D I P L O M A R B E I T

Optimal Dynamic Allocation of Prevention and Treatment in a Model of the Australian Heroin Epidemic

ausgeführt am Institut für

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der Technischen Universität Wien

unter der Anleitung von

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Chapter 1

Introduction

Nowadays, the drugsyndicates represent a huge problem worldwide, which can be seen at the thousands of drug deads per year. The transaction volume of drugs amounts to 500 billions USD yearly. This is an unthinkable amount because it's greater than the GDP of many states. Law enforcement, imprisonment of dealers, border controls, controls at airports or other police measures are puny and disabled in the "drug war" but essential attempts to fight against the destructive global power "drug". In this war the antagonist is from superiour dimension and so it's very hard to win this campaign.

In the drug scene the abbreviation "H" means heroin, which is one of the most hazardous opiates in the whole drug-catalogue. It's very easy to produce this drug, but since it represents a derivative of the opium (cleaned opium), the effect is stronger than from the ordinary opium. The name of heroin is derived from "heros" and "heroic" because the effect is amazingly strong. The impact of heroin is six-times as strong as that of opium. While opium is six to twelve hours active in the human body, heroin will be abolished in two to four hours.

The several forbiddances achieved just as little as the alcohol prohibition in the USA in the twenties. On the contrary, the illegal side found ways at all times and intends to offer the illegal drugs to the consumer. So this is one of the most discussed topics in the last years.

In my master thesis I discuss the actual heroin problem in Australia. To analyse this problem I use optimal control theory. The problem is formulated as a one-state two-control optimization model. The state variable is the number of heroin users, $A(t)$, with t denoting the time argument. The two control variables represent treatment spending, $u(t)$, and prevention spending, $w(t)$.

Both controls ($u(t)$, $w(t)$) have comparable effects on the number of drug users, by reducing their actual number. The difference is that treatment increases the outflow of drug users while prevention decreases initiation into drug use.

The data used to analyse this problem are from different sources (see Chapter 3). Since some data are not available, they had to be estimated. For instance, the budget is known, but the fractions which have been spent for prevention and for treatment, respectively, are unknown. In the first chapters we suggest that one half of the users quitted because of treatment, and the other half had other reasons for quitting, the so-called natural outflow. Moreover, significant parameters are unknown, and therefore some of them are taken from Caulkins et al. (2000) and from Tragler et al. (2001), who analysed similar models of the current U.S. cocaine epidemic.

In the next paragraphs I describe the structure of my master thesis and explain in a few sentences what I have analysed.

In Chapter 2 the model is formulated and described. We distinguish between three model types. The first model is the so-called model without controls. That means that nothing is spent for prevention or for treatment. The next one is the model with constant controls, i.e. a constant amount is spent each year for prevention and treatment. These two models are those with less work because they don't describe an optimal control model, and therefore can be solved very easily. The last one is the optimal control model. In this model, the prevention and treatment spendings are chosen in a way that the total costs of the drug problem are minimized. This model is the most interesting one because it's really ambitious mathematically.

In Chapter 3 the parameters are estimated. Some of them are taken from Caulkins et al. (2000) and from Tragler et al. (2001). Some of the others are estimated in that way that the error of the formulated function to the data is minimized over the time period. The estimation of the two parameters α (initiation term exponent) and z (treatment function exponent) is of great interest and is explained in Chapter 3 in detail.

Chapter 4 shows the solutions of the three variations of the model. The model without controls and the model with constant controls can be solved very easily. The optimal control model is solved by using Pontryagin's maximum principle (cf. Feichtinger et al. (1986) and Pontryagin et al. (1964)). This results in the investigation of a nonlinear system of two differential equations. The solutions are discussed and plotted for better understanding.

In Chapter 5 a sensitivity analysis is presented. First of all, the parameters measuring the prevention's efficiency are changed. The next section deals with the effects of changing the ratio between quitting due to treatment and quitting due to other reasons. Further, the analysis of a model with slightly different budget data is presented to conclude Chapter 5.

In the final Chapter 6 the results of the previous chapters are discussed and some motivations for further studies are given.

The Appendix is used for some detailed mathematical calculations and the analysis of a model variation.

Chapter 2

Formulation of the Base Model

The model investigated in this thesis is based on Caulkins et al. (2000), Tragler et al. (2001), Kaya (2002), and Mautner (2002). A one-state two-control dynamic optimization problem is considered where the state variable describes the number of drug users at time t, $A(t)$, and the two control variables are treatment spending, $u(t)$, and prevention spending, $w(t)$. The decision maker wants to minimize the discounted stream of the total social costs which are caused by the drug problem. These total costs consist of the social costs arising from the drug consumption and the budget spendings for treatment and prevention.

The control variables are subject to the non-negativity constraints

$$u(t) \geq 0, w(t) \geq 0 \quad \forall t. \tag{2.1}$$

The function $\vartheta(w)$ describes the effect of prevention spending on initiation. More precisely, the fraction of susceptibles who start using drugs although w is spent on prevention is given by the function $\vartheta(w)$. The function $\vartheta(w)$ is specified as follows:

$$\vartheta(w(t)) := h + (1-h)e^{-mw(t)}, \tag{2.2}$$

where m is the prevention's rate of decay and h the minimum level to which the prevention may decrease the initiation:

$$h := \min_{w(t)} I_{prev}(w(t)).$$

The condition $0 \leq h \leq 1$ must be guaranteed. It is assumed that the following equations

$$\vartheta(0) = 1, \tag{2.3}$$

$$\lim_{w \to \infty} \vartheta(w) = h, \tag{2.4}$$

and

$$\frac{\partial \vartheta(w)}{\partial w} = -m(1-h)e^{-mw} \leq 0 \tag{2.5}$$

are satisfied.

Equation (2.3) indicates that if nothing is spent on prevention, all susceptibles will start using drugs. For $w \to \infty$, $\vartheta(w)$ converges to the minimum level h and cannot be less this barrier, which is shown in equation (2.4). Equation (2.5) indicates that $\vartheta(w)$ is a monotonously decreasing function, which means the more is spent on prevention, the less people will start using drugs.

The proportion of users who stop using drugs due to treatment spending $u(t)$ is given by $c\beta$. It is assumed that $c\beta$ depends not only on treatment spending but also on the actual number of drug users $A(t)$. c describes the treatment proportionality constant. The treatment function β is given as follows:

$$\beta(A(t), u(t)) := \left(\frac{u(t)}{A(t)}\right)^z, \quad 0 \leq z \leq 1, \tag{2.6}$$

where z is the marginal efficiency of treatment.

The following conditions must be guaranteed for the function $\beta(A, u)$:

$$\frac{\partial \beta(A,u)}{\partial A} \leq 0, \tag{2.7}$$

$$\frac{\partial \beta(A,u)}{\partial u} \geq 0, \tag{2.8}$$

and

$$\beta(A, 0) = 0. \tag{2.9}$$

Equation (2.7) implies that for a constant treatment spending less users will stop using drugs because the amount has to be divided under more users. In equation (2.8) the number of drug users is held constant, and now the function is monotonously increasing in treatment spending. If nothing is spent on treatment, the value of β is equal to zero according to equation (2.9).

There exists also a "natural outflow", and it is assumed that it depends only on the drug price. The function Θ is specified as follows:

$$\Theta := \tilde{\mu} p^b, \tag{2.10}$$

where $\tilde{\mu}$ is the desistance proportionality constant, p the heroin price and $b \geq 0$ the elasticity of desistance.

The baseline initiation without prevention, I_{base}, is modelled as

$$I_{base} = \tilde{k}p^{-a}A^{\alpha}$$

(cf. Tragler et al. (2001)), where $\tilde{k}$ is the initiation proportionality constant, p the heroin price, $a \geq 0$ the elasticity of participation, and $0 \leq \alpha \leq 1$ the initiation term exponent.

The following properties are assumed for I_{base}:

$$\frac{\partial I_{base}}{\partial A} = \tilde{k}p^{-a}\alpha A^{a-1} \geq 0$$

and

$$\frac{\partial^2 I_{base}}{\partial A^2} = \tilde{k}p^{-a}\alpha(\alpha - 1)A^{\alpha-2} \leq 0.$$

The differential equation describing the dynamics of the number of heroin users consists of terms for initiation ($I(t)$), outflow due to treatment ($Q_{treat}(t)$), and natural outflow ($Q_{nat}(t)$) (cf. Figure 2.1).

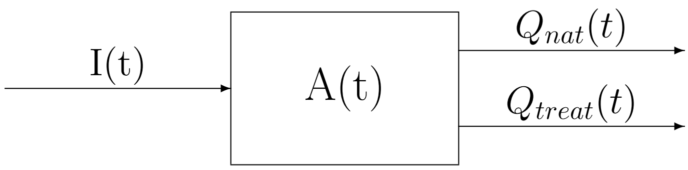

Figure 2.1: Flow diagram for the model by Tragler et al. (2001).

Now, the differential equation $\dot{A}(t)$ can be described as follows:

$$\dot{A}(t) = \tilde{k}p^{-a}A(t)^{\alpha}\vartheta(w(t)) - \Big[c\beta(A(t), u(t)) + \Theta\Big]A(t), \tag{2.11}$$

where $\vartheta(w)$, $\beta(A, u)$ and Θ are from (2.2), (2.6), and (2.10), respectively.

The objective function, J, which describes the total costs of drug use, does not only depend on $u(t)$ and $w(t)$, but also on the social costs caused by the use of illicit drugs. So the function is specified as follows:

$$J = \int_0^{\infty} e^{-rt}\Big(\rho A(t) + u(t) + w(t)\Big)\,\mathrm{d}t, \tag{2.12}$$

where ρ are the social costs per user and year and r is the discount rate.

The aim of the decision maker is to minimize the total costs of drug use and can be specified as follows:

$$\min_{u(t),w(t)\geq 0} J = \min_{u(t),w(t)\geq 0} \int\limits_0^\infty \underbrace{e^{-rt}\Big(\overbrace{\rho A(t)}^{social} + \overbrace{u(t)}^{treatment} + \overbrace{w(t)}^{prevention}\Big)}_{discounted\ costs} \,\mathrm{d}t.$$

2.1 The Model without Controls

This variant is characterized by a decision maker who doesn't care about the drug problem and therefore doesn't spend money for it. The development of the drug problem is just watched by the decision maker. So the model is specified as follows:

$$u(t) \equiv 0, w(t) \equiv 0 \quad \forall t.$$

From this it follows that

$$\vartheta(w) = \vartheta(0) = 1 \quad \text{and} \quad \beta(A,u) = \beta(A,0) = 0$$

(cf. (2.3) and(2.9)).

Substituting into the differential equation (2.11) yields

$$\dot{A}(t) = \tilde{k}p^{-a}A(t)^{\alpha} - \Theta A(t). \tag{2.13}$$

This characterizes no optimization problem because the two controls are not included in this model.

That's why the objective function J has the following form:

$$J = \int\limits_0^\infty e^{-rt}\rho A(t)\,\mathrm{d}t,$$

which depends only on the number of drug users (cf. (2.13)). The model without controls requires the least work, because the decision maker is confined to watch the problem.

2.2 The Model with Constant Controls

This model assumes that the decision maker spends a constant amount every time period for prevention and treatment, so the following properties have to be satisfied:

$$u(t) \equiv \bar{u}, w(t) \equiv \bar{w} \quad \forall t.$$

Therefore the differential equation (2.11) has the following form:

$$\dot{A}(t) = \tilde{k}p^{-a}A(t)^{\alpha}\vartheta(\bar{w}) - \Big[c\beta(A(t), \bar{u}) + \Theta\Big]A(t). \tag{2.14}$$

This problem is again no optimization problem because the two controls are constant and so they cannot be varied. The decision maker spends the same constant amount every time period and watches how the problem develops. The objective function J can be written as follows:

$$J = \int_0^{\infty} e^{-rt}(\rho A(t) + \bar{u} + \bar{w})\,\mathrm{d}t.$$

2.3 The Optimal Control Model

This is a real optimization problem because the decision maker wants to minimize the utility functional J subject to the differential equation (2.11) and the non-negativity constraints $u(t) \geq 0$ and $w(t) \geq 0 \quad \forall t$ (cf. (2.1)). The objective function J has the following form:

$$J = \int_0^{\infty} e^{-rt}(\rho A(t) + u(t) + w(t))\,\mathrm{d}t$$

(cf. (2.12)) and the decision maker aims to minimize this function.

Chapter 3

Determination of the Parameters

In what follows we describe the data that were used for the determination of the model parameters as well as the sources for these data. One of the key characteristics of the model investigated in this thesis is that only those heroin users are considered whose drug career length is greater than one year; these data will hence be refered to as "reduced" data as opposed to the "complete" data including also those users with career length of one year or less. The following data were used:

1. User Data: complete and reduced (yearly from 1971 to 1996); cf. Figure 3.1
2. Initiation Data: complete and reduced (yearly from 1971 to 1996); cf. Figure 3.2
3. Quitting Data: complete and reduced (yearly from 1971 to 1996); cf. Figure 3.3
4. Heroin Price Data (yearly from 1989 to 1999); cf. Figure 3.4
5. Budget Data: the total budget for treatment and prevention spending (yearly from 1986 to 1997); cf. Figure 3.5

User, initiation, and quitting data were provided by Kaya et al. (2002) (complete data) and Kaya and Agrawal (2002) (reduced data) and are based on data from the 1998 National Drug Strategy Household Survey. Heroin price data are from Caulkins (2001). Budget figures are partially from the Ministerial Council on Drug Strategy (1992) and from emails between the National

Drug Strategy Unit and our collaborators from the School of Mathematics, University of South Australia (UniSA), Adelaide (C.Y. Kaya, Y. Tugai, J.A. Filar, M.R. Agrawal).

Based on an observation of the UniSA group, the original budget figure for year 1992 provided by the National Drug Strategy Unit ($ 33.485 million) was adjusted to $ 50.000 million to yield a smoother curve for the budget data. Further the UniSA group worked on the decomposition of the budget into treatment and prevention components using additional sources. The shares of treatment and prevention do not seem to vary much from one year to another, by looking what's happening in the 1990s. Based on their compilation, on the average the share of treatment can be suggested to be taken as 60 %.

The following parameter values are taken directly from Tragler et al. (2001):

- $a = 0.25$
- $b = 0.25$
- $h = 0.84$
- $r = 0.04$
- $\rho = 42,000$

3.1 Estimation of m

The function $\vartheta(w) = h + (1-h)e^{-mw}$ (cf. (2.2)) provides information about the influence of prevention spending on initiation. That's why h can be described as the percentage of the susceptibles who will without fail start using drugs. So $(1-h)$ can be explained as the part of the susceptibles who can be influenced in using drugs. According to the U.S. prevention programs it is known that up to two thirds of $(1-h)$ will not start using drugs after participating in a conventional prevention program. Such programs are made for all kids in a special age group. The costs of an ordinary prevention program are approximately 150 USD per kid. Roughly 1.44 percent of the Australian population (about 18,000,000) are in such an age group, and so the costs of the prevention program are 38,880,000 USD. The conversion factor from USD to AUD is roughly 2, and so the spendings are 77,760,000 AUD per year. This means that every year this amount has to be spent. The

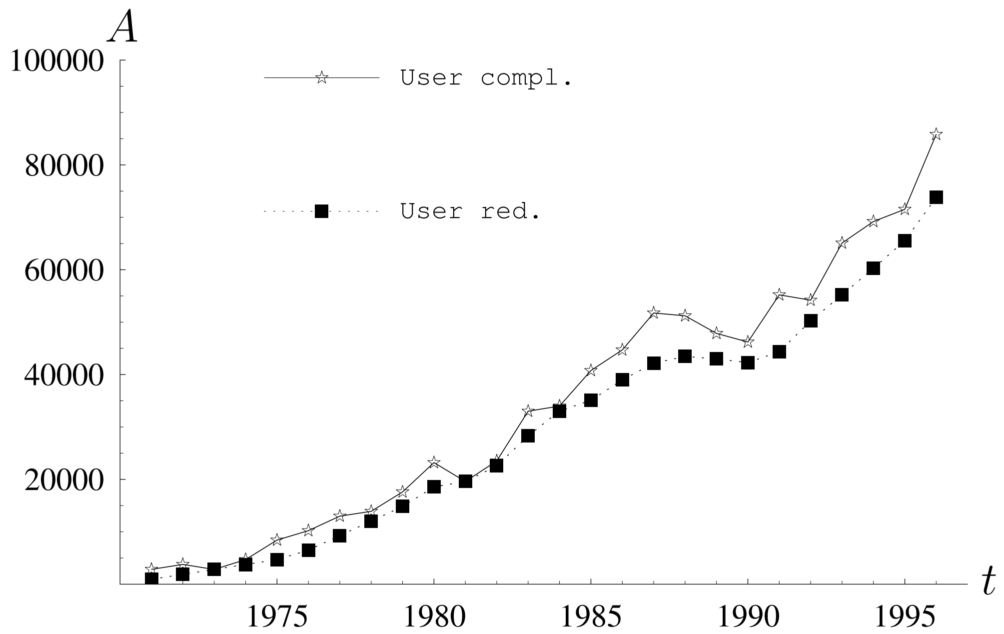

Figure 3.1: User Data: complete and reduced.

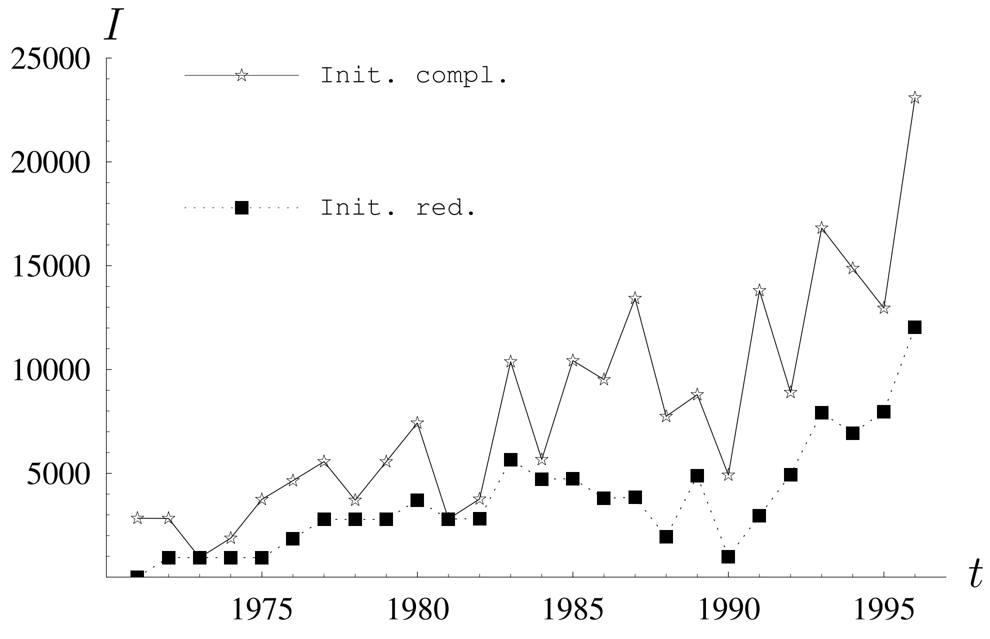

Figure 3.2: Initiation Data: complete and reduced.

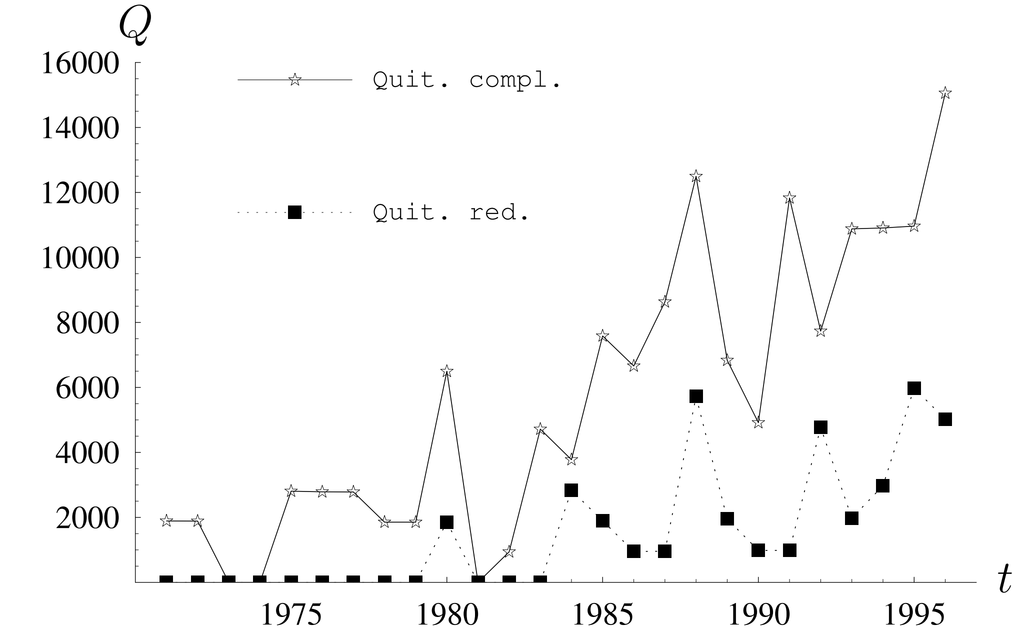

Figure 3.3: Quitting Data: complete and reduced.

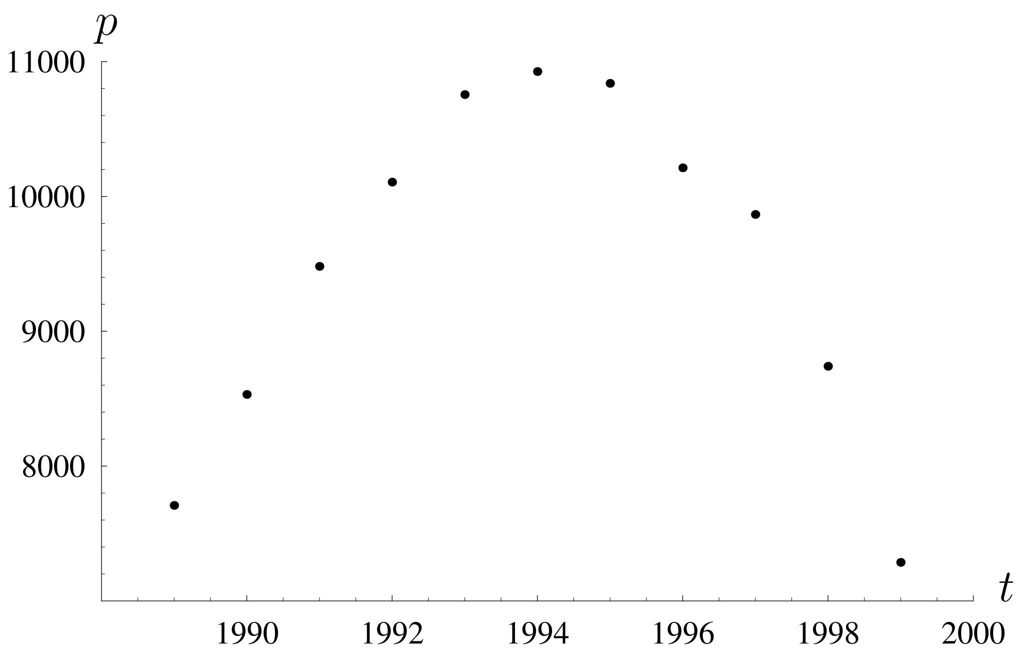

Figure 3.4: Price Data.

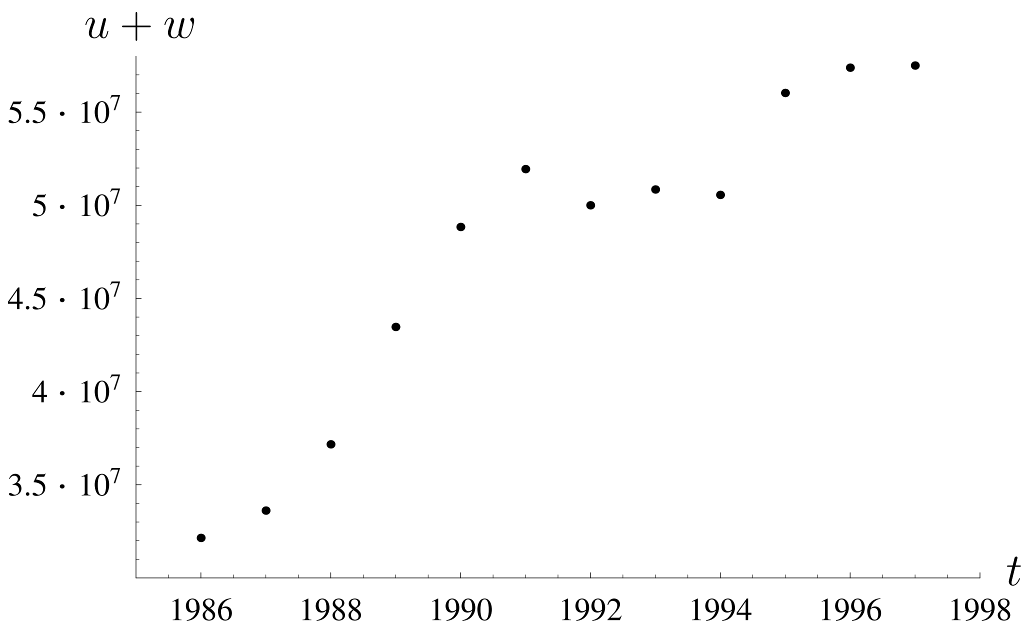

Figure 3.5: Budget Data.

term e^{-mw} gives information about the efficiency of a prevention program because it is the part of the influenceable persons who start using drugs after participating in a program. In our problem this means that:

$$e^{-mw} = \frac{1}{3}.$$

With the prevention spending of 77,760,000 AUD, m receives the following value of $m = 1.41282 \cdot 10^{-8}$. That means, the value of m is received by estimating the costs of a prevention program for 1.44 percent of the Australian population.

3.2 Estimation of $\tilde{k}$

To estimate $\tilde{k}$ the initiation function $I(t)$ is required. Equations (2.2) and (2.11) are used to specify the initiation function as follows:

$$I(t) := \tilde{k}p(t)^{-a}A(t)^{\alpha}\Big[h + (1-h)e^{-mw(t)}\Big].$$

In $I(t)$ there are two unknown parameters. In addition to $\tilde{k}$, α is also undetermined. The best value of $\tilde{k}$ is calculated by setting α constant. Consequently, the quadratic difference of the initiation function and the initiation data over the time period $t = 1989, \ldots, 1996$ is minimized:

$$\min_{\tilde{k}} \sum_{t=1989}^{1996} (I(t) - InitiationData(t))^2.$$

3.3 Estimation of c

For estimating c, the outflow due to treatment function, Q_{treat}, is required, which describes how many persons stop using drugs after being treated. To specify this function the function $\beta(A, u)$ (cf. (2.6)) is needed as well as the differential equation $\dot{A}(t)$ (cf. (2.11)). The function is given as follows:

$$Q_{treat}(t) := c\left(\frac{u(t)}{A(t)}\right)^z A(t).$$

As in the previous section, there exist two undetermined parameters, i.e. c and z. This problem will be solved like in the previous section. z is set constant and then the quadratic difference from the outflow due to treatment function and the half of the quitting data is minimized over c for the time period $t = 1989, \ldots, 1996$:

$$\min_{c} \sum_{t=1989}^{1996} (Q_{treat}(t) - \frac{1}{2}QuittingData(t))^2.$$

The half of the quitting data is taken because it is assumed that the other half of the persons who stop using drugs have other reasons for quitting.

3.4 Estimation of $\tilde{\mu}$

In the next steps the function of those users whose quitting has other reasons than treatment is required. This function is denoted as Q_{nat} and describes the natural outflow. It is specified as follows:

$$Q_{nat}(t) := \tilde{\mu} p(t)^b A(t).$$

In this function only the parameter $\tilde{\mu}$ is unknown. The best value of $\tilde{\mu}$ is calculated by minimizing the quadratic error of the natural outflow function and the half of the quitting data over the time period $t = 1989, \ldots, 1996$:

$$\min_{\tilde{\mu}} \sum_{t=1989}^{1996} (Q_{nat}(t) - \frac{1}{2}QuittingData(t))^2.$$

The quitting data are also divided by two because it is assumed that one half quits after being treated and the other half has other reasons, the so-called natural outflow.

3.5 Calculation of the Error

A decision has to be made, which combination of the parameters describes the model in the best way. In the calculations the values for α and z must be fixed and so the optimal values for $\tilde{k}$ and c are received. To get the number of users $A(t)$ with the specified parameters, the differential equation (2.14) with the condition $A(1971) = 945$ (the number of users with career length greater than 1 year in 1971) has to be solved. Additionally, the following assumption is made to solve the differential equation:

- The average drug price over the time period $t = 1989, \ldots, 1999$ is used for p.

At the beginning the budget data are used in the differential equation. To get the best combination of the parameters for given values of $\alpha = 0.1, (0.05), 0.9$ and $z = 0.1, (0.05), 0.9$ the optimal $\tilde{k}$ and z are calculated. To decide which combination of (α, z) is the best, a determination of the average error between the calculated drug users and drug users from the user data in the time period $t = 1971, \ldots, 1996$ is made as follows:

$$error = \frac{1}{t_N - t_1 + 1} \sum_{t=1971}^{1996} (A(t) - UserData(t))^2. \tag{3.1}$$

Figures 3.6 and 3.7 show the error as a function of the parameters α and z. The only difference of the two figures is the label of the vertical axes (i.e., the error).

In Figure 3.6 the error can be seen as a two-dimensional function of α and z. The first observation which can be made is that the error is very large for big values of α and z. To minimize the error a move to smaller values of α has to be made. It can be seen that the value of z is not so important for the size of the error.

Figure 3.7 shows the same problem but it is zoomed in and so better statements can be made. A small cleft between $\alpha = 0.4, \ldots, 0.6$ can be seen and the value of z is again not as important as the value of α. The region outside

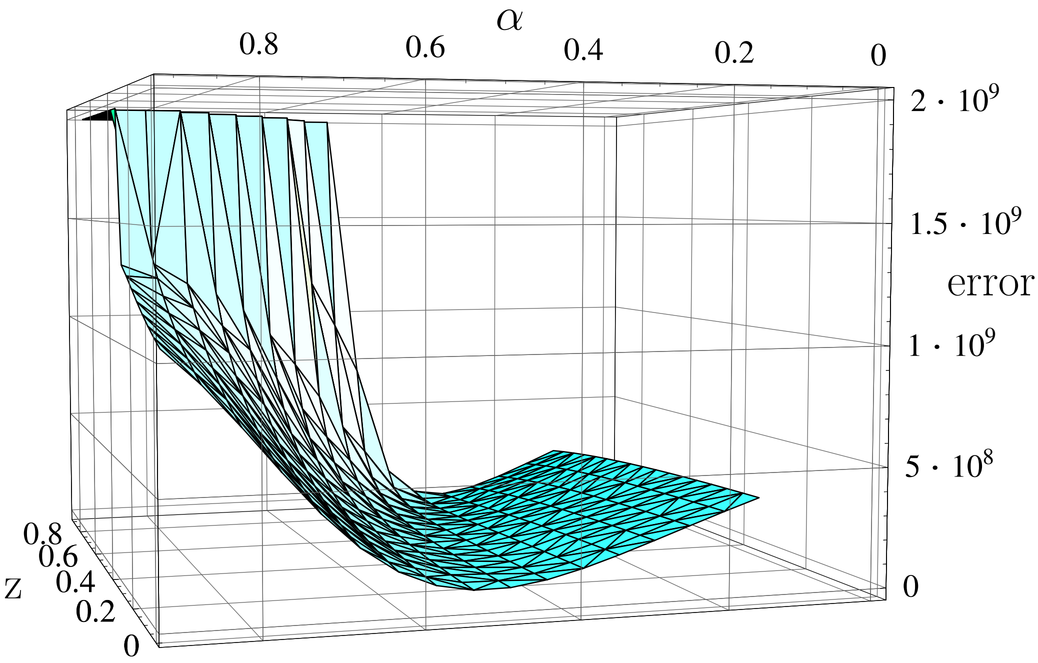

Figure 3.6: 3D-plot of the error (cf. (3.1)) with the maximal error $2 \cdot 10^9$.

the cleft can be excluded to receive optimal combinations of the parameters.

The minimal error in the model amounts to $8.50038 \cdot 10^6$. So the parameters of this variant will be used in the next chapter where the model is analysed. In Figure 3.8 the function of the users, $A(t)$, is plotted with the optimal parameters. To get a chance to see how good the function approximates the user data, the user data are also displayed in Figure 3.8.

In Table 3.1 the base parameter values are subsumed. These values will be used in the next chapter where the analysis of the three variants of the model is carried out.

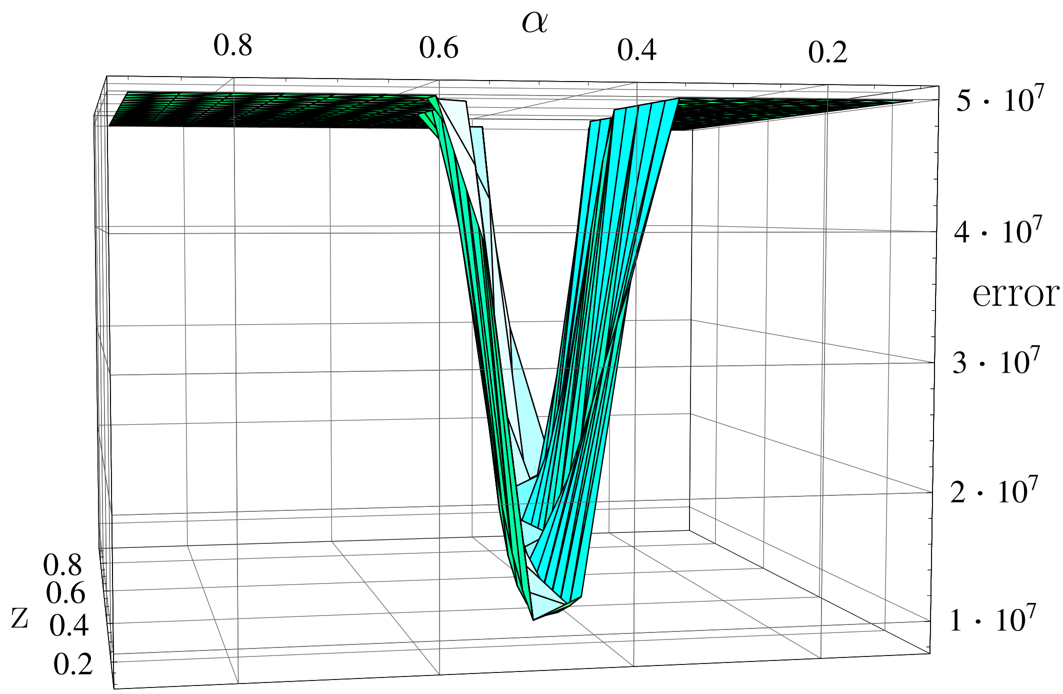

Figure 3.7: 3D-plot of the error (cf. (3.1)) with the maximal error $5 \cdot 10^7$.

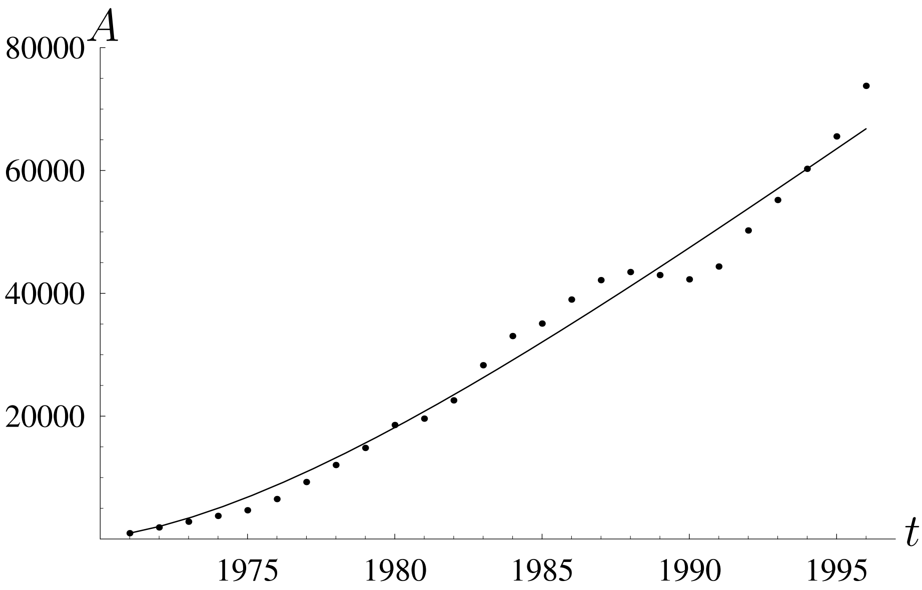

Figure 3.8: User data versus function of $A(t)$ for the base parameter values - budget data with $\bar{u} \equiv 2.79 \cdot 10^7$ and $\bar{w} \equiv 1.86 \cdot 10^7$.

Parameter	Description	Base Value
a	elasticity of participation	0.25
b	elasticity of desistance	0.25
c	treatment proportionality constant	0.000659075
h	minimum of $I_{prev}(t)$	0.84
$\tilde{k}$	initiation proportionality constant	828.347
m	prevention's rate of decay	$1.41282 \cdot 10^{-8}$
p	heroin price	9,495.86
r	discount rate	0.04
z	treatment function exponent	0.6
α	initiation term exponent	0.4
$\tilde{\mu}$	desistance proportionality constant	0.00297822
ρ	social costs per user per year	42,000.00

Table 3.1: Base paramter values.

Chapter 4

Analysis

To simplify the further calculations we set $k := \tilde{k}p^{-a}$ and $\mu := \tilde{\mu}p^b$. Furthermore, the time argument t will mostly be omitted.

4.1 The Model without Controls

The differential equation (2.13) characterizes a typical Bernoulli differential equation (cf. Boyce and DiPrima (1992)), and with Θ from (2.10) it follows:

$$\dot{A} = kA^{\alpha} - \mu A. \tag{4.1}$$

To find the steady states, the equation $\dot{A} = 0$ has to be solved. From this it follows that $\hat{A}_1 = 0$ and

$$\hat{A}_2 = \left(\frac{k}{\mu}\right)^{\frac{1}{1-\alpha}} \tag{4.2}$$

are the steady states.

To receive the number of drug users at any point in time t the differential equation (4.1) has to be solved. This can be done analytically (cf. Appendix A.1). With the initial value $A(0) = A_0$ it follows:

$$A(t) = \left[\frac{k}{\mu} + (A_0^{1-\alpha} - \frac{k}{\mu})e^{(\alpha-1)\mu t}\right]^{\frac{1}{1-\alpha}}. \tag{4.3}$$

The limit of this particular solution for $t \to \infty$ can be easily calculated because we have $0 \leq \alpha \leq 1$, from wich it follows that:

$$\lim_{t\to\infty} A(t) = \left(\frac{k}{\mu}\right)^{\frac{1}{1-\alpha}}.$$

This means that for any initial value of drug users $A_0 > 0$, $A(t)$ converges to the steady state value $\hat{A}_2$ given in (4.2). The convergence of the function $A(t)$ can be seen in Figure 4.1. The parameters are taken from Table 3.1.

It results in a steady state value of $\hat{A}_2 = 574,312$, and with the initial value $A_0 = 5,000$ and (4.3), the utility functional J amounts to $J = 1.23725 \cdot 10^{11}$ (see Table 4.1).

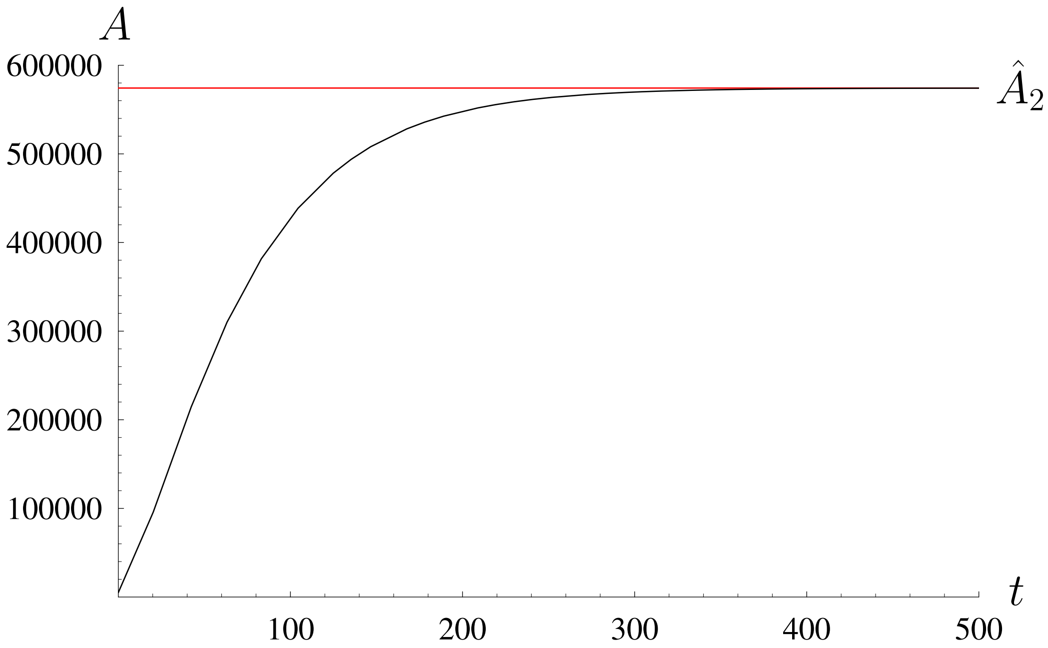

Figure 4.1: Model without controls: A(t) converges to the steady state value $\hat{A}_2$ (red line).

$\hat{A}_2$	$\bar{u}$	$\bar{w}$	J
574,312	0	0	$1.23725 \cdot 10^{11}$

Table 4.1: Model without controls: steady state value of $\hat{A}_2$, $u(t) = w(t) \equiv 0$, and value of the utility functional J.

4.2 The Model with Constant Controls

To solve this problem, the average of the budget data over the time period $t = 1986, \ldots, 1997$ is calculated. The total costs for treatment spending, $u(t)$, and prevention spending, $w(t)$, correspond to this amount. First of all, the condition that the decision maker spends 60 % for treatment and 40 % for prevention is set up. The both controls are constant at $u(t) \equiv 2.79 \cdot 10^7$ and $w(t) \equiv 1.86 \cdot 10^7$. To get the differential equation for this model, the functions $\vartheta(\bar{w})$, $\beta(A, \bar{u})$ and Θ are required (cf. (2.2), (2.6) and (2.10), respectively). Hence, the differential equation becomes

$$\dot{A} = kA^{\alpha}\Big(h + (1-h)e^{-m\bar{w}}\Big) - \bigg(c\Big(\frac{\bar{u}}{A}\Big)^z + \mu\bigg)A.$$

To get the steady state values, the parameters are taken from Table 3.1. Furthermore, the initial value is set to $A_0 = 5,000$. This yields to $\hat{A} = 341,946$ and for the utility functional $J = 7.85096 \cdot 10^{10}$. The convergence of the function $A(t)$ to the steady state and the steady state itself is plotted in Figure 4.2. The summary of the calculated values is given in Table 4.2.

$\hat{A}$	$\bar{u}$	$\bar{w}$	J
341,946	$2.79 \cdot 10^7$	$1.86 \cdot 10^7$	$7.85096 \cdot 10^{10}$

Table 4.2: Model with constant controls: steady state value of $\hat{A}$, constant values of $u(t)$ and $w(t)$ and value of the utility functional J.

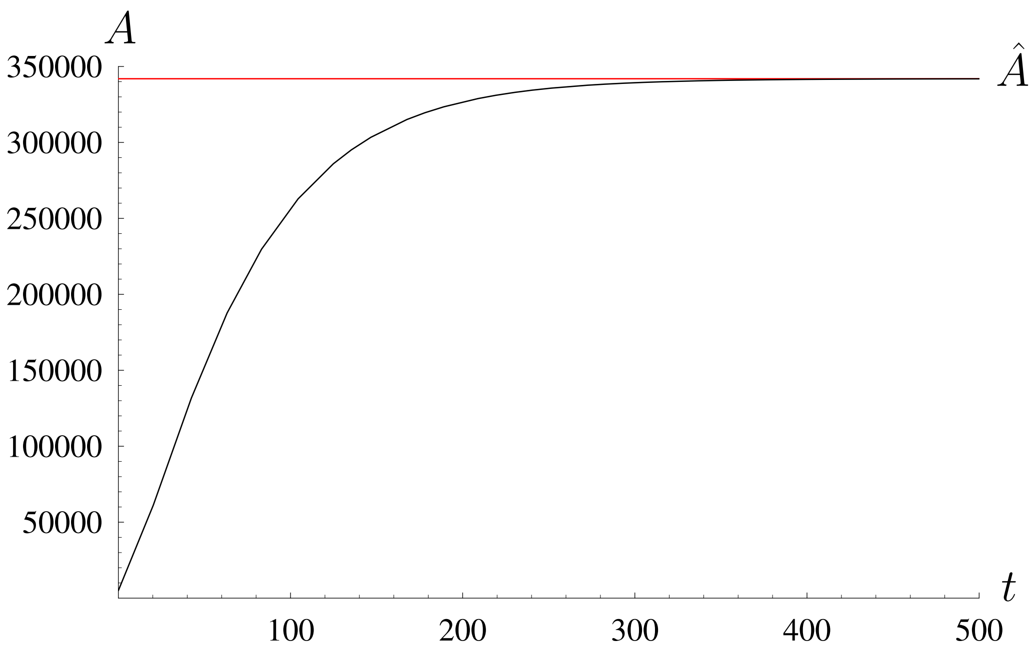

Figure 4.2: Model with constant controls: A(t) converges to the steady state value $\hat{A}$ (red line).

4.3 The Optimal Control Problem

This model is the most demanding one from a mathematical point of view because the use of Pontryagin's maximum priniciple is inevitable (cf. Feichtinger et al. (1986) and Leonard and Long (1996)). The optimization problem is to minimize the total costs of drug use over the time $t = 0, \ldots, \infty$. So the problem has to be converted to a maximum problem. This can be done by multiplying the function J with the factor -1. That's why the maximum problem has the following form:

$$\max_{u(t),w(t)} -J. \tag{4.4}$$

The function J is taken from equation (2.12). To get the differential equation $\dot{A}$, (2.2), (2.6) and (2.10) have to be inserted into equation (2.11) to get the following state equation

$$\dot{A} = kA^{\alpha}(h + (1-h)e^{-mw}) - \left(c\left(\frac{u}{A}\right)^{z} + \mu\right)A. \tag{4.5}$$

Now, the current-value Hamiltonian H for the optimization problem (4.4) subject to (4.5) can be formulated as follows:

$$H = \lambda_0(-\rho A - u - w) - \lambda\Bigg(kA^\alpha(h + (1-h)e^{-mw}) - \\ -\Big(c\Big(\frac{u}{A}\Big)^z + \mu\Big)A\Bigg)$$

where λ_0 and λ describe the costate variables [1]. The use of Pontryagin's maximum principle on this problem leads to two necessary optimality conditions

$$u = \arg\max_u H \tag{4.6}$$

and

$$w = \arg\max_w H. \tag{4.7}$$

Since H is concave with respect to (u, w), $H_u = 0$, $H_w = 0$ give the unrestricted extremum, but because of the non-negativity constraints $u \geq 0$, $w \geq 0$, the border solutions have to be take into account. That means, if the extremum is situated below 0, $u = 0$ and/or $w = 0$ have to be chosen as the extremum, respectively.

From the costate equation

$$\dot{\lambda} = r\lambda - H_A$$

the differential equation for $\dot{A}$ (cf. (4.5)), and the Hamiltonian maximizing conditions (4.6) and (4.7), a differential equation for $\dot{u}$ can be calculated (cf. Appendix A.2):

$$\dot{u} = u\Bigg[\frac{r - \alpha k A^{\alpha-1} + \mu}{1-z} - \frac{c\rho z u^{z-1}}{(1-z)A^{z-1}} - \frac{\alpha c z u^{z-1}}{(1-z)mA^z} + \\ + khA^{\alpha-1} + \frac{czu^{z-1}}{mA^z} - \mu\Bigg].$$

The concavity of the current-value Hamiltonian H w.r.t. A cannot be guaranteed. That's why the optimality of the solution can only be shown by means of the proof that the existing solution is the only one which satisfies the necessary optimality conditions.

[1] λ_0 can be set to one w.l.o.g., cf. Appendix A.3

To get the steady state values of this problem, the two equations $\dot{A} = 0$ and $\dot{u} = 0$ have to be solved simultaneously. Now, the parameters from Table 3.1 have to be used to show the phase portrait. The two necessary optimality conditions (4.6) and (4.7) allow to describe two of the four variables A, u, w, λ as functions of the others. Therefore, the value of $\hat{w}$ can be calculated by inserting the two solutions $\hat{A}$ and $\hat{u}$ in the equation for w_{opt} (cf. (A.9)).

The intersection of the both isoclines $\dot{A} = 0$ and $\dot{u} = 0$ represents a saddle point equilibrium $(\hat{A}, \hat{u})$ (cf. Figure 4.3), because the two eigenvalues are real numbers and have different signs. The green curves are the stable manifolds which give the optimal trajectories. The two grey lines represent the two isoclines $\dot{A} = 0$ and $\dot{u} = 0$. The essential values of this problem are subsumed in Table 4.3.

$\hat{A}$	$\hat{u}$	$\hat{w}$	$J^\star$
216	$3.04201 \cdot 10^8$	$1.20952 \cdot 10^7$	$9.95043 \cdot 10^9$

Table 4.3: Optimal control model: steady state values of $\hat{A}$, $\hat{u}$ and $\hat{w}$ and value of the utility functional J.

Figure 4.4 shows the optimal trajectories of treatment spending, $u(t)$, and of prevention spending, $w(t)$, which are plotted as functions of the number of drug users $A(t)$. The vertical line represents the steady state value $\hat{A}$. This means if you're on the optimal trajectories of u and w, you move to the steady state value. The only question is, from which side of $\hat{A}$ the motion to the steady state takes place. It can be seen that for any value of A treatment spending is greater than prevention spending.

In Figure 4.5 the prevention function $\theta(w)$ is plotted as a function of w. The horizontal line (violet) represents the value of $\theta(w)$ in the steady state $\hat{w}$. The optimal value amounts to 0.974867, while the function converges to the limit of $\theta(w)$ for $w \to \infty$.

Figure 4.6 shows the optimal trajectories of u and w as functions of the time argument t. It was supposed that the initial value of drug users is $A_0 = 5,000$. It can be seen that treatment spending $u(t)$ is always greater than prevention spending $w(t)$. Furthermore, the values of the two functions after 15 years are almost constant, and both spendings remain at this value.

From Figure 4.7 the optimal trajectory of drug users $A(t)$ can be derived as

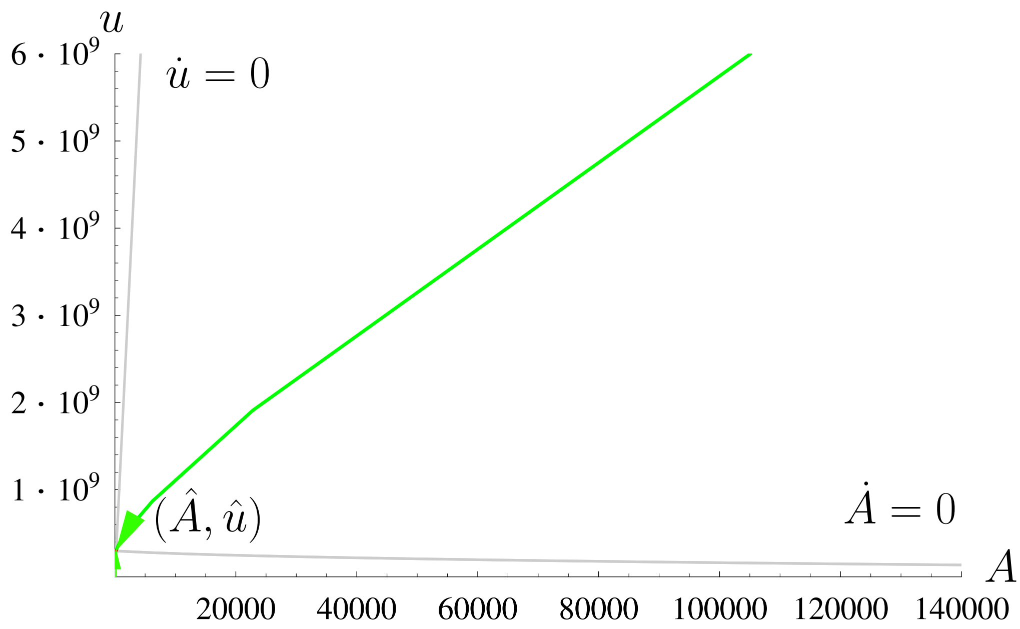

Figure 4.3: Phase portrait in the $A-u$-plane for the optimal control problem. The grey curves represent both isoclines $\dot{A} = 0$ and $\dot{u} = 0$, the green curves are the stable manifolds.

a function of the time argument t. The same initial value $A_0 = 5,000$ was supposed. The trajectory converges to the steady state value $\hat{A}$ (red line). The optimal trajectory of $A(t)$ represents also a monotonously decreasing function due to a higher initial value of drug users compared with the low steady state $\hat{A} = 216$. So one is basically always situated on the second stable manifold (cf. Figure 4.3, right green function).

Figure 4.8 shows the functions $I(t)$, $Q_{treat}(t)$ and $Q_{nat}(t)$ which are evaluated along the optimal paths of $u(t)$ and $w(t)$. It can be seen that in the first years the quitting due to treatment is greater than the initiation. This can be explained as follows: The optimal trajectory of drug users is a monotonously decreasing function and so the value of drug users has to decrease which can happen if $Q_{treat}(t) \geq I(t) \quad \forall t$. After few years the steady state is reached and now the people who initiate in the drug scene, quit after being treated. This can be explained mathematically as follows: $\dot{A}(t)|_{\hat{A}} = 0$. That means if you're staying in the steady state, the number of drug users doesn't increase.

In Figure 4.9 one can see the function $\theta(w)$ as a function of t where the opti-

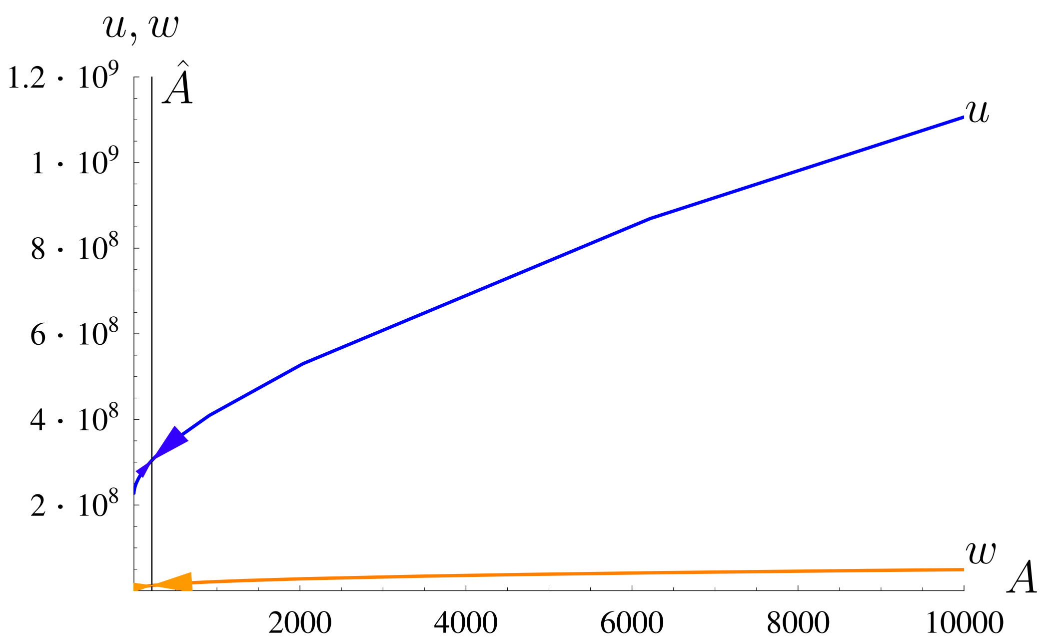

Figure 4.4: Optimal trajectories for u (blue) and w (orange) as functions of A. The vertical line represents the steady state value $\hat{A}$.

mal prevention spending has been taken. The initial value of the drug users is taken as $A_0 = 5,000$. The initial value of the function $\theta(w(0))$ amounts to 0.932601, and for $t \to \infty$ the function converges to the steady state value $\theta(\hat{w}) = 0.974867$. This means that the effect of prevention decreases, and after 15 years the value of $\theta(w)$ is almost constant.

A very interesting question can be formulated as follows: What happens, if the decision maker starts the optimal control with a delay?

This problem is analysed because it is possible that the decision maker starts the optimal control too late. This may have many causes. For instance, the spendings for the drug problem have to be granted from the parliament, since the financial plans are made one or two years before. The next possibility could be that the decision maker realizes too late that there already exists a drug problem or she/he doesn't want to realize that there is a drug problem and so many years pass until control measures start.

From Figure 4.10 it can be seen what happens to the utility functional J, if the optimal control starts with delay. That means, if the delay τ is x years,

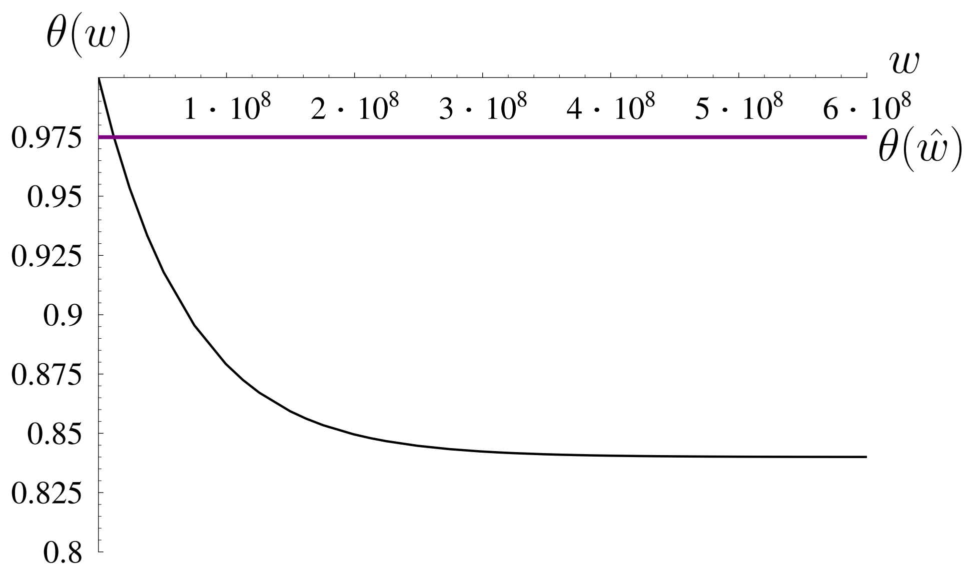

Figure 4.5: The prevention function $\theta(w)$ in addition to the horizontal line (violet) which represents θ in the steady state.

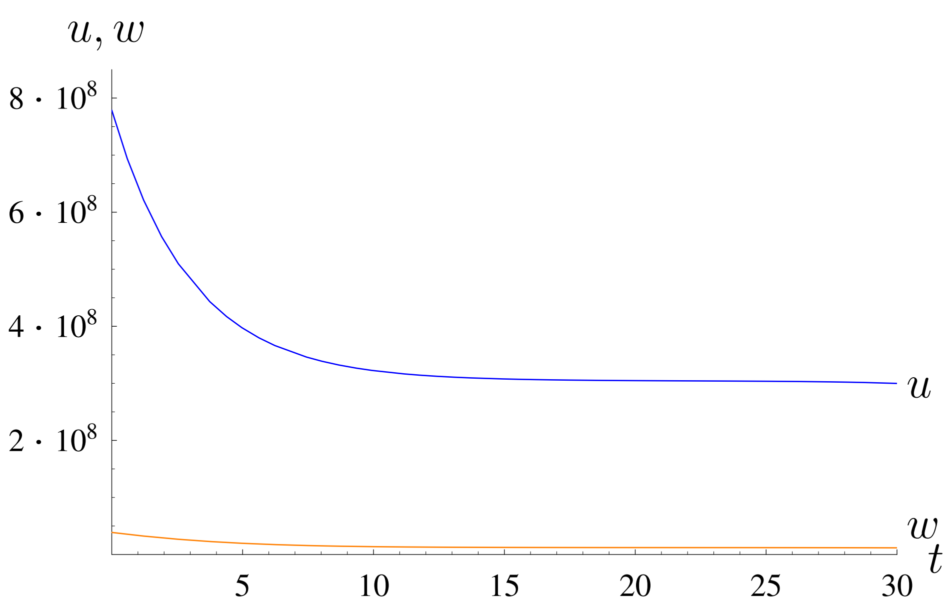

Figure 4.6: Optimal trajectories u (blue) and w (orange) as functions of t.

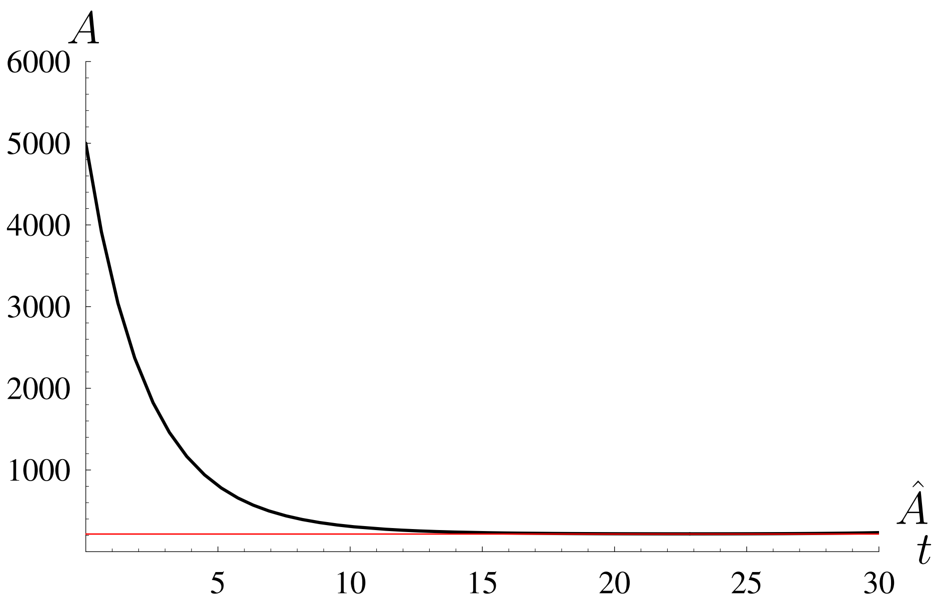

Figure 4.7: Optimal trajectory A converges to the steady state value $\hat{A}$ (red line).

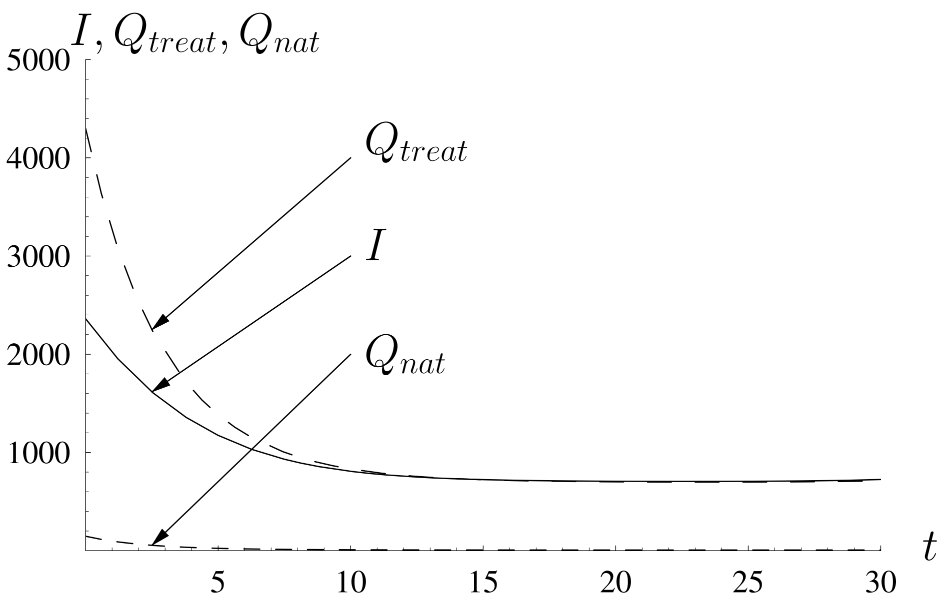

Figure 4.8: The functions $I(t)$, $Q_{treat}(t)$ and $Q_{nat}(t)$ evaluated along the optimal trajectories of u and w as functions of the time argument t.

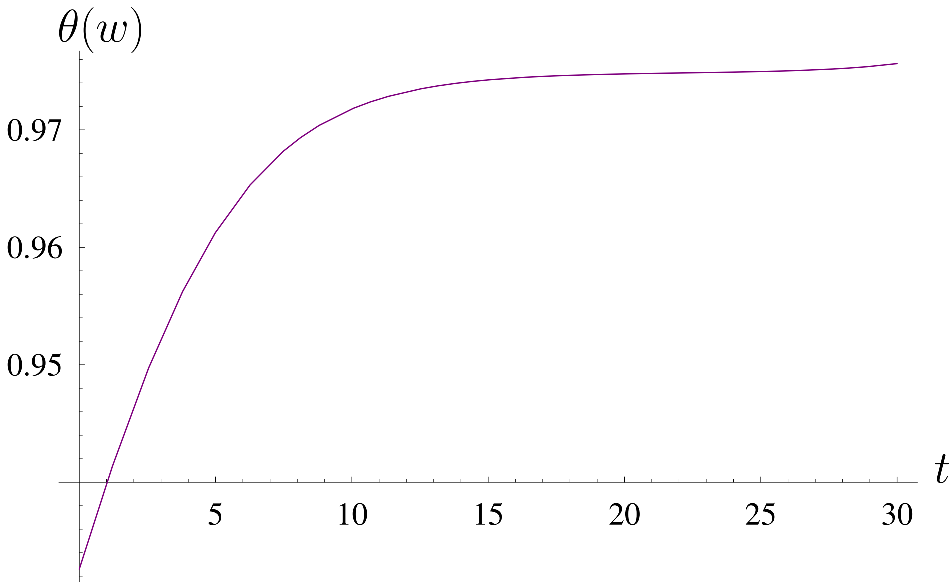

Figure 4.9: The prevention function $\theta(w)$ evaluated along the optimal trajectory of w as a function of t.

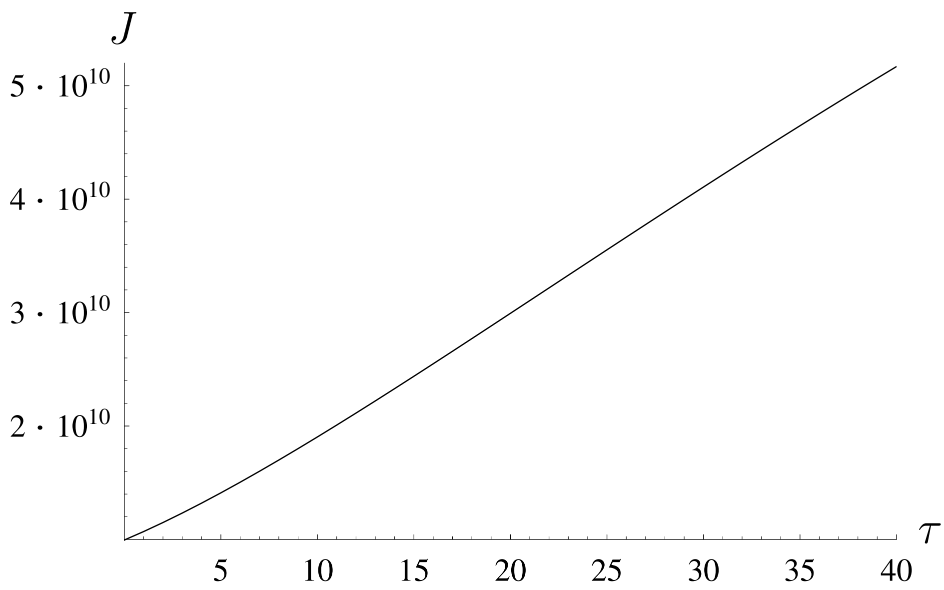

Figure 4.10: The total costs J as a function of the delay τ before the optimal control starts.

the number of drug users with the initial value of $A_0 = 5,000$ first increases just as in the model without controls. After x years, the optimal control starts and the function of the drug users $A(t)$ moves further on the optimal trajectory. So if $\tau = 0$ the optimal control model is used. We learn that a prompt spending is essential and saves a lot of money.

Chapter 5

Sensitivity Analysis

The sensitivity analysis is a very important part of analysing a problem like this, because many parameters have to be estimated (cf. Chapter 3), which produces inaccuracy. Note, that some paramters are taken from Tragler et al. (2001). In the first part, the effects of efficiency parameter changes have been observed. The second part deals with the consequences of changing the ratio between quitting due to treatment and the natural outflow. In the next one, the solutions for the case with the original budget data point from 1992 with \$ 33.485 million and with the ratio between $u(t)$ and $w(t)$ from 50:50 have been calculated. Finally, we refer to Appendix B, which deals with the introduction of a constant γ in the function $\beta(A(t), u(t))$ and the consequences which follow from introducing this constant.

5.1 Effects of Changes of Prevention's Efficiency Parameters

5.1.1 Changing the Minimum h of $I_{prev}(t)$

The value of the parameter h is taken from Tragler et al. (2001). It describes how effective prevention may be, because it describes the minimum level to which prevention may decrease the initiation. This means, if prevention is very effective, the value of h can get smaller, because less susceptibles will start using drugs. The value of h is only estimated and so it makes sense to alter the value of the parameter. Nobody knows how effective prevention really is and so the effects of changing h are very interesting to know.

In Figure 5.1 the changes of the steady state value of $\hat{A}$ for different values of h can be seen. The value of h is varied from 0.7 to 0.87. The other parameter values are taken from Table 3.1. The plot shows an almost linearly increasing function of h. This means, the more effective the prevention is (lower values of h) the smaller is the value of the steady state. The explanation for this can be seen in the differential equation (2.11). This equation includes a smaller flow of initiation for smaller values of h. The consequence is a lower steady state value of $\hat{A}$.

Figure 5.2 shows the effect of changing h on the steady state values of treatment spending $\hat{u}$ and prevention spending $\hat{w}$. We see that the more effective prevention is, the higher is the steady state value of $\hat{w}$. On the other hand, the higher the steady state value of the prevention spending, the lower is the steady state value of treatment spending.

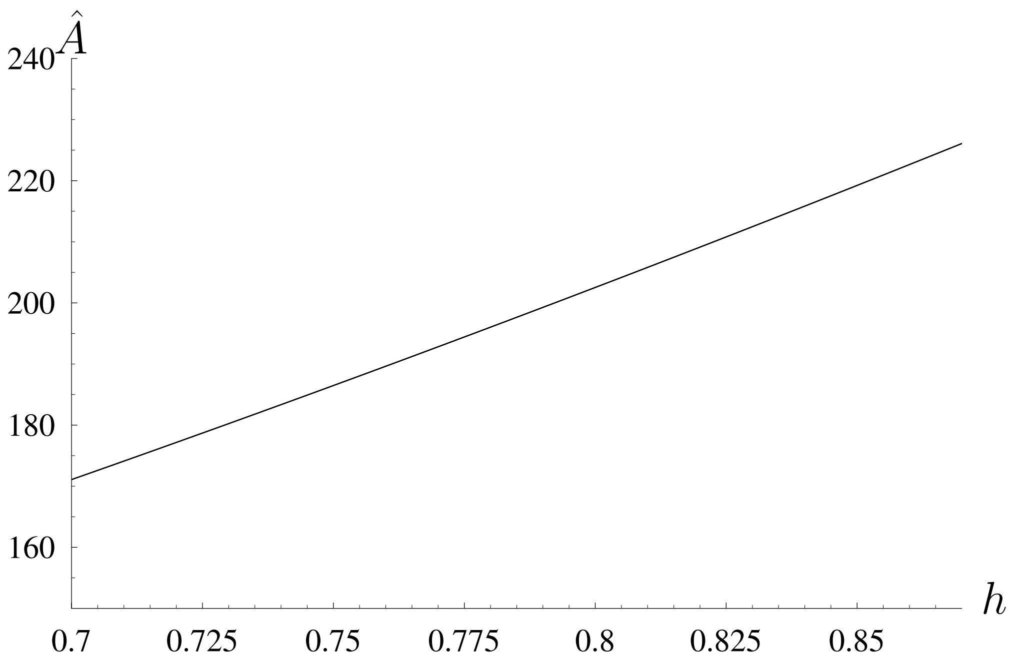

Figure 5.1: Steady state value of the number of users $\hat{A}$ as a function of h.

5.1.2 Changing Prevention's Rate of Decay m

The value of m is estimated in Chapter 3.1 and describes how fast the function $\vartheta(w) = h + (1-h)e^{-mw}$ (cf. (2.2)) reaches the level h. So the value m can be described also as the effectiveness of prevention but in another sense. The parameter h describes how high the minimal percentage of the susceptibles is who start using drugs although w is spent for prevention. The value

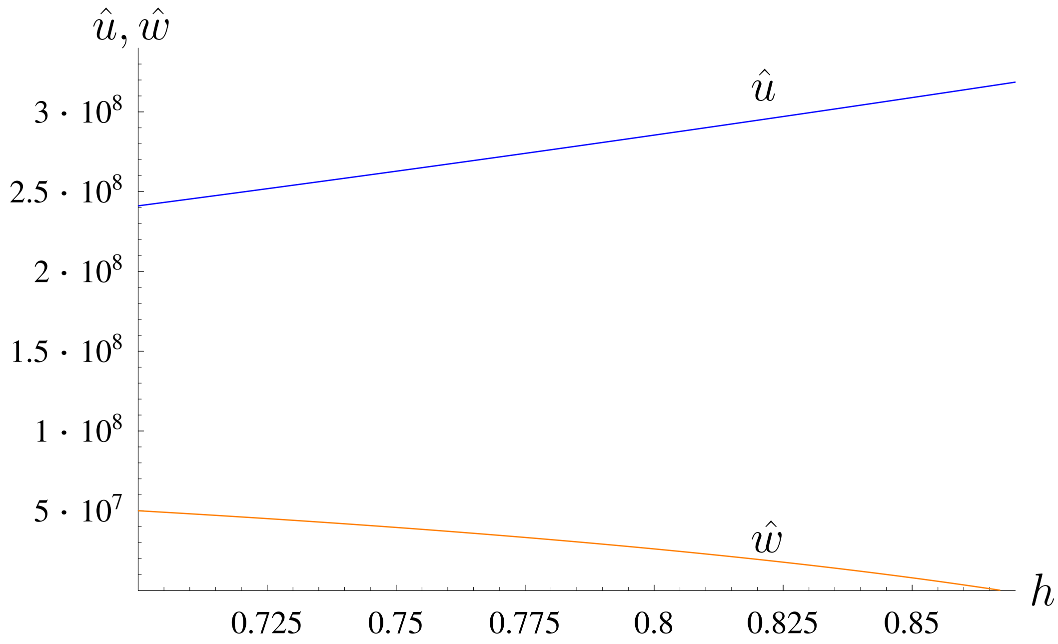

Figure 5.2: Steady state values of treatment spending $\hat{u}$ (blue) and prevention spending $\hat{w}$ (orange) as functions of h.

of m describes only the velocity to reach this minimal level. This means, the higher the value of m is, the faster the initiation reaches the minimum level.

In Figure 5.3 the steady state value $\hat{A}$ can be seen as a function of m. The function describes a convex, monotonously decreasing function of m. The values of m varied from $1.18 \cdot 10^{-8}$ to $2.9 \cdot 10^{-8}$. It can be seen that the steady state value of $\hat{A}$ decreases for higher values of m. The higher the value of m is, the faster $\vartheta(w)$ reaches the minimum level of h. Consequently, the term of initiation in the differential equation (2.11) is smaller, and so a lower steady state is reached.

Figure 5.4 shows the effect of different values of m on the steady state values of $\hat{u}$ and $\hat{w}$. $\hat{u}$ as a function of m describes a convex, monotonously decreasing function. On the other hand, $\hat{w}$ shows a concave, monotonously increasing function of m. The faster the minimum h can be reached, the more is spent on prevention and the less on treatment.

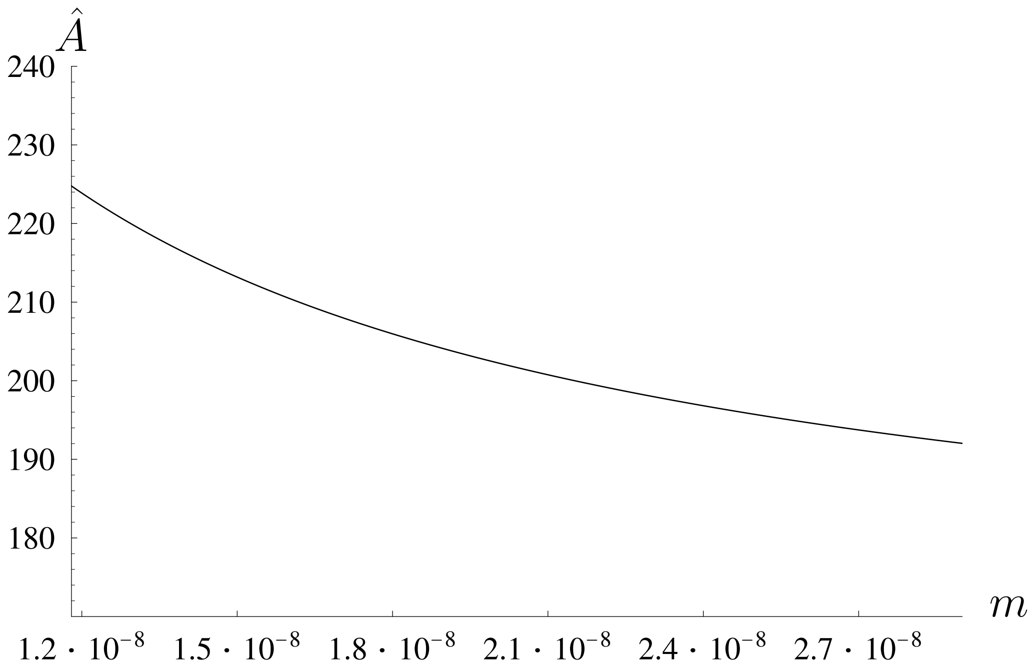

Figure 5.3: Steady state value of the number of users $\hat{A}$ as a function of m.

5.2 Changing the Ratio between Outflow Due to Treatment and Natural Outflow

In this section the ratio between the different outflows is changed. This has to be done since nothing about this ratio is well known. In Chapter 3 this ratio is set to 50:50. That means, 50 percent of the outflow are due to treatment and the other 50 percent quit due to other reasons (the so-called natural outflow). This proportion has an effect on the estimation of the parameters, and so the parameters have to be re-estimated. The first step of changing the ratio is to calculate the new parameters in the same way as in Chapter 3. The next step is to solve the optimal control model with Pontryagin's maximum principle (cf. Section 4.3). The resulting optimal trajectories are plotted. Big focus was set on the accuracy of the parameter changes, which has been increased.

5.2.1 Determination of the Parameters

In this subsection the estimation of the parameters will be discussed. The accuracy of the parameter changes has been increased because the former

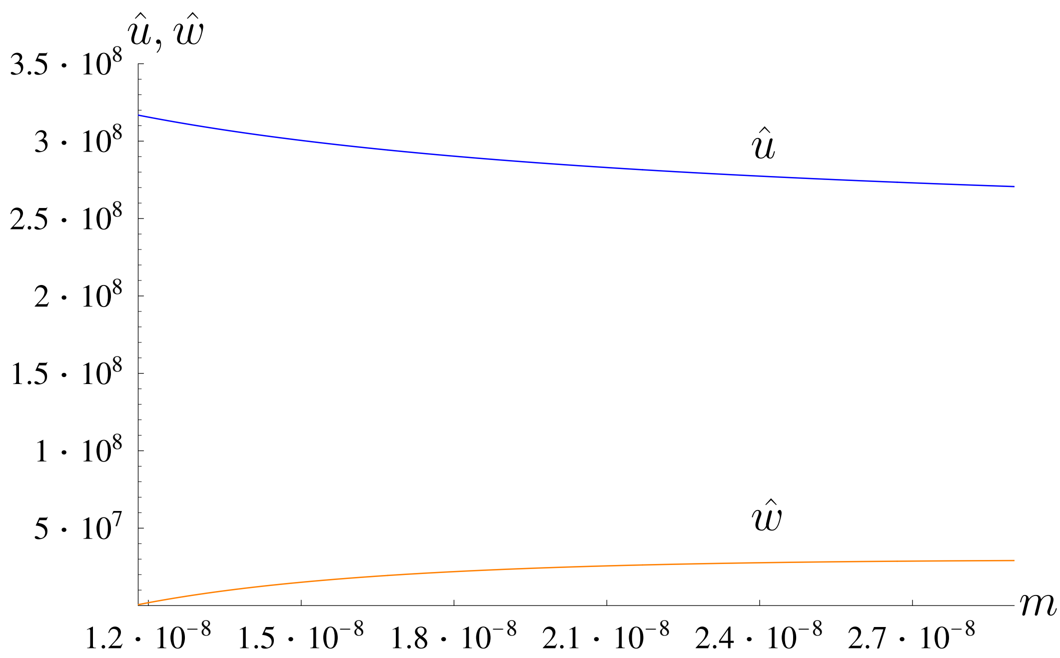

Figure 5.4: Steady state values of treatment spending $\hat{u}$ (blue) and prevention spending $\hat{w}$ (orange) as functions of m.

changes from $\alpha = z = 0.1, (0.05), 0.9$ has been altered to $\alpha = z =$0.1, (0.01), 0.9. The ratio changes have been made also in 0.01 steps from 0.1 to 0.9. The optimal parameters have been plotted for the different ratios. The only problem is represented through the following inequality

$$\alpha + z \leq 1. \tag{5.1}$$

If α and z fulfil this inequality, both isoclines $\dot{A} = 0$ and $\dot{u} = 0$ have only one intersection (cf. Mautner (2002)). This means that a saddle point equilibrium $(\hat{A}, \hat{u})$ has been reached. If the inequality (5.1) does not hold the two isoclines have no point of intersection and the optimal solution is a trajectory converging to the origin $(0, 0)$.

With the new parameter values, some interesting changes appear. The value of the ratio (abbreviated by R) has to be understood in the following way: The value of R is the fraction of users who quit because of treatment. The estimation shows that the sum of α and z doesn't fulfil the inequality (5.1) for the following values $R = 0.1, (0.01), 0.59$ and 0.61.

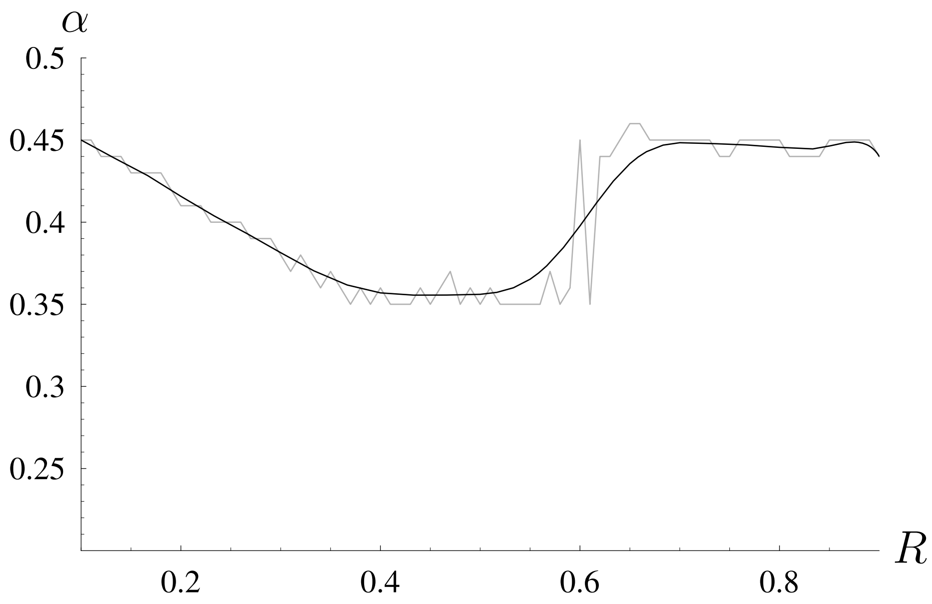

Figure 5.5: Optimal values of α for different outflow ratios, connected by lines (grey) and interpolated by bezier splines (black).

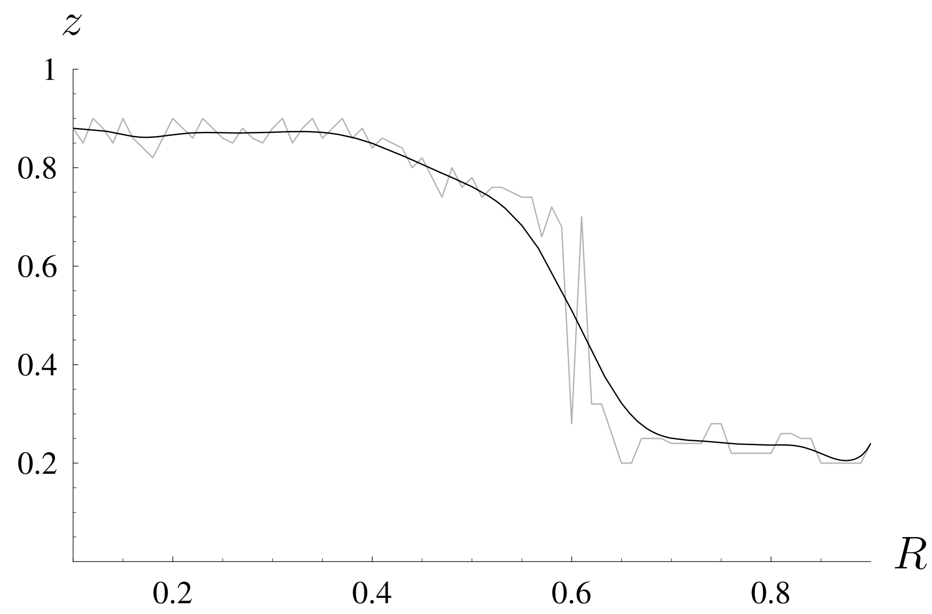

Figure 5.6: Optimal values of z for different outflow ratios, connected by lines (grey) and interpolated by bezier splines (black).

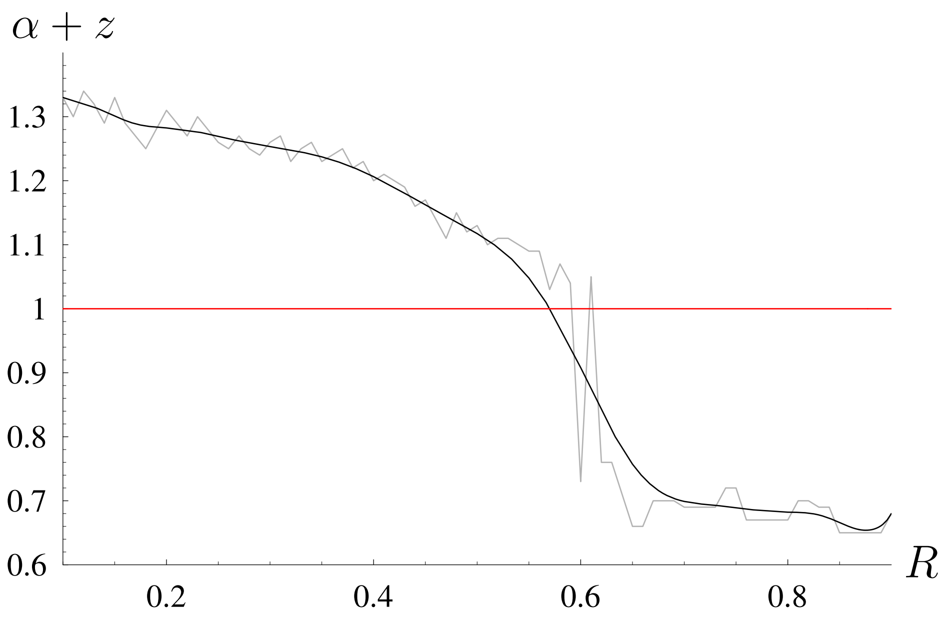

Figure 5.7: Sum of the optimal values of α and z for different outflow ratios, connected by lines (grey) and interpolated by bezier splines (black).

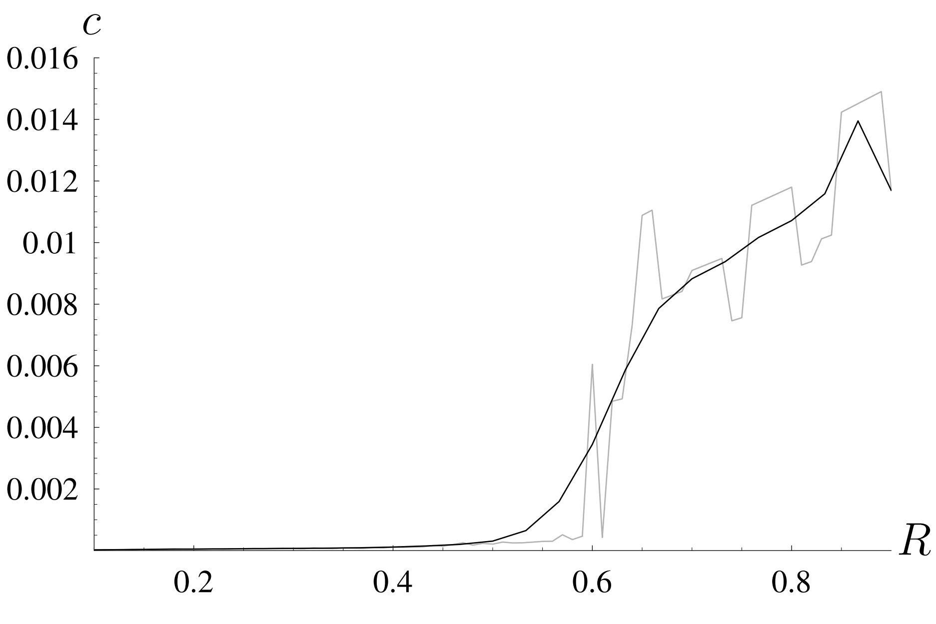

Figure 5.8: Optimal values of c for different outflow ratios, connected by lines (grey) and interpolated by bezier splines (black).

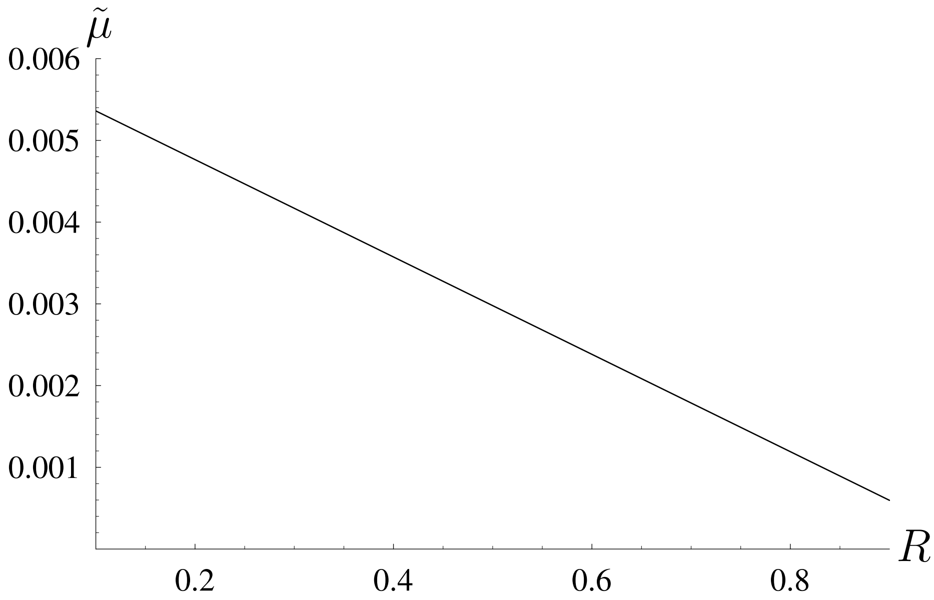

Figure 5.9: Optimal values of $\tilde{\mu}$ for different outflow ratios.

Figures 5.5-5.8 and 5.10-5.12 show optimal parameter values and steady state values for different values of R. To remove fluctuations in the plots, the values are also interpolated by bezier splines (i.e., cubic parables are stringed together with the same bend).

Figure 5.5 shows the optimal parameter α for the different ratios of outflow due to treatment and the natural outflow. It can be seen that the parameter moves only in the area $(0.35, 0.45)$. In the first part the function is monotonously decreasing and around $R = 0.5$, α is increasing. For values of R bigger than 0.65, the parameter α amounts always to 0.45.

In Figure 5.6 the optimal values for the paramter z can be seen. In the first part the value of z is very high (around 0.9) but it falls down around $R = 0.6$. This ratio represents the first one at which the sum of α and z is smaller than 1 (cf. (5.1)). The ratios $R = 0.62, (0.01), 0.9$ show z in the lower area.

Figure 5.7 represents the sum of both essential paramters α and z. As described before, for a sum greater than 1 the isoclines have no intersection point and the optimal solution is a trajectory converging to the origin $(0, 0)$. It can be seen that the ratios at which the isoclines have an intersection point are given by values of R above 0.6. The red line describes the step from no to

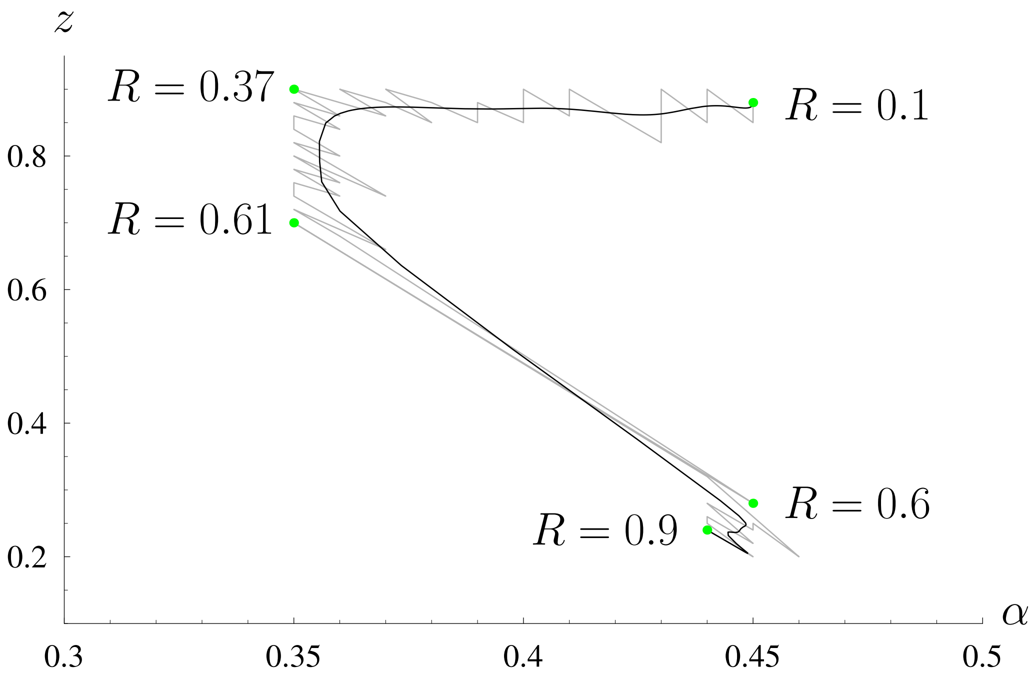

Figure 5.10: Optimal combination of the parameters α and z in the $\alpha - z$-plane, connected by lines (grey) and interpolated by bezier splines (black).

one intersection point. The bezier splines describe a monotonously decreasing function.

Figure 5.8 shows how the treatment proportionality constant c changes for different outflow ratios. At first glance it can be seen that the value of c is very small for ratios $R = 0.1, (0.01), 0.59$. The first increase takes place at $R = 0.6$, which is the first case where the inequality (5.1) is fulfilled. From then on the value of c increases very strongly.

In Figure 5.9 the desistance proportionality constant $\tilde{\mu}$ can be seen. The value of $\tilde{\mu}$ decreases with an increasing ratio R.

Figure 5.10 shows the changes of the combinations which occur if the ratio between quitting due to treatment and natural outflow will be changed. It can be seen that the changes of the values are in the first part only in the α-area. At the ratio value $R = 0.6$ a jump to the lower z-values takes place (this value represents the first case with a saddle-point equilibrium). Afterwards the combination of (α, z) stagnates at low z values. The same observations can be made by looking to the bezier splines but there the small fluctuations are removed.

5.2.2 Steady State Values after Ratio Changes

Comparison of R =0.1,(0.01),0.9

This part is used to describe which effect the ratio changes have on the steady state value of drug users. The first part ($R = 0.1, (0.01), 0.59$) was excluded since this one has two isoclines with no intersection point. This one has the long-run optimal solution $(\hat{A}, \hat{u}) = (0, 0)$. The ratio $R = 0.61$ produces also no saddle-point equilibrium and that's why this ratio has always the value 0 on the following plots. This phenomenon was described precisely in the subsection 5.2.1 and the reason for this is lying in inequality (5.1).

Figures 5.11 and 5.12 show the changes of the steaty state values of $\hat{A}$, $\hat{u}$ and $\hat{w}$ for different outflow ratios.

In Figure 5.11 it can be seen how the steady state value $\hat{A}$ changes for different values of R. The different values of $\hat{A}$ describe a non-monotonous function but values with small differences are always in the same area. The jumps from plane to plane which are characterized by differences in altitude are caused by the different values of c. Figure 5.8 shows the different values of c for different outflow ratios and this figure shows strong similarities with Figure 5.11. The bezier splines remove the high fluctations in the plot, which is why the black function has only small oscillations.

Figure 5.12 shows the steady state values $\hat{u}$ and $\hat{w}$ for different values of R. The optimal solutions of these two functions are almost constant except the value for $R = 0.61$ (cf. subsection 5.2.1).

Comparison of R =0.6, 0.65, 0.75 and 0.85 in detail

First and foremore 4 cases were lifted out. The ratios are $R = 0.6$, 0.65, 0.75 and 0.85. The optimal parameters for these cases are calculated in the same way as in Chapter 3 and are subsumed in Table 5.1. The remaining parameters are taken from Table 3.1. The optimal parameters are used to solve the differential equations (2.11) and (A.15) to receive the steady state values $(\hat{A}, \hat{u}, \hat{w})$ (cf. Table 5.2). It can be seen that the values of the essential parameters α and z are always in the same area.

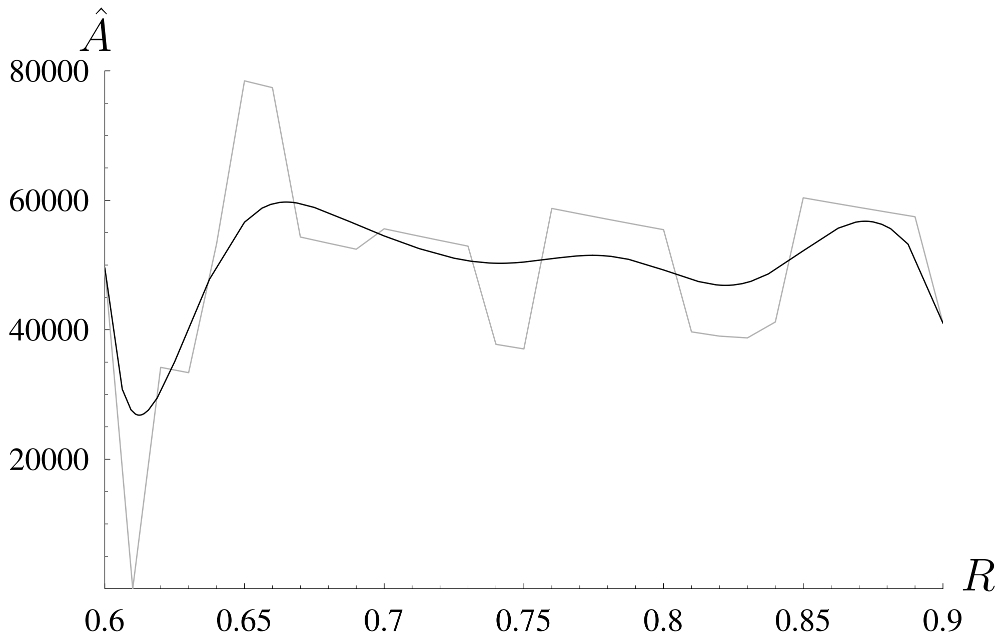

Figure 5.11: Steady state values $\hat{A}$ for different ratios $R = 0.1, (0.01), 0.9$, connected by lines (grey) and interpolated by bezier splines (black).

Figure 5.13 shows the phase portrait in the $A-u$-plane for the four different ratios. The blue curves are the stable manifolds for the different ratios, which represent the optimal trajectories. It can be seen that these curves have always the same appearance - the only difference is the ascent. The vertical lines represent the steady state values for the tested ratios.

In Figure 5.14 the phase portrait in the $A-w$-plane can be seen. The orange curves are the optimal trajectories for the different ratios. The functions look very similar but they have all different altitudes. The vertical lines represent the different steady state values.

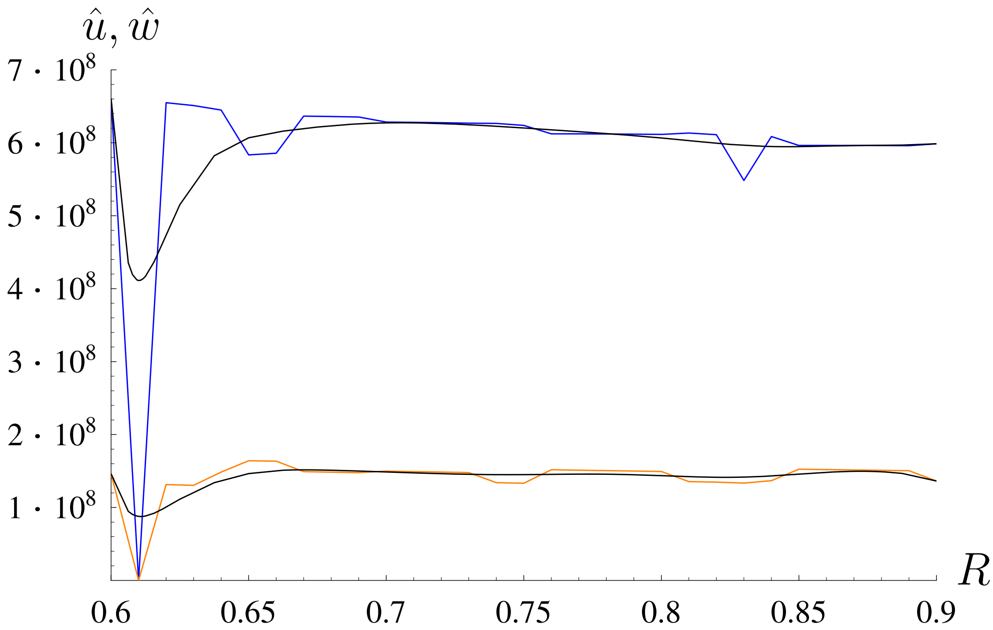

Figure 5.12: Steady state values $\hat{u}$ (blue) and $\hat{w}$ (orange) for different ratios $R = 0.1, (0.01), 0.9$, connected by lines and interpolated by bezier splines (black).

Par.	R =0.6	R =0.65	R =0.75	R =0.85
c	0.006048169	0.010884874	0.007560211	0.014234067
$\tilde{k}$	482.115	432.636	537.247	482.115
z	0.28	0.2	0.28	0.2
α	0.45	0.46	0.44	0.45
$\tilde{\mu}$	0.002382578	0.002084756	0.001489111	0.000893467

Table 5.1: Base paramter values for R =0.6, 0.65, 0.75 and 0.85.

5.3 Corrected Budget Data

This section has been included because the drug user data used in this master thesis include only users with a drug career length greater than one year. First, the original budget data point \$ 33.485 million for 1992 was used because this amount was confirmed by the National Drug Strategy Unit. Second, the ratio between treatment spending, $u(t)$, and prevention spending, $w(t)$, is 50:50. Third, oberservations were made that the budget data should be reduced, because if less drug users have to be treated, less money

R	$\hat{A}$	$\hat{u}$	$\hat{w}$	$J^{\star}$
0.6	49,564	$6.60338 \cdot 10^8$	$1.46147 \cdot 10^8$	$4.32116 \cdot 10^{10}$
0.65	78,458	$5.83398 \cdot 10^8$	$1.64068 \cdot 10^8$	$5.06296 \cdot 10^{10}$
0.75	37,040	$6.23845 \cdot 10^8$	$1.33240 \cdot 10^8$	$3.83275 \cdot 10^{10}$
0.85	60,388	$5.96310 \cdot 10^8$	$1.52492 \cdot 10^8$	$4.59194 \cdot 10^{10}$

Table 5.2: Steady state values of $\hat{A}$, $\hat{u}$, $\hat{w}$ and value of the utility functional J for R =0.6, 0.65, 0.75 and 0.85.

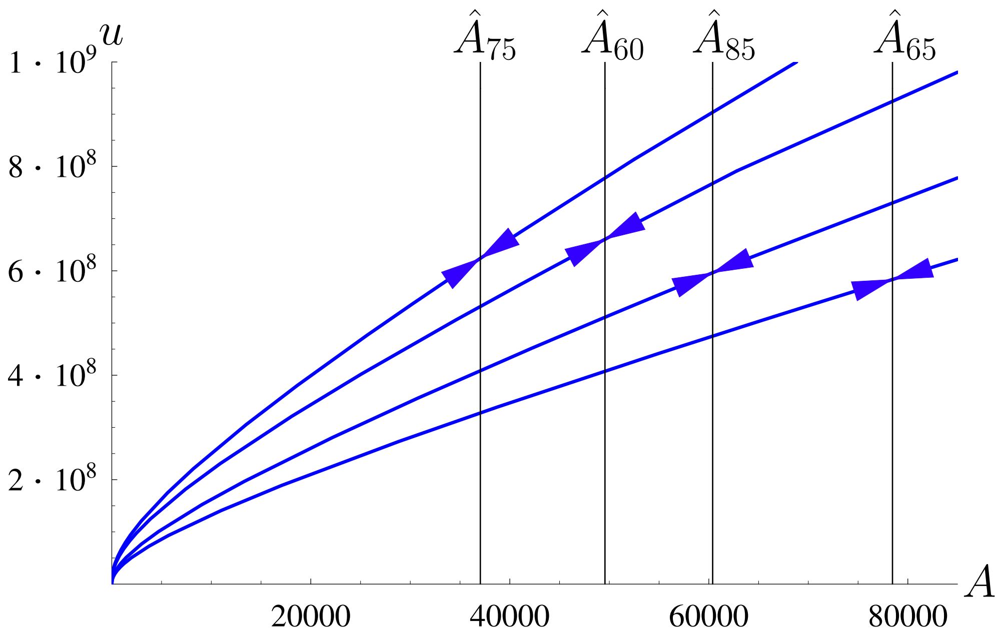

Figure 5.13: The optimal policies for treatment spending u as a function of A for different outflow ratios. The vertical lines represent the different values of $\hat{A}$ for different outflow ratios.

will be needed. That's why the budget data of every year in this section were reduced to 85 percent. At the beginning the optimal parameters have to be estimated. The determination runs like in Chapter 3 and the parameters are given in Table 5.3.

Figure 5.15 shows how good the determined parameters estimate the user data. This was done by solving the differential equation (2.11) with the new parameters and then plotting the function $A(t)$ with the user data points. It can be seen that the approximation is accurate.

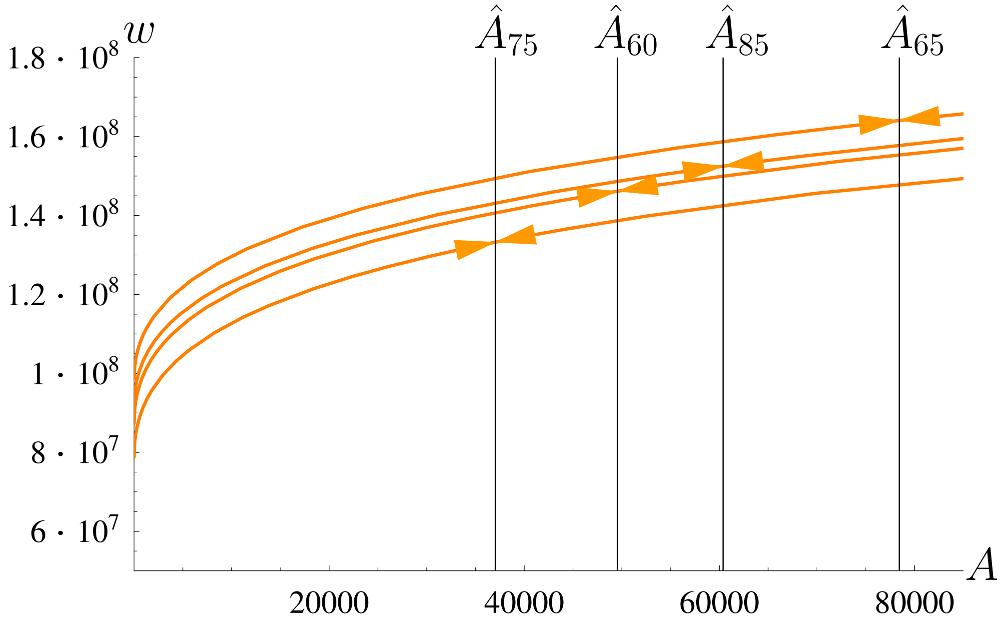

Figure 5.14: The optimal policies for treatment spending w as a function of A for different outflow ratios. The vertical lines represent the different values of $\hat{A}$ the different outflow ratios.

Parameter	Description	Base Value
a	elasticity of participation	0.25
b	elasticity of desistance	0.25
c	treatment proportionality constant	0.00494256
h	minimum of $I_{prev}(t)$	0.84
$\tilde{k}$	initiation proportionality constant	482.326
m	prevention's rate of decay	$1.41282 \cdot 10^{-8}$
p	heroin price	9,495.86
r	discount rate	0.04
z	treatment function exponent	0.3
α	initiation term exponent	0.45
$\tilde{\mu}$	desistance proportionality constant	0.00297822
ρ	social costs per user per year	42,000.00

Table 5.3: Base paramter values - corrected budget data.

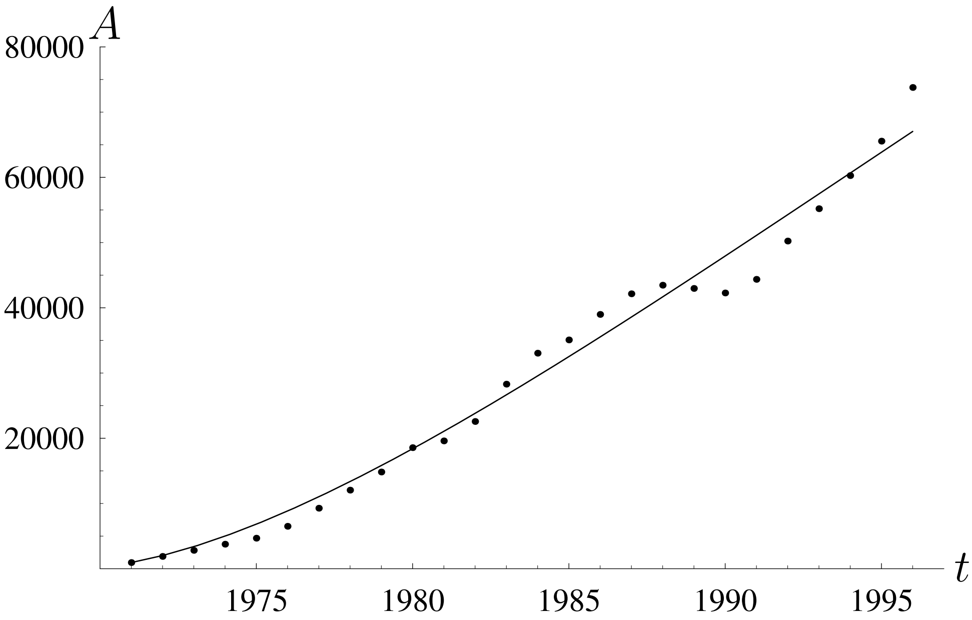

Figure 5.15: User data versus function of $A(t)$ for the parameter values from Table 5.3 - budget data with $\bar{u} = \bar{w} \equiv 1.95 \cdot 10^7$.

In the optimal control problem, the intersection of the both isoclines $\dot{A} = 0$ and $\dot{u} = 0$ represents a saddle point equilibrium $(\hat{A}, \hat{u})$ (cf. Figure 5.16), because the two eigenvalues are real numbers and have different signs. The green curves are the stable manifolds which give the optimal trajectories. The red curves are the instable manifolds, i.e. if you're on this curve you move away from the point of intersection. The two grey lines represent the two isoclines $\dot{A} = 0$ and $\dot{u} = 0$. The essential values of this problem are subsumed in Table 5.4.

$\hat{A}$	$\hat{u}$	$\hat{w}$	$J^\star$
45,019	$6.31873 \cdot 10^8$	$1.41658 \cdot 10^8$	$3.45959 \cdot 10^{10}$

Table 5.4: Steady state values of $\hat{A}$, $\hat{u}$ and $\hat{w}$ and value of the utility functional J - corrected budget data.

In Figure 5.17 the optimal functions of prevention spending, $w(t)$, and treatment spending, $u(t)$, are plotted as functions of the number of drug users $A(t)$. The vertical line at $\hat{A}$ represents the steady state value of the number

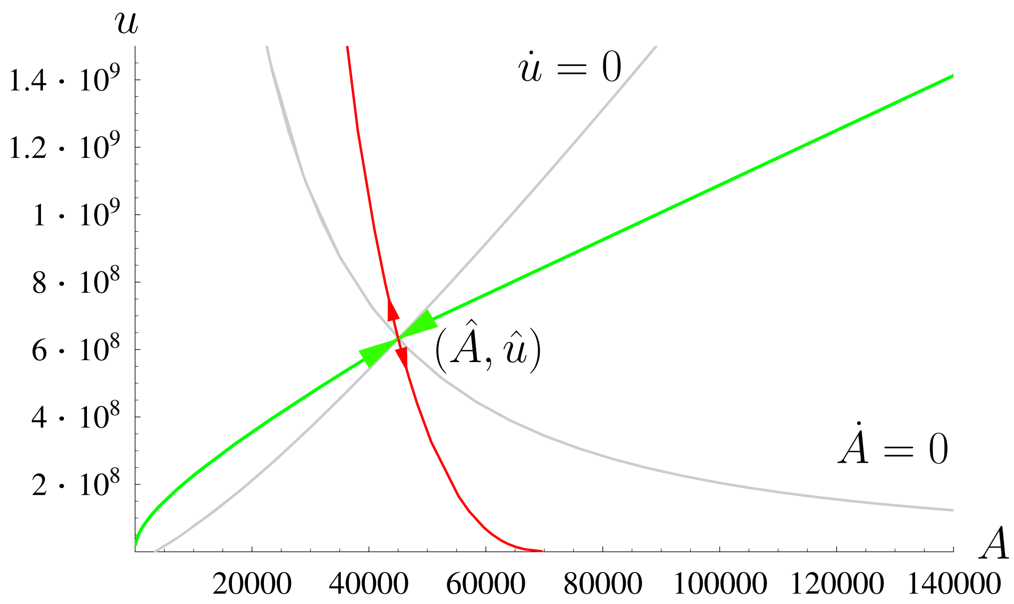

Figure 5.16: Phase portrait in the $A - u$-plane for the optimal control problem. The grey curves represent the both isoclines $\dot{A} = 0$ and $\dot{u} = 0$, the green curves are the stable manifolds and the red curves are the instable manifolds - corrected budget data.

of drug users. This can be explained as follows: if you're on the optimal trajectory of u and w, you move to the steady state value $\hat{A}$. If the number of drug users is small (smaller than 2,494 drug users) then the spending for prevention is greater than the spending for treatment. The optimal trajectory for u has a very high slope in contrast to the optimal trajectory for w which is nearly constant.

Figure 5.18 shows the prevention function $\theta(w)$ as a function of w and a horizontal line (violet), which represents the optimal value at the steady state $\theta(\hat{w})$. It can be seen very clearly that the function has a high descent in the first part because the limit of the function for high w is nearly h (cf. (2.4)). Besides it can be descerned that the horizontal line has the intersection point with $\theta(w)$ at $(0.861624, 1.41658 \cdot 10^8)$.

In Figure 5.19 the optimal trajectories of u and w can be seen as functions of the time argument t. It was presupposed that the number of drug users at time $t = 0$ is 5,000. Treatment spending $u(t)$ is always greater than preven-

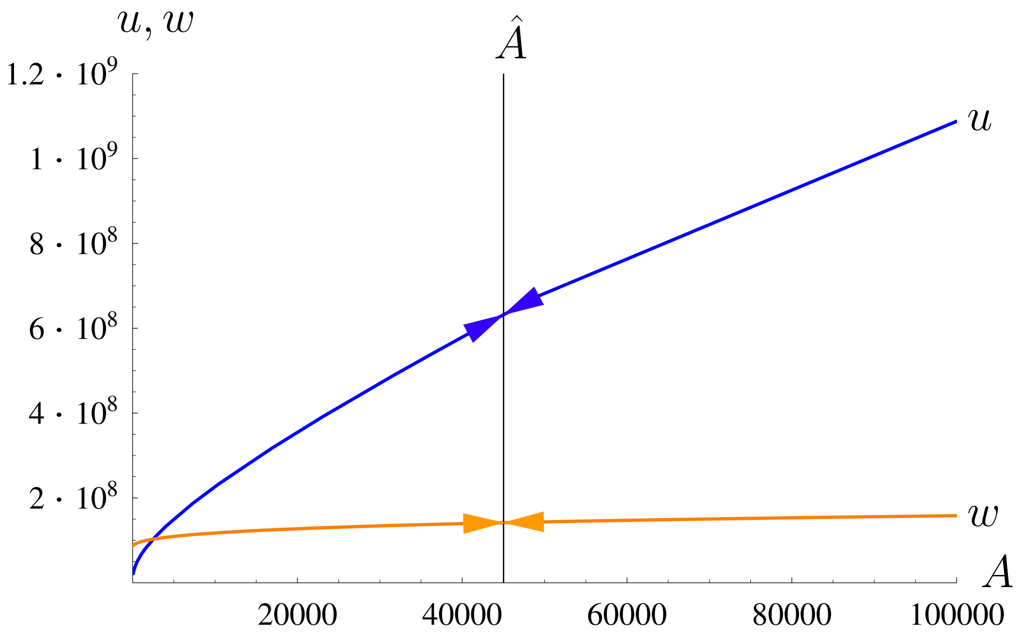

Figure 5.17: Optimal trajectories u (blue) and w (orange) as functions of A. The vertical line represents the steady state value $\hat{A}$ - corrected budget data.

tion spending $w(t)$ because the initial value is too high for another situation. If the initial value A_0 is smaller than 2,494 drug users, $w(t)$ is higher than $u(t)$ until the number of drug users exceed this limit.

Figure 5.20 shows the optimal trajectory of the number drug users A as a function of the time argument t. The trajectory converges to the steady state value $\hat{A}$ (red line). The initial value amounts to $A_0 = 5,000$ and the function has a high ascent but reaches the steaty state value approximately after 80 years.

In Figure 5.21 you can see the functions $I(t)$, $Q_{treat}(t)$ and $Q_{nat}(t)$ evaluated along the optimal paths of treatment and prevention spending. This yields to the "optimal trajectories" of initiation, the number of users treated successfully and natural quitting. It can be seen that the initiation is the function with the highest values, followed by the function Q_{treat} and than Q_{nat}. After 80 years the three functions are nearly constant and this can be explained by watching Figure 5.20. Since the number of drug users is nearly constant after 80 years, the three functions in Figure 5.21 have to be also nearly constant because A is defined through these functions. This can

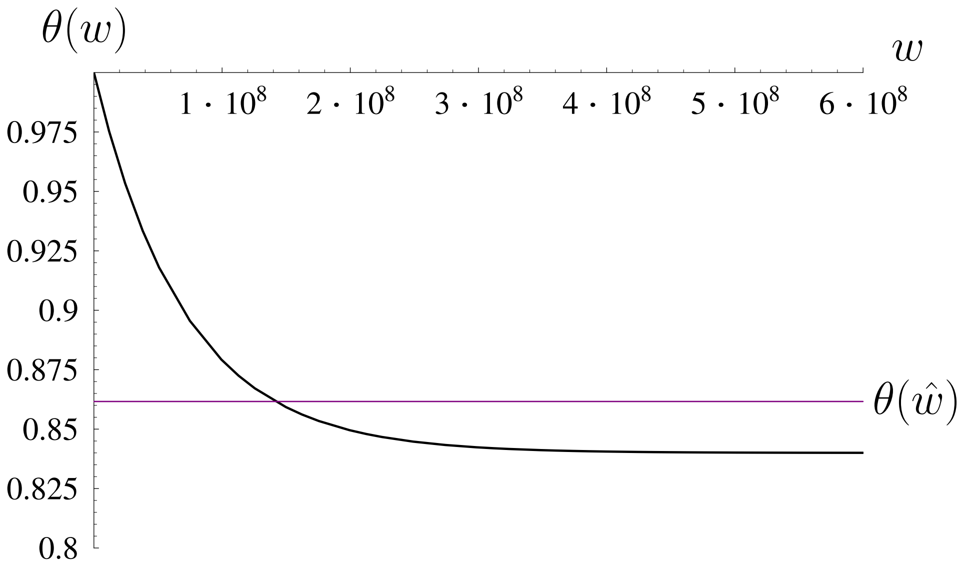

Figure 5.18: The prevention function $\theta(w)$ in addition to the horizontal line (violet) which represents θ in the steady state - corrected budget data.

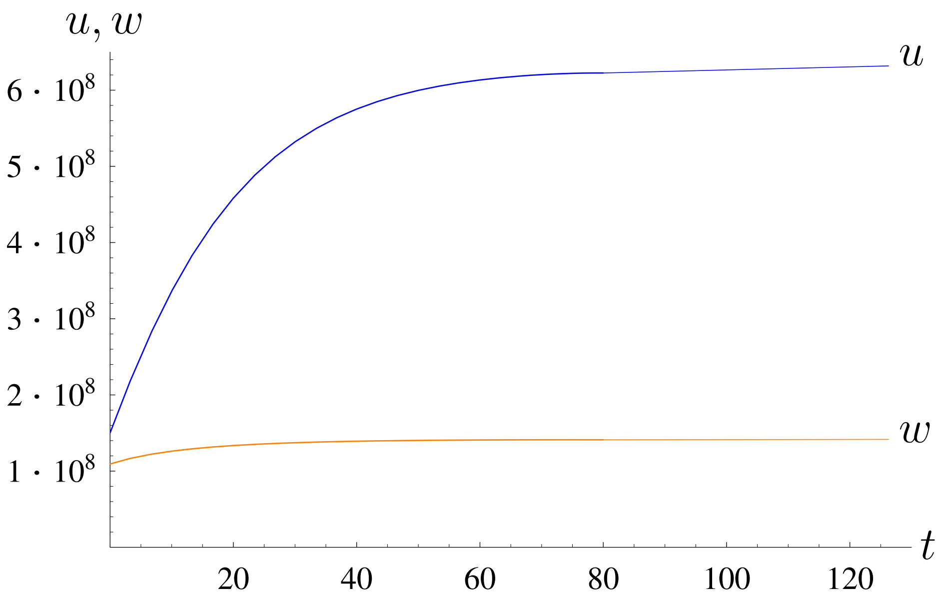

Figure 5.19: Optimal trajectories u (blue) and w (orange) as functions of t - corrected budget data.

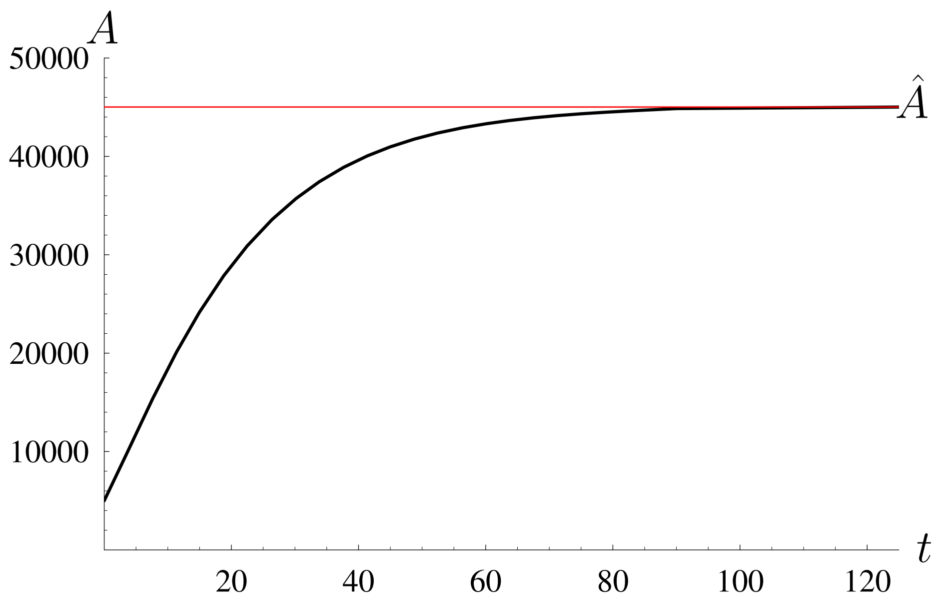

Figure 5.20: Optimal trajectory $A(t)$ converges to the steady state value $\hat{A}$ (red line) - corrected budget data.

be explained mathematically as follows: $\dot{A}(t)|_{\hat{A}} = 0$. That means if you're staying in the steady state, the number of drug users doesn't increase.

Figure 5.22 shows the prevention function $\theta(w)$ where the optimal prevention spending has been taken. It was assumed that the number of drug users at time $t = 0$ is 5,000. At the beginning, the value of this function is 0.874121. The descent in the first 35 years is very high, but after that the function converges to the steady state value. This means that at time $t = 0$, 12.59 percent of the potential initiates don't start using drugs, while 13.84 percent stay away in the steady state.

In Figure 5.23 it can be seen what happens to the utility functional J if the optimal control starts with delay. That means, if the delay τ is x years, the number of drug users with the initial value of $A_0 = 5,000$ increases just as in the model without controls for x years. After x years the optimal control starts and the function of the drug users $A(t)$ moves on the optimal trajectory. So if $\tau = 0$, the optimal control model is used. Notice that the number of drug users in the uncontrolled model after 12 years is greater than the steady state number of users in the optimal control model. That means, if $\tau \geq 12$, the optimal trajectories on the right side of the steady state value

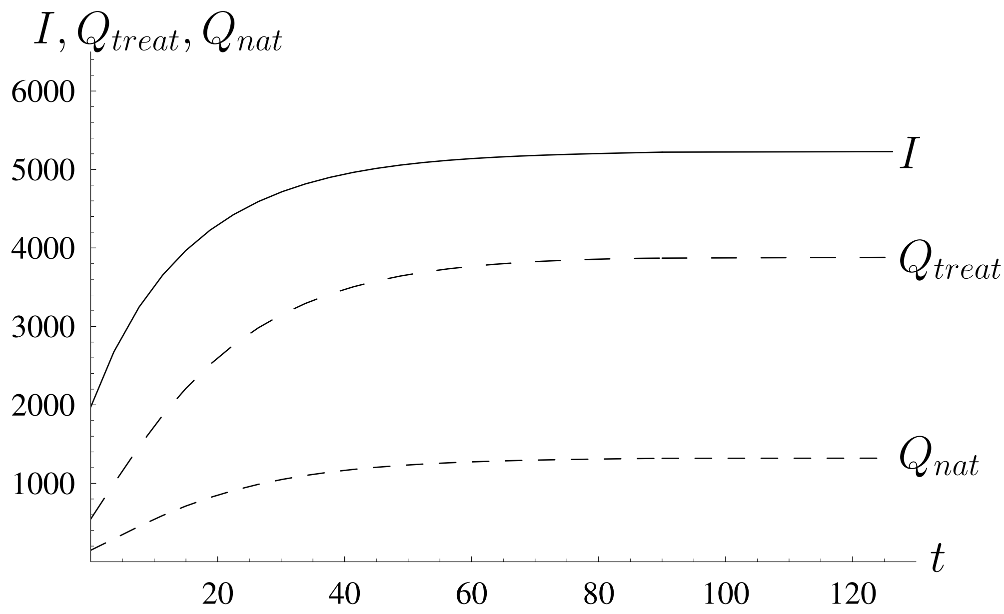

Figure 5.21: The functions $I(t)$, $Q_{treat}(t)$ and $Q_{nat}(t)$ evaluated along the optimal trajectories of u and w as functions of the time argument t - corrected budget data.

have to be used. Figure 5.23 demonstrates that the costs increase almost linearly with the delay. So each delayed start of the optimal control is very expensive implying that a prompt realization of the drug problem and a quick start of the spendings is advisable.

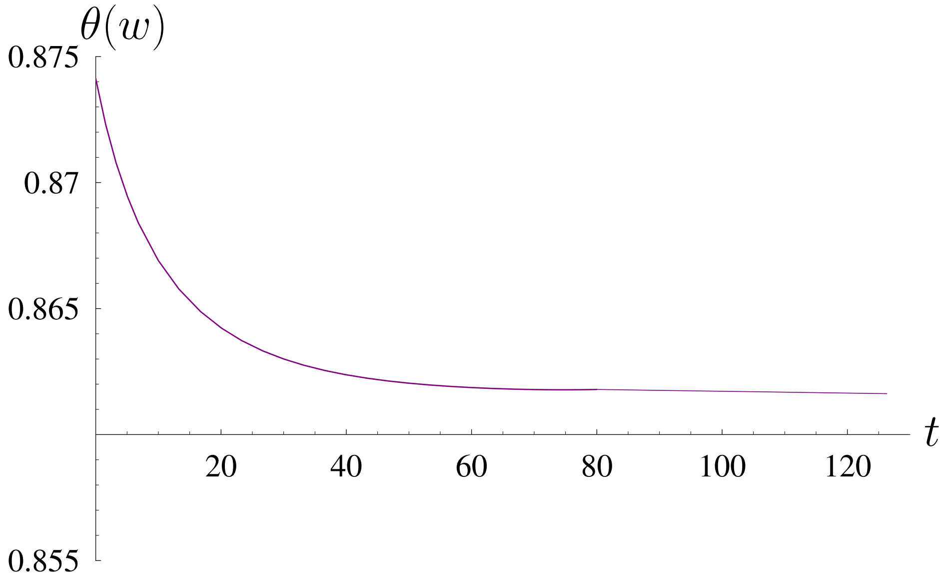

Figure 5.22: The prevention function $\theta(w)$ evaluated along the optimal trajectory of w as a function of t - corrected budget data.

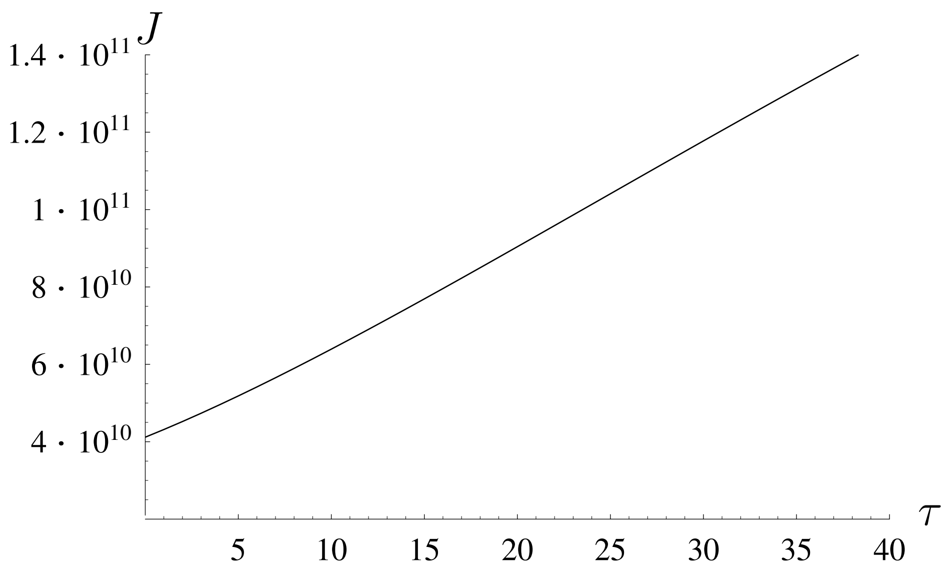

Figure 5.23: The total costs J as a function of the delay τ before the optimal control starts - corrected budget data.

Chapter 6

Conclusions and Extensions

To carry out the calculations in this thesis several data were needed. These data were provided to us by Kaya et al. (2002), Kaya and Agrawal (2002), Caulkins (2001), the Ministerial Council on Drug Strategy (1992), and from emails between the National Drug Strategy Unit and our collaborators from the School of Mathematics, University of South Australia, Adelaide (C.Y. Kaya, Y. Tugai, J.A. Filar, M.R. Agrawal). Tragler et al. (2001) used a similar model to analyze the current cocaine epidemic in the United States of America. Mautner (2002) used the same model but a different data set. In this analysis only those users were considered who have a drug career length greater than one year. That means that the user, quitting and initiation data include only drug users who appear in a period greater than one year.

In some sense, this analysis hence makes a difference between "light" and "heavy" users. In former analyses of the Australian heroin epidemic (e.g., Mautner (2002)) this distinction wasn't take into account. The work by Behrens et al. (1999), (2000) on U.S. cocaine makes differences between light and heavy users. The light users are people who consume drugs only occasionally and don't get addicted to them. These users have a positive influence on initiation because they make a positive impression on the non-users. On the other hand, the heavy users are people who take the drug regularly and often are addicted to them. These people have a negative influence on initiation because they are a deterrence for people who think about using drugs.

The determination of some of the model parameters was an important part of this work. For the estimation of these parameters the initiation function $I(t)$, the quitting due to treatment function $Q_{treat}(t)$ and the natural outflow function $Q_{nat}(t)$ were needed. These three functions are taken from the differential equation for $\dot{A}(t)$ (2.11). To determine the missing parameters, the exponents α and z were kept fixed while $\tilde{k}$ and c, respectively, were calculated so as to minimize the quadratic difference between $I(t)$ and $Q_{treat}(t)$ and the iniation data and half of the quitting data, respectively. The parameter $\tilde{\mu}$ was estimated by minimizing the quadratic error between $Q_{nat}(t)$ and half of the quitting data. In Chapter 3 the differences in this estimation between the α or z steps amount to 0.05. In the sensitivity analyis (cf. Section 5.2) this value was decreased to 0.01 to increase the accuracy of the calculations. In Chapter 3 the assumption was made that one half of the quitting people stop using drugs because of treatment and the other half has other reasons for quitting. This ratio was changed in Section 5.2 from 10:90 to 90:10 in 1 percent steps.

One of the problems with estimating the parameters is shown in Figure 5.10 which shows the changes of the parameters α and z in the $\alpha - z$-plane. The ratio between quitting due to treatment and natural outflow is unknown. In Chapter 3 the ratio is set to 50:50. Figure 5.11 and Figure 5.12 describe the steady state values of $\hat{A}$, $\hat{u}$ and $\hat{w}$ for different ratios. The fluctuations are very severe and so the real ratio of quitting is important to know to find the real optimal solution of the underlying problem.

The calculations of Chapter 4 distinguish between three variations of the model. The first one presupposed that nothing was spent for treatment or prevention. The long-run steady state in this case are 570,000 heroin users with total costs at \$ $1.23 \cdot 10^{11}$. The second variation demanded that treatment and prevention spending are at constant levels. More precisely, the average spendings from the budget data were shared out in the ratio 60:40 between $\bar{u}$ and $\bar{w}$. The steady state in this case are 340,000 users with total costs at \$ $7.85 \cdot 10^{10}$. The optimal control problem which uses Pontryagin's maximum principle (cf. Feichtinger et al. (1986) and Pontryagin et al. (1964)) yields 216 drug users as steady state level and total costs at \$ $9.95 \cdot 10^{9}$. One problem with this solution is the annual budget for the drug problem with \$ $3.16 \cdot 10^{8}$ in the steady state. The current budget for the Australian heroin problem amounts to \$ $5.75 \cdot 10^{7}$. The spending for the drug problem would have to increase to 550 percent to reach the optimal steady state value.

In Section 5.1 the efficiency parameters h and m were changed without reestimation of the other parameters. These two parameters influence the effect of prevention spending on initiation. The value of h describes the minimal percentage of the susceptibles who start using drugs although w was spent for prevention. The result is that the higher the value of h, the higher is the steady state level of drug users. Treatment and prevention spending run in opposite ways for increasing values of h. Spending for treatment increases with increasing values of h, while prevention spending decreases. The value of m describes how fast the minimum level h is reached. That's why for higher values of m the steady state values $\hat{A}$ decrease. The values of treatment spending go down for a higher m in contrast to the prevention spending which rises with increasing values of m.

Section 5.3 deals with the original budget data point from 1992 with \$ 33.485 million which was altered in the other part of the work to \$ 50 million on the recommendation of our colleagues in Australia. The ratio between treatment spending, $u(t)$, and prevention spending, $w(t)$, was also changed to 50:50. These calculations led to further interesting results.

Concerning possible extensions, the model could be expanded if a third control variable, i.e. law enforcement, is included. Such models have been investigated by Caulkins et al. (2000) and Tragler et al. (2001). This control has an effect on the drug price, because the more is spent on law enforcement, the more risky is drug dealing and hence the price is raising. If the price is higher, consumption decreases, more people will quit, and less people will initiate.

Appendix A

Technical Details

A.1 Solution of the Bernoulli Differential Equation (2.13)

To solve the differential equation (2.13), the fact is used that this is a Bernoulli differential equation (cf. Boyce and DiPrima (1992)). With Θ from (2.10) the following equation is obtained

$$\dot{A} = kA^{\alpha} - \mu A, \quad 0 < \alpha < 1, k, \mu \in \mathbb{R}^{+}. \tag{A.1}$$

The substitution $z^{\frac{1}{1-\alpha}} = A$ is made which leads to the differential equation

$$\dot{A} = \Big(\frac{1}{1-\alpha}\Big) z^{\frac{\alpha}{1-\alpha}} \dot{z}. \tag{A.2}$$

Now (A.2) is substituted into the differential equation (A.1) resulting in

$$\begin{aligned} \frac{1}{1-\alpha} z^{\frac{\alpha}{1-\alpha}} \dot{z} + \mu z^{\frac{1}{1-\alpha}} &= k z^{\frac{\alpha}{1-\alpha}} \\ \dot{z} + (1-\alpha)\mu z &= k(1-\alpha). \end{aligned}$$

With

$$m(t) = e^{\int (1-\alpha)\mu \, dt} = e^{(1-\alpha)\mu t}$$

z can be expressed as:

$$\begin{aligned} z = \frac{\int e^{(1-\alpha)\mu t}(1-\alpha)k \, dt + c}{e^{(1-\alpha)\mu t}} &= \frac{e^{(1-\alpha)\mu t}\frac{k}{\mu} + c}{e^{(1-\alpha)\mu t}} = \\ &= \frac{k}{\mu} + c e^{-(1-\alpha)\mu t}. \end{aligned}$$

By backsubstitution, the general solution of the differential equation (2.13) looks as follows:

$$A(t) = \left(\frac{k}{\mu} + ce^{(\alpha-1)\mu t}\right)^{\frac{1}{1-\alpha}}.$$

From the initial value $A(0) = A_0$ it follows that

$$A_0 = \left(\frac{k}{\mu} + c\right)^{\frac{1}{1-\alpha}}$$

and consequently

$$c = A_0^{1-\alpha} - \frac{k}{\mu}.$$

Finally, this yields to the solution (4.3) of the initial value problem of the differential equation.

A.2 Derivation of the Optimality Conditions in the Optimal Control Problem

The current-value Hamiltonian of the optimal control problem is specified as follows:

$$H = -\rho A - u - w + \lambda \Big\{ kA^{\alpha}[h + (1-h)e^{-mw}] - \\ -\Big[c\Big(\frac{u}{A}\Big)^{z} + \mu\Big]A\Big\}. \tag{A.3}$$

We have two necessary optimality conditions

$$H_u = 0 \quad \text{and} \quad H_w = 0$$

and the costate equation

$$\dot{\lambda} = r\lambda - H_A. \tag{A.4}$$

From this it follows that

$$H_u = -1 - \lambda cz\Big(\frac{u}{A}\Big)^{z-1} = 0 \tag{A.5}$$

and

$$H_w = -1 - \lambda mkA^{\alpha}(1-h)e^{-mw} = 0. \tag{A.6}$$

From (A.5) we get

$$\lambda(A,u) = -\frac{A^{z-1}}{czu^{z-1}} \tag{A.7}$$

and from (A.6) we have

$$\lambda(A,w) = -\frac{e^{mw}}{mk(1-h)A^{\alpha}}. \tag{A.8}$$

By equating (A.7) with (A.8) we get

$$w_{opt} = \frac{1}{m}\ln\Big(\frac{mk(1-h)A^{z+\alpha-1}}{czu^{z-1}}\Big). \tag{A.9}$$

Further, if $\lambda = \lambda(A,u)$ is differentiated with respect to time and the chain rule for two independent variables is used this yields to

$$\dot{\lambda} = \lambda_A \dot{A} + \lambda_u \dot{u}. \tag{A.10}$$

Now, if (A.10) is equated with (A.4) this leads to

$$\dot{u} = \frac{r\lambda - H_A - \lambda_A \dot{A}}{\lambda_u}. \tag{A.11}$$

Differentiating (A.7) with respect to A and u and (A.3) with respect to A yields

$$\lambda_A = -\frac{(z-1)A^{z-2}}{czu^{z-1}}, \tag{A.12}$$

$$\lambda_u = -\frac{(1-z)A^{z-1}}{czu^{z}} \tag{A.13}$$

and

$$H_A = -\rho + \lambda[\alpha k A^{\alpha-1}\vartheta(w_{opt}) - \mu - c(1-z)u^z A^{-z}]. \tag{A.14}$$

Finally, through the substitution (A.12)-(A.14) into (A.11) the differential equation for $\dot{u}$ follows:

$$\dot{u} = u\Big[\frac{r - \alpha k A^{\alpha-1} + \mu}{1-z} - \frac{c\rho z u^{z-1}}{(1-z)A^{z-1}} - \frac{\alpha c z u^{z-1}}{(1-z)mA^{z}} + \\ + khA^{\alpha-1} + \frac{czu^{z-1}}{mA^{z}} - \mu\Big]. \tag{A.15}$$

A.3 Proof that $\lambda_0 \neq 0$ in the Current-Value Hamiltonian

Generally, the current-value Hamiltonian H is specified as follows:

$$H = \lambda_0(-\rho A - u - w) + \lambda[kA^\alpha \vartheta(w) - (c\beta^z + \mu)A],$$

where λ_0 represents a constant.

Assume that $\lambda_0 = 0$. Then the current-value Hamiltonian reduces to

$$H = \lambda[kA^\alpha \vartheta(w) - (c\beta^z + \mu)A].$$

Through the necessary condition for optimality $H_u = 0$ it follows that:

$$H_u = -\lambda cz \frac{A^{1-z}}{u^{1-z}} = 0. \tag{A.16}$$

Due to $(\lambda_0, \lambda) \neq 0$ it follows $\lambda \neq 0$. Now there exists only one possible solution of (A.16) and this is $A = 0$. But this solution creates no control problem because this converges to the asymptotic steady stable solution $\hat{A} = 0$. Contradiction! W.l.o.g. $\lambda_0 = 1$ can be set.

A.4 Proof of the Concavity of the Current-Value Hamiltonian with Respect to (u, w)

In order to guarantee that the conditions (A.5) and (A.6) yield to a maximum, it is necessary that H is strictly concave in (u, w), i.e. the following conditions have to be satisfied:

$$H_{uu} < 0, H_{ww} < 0$$

and

$$H_{uu}H_{ww} > H_{uw}^2. \tag{A.17}$$

Differentiation of (A.5) and (A.6) with respect to u and w yields to

$$H_{uu} = -\lambda cz(z-1)\frac{u^{z-2}}{A^{z-1}} \tag{A.18}$$

and

$$H_{ww} = \lambda k A^\alpha (1-h) e^{-mw}. \tag{A.19}$$

Both expressions (A.18) and (A.19) are negative, because $\lambda < 0$ (cf. (A.7)), $c > 0$, $0 \leq z \leq 1$, $k > 0$ and $0 \leq h \leq 1$. The condition (A.17) is satisfied, because $H_{uw} = 0$. So it can be seen that the Hamiltonian H is really concave.

A.5 Solution of the Utility Functional J (2.12)

There exist two ways to calculate the utility functional J for the different models. The first is to calculate the value of the integral numerically. For the model without controls and the model with constant controls, the value of J has to be calculated this way.

For the optimal control problem we define

$$\begin{aligned} F(A(t),u(t),w(t),t) &= \rho A(t)+u(t)+w(t), \\ \dot{A}(t) &= f(A(t),u(t),w(t),t) \qquad \text{and} \\ H^0(A,\lambda,t) &= \max_{u(t),w(t)\geq 0} H(A,u,w,\lambda,t). \end{aligned}$$

Then the value of

$$J = \int_0^\infty e^{-rt}(\rho A(t)+u(t)+w(t))\,\mathrm{d}t$$

can be determined through a formula (cf. Feichtinger and Hartl (1986)).

Theorem A.5.1. Consider the optimal control problem

$$\max \int_0^\infty e^{-rt}F(A(t),u(t),w(t),t)\,\mathrm{d}t$$

$$\text{s.t.} \qquad \dot{A}(t) = f(A,u,w,t), \qquad A(0)=A_0, \qquad u(t),w(t)\geq 0.$$

F and f are autonomous (i.e. $F_t = 0$ and $f_t = 0$). For all trajectories which satisfy the necessary optimality conditions and which fulfil

$$\lim_{t\to\infty} e^{-rt}H^0(A(t),\lambda(t)) = 0,$$

the value of the utility functional J can be calculated through the formula

$$\begin{aligned} \int_0^\infty e^{-rt}F(A(t),u(t),w(t))\,\mathrm{d}t = \frac{1}{r}H(A(0),u(0),w(0),\lambda(0)) = \\ = \frac{1}{r}H^0(A(0),\lambda(0)). \end{aligned}$$

Proof. The complete proof can be gleaned from Feichtinger and Hartl (1986) (Theorem 4.13).

In our case the solution of this formula has to be multiplied with the factor -1 because the utility functional J has to be minimized.

Appendix B

Analysis of Model Variation

B.1 Formulation of the Model

The model variation has a slightly different treatment function $\beta(A, u)$ (cf. (2.6)), in which the constant γ is introduced in denominator. This constant γ should have the consequence that for small values of A, treatment is not so effective anymore because γ enlarges the denominator of β and hence reduces its value for small amounts of A. β is defined as follows:

$$\beta(A(t), u(t), \gamma) := \Big(\frac{u(t)}{A(t)+\gamma}\Big)^z. \tag{B.1}$$

The remaining functions stay as they are. The differential equation $\dot{A}(t)$ (cf. (2.11)) can be described with (B.1) as

$$\dot{A}(t) = \tilde{k}p^{-a}A(t)^{\alpha}\vartheta(w(t)) - \Big[c\Big(\frac{u(t)}{A(t)+\gamma}\Big)^z + \tilde{\mu}p^b\Big]A(t). \tag{B.2}$$

The differential equation $\dot{u}(t)$ was calculated in the same way as in Section A.2.

B.2 Determination of the Parameters

The parameters m, $\tilde{\mu}$, $\tilde{k}$ and α are calculated as in the Sections 3.1, 3.2 and 3.4. The optimal values of c and z are estimated as in Section 3.3 (but the outflow ratio R was changed as given 0.1,(0.01),0.9). The quitting due to treatment function $Q_{treat}(t)$ is defined as

$$Q_{treat}(t) := c\Big(\frac{u(t)}{A(t)+\gamma}\Big)^z A(t). \tag{B.3}$$

To determine the values of γ for the different outflow ratios the quadratic error between (B.3) and the quitting data was minimized over γ for the time period $t = 1989, \ldots, 1996$:

$$\min_{\gamma} \sum_{t=1989}^{1996} (Q_{treat}(t) - R \quad QuittingData(t))^2 \qquad \forall R \in [0.1, 0.9].$$

For a given value of z, values of γ are always the same for different ratios R (trivial proof). In Figure B.1 the optimal values for the constant γ can be seen.

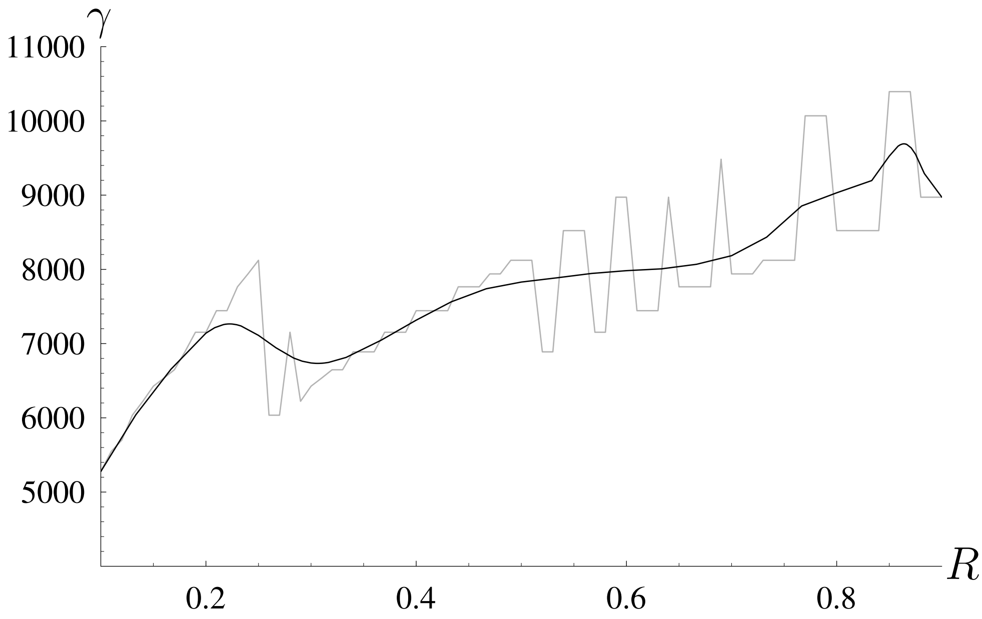

Figure B.1: Optimal values of γ for different outflow ratios, connected by lines (grey) and interpolated by bezier splines (black) - model variation.

The parameter changes for different values of R can be seen in Figure B.2, where the optimal values of α and z for different outflow ratios are illustrated in the $\alpha - z$-plane. The fluctuations of α are small but the optimal value of z reacts sensitively to changes of R. To remove the oscillations from Figure B.2 the optimal combinations of the parameters are interpolated by bezier splines. It can be seen in this figure that the true value of the outflow ratio R would be interesting to know because the fluctuations in particular in z are very high.

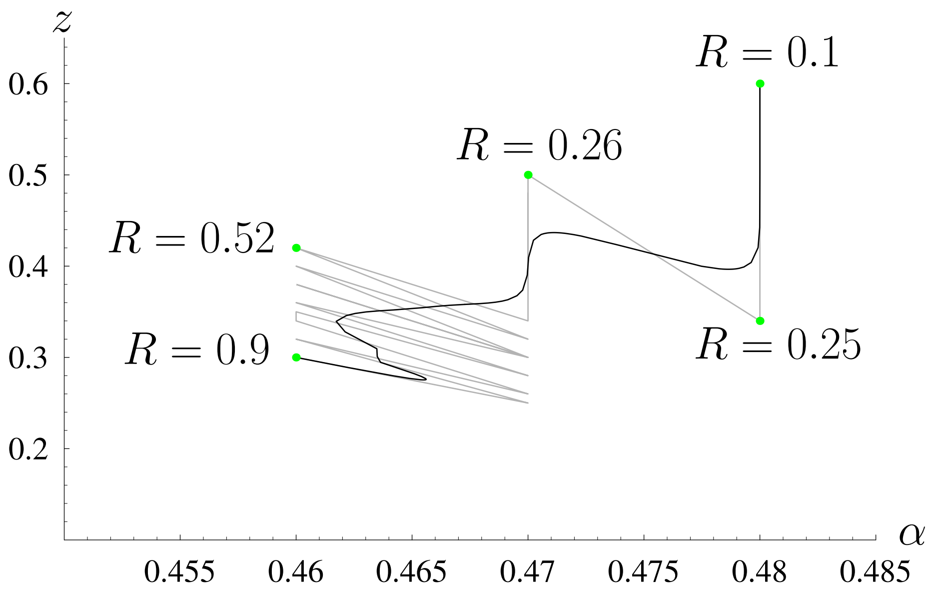

Figure B.2: Optimal combination of the parameters α and z in the $\alpha - z$-plane, connected by lines (grey) and interpolated by bezier splines (black) - model variation.

B.3 Steady State Values for Different Outflow Ratios R

The values of the steady states $\hat{A}$, $\hat{u}$ and $\hat{w}$ for the different ratios R are calculated and plotted in Figures B.3 and B.4. To remove small fluctuations, in Figure B.3 and B.4 the bezier splines are calculated and plotted.

Figure B.3 shows the steady state values $\hat{A}$ for different values of R. The bezier spline describes a - more or less - monotonously decreasing function from $R = 0.2$. Only the value $R = 0.28$ represents a "runaway" because this value lies far from the other steady states.

In Figure B.4 the steady state values $\hat{u}$ and $\hat{w}$ can be seen. $\hat{u}$ (blue) shows a monotonously decreasing function in the first part and from $R = 0.3$ the function is almost constant. The steady state values $\hat{w}$ (orange) are approximately constant over the whole range.

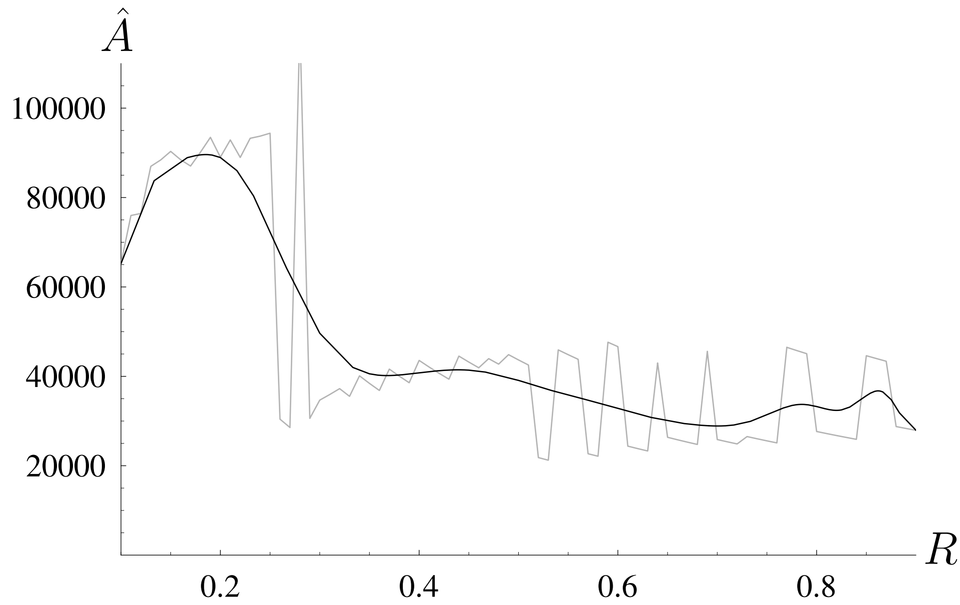

Figure B.3: Steady state values $\hat{A}$ for different ratios $R = 0.1, (0.01), 0.9$, connected by lines (grey) and interpolated by bezier splines (black) - model variation.

B.4 Stable Manifolds for some Outflow Ratios R

This section deals with the optimal trajectories of the new model (cf. (B.2) and the corresponding differential equation $\dot{u}(t)$). The stable manifolds represent the optimal trajectories converging to a saddle point equilibrium ($\hat{A}$,$\hat{u}$).

Figure B.5 shows the optimal trajectories of treatment spending $u(t)$ in the $A - u$-plane for different outflow ratios $R = 0.15$, 0.5 and 0.75. The vertical lines represent the steady state values $\hat{A}$ for different ratios. The main difference between these optimal trajectories is represented by the ascent of the functions.

In Figure B.6 the stable manifolds of prevention spending $w(t)$ can be seen. The stable manifolds are plotted in the $A - w$-plane and the vertical lines represent the steady state values $\hat{A}$. In all cases, w is negative for values of A below 2500 (approximately).

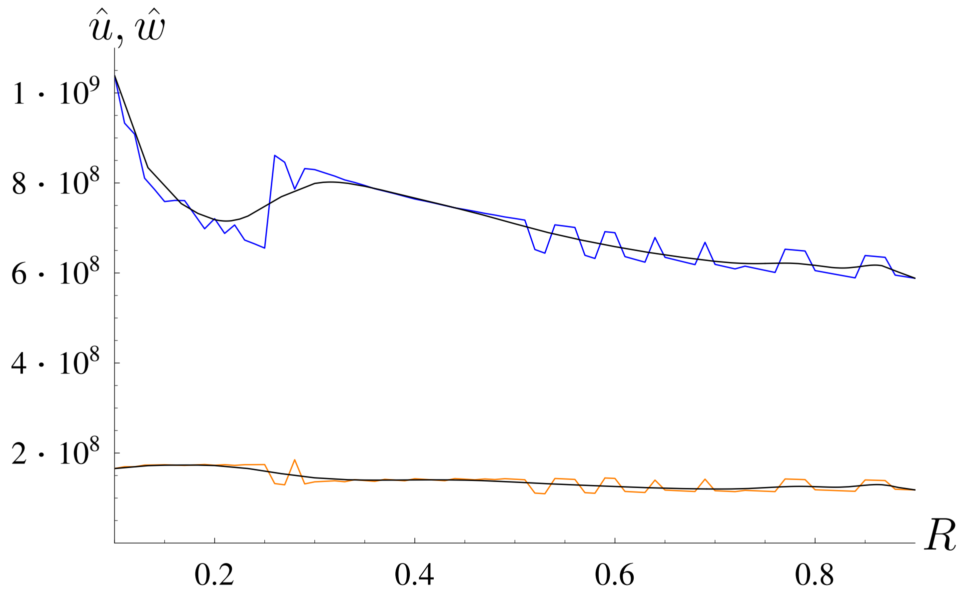

Figure B.4: Steady state values $\hat{u}$ (blue) and $\hat{w}$ (orange) for different ratios $R = 0.1, (0.01), 0.9$, connected by lines and interpolated by bezier splines (black) - model variation.

The optimal trajectories for $w(t)$ don't fulfil the inequality $w(t) \geq 0 \quad \forall t$ (cf. (2.1)). That's why the solution presented here is not fully correct and the optimal trajectories for $u(t)$ and $w(t)$ have to be determined with another approach, which is part of future research.

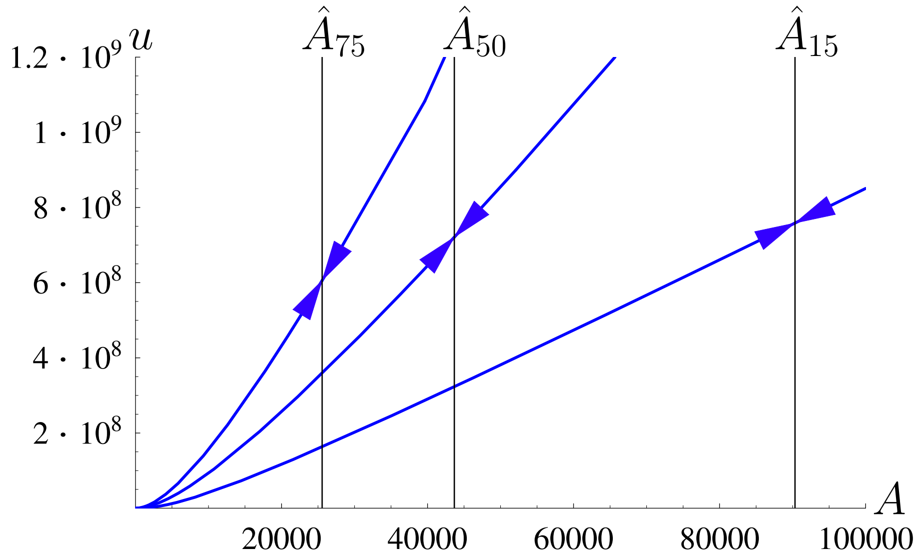

Figure B.5: The optimal policies for treatment spending u as a function of A for different outflow ratios. The vertical lines represent the values of $\hat{A}$ for different outflow ratios - model variation.

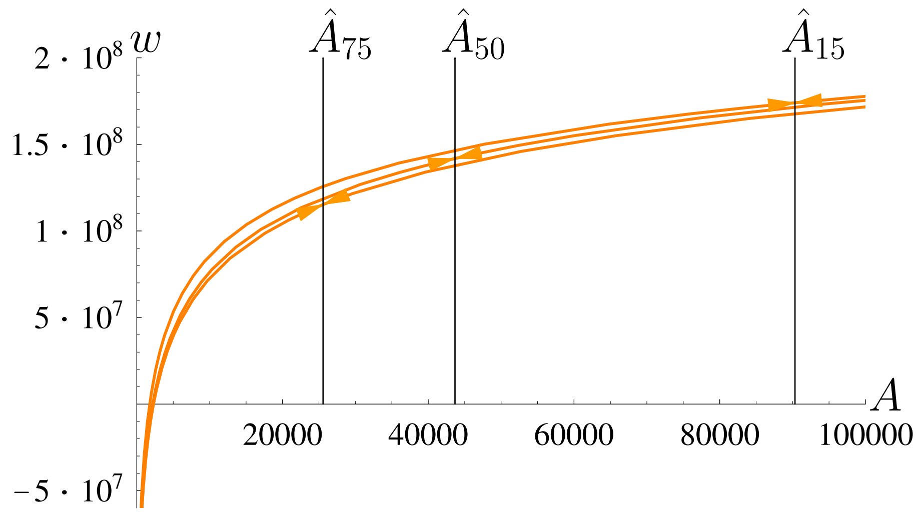

Figure B.6: The optimal policies for the prevention spending w as a function of A for different outflow ratios. The vertical lines represent the values of $\hat{A}$ for different outflow ratios - model variation.

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