k} \right| \leq \max(V^{\widehat{\varphi},x}_{T_k}, V^{\widehat{\varphi},x}_{T_k-}) \leq k + V^{\widehat{\varphi},x}_{T_k},$$

which yields

$$\sup_{t \in [0,T]} \left| (\chi_D \cdot \widetilde{S}^i)^{T_k}_t \right| \leq 2k + V^{\widehat{\varphi},x}_{T_k}.$$

Note that $P^*(T_k = T) \to 1$, since $(\chi_D \cdot \widetilde{S}^i)$ and $V^{\widehat{\varphi},x}$ are pathwise càdlàg. Fix $k \in \mathbb{N}$, $s, t \in [0, T]$ with $s \leq t$, and $F \in \mathcal{F}_s$. Define an admissible strategy $\psi \in \Phi(\Gamma)$ in the market $\widetilde{S} := (S^1, \dots, S^m, \widetilde{S}^{m+1}, \dots, \widetilde{S}^{m+n})$ by

$$\psi := \psi_1 + \psi_2, \tag{6.22}$$

where

$$\psi^j_1 := \begin{cases} 0 & \text{for } j \neq i \\ -\frac{\varepsilon}{4}\left(\frac{1}{2k \vee x}\right) \chi_{D \cap [0,T] \cap (F \times (s,t])} & \text{for } j = i \end{cases}. \tag{6.23}$$

$$\psi_2 := \frac{\varepsilon}{4}\left(\frac{1}{2k \vee x}\right)\widehat{\varphi}^{T_k}. \tag{6.24}$$

One can easily check, that this strategy ψ is indeed admissible. Lemma 5.5 and the fact that $\overline{\varphi}^T \cdot \widetilde{S} = \varphi^T \cdot (S^1, \ldots, S^m)$ is a P^*-martingale (according to Step 4 of the existence proof) yield that

$$\begin{aligned} & -\tfrac{\varepsilon}{4k}\left(\tfrac{1}{k\vee x}\right) E_{P^*}(((V_t^{\widehat{\varphi},x})^{T_k} + (\chi_D \cdot \widetilde{S}^i)_t^{T_k} - (V^{\widehat{\varphi},x})_s^{T_k} - \\ & \qquad -(\chi_D \cdot \widetilde{S}^i)_s^{T_k})\chi_F) = \qquad (6.25)\\ = \;& E_{P^*}((\psi - \overline{\varphi})^T \cdot \widetilde{S}_T) + E_{P^*}(\overline{\varphi}^T \cdot \widetilde{S}_T) \\ = \;& E\left(U'(V_T(\overline{\varphi}))\right)^{-1} E(U'(V_T(\overline{\varphi}))((\psi - \overline{\varphi})^T \cdot \widetilde{S}_T)) \\ \leq \;& 0. \end{aligned}$$

Therefore, $(\chi_D \cdot \widetilde{S}^i)^{T_k}$ is a P^*-submartingale, which implies that $\chi_D \cdot \widetilde{S}^i$ is a local P^*-submartingale. Similarly by replacing ψ with $-\psi$, it follows $\chi_D \cdot \widetilde{S}^i$ is a P^*-σ-supermartingale for any predictable subset D of $\{\widetilde{S}^i_- < U^i_-\}$.

Define stopping times

$$\tau_{t_0,k} := \inf\{t \geq t_0 : S_t^i \leq \widetilde{S}_t^i + \tfrac{1}{k}\} \quad \text{for any } t_0 \in [0,T],\, k \in \mathbb{N}. \tag{6.26}$$

Since S_t^i and $\widetilde{S}_t^i$ coincide at $t = T$, we always have $\tau_{t_0,k} \leq T$ for any $t_0 \in [0,T]$, $k \in \mathbb{N}$. Note that

$$\{S^i_- > \widetilde{S}^i_-\} \cap (0,T] = \cup_{t_0 \in \mathbb{Q}\cap[0,T]} \cup_{k\in\mathbb{N}} (t_0, \tau_{t_0,k}]. \tag{6.27}$$

Fix $t_0 \in [0,T]$, $k \in \mathbb{N}$. Since

$$\{L^i_- < S^i_-\} \cap \{\widetilde{S}^i_- < U^i_-\} \supseteq \{S^i_- > \widetilde{S}^i_-\}, \tag{6.28}$$

because of the constraints 6.1, we have that $\chi_{(t_0,\tau_{t_0,k}]} \cdot S^i$ and hence also $((S^i)_t^{\tau_{t_0,k}})_{t\in[t_0,T]}$ is a P^*-σ-submartingale. By Proposition 1.73 this process is even a P^*-submartingale. Similarly, it follows that $((S^i)_t^{\tau_{t_0,k}})_{t\in[t_0,T]}$ is a P^*-supermartingale. Since $(S^i)_T^{\tau_{t_0,k}} \leq (\widetilde{S}^i)_T^{\tau_{t_0,k}} + \frac{1}{k}$ we have $(S^i)_{t_0}^{\tau_{t_0,k}} \leq (\widetilde{S}^i)_{t_0}^{\tau_{t_0,k}} + \frac{1}{k}$ P-almost surely for any $k \in \mathbb{N}$. To see this, assume that the P^*-martingale $(\widetilde{S}^i - S^i)_t^{\tau_{t_0,k}} + \frac{1}{k}$ is not non-negative.

Consequently, $S^i_{t_0} \leq \widetilde{S}_{t_0}$ P-almost surely. Since this holds for any $t_0 \in \mathbb{Q} \cap [0,T]$, we have that $S^i \leq \widetilde{S}^i$ by right-continuity. Similarly, it is shown that $\{S^i < \widetilde{S}^i\}$ is a null-set, which yields the uniqueness of neutral price processes for terminal wealth.

NFLVR CONDITION: The NFLVR property of the price process S is shown in the usual way: Let $\psi \in \Phi(\Gamma)$. In Step 4 of the existence proof it is shown that $\psi^T \cdot S$ is a P^*-supermartingale and hence $E_{P^*}(f) \leq E_{P^*}(\psi^T \cdot S) \leq 0$ for any $f \in C$. Therefore $f = 0$ is the only non-negative element in the $L^\infty(\Omega, \mathcal{F}, P^*)$-closure of C. Since $P^* \sim P$ this is also true for any f in the $L^\infty(\Omega, \mathcal{F}, P)$-closure of C. Thus $f = 0$ P-almost surely for any such f with $f \geq 0$.

Remark 6.5 *(i) If one additionally assumes that $L^i_{T-} < U^i_{T-}$, then it is much easier to proof that S^i is a semimartingale for $i = m+1, \ldots m+n$. In* *this case, step 2 and step 3 of the existence proof simplify to the following argumentation: Define again* $T_\infty := \lim_{l\to\infty} T_l$. *Since* S^i *is càdlàg, we necessarily have that* $T_\infty = T_k$ *for some* $k \in \mathbb{N}$. *This implies that* S^i *is a local semimartingale and therefore a semimartingale.*

(ii) If the processes $L^{m+1}, \dots, L^{m+n}$ *and* $U^{m+1}, \dots U^{m+n}$ *are bounded, then condition 6.4 is trivially satisfied and the neutral derivative price processes defined in equation 6.2 are unique.*

(iii) European options with bounded discounted terminal payoff R^i *at time* T *may be considered as special cases of game contingent claims by letting*

$$L_t^i := \begin{cases} essinf\ R^i - 1 & \text{if } t < T \\ R^i & \text{if } t = T \end{cases} \tag{6.29}$$

and

$$U_t^i := \begin{cases} esssup\ R^i + 1 & \text{if } t < T \\ R^i & \text{if } t = T \end{cases} . \tag{6.30}$$

If we assume the absence of arbitrage, the price of the European claim will never leave the interval $[essinf\ R^i, esssup\ R^i]$. *Therefore, the additional right to cancel the contract prematurely is worthless. Equation 6.2 reduces to*

$$S_t^i = E_{P^*}(R^i|\mathcal{F}_t) \tag{6.31}$$

for European options.

(iv) American options with bounded exercise process L^i *and final payoff* L_T^i *are treated similarly by defining*

$$U_t^i := \begin{cases} esssup\ (\sup_{t\in[0,T]} L_t^i) + 1 & \text{if } t < T \\ L_T^i & \text{if } t = T \end{cases} . \tag{6.32}$$

The neutral price process S^i *in Equation 6.2 now has the form of a Snell envelope:*

$$S_t^i = esssup_{\tau\in\mathcal{T}(t,T)} E_{P^*}(L_\tau^i|\mathcal{F}_t). \tag{6.33}$$

6.2 Local Utility

6.2.1 The Neutral Pricing Measure

The general setting is as in the previous section. Fix a utility function U. We distinguish two kinds of securities: underlyings $1, \dots, m$ and derivatives $m+1, \dots, m+n$. The underlyings are given in terms of theirs discounted terminal price process $S = (S^1, \dots, S^m)$. At this stage, the only information on the derivatives is their discounted terminal payoffs $R^{m+1}, \dots, R^{m+n}$ at time T, which are supposed to be $\mathcal{F}_T$-measurable random variables. As explained above, our goal is to determine neutral price processes in the sense of the following

Definition 6.6 *We call special semimartingales* $S^{m+1}, \dots, S^{m+n}$ *neutral derivative price processes if*

(i) $S_T^{m+i} = R^{m+i}$ *a.s. for* $i = 1, \dots, n$.

(ii) *There exists a U-optimal portfolio $\bar{\varphi}$ in the extended market $(S^1, \ldots, S^{m+n})$ with $\bar{\varphi}^{m+i} = 0$ for $i = 1, \ldots n$.*

Definition 6.7 (i) *By a simple strategy we refer to a predictable $\mathbb{R}^m$-valued process of the form $\sum_{i=1}^{k} \psi_i \chi_{(T_{i-1}, T_i]}$ where $k \in \mathbb{N}$, $0 \leq T_1 \leq \ldots \leq T_k \leq T$ are stopping times, and ψ_i is a bounded $\mathcal{F}_{T_{i-1}}$-measurable, $\mathbb{R}^m$-valued random variable for $i = 1, \ldots k$.*

(ii) *By simple arbitrage we refer to a $\mathbb{R}^{m+n}$-valued simple strategy ξ such that*

$$V_T(\xi) = \int_0^T \xi_t^T d(S^1, \ldots, S^{m+n}) \geq 0 \quad a.s. \quad and \quad P[V_T(\xi) > 0] > 0. \tag{6.34}$$

In the subsequent we are going to need the following

Assumptions 6.8 (i) *There exists a U-optimal strategy $\varphi \in L(S)$ for the market $(S^1, \ldots, S^m)$.*

(ii) *The local martingale $\mathcal{E}(N)$ is a martingale, where*

$$N := U''(0) \int_0^{\cdot} \varphi_t^T dS_t^c + \frac{U'(\varphi^T x) - 1}{1 + V} * (\mu^S - \nu) \tag{6.35}$$

and

$$V_t := \int (U'(\varphi_t^T x) - 1) \nu(\{t\} \times dx) \quad \text{for } t \in [0, T]. \tag{6.36}$$

We define the probability measure $P^* \sim P$ by $\frac{dP^*}{dP} = \mathcal{E}(N)_T$.

(iii) *The P^*-local martingales $S^1, \ldots S^m$ are P^*-martingales.*

(iv) *We assume that $R^{m+1}, \ldots, R^{m+n}$ can be superhedged by simple trading strategies, i.e. for $i = 1, \ldots n$ there exists a $M_i \in \mathbb{R}$ and simple strategies ξ_i, η_i such that*

$$-M_i + \int_0^T \xi_t^T dS_t \leq R^{m+i} \leq M_i + \int_0^T \eta_t^T dS_t. \tag{6.37}$$

PROOF OF THE STATEMENTS IN THE ASSUMPTIONS.
Step 1: First of all we show that the density process of P^* can also be written as

$$\mathcal{E}(N) = \exp(X - K^X),$$

where $X := U''(0) \int_0^{\cdot} \varphi_t^T dS_t + \sum_{t \leq \cdot} (\log(U'(\varphi_t^T \Delta S_t)) - U''(0) \varphi_t^T \Delta S_t)$ and K^X denotes the modified Laplace cumulant process. In the proof of Theorem 5.8 it is shown that $\varphi \in L(S)$ implies that $U''(0)\varphi^T \cdot S$ is a well-defined semimartingale. Note that $\log(U'(x)) - U''(0)x \leq Mx^2$ for $x \in [-1, 1]$ and some $M \in \mathbb{R}$ that does not depend on x, because $f'(x)x - f'(x)\frac{x^2}{2} + \ldots$ is the Taylor expansion of $\log(f(x))$ at $x = 0$. Since

$$\sum_{t\leq .}(\Delta X_t)^2 \chi_{\{|\Delta X_t|\leq 1\}} \in \mathcal{V} \tag{6.38}$$

for any semimartingale X, because X is càdlàg and the process defined in (6.38) is increasing. It follows that

$$X := U''(0)\varphi^T \cdot S + \sum_{t\leq .}(\log(U'(\varphi_t^T \Delta S_t)) - U''(0)\varphi_t^T \Delta S_t) \tag{6.39}$$

is a well-defined semimartingale as well. We apparently have that $\Delta X = \log(U'(\varphi^T \Delta S))$ and therefore that $e^{\Delta X}$ is bounded by some constant, because U' was assumed to be bounded. Lemma 2.13 from (Kallsen und Shiryaev 2002) now implies X is exponentially special and therefore e^X a special semimartingale.

Step 2: Let $\hat{W}_t := \int (e^x - 1)\nu^X(\{t\} \times dx)$ for $t \in [0,T]$. From Theorem 1.55 if follows for $\varphi = 1$ that the local martingale $\exp(X - K^X)$ equals $\mathcal{E}(N)$ with

$$N := X^c + \frac{e^x - 1}{1 + \hat{W}} * (\mu^X - \nu^X). \tag{6.40}$$

It remains to show that N can be written as in the Assumptions. Obviously, we have $X^c = U''(0)\varphi^T \cdot S^c$ since the second term in (6.39) is a pure jump process. Moreover, $\Delta X = \log(U'(\varphi^T \Delta S))$ implies that

$$\hat{W}_t = \int \left(U'\left(\varphi_t^T x\right) - 1\right)\nu(\{t\} \times dx) = V_t \tag{6.41}$$

and

$$\frac{e^x - 1}{1 + \hat{W}} * (\mu^X - \nu^X) = \frac{U'(\varphi^T x) - 1}{1 + V} * (\mu^S - \nu).$$

Step 3: Let $Z := \mathcal{E}(N)$. From

$$\Delta N = \frac{U'(\varphi_t^T \Delta S) - 1}{1 + V} - \frac{V}{1 + V} = \frac{U'(\varphi_t^T \Delta S)}{1 + \hat{W}} - 1 \tag{6.42}$$

it follows with Equation 1.16 that $Z = Z_-(1 + \Delta N) = Z_- \frac{U'(\varphi^T \Delta S)}{1 + \hat{W}}$. Define the predictable process $\beta := U''(0)\varphi$ and $Y : \Omega \times [0,T] \times \mathbb{R}^m \to \mathbb{R}_+$ by $Y(t,x) := \frac{U'(\varphi_t^T x)}{1 + \hat{W}_t}$. Since $x = \Delta S_t(\omega)$ for $M^P_{\mu^S}$-almost all $(\omega, t, x) \in \Omega \times [0,T] \times \mathbb{R}^m$, we have

$$UZ = UZ_- \frac{U'(\varphi^T \Delta S)}{1 + \hat{W}} = UZ_- Y \quad M^P_{\mu^S}\text{-almost everywhere} \tag{6.43}$$

for any $\widetilde{\mathcal{P}}$-measurable function U and particularly for Y. This is equivalent to the statement $YZ_- = M^P_{\mu^X}[Z|\widetilde{\mathcal{P}}]$. Moreover,

$$Z^c = Z_-(U''(0)\varphi^T \cdot S^c) = (Z_-\beta)^T \cdot S^c \tag{6.44}$$

implies that

$$\left\langle Z^c, S^{i,c} \right\rangle = \left(Z_- c^{i\cdot}\beta\right) \cdot A \quad \text{for } i = 1, \ldots, m, \tag{6.45}$$

since the drift B for local martingales is identical zero, i.e. also for Z^c (see Lemma 1.71). From Girsanov's theorem for semimartingales (see Theorem 1.66), it follows that the P^*-characteristics (B^*, C^*, ν^*) of S are given by $C^* = C$, $\nu^* = Y \cdot \nu$, and

$$B^{*,i} = B^i + (U''(0)c^{i\cdot}\varphi) \cdot A + h^i(x)(Y-1) * \nu \quad \text{for } i = 1, \ldots, m. \tag{6.46}$$

Step 4: Fix $i \in \{1, \ldots, m\}$. Note that condition (5.14) implies

$$b_t^i + \int (h^i(x) - x^i) F_t(dx) = -U''(0)c_t^{i\cdot}\varphi_t - \int (x^i U'(\varphi_t^T x) - h^i(x)) F_t(dx) \tag{6.47}$$

since we have no trading constraints. On the set $\{\widehat{W}_t = 0\}$ this equals $-U''(0)c_t^{i\cdot}\varphi_t - \int (x^i Y(t,x) - h^i(x)) F_t(dx)$. If $\widehat{W}_t \neq 0$, then $\Delta A_t \neq 0$ and $c_t = 0$, because $\Delta A_t = 0$ together with $\nu = A \otimes F$ would imply that $\nu_t(dx) = \nu(\{t\} \times dx) = F(dt, dx)\Delta A_t$ is identical zero and therefore also $\widehat{W}_t = 0$. Moreover, because of (1.22) $C^{ij} = c^{ij} \cdot A$ is continuous and hence $c^{ij} = 0$ for all $1 \leq i, j \leq m$. On $\{\widehat{W}_t \neq 0\}$ we therefore have with (1.25)

$$\begin{aligned} \int h^i(x) F_t(dx) \Delta A_t &= \int h^i(x) \nu(\{t\} \times dx) = \Delta B_t^i = \\ &= (b_t^i + \int (h^i(x) - x^i) F_t(dx)) \Delta A_t, \end{aligned} \tag{6.48}$$

which implies

$$b_t^i + \int (h^i(x) - x^i) F_t(dx) = \int h^i(x) F_t(dx). \tag{6.49}$$

Therefore $\int x^i U'(\varphi_t^T x) F_t(dx) = 0$ and hence $\int x^i Y(t,x) F_t(dx) = 0$. This in turn implies that

$$b_t^i + \int (h^i(x) - x^i) F_t(dx) = -U''(0)c_t^{i\cdot}\varphi_t - \int (x^i Y(t,x) - h^i(x)) F_t(dx) \tag{6.50}$$

holds on the set $\{\widehat{W}_t \neq 0\}$ as well. Since $B^i = (b_t^i + \int (h^i(x) - x^i) F_t(dx)) \cdot A$, it follows from the previous step together with (6.46) that

$$\begin{aligned} B^{*,i} &= \left(-U''(0)c_t^{i\cdot}\varphi_t - \int (x^i Y(t,x) - h^i(x)) F_t(dx) \right) \cdot A \\ &\quad + (U''(0)c^{i\cdot}\varphi) \cdot A + h^i(x)(Y-1) * \nu = \\ &= -(x^i - h^i(x))(Y \cdot \nu) = -(x^i - h^i(x)) * v^*. \end{aligned} \tag{6.51}$$

By Proposition 1.63 $A^* = B^* + (x - h(x)) * \nu^* \equiv 0$, which means $B^* \equiv 0$ and therefore S^i is a P^*-local martingale according to the canonical semimartingale decomposition (1.21).

■

Definition 6.9 *We call the above probability measure P^* neutral pricing measure for local utility.*

The following theorem treats existence and uniqueness of neutral derivative prices. Moreover, it shows that these prices are obtained via conditional expectation relative to some equivalent martingale measure. This implies that the corresponding securities market allows no arbitrage according to Lemma 2.26.

Theorem 6.10 *Suppose that Assumptions 6.8 hold. Then the semimartingales* $S^{m+1}, \dots, S^{m+n}$ *defined by*

$$S_t^{m+i} := E_{P^*}(R^{m+i}|\mathcal{F}_t) \quad \text{for } t \in [0,T],\ i = 1,\dots,n$$

are neutral derivative price processes, where E_{P^*} *denotes (conditional) expectation relative to* P^**. These are up to indistinguishability the only neutral derivative price processes that do not lead to simple arbitrage opportunities.*

Proof. *Step 1:* For any simple strategy $\psi = \sum_{i=1}^{k} \psi_i \chi_{(T_{i-1},T_i]}$ and $T_{r-1} < s \leq T_r$ we have due to the P^*-martingale property of S

$$\begin{aligned} E((\psi^T \cdot S)_t|\mathcal{F}_s) &= E((\sum_{i=1}^{k} \psi_i \chi_{(T_{i-1},T_i]}) \cdot S_t|\mathcal{F}_s) = \\ &= \sum_{i=1}^{k} E((\psi_i \chi_{(T_{i-1},T_i]}) \cdot S_t|\mathcal{F}_s) = \\ &= \sum_{i=1}^{r} (\psi_i \chi_{(T_{i-1},T_i \wedge s]}) \cdot S_s + \sum_{i=r}^{k} E((\psi_i \chi_{(T_{i-1}\wedge s,T_i]}) \cdot S_t|\mathcal{F}_s) = \\ &= \sum_{i=1}^{r} (\psi_i \chi_{(T_{i-1},T_i \wedge s]}) \cdot S_s + \sum_{i=r}^{k} \psi_i E(\chi_{(T_{i-1}\wedge s,T_i]} \cdot S_t|\mathcal{F}_s) = \\ &= \sum_{i=1}^{r} (\psi_i \chi_{(T_{i-1},T_i \wedge s]}) \cdot S_s + 0 = \\ &= (\psi^T \cdot S)_s. \end{aligned} \tag{6.52}$$

It follows that $V(\psi) = \psi^T \cdot S$ is a P^*-martingale for any simple strategy ψ. This implies

$$-M_i + \xi_i^T \cdot S \leq S^{m+i} \leq M_i + \eta_i^T \cdot S \tag{6.53}$$

if M, ξ, η correspond to R^{m+i} as in the fourth part of Assumptions 6.8. In view of (Jacod 1979), (2.51) and Proposition 1.27, we conclude that S^{m+i} is a P-special semimartingale for $i = 1, \dots n$.
Step 2: Without loss of generality the characteristics of $\overline{S} := (S^1, \dots, S^{m+n})$ are given in the form (5.8), but with $(\overline{b}, \overline{c}, \overline{F})$ instead of (b, c, F). Let us repeat the steps leading to the measure P^* with the $\mathbb{R}^{m+n}$-valued processes $\overline{S}$ and $\overline{\varphi} := (\varphi, 0, \dots, 0) \in L(\overline{S})$ instead of S and φ. Obviously, the definition of N and therefore P^* is not affected by this alternative choice, because e.g.

$$\int_0^{\cdot} \overline{\varphi}_t^T d\overline{S}_t^c = \int_0^{\cdot} (\varphi_t, 0, \dots, 0)^T d(S^1, \dots, S^{m+n})_t^c = \int_0^{\cdot} \varphi_t^T dS_t^c \tag{6.54}$$

in (6.35). In step 4 of the last proof, we obtained the P^*-local martingale property of S^i for $i = 1, \dots m$ from the equation

$$b_t^i + U''(0) c_t^{i\cdot} \varphi_t + \int x^i (U'(\varphi_t^T \cdot x) - 1) F_t(dx) = 0 \tag{6.55}$$

or equivalently

$$\overline{b}_t^i + U''(0) \overline{c}_t^{i\cdot} \overline{\varphi}_t + \int x^i (U'(\overline{\varphi}_t^T \cdot x) - 1) \overline{F}_t(dx) = 0. \tag{6.56}$$

By reversing the argumentation in that step, we obtain the corresponding equation for $i = m+1, \ldots, m+n$ from the P^*-local martingale property of $S^{m+1}, \ldots, S^{m+n}$ (assume that (6.56) does not hold for any $m+1 \leq i \leq m+n$, then S^i would not be a local martingale). In view of Theorem 5.16, $\overline{\varphi}$ is a U-optimal strategy for $\overline{S}$, which implies that $S^{m+1}, \ldots, S^{m+n}$ are neutral derivative price processes.
Step 3: As in Step 1 we have due to the P^*-martingale property of $\overline{S}$ that $V(\psi) = \psi^T \cdot \overline{S}$ is a P^*-martingale for any simple strategy ψ. Hence there exists no simple arbitrage in this extended market.
Step 4: For the uniqueness part assume that $\widetilde{S}^{m+1}, \ldots, \widetilde{S}^{m+n}$ are neutral derivative prices corresponding to some U-optimal portfolio $\widetilde{\varphi} = (\widetilde{\varphi}^1, \ldots, \widetilde{\varphi}^m, 0, \ldots, 0)$ in the extended market $(S^1, \ldots, S^m, \widetilde{S}^{m+1}, \ldots, \widetilde{S}^{m+n})$. Since $\widetilde{\varphi}$ does not contain any derivative, we have that $(\widetilde{\varphi}^1, \ldots, \widetilde{\varphi}^m)$ is in particular an optimal strategy for the market S with the same local utility. Similarly, the local utility of φ in the market S and of $\overline{\varphi} := (\varphi, 0, \ldots, 0)$ in the market $(S^1, \ldots, S^m, \widetilde{S}^{m+1}, \ldots, \widetilde{S}^{m+n})$ tally.
Since also φ is U-optimal in the market S, it follows with Remark 5.13 that $(\widetilde{\varphi}^1, \ldots, \widetilde{\varphi}^m)$ and φ have the same local utility in the market S. This shows that $\overline{\varphi}$ is optimal for $(S^1, \ldots, S^m, \widetilde{S}^{m+1}, \ldots, \widetilde{S}^{m+n})$. Hence we may w.l.o.g. assume $\widetilde{\varphi} = \overline{\varphi}$.
Step 5: Similar as above we repeat the steps leading to the measure P^* with the $\mathbb{R}^{m+n}$-valued processes $(S^1, \ldots, S^m, \widetilde{S}^{m+1}, \ldots, \widetilde{S}^{m+n})$ and $\overline{\varphi}$ instead of S and φ. As before, the resulting measure P^* remains the same. As in Step 4 of the last proof, we conclude that $S^1, \ldots, S^m, \widetilde{S}^{m+1}, \ldots, \widetilde{S}^{m+n}$ are P^*-local martingales.
Step 6: Fix $i \in \{1, \ldots, n\}$ and let M_i, ξ_i, η_i be chosen for derivative $m+i$ as in the fourth part of Assumptions 6.8. The absence of simple arbitrage implies that

$$-M_i + \xi^T \cdot S \leq \widetilde{S}^{m+i} \leq M_i + \eta^T \cdot S \tag{6.57}$$

almost surely. Analogous as in Step 1 it follows that $\psi^T \cdot S$ is a P^*-martingale for any simple strategy ψ. Moreover, $\psi^T \cdot S$ trivially admits a terminal variable (see Theorem 1.14) and thus is uniformly integrable. Therefore, $\widetilde{S}$ is bounded from below and above by uniformly integrable P^*-martingales, which implies that is is of class (D) relative to P^* due to Lemma 1.17. Hence Proposition 1.16 yields that $\widetilde{S}^{m+i}$ is a P^*-martingale with the same terminal value R^{m+i} as S^{m+i}. This yields the uniqueness, since the positive P^*-submartingale $\left|\widetilde{S}^{m+i} - S^{m+i}\right|$ has terminal value zero and therefore is identical zero. ■

Remark 6.11 *(i) It may seem counterintuitive that simple arbitrages are not automatically excluded if derivatives are neutrally priced. On the mathematical side, this phenomenon corresponds to the fact that local martingales are not necessarily martingales. Put differently, some games e.g. the doubling or the suicide strategy are locally fair but turn out to be unfair on a global level.*

(ii) Note that P^* is an equivalent martingale measure (EMM) for the extended market $(S^1, \ldots, S^{m+n})$. In particular, neutral derivative prices coincide with the unique arbitrage-based prices in complete models.

6.2.2 The Pricing Formula

Here we assume that derivative speculators maximize their local utility in the sense of Section 5.3. In contrast to more common forms of utility maximization, this can lead to relatively explicit results to diverse models. From the theoretical point of view one may criticize that neutral derivative values depend on the utility function. However, the numerical differences are often small in practice. In models with continuous paths, the neutral prices do not depend on the utility function at all.

Similarly as above, we assume that the neutral pricing measure for local utility P^* exists for the underlyings's market $S^1,\dots,S^m$, which is for example the case if Assumptions 6.8 are satisfied.

Definition 6.12 *We call derivative price processes $S^{m+1},\dots,S^{m+n}$ neutral for local utility if there exists a strategy $\overline{\varphi}$ in the extended market $S^1,\dots,S^{m+n}$ which is locally optimal under the constraints Γ an satisfies $\overline{\varphi}^{m+1}=\dots=\overline{\varphi}^{m+n}=0$.*

The following result corresponds to Theorem 6.3 in the local utility setting.

Theorem 6.13 *Suppose that L^i,U^i are special semimartingales and furthermore that $\sup_{t\in[0,T]}\left|L^i_t\right|$ and $\sup_{t\in[0,T]}\left|U^i_t\right|$ are P^*-integrable for $i=m+1,\dots,$ $m+n$. Then there exits unique neutral derivative price processes. These are given by*

$$\begin{aligned} S^i_t &= essinf_{\tau^U\in\mathcal{T}(t,T)}\ esssup_{\tau^L\in\mathcal{T}(t,T)} E_{P^*}(R^i(\tau^L,\tau^U)|\mathcal{F}_t) \\ &= esssup_{\tau^L\in\mathcal{T}(t,T)}\ essinf_{\tau^U\in\mathcal{T}(t,T)} E_{P^*}(R^i(\tau^L,\tau^U)|\mathcal{F}_t) \end{aligned} \tag{6.58}$$

for $t\in[0,T]$, $i=m+1,\dots,m+n$, where $\mathcal{T}(t,T)$ and $R^i(\tau^L,\tau^U)$ are defined as in Theorem 6.3. Moreover, the extended market $S^1,\dots,S^{m+n}$ satisfies the NFLVR condition in sense of Definition 6.4.

PROOF OF THEOREM 6.13: Steps 1-3 for the existence and the NFLVR condition are literally shown as in the proof of Theorem 6.3. Only Step 4 and the uniqueness part have to be modified slightly.

EXISTENCE:
Step 4: Since $L^i\le S^i\le U^i$, we have that S^i is a special semimartingale for $i=m+1,\dots,m+n$ according to Proposition 1.27. Similarly as in Step 4 of the proof of Theorem 6.3 we want to show that $\overline{\varphi}:=(\varphi,0)\in\Phi'(\Gamma)$ is a locally optimal strategy for $S=(S^1,\dots,S^{m+n})$, where φ denotes a locally optimal strategy in the small market $S^1,\dots,S^m$. Denote by (b,c,F,A) the P-differential characteristics of S relative to $h(x)=x$. In view of Theorem 5.21 we have to show that

$$b+U''(0)c\varphi+\int x(U'(\overline{\varphi}^Tx)-1)F(dx)\in\Gamma^\circ. \tag{6.59}$$

Note that

$$\Gamma^\circ_t=\begin{array}{l}\{y\in\{0\}^m\times\mathbb{R}^m: \text{For } i=m+1,\dots,m+n \text{ we have } y^i\ge 0\\ \text{ if } L^i_{t-}<S^i_{t-} \text{ and } y^i\le 0 \text{ if } S^i_{t-}<U^i_{t-}\}.\end{array} \tag{6.60}$$

From the Girsanov Theorem 1.66 it follows that the P^*-differential characteristics (b^*, c^*, F^*, A) of S relative to some truncation function $h : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$ satisfy the equation

$$\begin{aligned} & b_t^{*,i} + \int (x^i - h^i(x))F_t^*(dx) \qquad (6.61) \\ = \quad & b_t^i + U''(0)c_t^{i\cdot}\overline{\varphi}_t + \int x^i \left(\frac{U'(\overline{\varphi}_t^T x)}{1+V_t} - 1 \right) F_t(dx) \\ = \quad & \frac{1}{1+V_t} \left(b_t^i + U''(0)c_t^{i\cdot}\overline{\varphi}_t + \int x^i (U'(\overline{\varphi}_t^T x) - 1) F_t(dx) \right) \end{aligned}$$

for $i = 1, \ldots, m+n$, where V_t is defined as in Equation 6.36 in Assumptions 6.8. The second equality clearly holds on the set $\{V_t = 0\}$. On $\{V_t \neq 0\}$ we have as in Step 4 in the proof of the Assumptions 6.8 that $c_t = 0$ and that

$$b_t^i - \int x^i F_t(dx) = 0 \qquad (6.62)$$

(see Equation 6.49). Since φ is optimal in the small market, Theorem 5.21 yields that expression 6.61 equals 0 for $i = 1, \ldots, m$. The same argument as in Step 2 of the proof of Theorem 6.3 shows that the left-hand side of equation 6.61 is non-negative on $\{L_{t-}^i < S_{t-}^i\}$ (respectively non-positive on $\{S_{t-}^i < U_{t-}^i\}$) for $i = 1, \ldots, m+n$. Together is follows that Condition 6.60 holds. Therefore $S^{m+1}, \ldots, S^{m+n}$ are neutral price processes for local utility.
Uniqueness: Assume that $\widetilde{S}^{m+1}, \ldots, \widetilde{S}^{m+n}$ are neutral derivative price process corresponding to some locally optimal strategy $\widetilde{\varphi} = (\widetilde{\varphi}^1, \ldots, \widetilde{\varphi}^m, 0, \ldots, 0)$ in the extended market $\widetilde{S} := (S^1, \ldots, S^m, \widetilde{S}^{m+1}, \ldots, \widetilde{S}^{m+n})$. As in Step 5 of the proof of Theorem 6.3 we may w.l.o.g. assume that $\widetilde{\varphi} = \overline{\varphi} = (\varphi, 0)$.

We denote by (b, c, F, A) the P-characteristics of $\widetilde{S}$ relative to $h(x) = x$. Since $\overline{\varphi}$ is an optimal strategy, Theorem 5.21 yields that Condition 6.59 holds $(P \otimes A)$-almost everywhere. As in the existence proof, we express this condition in terms of the P^*-differential characteristics (b^*, c^*, F^*, A) of $\widetilde{S}$ relative to some truncation function $h : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$. Fix $i \in \{m+1, \ldots, m+n\}$. Then the P^*-drift $b^{*,i} + \int (x^i - h^i(x))F^*(dx)$ of $\widetilde{S}^i$ is non-negative on $\{L_{t-}^i < \widetilde{S}_{t-}^i\}$ respectively non-positive on $\{\widetilde{S}_{t-}^i < U_{t-}^i\}$. Due to Lemma 1.67 and Lemma 1.71, this means that $\chi_D \cdot \widetilde{S}^i$ is a P^*-σ-submartingale for any predictable subset D of $\{L_{t-}^i < \widetilde{S}_{t-}^i\}$ and $\chi_D \cdot \widetilde{S}^i$ is a P^*-σ-supermartingale for any predictable subset D of $\{\widetilde{S}_{t-}^i < U_{t-}^i\}$. The uniqueness of neutral price processes follows now as in the proof of the uniqueness in Theorem 6.3.
■

Remark 6.5 following Theorem 6.3 holds accordingly in the setting of local utility.

Appendix A

Convex Optimization

This chapter gives a very short excerpt about convex optimization problems. The reader is referred to (Rockafellar 1970) and (Rockafellar and Wets 1998) for a detailed description of this subject.

Definition A.1 *(i) A set $K \subseteq \mathbb{R}^n$ is said to be a polyhedral set if it can be expressed as the intersection of a finite family of closed half-spaces or hyperplanes, or equivalently, can be specified by finitely many linear constraints, i.e. constraints $g^j(\psi) \leq 0$ or $g^j(\psi) = 0$ where g^j is affine.*

(ii) A set $K \subseteq \mathbb{R}^n$ is called a cone if $0 \in K$ and $\lambda x \in K$ for all $x \in K$ and $\lambda \geq 0$.

Remark A.2 *It is easy to show that $K \subseteq \mathbb{R}^n$ is a convex cone if and only if K is nonempty and contains $\sum_{i=1}^{m} \lambda_i x_i$ whenever $x_i \in K$ and $\lambda_i \geq 0$.*

Theorem A.3 *A cone K is polyhedral if and only if it can be expressed as the set of all $\sum_{i=1}^{m} \lambda_i a_i$ with $\lambda_i \geq 0$ and some finite collection of vectors $a_1, \ldots, a_m$.*

Lemma A.4 (Farkas) *The polar of a cone of form $K = \{x : a_i^T x \leq 0$ for $i = 1, \ldots, m\}$ is the cone consisting of all linear combinations $\sum_{i=1}^{m} \lambda_i a_i$ with $\lambda_i \geq 0$.*

Definition A.5 *By an ordinary convex program (P) we shall mean a problem of the following form: minimize the convex function $h(x) : \mathbb{R}^n \to \mathbb{R}$ subject to constraints*

$$g^j(\psi) \leq 0 \text{ for } j = 1, \ldots, p \text{ and } g^j(\psi) = 0 \text{ for } j = p+1, \ldots, q,$$

where $g_j : \mathbb{R}^n \to \mathbb{R}$ are finite convex functions for $j = 1, \ldots, p$, and $g_j : \mathbb{R}^n \to \mathbb{R}$ are affine functions for $j = p+1, \ldots, q$. Included here are the special cases where $p = q$ (i.e. no equality constrains) and $p = 0$ (no inequality constrains).

Theorem A.6 *Let (P) be an ordinary convex program, and let J be the set of indices such that g_j is not affine. Assume that the optimal value in (P) is* *not $-\infty$, and that (P) has at least one feasible solution which satisfies with strict inequality all the inequality constraints for $j \in J$. Then there exits a so called Kuhn-Tucker vector (not necessarily unique) for (P), i.e. there exit $\lambda_1, \dots, \lambda_q$ with $\lambda_j \geq 0$ for $j = 1, \dots, p$ such that the infimum of the convex function*

$$L := h + \lambda_1 g_1 + \cdots + \lambda_q g_q \tag{A.1}$$

is finite and equal to the optimal value of (P).

Remark A.7 *The existence of a Kuhn-Tucker vector particularly implies that*

$$\frac{\partial L}{\partial x_i} = D_i h + \lambda_1 D_i g_1 + \cdots + \lambda_q D_i g_q = 0 \quad \text{for } i = 1, \dots, n \tag{A.2}$$

if L is differentiable.

Theorem A.8 (Kuhn-Tucker) *Let (P) be an ordinary convex program in the notation above. Let $x^* \in \mathbb{R}^n$ be a given vector. In order that x^* be an optimal solution to (P), it is necessary and sufficient that there exits a vector $\lambda^* = (\lambda_1, \dots, \lambda_q)$ such that (λ^*, x^*) is a saddle-point of the Lagrangian function L of (P) defined in (A.1). Equivalently, x^* is an optimal solution if and only if there exist Lagrange multiplier values λ_i, which together with x^* satisfy the Kuhn-Tucker conditions for (P):*

a) $\lambda_i \geq 0$, $g_j(x^*) \leq 0$ *and* $\lambda_j g_j(x^*) = 0$ *for* $j = 1, \dots, p$,

b) $g_j(x^*) = 0$ *for* $j = p+1, \dots, q$,

c) $0 \in [\partial h + \lambda_1 \partial g_1 + \cdots + \lambda_q \partial g_q = 0]$.

Appendix B

Dynkin Games

This chapter gives a short excerpt about the valuation of Dynkin games. For a detailed treatment of the statements below see (Lepeltier et Maingueneau 1984).

Definition B.1 *A Dynkin game is a zero-sum game defined on* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P, \mathcal{T}, \mathcal{T}, J(S, S'))$ *such that*

(i) $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$ *is a filtered probability space satisfying the usual conditions, where* $\mathcal{F}_0$ *is the trivial* σ*-algebra.*

(ii) $\mathcal{T}$ *is the set of all game strategies, i.e. all stopping times.*

(iii) The payoff function $J(S, S')$ *is defined on* $\mathcal{T} \times \mathcal{T}$ *according to*

$$(S, S') \mapsto E(X_S \chi_{\{S \leq S'\}} - X'_{S'} \chi_{\{S' < S\}}). \tag{B.1}$$

where X *and* X' *are optional, bounded and right-continuous processes, that converge to* 0 *for* $t \to \infty$*. Moreover we suppose* $X \leq -X'$*.*

Definition B.2 *We define the upper (respectively lower) conditional limiting process* $\overline{X}$ *(respectively* $\underline{X}$*) for every stopping time* T *as*

$$\overline{X}_T \; : \; = essinf_{S' \geq T} \; esssup_{S \geq T} E_P(X_S \chi_{\{S \leq S'\}} - X'_{S'} \chi_{\{S' < S\}} | \mathcal{F}_T) \tag{B.2}$$
$$\underline{X}_T \; : \; = esssup_{S \geq T} \; essinf_{S' \geq T} E_P(X_S \chi_{\{S \leq S'\}} - X'_{S'} \chi_{\{S' < S\}} | \mathcal{F}_T).$$

Remark B.3 *We obviously have that* $\underline{X}_T \leq \overline{X}_T$*. Moreover the following holds true for every stopping time* T

$$\overline{X}_T \; : \; = essinf_{S' \geq T} \; esssup_{S \geq T} E_P(X_S \chi_{\{S < S'\}} - X'_{S'} \chi_{\{S' \leq S\}} | \mathcal{F}_T) \quad a.s.$$
$$\underline{X}_T \; : \; = esssup_{S \geq T} \; essinf_{S' \geq T} E_P(X_S \chi_{\{S < S'\}} - X'_{S'} \chi_{\{S' \leq S\}} | \mathcal{F}_T) \quad a.s.,$$

i.e. we can use either X *or* $-X'$ *as the payoff when the two players stop exactly at the same time, if we are only interested in the processes* $\underline{X}$ *and* $\overline{X}$*.*

Definition B.4 *For every* $\epsilon > 0$ *and stopping time* T *we define the stopping times*

$$D_T^{\varepsilon} \; : \; = \inf(t \geq T | \overline{X}_t \leq X_t + \varepsilon) \tag{B.3}$$
$$D_T'^{\varepsilon} \; : \; = \inf(t \geq T | \underline{X}_t \geq -X'_t - \varepsilon).$$

Theorem B.5 *Suppose there is given a Dynkin game according to Definition (B.1). Then the processes $\underline{X}$ and $\overline{X}$ are right-continuous and for all stopping times T satisfy a.s. the inequalities*

$$X_T \leq \overline{X}_T \leq -X'_T \text{ and } X_T \leq \underline{X}_T \leq -X'_T. \tag{B.4}$$

In addition, for all stopping times $U \geq T$ we a.s. have

$$\overline{X}_T \leq E(\overline{X}_{D_T^\varepsilon \wedge U}|\mathcal{F}_T) \text{ and } \underline{X}_T \geq E(\underline{X}_{D_T'^\varepsilon \wedge U}|\mathcal{F}_T). \tag{B.5}$$

In other words, $\underline{X}_{D_T^\varepsilon}$ is a supermartingale and $\overline{X}_{D_T'^\varepsilon}$ a submartingale.

Corollary B.6 *For every stopping time T, $\underline{X}_T = \overline{X}_T$ a.s. and in particular for $T = 0$*

$$\underline{X}_0 = \sup_{T\in\mathcal{T}} \inf_{T'\in\mathcal{T}} J(T,T') = \overline{X}_0 = \sup_{T\in\mathcal{T}} \inf_{T'\in\mathcal{T}} J(T,T') \tag{B.6}$$

is the unique value of the Dynkin game.

Appendix C

Frequently Used Notations and Symbols

$\mathbb{R}^n$	n-dimensional Euclidean space
$\mathbb{R}_+, \overline{\mathbb{R}}_+$	$= [0, \infty), [0, \infty]$ respectively
$C^n(\mathbb{R}^d)$	the space of n-times continuously differentiable functions on $\mathbb{R}^d$
b^T	the transposed of a vector b
$\underline{\lim}, \overline{\lim}$	limes inferior, limes superior
$\sigma \wedge \tau$	$= \min(\sigma, \tau)$
$\sigma \vee \tau$	$= \max(\sigma, \tau)$
$D_i f$	the i-th partial derivative of a function $f : \mathbb{R}^{d+1} \to \mathbb{R}$
χ_A	the characteristic function of the set A
ε_a	the Dirac measure sitting at point a
B_t	Brownian motion
$\mathcal{B}$	the Borel σ-algebra
a.a., *a.s.*, *a.e.*	almost all, almost surely, almost everywhere
$P \ll Q$	the measure P is absolutely continuous w.r.t. the measure Q
$P \sim Q$	P is equivalent to Q, i.e. $P \ll Q$ and $Q \ll P$
$P' \overset{loc}{\ll} P$	see Definition 1.64
$\mathcal{M}, \mathcal{M}_{loc}, \mathcal{M}^c_{loc}, \mathcal{L}$	see Definition 1.10
$\mathcal{H}^2, \mathcal{H}^2_{loc}$	see Definition 1.10
$\mathcal{V}, \mathcal{V}^+$	see Definition 1.22
$\mathcal{A}^+, \mathcal{A}^+_{loc}$	see Definition 1.23
$\mathcal{S}, \mathcal{S}_p$	see Definition 1.24
$H^T \cdot X$	the stochastic integral of H with respect to X
$L^0(X), L^0_{loc}(X)$	see Definition 1.32
$L(X)$	see Definition 1.34
A^p	the compensator of a process, see Theorem 1.39
$\langle ., . \rangle$	see Theorem 1.21
$[., .]$	see Definition 1.36

$\mathcal{O}, \mathcal{P}$	see Definition 1.5
$\widetilde{\Omega}, \widetilde{\mathcal{O}}, \widetilde{\mathcal{P}}$	see discussion following Definition 1.42
$W * \mu$	see discussion following Definition 1.42
G_{loc}	see Definition 1.47
$\mathcal{E}(X)$	the exponential semimartingale of X
$\mathcal{C}_t^d$	a truncation function, see Definition 1.57
$M_\mu^P, M_\mu^P(W\|\widetilde{\mathcal{P}})$	see Definition 1.64
$\varphi_t = (\varphi_t^0, \ldots, \varphi_t^n)$	a portfolio held at time t
$\mathcal{L}^1(S)$	see Definition 5.6
$\Phi(\Gamma)$	see Definition 5.3
$\Phi'(\Gamma)$	see Definition 5.18
$\mathcal{K}(S,T)$	see Definition 2.1
$\mathcal{K}(S,T), \mathcal{K}^m(S,T), \mathcal{K}^{n\times m}(S,T)$	see Definition 2.8
$\mathcal{K}'(S,T), \mathcal{K}'^m(S,T), \mathcal{K}'^{n\times m}(S,T)$	see Definition 2.10
$\overline{X}_t$	the normalized price vector
V_t^φ	$= \varphi_t^T X_t$, the value process
$V_t^{\varphi,\varepsilon}$	$= \varepsilon + \varphi_t^T X_t$, the value process with initial fortune ε
$\overline{V}_t^\varphi$	$= \varphi_t^T \overline{X}_t$, the normalized value process
$\mathcal{T}(0,N)$	set of stopping times with values in $\{0, \ldots, N\}$
$\mathcal{T}(0,T)$	set of stopping times with values in $[0,T]$

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