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MSc Economics

Club Voting with Probabilistic Preferences

A Master's Thesis submitted for the degree of
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supervised by

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Abstract

The existence of groups of agents in assemblies and other legislative bodies who commit to vote unanimously is empirically well reported. Examples encompass factions in parliament and blocs in assemblies. This paper analyzes a cooperative model of a simple majority voting game with two alternatives. In giving her vote, a player is bound to the alternative that is preferred by a majority within the club she is a member of. It is analyzed what conditions club structures have to satisify to be stable with respect to selected deviations. Players know the joint distribution over preference profiles of all players in the assembly but not their own preferences. Sufficient and necessary conditions for a club structure to be stable with respect to players quitting their clubs are established. When players are free to propose clubs to other players, the existence of a club comprising 50%+1 players of the assembly is necessary for a club structure to be stable. When players contrast the situation in which their club is enforced to the situation in which no member is constrained in giving her vote, we find that the number of players who are better off under enforcement is related to the absolute difference in votes when the club becomes pivotal and to the correlation of preferences of players within the club.

1 Introduction

A club is a coalition of agents in a legislative assembly whose members accord within the club to vote unanimously. Hence before actually making their votes, members of a club agree on one alternative that is subsequently supported by all. Real world examples of clubs encompass factions in parliaments and blocs in assemblies of international organisations. Since an agent may find himself obliged to vote for an alternative other than her top preference, the question is: what are the conditions so that an agent is willing to commit to a club in the first place? Arguments for forming clubs heard in reality and found in the academic Political Science literature are vague; when being asked why Austria always votes in line with all other EU-countries in the United Nations Assembly, Austrian public officals routinely answer that "a small country like Austria can't achieve anything on its own". Shepsle and Laver (1999) merely state that "as long as lack of ideological cohesion is not great, factions will probably have an incentive to subject to party discipline". McCarty, Poole and Rosenthal (2001) describe parties in parliament as "cartels, that direct legislative activities to enhance the collective electoral fortunes of their members". There is a prevalent credo that in order to exert influence, groups have to emerge in unity. Buf if members of such a club benefit from overturning the votes of other members in the club, need not there be an asymmetry in the incentives to form a club? Is it consistent for all agents in a club to benefit from club discipline? Is it consistent for all members of a club to stay in the club as opposed to quitting? Furthermore, it is an often observed fact that in the real world, clubs comprise agents that are perceived as similar in their preferences, like the club of EU-countries in the UN-Assembly. This motivates the following question: What do the gains from being in a club depend on? Would it not make more sense for Austria and Germany to form a club with Saudi-Arabia to overrule Saudi-Arabia's vote? But then, why would Saudi-Arabia want to form such a club?

The model in this paper formalizes the gains from club discipline and establishes criteria for the desireability of maintaining a given club structure. Agents in an assembly who draw their preference from a commonly known probablility distribution have the opportunity to commit to a club whose members vote unanimously[1]. There are two alternatives and the winner of the ballot is determined by simple majority

[1] The issue *how* players can commit to a club is not addressed in this paper.

voting with a status-quo alternative. However, players may deviate from a proposed club structure by taking individually feasible actions. Therefore, we are interested in club structures that are stable with respect to such actions. The agenda is that different sets of feasible actions are considered and their implications are analyzed. Section 3 analyzes stability when agents are given the option to quit their club and vote on their own, Section 4 considers stability when players may propose new clubs, and Section 5 is devoted to the gains of club discipline, henceforth called club enforcement. Those club structures in which no player would want to quit her club are described and two results that relate the number of players in a club who benefit from club enforcement to the difference in votes when the club becomes pivotal and to the similarity in preferences are presented.

1.1 Relations to the Literature

This paper connects to three branches of existing literature. From a methodological point of view, the present model is clearly cooperative in that it uses a characteristic partition function with non-transferable utility. The approach of this paper to model the characteristic function with probabilistic preferences is novel. The seminal contribution on stability in cooperative games is due to von Neumann and Morgenstern (1944). Unlike "von Neumann-Morgenstern stable sets", the solution concept in this paper, "stability with respect to individually feasible actions", does not require deviations from a proposed solution to be stable against counter deviations. "Stability with respect to individually feasible actions" is very similar to the solution concept "pairwise stability" in network games which also does not consider counter reactions. This concept was introduced by Jackson and Wolinsky (1996). Cooperative models of pairwise majority voting games with ordinal preferences were considered by Rubinstein (1990), Le Breton and Salles (1990) and Chakravorti (1999). In these models, agents are assumed to vote far-sighted[2]. Cooperative Models of Voting with more general voting rules, ordinal preferences and non-transferable utility were studied by Dummett and Farquarson (1961) and Nakamura (1975). They adopt a myopic notion of stability: a combination of votes is stable if there is no subgroup of agents who could obtain a strictly better outcome by altering their votes while all other agents hold their votes constant.

[2]The degree of far-sightedness varies: In Rubinstein (1990) and Le Breton and Salles (1990) agents think one step ahead whereas in Chakravorti (1999), agents think arbitrarily far ahead.

There are obvious links to the literature in Political Science on party discipline and blocs. Various articles provide empirical case studies of party discipline. The studies of McCarty, Poole and Rosenthal (2001) and Snyder and Groseclose (2000) on the votes of representatives in the US congress are the two most promeninent examples. They conceptualize party discipline as a perturbation of a representative's individual preference in a spatial model of preference. Bloc Voting in the General Assembly of the United Nations was noted in the very early contribution of Ball (1951). In contrast, theoretical papers on party discipline are scarce; Laver and Shepsle (1999) give an informal discussion of the factors that facilitate party cohesion, Mitchell (1999) discusses how agents can be incentivized to adhere to party discipline.

Last, this paper relates to non-cooperative models of coalition formation in legislature and models in which the electorate is endogenously chosen. Baron (1989) and Norman (2002) study the problem of coalition formation when agents decide upon the division of a perfectly divisible "pie" by simple majority voting. Jackson and Moselle (2002) consider coalition formation in parliaments when types can be located on a two-dimensional scale. The papers by Barbera, Maschler and Shalev (2003) and Berga, Bergantinos, Masso and Neme (2006) analyze the strategic considerations that agents, who belong to some social entity such as a club, undertake when they decide whether or not to admit new members to the social entity or quit the social entity.

2 The Model

Definition 1. *Let $I = \{1, .., m\}$ be the set of players, $D = \{A, B\}$ be the set of alternatives and π be a probability densitiy function over the set of strict preference profiles of players in I over D, i.e. $\pi \in \Delta^{2^m - 1}$. Let Ω be the set of all partitions on I. For every Player $i \in I$ there is a correspondence of individually feasible actions $A_i : \Omega \rightrightarrows \Omega$. Then $(I, D, \pi, (A_i)_{i \in I})$ defines a* ***club voting game****.*

The club voting game unfolds as follows: first, players settle on a partition $P \in \Omega$. This partition constitutes the club structure of the assembly. Then, nature draws a preference profile from the set of strict preference profiles of players in I over alternatives in D. For the drawn preference profile and the previously chosen partition, the winning alternative of the ballot is determined. The sequence of events is visualized in figure 1. The winning alternative is determined as follows: in each club, the alternative that is preferred by more players wins the "within-club" ballot. If both alternatives are preferred by the same number of players, the status-quo alternative A wins. Then each player votes for the alternative that won the "within-club" ballot of the club she is a member of. The alternative that receives more votes wins. If votes are tied A wins. Henceforth, votes will be distinguished from players' preferences: a player's vote depends on the outcome of the within-club ballot of the club she is a member of, while a player's preference only depends on the draw of the preference profile but not on the club structure.

The model seeks to explain on which partitions players will settle. Players know the probability distribution over all preference profiles but they do not know their own preference when they choose the partition. Hence, there is no private information. Only those partitions are reasonable upon which players cannot improve by taking individually feasible actions[3]. Players are assumed to care only about their probability to win. The probability to win for any player for a given partition is computed in a straightforward manner: a player runs through all preference profiles with strictly positive probability and adds the probabilities of those profiles to her probability to win in which her preferred alternative would be equal to the winning

[3]We do require that a coalition is stable w.r.t to actions which could be countered by subsequent reactions. The story is that the actual ballot in the assembly marks an endpoint of the club formation phase and that there is a short period immediately before the ballot in which players may deviate and other player have no time to react. Therefore, it is required that club structures are stable w.r.t to such individually feasible actions.

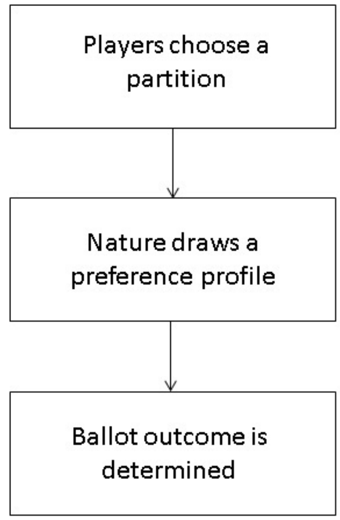

Figure 1: Sequence of events

alternative. Hence, every player there is a function $p_i : \Omega \times \Delta^{2^m-1} \to \mathbb{R}_+$.

Example 1. Consider a club voting game with three players and the partition in which all players belong to the same club (1 2 3). Suppose that π is such that $\pi(AAB) = 0.5$ and $\pi(ABB) = 0.5$, where AAB stands for the profile in which player 1 and 2 prefer A, while player 3 prefers B. Then players 1 and 3 win with probability 0.5 whereas player 2 wins with probability 1. If instead the partition (1 2)(3) is considered, player 1 wins with probability 1, player 2 with probability 0.5 and player 3 never wins.

Definition 2. *A solution to some club voting game $(I, D, \pi, (A_i)_{i\in I})$ is a subset of Ω. A $P \in \Omega$ is **stable** if $\nexists i \in I$ s.t. $\exists P' \in A_i(P)$ s.t. $p_i(P', \pi) > p_i(P, \pi)$.*

3 Stability with one feasible action: quitting her club

Let the individually feasible action correspondence be such that player i quits the club she is a member of and becomes a singleton-club, i.e. $A_i^q(P) = \{c \setminus \{i\} \mid c \in P\} \cup \{\{i\}\}$. If the player is in a singleton-club, the correspondence maps to the same partition. Since A_i^q maps to precisely one partition, stability is characterized the following way:

Definition 3. *A $P \in \Omega$ is* ***stable w.r.t quitting*** *if for every player i it holds that $p_i(P, \pi) - p_i(A_i^q(P), \pi) \geq 0$.*

When assessing the above inequality, every player analyzes the probability of those preference profiles in which the ballot outcome would be different if she quit her club. In this regard, it is useful to introduce some vocabulary; for a given preference profile, a club is said to be *pivotal* if the ballot outcome would have been different if the within-club ballot of this particular club had had a different outcome. A player i is said to "turn her club" if the alternative that wins the within-club ballot of player i's club is different from the alternative that would win the within-club ballot of a club made up by all players of player i's club *except player i himself*[4]. Profiles such that some player i would win if she stayed in her club, but would lose if she quit, are called "profiles in which player i gains from staying". The probability of all these profiles is called "player i's probability to gain from staying". "Profiles in which player i gains from quitting" and "player i's probability to gain from quitting" are defined in an analogous way.

For any player i there can only be a difference between staying and quitting for preference profiles in which player i's club is pivotal. Apart from that, there are two aspects that matter: the number of votes by which player i's preferred alternative is ahead outside of the club[5] and whether player i turns her club. Generally, some

[4] There is a close relation to player i being pivotal within her club; a player prefering $A(B)$ is pivotal whenever A is tied or one ahead (B is one or two ahead) within her club. However, a player prefering $A(B)$ "turns her club" iff A is tied (B is one ahead) within her club. For example, for a club of 5 players all players who prefer A in the profile (A A A B B) are pivotal, but no player turns the club if he quits since (A A B B) and (A A A B) have the same outcome as (A A A B B). In the 4 player club (A A B B), all players prefering A turn the club.

[5] These are the votes of all clubs except player i's club

player i gains from quitting only if her preferred alternative is the size (or almost the size) of player i's club ahead in votes outside of the club[6]. This is quite obvious: if this condition is not met, player $i's$ vote becomes irrelevant if she quits her club.

Example 2. Consider the partition (1 2 3 4 5)(6 7 8 9)(10 11 12)(13 14 15). In the profile (A B B B B)(A A A A)(A A A)(B B B) player 1's preferred alternative is 4 votes ahead outside of player 1's club. If she stays and this profile obtains, she will be defeated. If player 1 instead quits her club and votes on her own, alternative A wins. In contrast, in the profile (A B B B B)(B B B B)(A A A)(A A A) player 1 does not gain from quitting because if she quit, the four remaining players in her club would determine the winning alternative.

The other aspect that matters is whether a player turns her club. A player gains from staying in her club only if she turns her club.

Example 3. Consider the partition of Example 2. Player 1 turns her club in the profile (B A A B B)(A A A A A)(B B B)(B B B). Note that player 1 gains from staying. In the profile (B A B B B)(A A A A A)(B B B)(B B B) player 1 does not gain from staying.

For any player i of some non-singleton club, table 1 gives a classification of preference profiles regarding how they enter player $i's$ considerations whether or not to quit her club. Of course, only preference profiles such that player $i's$ club is pivotal are considered. Table 1 underlies the logic of everything that follows. However, to make precise what has been said so far it is necessary to introduce some notation. Firstly, consider the probability space with the sample space given by all strict preference profiles of players in I over D (which is finite) and the probability function given by π. On this probability space, and for a given partition P, define the random variable Δ_i^I that records the number of votes that A is ahead in the assembly, not counting votes of members of player $i's$ club (i.e. the number of votes that A is ahead outside of the club). Analogously, let Δ_i^c be equal to the number of players in player i's club who prefer A minus the number of players in player i's club who prefer B. Note that these random variables require the identity of player i and that $\Delta_i^I = \Delta_j^I$ and

[6]If player i prefers the status-quo alternative A, she gains only if alternative A is $n-1$ or $n-2$ votes ahead (n denotes the club size of player i's club) outside of player i's club. If she prefers B instead, she gains only if B is n or $n-1$ votes ahead.

	Number of votes by which player i's preferred alternative is ahead outside of player i's club	
	Size (or almost the size) of player i's club	Less than the size of player i's club
player i turns her club	no difference	gains from staying
player i does not turn the club and is defeated in the club	gains from quitting	no difference
player i does not turn her club but wins in the club	no difference	no difference

Figure 2: Classification of preference profiles

$\Delta_i^c = \Delta_j^c$ if players i and j belong to the same club. Since player i's own preference also matters, let D_i be player i's preferred alternative. It is convenient to depict the preference profiles that matter for player $i's$ decision in terms of these three variables.

Example 4. Given the partition from example 2, the event $(D_1 = A \wedge \Delta_1^c = -3 \wedge \Delta_I^1 = -2)$ pertains to all preference profiles of the form (A B B B B)(A wins)(B wins)(B wins).

If n denotes the club size of player $i's$ club, the club is pivotal if and only if $-n \leq \Delta_i^I \leq +(n-1)$. In the same vein, player i turns her club if and only if $D_i = A \wedge \Delta_i^c = 0$ or $D_i = B \wedge \Delta_i^c = -1$. If all events are measured with respect to π, the difference in probability to win is given by the following expression:

(1)

$$
\begin{aligned}
p_i(P,\pi) - p_i(A_i^q(P),\pi) = &\sum_{t\in\{-n,..,n-3\}} \pi(D_i = A \wedge \Delta_i^c = 0 \wedge \Delta_i^I = t)\\
&+ \sum_{t\in\{-(n-2),..,n-1\}} \pi(D_i = B \wedge \Delta_i^c = -1 \wedge \Delta_i^I = t)\\
&- \sum_{j\in\{n-1,n-2\}} \pi(D_i = A \wedge \Delta_i^c \leq -1 \wedge \Delta_i^I = j)\\
&- \sum_{j\in\{-n,-(n-1)\}} \pi(D_i = B \wedge \Delta_i^c \geq 0 \wedge \Delta_i^I = j)
\end{aligned}
$$

if $n \neq 2$, or

(2)

$$
\begin{aligned}
p_i(P,\pi) - p_i(A_i^q(P),\pi) = &\sum_{t\in\{-1,-2\}} \pi(D_i = A \wedge \Delta_i^c = 0 \wedge \Delta_i^I = t)\\
&- \sum_{t\in\{-1,-2\}} \pi(D_i = B \wedge \Delta_i^c = 0 \wedge \Delta_i^I = t)
\end{aligned}
$$

if $n = 2$

It is easy to relate expression (1) to table 1. That a player gains from quitting only if her preferred alternative is the size (or almost the size) of the player's club ahead outside of the club is reflected in the fact that all terms which appear with a negative sign in (1) pertain to events such that $\Delta_i^I \in \{n-1, n-2\}$ if $D_i = A$ or $\Delta_i^I \in \{-n, -(n-1)\}$ if $D_i = B$. Likewise, all terms that enter with positive sign pertain to events such that player i turns her club, i.e. $\Delta_i^c = 0$ if $D_i = A$ or $\Delta_i^c = -1$ if $D_i = B$.

Generally, the question whether the inequality $p_i(P,\pi) - p_i(A_i^q(P),\pi) \geq 0$ will be satisified for all players, thereby making P stable, cannot be answered beforehand and depends on the exact specification of π. However, in the next section the structures of partitions are exploited to derive a number of necessary and sufficient conditions for stability that are independent or almost independent of the specification of π.

3.1 Properties of partitions

In this section, only reference to the set of players I and partitions of this set is made. The most obvious way to derive a sufficient condition for stability w.r.t quitting is to rule out events belonging to the first column of table 1. According to expression (1), all terms that have negative sign pertain to events such that the absolute difference in votes outside of the club is at least $n-2$. If the structure of the partition does not permit such events, no further conditions on the distribution will be required. The definition below makes this idea precise.

Definition 4. *Consider a set of players $|I| \geq 3$ and a partition P that contains at least one club of club size greater or equal 3. Let l be the number of clubs in P, $c \in \mathbb{N}^l$ be a vector containing the club sizes and let $a \in \{+1, -1\}^l$. Then, P is said to be* ***robust*** *if there does not exist a club i of size c_i such that $a_{-i}^T c_{-i} \in \{c_i - 2, c_i - 1, c_i\}$ for some $a_{-i} \in \{+1, -1\}^{l-1}$.*

Lemma 1. *Let P contain at least one club of club size greater or equal 3. Then, P is robust if and only if $\nexists a \in \{+1, -1\}^l$ such that $c^T a \in \{0, 1, 2\}$.*

Proof. Suppose $c^T a = r$ where $r \in \{0, 1, 2\}$, then there has to exist an i so that $a_i = +1$. Then $a_{-i}^T c_{-i} = r - c_i$ and $-a_{-i}^T c_{-i} = c_i - r$. Suppose P is not robust, then $a_{-i}^T c_{-i} = c_i - r$ and $-a_{-i}^T c_{-i} + c_i = r$.

Loosely speaking, a partition is robust if it cannot happen that all but one club team up in such a way that the absolute difference in votes equals or almost equals the size of the remaining club. In contrast, when there is a club that becomes pivotal only if the other clubs team up in such a way that the absolute difference in votes equals or almost equals the club's size, the partition will be called fragile.

Definition 5. *Consider a set of players $|I| \geq 3$ and a partition P. Let l be the number of clubs in P, $c \in \mathbb{N}^l$ be a vector containing the clubsizes of the clubs in P and let $a \in \{+1, -1\}^l$. Then, P is said to be* ***fragile*** *if there exists a club i of size $c_i \geq 2$ such that $a_{-i}^T c_{-i} \in \{c_i, c_i - 1\}$ for some a_{-i} but $\nexists \bar{a}_{-i}$ such that $0 \leq \bar{a}_{-i}^T c_{-i} < c_i - 1$. The clubs for which this holds true are called* ***fragile clubs****.*

It should be noted that fragile and robust partitions are *everything but exceptional.*

Indeed, they are very common.

Example 5. (1 2 3)(4 5 6)(7 8 9) is robust while (1 2 3)(4 5 6)(7 8 9)(10 11 12) is fragile.

An important type of robust partitions are partitions with dominant clubs - these are clubs which comprise more than half of the players in the assembly.

3.2 Stability for very general distributions

Under very general conditions, robust partitions are stable while fragile partitions are not stable. However, the precise statements turn out to be a bit cumbersome.

Proposition 1. *Consider a club voting game* $(I, D, \pi, (A_i^q)_{i\in I})$ *where* $|I| \geq 3$*. Then, every robust partition is stable w.r.t quitting.*

Proof. Immediate.

The following proposition presents a key insight. Suppose that a partition is fragile, then there is a club which is pivotal only if the absolute difference in votes outside of the club equals the club size or the club size minus one. A profile such that some player in the fragile club gains from staying must be a profile such that this player turns the fragile club when her preferred alternative is behind outside of the club. However, such a profile must be a profile such that the preferred alternative of some *other player in the fragile club* is defeated while this other player's preferred alternative is ahead outside of the club. Hence, this profile is a profile such that the other player gains from quitting. Thus, the incentives of the players in the fragile club are asymmetric in the sense that what is good for one player has to be bad for some other player in the club. Hence, the incentives are not competible.

Proposition 2. *Consider a club voting game* $(I, D, \pi, (A_i^q)_{i\in I})$ *where* $|I| \geq 3$ *and* $\pi >> 0$*. Let* P *be a fragile partition and consider any fragile club. Let* n *denote the club size of the fragile club. Let* Δ^I *be the number of votes by which* A *is ahead outside of the fragile club and* Δ^c *be the number of players by which alternative* A *is ahead in preferences within the fragile club. First, consider the case when the fragile club is pivotal only if the absolute difference in votes equals its club size. Then, the* *following is true:*

- *if* $n \geq 3$, *P is not stable w.r.t quitting*

Now, consider the case when the fragile club is pivotal only if the absolute difference in votes equals its club size minus one. Then, the following is true:

- *if n is even and $n \neq 2$, P is not stable w.r.t quitting*
- *if n is odd, P is not stable w.r.t quitting if*

$$\begin{aligned}\pi(\Delta^c = -1 \wedge \Delta^I = n-1) < &\pi(-n < \Delta^c < -1 \wedge \Delta^I = n-1) + \\ &\pi(0 < \Delta^c < n \wedge \Delta^I = -(n-1))\end{aligned}$$

Proof. see Appendix.

Remark. If the fragile clubs is of size $n = 2$, then P is not stable w.r.t quitting if $\pi(AB \mid \Delta^I = -2) \neq \pi(BA \mid \Delta^I = -2)$ or $\pi(AB \mid \Delta^I = -2) \neq \pi(BA \mid \Delta^I = -2)$.

Admittedly, the fact that in some cases very adverse and unlikely exceptions exist make the statements look quite inelegant. To summarize, for very general distributions we have derived a sufficient condition for a partition to be stable with respect to quitting, namely that it is robust, and a necessary condition, namely that it is not fragile. Thus, the intuition, that incentives to stay in a club are contradictory, is correct for a special class of club structures. When partitions are neither robust nor fragile, knowledge of the exact specification of π is necessary to tell stable from unstable partitions.

3.3 Stability for partitions with large clubs

As we have seen in the previous sections, players gain from staying only if they turn their club. This gives rise to the following idea: if clubs are large in absolute size, one suspects that the probability to turn the club for an individual player is relatively small. In contrast, the probability to win by quitting her club and voting independently does not depend on the *absolute*, but on the *relative sizes* of the clubs. Hence, for a non-robust partition consisting of large clubs, one has good reason to believe that this partition is not stable w.r.t quitting.

Example 6. Consider a partition consisting of five clubs of the sizes 3,4,4,5,5 respectively. This partition is neither robust nor fragile. A player in the 3-player club gains from staying in the club only if and only if the other players in the club and the other clubs in the assembly tie. She gains from quitting if and only if she is defeated in the club and her preferred alternative is supported by the two 5-player clubs while the two 4-player clubs favor the other alternative. Now consider the very similar partition consisting of clubs of sizes 31,40,40,55,55 respectively. This partition is neither robust nor fragile just like the previous partition. A player in the 31-player club gains from quitting if and only if she is defeated in the club while the two largest clubs support her preferred alternative and the smaller clubs support the other alternative. She gains from staying if and only if the other players in the club and the other clubs in the assembly tie. However, compared to the set of all preference profiles, the set of profiles such that the other 30 players in the club tie is marginal.

To make the argument precise, I will present a limit result for club sizes going to infinity.

Proposition 3. *Consider a sequence of club voting games* $((I_m, D, \pi, (A_i^q)_{i\in I})_{m\in\mathbb{N}})$ *that satisfies the condition that there exist* $\epsilon, M > 0$ *s.t.* $\frac{\epsilon}{2^m} \leq \pi(\textit{any single profile}) \leq \frac{M}{2^m}$. *Then for any fixed number of clubs* $s \in \mathbb{N}$, *there exists a minimum club size* $c_{min} \in \mathbb{N}$, *such that in any game with at least* $c_{min}s$ *players, any non-robust partition* P, *containing precisely* s *clubs of size greater or equal* c_{min}, *is not stable w.r.t quitting.*

Proof. see Appendix.

So, for partitions with large clubs there is a characterization of the set of partitions that are stable w.r.t quitting, namely robust partitions.

3.4 Remarks on alternative tie-breaking rules

There are several natural ways to specify the tie-breaking rule in the club voting game. In fact, Propositions 2 and 3 go through even if a fair coin-toss is used as tie-breaker. Using the fair coin-toss rule yields identical definitions of robustness and fragility.

Another tie-breaking rule that has been considered is "chairman tie-breaking". In every club and in the overall assembly there is a chairman who has the decisive vote (or preference respectively). To analyze the chairman's decision whether or not to quit her club, a vice-chairman has to be appointed who does take over in case that the chairman quits. Nevertheless, robust partitions turn out to be stable. Proposition 3 can be also proven for "chairman tie-breaking", but fragile partitions are not necessarily unstable. But as a matter of fact, if $\pi >> 0$ exceptions are hard to construct. Thus, the results presented so far do not hinge too much on status-quo tie-breaking.

4 Stability with respect to proposing new clubs

The previous section considered stability under a minimal action set of feasible actions. In contrast, in this section stability under a very rich set of feasible actions is studied. Let $A_i^p : \Omega \rightrightarrows \Omega$ be the individually feasible action correspondence where player i proposes a new club. If all members of the new club, including player i herself, strictly improve over the current partition, the proposal is accepted. All other players stay in their current clubs. More precisely, any $P' \in A_i^p(P)$ that is accepted contains a club c, which player i is a member of, made up by players who all satisfy $p_i(P', \pi) > p_i(P, \pi)$. All other clubs in P' are constructed s.t. $\forall c' \in P, c' \setminus c \in P'$ if $c' \setminus c \neq \emptyset$.

If it is required that a partition P is stable with respect to proposing new clubs, the result is negative. If $\pi >> 0$, it turns out that virtually nothing is stable [7]. Unfortunately, the problem is so overwhelmingly complex that there are no analytical conclusions to be drawn. However, the following conjecture has not been falsified in repeated simulations. The conjecture pertains to "minimally dominant clubs" which are clubs of size $\frac{n}{2}+1$ ($\frac{n+1}{2}$) if n denotes the number of players and if n is even (odd).

Conjecture 1. *Consider a club voting game $(I, D, \pi, (A_i^p)_{i \in I})$ where $|I| \geq 4$ and $\pi >> 0$. Consider any partition P that does not contain a minimally dominant club, i.e. a club containing $\frac{n+1}{2}$ or $\frac{n}{2}+1$ players where n denotes the number of players in the assembly. Then P is not stable.*

The condition that there is a minimally dominant club is not sufficient. Most often, minimally dominant clubs are not stable. Hence it is not clear whether there are *any* stable partitions for arbitrary $\pi >> 0$.

[7] Of course, there are trivial examples when the probability distribution does not satisfy $\pi >> 0$

5 Club Enforcement

In this section, a different action is considered. Let $A_i^{ne} : \Omega \to \Omega$ be a function that maps a given partition P to the partition P' in which player $i's$ club consists of singletons only, i.e. $A_i^{ne}(P) = \{c \in P \mid i \notin c\} \cup \{\{j\} \mid j \in c' \wedge i \in c' \mid c' \in P\}$. P' is interpreted as the partition in which player $i's$ club is not enforced. A player i is in favor of enforcing the club iff $p_i(P, \pi) - p_i(A_i^{ne}(P), \pi) \geq 0$. If this condition holds, we say that player i benefits from club-enforcement. The precarious question arises whether it can be consistent for all players to benefit from club enforcement.

To analyze whether a player benefits from club enforcement, preference profiles in which the ballot outcome differs are studied. Again, there can only be a difference for profiles in which the player's club is pivotal. Secondly, if the alternative that has a majority among the members of the club is at least tied in votes outside of the club, there is no difference whether the club is enforced or not. A difference does emerge only if the club favors the alternative that is behind in votes outside of the club. In this case, whenever the absolute difference in the number of players in the club, who prefer either alternative, is smaller than the absolute difference in votes outside of the club, those players preferring the alternative that has the club-majority gain from enforcement and those who prefer the other lose from enforcement.

Example 7. Consider a club of 10 players. In profiles such that alternative A is 7 votes ahead outside of the club while B is ahead 6:4,7:3 or 8:2 in the club, all players favoring B gain from enforcement whereas the players favoring A lose. When B is ahead 9:1 or 10:0 in the club, there are no gains/losses from club enforcement.

It is important to realize that Example 7 does not suggest that incentives have to be incompetible. Consider for example all profiles such that 6 players in the club favor B and 4 players prefer A. Any player will find herself in the 6-player group in 60% of all profiles and in the 4-player group in 40% of all profiles. Hence, for any player there are more profiles in which she gains from enforcement than profiles in which she loses from enforcement. Thus, if one assumes all profiles to have uniform probability, every player in every club is better off when her club is enforced. This example shows that, under suitable circumstances, it might very well be consistent for every player to benefit from club enforcement. However, if more general distribution are considered some players may not benefit from club enforcement.

Comparative statics for a single club

To study under what conditions players are more likely to benefit from club enforcement, we study a single club in isolation. Consider a club (that is part of an unspecified assembly) of n players that becomes pivotal only if the absolute difference in votes outside of the club is d. Suppose that the probability distribution over preference profiles of players in this club π^c is independent from the probability distribution over preference profiles of players in the rest of the assembly. The only aspect of the latter distribution that matters is the probability that A is ahead when the club becomes pivotal and the probability that B is ahead when the club becomes pivotal. Denote these probabilities as p_A and $1 - p_A$. The quadruple (n, π^c, d, p_A) captures all relevant aspects. Let the random variable Δ^c, defined on the probability space formed by the set of strict preference profiles of players in the club and π^c, count the number of players by which A is ahead in the club (just like in section 3). So, for example, in a 10-player club $\Delta^c = -2$ pertains to all profiles in which B is preferred by 6 players of the club while A is preferred by 4 players. Also define D_i that records player i's preference. Let e_i be the function that computes player i's probability to win when the club is enforced and let $\bar{e}_i$ be the function that computes player i's probability to win when the club is not enforced. Then, as in section 3, the difference in probability to win for player i is:

(3)

$$
\begin{aligned}
e_i(d, p_A, \pi^c) - \bar{e}_i(d, p_A, \pi^c) = \\
&\sum_{t \in E} (\pi^c(D_i = A \mid \Delta^c = t) - \pi^c(D_i = B \mid \Delta^c = t))\pi^c(\Delta^c = t)(1 - p_A) \\
&+ \sum_{j \in F} (\pi^c(D_i = B \mid \Delta^c = j) - \pi^c(D_i = A \mid \Delta^c = j))\pi^c(\Delta^c = j)p_A
\end{aligned}
$$

where $E = \{\text{all integers } x \text{ such that } 0 \leq x < d\}$
and $F = \{\text{all integers } y \text{ such that } -d \leq y < 0\}$

Expression (3) establishes that a player benefits from enforcement if and only if the (weighted) sum of conditional probabilities to prefer the alternative that wins in the club is greater than the (weighted) sum of conditional probabilities to prefer the alternative that is defeated in the club, when the alternative that wins in the club is behind in votes outside of the club and would not win in the overall ballot if the club was not enforced, i.e. if $-d \leq \Delta^c < d$. Profiles such that $-d \leq \Delta^c < d$ for a quadruple (n, π^c, d, p_A) are called *relevant profiles*. For a given d and for some x such that $0 \leq x < d$ consider the probability to prefer A conditional on $\Delta^c = x$. For any player such that $\pi(D_i = A \mid \Delta^c = x) - \pi(D_i = B \mid \Delta^c = x) \geq 0$, or equivalently $\pi(D_i = A \mid \Delta^c = x) \geq 0.5$, the player's difference in probability to win increases over profiles such that $\Delta^c = x$. Conversely, for some y such that $-d \leq y < 0$ consider the probability to prefer B conditional on $\Delta^c = y$. For any player such that $\pi(D_i = A \mid \Delta^c = y) \leq 0.5$ her difference in probability to win increases over profiles such that $\Delta^c = y$.

In figure 3, we have drawn an example of a profile of probabilities to prefer A conditional on Δ^c for some player i (for a club with an even number of players). For all values of Δ^c such that the conditional probability lies in the grey shaded region, the player's difference in probability to win increases (provided that these profiles are relevant). On the other hand, if the conditional probability lies in the white region, her difference in probability to win decreases. In principle, we could draw such a profile for every player. However, the profiles have to be *consistent* in the sense that the mean conditional probability to prefer A over all players has to equal $(n + \Delta^c)/(2n)$ for every Δ^c. The mean conditional probabilites to prefer A are also drawn in figure 3. But how does d enter in figure 3? If the club becomes pivotal only if $d = 1$, the only relevant profiles are those in which $\Delta^c = 0$. If the club becomes pivotal only if $d = 2$ also profiles such that $\Delta^c = -2$ are relevant[8]. If d increases, more and more profiles become relevant.

However, it is not quite clear whether $e_i(d, p_A, \pi^c) - \bar{e}_i(d, p_A, \pi^c) \geq 0$ will be satisfied for all players. In particular, we would like to know how the number of players who benefit from club enforcement depends on π^c, d and p_A. For this purpose define

$$L : \mathbb{N} \times [0, 1] \times \Delta^{2^n - 1} \to \mathbb{N}$$

where

$$L(d, p_A, \pi^c) = \{\text{number of players such that } e_i(d, p_A, \pi) - \bar{e}_i(d, p_A, \pi) \geq 0\}$$

[8]Profiles such that $\Delta^c = +2$ are not relevant because then A wins irrespective of whether the club is enforced.

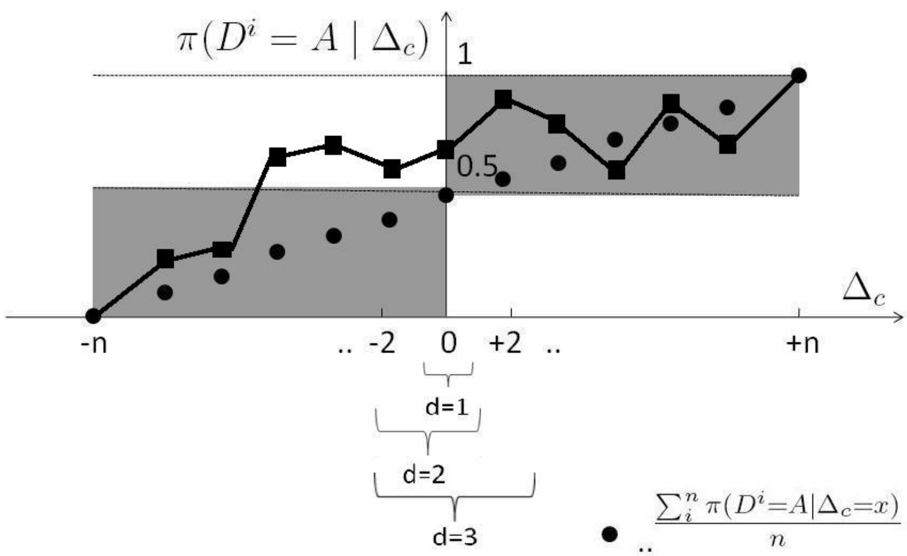

Figure 3: Conditional probabilities

L computes for every probability distribution and absolute difference in votes the number of players who benefit from club enforcement.

5.1 Club Enforcement and the difference in votes outside of the club

It easy to motivate why it is to be expected that many players benefit from enforcement if d is large. If the club is not enforced, the players in the club overturn the ballot outcome only in profiles such that $\Delta^c < -d$ or $\Delta^c \geq d$. If d is large relative to the club size, these events are rare. In contrast, when the club is enforced and d is large, profiles in which Δ^c is large in absolute terms are relevant. The mean probability to prefer the winning alternative conditional on Δ^c over all players is large for large absolute values of Δ^c. Hence,it is to be suspected that in profiles such that Δ^c is large in absolute terms, players are more likely to gain from enforcement.

Even though this fact is indicative, it proves hard to obtain a precise statement. The following proposition establishes monotonicity of the number of players who benefit from club-enforcement in d. However, the presented sufficient conditions are very restrictive.

Definition 6. *("Winners remain winners") Consider a quadruple* (n, π^c, p_A, d).

Then π^c is said to have the "winners remain winners" property if for all players i it holds that $\pi^c(D_i = A \mid \Delta^c = t) < 0.5$ for some $t > 0$ implies that $\pi^c(D_i = A \mid \Delta^c = t') < 0.5$ $\forall t'$ such that $0 \leq t' \leq t$, and $\pi^c(D_i = B \mid \Delta^c = t) < 0.5$ for some $t < 0$ implies that $\pi^c(D_i = B \mid \Delta^c = t') < 0.5$ $\forall t'$ such that $t \leq t' < 0$.

Example 8. Suppose that for the 10-player club of Example 7 the "winners remain winners" property is satisfied. Then, a player who is more likely to prefer B for profiles in which A is 7:3 ahead must be more likely to prefer B in profiles such that A is 6:4 ahead or A and B are tied. Conversely, a player who is more likely to prefer A when A is 6:4 ahead is more likely to prefer A when A is either 7:3, 8:2 or 9:1 ahead.

Definition 7. *(Symmetric conditional probabilities) Consider a quadruple (n, π^c, p_A, d). Then (π^c, p_A) is said to have the "symmetric conditional probabilities" property if $p_A = 0.5$ and if for all players i it holds that $\forall t$ such that $0 < t \leq n$, $\pi^c(D_i = A \mid \Delta^c = t) = \pi^c(D_i = B \mid \Delta^c = -t)$.*

Proposition 4. *Consider a quadruple $(n, \bar{\pi}_c, \bar{p}_A, d)$. Let $\bar{\pi}_c$ have the "winners remain winners" property and $(\bar{\pi}_c, \bar{p}_A)$ have the symmetric conditional probabilities property. Then, $L(d, \bar{p}_A, \bar{\pi}_c)$ is increasing in d.*

Proof. see Appendix.

5.2 Club Enforcement and club-homogeneity in preferences

In the introductory section, it was remarked that clubs in the real world comprise agents whose preferences are similar, i.e. correlated. In the context of the model, if $|\Delta^c|$ is large, that means that many players in the club prefer the same alternative. Conversely, if $|\Delta^c|$ is small either alternative is preferred by a significant number of players. Thus, it seems intuitive to conceive clubs in which much probability weight is put on profiles such that many players prefer the same alternative, i.e. profiles such that Δ^c is large in absolute values, as "club-homogenous in preferences". Henceforth, we say that for a given triple (n, p_A, d), "π^c is more club-homogenous than $\pi^{c'}$" if the cumulative distribution function of $|\Delta^c|$, $F_{|\Delta^c|}$, first-order stochastically dominates $F_{|\Delta^{c'}|}$.

However, it is known that for a given difference in the rest of the assembly d, all profiles such that $\Delta^c \geq d$ or $\Delta^c < -d$ are not relevant. Hence, define for a given triple (n, p_A, d) and a given π^c the conditional cdf that only includes profiles that are relevant as

$$F_{|\Delta^c|,d}(m) = \frac{\pi^c(|\Delta^c| \leq m \wedge \Delta^c \in J)}{\sum_{t \in J} \pi^c(\Delta^c = t)}$$

where

$$J = \{\text{all even(odd) integers } t \text{ such that } -d \leq t < d\} \text{ if n is even(odd)}$$

We then may say that for a given triple (n, p_A, d), "π^c is more club-homogenous conditional on d than $\pi^{c'}$" if $F_{|\Delta^c|,d}$ first-order stochastically dominates $F_{|\Delta^{c'}|,d}$.

Definition 8. *(Monotonic conditional probabilities) Consider a quadruple (n, π^c, p_A, d). Then π^c is said to have the monotonic conditional probabilities property if for all players i it holds that $\forall t, t'$ such that $n \geq t > t' \geq -n$, $\pi^c(D_i = A \mid \Delta^c = t) \geq \pi^c(D_i = A \mid \Delta^c = t')$.*

Lemma 2. *If π^c has the monotonic conditional probabilities property then it has the "winners remain winners" property.*

Proof. Immediate.

Proposition 5. *Consider a quadruple $(n, \pi^c, \bar{p}_A, \bar{d})$. Let π^c have the monotonic conditional probabilities property and $(\pi^c, \bar{p}_A)$ have the symmetric conditional probabilities property. Consider a quadruple $(n, \pi^{c'}, \bar{p}_A, \bar{d})$ where $\pi^{c'}(D_i = A \mid \Delta^{c'}) = \pi^c(D_i = A \mid \Delta^c)$ $\forall i$, i.e. each players conditional probabilities to prefer A are the same. Then, if π^c is more club-homogenous conditional on $\bar{d}$ than $\pi^{c'}$, $L(\bar{d}, \bar{p}_A, \pi^c) \geq L(\bar{d}, \bar{p}_A, \pi^{c'})$.*

Proof. see Appendix.

However, it should be noted that π^c being more club-homogenous than $\pi^{c'}$ does not imply that π^c is more club-homogenous conditional on d than $\pi^{c'}$ for any d that is smaller than the club's size. The converse statement is also false. Having probability weight on profiles such that $|\Delta^c|$ is large is favorable for club enforcement only if these profiles are relevant, i.e. only if $-d \leq \Delta^c < d$. For example, if π^c happens to be the perfect correlation case, i.e. the case when all players in the club always happen to prefer the same alternative, any incentive to enforce the club ceases to exist. Hence, the very general statement "the more similar the preferences of the players in the club, the more players benefit from enforcement" is wrong. We have to resort to the weaker version that only pertains to *relevant profiles.* Put differently, players should be similar in preferences in order to benefit from club enforcement but not too similar.

Example 9. Consider again the club of Example 7 and 8 consisting of 10 player that become pivotal only if the absolute difference in votes outside of the club is 7. Suppose that the probability distribution on preference profiles of players in this club π^c and p_A satisify the monotonic and the symmetric conditional probabilities properties. Suppose that a distribution $\pi^{c'}$ is constructed that differs from π^c only in that all probability mass from profiles such that one alternative is 7:3 ahead is shifted towards profiles such that one alternative is 8:2 ahead. Clearly, preferences are more correlated and $\pi^{c'}$ is more club-homogenous conditional on d than π^c. Hence, the number of players who benefit from club enforcement has to be weakly greater for $\pi^{c'}$. Then, consider a probability distribution $\pi^{c''}$ that differs from $\pi^{c'}$ only in that all probability mass from profiles such that one alternative is 8:2 ahead is shifted towards profiles such that one alternative is 9:1 ahead. Then, preferences are even more correlated but $\pi^{c'}$ is more club-homogenous conditional on d than $\pi^{c''}$ and thus, the number of players who benefit from club enforcement has to be weakly smaller for $\pi^{c''}$.

6 Conclusion

This paper analyzes a simple majority voting game between two alternatives in which players may commit to a club whose members votes unanimously. Players know the joint probability distribution over preference profiles but not their own preference. We find that in a club structure that satisfies the robustness property (defined in section 3), no player has an incentive to quit her club. In contrast, if the club structure has the fragility-property there must be a player who wants to quit her club. When the number of clubs is held constant and the size of the clubs is increased, we find that robustness is not only sufficient but also necessary for a club structure to be stable with respect to quitting. Simulation results suggest that, when players are given the opportunity to propose new clubs to other players, the existence of a minimally dominant club (defined in section 4) is necessary for a club structure to be stable with respect to proposing new clubs. However, this condition is not sufficient. Last, when players in a club contrast the situation in which their club is enforced (i.e. they have to vote unanimously) with the situation in which every player votes on her own, we find (under restrictive symmetry and monotonicity assumptions) that the number of players who are better off under enforcement increases in the absolute difference in votes when the club becomes pivotal. Furthermore, the relation of the number of players in the club who are better off under enforcment and the similarity in preferences of players in the club is non-monotonic. More similarity favors club enforcement only up to certain point.

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8 Appendix

Proof of Proposition 2

Consider the first case, when the club becomes pivotal only if the difference in votes in the rest of the assembly equals the club size. Note that for any two players i, j who belong to the same club $\Delta_i^I = \Delta_j^I$ and $\Delta_i^c = \Delta_j^c$. Hence, we can drop the individual-index for these random variables if only players from the fragile club are considered.

Subcase $n = 2$: If both players are better off staying, expression (2) yields ,

$$\pi(D_1 = A \wedge \Delta^c = 0 \wedge \Delta^I = -2) \geq \pi(D_1 = B \wedge \Delta^c = 0 \wedge \Delta^I = -2)$$

$$\pi(D_2 = A \wedge \Delta^c = 0 \wedge \Delta^I = -2) \geq \pi(D_2 = B \wedge \Delta^c = 0 \wedge \Delta^I = -2)$$

A profile such that $\Delta^c = 0$ is a profile of the form (B A) or (A B). Hence it is equivalent to say that

$$\pi(AB \mid \Delta^I = -2) \geq \pi(BA \mid \Delta^I = -2)$$

$$\pi(BA \mid \Delta^I = -2) \geq \pi(AB \mid \Delta^I = -2)$$

This can only be the case if $\pi(BA \mid \Delta^I = -2) = \pi(AB \mid \Delta^I = -2)$.

Subcase $n \geq 3$: If n is odd, every player has to satisfy the condition

$$-\pi(D_i = B \wedge \Delta^c \geq 0 \wedge \Delta^I = -n) \geq 0$$

By the assumption that $\pi >> 0$, there has to be a profile that appears on the lhs of this inequality. Note that there are is nothing to be gained from staying: if the club becomes pivotal only if the difference in votes outside of the club equals the club size, players may only hope to gain when they turn the fragile club in favor of the status-quo alternative A. However, if the club size is odd, there are no such profiles. If n is even, they have to satisfy

$$\pi(D_i = A \wedge \Delta^c = 0 \wedge \Delta^I = -n) \geq \pi(D_i = B \wedge \Delta^c \geq 0 \wedge \Delta^I = -n)$$

or equivalently

$$\begin{aligned}\pi(D_i = A \wedge \Delta^c = 0 \wedge \Delta^I = -n) \geq &\pi(D_i = B \wedge \Delta^c = 0 \wedge \Delta^I = -n) + \\ &\pi(D_i = B \wedge \Delta^c > 0 \wedge \Delta^I = -n)\end{aligned}$$

implying that

$$\begin{aligned}\sum_i \pi(D_i = A \wedge \Delta^c = 0 \wedge \Delta^I = -n) \geq &\sum_i \pi(D_i = B \wedge \Delta^c = 0 \wedge \Delta^I = -n) + \\ &\sum_i \pi(D_i = B \wedge \Delta^c > 0 \wedge \Delta^I = -n)\end{aligned}$$

Now, note that the term $\sum_i \pi(D_i = B \wedge \Delta^c > 0 \wedge \Delta^I = -n)$ has to be strictly positive if $\pi >> 0$. Consider any profile that enters the lhs of this inequality. Since $\Delta^c = 0$, there have to be equally many players in the fragile club prefering either alternative. Thus any such profile must enter for the $n/2$ players who prefer A on the lhs and for the $n/2$ players who prefer B on the rhs of this inequality. Hence, these terms cancel out and the inequality must be violated.

For the second part of the proposition suppose the fragile club becomes pivotal only if the difference in votes equals the club size minus one. The case when $n = 2$ is precisely as above.

Subcase $n \geq 3$, n is even: For every player it has to hold that

$$\begin{aligned}\pi(D_i = A \wedge \Delta^c = 0 \wedge \Delta^I = -(n-1)) \geq &\pi(D_i = B \wedge \Delta^c = 0 \wedge \Delta^I = -(n-1)) + \\ &\pi(D_i = A \wedge \Delta^c \leq -1 \wedge \Delta^I = n-1)\end{aligned}$$

implying that

$$\begin{aligned}\sum_i \pi(D_i = A \wedge \Delta^c =&0 \wedge \Delta^I = -(n-1)) \geq \\ &\sum_i \pi(D_i = B \wedge \Delta^c = 0 \wedge \Delta^I = -(n-1)) + \\ &\sum_i \pi(D_i = A \wedge \Delta^c \leq -1 \wedge \Delta^I = n-1)\end{aligned}$$

Again, $\pi >> 0$ implies $\sum_i \pi(D_i = A \wedge \Delta^c \leq -1 \wedge \Delta^I = n-1) > 0$ and like before, all profiles that enter on the lhs of this inequality enter precisely as often as on the rhs. Thus, this inequality must be violated.

Subcase n is odd: For every player it has to hold that

$$\begin{aligned}\pi(D_i = B \wedge \Delta^c = -1 \wedge \Delta^I = n-1) \geq & \pi(D_i = A \wedge \Delta^c \leq -1 \wedge \Delta^I = n-1) + \\ & \pi(D_i = B \wedge \Delta^c \geq 0 \wedge \Delta^I = -(n-1))\end{aligned}$$

or equivalently

$$\begin{aligned}\pi(D_i = B \wedge \Delta^c = -1 \wedge \Delta^I = n-1) \geq & \pi(D_i = A \wedge \Delta^c = -1 \wedge \Delta^I = n-1) + \\ & \pi(D_i = A \wedge \Delta^c < -1 \wedge \Delta^I = n-1) + \\ & \pi(D_i = B \wedge \Delta^c \geq 0 \wedge \Delta^I = -(n-1))\end{aligned}$$

implying that

$$\begin{aligned}\sum_i \pi(D_i = B \wedge \Delta^c = -1 \wedge \Delta^I = n-1) \geq & \sum_i \pi(D_i = A \wedge \Delta^c = -1 \wedge \Delta^I = n-1) + \\ & \sum_i \pi(D_i = A \wedge \Delta^c < -1 \wedge \Delta^I = n-1) + \\ & \sum_i \pi(D_i = B \wedge \Delta^c \geq 0 \wedge \Delta^I = -(n-1))\end{aligned}$$

Like before, every profile that enters the lhs of the inequality appears $(n+1)/2$ times on the lhs and $(n-1)/2$ times on the rhs. Thus, every profile such that $(\Delta^c = -1 \wedge \Delta^I = n-1)$ appears precisely once on the lhs after cancelling terms. Furthermore note that $\sum_i \pi(D_i = A \wedge \Delta^c < -1 \wedge \Delta^I = n-1) \geq \pi(-n < \Delta^c < -1 \wedge \Delta^I = n-1)$ and $\sum_i \pi(D_i = B \wedge \Delta^c \geq 0 \wedge \Delta^I = -(n-1)) \geq \pi(0 < \Delta^c < n \wedge \Delta^I = -(n-1))$. Hence, we conclude that

$$\begin{aligned}\pi(\Delta^c = -1 \wedge \Delta^I = n-1) \geq & \pi(-n < \Delta^c < -1 \wedge \Delta^I = n-1) + \\ & \pi(0 < \Delta^c < n \wedge \Delta^I = -(n-1))\end{aligned}$$

Proof of Proposition 3

Before going to the actual proof, some remarks are in order. If m is the number of players in the club voting game, the set of strict preference profiles has 2^m elements. For each club i in a given partition, one may define the "club-profiles of club i" as the set of strict preference profiles of players in club i. If the club contains c_i players, the set of "club profiles of club i" has 2^{c_i} elements. If the partition P contains s

clubs with club sizes given by $(c_1, .., c_s)$, the {set of all preference profiles} equals {club profiles of club 1} $\times .. \times$ {club profiles of club s} and $2^m = 2^{c_1} 2^{c_2} .. 2^{c_s}$.

Another fact that will be of particular importance pertains to the set of preference profiles in which a player turns her club. A player turns her club only in club profiles such that A and B are tied if the club size is even or in club profiles such that B is one ahead if the club size is odd. If n denotes the club size, any player turns her club in precisely $\frac{\binom{n}{n/2}}{2}$ club profiles[9] if n is even and precisely $\binom{n}{(n+1)/2}(\frac{1}{2} + \frac{1}{2n})$ club profiles if n is odd. As there are 2^n club profiles in total, one can define the fraction of club profiles such that any player turns her club as

$$S_n = \begin{cases} \frac{\binom{n}{\frac{n}{2}}}{2^{n+1}} & \text{if n is even} \\ \frac{\binom{n}{\frac{n+1}{2}}(\frac{1}{2}+\frac{1}{2n})}{2^n} & \text{if n is odd} \end{cases}$$

This sequence converges to 0 and is decreasing. This implies that the fraction of club profiles such that A wins and the fraction of club profiles such that B wins both converge to 0.5 - denote these two sequences as W_n^A and W_n^B. Define the fraction of club profiles such that player i prefers A and is defeated as

$$D_n^A = \begin{cases} \frac{\frac{2^{n-1}}{2} - \binom{n-1}{\frac{n}{2}}}{2^n} & \text{if n is even} \\ \frac{\frac{2^{n-1}}{2} - \binom{n-1}{(n-1)/2}}{2^n} & \text{if n is odd} \end{cases}$$

and the fraction of club profiles such that a player prefers B and is defeated as

$$D_n^B = \frac{1}{4} \text{ for all n}$$

Both sequences converge to 1/4. No index for the player is necessary since both sequences are identical for all players. Furthermore, note that for any player i

$$\{\text{profiles such that player i gains from staying}\} \subseteq \{\text{profiles such that player i turns her club}\}$$

since the latter also contains profiles in which player i's club is not pivotal.

[9] The binomial coefficient is divided by 2 since any player prefers A in only one half of these profiles. The players prefering A are the only ones who turn the club when A and B are tied.

Now fix any s and consider any partition P containing s clubs and m players. Let $(c_1,..,c_s) \in \mathbb{N}^s$ denote the sizes of the clubs and suppose that for some c_{min}, $c_j \geq c_{min}$ for all j. Now consider the probability to gain from staying for some player i who is a member of some club j, that is, the probability of all profiles such that player i gains from staying.

$$\begin{aligned}
&\pi(\{\text{profiles such that i gains from staying}\}) \\
&\qquad \leq \pi(\{\text{profiles such that i turns her club}\}) \\
&\qquad \leq \#(\{\text{profiles such that i turns her club}\})\frac{M}{2^m} \\
&\qquad = \#(\{\text{club profiles of club j such that i turns her club}\})\frac{2^{m-c_j}M}{2^m} \\
&\qquad = \#(\{\text{club profiles of club j such that i turns her club}\})\frac{M}{2^{c_j}} \\
&\qquad = S_{c_j}M \\
&\qquad \leq S_{c_{min}}M \\
&\qquad \to 0
\end{aligned}$$

Hence, the probability to gain from staying converges to zero when $c_{min} \to \infty$.

Furthermore, for this P with club sizes $(c_1,..,c_s)$, the fraction of profiles such that player i prefers A and is defeated in her club j while A wins in all other clubs is given by

$$D^A_{c_j}W^A_{c_1}..W^A_{c_s}$$

Now consider the limit of $D^A_{c_{min}}W^A_{c_{min}}..W^A_{c_{min}}$. If $c_{min} \to \infty$,

$$D^A_{c_{min}}W^A_{c_{min}}..W^A_{c_{min}} \to \frac{1}{4}\frac{1}{2^{s-1}} = \frac{1}{2^{s+1}}$$

Trivially,

$$\forall \gamma > 0, \exists c_{min} \in \mathbb{N} such that \forall (c_1,..,c_s) \geq (c_{min},..,c_{min}), |D^A_{c_j}W^A_{c_1}..W^A_{c_s} - \frac{1}{2^{s+1}}| < \gamma$$

This, of course, also holds for arbitrary combinations of A's and B's like $D^B_{c_j}W^A_{c_1}W^B_{c_2}..W^A_{c_s}$. Now, take any $a_{-j} \in \{+1,-1\}^{s-1}$ and let $r \in \mathbb{N}$ be such such that $c_j - r = a^T_{-j}c_{-j}$

is satisfied. Let the set of *enemies* be the indices of those clubs whose $a_t = -1$ and let the set of *allies* be set the indices of those clubs whose $a_t = +1$.

Then, for any player i belonging to some club j

$$\begin{aligned}
&\pi(\{\text{profiles such that i prefers A and is} \\
&\quad \text{defeated in his club while A(B)} \\
&\quad \text{wins in all ally(enemy)-clubs}\}) \geq \#(\{\text{profiles such that i prefers A and} \\
&\qquad \text{is defeated in his club while A(B)} \\
&\qquad \text{wins in all ally(enemy)-clubs}\})\frac{\epsilon}{2^m} \\
&\qquad = (D^A_{c_j} \prod_{t \in allies} W^A_{c_t} \prod_{t \in enemies} W^B_{c_t})\epsilon \\
&\qquad = (\frac{1}{2^{s+1}} - \gamma_{c_{min}})\epsilon \\
&\qquad \to \frac{\epsilon}{2^{s+1}}
\end{aligned}$$

if $c_{min} \to \infty$.

To summarize, in the first step we obtained a sequence converging to 0 that is bounding the probability to gain from staying for any player from above. In the second step we obtained a sequence that converges to $0.5^{s+1}\epsilon$. Now, for a fixed s choose c_{min} so large that the first sequence is well below $0.5^{s+1}\epsilon$ and that the second sequence is very close to $0.5^{s+1}\epsilon$. Now consider any game with at least $c_{min}s$ players and any non-robust partition with precisely s club that all satisfy the minimum club-size requirement. Let club sizes be given by $(c_1, .., c_s)$. Since the partition is not robust, $\exists c_j, r, a{-}j$ such that $c_j - r = c^T_{-j}a_{-j}$ where $r \in \{0, 1, 2\}$. When the clubs team up according to a_{-j}, players in club j may gain from quitting when they are defeated in the club. For all players in this club, the probability to gain from staying is close to zero whereas the probability to gain from quitting is bounded from below by $0.5^{s+1}\epsilon$.

Proof of Proposition 4

Suppose a player has a conditional probability to prefer A of greater or equal 0.5 for any $\Delta^c = t$ such that $t \geq 0$, then her conditional probability to prefer A must be greater or equal 0.5 for all $\Delta^c = t'$ where $t' \geq t$ by the "winner remains winner" property. By symmetry, the conditional probability to prefer B must be greater or equal 0.5 for all $\Delta^c = t''$ where $t'' \leq -t$. Similarly, suppose a player has a conditional probability to prefer B of greater or equal 0.5 for any $\Delta^c = t$ such that $t < 0$, then her conditional probability to prefer B must be greater or equal 0.5 for all $\Delta^c = t'$ where $t' \leq t$. Again by symmetry, the conditional probability to prefer A must be greater or equal 0.5 for all $\Delta^c = t''$ where $t'' \geq -t$. Thus if, a player has a conditional probability of greater or equal 0.5 to prefer the alternative that wins within the club for any $\Delta^c = t$, then her conditional probability to prefer the winning alternative must be greater or equal 0.5 for all $\Delta^c = t'$ where $|t'| \geq |t|$.

Now consider a player who benefits from club enforcement for a given $0 \leq d \leq n$. This implies that for some t her conditional probability to prefer the winning alternative must be greater or equal 0.5 if $\Delta^c = t$. Then, for all t' such that $|t'| \geq |t|$, the conditional probability to prefer the winning alternative must be greater or equal 0.5 if $\Delta^c = t'$. For any $d' > d$, the additional profiles which enter this player's considerations have to be profiles such that $\Delta^c = t''$ where $|t''| \geq |t|$. Hence, her probability to gain from enforcement can only increase if $d' > d$. If $d' \geq n$, all players gain from enforcement by convention.

Proof of Proposition 5

Consider $\pi_{c'}$ that is more club-homogenous conditional on d than π_c. Consider a player i that benefits from club enforcement for the distribution π_c. By symmetry, $\forall t > 0$,

$$\pi^c(D_i = A \mid \Delta^c = t) - \pi^c(D_i = B \mid \Delta^c = t)$$
$$= \pi^c(D_i = B \mid \Delta^c = -t) - \pi^c(D_i = A \mid \Delta^c = -t)$$

and let

$$\omega_t = \pi^c(D_i = A \mid \Delta^c = t) - \pi^c(D_i = B \mid \Delta^c = t) \; \forall t$$

Since player i benefits from club enforcement, expression (3) yields

$$\omega_0 \pi^c(\Delta^c = 0)(1 - p_A) + \sum_{0<t\leq d} \omega_t \pi^c(\Delta^c = t)(1 - p_A) + \sum_{0<t<d} \omega_t \pi^c(\Delta^c = -t) p_A \geq 0$$

Since $p_A = 0.5$,

$$\omega_0 \pi^c(\Delta^c = 0) + \sum_{0<t<d} \omega_t \pi^c(\Delta^c = t) + \sum_{0<t\leq d} \omega_t \pi^c(\Delta^c = -t) \geq 0$$

$$\omega_0 \pi^c(\Delta^c = 0) + \sum_{0<t<d} \omega_t(\pi^c(\Delta^c = t) + \pi^c(\Delta^c = -t)) + \omega_d \pi^c(\Delta^c = -d) \geq 0$$

$$\sum_{0\leq t<d} \omega_t \pi^c(|\Delta^c| = t) + \omega_d \pi^c(\Delta^c = -d) \geq 0$$

$$\sum_{0\leq t<d} \omega_t \frac{\pi^c(|\Delta^c| = t)}{\pi^c(-d \leq \Delta^c < d)} + \omega_d \frac{\pi^c(\Delta^c = -d)}{\pi^c(-d \leq \Delta^c < d)} \geq 0$$

The lhs can be interpreted as the "expected increase in the probability to win conditional on profiles being relevant". Since the w_t increase in t by monotonicity and because this conditional distribution is first-order stochastically dominated by $F_{|\Delta^{c'}|,d}$, this "expectation" has to be greater for $\pi^{c'}$. Hence,

$$\begin{aligned}
\sum_{0\leq t<d} \omega_t \frac{\pi^{c'}(|\Delta^{c'}| = t)}{\pi^{c'}(-d \leq \Delta^{c'} < d)} + \omega_d \frac{\pi^{c'}(\Delta^{c'} = -d)}{\pi^{c'}(-d \leq \Delta^{c'} < d)} & \\
\geq \sum_{0\leq t<d} \omega_t \frac{\pi^c(|\Delta^c| = t)}{\pi^c(-d \leq \Delta^c < d)} + \omega_d \frac{\pi^c(\Delta^c = -d)}{\pi^c(-d \leq \Delta^c < d)} & \\
\geq 0 &
\end{aligned}$$

implying

$$\sum_{0\leq t<d} \omega_t \pi^{c'}(|\Delta^{c'}| = t) + \omega_d \pi^{c'}(\Delta^{c'} = -d) \geq 0$$

meaning that player i benefits from club enforcement for $\pi^{c'}$.