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# DIPLOMARBEIT

## Combinatorial $r$ -Species and their Substitution

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unter der Anleitung von

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## Kurzfassung

Kombinatorische Spezies wurden von Joyal als Methode zur konzeptuellen Behandlung von kombinatorischen Strukturen entwickelt. Wir betrachten eine Verallgemeinerung davon für die Wirkung des Kranzproduktes einer zyklischen Gruppe der Ordnung  $r$  und einer symmetrischen Gruppe, auch  $r$ -Spezies genannt. Einige Aspekte von  $r$ -Spezies wurden bereits von Henderson, Hetyei und Choquette untersucht. In dieser Arbeit geben wir einen Überblick über das Thema, wobei wir dem Verhalten der Zyklenindikatorreihe unter verschiedenen Operationen von  $r$ -Spezies, wie zum Beispiel Produkt und Substitution, besondere Aufmerksamkeit schenken. Das letzte Kapitel beschäftigt sich mit der Berechnung der Zyklenindikatorreihe von drei verschiedenen Arten der Substitution, von denen eine neu ist.



## Abstract

Combinatorial species were introduced by Joyal as a device for computing conceptually with combinatorial structures. We study a generalization of this concept for actions of the wreath product of a cyclic group of order  $r$  and a symmetric group, called  $r$ -species. Some aspects of  $r$ -species were previously considered by Henderson, Heteyi and Choquette. We give an overview of the topic, concentrating on the behavior of the cycle index series and its specializations under various operations on  $r$ -species, including product and substitution. The final chapter is concerned with the computation of the cycle index series of three kinds of substitution, one of them is new.





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## CHAPTER 1

# Introduction

In enumerative combinatorics species of structures are used to work with labeled (and unlabeled) objects, that are called structures. They are defined as a functor from the category of sets with bijections to itself. What may be a very abstract definition turns out to be very natural: We get labeled structures and an action of the symmetric group  $\mathfrak{S}_n$  on them.

Now  $r$ -species are generalized species of structures where we not longer only consider a permutational action but a signed permutational and our labels now may have signs. In the case  $r = 2$  the used group is the hyperoctahedral group, the group of symmetries of an  $n$ -dimensional octahedron, more generally we use  $\mathfrak{W}_{r,n} := \mathfrak{S}_n \wr \mathbf{C}_r$ , the wreath product of the symmetric and the cyclic group  $\mathbf{C}_r$ . What we get then is a generalized theory of species of structures.

### 1.1. Related work

**1.1.1. Ordinary Species.** It was André Joyal ([Joy81] and [Joy86]) who developed the combinatorial theory of species of structures in the eighties. Since then a wide theory was built. For a textbook treatment see [BLL98].

**1.1.2. Hyperoctahedral Species (2-Species).** In [HLL98] Gábor Hetyei, Gilbert Labelle and Pierre Leroux introduce hyperoctahedral species in a very geometrical way, using hypercube and hyperoctahedron (therefore they call them cubical species). They develop a geometrical language and define for example species of vertices of a hypercube, which of course is interesting under the group of symmetries of a hypercube (which is the same as the one of a hyperoctahedron).

A more algebraic approach is presented by Philippe Choquette and Nantel Bergeron in [BC08] and Philippe Choquette in [Cho10], where species over vector spaces are considered and some more examples are given. However the cycle indicator series (or Frobenius character) is not introduced.

**1.1.3.  $r$ -Species.** The generalization of hyperoctahedral species to  $r$ -species is done by Anthony Henderson [Hen04]. He also introduces the cycle indicator series for hyperoctahedral species and introduce two kinds of substitutions, however he only states few examples.

### 1.2. Objectives

This work considers  $r$ -species and the operations sum  $+$ , product  $\cdot$  and three kinds of substitution (all denoted by  $\circ$ ): One between an ordinary species and an  $r$ -species, one between an  $r$ -species and an ordinary species, like they are introduced in [Hen04] and one between two  $r$ -species as introduced in [HLL98]. The objectives of this work are:

- Give a combinatorial prove, that for the cycle indicator series  $Z_F$  of a species  $F$  and the two substitutions introduced by [Hen04] holds  $Z_{F \circ G} = Z_F \circ Z_G$ .
- Consider the substitution between two  $r$ -species as in [HLL98], and establish a similar identity.
- Understand the three kinds of substitution better by giving examples and decompose already known species with the help of the substitution to get a better understanding and easier calculation.

### 1.3. Thesis outline

- In Chapter 2 we state basic definitions and results we will need later on.
- In Chapter 3 we give an introduction to the theory of  $r$ -species and define the cycle indicator series. Moreover we give some examples.
- In Chapter 4 we define three types of substitution, give two proofs for  $Z_{F \circ G} = Z_F \circ Z_G$  for the substitutions of [Hen04] and find a way to describe  $Z_{F \circ G} = Z_F \circ Z_G$  for the substitution of [HLL98]. Furthermore we give examples of species that arise from substitution.

## Basic Definitions and Results

We start with some notation:

- $r, n \in \mathbb{N}$  and  $\zeta$  is a primitive  $r^{\text{th}}$  root of unity;
- $\mathbf{C}_r := (\{\zeta^j : j = 0, 1, \dots, r-1\}, \cdot)$  is a cyclic group of order  $r$ ;
- $[n]$  is the set  $\{1, 2, 3, \dots, n\}$ ;
- $\mathfrak{S}_n$  is the symmetric group of  $n$  elements and  $\mathfrak{S}_S$  the group of permutations on a set  $S$  (Note that we do not distinguish between  $\mathfrak{S}_n$  and  $\mathfrak{S}_{[n]}$ );
- $M_{\mathbf{C}_r} := \{\zeta^j m : m \in M, j \in \{0, 1, \dots, r-1\}\}$  for a set  $M$ .

### 2.1. Generalized Signed Permutations

**2.1.1. Wreath Product.** The wreath product  $\wr$  of two groups is a certain kind of a semi direct product  $\rtimes$ , as we are only interested in the group  $\mathfrak{S}_n \wr \mathbf{C}_r$  and the definitions of  $\wr$  and  $\rtimes$  are quite general, we just define it for  $\mathfrak{S}_n \wr G$  (compare with [Mac95]):

DEFINITION 2.1.1. We define the wreath product of  $\mathfrak{S}_n \wr G$ ,  $G$  a group, as a group on the set  $G^n \times \mathfrak{S}_n$  with the action  $(g, \sigma)(h, \tau) := (g \cdot \sigma(h), \sigma\tau)$  where  $G^n = G \times G \times \dots \times G$  is the direct product of  $n$  copies of  $G$  and  $\mathfrak{S}_n$  acts on  $G^n$  by permuting the factors  $\sigma(g_1, g_2, \dots, g_n) = (g_{\sigma^{-1}(1)}, g_{\sigma^{-1}(2)}, \dots, g_{\sigma^{-1}(n)})$ .

DEFINITION 2.1.2. We define  $\mathfrak{W}_{r,n} := \mathfrak{S}_n \wr \mathbf{C}_r$ .

REMARK 2.1.3. An element  $\omega = ((\xi_{\sigma^{-1}(1)}, \xi_{\sigma^{-1}(2)}, \dots, \xi_{\sigma^{-1}(n)}), \sigma)$  of  $\mathfrak{W}_{r,n}$  can be identified with a bijection of  $[n]_{\mathbf{C}_r}$  to itself satisfying:  $\omega(\eta j) = \eta \xi_j \sigma(j)$ .

Note that now  $\omega \cdot \tilde{\omega}$  is exactly  $\omega \circ \tilde{\omega}$ :

$$\begin{aligned} \omega \cdot \tilde{\omega}(\eta j) &= ((\xi_{\sigma^{-1}(1)}, \xi_{\sigma^{-1}(2)}, \dots, \xi_{\sigma^{-1}(n)})\sigma((\tilde{\xi}_{\tilde{\sigma}^{-1}(1)}, \tilde{\xi}_{\tilde{\sigma}^{-1}(2)}, \dots, \tilde{\xi}_{\tilde{\sigma}^{-1}(n)})), \sigma\tilde{\sigma})(\eta j) \\ &= ((\xi_{\sigma^{-1}(1)}\tilde{\xi}_{\sigma^{-1}\tilde{\sigma}^{-1}(1)}, \xi_{\sigma^{-1}(2)}\tilde{\xi}_{\sigma^{-1}\tilde{\sigma}^{-1}(2)}, \dots, \xi_{\sigma^{-1}(n)}\tilde{\xi}_{\sigma^{-1}\tilde{\sigma}^{-1}(n)}), \sigma\tilde{\sigma})(\eta j) \\ &= \eta \xi_{\tilde{\sigma}(j)} \tilde{\xi}_j \sigma\tilde{\sigma}(j) \\ \omega(\tilde{\omega}(\eta j)) &= \omega(\eta \tilde{\xi}_j \tilde{\sigma}(j)) = \eta \tilde{\xi}_j \xi_{\tilde{\sigma}(j)} \sigma(\tilde{\sigma}(j)) \end{aligned}$$

On the other hand every such bijection  $[n]_{\mathbf{C}_r}$  to itself with  $\omega(\eta j) = \eta \xi_j \sigma(j)$  is an element of  $\mathfrak{W}_{r,n}$  by  $((\xi_{\sigma^{-1}(1)}, \xi_{\sigma^{-1}(2)}, \dots, \xi_{\sigma^{-1}(n)}), \sigma)$ . From now on we will use this new notation as bijection. For an example see subsection 2.1.2.

### 2.1.2. The Group $\mathfrak{W}_{r,n}$ .

DEFINITION 2.1.4. For  $\mathfrak{W}_{r,n}$ , we call an element of  $\mathfrak{W}_{r,n}$  (generalized) signed permutation and every element of  $\mathbf{C}_r$  a (generalized) sign.

REMARK 2.1.5. • Note that those signs have nothing to do with the sign of a permutation in  $\mathfrak{S}_n$ .

- Note that for  $r = 1$  we have  $\mathfrak{W}_{r,n} = \mathfrak{S}_n$ . Therefore everything stated here also holds for the symmetric group.

- $\mathfrak{W}_{2,n}$  is also called the hyperoctahedral group as it is (isomorphic to) the group of symmetries of an  $n$ -dimensional octahedron.
- The term ‘signed permutation’ is common for  $r = 2$  as there  $\mathbf{C}_r = (\{1, -1\}, \cdot)$  so we have signs  $+$  and  $-$ . Through this thesis we will use the term also for general  $r$  and omit the term ‘generalized’.

Often we do not only consider a  $\mathfrak{W}_{r,n}$ -action only on the set  $[n]_{\mathbf{C}_r}$  but on more general sets, we therefore define:

DEFINITION 2.1.6. On every set of size  $n \cdot r$  together with a permutation  $\eta$  that consists of  $n$  cycles with length  $r$ , we can define a  $\mathfrak{W}_{r,n}$ -action on it in that way that the permutation part of  $\mathfrak{W}_{r,n}$  is defined by a permutation of a set that consists of one element per cycle, and the  $\mathbf{C}_r$ -action of  $\zeta$  is  $\eta$ . This action is equivariant.

We will call these sets signed sets.

REMARK 2.1.7. We often identify such a signed set with  $[n]_{\mathbf{C}_r}$ .

EXAMPLE 2.1.8. (1) For any set  $M$  the set  $M_{\mathbf{C}_r}$  is a signed set with a natural  $\mathfrak{W}_{r,n}$ -action.  
(2) Let  $r = 3$ ,  $n = 2$ ,  $S = \{\alpha, 1, a, s, N, 5\}$  and  $\eta = (\alpha 5 s)(1 a N)$ . A bijection respecting the  $\mathfrak{W}_{r,n}$ -action to  $[2]_{\mathbf{C}_3}$  could be

$$\alpha \mapsto 1, a \mapsto \zeta 1, N \mapsto \zeta^2 1, 1 \mapsto 2, a \mapsto \zeta 2, N \mapsto \zeta^2 2.$$

Note that this bijection is already defined by  $\alpha \mapsto 1$  and  $1 \mapsto 2$ .

We therefore identify  $S$  with  $[2]_{\mathbf{C}_3}$ .

REMARK 2.1.9. A signed permutation can be also written in a two-line notation:

$$\omega = \begin{pmatrix} 1 & 2 & \dots & n \\ \omega(1) & \omega(2) & \dots & \omega(n) \end{pmatrix}$$

(or in a one-line notation:  $(\omega(1)\omega(2)\dots\omega(n))$ ) or in a (generalized) cycle notation:

We consider the cycles  $c$  of  $\omega$  with  $x_1, x_2, \dots, x_{l_c}$  with  $x_i \mapsto \xi_i x_{i+1}$  and  $x_{l_c} \mapsto \xi_{l_c} x_1$ . We write  $(x_1 \xi_1 x_2 \xi_2 \dots x_{l_c} \xi_{l_c})$  for this cycle and write the terms of the cycles in a row.

When working with general signed sets  $S \neq M_{\mathbf{C}_r}$  we define the absolute value of an element as one arbitrary but fixed  $\mathbf{C}_r$ -orbit and use it for the cycle notation. Note that this definition of the absolute value is not unique and just brings easier notation sometimes.

EXAMPLE 2.1.10. (1)  $r = 3$ ,  $n = 5$ ,  $\omega(1) = \zeta 3$ ,  $\omega(2) = 4$ ,  $\omega(3) = 5$ ,  $\omega(4) = \zeta 2$ ,  $\omega(5) = \zeta^2 1$  we have:

$$\omega = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \zeta 3 & 4 & 5 & \zeta 2 & \zeta^2 1 \end{pmatrix}$$

or

$$\omega = (1\zeta 3\zeta^0 5\zeta^2)(2\zeta^0 4\zeta)$$

This appears more convenient than the traditional notation:

$$(1\zeta 3\zeta 5)(\zeta 1\zeta^2 3\zeta^2 5)(\zeta^2 1 3 5)(2 4 \zeta 2 \zeta 4 \zeta^2 2 \zeta^4)$$

where we write the whole cycles, as the traditional notation contains redundant information.

(2)  $r = 2$ ,  $S = \{\{1, -2\}, \{-1, 2\}, \{3, 4\}, \{-3, -4\}, \{5, -6\}, \{-5, 6\}\}$ .

The  $\mathbf{C}_2$ -action is defined point wise by the elements of the sets of  $S$ . We define (for example):

- $|\{1, -2\}| = |\{-1, 2\}| = \{-1, 2\}$
- $|\{3, 4\}| = |\{-3, -4\}| = \{-3, -4\}$
- $|\{5, -6\}| = |\{-5, 6\}| = \{5, -6\}$



Now we can write  $(\{1, -2\} + \{-3, -4\} - \{5, -6\} - )$  instead of

$$\sigma = \begin{pmatrix} \{1, -2\} & \{-3, -4\} & \{5, -6\} \\ \{-3, -4\} & \{-5, 6\} & \{1, -2\} \end{pmatrix}.$$

**2.1.3. Cycle Type.** In analogy to permutations of  $\mathfrak{S}_n$  where the cycle type is defined by the numbers of cycles with a certain length, we define the cycle type of a signed permutation the following way: As we will see later, exactly the elements of a conjugacy class have the same cycle type. Therefore the cycle type is sometimes defined as the conjugacy class.

DEFINITION 2.1.11. For  $\omega \in \mathfrak{W}_{r,n}$  we define the cycle type as a tuple

$$(\omega_1(1), \omega_1(\zeta), \dots, \omega_1(\zeta^{r-1}); \omega_2(1), \omega_2(\zeta), \dots, \omega_2(\zeta^{r-1}); \omega_3(1), \dots)$$

where  $\omega_l(\xi)$  is the number of cycles of  $\omega$  with length  $l$  and type  $\xi$ .

Length and type of a cycle  $c$  can be defined via the smallest number  $l_c$  with  $\exists \xi_c$  and  $\omega^{l_c}(x) = \xi_c x$ .

REMARK 2.1.12. The type of a cycle of a signed permutation on a set  $M_{\mathbf{C}_r}$  can be equivalently defined as the product of signs of the elements in this cycle.

LEMMA 2.1.13. *For a signed permutation on a set  $M_{\mathbf{C}_r}$  the two definitions of the type of a cycle are indeed equivalent.*

PROOF. Let  $x$  be an arbitrary element of  $M_{\mathbf{C}_r}$ . Then  $x = \xi \tilde{x}$  with  $\tilde{x} = |x|$ .  $\tilde{x}$  is mapped under  $\omega$  to  $\xi_1 x_1$  where  $x_1 = |x_1|$ ,  $x_1$  is mapped under  $\omega$  to  $\xi_2 x_2$ , so  $\tilde{x}$  is mapped under  $\omega^2$  to  $\xi_1 \cdot \xi_2 x_2$ . Iterating this shows that  $\tilde{x}$  is mapped under  $\omega^l$  to  $\prod_i \xi_i x_l$ . As  $x_l$  and  $\tilde{x}$  are in the same  $\mathbf{C}_r$ -orbit ( $l$  is the length of the cycle) and are their absolute value they are equal. Thus  $\omega^l(x) = (\prod_i \xi_i)x$ .  $\square$

REMARK 2.1.14. There is another way to define the cycle type of an unsigned permutation: The cycle type of a  $\sigma \in \mathfrak{S}_n$  is a partition of  $n$ , so that the number of  $l$  in this partition is the number of cycles with length  $l$ . (Remember that a partition  $\lambda$  of  $n$  is a sequence  $(\lambda_1, \lambda_2, \dots)$  with  $\lambda_1 \leq \lambda_2 \leq \dots$  and  $\sum \lambda_i = n$ .) We therefore sometimes write  $\sigma \vdash n$  for  $\sigma$  is a cycle type of a permutation in  $\mathfrak{S}_n$  even when we use the original definition of a cycle type.

We can analogously define the cycle type of an  $\omega \in \mathfrak{W}_{r,n}$  by a multi-set of elements in  $[n]_{\mathbf{C}_r}$  where we have the number of  $\xi l$  in this multi-set is the number of cycles with length  $l$  and cycle type  $\xi$ . Thus  $|\xi l|$  build a partition of  $n$ . We then have the following two identities:

- For  $(a_i(\zeta^j)_{i,j})$  being the ordinary definition of a cycle type it holds that  $\sum a_i(\zeta^j) \cdot i = n$ .
- For  $(\zeta^{j(i)} a_i)_i$  being a cycle type of the alternative definition it holds that  $\sum a_i = n$ .

In analogy we sometimes write  $\omega \vdash_r n$  for  $\omega$  is a cycle type of a permutation in  $\mathfrak{W}_{r,n}$  and call  $(\zeta^{j(i)} a_i)_i$  with  $\sum a_i = n$  an  $r$ -partition of  $n$ .

We furthermore define the size as  $|\omega| = n$  and the length  $l(\omega)$  as the number of cycles.

EXAMPLE 2.1.15. Let  $r = 2$ ,  $n = 5$  and  $\omega \in \mathfrak{W}_{r,n}$  with

$$\omega = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ -2 & -5 & -4 & 3 & 1 \end{pmatrix}$$

then we have  $\omega_2(-1) = 1$ ,  $\omega_3(1) = 1$  and  $\omega_i(\zeta^j) = 0$  otherwise. Thus the cycle type is  $(0, 0; 0, 1; 1, 0; , 0, 0; \dots)$ . The cycle type in the alternative definition is  $(-2, 3)$ . Its size is 5 and its length 2.

As for  $\mathfrak{S}_n$  we have the following characterization of the conjugacy classes of  $\mathfrak{W}_{r,n}$ :

LEMMA 2.1.16. *The conjugacy classes of  $\mathfrak{W}_{r,n}$  are exactly the signed permutations with the same cycle type.*

PROOF. We start by showing that for  $\tau$  and  $\omega \in \mathfrak{W}_{r,n}$  with the same cycle type there exists a  $\sigma \in \mathfrak{W}_{r,n}$  with  $\tau = \sigma\omega\sigma^{-1}$ . Note that  $\sigma$  is not unique, we just construct one possible  $\sigma$ :

There exists a bijection between the cycles  $c$  of  $\tau$  length  $l$  and type  $\xi$  and the ones of  $\omega$  with the same properties as the cycle type is the same. As  $\sigma c_1 c_2 \sigma^{-1} = \sigma c_1 \sigma^{-1} \sigma c_2 \sigma^{-1}$  it suffices to consider only one cycle:

A cycle  $c$  of  $\tau$  consists of the elements  $x_1, x_2, \dots, x_l$  with  $x_i \mapsto \xi_i x_j$  and the one in  $\omega$  consists of the elements  $y_1, y_2, \dots, y_l$  with  $y_i \mapsto \eta_i y_j$ . We define

$$\sigma^{-1}(x_i) := \frac{\eta_1 \cdot \eta_2 \cdots \eta_{i-1}}{\xi_1 \cdot \xi_2 \cdots \xi_{i-1}} y_i.$$

Thus

$$\sigma(y_i) = \left( \frac{\eta_1 \cdot \eta_2 \cdots \eta_{i-1}}{\xi_1 \cdot \xi_2 \cdots \xi_{i-1}} \right)^{-1} x_i = \frac{\xi_1 \cdot \xi_2 \cdots \xi_{i-1}}{\eta_1 \cdot \eta_2 \cdots \eta_{i-1}} x_i.$$

Now, for  $i < l$  we consider

$$\begin{aligned} \sigma\omega\sigma^{-1}(x_i) &= \sigma\omega\left(\frac{\eta_1 \cdot \eta_2 \cdots \eta_{i-1}}{\xi_1 \cdot \xi_2 \cdots \xi_{i-1}} y_i\right) = \sigma\left(\eta_i \frac{\eta_1 \cdot \eta_2 \cdots \eta_{i-1}}{\xi_1 \cdot \xi_2 \cdots \xi_{i-1}} y_{i+1}\right) \\ &= \frac{\xi_1 \cdot \xi_2 \cdots \xi_i}{\eta_1 \cdot \eta_2 \cdots \eta_i} \eta_i \frac{\eta_1 \cdot \eta_2 \cdots \eta_{i-1}}{\xi_1 \cdot \xi_2 \cdots \xi_{i-1}} x_{i+1} = \xi_i x_{i+1}. \end{aligned}$$

For  $i = l$  we have

$$\sigma\omega\sigma^{-1}(x_l) = \sigma\omega\left(\frac{\eta_1 \cdot \eta_2 \cdots \eta_{l-1}}{\xi_1 \cdot \xi_2 \cdots \xi_{l-1}} y_l\right).$$

As the cycle types of  $\tau$  and  $\omega$  are the same (and thus the fraction of the products of the signs is 1) we can rewrite this as

$$\sigma\omega\left(\frac{\xi_l}{\eta_l} y_l\right) = \sigma\left(\eta_l \frac{\xi_l}{\eta_l} y_l\right) = \sigma(\xi_l x_1) = \xi_l x_1.$$

For the other direction we need to show that for  $\tau$  and  $\omega \in \mathfrak{W}_{r,n}$  with  $\tau = \sigma\omega\sigma^{-1}$  and  $\sigma \in \mathfrak{W}_{r,n}$  have the same cycle type. Therefore we show that a cycle  $c$  of  $\omega$  with length  $l$  and type  $\xi$  has the same properties as  $\sigma(c)$  of  $\tau$ : Let  $c$  be  $(x_1 \xi_1 x_2 \xi_2 \dots x_l \xi_l)$  and let  $\prod_{i=1}^l \xi_i = \xi$ . We further define  $y_i = \sigma(x_i)$ . Now we have

$$\tau(y_i) = (\sigma\omega\sigma^{-1})(\sigma(x_i)) = \sigma(\omega(x_i)) = \sigma(\xi_i x_{i+1}) = \xi_i y_{i+1}$$

what shows the lemma. □

The corresponding result for  $G \wr \mathfrak{S}_n$  can be found in [Mac95] (Chapter I, Appendix B).

In analogy to ordinary permutations we can calculate the number of signed permutations with a given cycle type:

LEMMA 2.1.17. *There are*

$$\frac{r^n n!}{\prod_{k=1}^n \prod_{j=0}^{r-1} (kr)^{\omega_k(\zeta^j)} \omega_k(\zeta^j)!}$$

*signed permutations with cycle type*

$$(\omega_1(1), \omega_1(\zeta), \dots, \omega_1(\zeta^{r-1}); \omega_2(1), \omega_2(\zeta), \dots, \omega_2(\zeta^{r-1}); \omega_3(1), \dots).$$

PROOF. We rather prove that

$$n! \prod_{k=1}^n \prod_{j=0}^{r-1} r^{(k-1)\omega_k(\zeta^j)} = n! r^n \prod_{k=1}^n \prod_{j=0}^{r-1} r^{-\omega_k(\zeta^j)}$$

is equal to the number of the permutations times

$$\prod_{k=1}^n \prod_{j=0}^{r-1} k^{\omega_k(\zeta^j)} \sigma_k(\zeta^j)!.$$

This implies the claim. (Note that  $\sum k\omega_k(\zeta^j) = n$ .)

Therefore we consider the cycle notation of  $\omega$ . There are  $n!$  ways to arrange  $[n]$ . When defining the first  $\omega_1(\zeta^0)$  numbers as being the  $\omega_1(\zeta^0)$  cycles with length 1 and type  $\zeta^0$ , the next  $\sigma_1(\zeta^1)$  numbers as being the cycles with length type  $\zeta^1$  and so on (note that for  $\sigma_l(\zeta^j)$  we need  $l$  positions per cycle) we get signed permutations of  $[n]$ . For the signs we have  $r^{k-1}$  ways to define them:  $r$  for each element, only the last one needs to be  $\zeta^j$  divided by the product of the other signs of this cycle as the cycle type is given with  $\zeta^j$ . All together the number of possible ways is

$$\prod_{k=1}^n \prod_{j=0}^{r-1} r^{(k-1)\sigma_k(\zeta^j)}.$$

However some of this arrangements define the same permutation: As we do not distinguish the order of the cycles of same type and it does not matter which of the elements begins in one cycle, the number of arrangements which define the same partitions is:

$$\prod_{k=1}^n \prod_{j=0}^{r-1} k^{\sigma_k(\zeta^j)} \sigma_k(\zeta^j)!.$$

□

## 2.2. Categories and Functors

We start with two general definitions on category theory we will need later on, for detailed information about categories see [Awo10] or [Hun80].

DEFINITION 2.2.1. A category  $\mathcal{C}$  is a class of objects (denoted by  $A, B, C, \dots$ ) together with a class of morphisms (denoted by  $f, g, h, \dots$ ) where:

- (1) for each morphism  $f$  there exist two objects  $A, B$  (called the domain and codomain, denoted by  $f : A \rightarrow B$ ).
- (2) for each object  $A$  there exists a distinguished morphism  $1_A : A \rightarrow A$ .
- (3) there exists a composition  $g \circ f : A \rightarrow C$  for any  $f : A \rightarrow B, g : B \rightarrow C$  with the following properties:
  - (a) Associativity: for  $f : A \rightarrow B, g : B \rightarrow C, h : C \rightarrow D$  it holds that  $h \circ (g \circ f) = (h \circ g) \circ f$ .
  - (b) Identity: for  $f : A \rightarrow B$  it holds that  $1_B \circ f = f$  and  $f \circ 1_A = f$ .

The following definition gives some examples of categories we will use frequently:

DEFINITION 2.2.2. We define  $\mathbb{B}$  as the category of sets with bijections. Hence, the morphisms going from a set to itself are permutations.

We define  $\mathbb{B}_r$  as the category of signed sets. In other words it is the category of sets, that have a  $\mathfrak{W}_{r,n}$ -action with bijections respecting this action.

REMARK 2.2.3. Note that  $\mathbb{B} = \mathbb{B}_1$ .

Another term of category theory which we will use soon is the one of a functor:

DEFINITION 2.2.4. For two given categories  $\mathcal{C}$  and  $\mathcal{D}$  we define a functor  $F$  as a mapping between the objects of  $\mathcal{C}$  and  $\mathcal{D}$  and the morphisms of  $\mathcal{C}$  and  $\mathcal{D}$  with the following properties:

- (1)  $F(f : A \rightarrow B) = F(f) : F(A) \rightarrow F(B)$

- (2)  $F(g \circ f) = F(f) \circ F(g)$
- (3)  $F(1_A) = 1_{F(A)}$

### 2.3. Symmetric Functions

**2.3.1. Definitions.** We state here only a very rough definition. A more formal definition and more information about symmetric functions can be found in [Sta99] or [Mac95].

DEFINITION 2.3.1. We define a symmetric function as formal power series on the set of indeterminates  $x = \{x_1, x_2, x_3, \dots\}$  over  $\mathbb{C}$  as a formal power series:

$$f(x) = \sum \alpha c_\alpha x^\alpha$$

Where

- $\alpha$  are infinite weak compositions of natural numbers  $n_\alpha$  (infinite tuples of numbers in  $\mathbb{N}$  with  $\sum_i \alpha_i = n_\alpha$ )
- $x^\alpha = \prod_i x_i^{\alpha_i}$  and  $c_\alpha \in \mathbb{C}$
- $f(x_{\sigma(1)}, x_{\sigma(2)}, \dots) = f(x_1, x_2, \dots)$  for all permutations  $\sigma$  of  $\mathbb{N}$ .

In other words a symmetric function is a formal power series in infinitely many variables with finite summands.

THEOREM 2.3.2. *The symmetric functions form a  $\mathbb{C}$ -algebra.*

We call this  $\mathbb{C}$ -algebra  $\Lambda$ . A proof can be found in [Sta99] or [Mac95].

We now define two different bases of  $\Lambda$ :

DEFINITION 2.3.3 (Power Sum Symmetric Functions). For  $\lambda$  a partition of  $n$  we define:

$$p_i = \sum_j x_j^i \text{ and } p_\lambda = p_{\lambda_1} p_{\lambda_2} \dots$$

DEFINITION 2.3.4 (Complete Homogeneous Symmetric Functions). For  $\lambda$  a partition of  $n$  we define:

$$h_i = \sum_{j_1 \leq \dots \leq j_i} x_{j_1} \dots x_{j_i} \text{ and } h_\lambda = h_{\lambda_1} h_{\lambda_2} \dots$$

**2.3.2. The  $\mathbb{C}$ -Algebra  $\Lambda(r)$ .** We define  $\Lambda(r)$  as in [Hen04] or [Mac95] without the use of symmetric functions. However we will see later that the point of view of symmetric functions is useful sometimes.

DEFINITION 2.3.5. We define  $\Lambda(r)$  as the  $\mathbb{C}$ -algebra generated by  $\{p_i(\zeta^j) : i, j \in \mathbb{N}\}$ .

DEFINITION 2.3.6. For an  $r$ -partition  $\lambda_r = (\xi_i a_i)_i$  we define  $p_{\lambda_r} := \prod p_{a_i}(\xi_i)$ .

EXAMPLE 2.3.7.  $r = 3$ ,  $\lambda_r = (\zeta 2, \zeta 2, \zeta^2 2, \zeta^0 1)$  then  $p_{\lambda_r} = p_2(\zeta)^2 p_2(\zeta^2) p_1(\zeta^0)$ .

DEFINITION 2.3.8. For a signed permutation  $\omega \in \mathfrak{W}_{r,n}$  we define  $p_\omega$  as  $p_{\rho(\omega)}$ , where  $\rho(\omega)$  is the alternative cycle type and thus an  $r$ -partition.

EXAMPLE 2.3.9. Once again we consider the case  $r = 2$ ,  $n = 5$  and  $\omega \in \mathfrak{W}_{r,n}$  with

$$\omega = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ -2 & -5 & -4 & 3 & 1 \end{pmatrix}.$$

We already know that the cycle type in the alternative definition is  $(-2, 3)$ , thus  $p_\omega = p_2(-1) \cdot p_3(1)$ .

REMARK 2.3.10. For  $r = 1$  we write  $p_i$  instead of  $p_i(1)$  and we identify  $\Lambda(1)$  with  $\Lambda$ , and  $p_i$  with the  $i^{\text{th}}$  power sum symmetric function.

One can, in analogy with this case, define  $p_i(\xi)$  as the  $i^{\text{th}}$  power sum symmetric function on indeterminates  $x_{j,\xi}$  and analogously for  $h_i(\xi)$ . (Note that then  $h_{i,\xi} = h_i \circ p_1(\xi)$ . We will define  $\circ$  later.) Here we stress the point of view of symmetric functions. In particular, using different bases of  $\Lambda(r)$  will be very convenient.

In the case  $r = 2$  we can consider not only  $p_i(1)$  and  $p_i(-1)$ , we also have power sum symmetric functions corresponding to the trivial and nontrivial character of  $\mathbf{C}_2$   $p_i(x)$  and  $p_i(y)$ . For an exact definition see [Mac95] (Chapter I, Appendix B). We will use them only in some examples, so for us it is sufficient to know how to perform a change of variables:

- $p_i(1) = p_i(x) + p_i(y)$
- $p_i(-1) = p_i(x) - p_i(y)$

More generally:

$$p_i(\gamma) = \sum_{\xi \in \mathbf{C}_r} \psi_\xi^{-1} \gamma(\xi) p_i(\xi)$$

Here  $\psi_\xi$  is the size of the centralizer of  $\xi$  in  $\mathbf{C}_r$  which is  $r$ , and  $\gamma(c)$  is the value of the character  $\gamma$  on the conjugacy class of  $c$ . In case of the trivial character  $x$ , this value is always 1 and we get:

$$p_i(x) = \sum_{\xi \in \mathbf{C}_r} \frac{1}{r} p_i(\xi)$$



## CHAPTER 3

### r-Species

#### 3.1. Definition and Examples

The definition of a species is a very theoretical one, however the concept is quite intuitive:

**DEFINITION 3.1.1.** We define an ordinary species as a functor  $F : \mathbb{B} \rightarrow \mathbb{B}$ .

In analogy we define an  $r$ -species as a functor  $F : \mathbb{B}_r \rightarrow \mathbb{B}_1$ .

**REMARK 3.1.2.** (1) When defining an  $r$ -species we often just state the objects as the morphisms are defined in a natural way. Morphisms from one  $r$ -set to another are less important than those of  $\mathfrak{W}_{r,n}$  on one  $r$ -set  $M$ . We call the way  $F[\omega]$  (for  $\omega \in \mathfrak{W}_{r,n}$ ) acts on  $F[M]$  the  $\omega$ -action, and the way  $F[\mathfrak{W}_{r,n}]$  acts the  $\mathfrak{W}_{r,n}$ -action on this species.

(2) Note that we often write  $F[n]_{\mathbf{C}_r}$  instead of  $F[[n]_{\mathbf{C}_r}]$  and  $F[n]$  instead of  $F[[n]]$ , so we sometimes skip the outer brackets.

**EXAMPLE 3.1.3** ( $r$ -sets). We define  $\mathcal{E}_n^r$ , the species of  $r$ -sets with  $n \cdot r$  elements ( $n \geq 0$ ), as

$$\mathcal{E}_n^r : \mathbb{B}_r \rightarrow \mathbb{B}_1, M \mapsto \begin{cases} \{M\} & |M| = r \cdot n \\ \emptyset & \text{otherwise} \end{cases}$$

We furthermore define  $\mathcal{E}^r$  as the species of all  $r$ -sets:  $\mathcal{E}^r : \mathbb{B}_r \rightarrow \mathbb{B}_1, M \mapsto \{M\}$ . This species corresponds to the trivial representation of  $\mathfrak{W}_{r,n}$ . We write  $\mathcal{E}$  for  $\mathcal{E}^1$  and  $\mathcal{E}_+$  for the species of non empty sets.

The unique  $\mathcal{E}_3^2$  structure is  $[3]_{\mathbf{C}_2} (= \{1, -1, 2, -2, 3, -3\})$ .

**EXAMPLE 3.1.4** (Vertices). We define  $\mathcal{V}_n^r$ , the species of vertices as:

$$\mathcal{V}_n^r[n]_{\mathbf{C}_r} = \{ \{ \zeta^{i_1} 1, \zeta^{i_2} 2, \dots, \zeta^{i_n} n \} : i_j \in \{0, 1, 2, \dots, r-1\} \}.$$

The  $\mathfrak{W}_{r,n}$ -action is defined point wise.

Note that in the case  $r = 2$  those are the vertices of an  $n$ -dimensional cube.

For example  $\mathcal{V}_n^r[3]_{\mathbf{C}_2} =$

$$\{ \{1, 2, 3\}, \{-1, 2, 3\}, \{1, -2, 3\}, \{1, 2, -3\}, \{-1, -2, 3\}, \{-1, 2, -3\}, \{1, -2, -3\}, \{-1, -2, -3\} \}.$$

For  $\omega = (1 + 2 - 3+)$  we have

$$\mathcal{V}_n^r[\omega] = (\{1, 2, 3\}\{1, 2, -3\}\{-1, 2, -3\}\{-1, -2, -3\}\{-1, -2, 3\}\{1, -2, 3\})(\{1, -2, -3\}\{-1, 2, 3\}).$$

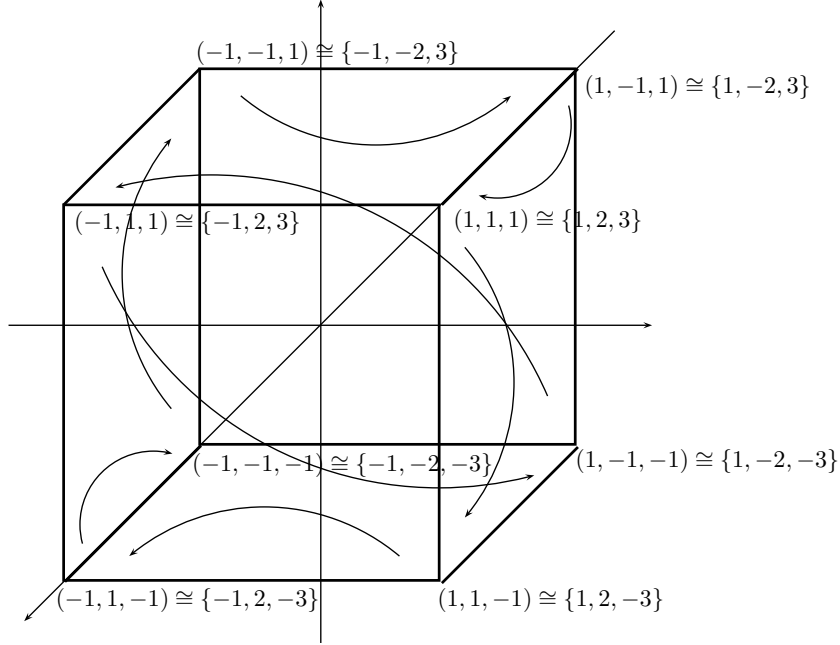
For a graphical representation see figure 1.

**EXAMPLE 3.1.5** (Signed Cycles). We define  $\mathcal{C}^r$ , the species of (oriented, nonempty) signed cycles, we also sometimes call them  $r$ -cycles, as

$$\mathcal{C}^r[M_{\mathbf{C}_r}] = \{ (j_1 \xi_1 j_2 \xi_2 \dots j_n \xi_n) : \xi_i \in \mathbf{C}_r, j_i \in M, j_{i_1} \neq j_{i_2} \text{ if } i_1 \neq i_2 \} / \sim$$

where

$$(j_1 \xi_1 j_2 \xi_2 \dots j_n \xi_n) \sim (k_1 \eta_1 k_2 \eta_2 \dots k_n \eta_n)$$



$$\omega = (1 + 2 - 3+)$$

$$\mathcal{V}^r[\omega] = ((1, 1, 1), (1, 1, -1), (-1, 1, -1), (-1, -1, -1), (-1, -1, 1), (1, -1, 1))((1, -1, -1), (-1, 1, 1))$$

FIGURE 1. Vertices of a 3-dimensional cube and an  $\omega$ -action.

if  $\exists l$  with

$$(j_l \xi_l j_{l+1} \xi_{l+1} \dots j_n \xi_n j_1 \xi_1 j_2 \xi_2 \dots j_{l-1} \xi_{l-1}) = (k_1 \eta_1 k_2 \eta_2 \dots k_n \eta_n).$$

This cycles can be represented as cycles where the positions are labeled with elements of  $M$  and the arcs between them are labeled with signs. Furthermore, we can identify such a cycle with a cycle of a signed permutation. The  $\mathfrak{W}_{r,n}$ -action is then defined as  $C^r[\omega](c) = \omega \circ c \circ \omega^{-1}$ . This corresponds to the adjoint representation of  $\mathfrak{W}_{r,n}$ . We also write  $\mathcal{C}$  for  $\mathcal{C}^1$ .

This action can be calculated by a re-labeling in the first step ( $a\xi b$  turns into  $\omega(a)\xi\omega(b)$ , note that  $\omega(a)$  and  $\omega(b)$  may have signs) and a changing of signs in a second step ( $\omega(a)\xi\omega(b)$  turns into  $|\omega(a)| \left( \text{sgn}(\omega(a))^{-1} \xi \text{sgn}(\omega(b)) \right) |\omega(b)|$ ).

An example of a structure in  $[5]_{\mathbf{C}_3}$  is  $(3\zeta 1\zeta^0 2\zeta^0 5\zeta^0 4\zeta^2)$ . The  $\mathcal{C}^3[\omega]$ -image for  $\omega = (1)(2\zeta 3\zeta^0)(4)(5\zeta)$  then is  $(2\zeta 1\zeta 3\zeta^0 5\zeta^2 4\zeta^2)$ .

A graphical representation can be seen in figure 2.

Note that we do not allow empty cycles.

When working with general signed sets  $S \neq M_{\mathbf{C}_r}$  we use the absolute values for labeling the positions.

There is an easy way to convert an  $r$ -species into a normal species (compare with [HLL98]). Therefore we need the natural embedding of  $\mathfrak{S}_n$  to  $\mathfrak{W}_{r,n}$ :

DEFINITION 3.1.6. We define a natural embedding  $e : \mathfrak{S}_n \rightarrow \mathfrak{W}_{r,n}$  by defining  $e(\sigma) \in \mathfrak{W}_{r,n}$  as the signed permutation  $\xi x \mapsto \xi\sigma(x)$ . We often write  $\sigma$  instead of  $e(\sigma)$ .

DEFINITION 3.1.7. For an  $r$ -species  $F$  we define the restriction  $\Delta F$  by:

- $\Delta F[M] = F[M_{\mathbf{C}_r}]$



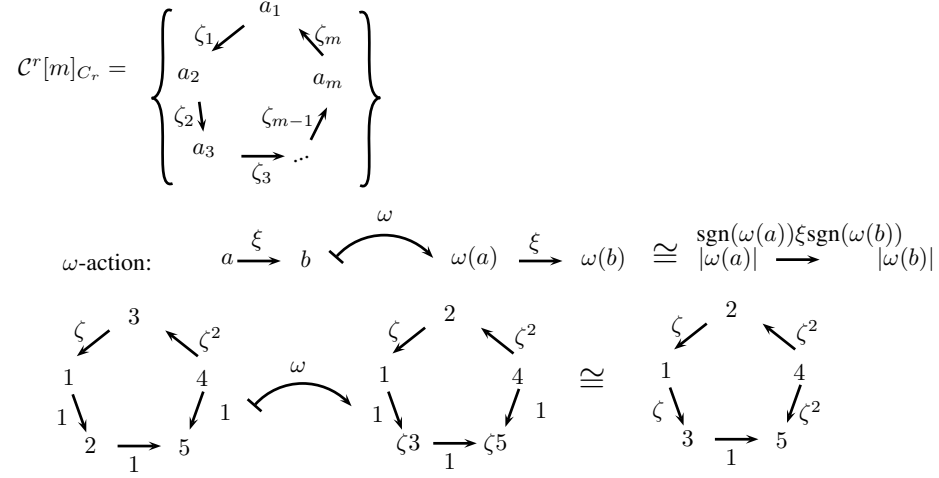


FIGURE 2. A general signed cycle and an example of  $C_3[5]_{C_3}$  together with an  $\omega$ -action.

- $\Delta F[\sigma] = F[\sigma]$

REMARK 3.1.8. In other words when creating  $\Delta F$  we just take the structures of  $F$  and ‘forget’ the  $\mathfrak{W}_{r,n}$ -action, that is not part of  $\mathfrak{S}_n$ .

EXAMPLE 3.1.9 (Vertices). We consider  $\Delta \mathcal{V}_n^r$ : The structures are the same as those of  $\mathcal{V}_n^r$ :

$$\Delta \mathcal{V}_n^r[n] = \{ \{ \zeta^{i_1} 1, \zeta^{i_2} 2, \dots, \zeta^{i_n} n \} : i_j \in \{0, 1, 2, \dots, r-1\} \}.$$

However we only have a  $\mathfrak{S}_n$ -action, so for example  $\{-1, 2, 3\}$  and  $\{1, 2, 3\}$  are not in the same orbit. More generally: exactly the elements that have the same number of the same signs are in one  $\mathfrak{S}_n$ -orbit.

### 3.2. The Cycle Indicator Series

One can regard the cycle indicator series as a generalization of generating functions as it has both, the information about the exponential generating function (egf) of labeled objects and the type generating function (tgf) for unlabeled objects.

DEFINITION 3.2.1. For an  $r$ -species  $F$  we define the cycle index series (also called Frobenius character) as:

$$Z_F = \sum_{n \geq 0} \frac{1}{n! r^n} \sum_{\omega \in \mathfrak{W}_{r,n}} |\text{fix } F[\omega]| p_\omega.$$

LEMMA 3.2.2. *The number of fixed points of  $F[\omega]$  depends only on the cycle type of the (signed) permutation  $\omega$ .*

PROOF. Exactly if  $\tau$  and  $\omega \in \mathfrak{W}_{r,n}$  have the same cycle type there exists a  $\sigma \in \mathfrak{W}_{r,n}$  so that  $\tau = \sigma \omega \sigma^{-1}$  (note that  $\sigma$  is not unique, compare with lemma 2.1.16). A fixed point  $f$  of  $\tau$  is thus fixed under  $\sigma \omega \sigma^{-1}$  so  $\sigma^{-1}(f)$  is fixed under  $\omega$ :  $\omega(\sigma^{-1}(f)) = (\sigma^{-1} \tau \sigma)(\sigma^{-1}(f)) = \sigma^{-1} \tau \sigma \sigma^{-1}(f) = \sigma^{-1} \tau(f) = \sigma^{-1}(f)$ . This relationship is bijective as  $\sigma$  is a bijection.

Now we do not consider fixed points of  $\tau$  and  $\omega$  but those of  $F[\tau]$  and  $F[\omega]$  for  $F$  being an  $(r-)$ species. But as  $F[\tau] = F[\sigma \omega \sigma^{-1}] = F[\sigma] F[\omega] F[\sigma]^{-1}$  with  $F[\tau], F[\sigma], F[\omega] \in \mathfrak{S}_{F[n]} = W_{1, |F[n]|}$  this holds.  $\square$

EXAMPLE 3.2.3 ( $r$ -Sets). We now calculate  $Z_{\mathcal{E}^r}$ . Therefore we need the number of fixed points under a signed permutation. As every set  $M$  is fixed under any signed permutation and there is only one structure  $(\{M\})$ , this number is 1 for every  $\omega$ .

$$Z_{\mathcal{E}^r} = \sum_{n \geq 0} \frac{1}{n!r^n} \sum_{\omega \in \mathfrak{W}_{r,n}} |\text{fix } \mathcal{E}^r[\omega]| p_\omega = \sum_{n \geq 0} \frac{1}{n!r^n} \sum_{\omega \in \mathfrak{W}_{r,n}} 1 \cdot p_\omega$$

We can use lemma 2.1.17 and 3.2.2 and just consider the cycle types of  $\omega$ . (Recall that  $w_k(\zeta^j)$  is the number of cycles of length  $k$  and type  $\zeta^j$ . Therefore, for  $k > n$  holds  $w_k(\zeta^j) = 0$ .)

$$\begin{aligned} &= \sum_{n \geq 0} \frac{1}{n!r^n} \sum_{\omega \vdash_r n} \frac{r^n n!}{\prod_{k=1}^n \prod_{j=0}^{r-1} (kr)^{\omega_k(\zeta^j)} \omega_k(\zeta^j)!} p_\omega \\ &= \sum_{n \geq 0} \sum_{\omega \vdash_r n} \frac{1}{\prod_{k=1}^n \prod_{j=0}^{r-1} \omega_k(\zeta^j)!} \prod_{k=1}^n \prod_{j=0}^{r-1} \left( \frac{p_k(\zeta^j)}{k \cdot r} \right)^{\omega_k(\zeta^j)} \end{aligned}$$

We now do some reordering to obtain an exponential form by considering  $m = \sum_{k,j} \omega_k(\zeta^j)$  instead of  $n = \sum_{k,j} k\omega_k(\zeta^j)$ :

$$\begin{aligned} &= \sum_{m \geq 0} \frac{1}{m!} \sum_{\sum_{k,j} \omega_k(\zeta^j) = m} \frac{m!}{\prod_k \prod_{j=0}^{r-1} \omega_k(\zeta^j)!} \prod_k \prod_{j=0}^{r-1} \left( \frac{p_k(\zeta^j)}{k \cdot r} \right)^{\omega_k(\zeta^j)} \\ &= \sum_{m \geq 0} \frac{1}{m!} \left( \sum_{\sum_{k,j} \omega_k(\zeta^j) = m} \left( \omega_1(\zeta^0) \omega_1(\zeta^1) \dots \omega_1(\zeta^r) \omega_2(\zeta^0) \dots \right) \prod_k \prod_{j=0}^{r-1} \left( \frac{p_k(\zeta^j)}{k \cdot r} \right)^{\omega_k(\zeta^j)} \right) \end{aligned}$$

Applying the multinomial theorem we obtain:

$$= \sum_{m \geq 0} \frac{1}{m!} \left( \sum_{k > 0} \sum_{j=0}^{r-1} \frac{p_k(\zeta^j)}{k \cdot r} \right)^m = \exp \left( \sum_{k > 0} \sum_{j=0}^{r-1} \frac{p_k(\zeta^j)}{k \cdot r} \right) = \exp \left( \sum_{j=0}^{r-1} \sum_{k > 0} \frac{p_k(\zeta^j)}{k \cdot r} \right)$$

In some cases this result is good for further calculations, however we will here calculate further on to get a result in complete homogeneous symmetric functions  $h_i$ , therefore, we interpret the  $p_k(\zeta^j)$  as  $\sum_i x_{i,j}^k$ . (Compare with remark 2.3.10.)

On this point we could also use  $p_k(x) = \frac{\sum_{j=0}^{r-1} p_k(\zeta^j)}{r}$  (compare with remark 2.3.10) and get a slightly different result, which we will state in the end of the calculation.

$$\begin{aligned} &= \prod_{j=0}^{r-1} \exp \left( \frac{1}{r} \sum_{k > 0} \sum_i \frac{x_{i,j}^k}{k} \right) = \prod_{j=0}^{r-1} \exp \left( \frac{1}{r} \sum_i \log \left( \frac{1}{1 - x_{i,j}} \right) \right) = \prod_{j=0}^{r-1} \exp \left( \log \left( \left( \prod_i \frac{1}{1 - x_{i,j}} \right)^{\frac{1}{r}} \right) \right) \\ &= \prod_{j=0}^{r-1} \left( \prod_i \frac{1}{1 - x_{i,j}} \right)^{\frac{1}{r}} = \prod_{j=0}^{r-1} \left( \prod_i \sum_k x_{i,j}^k \right)^{\frac{1}{r}} = \prod_{j=0}^{r-1} \left( \sum_n \sum_{i_1 \leq \dots \leq i_n} x_{i_1,j} \dots x_{i_n,j} \right)^{\frac{1}{r}} \\ &= \prod_{j=0}^{r-1} \left( \sum_n h_n(\zeta^j) \right)^{\frac{1}{r}} \end{aligned}$$

Alternatively: (The steps we are skipping are analogous to the ones we did before.)

$$= \exp \left( \sum_{k \geq n} \sum_{j=0}^{r-1} \frac{p_k(\zeta^j)}{k \cdot r} \right) = \exp \left( \sum_{k \geq n} \frac{p_k(x)}{k} \right) = \dots = \sum_n h_n(x)$$

EXAMPLE 3.2.4 (Vertices). To calculate  $Z_{\mathcal{V}_n^r}$  we need to know how many fixed points  $\mathcal{V}_n^r[n]_{\mathcal{C}^r}$  has under  $\mathcal{V}_n^r[\omega]$ . Therefore we consider the cycles of  $\omega$ .

First we consider a cycle of length  $l$  and type 1. We only need to consider the cycle that only has  $\zeta^0$  as signs: The number of fixed points only depends on the type (compare with lemma 3.2.2) and therefore is the same for all cycles with type  $\zeta^0$ : If and only if the  $l$  elements of this cycle have the same sign in a vertex, this vertex is a fixed point. Therefore, we get  $r$  fixed points, for we can choose one sign for all elements in this cycle.

If we consider a fixed point of any other cycle  $c$  with length  $l$  and type  $\xi$ , it will need to be a fixed point under  $c^l = \xi \text{id}$  too. As the  $\omega$ -action is defined point wise, no element can be a fixed point of  $c^l = \xi \text{id}$ .

Therefore the number of fixed points of  $\omega$  is  $\prod_{k=1}^n r^{\omega_k(1)}$  if it consists only of cycles with type 1. Otherwise there are no fixed points. Now we can calculate  $Z_{\mathcal{V}_n^r}$ :

$$\begin{aligned} Z_{\mathcal{V}_n^r} &= \frac{1}{n!r^n} \sum_{\omega \in \mathbb{W}_{r,n}} |\text{fix } F[\omega]| p_\omega = \frac{1}{n!r^n} \sum_{\omega \vdash n} \frac{r^n n!}{\prod_{k=1}^n k^{\omega_k(1)} r^{\omega_k(1)} \omega_k(1)!} \prod_k r^{\omega_k(1)} p_\omega(1) \\ &= \sum_{\omega \vdash n} \frac{1}{\prod_{k=1}^n \omega_k(1)!} \prod_k \frac{p_k(1)}{k} \end{aligned}$$

Now we can do the same calculations as we did with the  $r$ -sets (example 3.2.3) and get (if we consider also the empty vertex):

$$Z_{\mathcal{V}^r} = \exp \left( \sum_k \frac{p_k(1)}{k} \right) = \sum_n h_n(1)$$

EXAMPLE 3.2.5 (Signed Cycles). For calculating  $Z_{\mathcal{C}^r}$  we once again need to consider fixed points under the  $\omega$ -action:

We start with considering a structure (a cycle) of  $\mathcal{C}^r$ . In the first step we only consider the elements, not their signs: For being a fixed point under  $\omega$ ,  $\omega$  needs to introduce a shift on the elements (then and only then image and pre-image are equivalent under  $\sim$  and therefore the same structure). The number of cycles,  $\omega$  can have, depends on the size of the shift. Let our cycle have length  $l_c$ , and let the shift be  $s$ , then the number of cycles of  $\omega$  is  $\gcd(l, s)$ , their length is  $\frac{l}{\gcd(l, s)}$ :  $s \cdot \frac{l}{\gcd(l, s)} = \text{lcm}(l, s)$  and therefore  $\omega^{\frac{l}{\gcd(l, s)}}$  is the smallest power of  $\omega$  whose action on the cycle is the identity, therefore this is the length of the cycle. The number of cycles follows immediately. (For graphical examples see figure 3.)

It follows that the only  $\omega$  that can have fixed points are those where all cycles have the same length  $l$ . Let the number of such cycles be  $k$ . We now count the possible ways to build an  $r$ -cycle that is a fixed point. Without loss of generality, we set 1 to position 0. We next consider  $|\omega^{-1}(1)|$ : We can put it on every  $k^{\text{th}}$ -position that is coprime to  $l$ . Exactly then  $\omega^l$  is the first power that is the identity. The rest of the elements of this cycle of  $\omega$  is determined, too. We so far have  $\phi(l)$  possible choices (where  $\phi$  denotes Euler's Phi-function). Next we consider the cycle with the smallest element not used so far: We have  $l \cdot (k-1)$  possible positions for it. The rest of this  $\omega$  cycle is determined by this position and the choice of  $|\omega^{-1}(1)|$ . The rest of the cycles is treated analogously. All together we have  $\phi(l)l^{k-1}(k-1)!$  possible fixed points (without signs so far). Note

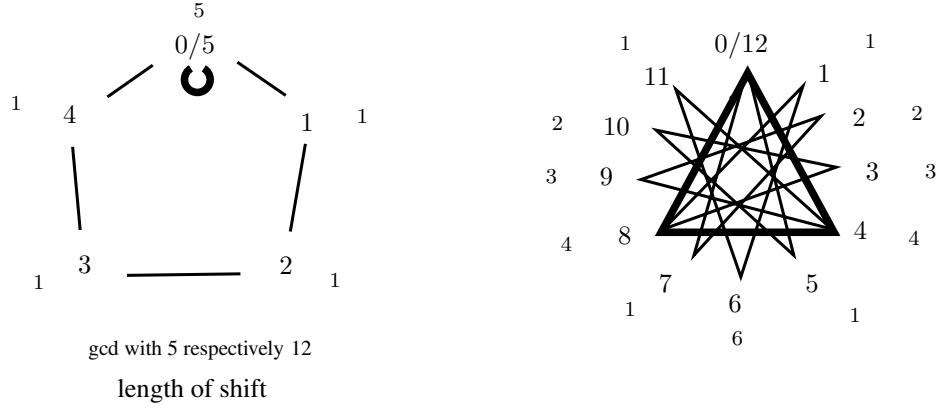


FIGURE 3. Two examples of cycles and an  $\omega$ -action, where those cycles are fixed points.

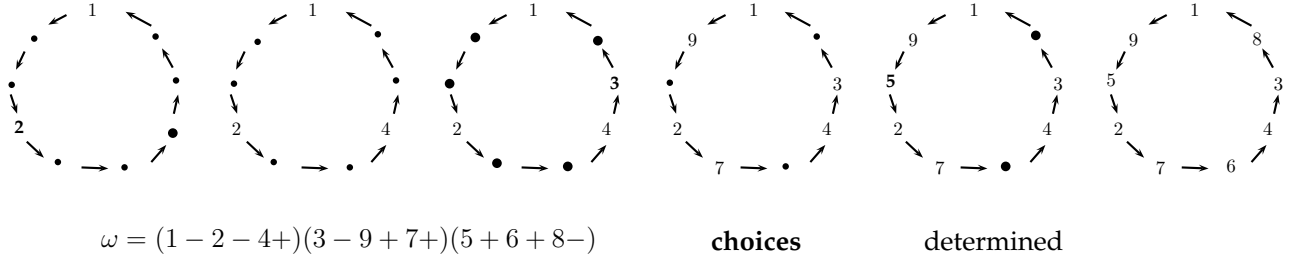


FIGURE 4. Example of constructing the elements of a fixed  $r$ -cycle.

that the cycles alternate in the sense that after an element of lets say cycle 1, there is always an element of cycle 2, then one from cycle 3 and so on, and after the one of cycle  $k$  there follows the next element of cycle 1. (For graphical examples see figure 4.)

In the second step we concentrate on the signs: For each  $\omega$ -cycle we can choose exactly one sign, the rest is determined, only  $\omega$  where all the cycles have the same sign can possibly have fixed points (this gives us  $r^k$  choices):

We start with choosing  $\xi_{a_1}$  and call the cycle of  $a_1$  cycle  $a$ , then  $a_1 \xi_1 b_1$  ( $b_1$  was chosen before and is from another cycle of  $\omega$ , we call it cycle  $b$ ) is part of our cycle  $c$ . Remember that  $c$  is our signed cycle (so this is the structure) and  $a$  and  $b$  are cycles of  $\omega$ , the morphism. As  $c = \omega c \omega^{-1}$  we know that  $c(\omega(a_1)) = \xi_1 \omega(b_1)$  so we define  $a_2 := |\omega(a_1)|$ ,  $\xi_2 := \text{sgn}(\omega(a_1))^{-1} \xi_1 \text{sgn}(\omega(b_1))$  and  $b_2 := |\omega(b_1)|$ . We iterate this until we have  $m$  with  $a_m = a_1$ . Now it is necessary that  $\xi_m = \xi_1$ . Due to construction we know that

$$\begin{aligned} \xi_m &= \text{sgn}(\omega(a_{m_1}))^{-1} \cdots \text{sgn}(\omega(a_1))^{-1} \xi_1 \text{sgn}(\omega(b_1)) \cdots \text{sgn}(\omega(b_{m-1})) \\ &= (\text{type of cycle } a)^{-1} \xi_1 (\text{type of cycle } b). \end{aligned}$$

Now exactly if cycle  $a$  and cycle  $b$  have the same type, this construction is feasible. (For graphical examples see figure 5.)

Now we can calculate  $Z_{C^r}$ :

$$Z_{C^r} = \sum_{n>0} \frac{1}{n! r^n} \sum_{\omega \in \mathfrak{W}_{r,n} \text{ with } p_\omega = (p_{l_\omega}(\xi_\omega))^{k_\omega}} \phi(l_\omega) r^{k_\omega} l_\omega^{k_\omega - 1} (k_\omega - 1)! p_{l_\omega}(\xi)^{k_\omega}$$

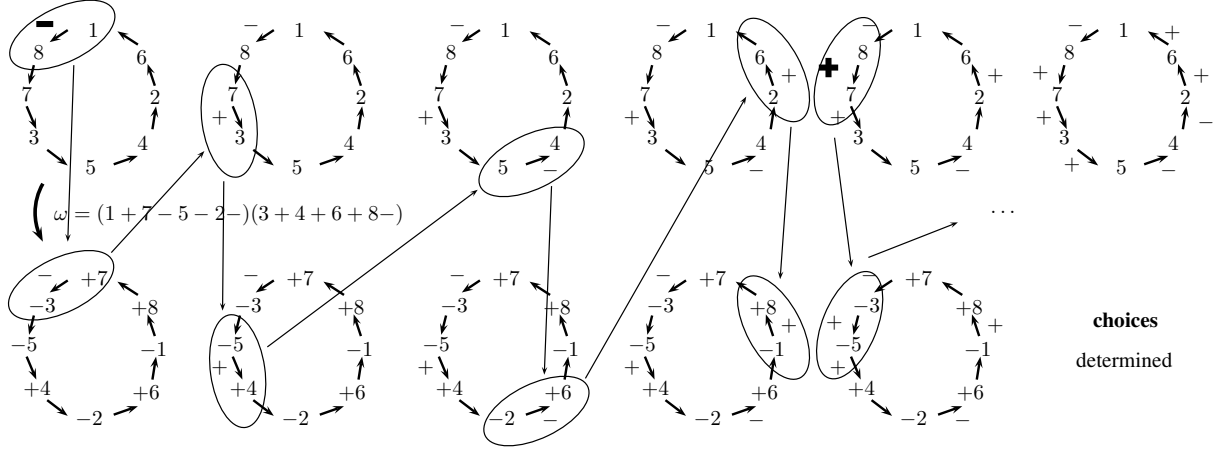


FIGURE 5. Example of constructing the signs of a fixed  $r$ -cycle.

$$\begin{aligned}
&= \sum_{\xi \in \mathbf{C}_r} \sum_{n>0} \frac{1}{n!r^n} \sum_{l,k \text{ with } l \cdot k = n} \binom{n!r^n}{l^k r^k k!} \phi(l) r^k l^{k-1} (k-1)! p_l(\xi)^k \\
&= \sum_{\xi \in \mathbf{C}_r} \sum_{n>0} \sum_{l,k \text{ with } l \cdot k = n} \frac{\phi(l)}{lk} p_l(\xi)^k = \sum_{\xi \in \mathbf{C}_r} \sum_{l,k>0} \frac{\phi(l)}{lk} p_l(\xi)^k \\
&= \sum_{\xi \in \mathbf{C}_r} \sum_{l>0} \frac{\phi(l)}{l} \sum_{k>0} \frac{p_l(\xi)^k}{k} = \sum_{\xi \in \mathbf{C}_r} \sum_{l>0} \frac{\phi(l)}{l} \log \left( \frac{1}{1 - p_l(\xi)} \right)
\end{aligned}$$

**3.2.1. Specializations.** As mentioned before, the cycle indicator series contains information about numbers of objects in the species.

In analogy to  $r = 1$ , we define an exponential generating series, that counts the (labeled) objects and a type generating series that counts the unlabeled objects, in other words the orbits of the  $\mathfrak{W}_{r,n}$ -action:

DEFINITION 3.2.6. We call  $\sum_n |F[n]_{\mathbf{C}_r}| \frac{z^n}{r^n n!}$  generalized exponential generating series (egs) and write it as  $F(z)$ . Furthermore, we call  $\sum_n |F[n]_{\mathbf{C}_r}/\mathfrak{W}_{r,n}| z^n$  type generating series (tgs) and write it as  $\tilde{F}(z)$ .

THEOREM 3.2.7 (Specializations). *By setting*

- (1)  $p_1(1) = z$  and  $p_i(\xi) = 0$  otherwise
- (2)  $p_i(\xi) = z^i$
- (3)  $p_1(\xi) = z$  and  $p_i(\xi) = 0$  otherwise
- (4)  $p_i(1) = \frac{z^i}{i-1}$  and  $p_i(\xi) = 0$  otherwise

the cycle indicator series transform into

- (1)  $\sum_n |F[n]_{\mathbf{C}_r}| \frac{z^n}{r^n n!} = F(z)$  which is the generalized exponential generating series
- (2)  $\sum_n |F[n]_{\mathbf{C}_r}/\mathfrak{W}_{r,n}| z^n = \tilde{F}(z)$  which is the type generating series (it counts  $\mathfrak{W}_{r,n}$ -orbits)
- (3)  $\sum_n |F[n]_{\mathbf{C}_r}/\mathbf{C}_r| \frac{z^n}{n!} =: \tilde{F}^{\mathbf{C}_r}(z)$
- (4)  $\sum_n |F[n]_{\mathbf{C}_r}/\mathfrak{S}_n| \frac{z^n}{r^n} =: \tilde{F}^{\mathfrak{S}_n}(z)$ .

For the proof of this theorem we need ‘Burnside’s Lemma’, which can be found (together with a proof) for example in [Sta99] or [BLL98].

LEMMA 3.2.8 (Cauchy-Frobenius). *Let  $M$  be a finite set and  $\Gamma$  be a subgroup of  $\mathfrak{S}_M$ . Then*

$$|M/\Gamma| = \frac{1}{|\Gamma|} \sum_{\sigma \in \Gamma} |\text{fix } \sigma|.$$

*In other words the number of  $\Gamma$ -orbits in  $M$  times  $|\Gamma|$  is the sum over all fixed points of elements in  $\Gamma$  (counted with multiplicities if they are fixed point of more than one element in  $\Gamma$ ).*

PROOF OF THEOREM 3.2.7. (1) By setting  $p_1(1) = z$  and  $p_i(\xi) = 0$  otherwise the only  $p_\omega \neq 0$  is  $p_{\text{id}}$ . What we get then is

$$\sum_n \frac{1}{n!r^n} |\text{fix } F[\text{id}]| z^n.$$

As every structure is a fixed point under the identity, we have indeed a generalized exponential generating function.

(2) By setting  $p_i(\xi) = z^i$  we get

$$\sum_n z^n \frac{1}{n!r^n} \sum_{\omega \in \mathfrak{W}_{r,n}} |\text{fix } F[\omega]|.$$

Now we interpret  $\mathfrak{W}_{r,n}$  as a subgroup of  $\mathfrak{S}_{[n]_{\mathbf{C}_r}}$ , use ‘Burnside’s Lemma’ and get

$$\sum_n (\text{number of } \mathfrak{W}_{r,n}\text{-orbits}) z^n.$$

- (3) The proof is a combination of the previous two points, note that those  $\omega$  with  $p_\omega = \prod_{j=0}^{r-1} p_1(\zeta^j)^{i_j}$  define the  $\mathbf{C}_r$ -action.  
(4) Analogous as before, note that we want only such  $\omega$  that define the  $\mathfrak{S}_n$ -action. Exactly one of the  $r^{i-1}$  cycles of length  $i$  with sign 1 defines it. (The one that consists of 1 as signs only.)

□

EXAMPLE 3.2.9 (Signed Cycles). We calculate the specializations of signed cycles (remember that  $Z_{\mathbf{C}^r} = \sum_{l,k>0} \frac{\phi(l)}{lk} p_l(1)^k$ ):

- (1) We get  $\mathcal{C}^r(z) = \sum_{k>0} \frac{\phi(1)}{1k} z^k = \sum_k (k-1)! r^k \frac{z^k}{k!r^k}$ . By calculating directly how many signed cycles we have, we get indeed  $(k-1)!r^k$  ( $k!$  possible arrangements with  $r^k$  signs, every shift defines the same cycle, there are  $k$  shifts).  
(2) We get  $\tilde{\mathcal{C}}^r(z) = \sum_{\xi \in \mathbf{C}_r} \sum_{l,k>0} \frac{\phi(l)}{lk} z^{l \cdot k} = r \sum_{n>0} \frac{z^n}{n} \sum_{l|n} \phi(l)$ . Gauß proved that  $\sum_{l|n} \phi(l) = n$  so we get  $\sum_n r z^n$  which is exactly the generating function of cycle types of cycles. (As  $\omega \circ c \circ \omega^{-1}$  always has the same type as  $c$ , the orbits of the  $\mathfrak{W}_{r,n}$ -action are those with the same type.)  
(3) We get  $\tilde{\mathcal{C}}^{r \mathbf{C}^r}(z) = \sum_{\xi \in \mathbf{C}_r} \sum_{k>0} \frac{\phi(1)}{1k} z^k = \sum_k r(k-1)! \frac{z^k}{k!}$ . There are indeed  $(k-1)!$  unsigned cycles for each type.  
(4) We get  $\tilde{\mathcal{C}}^{r \mathfrak{S}^\bullet}(z) = \sum_{l,k>0} \frac{\phi(l)}{lk} \left( \frac{z^l}{r^{l-1}} \right)^k = \sum_{l,k>0} \frac{\phi(l)}{lk} \frac{z^{l \cdot k}}{r^{l \cdot k - k}} = \sum_n \frac{z^n}{r^n} \frac{1}{n} \sum_{l|n} \phi(l) \cdot r^{\frac{n}{l}}$ . This is the number of necklaces with  $n$  beads and  $r$  colors.

EXAMPLE 3.2.10 ( $r$ -Sets). The specializations of  $r$ -Sets are ( $Z_{\mathcal{E}^r} = \exp \left( \sum_{j=0}^{r-1} \sum_{k>0} \frac{p_k(\zeta^j)}{k \cdot r} \right)$ ):

- (1)  $\mathcal{E}^r(z) = \sum_n \frac{z^n}{r^n n!}$   
(2)  $\tilde{\mathcal{E}}^r(z) = \sum_n z^n$   
(3)  $\tilde{\mathcal{E}}^{r \mathbf{C}^r}(z) = \sum_n \frac{z^n}{n!}$

$$(4) \tilde{\mathcal{E}}^r{}^{\mathfrak{S}^\bullet}(z) = \sum_n \frac{z^n}{r^n}$$

EXAMPLE 3.2.11 (Vertices). The specializations of vertices are ( $Z_{\mathcal{V}^r} = \exp(\sum_k \frac{p_k(1)}{k})$ ):

$$(1) \mathcal{V}^r(z) = \sum_n \frac{z^n}{n!}$$

$$(2) \tilde{\mathcal{V}}^r(z) = \sum_n z^n$$

$$(3) \tilde{\mathcal{V}}^r{}^{\mathbf{C}^r}(z) = \sum_n \frac{z^n}{n!}$$

$$(4) \tilde{\mathcal{V}}^r{}^{\mathfrak{S}^\bullet}(z) = \sum_n \frac{(n+1)z^n}{r^n}$$

Another kind of specialization is the way we convert a cycle indicator series  $Z_F$  into  $Z_{\Delta F}$ :

LEMMA 3.2.12.

$$\begin{aligned} Z_{\Delta F} &= \sum_n r^n Z_{F_n} \Big|_{p_j(\zeta^k) = \begin{cases} 0 & \text{if } k \neq 0 \\ \frac{1}{r^{j-1}} p_j & \text{otherwise} \end{cases}} \\ &= Z_F \Big|_{p_\lambda(1) = r^{l(\lambda)} p_\lambda, p_j(\xi) = 0 \text{ for } \xi \neq 1} \\ &= Z_F \Big|_{p_k(1) = r p_k, p_j(\xi) = 0 \text{ for } \xi \neq 1} \\ &= Z_F \Big|_{p_j(x) = p_j(y) = \dots = p_j} \end{aligned}$$

PROOF. We compare  $Z_{\Delta F}$  and  $Z_F$ :

$$Z_{\Delta F} = \sum_n \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} |\text{fix } \sigma| p_\sigma, \quad Z_F = \sum_n \frac{1}{r^n n!} \sum_{\omega \in \mathfrak{W}_{r,n}} |\text{fix } \omega| p_\omega.$$

As for every  $\sigma \in \mathfrak{S}_n$  the number of fixed points is the same when we consider it as an element of  $\mathfrak{W}_{r,n}$  we need to extract the terms of  $Z_F$  that come from such a  $\sigma$  and multiply then by  $r^n$ . Every such  $p_\sigma$  has only type 1, so we set  $p_j(\xi) \mapsto 0$  for  $\xi \neq 1$ .

Now we consider a cycle  $c$  with length  $l$  of  $\sigma \in \mathfrak{S}_n$ . By choosing signs for  $c$  we get a cycle of an element of  $\mathfrak{W}_{r,n}$ . As we want type 1 only, there are  $r^{l-1}$  ways to choose it. Every element of  $\mathfrak{W}_{r,n}$  with only type 1 cycles can be built this way from a unique  $s$ . So we set  $p_c(1) \mapsto p_c \frac{1}{r^{l-1}}$ .

This proves the first equality. For the second and third one we calculate what happens with a  $p_\lambda(1)$ :

$$p_\lambda(1) = \prod_{k=1}^{l(\lambda)} p_k^{\lambda_k}(1) \mapsto r^{|\lambda|} \prod_{k=1}^{l(\lambda)} p_k^{\lambda_k} \frac{1}{r^{\lambda_k-1}} = \prod_{k=1}^{l(\lambda)} p_k^{\lambda_k} r = r^{l(\lambda)} p_\lambda$$

This proves the second and third equality. The fourth one we only prove for  $r = 2$ :

$$p_\lambda(1) = \prod_{k=1}^{l(\lambda)} p_k^{\lambda_k}(1) = \prod_{k=1}^{l(\lambda)} p_k^{\lambda_k}(x) + p_k^{\lambda_k}(y) \mapsto \prod_{k=1}^{l(\lambda)} 2p_k^{\lambda_k} = r^{l(\lambda)} p_\lambda$$

So for only type 1 signed permutation this holds. Now we consider any  $p_k(-1)$ :

$$p_k(-1) = p_k(x) - p_k(y) \mapsto p_k - p_k = 0$$

This shows the fourth equality. □

EXAMPLE 3.2.13 (Vertices). We calculate  $Z_{\Delta \mathcal{V}^r}$ :

$$\begin{aligned} Z_{\Delta \mathcal{V}^r} &= Z_{\mathcal{V}^r} \Big|_{p_k(1) = r p_k, p_j(\zeta^k) = 0 \Leftrightarrow k \neq 0} \\ &= \left( \exp \left( \sum_k \frac{p_k(1)}{k} \right) \right) \Big|_{p_k(1) = r p_k, p_j(\zeta^k) = 0 \Leftrightarrow k \neq 0} \end{aligned}$$

$$= \exp\left(\sum_k \frac{rP_k}{k}\right)$$

We can also calculate specializations of  $\Delta$ : For example for  $r = 2$  we have:

$$(\Delta \mathcal{V}^r)(z) = \exp(2z).$$

### 3.3. Sum and Product

In analogy to ordinary species we define sum and product of  $r$ -species:

**DEFINITION 3.3.1.** For  $F, G$  being  $r$ -species, we define  $(F + G)[M] = F[M] \cup G[M]$  and the  $\omega$ -action point wise. Therefore an  $(F + G)$ -structure is either an  $F$ -structure or a  $G$ -structure.

**REMARK 3.3.2.** As it follows from the definition that the sum is associative (and commutative), we can consider sums of  $k$   $r$ -species too.

The neutral element regarding  $+$  is the empty species  $0[M] := \emptyset$ .

**THEOREM 3.3.3.** For  $F, G$  being  $r$ -species, it holds that  $Z_{F+G} = Z_F + Z_G$  as well as:

- $(F + G)(x) = F(x) + G(x),$
- $(\widetilde{F + G})(x) = \tilde{F}(x) + \tilde{G}(x),$
- $(F + G)^{C_r}(x) = \tilde{F}^{C_r}(x) + \tilde{G}^{C_r}(x),$
- and  $(\widetilde{F + G})^{\mathfrak{S}_\bullet}(x) = \tilde{F}^{\mathfrak{S}_\bullet}(x) + \tilde{G}^{\mathfrak{S}_\bullet}(x).$

**PROOF.** For calculating  $Z_{F+G}$  we need to consider fixed points of  $F + G$ . As every structure of  $F + G$  is either in  $F$  or in  $G$ , it is a fixed point if and only if it is a fixed point in  $F$  or  $G$ , therefore, the number of fixed points of  $F + G$  is exactly the sum of the fixed points of  $F$  and  $G$ , so  $Z_{F+G} = Z_F + Z_G$ .

In the same way the number of structures and their  $\mathfrak{W}_{r,n}$ -orbits sum up and we get the result for the exponential generating function and the type generating function as well.  $\square$

**EXAMPLE 3.3.4.** We consider  $\mathcal{E}_{\leq k}^r = \sum_{n=0}^k \mathcal{E}_n^r$ . The species of  $r$ -sets with a maximum of  $r \cdot k$  elements.

Note that  $\mathcal{E}^r$  can be seen as  $\sum_{n \geq 0} \mathcal{E}_n^r$ , however, we will not discuss this further, for more information about limit values and convergence of species see [BLL98].

**DEFINITION 3.3.5.** We define  $(F \cdot G)[M] = \bigcup_{M_1 \cup M_2 = M, M_1 \cap M_2 = \emptyset, M_1, M_2 \in \mathbb{B}_r} F[M_1] \times G[M_2]$  and the  $\omega$ -action point wise. Therefore an  $(F \cdot G)$ -structure is a tuple of an  $F$ -structure and a  $G$ -structure.

**REMARK 3.3.6.** It follows that  $\cdot$  is associative, commutative (up to isomorphism) and distributive over  $+$ .

The neutral element regarding  $\cdot$  is the species  $1 := \mathcal{E}_0^r$ .

**THEOREM 3.3.7.** For  $F, G$  being  $r$ -species, it holds that  $Z_{F \cdot G} = Z_F \cdot Z_G$  as well as:

- $(F \cdot G)(x) = F(x) \cdot G(x),$
- $(\widetilde{F \cdot G})(x) = \tilde{F}(x) \cdot \tilde{G}(x),$
- $(F \cdot G)^{C_r}(x) = \tilde{F}^{C_r}(x) \cdot \tilde{G}^{C_r}(x),$
- and  $(\widetilde{F \cdot G})^{\mathfrak{S}_\bullet}(x) = \tilde{F}^{\mathfrak{S}_\bullet}(x) \cdot \tilde{G}^{\mathfrak{S}_\bullet}(x).$

**PROOF.** We start with calculating the fixed points of  $(F \cdot G)[\omega]$ : For a fixed point, both the  $F$ -part and the  $G$ -part need to be fixed. Therefore, it is necessary that each cycle of  $\omega$  permutes



only elements that are all contained in either  $F$  or  $G$ . So we can construct every possible fixed point by choosing cycles  $\omega^F = c_1, \dots, c_j$  and define  $\omega^G$  as the rest of the other cycles:

$$Z_{F \cdot G} = \sum_{n, \omega \vdash_r n} |\text{fix } \omega| p_\omega \frac{1}{\prod_{k, \xi} \omega_k(\xi)! (rk)^{\omega_k(\xi)}}$$

As there are  $\binom{\omega_k(\xi)}{\omega_k^F(\xi)}$  possible ways to choose  $\omega_k^F(\xi)$  cycles of length  $k$  and type  $\xi$  for  $F$  we get:

$$= \sum_{n, \omega \vdash_r n} \left( \sum_{\omega^F \omega^G = \omega} |\text{fix } \omega^F| |\text{fix } \omega^G| \prod_{k, \xi} \binom{\omega_k(\xi)}{\omega_k^F(\xi)} \right) p_\omega \frac{1}{\prod_{k, \xi} \omega_k(\xi)! (rk)^{\omega_k(\xi)}}$$

Note that  $\omega_k(\xi) = \omega_k^F(\xi) + \omega_k^G(\xi)$  holds.

$$\begin{aligned} &= \sum_{n, \omega \vdash_r n} \sum_{\omega^F \omega^G = \omega} |\text{fix } \omega^F| |\text{fix } \omega^G| p_{\omega^F} p_{\omega^G} \frac{1}{\prod_{k, \xi} \omega_k^F(\xi)! \omega_k^G(\xi)! (rk)^{\omega_k^F(\xi) + \omega_k^G(\xi)}} \\ &= \left( \sum_{l, \omega^F \vdash_r l} |\text{fix } \omega^F| p_{\omega^F} \frac{1}{\prod_{k, \xi} \omega_k^F(\xi)! (rk)^{\omega_k^F(\xi)}} \right) \cdot \left( \sum_{l, \omega^G \vdash_r l} |\text{fix } \omega^G| p_{\omega^G} \frac{1}{\prod_{k, \xi} \omega_k^G(\xi)! (rk)^{\omega_k^G(\xi)}} \right) \\ &= Z_F \cdot Z_G \end{aligned}$$

□

EXAMPLE 3.3.8 ( $k$ -Faces). We consider the species of  $k$ -faces of an  $n$ -dimensional cube  $\mathcal{F}_{k,n}^r$  (compare with [HLL98]): We already know the 0-faces which are the vertices. Now we can represent a  $k$ -face of a cube as a set of vertices that share  $n - k$  signs, we can write it as a set of  $k$  labels with free signs together with an  $(n - k)$ -dimensional vertex.

For example an edge in a three dimensional cube (in  $\mathcal{F}_{1,3}^2$ ) could be  $\{\{1, 2, -3\}, \{-1, 2, -3\}\}$  but not  $\{\{1, 2, -3\}, \{1, -2, 3\}\}$ . We can represent  $\{\{1, 2, -3\}, \{-1, 2, -3\}\}$  also as  $(\{\pm 1\}, \{2, -3\})$ . So it holds that:

$$\mathcal{F}_{k,n}^r = \mathcal{E}_k^r \cdot \mathcal{V}_{n-k}^r$$

We therefore can calculate the cycle indicator series easily:

$$Z_{\mathcal{F}_{k,n}^r} = Z_{\mathcal{E}_k^r} \cdot Z_{\mathcal{V}_{n-k}^r} = h_k(x) h_{n-k}(1)$$

Furthermore we can calculate the exponential generating series and the type generating series:

$$\mathcal{F}_{k,n}^r(z) = \mathcal{E}_k^r(z) \cdot \mathcal{V}_{n-k}^r(z) = \frac{z^k}{r^k k!} \cdot \frac{z^{n-k}}{(n-k)!} = \frac{z^n}{r^n n!} r^{n-k} \binom{n}{k}$$

$$\tilde{\mathcal{F}}_{k,n}^r(z) = \tilde{\mathcal{E}}_k^r(z) \cdot \tilde{\mathcal{V}}_{n-k}^r(z) = z^k \cdot z^{n-k} = z^n$$

Note that a  $k$ -face of a cube can be associated with an  $(n - k)$ -face with a hyperoctahedron as they are dual (for a graphical example see figure 6)

EXAMPLE 3.3.9 (Fixed Point Free signed Permutations). Consider the Species  $\mathcal{W}_{FPF}^r$  of fixed point free signed permutations. The calculation of its associated series may be difficult to calculate, however we have  $\mathcal{W}^r = \mathcal{E}^r \cdot \mathcal{W}_{FPF}^r$  which makes calculating easier. Here  $\mathcal{W}^r$  is the species of signed permutations as it will be introduced in example 3.4.3.

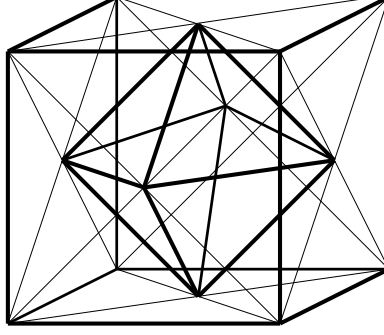


FIGURE 6. A cube and an octahedron.

### 3.4. Examples of $r$ -Species

EXAMPLE 3.4.1 (Type C Parking Functions). For more information about type C parking functions see [ST14].

A type C parking function can be defined as a vector  $f = (f_1, f_2, \dots, f_n)$  where we have  $f_i \in \{-n, -n+1, \dots, -1, 0, 1, 2, \dots, n\}$ . They can also be represented by a labeled lattice path from  $(0, 0)$  to  $(n, n)$  with only north and east steps where the north steps are labeled by  $i_1, i_2, \dots, i_n \in \{\pm 1, \pm 2, \dots, \pm n\}$  with  $|i_k| \neq |i_l|$ ,  $i_1 = |i_1|$  and  $|i_j| > |i_{j+1}|$  if there is no east step between the  $j^{\text{th}}$  and the  $(j+1)^{\text{th}}$  north step. For  $f_l = \pm k$  we know that  $\pm l$  occurs after  $k$  east steps, with this information we can build the path (by sorting numbers in the same row we get the last condition). The  $\omega$ -action is defined by permuting the positions and changing the signs. (Compare with figure 7.) All together we get  $P[n]_{\mathcal{C}_2} = \{(f_1, f_2, \dots, f_n) \text{ with } f_i \in \{-n, -n+1, \dots, -1, 0, 1, 2, \dots, n\}\}$ . For calculating  $Z_P$  we need to consider fixed points: Only numbers with the same absolute value can be interchanged in the vector notation and only numbers in the same row can be mapped to each other in the path notation.

We now consider a cycle with length  $l$  and type 1. As the number of fixed points only depends on the cycle type we only consider the cycle where all signs are 1: A parking function is a fixed point if on all positions of this cycle there is the same number with the same sign. Therefore, we get  $2n+1$  fixed points.

Now we consider a cycle with length  $l$  and type  $-1$ . Once more we only consider particular cycles, those where exactly one  $-1$  occurs as a sign. Under such a permutation exactly one of the signs is changed, so we only get a fixed point if all positions are 0.

Now we can calculate  $Z_P$ :

$$\begin{aligned}
Z_P &= \sum_n \frac{1}{2^n n!} \sum_{\omega \in \mathfrak{W}_{2,n}} (2n+1)^{\sum_{k=1}^n \omega_k(1)} p_\omega \\
&= \sum_n \frac{1}{2^n n!} \sum_{\omega \in \mathfrak{W}_{2,n}} \frac{2^n n!}{\prod_{k=1}^n (2k)^{\omega_k(1) + \omega_k(-1)} \omega_k(1)! \omega_k(-1)!} (2n+1)^{\sum_{k=1}^n \omega_k(1)} p_\omega \\
&= \sum_{n, \omega \in \mathfrak{W}_{2,n}} \frac{(2n+1)^{\sum_{k=1}^n \omega_k(1)}}{\prod_{k=1}^n (2k)^{\omega_k(1) + \omega_k(-1)} \omega_k(1)! \omega_k(-1)!} \prod_{k=1}^n p_k(1)^{\omega_k(1)} p_k(-1)^{\omega_k(-1)} \\
&= \sum_{\lambda, \mu} (2n+1)^{|\lambda|} \frac{p_\lambda(1) \cdot p_\mu(-1)}{\prod_k (2k)^{\mu_k + \lambda_k} \lambda_k! \mu_k!}
\end{aligned}$$

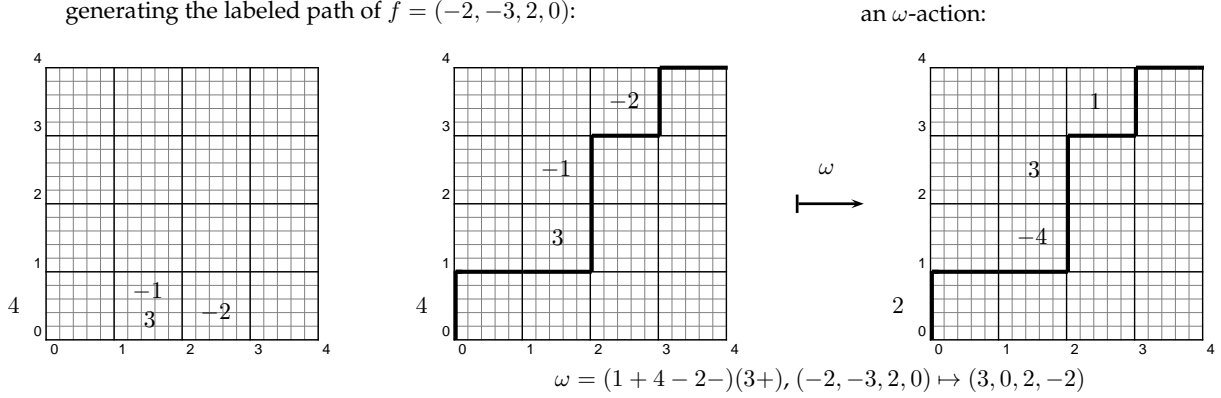


FIGURE 7. A parking function and an  $\omega$ -action.

The exponential generating function is given through:

$$P(x) = \sum_n \frac{z^n}{r^n n!} (2n + 1)^n$$

EXAMPLE 3.4.2 (Diagonals). Similar to the vertices we define the species of diagonals as the diagonals of  $\mathcal{V}^2$  the following way:

$$\mathcal{D}[M_{\mathbf{C}_2}] = \{ \{i_1, i_2, \dots, i_{|M|}\} : i_l \in M_{\mathbf{C}_2}, |i_l \neq i_k| \} / (-M)$$

The  $\mathfrak{W}_{2,n}$ -action is defined point wise. This is well defined as  $-\omega(i) = \omega(-i)$ .

When describing objects of  $\mathcal{D}[n]_{\mathbf{C}_2}$  we use the representative of an equivalence class that contains 1 for describing it.

For example:

$$\mathcal{D}[3]_{\mathbf{C}_2} = \{ \{1, 2, 3\}, \{1, 2, -3\}, \{1, -2, 3\}, \{1, -2, -3\} \}$$

For  $\omega = (1 + 2 - 3+)$  we get:

$$\mathcal{D}[\omega] = (\{1, 2, 3\}\{1, 2, -3\}\{1, -2, 3\}) (\{1, -2, -3\})$$

For a graphical representation see figure 8.

For calculating  $Z_{\mathcal{D}}$  we once again need to consider the fixed points: However there are two different kinds of fixed points: There are those where even the vertices are fixed and those where the vertices of one diagonal are interchanged. The first ones are already analyzed by example 3.2.4. Note that as we have two vertices per diagonal, the number of fixed points is exactly the half of it. The other ones occur under exactly the same signed permutations composed with the mapping  $\psi : i \mapsto -i \forall i$ . Composing a cycle  $c$  with  $\psi$  simply changes all the signs (in both notations). Therefore, when the number of elements is even, the cycle type will be the same, and if it is odd, it will have the same length but a different (as there are only two, the other) sign:

For example  $(1 + 2 + 3+)$  has cycle type  $(0, 0; 0, 0; 1, 0)$ , this goes composed with  $\psi$  to  $(1 - 2 - 3-)$  and has now cycle type  $(0, 0; 0, 0; 0, 1)$  and  $(1 + 2+)$  goes to  $(1 - 2-)$  where both mappings have cycle type  $(0, 0; 1, 0)$ .

Therefore, we get the following numbers of fixed points: For the first type we get:

$$\frac{1}{2} \prod_{k=1}^n 2^{\omega_k(1)}$$

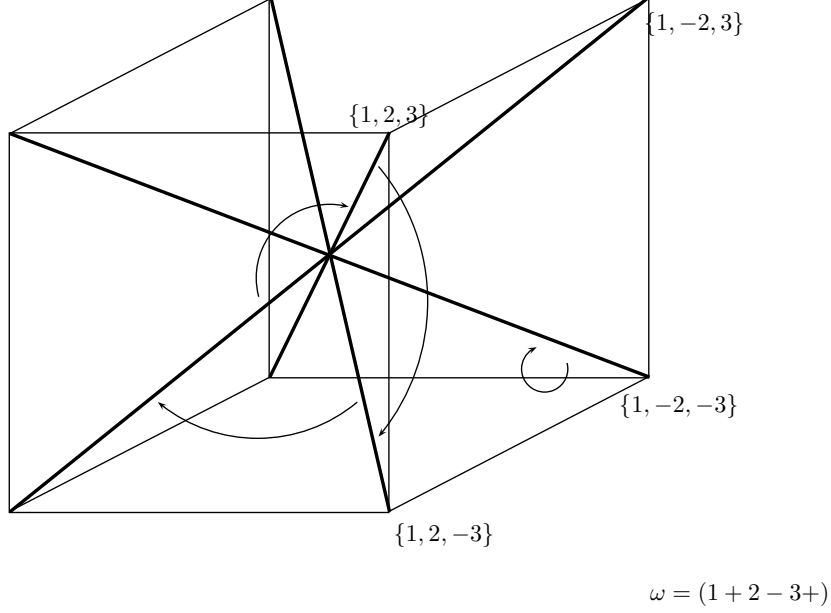


FIGURE 8. Diagonals in a 3-dimensional space and an  $\omega$ -action.

for all  $\omega$  consisting of only cycles with type 1 and for the second we get:

$$\frac{1}{2} \prod_{k=1}^n 2^{\omega_k((-1)^k)}$$

for all  $\omega$  consisting of only even-length cycles with type 1 and odd-length cycles with type  $-1$ . Now we can calculate  $Z_{\mathcal{D}[n]_{\mathcal{C}_2}}$ :

$$Z_{\mathcal{D}} = \sum_n \frac{1}{2^n n!} \frac{1}{2} \left( \left( \sum_{\substack{\omega \in \mathfrak{W}_{r,n} \\ p_\omega = \prod_k (p_k(1))^{\omega_k(1)}}} \prod_{k=1}^n 2^{\omega_k(1)} p_\omega \right) + \left( \sum_{\substack{\omega \in \mathfrak{W}_{r,n} \\ p_\omega = \prod_k (p_k((-1)^k))^{\omega_k((-1)^k)}}} \prod_{k=1}^n 2^{\omega_k((-1)^k)} p_\omega \right) \right)$$

We now only consider the cycle type of  $\omega$ . Through the even/odd condition all we need to know is how many cycles of  $\omega$  with length  $k$  there are. Therefore, we consider  $\omega \vdash n$ . Note that the second product may arise from a different  $\omega$ :

$$= \sum_n \frac{1}{2^n n!} \frac{1}{2} \sum_{\omega \vdash n} \frac{n! 2^n}{\prod_{k=1}^n (2k)^{\omega_k} \omega_k!} \left( \left( \prod_{k=1}^n 2^{\omega_k} p_k(1)^{\omega_k} \right) + \left( \prod_{k=1}^n 2^{\omega_k} p_k((-1)^k)^{\omega_k} \right) \right)$$

Now we use  $p_k(1) = p_k(x) + p_k(y)$  and  $p_k(-1) = p_k(x) - p_k(y)$  (compare with Remark 2.3.10), and cancel some terms:

$$= \sum_n \frac{1}{2} \sum_{\omega \vdash n} \frac{1}{\prod_{k=1}^n (k)^{\omega_k} \omega_k!} \left( \left( \prod_{k=1}^n (p_k(x) + p_k(y))^{\omega_k} \right) + \left( \prod_{k=1}^n (p_k(x) + (-1)^k p_k(y))^{\omega_k} \right) \right)$$

$$= \frac{1}{2} \left( \left( \sum_{n, \omega \vdash n} \prod_{k=1}^n \frac{1}{\omega_k!} \left( \frac{p_k(x) + p_k(y)}{k} \right)^{\omega_k} \right) + \left( \sum_{n, \omega \vdash n} \prod_{k=1}^n \frac{1}{\omega_k!} \left( \frac{p_k(x) + (-1)^k p_k(y)}{k} \right)^{\omega_k} \right) \right)$$

Now we use the same calculations for both sums as we used for  $r$ -sets (example 3.2.3) and get:

$$\begin{aligned} &= \frac{1}{2} \left( \exp \left( \sum_k (p_k(x) + p_k(y)) \right) + \exp \left( \sum_k (p_k(x) + (-1)^k p_k(y)) \right) \right) \\ &= \frac{1}{2} \left( \exp \left( \sum_k p_k(x) \right) \cdot \exp \left( \sum_k p_k(y) \right) + \exp \left( \sum_k p_k(x) \right) \cdot \exp \left( \sum_k (-1)^k p_k(y) \right) \right) \end{aligned}$$

Once again we use the same calculations as we used for  $r$ -sets (example 3.2.3) and get the following. We therefore use  $(-1)^k p_k(y) = p_k(-y)$  and  $(-1)^k h_k(y) = h_k(-y)$ :

$$\begin{aligned} &= \frac{1}{2} \left( \left( \sum_k h_k(x) \right) \left( \sum_k h_k(y) \right) + (-1)^k \left( \sum_k h_k(x) \right) \left( \sum_k h_k(y) \right) \right) \\ &= \frac{1}{2} \left( \left( \sum_n \sum_k h_{n-k}(x) h_k(y) \right) + (-1)^k \left( \sum_n \sum_k h_{n-k}(x) h_k(y) \right) \right) \\ &= \sum_n \sum_k h_{n-2k}(x) h_{2k}(y) \end{aligned}$$

The specializations for diagonals are:

- (1)  $\mathcal{D}(z) = \sum_n \frac{z^n}{rn!}$
- (2)  $\tilde{\mathcal{D}}(z) = \sum_n z^n$
- (3)  $\tilde{\mathcal{D}}^{\mathbf{C}^r}(z) = \sum_n \frac{z^n}{n!}$
- (4)  $\tilde{\mathcal{D}}^{\mathfrak{S}^\bullet}(z) = \sum \lceil \frac{n+1}{2} \rceil \frac{z^n}{r^n}$

EXAMPLE 3.4.3 (Signed Permutations). Note that we also call them  $r$ -permutations. We consider  $\mathfrak{W}_{r,n}$  as an example of an  $r$ -species:

$$\mathcal{W}^r[n]_{\mathbf{C}^r} = \mathfrak{W}_{r,n}, \quad \mathcal{W}^r[\omega](\tau) = \omega \circ \tau \circ \omega^{-1}$$

If we use the cycle notation we can interpret a  $\mathcal{W}^r$ -structure as a set of  $\mathbf{C}^r$ -structures, what we will do later when we consider substitution of  $r$ -species.

Now we are interested in the cycle indicator series, so we need to consider fixed points under an  $\omega \in \mathfrak{W}_{r,n}$ : For a  $\tau \in \mathcal{W}^r$  being a fixed point, it is necessary that all elements of one cycle are mapped to the elements of another (or the same) cycle with same size, therefore, elements in one cycle of  $\tau$  need to be in a cycle of  $\omega$  with same length. Analogous as in  $\mathbf{C}^r$ , we need the same types for cycles of  $\omega$  that occur in such of  $\tau$ .

We now construct any possible fixed point of  $\omega$ : We consider the elements of  $\omega$ -cycles with length  $l$  and type  $\xi$  (elements of other  $\omega$ -cycles need to be in different  $\tau$ -cycles), their number is  $\omega_l(\xi)$ :

Therefore, we sort the  $\omega$ -cycles by their smallest element and call these elements  $e_1, e_2, \dots, e_{\omega_l(1)}$ . Then we start with element  $e_1$ , choose the first element of the  $\omega$ -cycle of  $e_1$  that occurs in the  $\tau$ -cycle of  $e_1$ , and in a second step we choose the sign  $\xi_{e_1}$ . There are  $l \cdot r$  possible choices (the  $l$ -st possibility is the one of  $e_1$  being a fixed point so far) and therewith the positions (and their signs) in  $\tau$  of the other elements of the  $\omega$ -cycle of  $e_1$  are determined. (Compare with the construction of fixed points in  $\mathbf{C}^r$ .) Note that if this element is  $k$  positions in the  $\omega$ -cycle away from  $e_1$ , the elements will be in  $\gcd(l, k)$  cycles.

Now consider  $e_2$ :  $e_2$  can either be in a new cycle ore in one of the cycles already constructed by

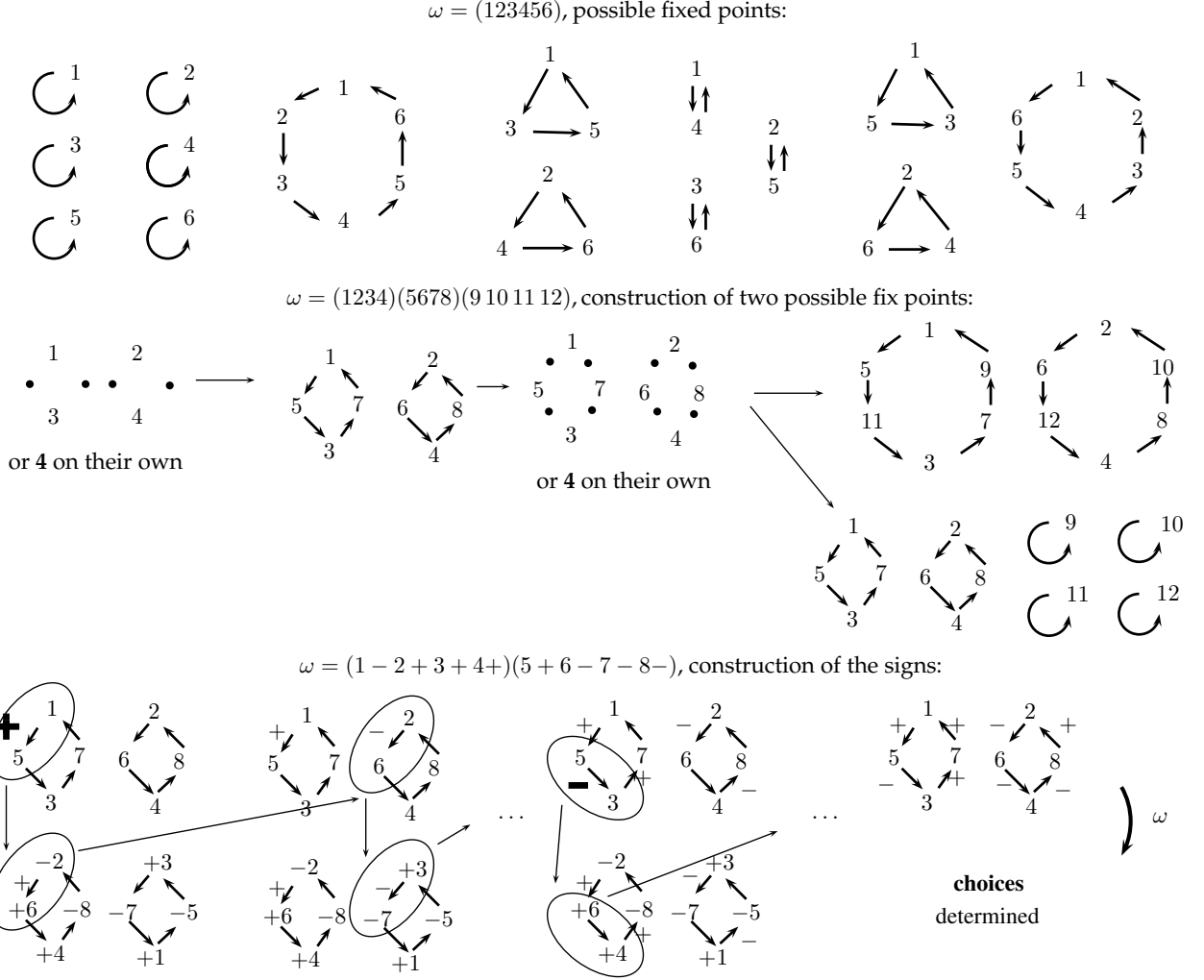


FIGURE 9. Examples of constructing fixed  $r$ -permutations.

$e_1$ , in the first case, we get, like before,  $l \cdot r$  possible choices, in the second case we can choose the element of the  $e_1$ - $\omega$ -cycle of which  $e_2$  is the nearest follower as well its sign. The rest is determined. All together we get  $2lr$  possibilities.

For  $e_3$ , we once more get  $l \cdot r$  possibilities for a new cycle, and  $2lr$  for an already constructed one ( $lr$  for being the follower of an  $e_1$ -cycle element and  $lr$  for those of the  $e_2$ -cycle, no matter if they share cycle or not), thus all together  $3lr$  possibilities.

Iterating this gives us  $\omega_l(\xi)!l^{\omega_l(\xi)}r^{\omega_l(\xi)}$ . Therefore, we get  $\prod_l \omega_l(\xi)!l^{\omega_l(\xi)}r^{\omega_l(\xi)}$  fixed points. (For graphical examples see figure 9.)

Now we can calculate  $Z_{\mathcal{W}^r}$ :

$$\begin{aligned}
 Z_{\mathcal{W}^r} &= \sum_{m \geq 0} \frac{1}{r^m m!} \sum_{\omega \in \mathbb{W}_{r,n}} \prod_l \omega_l(\xi)!l^{\omega_l(\xi)}r^{\omega_l(\xi)} p_\omega \\
 &= \sum_{m \geq 0} \frac{1}{r^m m!} \sum_{\omega \vdash_r m} \prod_{\xi, l} \omega_l(\xi)!l^{\omega_l(\xi)}r^{\omega_l(\xi)} p_\omega \frac{m!r^m}{\prod_{l, \xi} (lr)^{\omega_l(\xi)} \omega_l(\xi)!}
 \end{aligned}$$

$$= \sum_{m \geq 0, \omega \vdash m} p_\omega$$

The exponential generating function is  $\mathcal{W}^r(Z) = \sum_n z^n$  and the type generating function  $\tilde{\mathcal{W}}^r(Z) = \sum_n z^n |\{w \vdash_r n\}|$ .

**3.4.1. Set Partitions.** We define two new kinds of set partitions we will need later as  $r$ -species.

DEFINITION 3.4.4. We define if  $r = 1$  and an ordinary set  $S$  and  $\text{Par}[S]$  as the species of all set partitions of  $S$ . For general  $r$  we define two kinds of partitions for a set  $M \in \mathbb{B}_r$ .

- (1) We define  $\text{Par}_{\mathbf{C}_r}[M]$  as the set of all partitions, where the sets are stable under the  $\mathbf{C}_r$ -action. Therefore, for a block  $b$  of  $\pi$   $x \in b, j \in \mathbb{N} \Rightarrow \zeta^j x \in b$ .
- (2) We define  $\text{Par}_{\mathbb{B}_r}[M]$  as the set of all partitions  $\pi$  which are preserved under the  $\mathbf{C}_r$ -action and where  $\pi$  as an object of  $\mathbb{B}_r$ . In other words, for every set  $N \in \pi$  the set  $\zeta \cdot N$  is also in  $\pi$ .

EXAMPLE 3.4.5. Let  $M = 5\mathbf{C}_3$ . A partition of  $[5]$  is for example  $\{\{1, 3\}, \{2, 4, 5\}\}$ .

- (1) A partition of  $M$  in  $\text{Par}_{\mathbf{C}_3}[M]$  is for example:

$$\{\{1, \zeta 1, \zeta^2 1, 3, \zeta 3, \zeta^2 3\}, \{2, \zeta 2, \zeta^2 2, 4, \zeta 4, \zeta^2 4, 5, \zeta 5, \zeta^2 5\}\}.$$

- (2) A partition of  $M$  in  $\text{Par}_{\mathbb{B}_3}[M]$  is for example:

$$\{\{1, \zeta 3\}, \{\zeta 1, \zeta^2 3\}, \{\zeta^2 1, 3\}, \{2, \zeta^2 4, \zeta^2 5\}, \{\zeta 2, 4, 5\}, \{\zeta^2 2, \zeta 4, \zeta 5\}\}.$$

As the previous example motivates and the following lemma shows, there is a natural bijection between  $\text{Par}[S]$  and  $\text{Par}_{\mathbf{C}_r}[S]$ , and that there is a strong connection between  $\text{Par}[S]$  and  $\text{Par}_{\mathbb{B}_r}[S]$ :

LEMMA 3.4.6. *Let  $S$  be an ordinary set, so that  $S_{\mathbf{C}_r} = M$  and let  $\pi$  be a set partition of  $S$ , then every partition of one of the two kinds of set partitions can be constructed with the help of  $\pi$ :*

- (1) We define  $\tilde{\pi} \in \text{Par}_{\mathbf{C}_r}[M]$  as the set of all sets  $b_{\mathbf{C}_r}$  with  $b \in \pi$ .
- (2) We choose arbitrary signs  $\xi_x \in \mathbf{C}_r$  for all  $x \in S$  and define  $\tilde{\pi} \in \text{Par}_{\mathbb{B}_r}[M]$  as the set of all  $\{\xi_x \cdot \zeta^i x : x \in b\}$  with  $b \in \pi, i = 0, 1, 2, \dots, (r-1)$ .

PROOF. We need to show that our construction leads indeed to such a partition and that every such partition can be constructed this way:

- (1) Any  $b_{\mathbf{C}_r}$  is by definition stable under the  $\mathbf{C}_r$ -action.

For a given  $\tilde{\pi} \in \text{Par}_{\mathbf{C}_r}[M]$ , we can define  $\pi$  as the set of all  $\{|x| : x \in b\}$ . As for  $|x| = |y|$  holds that  $\exists \xi \in \mathbf{C}_r : x = \xi y$ , and therefore,  $x$  and  $y$  are in the same block, this leads to a partition of  $S$ . ( $|x|$  is here the element of  $\{\xi x : \xi \in \mathbf{C}_r\}$  with sign 1.)

- (2) Any  $\{\xi_x \cdot \zeta^i x : x \in b\}$  under the action of  $\zeta^j \in \mathbf{C}_r$  is  $\{\xi_x \cdot \zeta^{i+j} x : x \in b\}$ , and therefore, it is still in the partition, and our partition is a  $\mathbb{B}_r$ -set.

For a given  $\tilde{\pi} \in \text{Par}_{\mathbf{C}_r}[M]$ , we can once more define  $\pi$  as the set of all  $\{|x| : x \in b\}$ . (Note that some of the  $b$  will define the same set, but as it is a set of sets this does not matter.)

□

DEFINITION 3.4.7. We define  $n_\pi$  as:

- (1)  $|\pi|$  if  $\pi \in \text{Par}[M]$  or  $\pi \in \text{Par}_{\mathbf{C}_r}[M]$ ;
- (2)  $\frac{|\pi|}{r}$  if  $\pi \in \text{Par}_{\mathbb{B}_r}[M]$ ,

and the sizes of the blocks  $b$  and the number of blocks with that size as:

- (1)  $|b|$  and their number if  $\pi \in \text{Par}[M]$ ,
- (2)  $\frac{|b|}{r}$  and their number if  $\pi \in \text{Par}_{\mathbf{C}_r}[M]$ ,

(3)  $|b|$  and their actual number times  $\frac{1}{r}$  if  $\pi \in \text{Par}_{\mathbb{B}_r}[M]$ .



## Substitution and Plethysm

### 4.1. Substitution

For the substitution of ordinary species there are three different kinds of generalizations, two of them (the first two ones), which are both between an ordinary and an  $r$ -species and where we know how the corresponding cycle index series behaves, can be found in [Hen04]. The third substitution can be found in [HLL98] and is between two  $r$ -species.

DEFINITION 4.1.1 (Type 1). Let  $F$  be an ordinary (1-)species and  $G$  an  $r$ -species. We define:

$$(F \circ G)[M] = \sum_{\pi \in \text{Par}_{\mathbf{C}_r}[M]} (F[\pi] \times \prod_{N \in \pi} G[N]).$$

The definition on morphisms is natural and will be formulated later.

EXAMPLE 4.1.2 (Signed Permutations). We want to analyze  $\mathcal{E} \circ \mathcal{C}^r$ :

Therefore, we need to consider an  $\mathcal{E} \circ \mathcal{C}^r$  structure on a set  $M$ . This structure is a tuple  $(\pi, f, g = (g_b)_{b \in \pi})$  where  $\pi$  is in  $\text{Par}_{\mathbf{C}_r}[M]$ ,  $f$  is the set of elements of  $\pi$  and therefore  $\pi$  itself, and the  $g_b$  are cycles of elements that are in one set of  $\pi$ . In other words we have a set of cycles, which is a signed permutation.

To see that  $\mathcal{W}^r = \mathcal{E} \circ \mathcal{C}^r$ , we consider a  $\mathcal{W}^r$ -structure  $w$  on a set  $M$ . This structure is a signed permutation, so it consists of various cycles  $c_1, c_2, \dots, c_k$ . A cycle  $c_i$  can be written as  $(e_{i,1} \xi_{i,1} e_{i,2} \xi_{i,2} \dots e_{i,j_i} \xi_{i,j_i})$ .

We now state an isomorphism  $\Phi : \mathcal{W}^r \rightarrow \mathcal{E} \circ \mathcal{C}^r$ :

Now  $\Phi(w) = (\pi_w, f_w, g_w)$  where:

- $\pi_w = \{\{e_{i,1}, e_{i,2}, \dots, e_{i,j_i}\}_{\mathbf{C}_r} : i = 1, 2, \dots, k\}$
- $f_w = \{\{e_{i,1}, e_{i,2}, \dots, e_{i,j_i}\}_{\mathbf{C}_r} : i = 1, 2, \dots, k\}$
- $g_w = \{c_1, c_2, \dots, c_k\}$

That  $\Phi$  is indeed an isomorphism (bijective, compatible with the  $\mathfrak{W}_{r,n}$ -action) can be seen easily by recalculating.

For a graphical example see figure 1. Detailed information about the  $\mathfrak{W}_{r,n}$ -action will be given later.

$$\begin{aligned} & \Phi((1\zeta^2 3\zeta^2)(2\zeta 5\zeta 4\zeta^0)) = (\pi, f, g) \text{ with:} \\ & f = \{\{1, \zeta 1, \zeta^2 1, 3, \zeta 3, \zeta^2 3\}, \{2, \zeta 2, \zeta^2 2, 4, \zeta 4, \zeta^2 4, 5, \zeta 5, \zeta^2 5\}\} \\ & g = \left\{ \begin{array}{cc} \begin{array}{ccc} & 2 & \\ \zeta \swarrow & & \searrow 1 \\ 5 & \longrightarrow & 4 \\ & \zeta & \end{array} & \begin{array}{ccc} \zeta^2 & \downarrow \uparrow & \zeta^2 \\ & 3 & \end{array} \end{array} \right\} \end{aligned}$$

FIGURE 1. A  $\mathcal{W}^r$ -structure and the associated  $\mathcal{E} \circ \mathcal{C}^r$ -structure.

$$\pi = \{\{1, \zeta 3\}, \{\zeta 1, \zeta^2 3\}, \{\zeta^2 1, 3\}, \{2, \zeta^2 4, \zeta^2 5\}, \{\zeta 2, 4, 5\}, \{\zeta^2 2, \zeta 4, \zeta 5\}\}$$

$$f = \{\{\zeta^2 1, 3\}, \{\zeta 2, 4, 5\}\}$$

$$g = \left\{ \begin{array}{ccc} & 2 & \\ & \swarrow \quad \searrow & \\ 5 & \longrightarrow & 4 \\ & & \\ & & \begin{array}{c} 1 \\ \downarrow \uparrow \\ 3 \end{array} \end{array} \right\}$$

FIGURE 2. A  $\mathcal{V}^r \circ \mathcal{C}$ -structure.

DEFINITION 4.1.3 (Type 2). Let  $F$  be an  $r$ -species and  $G$  an ordinary (1)-species. We define:

$$(F \circ G)[M] = \sum_{\pi \in \text{Par}_{\mathbb{B}_r}[M]} (F[\pi] \times \prod_{\mathcal{O} \in \pi_{\mathbf{C}_r}} G[\mathcal{O}])$$

where:

- $\mathcal{O} \in \pi_{\mathbf{C}_r}$  are the orbits of the  $\mathbf{C}_r$ -action in  $\pi$ .
- $G[\mathcal{O}] := \{(g_N)_{N \in \mathcal{O}} : b_\xi \dot{N} = G[\xi](b_N), \forall N \in \mathcal{O}, \xi \in \mathbf{C}_r\}$ .

The definition on morphisms is natural and will be formulated later.

EXAMPLE 4.1.4 (Vertices of Cycles). We analyze  $\mathcal{V}^r \circ \mathcal{C}$ :

A structure of  $\mathcal{V}^r \circ \mathcal{C}[n]_{\mathbf{C}_r}$  is a tuple  $(\pi, f, (g_N)_{N \in \mathcal{O}})$  where:

- $\pi \in \text{Par}_{\mathbb{B}_r}[[n]_{\mathbf{C}_r}]$
- $f \in \mathcal{V}^r[\pi]$
- $g_N = \mathcal{C}[N]$

An example for a structure  $h$  in  $F \circ G[5]_{\mathbf{C}_3}$  is the tuple  $h = (\pi, f, g = (g_b)_{b \in \pi})$  where

- $\pi = \{\{1, \zeta 3\}, \{\zeta 1, \zeta^2 3\}, \{\zeta^2 1, 3\}, \{2, \zeta^2 4, \zeta^2 5\}, \{\zeta 2, 4, 5\}, \{\zeta^2 2, \zeta 4, \zeta 5\}\}$
- $f = \{\{\zeta^2 1, 3\}, \{\zeta 2, 4, 5\}\}$
- $g = \{(254), (13)\}$

Note, that for shorter notation we write here  $\{(13)\}$  instead of  $\{(1\zeta^0 3\zeta), (1\zeta 3\zeta^2), (1\zeta^2 3\zeta^0)\}$  as we can construct the latter with help of  $\pi$  and  $f$ .

Note that this  $r$ -species is not the same as  $\mathcal{W}^r$ , even though one can associate these structures. Nevertheless, the  $\mathfrak{W}_{r,n}$ -action is different as we will see later on. Detailed information about the  $\mathfrak{W}_{r,n}$ -action will be given later.

For a graphical example see figure 2.

The third substitution can be found in [HLL98] and is between two  $r$ -species. Note that [HLL98] considers only the case of hyperoctahedral species ( $r = 2$ ). His definition, however, can be easily generalized:

DEFINITION 4.1.5 (Type 3). Let  $F$  and  $G$  be  $r$ -species. We define:

$$(F \circ G)[M_{\mathbf{C}_r}] = \sum_{\pi \in \text{Par}[M]} (F[\pi_{\mathbf{C}_r}] \times \prod_{N \in \pi} G[N_{\mathbf{C}_r}]).$$

The definition on morphisms is natural and will be formulated later.

EXAMPLE 4.1.6 (Vertices of Signed Cycles). We analyze  $\mathcal{V}^r \circ \mathcal{C}^r$  (this species can be seen as signed permutations that have an extra sign):

A structure of  $\mathcal{V}^r \circ \mathcal{C}^r[n]_{\mathbf{C}_r}$  is a tuple  $(\pi, f, (g_b)_{b \in \pi})$  where:

- $\pi \in \text{Par}[[n]]$

$$\pi = \{\{1, 3\}, \{2, 4, 5\}\}$$

$$f = \{\zeta^2\{1, 3\}, \{2, 4, 5\}\} \quad g = \left\{ \begin{array}{ccc} & 2 & \\ \zeta \swarrow & & \searrow 1 \\ 5 & \longrightarrow & 4 \\ & \zeta & \end{array} \quad \zeta^2 \begin{array}{c} 1 \\ \downarrow \uparrow \\ 3 \end{array} \zeta^2 \right\}$$

FIGURE 3. A  $\mathcal{V}^r \circ \mathcal{C}$ -structure.

- $f \in \mathcal{V}^r[\pi_{\mathbf{C}_r}]$
- $g_b = \mathcal{C}^r[b_{\mathbf{C}_r}]$

An example for a structure  $h$  in  $\mathcal{V}^r \circ \mathcal{C}^r$  is the tuple  $h = (\pi, f, g = (g_b)_{b \in \pi})$  where:

- $\pi = \{\{1, 3\}, \{2, 4, 5\}\}$
- $f = \{\zeta^2\{1, 3\}, \{2, 4, 5\}\}$
- $g = \{(2\zeta 5\zeta 4\zeta^0 2), (1\zeta^2 3\zeta^2)\}$

Detailed information about the  $\mathfrak{W}_{r,n}$ -action will come later.

For a graphical example see figure 3.

LEMMA 4.1.7. *Any substitution of type 1 and 2 is associative in that sense that  $(F \circ G) \circ H = F \circ (G \circ H)$  holds no matter what type(s) of substitution we have (up to isomorphism).*

PROOF. We proof it by example for the case that  $F, G$  are normal species and  $H$  is an  $r$ -species and only for the set of structures. That it also holds for the morphisms will be clear when defining them precisely. The other cases are analogous.

$$\begin{aligned} (F \circ (G \circ H))[M] &= \sum_{\pi_1 \in \text{Par}_{\mathbf{C}_r}[M]} F[\pi_1] \prod_{b_1 \in \pi_1} (G \circ H)[b_1] \\ &= \sum_{\pi_1 \in \text{Par}_{\mathbf{C}_r}[M]} F[\pi_1] \prod_{b_1 \in \pi_1} \left( \sum_{\pi_2 \in \text{Par}_{\mathbf{C}_r}[b_1]} G[\pi_2] \prod_{b_2 \in \pi_2} H[b_2] \right) \\ &= \sum_{\pi_1 \in \text{Par}_{\mathbf{C}_r}[M]} \sum_{\pi_2 \in \text{Par}_{\mathbf{C}_r}[b_1]} F[\pi_1] \prod_{b_1 \in \pi_1} G[\pi_2] \prod_{b_2 \in \pi_2} H[b_2] \end{aligned}$$

Now  $\pi_1$  consists of  $\mathbf{C}_r$ -parts of  $M$  and  $\pi_2$  of parts of these. We define  $\pi_3$  as union of  $\pi_2$  that come from the same  $\pi_1$  and get  $\pi_3 \in \text{Par}_{\mathbf{C}_r}[M]$ . We define  $\pi_4$  as partition of  $\pi_3$  in that way that the union of the parts are the parts of  $\pi_1$ . Then  $F[\pi_1] \cong F[\pi_4]$  and the former  $\pi_2$  are now the parts of  $\pi_4$ . (For a graphical example see figure 4.) Then we get:

$$\begin{aligned} &= \sum_{\pi_3 \in \text{Par}_{\mathbf{C}_r}[M]} \left( \sum_{\pi_4 \in \text{Par}[\pi_3]} F[\pi_4] \prod_{b_4 \in \pi_4} G[b_4] \right) \prod_{b_3 \in \pi_3} H[b_3] \\ &= \sum_{\pi_3 \in \text{Par}_{\mathbf{C}_r}[M]} (F \circ G)[\pi_3] \prod_{b_3 \in \pi_3} H[b_3] \\ &= ((F \circ G) \circ H)[M] \end{aligned}$$

□

REMARK 4.1.8.

- $X := \mathcal{E}_1^1$  is the neutral element regarding  $\circ$ .

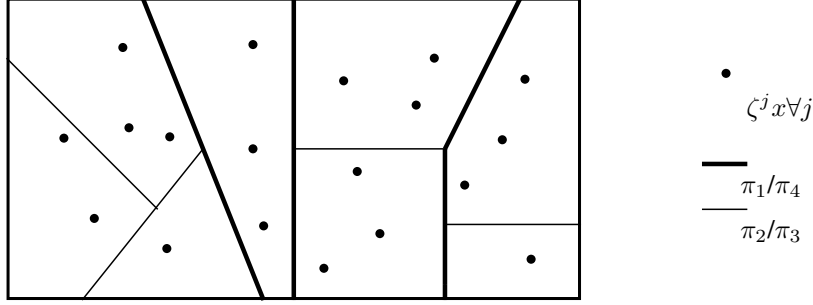


FIGURE 4. Partitions of  $F \circ G \circ H$ .

- $\circ$  is linear in the first argument.
- For any ordinary species  $F$  it holds that  $\mathcal{V}_1^r \circ F = F \circ \mathcal{V}_1^r$ .
- We have  $\Delta(G \circ F) = G \circ \Delta F$  whenever  $G$  is an ordinary species and  $F$  an  $r$ -species but not necessarily  $\Delta(G \circ F) = \Delta G \circ F$  whenever  $G$  is an  $r$ -species and  $G$  and ordinary species.

In the case  $r = 1$  the literature (see for example in [BLL98]) tells us that not only the following lemma for the exponential generating series holds, but also a similar result for the cycle indicator series, to which we will come later.

**THEOREM 4.1.9.** *For  $F(x), G(x)$  generating functions of  $F, G$  being ( $r$ -)species, it holds that*

- $(F \circ G)(x) = F(G(x))$  in type 1 or type 2
- and  $(F \circ G)(x) = F(rG(x))$  in type 3.

**REMARK 4.1.10.** Note that the third equation is new. ([HLL98] defines the exponential generating function in a slightly different way (skipping  $\frac{1}{r^n}$ ) and does not consider cycle indicator series.)

**EXAMPLE 4.1.11 (Signed Permutations).** We want to show that  $\mathcal{E}(\mathcal{C}^r(x)) = \mathcal{W}^r(x)$ : Therefore, we need the exponential generating functions (compare with theorem 3.2.7):

- (1) We start with the species of sets:

$$\mathcal{E}(x) = \exp \left( \sum_{k \geq 0} \frac{p_k}{k} \right)_{|p_1(1)=x, p_i(\xi)=0} = \exp(x)$$

which states that there is exactly one set for each size in  $\mathcal{E}$  which is indeed true.

- (2) For the signed cycles we already know (example 3.2.9) that:

$$\mathcal{C}^r(x) = \sum_n \frac{z^n}{n}$$

- (3) For the signed permutations we get:

$$\mathcal{W}^r(x) = \sum_{n, \omega \vdash_r n} p_{\omega|p_1(1)=x, p_i(\xi)=0} = \sum_n x^n$$

which states that we have  $n!2^n$  signed permutations, which is indeed true.

Now we can calculate  $\mathcal{E}(\mathcal{C}^r(x))$ :

$$\mathcal{E}(\mathcal{C}^r(x)) = \exp \left( \sum_n \frac{x^n}{n} \right) = \exp(-\log(1-x)) = \frac{1}{1-x} = \sum_n x^n = \mathcal{W}^r(x)$$

PROOF OF THEOREM 4.1.9. Although this will follow immediately out of the theorems about the cycle indicator series (4.3.1, 4.3.2, 4.4.21) and through the fact, that the exponential generating function is a specialization of it (subsection 3.2.1), we will proof this directly by counting the number of  $F \circ G$ -structures.

Let  $F(x) = \sum f_k x^k \frac{1}{k!r^k}$  (and  $r = 1$  at type 1) and  $G(x) = \sum g_k x^k \frac{1}{k!r^k}$  (and  $r = 1$  at type 2). Let further be  $j_i$  the number of blocks with size  $i$  as defined before. For a fixed partition  $\pi$  of  $[k]$  or  $[k]_{\mathbb{C}_r}$ , depending on the case the number, of such structures is  $f_k (g_1)^{j_1} (g_2)^{j_2} \dots (g_m)^{j_m}$ . These structures have size  $m = \sum i j_i$  (note that  $k = \sum j_i$ ).

The number of partitions with  $j_i$  blocks of size  $i$  is (compare with lemma 4.3.25):

- Type 1:  $\frac{m!}{\prod_i (i!)^{j_i} j_i!}$
- Type 2:  $\frac{m!}{\prod_i (i!)^{j_i} j_i!} \cdot \prod_i r^{(i-1) \cdot j_i} = \frac{m!}{\prod_i (i!)^{j_i} j_i!} \cdot r^{m-k}$
- Type 3:  $\frac{m!}{\prod_i (i!)^{j_i} j_i!}$

Therefore, the term with  $x^m$  is:

- Type 1:  $f_k (g_1)^{j_1} (g_2)^{j_2} \dots (g_m)^{j_m} \frac{m!}{\prod_i (i!)^{j_i} j_i!} \frac{x^m}{m!r^m}$
- Type 2:  $f_k (g_1)^{j_1} (g_2)^{j_2} \dots (g_m)^{j_m} \frac{m!}{\prod_i (i!)^{j_i} j_i!} \cdot r^{m-k} \frac{x^m}{m!r^m}$
- Type 3:  $f_k (g_1)^{j_1} (g_2)^{j_2} \dots (g_m)^{j_m} \frac{m!}{\prod_i (i!)^{j_i} j_i!} \frac{x^m}{m!r^m}$

Now we calculate  $(F(G(x)))$  for type 1 or type 2 and  $F(rG(x))$  for type 3:

- Type 1:

$$F(G(x)) = \sum_k \frac{f_k}{k!} \left( \sum_l g_l \frac{x^l}{r^l l!} \right)^k = \sum_k \frac{f_k}{k!} \sum_{\sum j_i = k} \frac{k!}{\prod_i j_i!} g_1^{j_1} \dots g_m^{j_m} \frac{x^m}{r^m \prod_i i!^{j_i}}$$

- Type 2:

$$F(G(x)) = \sum_k \frac{f_k}{k!r^k} \left( \sum_l g_l \frac{x^l}{l!} \right)^k = \sum_k \frac{f_k}{k!r^k} \sum_{\sum j_i = k} \frac{k!}{\prod_i j_i!} g_1^{j_1} \dots g_m^{j_m} \frac{x^m}{\prod_i i!^{j_i}}$$

- Type 3:

$$F(rG(x)) = \sum_k \frac{f_k}{k!r^k} \left( r \sum_l g_l \frac{x^l}{l!r^l} \right)^k = \sum_k \frac{f_k}{k!r^k} r^k \sum_{\sum j_i = k} \frac{k!}{\prod_i j_i!} g_1^{j_1} \dots g_m^{j_m} \frac{x^m}{r^m \prod_i i!^{j_i}}$$

which is the same. □

## 4.2. Plethysm

As in the case of ordinary (1-)species, there are plethystic operations for the cycle index series of substituted species for the two cases which can be found in [Hen04]:

[Hen04] claims that it is not possible to define a plethysm  $\circ : \Lambda(r) \times \Lambda(r) \rightarrow \Lambda(r)$ , however, he defines a plethysm  $\circ : \Lambda(1) \times \Lambda(r) \rightarrow \Lambda(r)$  and one  $\circ : \Lambda(r) \times \Lambda(1) \rightarrow \Lambda(r)$ :

DEFINITION 4.2.1 (Type 1). We define  $\circ : \Lambda(1) \times \Lambda(r) \rightarrow \Lambda(r)$  uniquely as follows:

- (1)  $\forall g \in \Lambda(r)$ , the map  $\Lambda(1) \rightarrow \Lambda(r) : f \mapsto f \circ g$  is a  $\mathbb{C}$ -algebra homomorphism:  
 $(\forall g \in \Lambda(r), f_1, f_2 \in \Lambda(1): (f_1 + f_2) \circ g = f_1 \circ g + f_2 \circ g$  and  $(f_1 \cdot f_2) \circ g = (f_1 \circ g) \cdot (f_2 \circ g))$
- (2)  $\forall i \in \mathbb{N}$ , the map  $\Lambda(r) \rightarrow \Lambda(r) : g \mapsto p_i \circ g$  is a  $\mathbb{C}$ -algebra homomorphism:  
 $(\forall i \in \mathbb{N}, g_1, g_2 \in \Lambda(r): p_i \circ (g_1 + g_2) = p_i \circ g_1 + p_i \circ g_2$  and  $p_i \circ (g_1 \cdot g_2) = (p_i \circ g_1) \cdot (p_i \circ g_2))$
- (3)  $p_i \circ p_j(\xi) = p_{ij}(\xi)$

This can be defined in another way (compare [BLL98]):

LEMMA 4.2.2.

$$\begin{aligned} & Z_F(p_1, p_2, p_3, \dots) \circ Z_G(p_1(1), p_1(\zeta), p_1(\zeta^2), \dots, p_1(\zeta^{r-1}), p_2(1), \dots, p_2(\zeta^{r-1}), p_3(1), \dots) \\ &= Z_F(Z_G(p_{1.1}(1), p_{1.1}(\zeta), \dots, p_{1.1}(\zeta^{r-1}), p_{2.1}(1), \dots), Z_G(p_{1.2}(1), \dots, p_{1.2}(\zeta^{r-1}), p_{2.2}(1), \dots), \\ & \quad Z_G(p_{1.3}(1), \dots, p_{1.3}(\zeta^{r-1}), p_{2.3}(1), \dots), \dots) \end{aligned}$$

In other words, we substitute for every  $p_i$  in  $Z_F$  a modified  $Z_G$ , where we have substituted for every  $p_j(\xi)$  a  $p_{i,j}(\xi)$ .

PROOF. We want to calculate  $Z_F \circ Z_G$ , for an easier reading we write  $Z_F = \sum_{n, \lambda \vdash n} a_\lambda p_\lambda$  and  $Z_G = \sum_{n, \mu \vdash r n} b_\mu p_\mu$ :

$$Z_F \circ Z_G = \left( \sum_{n, \lambda \vdash n} a_\lambda p_\lambda \right) \circ \left( \sum_{n, \mu \vdash r n} b_\mu p_\mu \right)$$

Now we use  $(f_1 + f_2) \circ g = f_1 \circ g + f_2 \circ g$  and  $(f_1 \cdot f_2) \circ g = (f_1 \circ g) \cdot (f_2 \circ g)$ . Note that we here use the cycle type definition of a partition:

$$= \sum_{n, \lambda \vdash n} (a_\lambda p_\lambda) \circ \left( \sum_{n, \mu \vdash r n} b_\mu p_\mu \right) = \sum_{n, \lambda \vdash n} a_\lambda \prod_{i \in \lambda} (p_i \circ \left( \sum_{n, \mu \vdash r n} b_\mu p_\mu \right))$$

Now we have on the left side of  $\circ$  only  $p_i$ , so we can use  $g_1, g_2 \in \Lambda(r)$ :  $p_i \circ (g_1 + g_2) = p_i \circ g_1 + p_i \circ g_2$  and  $p_i \circ (g_1 \cdot g_2) = (p_i \circ g_1) \cdot (p_i \circ g_2)$ :

$$\begin{aligned} &= \sum_{n, \lambda \vdash n} a_\lambda \prod_{i \in \lambda} \left( \sum_{n, \mu \vdash r n} p_i \circ (b_\mu p_\mu) \right) = \sum_{n, \lambda \vdash n} a_\lambda \prod_{i \in \lambda} \sum_{n, \mu \vdash r n} b_\mu \prod_{j(\xi) \in \mu} p_i \circ p_j(\xi) \\ &= \sum_{n, \lambda \vdash n} a_\lambda \prod_{i \in \lambda} \sum_{n, \mu \vdash r n} b_\mu \prod_{j(\xi) \in \mu} p_{i,j}(\xi) \end{aligned}$$

This is exactly the formula stated above.  $\square$

Now we do the same for the second case:

DEFINITION 4.2.3 (Type 2). We define  $\circ : \Lambda(r) \times \Lambda(1) \rightarrow \Lambda(r)$  uniquely as follows:

- (1)  $\forall g \in \Lambda(1)$ , the map  $\Lambda(r) \rightarrow \Lambda(r) : f \mapsto f \circ g$  is a  $\mathbb{C}$ -algebra homomorphism.  
 $(\forall g \in \Lambda(1), f_1, f_2 \in \Lambda(r) : (f_1 + f_2) \circ g = f_1 \circ g + f_2 \circ g$  and  $(f_1 \cdot f_2) \circ g = (f_1 \circ g) \cdot (f_2 \circ g))$
- (2)  $\forall i \in \mathbb{N}, \xi \in \mathbf{C}_r$ , the map  $\Lambda(1) \rightarrow \Lambda(r) : g \mapsto p_i(\xi) \circ g$  is a  $\mathbb{C}$ -algebra homomorphism.  
 $(\forall i \in \mathbb{N}, \xi \in \mathbf{C}_r, g_1, g_2 \in \Lambda(r) : p_i(\xi) \circ (g_1 + g_2) = p_i(\xi) \circ g_1 + p_i(\xi) \circ g_2$  and  $p_i(\xi) \circ (g_1 \cdot g_2) = (p_i(\xi) \circ g_1) \cdot (p_i(\xi) \circ g_2))$
- (3)  $p_i(\xi) \circ p_j = p_{ij}(\xi^j)$

This also can be defined in another way (compare [BLL98]):

LEMMA 4.2.4.

$$\begin{aligned} & Z_F(p_1(1), p_1(\zeta), p_1(\zeta^2), \dots, p_1(\zeta^{r-1}), p_2(1), \dots, p_2(\zeta^{r-1}), p_3(1), \dots) \circ Z_G(p_1, p_2, p_3, \dots) \\ &= Z_F(Z_G(p_1(1^1), p_2(1^2), p_3(1^3), \dots), Z_G(p_1(\zeta^1), p_2(\zeta^2), p_3(\zeta^3), \dots), \dots, \\ & \quad Z_G(p_1(\zeta^{(r-1) \cdot 1}), p_2(\zeta^{(r-1) \cdot 2}), p_3(\zeta^{(r-1) \cdot 3}), \dots), \\ & \quad Z_G(p_{1.2}(1^1), p_{2.2}(1^2), \dots), Z_G(p_{1.2}(\zeta^1), p_{2.2}(\zeta^2), \dots), \dots, Z_G(p_{1.2}(\zeta^{(r-1) \cdot 1}), p_{2.2}(\zeta^{(r-1) \cdot 2}), \dots), \\ & \quad Z_G(p_{1.3}(1^1), p_{2.3}(1^2), \dots), Z_G(p_{1.3}(\zeta^1), p_{2.3}(\zeta^2), \dots), \dots) \end{aligned}$$

In other words, we substitute for every  $p_i(\xi)$  in  $Z_F$  a modified  $Z_G$ , where we have substituted for every  $p_j$  a  $p_{i,j}(\xi^j)$ .

PROOF. The proof is analogous to that of type 1 (lemma 4.2.2).  $\square$

Examples to the plethystic substitution will come in section 4.5.

REMARK 4.2.5. For the third type we have no classical plethysm, however, we will find a new way to describe the substitution of the cycle indicator series later.

### 4.3. Theorems of Type 1 and 2

THEOREM 4.3.1 (Type 1). *Let  $F$  be an ordinary (1-)species and  $G$  an  $r$ -species. Then  $Z_{F \circ G} = Z_F \circ Z_G$ .*

THEOREM 4.3.2 (Type 2). *Let  $F$  be an  $r$ -species and  $G$  an ordinary (1-)species. Then  $Z_{F \circ G} = Z_F \circ Z_G$ .*

For examples see section 4.5.

REMARK 4.3.3. This is already proven in [Hen04] by means of polynomial functors and characters. We present a direct proof by computation. For  $r = 1$ , another proof can be found in [BLL98] (chapter 4.3), using the type generating function.

We will proof the two theorems simultaneously, as the proofs are quite similar, and some of the arguments are even the same.

For the proofs of theorem 4.3.1 and theorem 4.3.2 we consider a typical  $F \circ G$  structure, similar as in [BLL98].

In type 1 such a structure  $h$  is a tuple  $h = (\pi, f, (g_b)_{b \in \pi})$  where:

- (1)  $\pi$  is a set partition in  $\text{Par}_{\mathbf{C}_r}[[n]_{\mathbf{C}_r}]$
- (2)  $f$  is an  $F$ -structure on  $\pi$
- (3)  $g_b$  are  $G$ -structures on  $b$ ;

In type 2 such a structure  $h$  is a tuple  $h = (\pi, f, (g_{\mathcal{O}})_{\mathcal{O} \in \pi_{\mathbf{C}_r}})$ :

- (1)  $\pi$  is a set partition in  $\text{Par}_{\mathbb{B}_r}[[n]_{\mathbf{C}_r}]$
- (2)  $f$  is an  $F$ -structure on  $\pi$
- (3)  $g_{\mathcal{O}} = (g_b)_{b \in \mathcal{O}}$  are tuples of  $G$ -structures, where  $\mathcal{O}$  is a  $\mathbf{C}_r$ -orbit of  $\pi$

Now it is time to define  $F \circ G[\omega]$  on a  $F \circ G$ -structure:

DEFINITION 4.3.4 (Type 1). We define

$$F \circ G[\omega]((\pi, f, (g_b)_{b \in \pi})) = (\omega(\pi), F[\sigma](f), (G[\omega|_b](g_b))_{b \in \pi})$$

where

- (1)  $\omega(\pi)$  is defined pointwise, so each block  $b \in \pi$  maps to  $\omega(b)$ . Thus, the result is a set partition with the same sizes of the blocks and, still in  $\text{Par}_{\mathbf{C}_r}[[n]_{\mathbf{C}_r}]$ .
- (2) We define  $\sigma$  as the bijection between  $\pi$  and  $\omega(\pi)$  where  $\sigma(b) := \omega(b)$ .

DEFINITION 4.3.5 (Type 2). We define

$$F \circ G[\omega]((\pi, f, (g_b)_{b \in \pi})) = (\omega(\pi), F[\sigma](f), (G[\omega|_b](g_{\mathcal{O}}))_{\mathcal{O} \in \pi_{\mathbf{C}_r}})$$

where:

- (1)  $\omega(\pi)$  is defined pointwise, so each block  $b \in \pi$  maps to  $\omega(b)$ . Thus, the result is a set partition with the same sizes of the blocks, and still in  $\text{Par}_{\mathbb{B}_r}[[n]_{\mathbf{C}_r}]$ .
- (2) We define  $\sigma$  as the bijection between  $\pi$  and  $\omega(\pi)$  where  $\sigma(b) := \omega(b)$ . Note that here  $\sigma$  is a signed permutation  $\sigma \in W_{r, n_{\pi}}$ .
- (3)  $G[\omega|_b]g_{\mathcal{O}} = (G[\omega|_b]g_b)_{b \in \mathcal{O}}$ . Note that, as a  $\mathbf{C}_r$ -orbit is mapped to a  $\mathbf{C}_r$ -orbit, the tuples of  $(G[\omega|_b]g_b)_{b \in \mathcal{O}}$  belong to the  $\mathbf{C}_r$ -orbit  $(G[\omega|_b]g_b)_{b \in \mathcal{O}}$ .

EXAMPLE 4.3.6. (1) Let  $F$  be the species of sets and  $G$  the species of  $r$ -cycles. An example for a structure  $h$  in  $F \circ G[5]_{\mathbf{C}_3} = \mathcal{E} \circ \mathcal{C}^r[5]_{\mathbf{C}_3}$  (we have seen that this is isomorphic to the species of signed permutations) is the tuple  $h = (\pi, f, g = (g_b)_{b \in \pi})$  where

- $\pi = \{\{1, \zeta 1, \zeta^2 1, 3, \zeta 3, \zeta^2 3\}, \{2, \zeta 2, \zeta^2 2, 4, \zeta 4, \zeta^2 4, 5, \zeta 5, \zeta^2 5\}\}$
- $f = \{\{1, \zeta 1, \zeta^2 1, 3, \zeta 3, \zeta^2 3\}, \{2, \zeta 2, \zeta^2 2, 4, \zeta 4, \zeta^2 4, 5, \zeta 5, \zeta^2 5\}\}$
- $g = \{(2\zeta 5\zeta 4\zeta^0), (1\zeta^2 3\zeta^2)\}$

Note that if  $\pi$  is obvious by knowing  $f$  and  $g$  we sometimes omit it. For a graphical representation see figure 5.

Now consider a signed permutation

$$\omega = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \zeta^2 1 & \zeta 3 & 2 & 5 & \zeta 4 \end{pmatrix} = (1\zeta^2)(2\zeta 3\zeta^0)(4\zeta^0 5\zeta),$$

then  $F \circ G[\omega]h$  is given through:

- $\pi' = \{\{1, \zeta 1, \zeta^2 1, 2, \zeta 2, \zeta^2 2\}, \{3, \zeta 3, \zeta^2 3, 4, \zeta 4, \zeta^2 4, 5, \zeta 5, \zeta^2 5\}\}$
- $f' = \{\{1, \zeta 1, \zeta^2 1, 2, \zeta 2, \zeta^2 2\}, \{3, \zeta 3, \zeta^2 3, 4, \zeta 4, \zeta^2 4, 5, \zeta 5, \zeta^2 5\}\}$
- $g' = \{(3\zeta 4\zeta^0 5\zeta 3), (1\zeta^0 2\zeta)\}$

(2) Let  $F$  be the species of vertices and  $G$  the species of ordinary cycles. Note that this  $r$ -species is not the species of signed permutations, even though it does have equivalent structures, as the  $\mathfrak{W}_{r,n}$ -action is different, as we will see in this example). An example for a structure  $h$  in  $F \circ G[5]_{\mathbf{C}_3}$  is the tuple  $h = (\pi, f, g = (g_b)_{b \in \pi})$  where

- $\pi = \{\{1, \zeta 3\}, \{\zeta 1, \zeta^2 3\}, \{\zeta^2 1, 3\}, \{2, \zeta^2 4, \zeta^2 5\}, \{\zeta 2, 4, 5\}, \{\zeta^2 2, \zeta 4, \zeta 5\}\}$
- $f = \{\{\zeta^2 1, 3\}, \{\zeta 2, 4, 5\}\}$
- $g = \{(254), (13)\}$

Remember, that we write here  $\{(13)\}$  instead of  $\{(1\zeta^0 3\zeta), (1\zeta 3\zeta^2), (1\zeta^2 3\zeta^0)\}$ , again.

For a graphical representation of this see figure 6.

Now consider once again a signed permutation:

$$\omega = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \zeta^2 1 & \zeta 3 & 2 & 5 & \zeta 4 \end{pmatrix} = (1\zeta^2)(2\zeta 3\zeta^0)(4\zeta^0 5\zeta)$$

Then  $F \circ G[\omega]h$  is given through:

- $\pi' = \{\{\zeta^2 1, \zeta 2\}, \{\zeta 1, \zeta^2 2\}, \{\zeta 1, 2\}, \{\zeta 3, \zeta^5 5, 4\}, \{\zeta^2 3, 5, \zeta 4\}, \{3, \zeta 5, \zeta^2 4\}\}$
- $f' = \{\{\zeta 1, 2\}, \{\zeta^3 3, 5, \zeta 4\}\}$
- $g' = \{(345), (12)\}$

REMARK 4.3.7. Note that  $\omega \in \mathfrak{W}_{r,n}$  with  $\omega(\pi) = \pi$  induces a permutation  $\sigma \in \mathfrak{S}_\pi$  of the blocks  $b$ : with  $\sigma(b) := \omega(b)$  we define a bijection between  $\pi$  and  $\omega(\pi)$ ; as  $\pi = \omega(\pi)$  we get a permutation. By convention we will always use the letter  $\sigma$  to denote the permutation induced by  $\omega$ .

EXAMPLE 4.3.8. Consider  $r = 2, n = 5$  and the partition  $\{\{1, 2\}, \{3, 4\}, \{5\}\}$  of  $[n]$ .

(1) Type 1: Let  $\pi = \{\{1, -1, 2, -2\}, \{3, -3, 4, -4\}, \{5, -5\}\}$  and  $\omega = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ -3 & -4 & -2 & 1 & -5 \end{pmatrix}$ , then we have  $\sigma = (\{1, -1, 2, -2\}\{3, -3, 4, -4\})(\{5, -5\})$ .

(2) Type 2: Let  $\pi = \{\{1, -2\}\{-1, 2\}, \{3, 4\}, \{-3, -4\}, \{5\}, \{-5\}\}$  and  $\omega = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ -3 & 4 & -2 & 1 & -5 \end{pmatrix}$ , then we have  $\sigma = \left( \begin{array}{ccc} \{1, -2\} & \{3, 4\} & \{5\} \\ -\{3, 4\}(=\{-3, -4\}) & \{1, -2\} & -\{5\}(=\{-5\}) \end{array} \right)$  that is a signed permutation of the blocks.

Note that a permutation that fixes a  $\mathbf{C}_r$ -partition does not necessarily fix a  $\mathbb{B}_r$ -partition that belongs to the same ordinary partition as we see in this example.



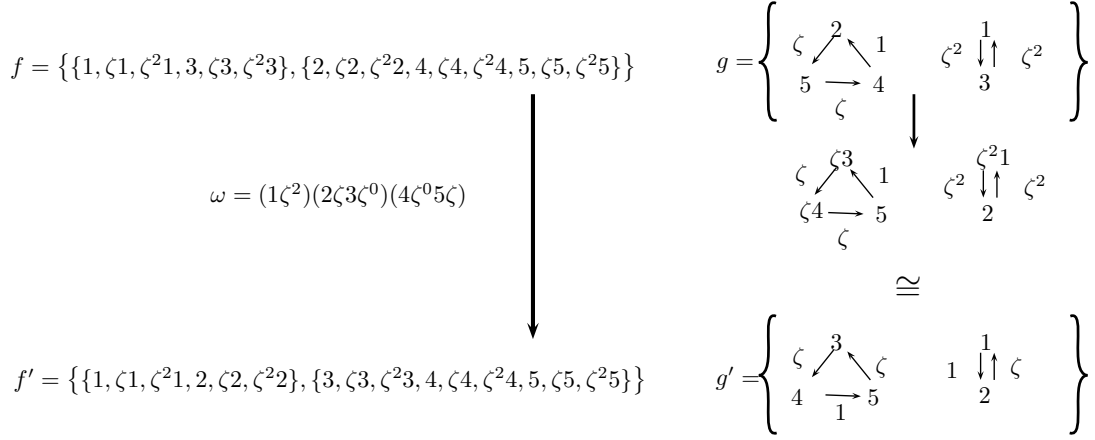


FIGURE 5.  $\omega$ -Action on a substitution of type 1.

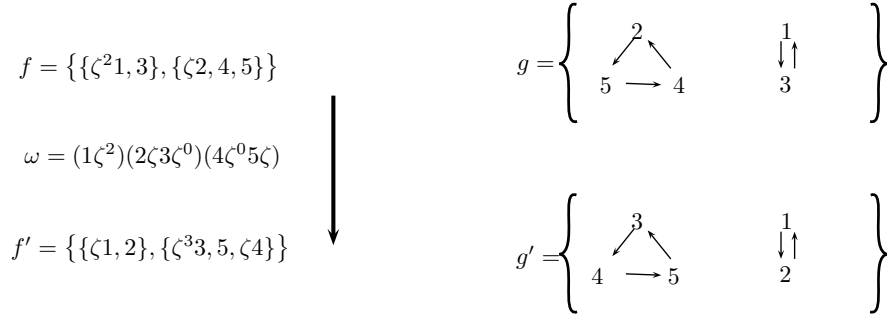


FIGURE 6.  $\omega$ -Action on a substitution of type 2.

For the cycle indicator series  $Z_{F \circ G}$ , we need to consider, under which conditions such a structure is a fixed point under a signed permutation:

LEMMA 4.3.9 (Type 1). *A structure  $h \in (F \circ G)[n]_{\mathbf{C}_r}$  is a fixed point of  $(F \circ G)[\omega]$  if and only if:*

- (1)  $\omega(\pi) = \pi$  (Therefore, the parts belonging to one cycle of  $\sigma$  need to have the same size.)
- (2)  $f$  needs to be a fixed point under  $F[\sigma]$ .
- (3)  $G[\omega](g_b) = g_{\omega(b)}$ .

PROOF. The first two conditions are obvious.

For the third we want  $(g_b)_{b \in \pi}$  to be fixed under the action of  $\omega$ . We know that  $\omega$  permutes the blocks  $b$ , so the  $G$ -structure with labels of  $\omega(b)$  is  $G[\omega](g_b)$ , and therefore,  $G[\omega](g_b) = g_{\omega(b)}$ .  $\square$

REMARK 4.3.10. For the case of type 2 this lemma and its proof is almost the same. Note that, as mentioned in lemma 4.3.5, the  $g_b$  of one  $\mathbf{C}_r$ -orbit are mapped to the same  $\mathbf{C}_r$ -orbit.

LEMMA 4.3.11 (Type 1). *Let  $\sigma$  be a permutation and  $\pi \in \text{Par}_{\mathbf{C}_r}[[n]_{\mathbf{C}_r}]$ . Let  $\omega$  be a signed permutation that induces  $\sigma$  on  $\pi$  and  $\omega(\pi) = \pi$ .*

*Then we can construct the fixed points of  $\omega$  in the following way:*

- (1) Let  $f$  be any fixed point of  $F[\sigma]$ .

- (2) For every cycle of  $\sigma$  with length  $l$  take an arbitrary block  $b$  belonging to this cycle, let  $g_b$  be an arbitrary fixed point of  $(\omega^l)|_b$  and  $g_\omega(b) = \omega(g_b) (= \sigma)$  for the other blocks belonging to the same cycle.

The number of fixed points is

$$|\text{fix } F[\sigma]| \cdot \prod_{c \text{ a cycle of } \sigma} |\text{fix } G[(\omega^{l_c})|_{b_c}]|$$

where  $p_c$  is an arbitrary block belonging to  $c$ , and  $l_c$  is the length of  $c$ .

PROOF. The construction is well defined: As block  $b$  belongs to a cycle of  $\sigma$  with length  $l$ ,  $(\omega^l)|_b : b \rightarrow b$ . Furthermore,  $\omega$  is a bijection of  $b$  and  $\omega(b)$ , hence,  $\omega(g_b)$  is indeed a  $G$ -structure. By construction,  $f$  and  $b$  are fixed under the action of  $\omega$ , and  $\omega(g_b) = g_{\omega(b)}$  holds. It remains to show, that every fixed point can be constructed this way: If we iterate  $G[\omega](g_b) = g_{\omega(b)}$  we get  $\omega^l(g_b) = g_{\omega^l(b)} = g_b$ , so being a fixed point of  $(\omega^l)|_b$  is necessary for any fixed point.  $\square$

REMARK 4.3.12. This lemma and its proof are identical for type 2 if one just changes  $\pi \in \text{Par}_{\mathbf{C}_r}[[n]_{\mathbf{C}_r}]$  into  $\pi \in \text{Par}_{\mathbb{B}_r}[[n]_{\mathbf{C}_r}]$  and  $b_c$  into  $\mathcal{O}_c$ .

DEFINITION 4.3.13 (Type 1). For  $\tau \in \mathfrak{W}_{r,n}$  with cycle type

$$(\tau_1(\zeta^0), \tau_1(\zeta^1), \dots, \tau_1(\zeta^{r-1}); \tau_2(\zeta^0), \dots, \tau_m(\zeta^{r-1}))$$

and a power sum symmetric function belonging to  $\tau$

$$p_\tau = p_1^{\tau_1(\zeta^0)}(\zeta^0) \cdot p_1^{\tau_1(\zeta^1)}(\zeta^1) \cdot \dots \cdot p_1^{\tau_1(\zeta^{r-1})}(\zeta^{r-1}) \cdot p_2^{\tau_2(\zeta^0)}(\zeta^0) \cdot \dots \cdot p_m^{\tau_m(\zeta^{r-1})}(\zeta^{r-1})$$

we define for  $l \in \mathbb{N}$ :

$$p_{\tau * 1 l} = p_{1,l}^{\tau_1(\zeta^0)}(\zeta^0) \cdot p_{1,l}^{\tau_1(\zeta^1)}(\zeta^1) \cdot \dots \cdot p_{1,l}^{\tau_1(\zeta^{r-1})}(\zeta^{r-1}) \cdot p_{2,l}^{\tau_2(\zeta^0)}(\zeta^0) \cdot \dots \cdot p_{m,l}^{\tau_m(\zeta^{r-1})}(\zeta^{r-1}) = p_l \circ p_\tau.$$

REMARK 4.3.14. This means that by transforming  $p_\tau$  into  $p_{\tau * 1 l}$  we simply replace every  $p_i(\xi)$  with  $p_{i,l}(\xi)$ .

DEFINITION 4.3.15 (Type 2). For  $\tau \in \mathfrak{S}$  with cycle type

$$(\tau_1, \tau_2, \tau_3 \dots, \tau_m)$$

and a power sum symmetric function belonging to  $\tau$

$$p_\tau = p_1^{\tau_1} \cdot p_2^{\tau_2} \cdot p_3^{\tau_3} \cdot \dots \cdot p_m^{\tau_m}$$

we define for  $l \in \mathbb{N}$  and  $\xi \in \mathbf{C}_r$ :

$$p_{\tau * 2 \xi l} = p_{1,l}^{\tau_1}(\xi^1) \cdot p_{2,l}^{\tau_2}(\xi^2) \cdot p_{3,l}^{\tau_3}(\xi^3) \cdot \dots \cdot p_{m,l}^{\tau_m}(\xi^m) = p_l(\xi) \circ p_\tau.$$

REMARK 4.3.16. This means that by transforming  $p_\tau$  into  $p_{\tau * 2 \xi l}$  we simply replace every  $p_i$  with  $p_{i,l}(\xi^i)$ .

REMARK 4.3.17. What may be a little bit confusing is, that we here multiply with  $l$ , respectively  $\xi l$  from right, as in fact  $p_{\tau * 1 l} = p_l \circ p_\tau$  respectively  $p_{\tau * 2 \xi l} = p_l(\xi) \circ p_\tau$  holds where  $p_l$  respectively  $p_l(\xi)$ , is multiplied from left. We could define it the other way around but later on it will be easier to read.

LEMMA 4.3.18 (Type 1). Let  $\omega \in W_{r,n}$ ,  $\sigma \in \mathfrak{S}_\pi$  induced by  $\omega$  on  $\pi$  and  $\pi \in \text{Par}_{\mathbf{C}_r}[[n]_{\mathbf{C}_r}]$ , then

$$p_\omega = \prod_{c \text{ cycle of } \sigma} p_{(\omega^{l_c})|_{b_c}} = \prod_{c \text{ cycle of } \sigma} p_{l_c} \circ p_{\omega|_{b_c}}$$

where  $l_c$  is the length of a cycle  $c$  and  $b_c$  an arbitrary block of  $\pi$  belonging to  $c$ .

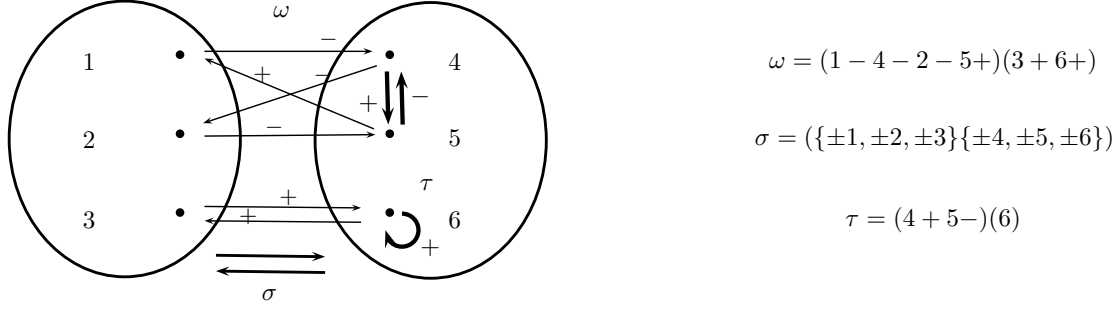


FIGURE 7. An example of a  $*_1$ -operation.

EXAMPLE 4.3.19. Let  $\omega = (1-4-2-5+)(3+6+)$  and  $\pi = \{\{\pm 1, \pm 2, \pm 3\}, \{\pm 4, \pm 5, \pm 6\}\}$ . Then we have  $p_\omega = p_2(1) \cdot p_4(-1)$ .  $\sigma$  consists of one cycle with length 2 ( $\sigma = (\{\pm 1, \pm 2, \pm 3\}\{\pm 4, \pm 5, \pm 6\})$ ). Let  $b_c = \{\pm 4, \pm 5, \pm 6\}$  and  $\tau = \omega_{|b_c}^2 = \begin{pmatrix} 4 & 5 & 6 \\ 5 & -4 & 6 \end{pmatrix}$ . Because,  $p_\tau = p_1(1) \cdot p_2(-1)$ , we have  $p_\omega = p_{\tau*_1 2} = p_{1.2}(1) \cdot p_{2.2}(-1)$ , which indeed coincides with  $p_\omega$ . For a graphical representation see figure 7.

PROOF. Let  $\xi_d \in \mathbf{C}_r$  be the type of the cycle  $d$ . As  $\omega$  induces  $\sigma$ , every cycle of  $\omega$  is fully contained in the blocks of one of the cycles of  $\sigma$ . Therefore, we can sort the  $p_i(\xi)$  according to the cycle of  $\sigma$  they belong to:

$$p_\omega = \prod_{d \text{ cycle of } \omega} p_{l_d}(\xi_d) = \prod_{c \text{ cycle of } \sigma} \prod_{\substack{d \text{ cycle of } \omega, \\ \text{part of } c}} p_{l_d}(\xi_d)$$

Now consider a cycle  $d$  of  $\omega$  with length  $l_d$ , part of a cycle  $c$  of  $\sigma$  with length  $l_c$ : The length of the cycle  $d^{l_c}$  is  $\frac{l_d}{l_c}$ . Now we have  $(d^{l_c})_{l_c}^{\frac{l_d}{l_c}}(x) = d^{l_d}(x) = \xi_d x$ , hence the type of  $d^{l_c}$  is the same as the one of  $d$ . It follows that:

$$p_{l_d}(\xi_d) = p_{\frac{l_d}{l_c}*_1 l_c}(\xi_d) = p_{d^{l_c}*_1 l_c}.$$

Furthermore,  $\omega_{|b_c}^{l_c}$  consists of all such cycles  $d$ , hence

$$p_{(\omega_{|b_c}^{l_c})*_1 l_c} = \prod_{\substack{d \text{ cycle of } \omega, \\ \text{part of } c}} p_{d^{l_c}*_1 l_c}.$$

□

Here, type 2 is a little more complicated and we need some more work preparing it:

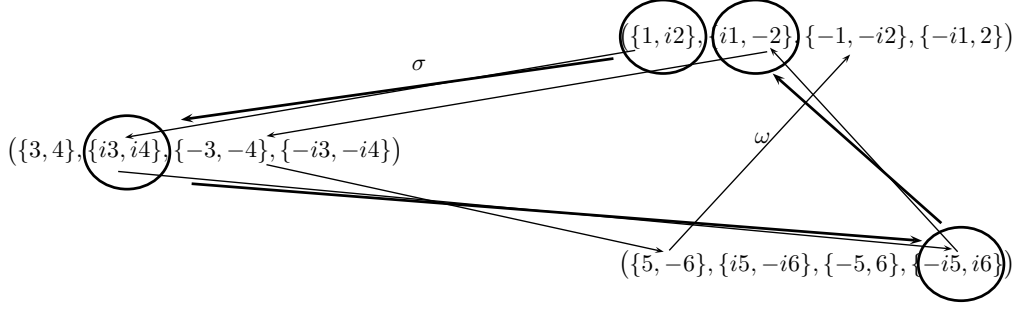
DEFINITION 4.3.20. Consider a cycle  $c$  of  $\sigma$  with length  $l$  and type  $\xi$ . Then  $\omega^l|_{\mathcal{O}}$  maps  $\mathcal{O}$  to  $\mathcal{O}$ . We define a permutation  $\tau_c$  on the set  $S_c$  of all tuples of elements that belong to one  $\mathbf{C}_r$ -cycle  $\tau_c((\zeta^i x)_{i=1,2,\dots,(r-1)}) := (\omega^l(\zeta^i x))_{i=1,2,\dots,(r-1)}$ .

REMARK 4.3.21. Note that  $\tau$  is an ordinary permutation and  $\omega^l$  is a signed one!

LEMMA 4.3.22. (1)  $|\text{fix } G[(\omega^l)_{|\mathcal{O}}]| = |\text{fix } G[\tau_c]|$ .

(2) The number of cycles  $c$  with length  $l$  is the same of  $\tau$  and  $\omega^l|_{\mathcal{O}}$ .

PROOF.



$$\begin{aligned}\omega &= (1i3 - 5 - i2 + 4 + 6+) \\ \sigma &= (\{1, i2\} + \{i3, i4\} + \{-i5, i6\}i) \\ \tau &= (12)\end{aligned}$$

FIGURE 8. An example of a  $*_2$ -operation.

This follows directly from the definition: As  $\tau$  is defined by the pointwise action of  $\omega$  a tuple is a fix point under  $\omega$  if and only if it is under  $\tau$ .

Every cycle of  $\tau$  comes from a cycle of  $\omega^l | \mathcal{O}$ . We just ‘forget’ the signs of  $\omega$ . The length of the cycle remains.  $\square$

LEMMA 4.3.23 (Type 2). *Let  $\omega \in W_{r,n}$ ,  $\sigma \in W_{r,n\pi}$  induced by  $\omega$  on  $\pi$  and  $\pi \in \text{Par}_{\mathbb{B}_r}[[n]_{C_r}]$ , then*

$$p_\omega = \prod_{c \text{ cycle of } \sigma} p_{\tau_c * 2\xi_c l_c} = \prod_{c \text{ cycle of } \sigma} p_{l_c}(\xi_c) \circ p_{\tau_c}$$

where  $l_c$  is the length and  $\xi_c$  the type of a cycle  $c$  and  $b_c$  an arbitrary block of  $\pi$  belonging to  $c$ .

EXAMPLE 4.3.24.  $r = 4$ ,  $n = 6$ :

$$\begin{aligned}\pi &= \{\{1, i2\}, \{i1, -2\}, \{-1, -i2\}, \{-i1, 2\}, \{3, 4\}, \{i3, i4\}, \{-3, -4\}, \{-i3, -i4\}, \\ &\quad \{5, -6\}, \{i5, -i6\}, \{-5, 6\}, \{-i5, i6\}\}\end{aligned}$$

and  $\omega = (1i3 - 5 - i2 + 4 + 6+)$ , or in other notation  $\omega = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ i3 & 4 & -5 & 6 & -i2 & 1 \end{pmatrix}$ . We therefore

have  $\tau = (12)$  and  $\sigma = \begin{pmatrix} \{1, i2\} & \{i3, i4\} & \{-i5, i6\} \\ \{i3, i4\} & \{-i5, i6\} & \{i1, -2\} \end{pmatrix}$ . Then lemma 4.3.23 claims that  $p_\omega = p_6(-1) = p_{\tau * i, 3} = p_{2 \cdot 3}(i^2)$ , as  $\sigma$  has one cycle of length 3 and sign  $i$ , and  $\tau$  one cycle with length 2. The  $2 \cdot 3$  has exactly the same reason as in type 1. The  $i^2$  is because  $\sigma$  is just one ‘round’ of  $\omega$ , however, the length of  $\tau$  tells us that there are two ‘rounds’.

For a graphical representation see figure 8.

PROOF. The proof is similar to that of type 1 (lemma 4.3.18). As  $\omega$  induces  $\sigma$ , every cycle of  $\omega$  is fully contained in the blocks of one of the cycles of  $\sigma$ . Therefore, we can sort the  $p_i(\xi)$  according to the cycle of  $\sigma$  they belong to:

$$p_\omega = \prod_{d \text{ cycle of } \omega} p_{l_d}(\xi_d) = \prod_{c \text{ cycle of } \sigma} \prod_{\substack{d \text{ cycle of } \omega, \\ \text{part of } c}} p_{l_d}(\xi_d)$$

Now consider a cycle  $d$  of  $\omega$  with length  $l_d$  and type  $\xi_d$ , part of a cycle  $c$  of  $\sigma$  with length  $l_c$  and type  $\xi_c$ : The length of the cycle  $d^{l_c}$  is  $\frac{l_d}{l_c}$ .

For calculating the type we consider the definition which does not use the product. The type of  $c$  is  $\xi_c$  with  $\xi_c b = \omega^{l_c}(b)$  where  $b$  is a block of  $\pi$ . The type of  $d$  can be seen as  $\xi_d x = \omega^{l_d}(x)$  with  $x \in b$ . If we consider  $\xi_c b = \omega^{l_c}(b)$  under the action of  $\omega^{\frac{l_d}{l_c}-1}$ , we get  $\xi_c^{\frac{l_d}{l_c}} b = \omega^{\frac{l_d}{l_c} l_c}(b) = \omega^{l_d}(b)$  as we have  $\xi_c^k(b) = \omega^{k l_c}(b)$ . Now the elements of  $b$  are mapped to ones of the same  $\mathbf{C}_r$ -cycle, thus  $\xi_c^{\frac{l_d}{l_c}} = \xi_d$ . It follows that:

$$p_{l_d}(\xi_d) = p_{\frac{l_d}{l_c} l_c}(\xi_c^{\frac{l_d}{l_c}}) = p_{\frac{l_d}{l_c} * 2 \xi l_c}.$$

Furthermore,  $\omega_{|b_c}^{l_c}$  consists of all such cycles  $d$ . What we need are the lengths of the cycles of  $\omega_{|b_c}^{l_c}$ , which are the same as those of  $\tau_c$ , hence

$$p_{\tau_c * 2 \xi l_c} = \prod_{\substack{d \text{ cycle of } \omega, \\ \text{part of } c}} p_{\frac{l_d}{l_c} * 2 \xi l_c}.$$

□

LEMMA 4.3.25. *The number of partitions of  $[n]$  with  $j_i$  blocks of size  $u_i$  is*

$$\frac{n!}{\prod_i (u_i!)^{j_i} j_i!}.$$

REMARK 4.3.26. This number looks similar to the number of permutations with the same cycle type and is proven similarly. There is one difference though: note that there is a  $u_i$  factorial. Note further, that the number of partitions in  $\text{Par}_{\mathbf{C}_r}[[n]_{\mathbf{C}_r}]$  is the same, and the number of partitions in  $\text{Par}_{\mathbb{B}_r}[[n]_{\mathbf{C}_r}]$  is the number of partitions in  $\text{Par}[[n]]$  times  $\prod_i r^{(u_i-1) \cdot j_i}$ , as we have  $r^{(u_i-1)}$  possibilities for the signs in the set of  $\text{Par}_{\mathbb{B}_r}[[n]_{\mathbf{C}_r}]$ , where 1 has sign 1. Then all the other sets are determined. (Compare with lemma 3.4.6.)

PROOF. We rather proof that  $n!$  is equal to the number of the partitions times  $\prod_i (u_i!)^{j_i} j_i!$ : There are  $n!$  ways to arrange  $[n]$ , when defining the first  $j_1$  numbers as being the  $j_1$  blocks of size one, the next  $j_2 \cdot 2$  numbers as being (each two of them) the blocks of size two and so on we get partitions of  $[n]$ .

However, some of this arrangements define the same partition: As we do not distinguish the order of the blocks of same size, nor the order of the elements in one block, there are  $\prod_i (u_i!)^{j_i} j_i!$  arrangements which define the same partitions. □

DEFINITION 4.3.27 (Type 1). We define  $s_{i,l}$  as the number of cycles of  $\sigma$  with length  $l$  and blocks of size  $u_i$ . Furthermore, we define  $t_{i,l,\tau}$  as the number of cycles with length  $l$ , with blocks of size  $u_i$ , and with  $\tau = (\omega^l)_{|b_c}$ , where  $b_c$  is one (arbitrary but fixed) part of the cycle.

LEMMA 4.3.28. *There are three identities following directly from definition 4.3.27:*

- (1)  $\sum_i s_{i,l} = \sigma_l$
- (2)  $\sum_{\tau} t_{i,l,\tau} = s_{i,l}$
- (3)  $\sum s_{i,l} \cdot l$  is the number of blocks with size  $u_i$  ( $= j_i$ ).

DEFINITION 4.3.29 (Type 2). We define  $s_{i,l,\xi}$  as the number of cycles of  $\sigma$  with length  $l$ , type  $\xi$  and blocks of size  $u_i$ .

Furthermore we define  $t_{i,l,\xi,\tau}$  as the number of cycles with length  $l$  type  $\xi$  with blocks of size  $u_i$ , and with  $\tau = (\omega^l)_{|b_c}$ , where  $b_c$  is one (arbitrary but fixed) part of the cycle.

LEMMA 4.3.30. *Again, there are three identities following directly from definition 4.3.29:*

- (1)  $\sum_i s_{i,l,\xi} = \sigma_l(\xi)$
- (2)  $\sum_{\tau} t_{i,l,\tau,\xi} = s_{i,l,\xi}$

(3)  $\sum_{l,\xi} s_{i,l,\xi} \cdot l$  is the number of blocks with size  $u_i (= j_i)$ .

REMARK 4.3.31. Note that if we have  $t_{i,l,\tau}$ , with a given  $\tau$  we also know  $i$ , so we sometimes omit it (with  $\tau$  we also know the size of blocks, of which  $\tau$  is a permutation).

LEMMA 4.3.32 (Type 1). *For fixed  $\sigma$  and  $\pi$  the number of different  $\omega$  which induce the same multiset of  $(\omega^{l_c})_{|b_c}$  for every cycle  $c$  (with length  $l_c$  and an arbitrary but fixed block  $b_c$  belonging to this cycle  $c$ ) where a signed permutation of this multiset can belong to any proper cycle  $c$  is:*

$$\prod_{i,l} u_i!^{s_{i,l}(l-1)} r^{u_i s_{i,l}(l-1)} \cdot \frac{s_{i,l}!}{\prod_{\tau} t_{i,l,\tau}!}.$$

REMARK 4.3.33. In other words we have given a multiset of signed permutations, lets call them  $\tau_i$ , we know  $\sigma$  and  $\pi$  and we want to know the number of  $\omega$  they could induce them. Then we have to choose how they act on ‘their’ cycle and on which cycle they act.

Note that the notation  $(\omega^{l_c})_{|b_c}$  may be a little bit confusing as it seems that for each signed permutation the cycle  $c$  is already chosen, however it is not!

PROOF. For a given  $\omega_{|b_c}^{l_c}$ , we can construct  $\omega$  on the blocks  $\omega^m(b_c)$ ,  $m = 1, 2, \dots, l-1$ , in  $u_c!^{(l_c-1)} r^{u_c(l_c-1)}$  different ways: We have  $u_c! r^{u_c}$  ways to define  $\omega|b_c : b_c \rightarrow \sigma(b_c)$ , another  $u_c! r^{u_c}$  ways to define  $\omega|\sigma b_c$ , and so on. Only for  $\omega|_{\sigma^{l-1}(b_c)}$  we are forced to define in a way that we get  $\omega_{|b_c}^l$ . All together we have  $u_c!^{(l_c-1)} r^{u_c(l_c-1)}$  for one cycle, considering all cycles, the factors multiply, and give us the first part of our formula.

Now we show the second part: We can permute the  $\omega_{|b_c}^{l_c}$  between the different cycles with same size, permuting some with same  $\omega_{|b_q}^{l_c} = \tau$  will make no difference, hence we have  $\frac{s_{i,l}!}{\prod_{\tau} t_{i,l,\tau}!}$  possible ways to do so. Again, multiplying up everything gives us the related part of this formula.  $\square$

LEMMA 4.3.34 (Type 2). *For fixed  $\sigma$  and  $\pi$  the number of different  $\omega$  which induce the same multiset of  $(\tau_c)$  for every cycle  $c$  (with length  $l_c$ ) where a permutation of this multiset can belong to any proper cycle  $c$  is:*

$$\prod_{i,l,\xi} u_i!^{s_{i,l,\xi}(l-1)} \cdot \frac{s_{i,l,\xi}!}{\prod_{\tau} t_{i,l,\tau,\xi}!}.$$

PROOF. Compare with remark 4.3.33! For one given  $\tau_c$ , we can construct  $\omega$  on a  $\mathbf{C}_r$ -orbit  $(b)_{b \in \mathcal{O}}$  of  $\pi$  in  $u_q!^{(l-1)}$  different ways: We have  $u_i!$  ways to define  $\omega_{|(b)_{b \in \mathcal{O}}}$  for an arbitrary tuple  $(b)_{b \in \mathcal{O}}$  (the signs are defined through the signs of  $\sigma$  and the partition  $\pi$ ), another  $u_q!$  ways to define  $\omega|_{\sigma(b_c)}$ , and so on. Only for  $\omega|_{\sigma^{l-1}(b_c)}$  we are forced to define in a way that we get  $\omega_{|b_c}^l$ . All together, we have  $u_q!^{(l-1)} r^{u_q(l-1)}$  for one cycle, considering all cycles, the factors multiply, and give us the first part of our formula.

The second part of the proof is exactly the same as the one for type 1.  $\square$

LEMMA 4.3.35. *The number of permutations  $\sigma$  with the same cycle type on  $\pi$  and the same term  $p_\omega$  (with notation as before) is:*

(1) for type 1

$$\frac{\prod_i j_i!}{\prod_{i,l} s_{i,l}! l^{s_{i,l}}}.$$

(2) and for type 2

$$\frac{\prod_i j_i!}{\prod_{i,l,\xi} s_{i,l,\xi}! l^{s_{i,l}}} \cdot \prod_{i,l,\xi} r^{s_{i,l,\xi}(l-1)}$$

PROOF. We just prove it for type 1, the proof for type 2 is very similar.

We have given partition, cycle type and  $p_\omega$ . The latter tells us how many cycles with length  $l$  and blocks of size  $u_i$  we need to have, so with that point of view  $s_{i,l}$  is also given.

Now we consider for any  $i$  only blocks with size  $u_i$ : The cycle type of  $\sigma$  restricted to those parts is  $(\frac{s_{i,1}}{1}, \frac{s_{i,2}}{2}, \frac{s_{i,3}}{3}, \dots)$  as  $s_{i,l}$  is the number of cycles with length  $l$  and size of the blocks  $u_i$ . Therefore, their number is:

$$\frac{j_i!}{s_{i,l}!l^{s_{i,l}}}.$$

Multiplying over all  $i$  leads to the desired formula.  $\square$

PROOF OF THEOREM 4.3.1 (TYPE 1). Now we are ready to transform  $Z_{F \circ G}$  into  $Z_F \circ Z_G$ .

$$\begin{aligned} Z_{F \circ G} &= \sum_{n \in \mathbb{N}, \omega \in \mathfrak{W}_{r,n}} \frac{1}{n!r^n} |\text{fix}(F \circ G)[\omega]| p_\omega \\ &= \sum_{n \in \mathbb{N}} \sum_{\pi \in \text{Par}_{\mathcal{C}_r}[[n]_{\mathcal{C}_r}]} \sum_{\sigma \in \mathfrak{S}_{|\pi|}} \sum_{\omega \text{ induces } \sigma \text{ on } \pi} \frac{1}{n!r^n} |\text{fix}_\pi(F \circ G)[\omega]| p_\omega \end{aligned}$$

Where  $\text{fix}_\pi(F \circ G)[\omega]$  denotes those fixed points that have  $\pi$  as their partition.

Lemma 4.3.11 tells us the number of fixed points appearing in this term, due to lemma 4.3.18 we can transform  $p_\omega$  appropriately. Here  $b_{l,m}$  is an arbitrary but fixed block belonging to the  $m^{\text{th}}$  cycle with length  $l$  of  $\sigma$ .

Note that if we sum over all  $\omega$ , we sum over all  $\omega$  in  $\mathfrak{W}_{r,n}$  for all possible  $n$ , analogous for  $\pi$  and  $\sigma$ .  $n$  is then determined by  $\sigma$ ,  $\pi$ , and  $\omega$ :

$$= \sum_{\pi} \sum_{\sigma} \sum_{\omega} \frac{1}{n!r^n} \prod_l \prod_{m=1}^{\sigma_l} |\text{fix } G[(\omega^l)_{|b_{l,m}}]| \cdot p_{(\omega^l)_{|b_{l,m}} * 1l} \cdot |\text{fix } F[\sigma]|$$

We use lemma 4.3.32 and consider from now on only  $\tau = (\omega^l)_{|b_{l,m}}$  and  $\sigma$ . For the manipulation of the next three lines we use lemma 4.3.28:

$$\begin{aligned} &= \sum_{\pi, \sigma} |\text{fix } F[\sigma]| \frac{1}{n!r^n} \prod_{i,l} u_i!^{s_{i,l}(l-1)} r^{u_i s_{i,l}(l-1)} s_{i,l}! \prod_{\tau} (|\text{fix } G[\tau]| p_{\tau * 1l})^{t_{i,l,\tau}} \frac{1}{t_{i,l,\tau}!} \\ &= \sum_{\pi, \sigma} |\text{fix } F[\sigma]| \frac{1}{n!r^n} \prod_{i,l} u_i!^{s_{i,l}l} r^{u_i(s_{i,l})} s_{i,l}! u_i!^{-\sum_{\tau} t_{i,l,\tau}} r^{-u_i!(\sum_{\tau} t_{i,l,\tau})} \prod_{\tau} (|\text{fix } G[\tau]| p_{\tau * 1l})^{t_{i,l,\tau}} \frac{1}{t_{i,l,\tau}!} \\ &= \sum_{\pi, \sigma} |\text{fix } F[\sigma]| \frac{1}{n!r^n} \prod_i u_i!^{j_i} r^{u_i j_i} \prod_l s_{i,l}! \prod_{\tau} (|\text{fix } G[\tau]| p_{\tau * 1l} \frac{1}{u_i! r^{u_i}})^{t_{i,l,\tau}} \frac{1}{t_{i,l,\tau}!} \end{aligned}$$

Next we use lemma 4.3.35 so from now on we only consider the cycle type of  $\sigma$ :

$$\begin{aligned} &= \sum_{k, \sigma \vdash k} |\text{fix } F[\sigma]| \sum_{\pi} \frac{1}{n!r^n} \frac{\prod_i j_i!}{\prod_{i,l} s_{i,l}! l^{s_{i,l}}} \prod_i u_i!^{j_i} r^{u_i j_i} \prod_l s_{i,l}! \prod_{\tau} (|\text{fix } G[\tau]| p_{\tau * 1l} \frac{1}{u_i! r^{u_i}})^{t_{i,l,\tau}} \frac{1}{t_{i,l,\tau}!} \\ &= \sum_{k, \sigma \vdash k} |\text{fix } F(\sigma)| \sum_{\pi} \frac{1}{n!r^n} r^n \prod_i j_i! u_i!^{j_i} \prod_l \frac{1}{l^{s_{i,l}}} \prod_{\tau} (|\text{fix } G[\tau]| p_{\tau * 1l} \frac{1}{u_i! r^{u_i}})^{t_{i,l,\tau}} \frac{1}{t_{i,l,\tau}!} \end{aligned}$$

Now we use lemma 4.3.25 (together with remark 4.3.26), and from now on just consider the sizes of the parts of  $\pi$ :

$$\begin{aligned}
&= \sum_{k, \sigma \vdash k} |\text{fix } F[\sigma]| \sum_{\pi \text{ with sizes } u_i} \prod_{i,l} \frac{1}{l^{s_{i,l}}} \prod_{\tau} (|\text{fix } G[\tau]| p_{\tau * 1 l} \frac{1}{u_i! r^{u_i}})^{t_{i,l,\tau}} \frac{1}{t_{i,l,\tau}!} \\
&= \sum_{k, \sigma \vdash k} |\text{fix } F[\sigma]| \sum_{\pi \text{ with sizes } u_i} \frac{\prod_l \sigma_l!}{\prod_l \sigma_l!} \prod_l \frac{1}{l^{\sum_i s_{i,l}}} \prod_i \prod_{\tau} (|\text{fix } G[\tau]| p_{\tau * 1 l} \frac{1}{u_i! r^{u_i}})^{t_{i,l,\tau}} \frac{1}{t_{i,l,\tau}!} \\
&= \sum_{k, \sigma \vdash k} |\text{fix } F[\sigma]| \sum_{\pi \text{ with sizes } u_i} \frac{1}{\prod_l \sigma_l!} \prod_l \frac{1}{l^{\sigma_l}} \frac{\sigma_l!}{\prod_{i,\tau} t_{i,l,\tau}!} \prod_{i,\tau} (|\text{fix } G[\tau]| p_{\tau * 1 l} \frac{1}{u_i! r^{u_i}})^{t_{i,l,\tau}} \\
&= \sum_{k, \sigma \vdash k} |\text{fix } F[\sigma]| \frac{1}{\prod_l \sigma_l! l^{\sigma_l}} \sum_{\pi \text{ with sizes } u_i} \prod_l \frac{\sigma_l!}{\prod_{i,\tau} t_{i,l,\tau}!} \prod_{\tau,i} (|\text{fix } G[\tau]| p_{\tau * 1 l} \frac{1}{u_i! r^{u_i}})^{t_{i,l,\tau}}
\end{aligned}$$

The sizes  $u_i$  and the number of their appearance are fully determined by the sums of the  $t_{i,l,\tau}$ :

$$= \sum_{k, \sigma \vdash k} |\text{fix } F[\sigma]| \frac{1}{\prod_l \sigma_l! l^{\sigma_l}} \sum_{\substack{\sum_{\tau} t_{1,\tau} = \sigma_1, \sum_{\tau} t_{2,\tau} = \sigma_2, \\ \dots, \sum_{\tau} t_{m,\tau} = \sigma_m}} \prod_l \frac{\sigma_l!}{\prod_{\tau} t_{l,\tau}!} \prod_{\tau} (|\text{fix } G[\tau]| p_{\tau * 1 l} \frac{1}{u_{i(\tau)}! r^{u_{i(\tau)}}})^{t_{l,\tau}}$$

Now we change the order of product and sum: (In the term above we have a sum over all possible products of terms, depending on  $l$ , and in the term below we have a product over sums of all possible terms, belonging to a fix  $l$ .)

$$= \sum_{k, \sigma \vdash k} |\text{fix } F[\sigma]| \frac{1}{\prod_l \sigma_l! l^{\sigma_l}} \prod_l \sum_{\sum_{\tau} t_{l,\tau} = \sigma_l} \frac{\sigma_l!}{\prod_{\tau} t_{l,\tau}!} \prod_{\tau} (|\text{fix } G[\tau]| p_{\tau * 1 l} \frac{1}{u_{i(\tau)}! r^{u_{i(\tau)}}})^{t_{l,\tau}}$$

We use the multinomial theorem and get:

$$= \sum_{k, \sigma \vdash k} |\text{fix } F[\sigma]| \frac{1}{k} \frac{k!}{\prod_l \sigma_l! l^{\sigma_l}} \prod_l \left( \sum_{\tau} |\text{fix } G[\tau]| p_{\tau * 1 l} \frac{1}{u_{i(\tau)}! r^{u_{i(\tau)}}} \right)^{\sigma_l}$$

As we have here the number of  $\sigma$  with the same cycle type we can go back to considering every different  $\sigma$ :

$$= \sum_{\sigma} |\text{fix } F[\sigma]| \frac{1}{n^{\pi}} \prod_l \left( \sum_{\tau} |\text{fix } G[\tau]| p_{\tau * 1 l} \frac{1}{u_{i(\tau)}! r^{u_{i(\tau)}}} \right)^{\sigma_l}$$

As  $p_{\tau * 1 l}$  is just another notation for replacing  $p_j(\xi)$  with  $p_{j,l}(\xi)$ , this is precisely the plethystic substitution.

$$\begin{aligned}
&= \left( \sum_{n, \sigma \in \mathfrak{S}_n} |\text{fix } F[\sigma]| \frac{1}{n!} p_{\sigma} \right) \circ \left( \sum_{n, \tau \in \mathfrak{W}_{r,n}} |\text{fix } G[\tau]| \frac{1}{n! r^n} p_{\tau} \right) \\
&= Z_F \circ Z_G
\end{aligned}$$

□

The proof of type 2 is very similar:

PROOF OF THEOREM 4.3.2 (TYPE 2). Now we are ready to transform  $Z_{F \circ G}$  into  $Z_F \circ Z_G$ .

$$Z_{F \circ G} = \sum_{n, \omega \in \mathfrak{W}_{r,n}} \frac{1}{n! r^n} |\text{fix}(F \circ G)[\omega]| p_{\omega}$$



$$= \sum_n \sum_{\pi \in \text{Par}_{\mathbb{B}_r}[[n]_{\mathbf{C}_r}]} \sum_{\sigma \in W_{r,|\pi|}} \sum_{\omega \text{ induces } \sigma \text{ on } \pi} \frac{1}{n!r^n} |\text{fix}_{\pi}(F \circ G)[\omega]| p_{\omega}$$

Where  $\text{fix}_{\pi}(F \circ G)[\omega]$  denotes those fixed points that have  $\pi$  as their partition.

Lemma 4.3.11 (together with 4.3.12) tells us the number of fixed points appearing in this term, due to lemma 4.3.23 and lemma 3.2.2 we can transform  $p_{\omega}$  appropriately. Here  $S_c$  is in definition 4.3.20 belonging to a cycle  $c$  of  $\sigma$ .

Note that if we sum over all  $\omega$ , we sum over all  $\omega$  in  $\mathfrak{S}_n$  for all possible  $n$ , analogous for  $\pi$  and  $\sigma$ .  $n$  is then determined by  $\pi, \sigma$  and  $\omega$ :

$$= \sum_{\pi} \sum_{\sigma} \sum_{\omega} \frac{1}{n!r^n} \prod_c |\text{fix } G[\tau_c]| \cdot p_{\tau_c * 2\xi^{j_l}} \cdot |\text{fix } F[\sigma]|$$

We use lemma 4.3.34 and consider from now on only  $\tau$  and  $\sigma$ . For the manipulation of the next three lines we use lemma 4.3.30:

$$\begin{aligned} &= \sum_{\pi, \sigma} |\text{fix } F[\sigma]| \frac{1}{n!r^n} \prod_{i, l, \xi} u_i!^{s_{i, l, \xi}^{(l-1)}} \cdot s_{i, l, \xi}! \cdot \prod_{\tau} (|\text{fix } G[\tau]| p_{\tau * 2\xi^l})^{t_{i, l, \xi, \tau}} \frac{1}{t_{i, l, \xi, \tau}!} \\ &= \sum_{\pi, \sigma} |\text{fix } F[\sigma]| \frac{1}{n!r^n} \prod_{i, l, \xi} u_i!^{s_{i, l, \xi}^l} u_i!^{-\sum_{\tau} t_{i, l, \xi, \tau}} \cdot s_{i, l, \xi}! \cdot \prod_{\tau} (|\text{fix } G[\tau]| p_{\tau * 2\xi^l})^{t_{i, l, \tau}} \frac{1}{t_{i, l, \xi, \tau}!} \\ &= \sum_{\pi, \sigma} |\text{fix } F[\sigma]| \frac{1}{n!r^n} \prod_{i, \xi} u_i!^{j_i} \prod_{l, \xi} s_{i, l, \xi}! \cdot \prod_{\tau} (|\text{fix } G[\tau]| p_{\tau * 2\xi^l} \cdot \frac{1}{u_i!})^{t_{i, l, \tau}} \frac{1}{t_{i, l, \xi, \tau}!} \end{aligned}$$

Next we use lemma 4.3.35 so from now on we only consider the cycle type of  $\sigma$ . Again we use lemma 4.3.30 for formula manipulation:

$$\begin{aligned} &= \sum_{k, \sigma \vdash_{\mathbf{C}_r} k} |\text{fix } F[\sigma]| \sum_{\pi} \frac{1}{n!r^n} \frac{\prod_i j_i!}{\prod_{i, l, \xi} s_{i, l, \xi}!^{j_i}} \prod_i u_i!^{j_i} \prod_{l, \xi} s_{i, l, \xi}! r^{s_{i, l, \xi}^{(l-1)}} \prod_{\tau} \frac{(|\text{fix } G[\tau]| \frac{p_{\tau * 2\xi^l}}{u_i!})^{t_{i, l, \tau}}}{t_{i, l, \xi, \tau}!} \\ &= \sum_{k, \sigma \vdash_{\mathbf{C}_r} k} |\text{fix } F[\sigma]| \sum_{\pi} \frac{1}{n!r^n} \prod_i j_i! u_i!^{j_i} \prod_{l, \xi} r^{s_{i, l, \xi}^l} r^{-s_{i, l, \xi}} \frac{1}{l^{s_{i, l, \xi}}} \prod_{\tau} \frac{(|\text{fix } G[\tau]| \frac{p_{\tau * 2\xi^l}}{u_i!})^{t_{i, l, \tau}}}{t_{i, l, \xi, \tau}!} \end{aligned}$$

Now we use lemma 4.3.25 (together with 4.3.26), and from now on just consider the sizes of the parts of  $\pi$ . Again we use lemma 4.3.30 for formula manipulation:

$$\begin{aligned} &= \sum_{k, \sigma \vdash_{\mathbf{C}_r} k} |\text{fix } F[\sigma]| \sum_{\pi \text{ with sizes } u_i} \frac{1}{r^n} \prod_i r^{(u_i-1) \cdot j_i} \prod_i r^{j_i} \prod_{l, \xi} r^{-s_{i, l, \xi}} \frac{1}{l^{s_{i, l, \xi}}} \prod_{\tau} \frac{(|\text{fix } G[\tau]| \frac{p_{\tau * 2\xi^l}}{u_i!})^{t_{i, l, \tau}}}{t_{i, l, \xi, \tau}!} \\ &= \sum_{k, \sigma \vdash_{\mathbf{C}_r} k} |\text{fix } F[\sigma]| \sum_{\pi \text{ with sizes } u_i} \prod_{i, l, \xi} r^{-s_{i, l, \xi}} \frac{1}{l^{s_{i, l, \xi}}} \prod_{\tau} \frac{(|\text{fix } G[\tau]| \frac{p_{\tau * 2\xi^l}}{u_i!})^{t_{i, l, \tau}}}{t_{i, l, \xi, \tau}!} \\ &= \sum_{k, \sigma \vdash_{\mathbf{C}_r} k} |\text{fix } F[\sigma]| \sum_{\pi \text{ with sizes } u_i} \frac{\prod_{i, l, \xi} \sigma_l(\xi)!}{\prod_{i, l, \xi} \sigma_l(\xi)!} \prod_{l, \xi} \frac{1}{(r \cdot l)^{\sum_i s_{i, l, \xi}}} \prod_{i, \tau} \frac{(|\text{fix } G[\tau]| \frac{p_{\tau * 2\xi^l}}{u_i!})^{t_{i, l, \tau}}}{t_{i, l, \xi, \tau}!} \\ &= \sum_{k, \sigma \vdash_{\mathbf{C}_r} k} |\text{fix } F[\sigma]| \frac{1}{\prod_{l, \xi} (r \cdot l)^{\sigma_l(\xi)} \sigma_l(\xi)!} \sum_{\pi \text{ with sizes } u_i} \prod_{l, \xi} \frac{\sigma_l(\xi)!}{\prod_{\tau, i} t_{i, l, \xi, \tau}!} \prod_{\tau, i} (|\text{fix } G[\tau]| \frac{p_{\tau * 2\xi^l}}{u_i!})^{t_{i, l, \tau}} \end{aligned}$$

The sizes  $u_i$  and the number of their appearance are fully determined by the  $t_{i,l,\xi,\tau}$ .

$$= \sum_{k, \sigma \vdash_{\mathbf{C}_r} k} |\text{fix } F[\sigma]| \frac{1}{\prod_{l,\xi} (r \cdot l)^{\sigma_l(\xi)} \sigma_l(\xi)!} \sum_{\forall l,\xi: \sum_{\tau} t_{l,\xi,\tau} = \sigma_l(\xi)} \prod_{l,\xi} \frac{\sigma_l(\xi)!}{\prod_{\tau,i} t_{i,l,\xi,\tau}!} \prod_{\tau,i} (|\text{fix } G[\tau]| \frac{p_{\tau * 2\xi l}}{u_i!})^{t_{i,l,\tau}}$$

Now we change the order of product and sum: (In the term above we have a sum over all possible products of terms, depending on  $l$  and  $\xi$  and in the term below we have a product over sums of all possible terms, belonging to fix  $l$  and  $\xi$ .)

$$= \sum_{k, \sigma \vdash_{\mathbf{C}_r} k} |\text{fix } F[\sigma]| \frac{1}{\prod_{l,\xi} (r \cdot l)^{\sigma_l(\xi)} \sigma_l(\xi)!} \prod_{l,\xi} \sum_{\sum_{\tau} t_{l,\xi,\tau} = \sigma_l(\xi)} \frac{\sigma_l(\xi)!}{\prod_{\tau} t_{l,\xi,\tau}!} \prod_{\tau} (|\text{fix } G[\tau]| \frac{p_{\tau * 2\xi l}}{u_i(\tau)!})^{t_{l,\tau}}$$

Now we use the multinomial theorem and get:

$$= \sum_{k, \sigma \vdash_{\mathbf{C}_r} k} |\text{fix } F[\sigma]| \frac{1}{k! r^k} \frac{k! r^k}{\prod_{l,\xi} (r \cdot l)^{\sigma_l(\xi)} \sigma_l(\xi)!} \prod_{l,\xi} \left( \sum_{\tau} (|\text{fix } G[\tau]| p_{\tau * 2\xi l} \frac{1}{u_i(\tau)!})^{\sigma_l(\xi)} \right)$$

As we have here the number of  $\sigma$  with the same cycle type we can go back to considering every different  $\sigma$ :

$$= \sum_{\sigma} |\text{fix } F[\sigma]| \frac{1}{k! r^k} \prod_{l,\xi} \left( \sum_{\tau} (|\text{fix } G[\tau]| p_{\tau * 2\xi l} \frac{1}{u_i(\tau)!})^{\sigma_l(\xi)} \right)$$

As  $p_{\tau * 2\xi l}$  is just another notation for replacing  $p_i(\xi)$  with  $p_{i:l}(\xi^k)$ , this is precisely the plethystic substitution:

$$\begin{aligned} &= \left( \sum_{n, \sigma \in \mathfrak{W}_{r,n}} |\text{fix } F[\sigma]| \frac{1}{n! r^n} p_{\sigma} \right) \circ \left( \sum_{n, \tau \in \mathfrak{S}_n} |\text{fix } G[\tau]| \frac{1}{n!} p_{\tau} \right) \\ &= Z_F \circ Z_G \end{aligned}$$

□

#### 4.4. Substitution of the Cycle Indicator Series of Type 3

We now analyze type 3 in the same way to find a possible definition of a substitution for type 3:

Again, we first consider a typical  $F \circ G$ -structure. Such a structure  $h$  is a tuple  $h = (\pi, f, (g_b)_{b \in \pi})$  where:

- (1)  $\pi$  is a set partition in  $\text{Par}[[n]]$
- (2)  $f$  is an  $F$ -structure on  $\pi_{\mathbf{C}_r}$
- (3)  $g_b$  are  $G$ -structures on  $b_{\mathbf{C}_r}$ ;

Now it is time to define  $F \circ G[\omega]$  for  $\omega \in \mathfrak{W}_{r,n}$ , acting on  $[n]_{\mathbf{C}_r}$  on an  $F \circ G$ -structure:

DEFINITION 4.4.1. We define

$$F \circ G[\omega]((\pi, f, (g_b)_{b \in \pi})) = (\omega(\pi), F[\sigma](f), (G[\omega|_{b_{\mathbf{C}_r}}](g_b))_{b \in \pi})$$

where

- (1)  $\omega(\pi)$  is defined pointwise via  $k \mapsto |\omega(k)|$ , so each block  $b \in \pi$  maps to  $\omega(b)$ . The result is thus a set partition with the same sizes of blocks.
- (2) We define  $\sigma(b) := \prod_{x \in b} \text{sgn}(\omega(x)) \cdot \omega(b)$  with  $\omega(\pi)$  as before, and  $\sigma(\xi b) = \xi \sigma(b)$ , so that we get a signed permutation on  $\pi_{\mathbf{C}_r}$ .

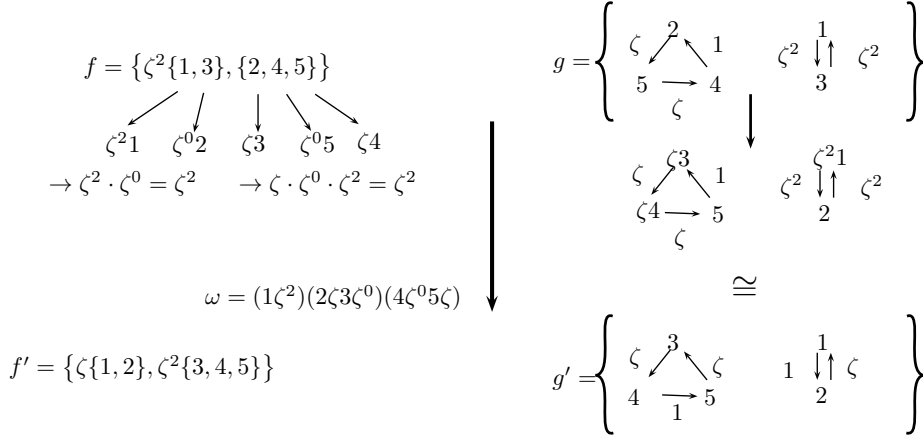


FIGURE 9.  $\omega$ -action on a substitution of type 3.

EXAMPLE 4.4.2. Let  $F$  be the species of  $r$ -vertices and  $G$  the species of  $r$ -cycles (this species can be seen as signed permutations that have an extra sign). An example for a structure  $h$  in  $F \circ G[[5]_{\mathbf{C}_3}]$  is the tuple  $h = (\pi, f, g = (g_b)_{b \in \pi})$  where

- $\pi = \{\{1, 3\}, \{2, 4, 5\}\}$
- $f = \{\zeta^2\{1, 3\}, \{2, 4, 5\}\}$
- $g = \{(2\zeta 5\zeta 4\zeta^0 2), (1\zeta^2 3\zeta^2)\}$

Now consider once again a signed permutation:

$$\omega = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \zeta^2 1 & \zeta 3 & 2 & 5 & \zeta 4 \end{pmatrix} = (1\zeta^2)(2\zeta 3\zeta^0)(4\zeta^0 5\zeta)$$

Then  $F \circ G[\omega]h$  is given through:

- $\pi' = \{\{1, 2\}, \{3, 4, 5\}\}$
- $f' = \{\zeta\{1, 2\}, \zeta^2\{3, 4, 5\}\}$
- $g' = \{(3\zeta 4\zeta^0 5\zeta), \{(1\zeta^0 2\zeta)\}$

For a graphical representation of this, see figure 9.

REMARK 4.4.3. Note that  $\omega \in \mathfrak{W}_{r,n}$  with  $\omega(\pi) = \pi$ , in the sense as before induces a signed permutation  $\sigma \in \mathfrak{W}_{r,n}$  on  $\pi_r$ . Remember that  $\omega$  and  $\sigma$  have this strong connection!

EXAMPLE 4.4.4.  $r = 2, n = 5$ : Let  $\pi = \{\{1, 2\}, \{3, 4\}, \{5\}\}$  and  $\omega = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ -3 & 4 & 2 & 1 & -5 \end{pmatrix}$ .

Then  $\sigma = \begin{pmatrix} \{1, 2\} & \{3, 4\} & \{5\} \\ -\{3, 4\} & \{1, 2\} & \{5\} \end{pmatrix}$ .

For the cycle indicator series  $Z_{F \circ G}$ , we need to consider under which conditions such a structure is a fixed point under a signed permutation (for a proof see lemma 4.3.9, the proofs are identical):

LEMMA 4.4.5. For a fixed point  $h \in F \circ G[n]_{\mathbf{C}_r}$  under  $\omega$  the following must hold:

- (1)  $\omega(\pi) = \pi$  (therefore the parts belonging to one cycle of  $\sigma$  need to have the same size)
- (2)  $f$  needs to be a fixed point under  $F[\sigma]$
- (3)  $G[\omega](g_b) = g_{\omega(b)}$

LEMMA 4.4.6. Let  $\sigma$  be a signed permutation and  $\pi \in \text{Par}[[n]_{C_r}]$ . Let  $\omega$  be a signed permutation that induces  $\sigma$  on  $\pi_r$  and  $\omega(\pi) = \pi$ .

Then we can construct the fixed points of  $\omega$  in the following way:

- (1)  $f$  be any fixed point of  $F[\sigma]$
- (2) for every cycle of  $\sigma$  with length  $l$  take an arbitrary block  $b$  belonging to this cycle, let  $g_b$  be an arbitrary fixed point of  $(\omega^l)_{|b_{C_r}}$  and  $g_\omega(b) = \omega(g_b) (= \sigma)$  for the other blocks belonging to the same cycle.

The number of fixed points is

$$|\text{fix } F[\sigma]| \cdot \prod_{c \text{ a cycle of } \sigma} |\text{fix } G[(\omega^{l_c})_{|b_c}]|$$

where  $p_c$  is an arbitrary block belonging to  $c$  and  $l_c$  is the length of  $c$ .

REMARK 4.4.7. The proof of this lemma is identical to the one of lemma 4.3.11. Note that it is necessary that  $\omega$  induces  $\sigma$ !

The definition of  $*_3$  might be a little unintuitive. The reason why we define it that way will become clear later on.

DEFINITION 4.4.8. For  $\tau \in \mathfrak{M}_{r,n}$  with cycle type

$$(\tau_1(\zeta^0), \tau_1(\zeta^1), \dots, \tau_1(\zeta^{r-1}), \tau_2(\zeta^0), \dots, \dots, \tau_m(\zeta^{r-1}))$$

and the power sum symmetric function associated with  $\tau$

$$p_\tau = p_1^{\tau_1(\zeta^0)}(\zeta^0) \cdot p_1^{\tau_1(\zeta^1)}(\zeta^1) \cdot \dots \cdot p_1^{\tau_1(\zeta^{r-1})}(\zeta^{r-1}) \cdot p_2^{\tau_2(\zeta^0)}(\zeta^0) \cdot \dots \cdot p_m^{\tau_m(\zeta^{r-1})}(\zeta^{r-1})$$

we define

$$p_{\tau*_3\xi l} := \begin{cases} p_{1.l}^{\tau_1(\zeta^0)}(\zeta^0) \cdot \dots \cdot p_{1.l}^{\tau_1(\zeta^{r-1})}(\zeta^{r-1}) \cdot p_{2.l}^{\tau_2(\zeta^0)}(\zeta^0) \cdot \dots \cdot p_{m.l}^{\tau_m(\zeta^{r-1})}(\zeta^{r-1}) & \text{if } \prod_{k,j} (\zeta^j)^{\tau_k(\zeta^j)} = \xi \\ 0 & \text{otherwise} \end{cases}.$$

REMARK 4.4.9. This means that by transforming  $p_\tau$  into  $p_{\tau*_3 l}$  we transform every  $p_i(\zeta^j)$  into a  $p_{i.l}(\zeta^j)$ , if the product of the signs of the elements of the signed permutation they are coming from are the same as  $\xi$  and otherwise we set it zero.

LEMMA 4.4.10. Let  $\omega \in W_{r,n}$ ,  $\sigma \in W_{r,n_\pi}$  induced by  $\omega$  on  $\pi$  and  $\pi \in \text{Par}[[n]]$ , then

$$p_\omega = \prod_{c \text{ cycle of } \sigma} p_{(\omega^{l_c})_{|b_c} *_3 \xi c l_c}$$

where  $l_c$  is the length of a cycle  $c$ , and  $b_c$  an arbitrary block of  $\pi$  belonging to  $c$ .

REMARK 4.4.11. Note that in the case of a species the condition  $\prod_{k,j} (\zeta^j)^{\tau_k(\zeta^j)} = \xi$  is always fulfilled as  $\tau$  and  $\sigma$  are induced by the same  $\omega$ .

We need it, however, to formulate theorem 4.4.21 in a reasonable way.

EXAMPLE 4.4.12. Let  $\omega = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ -4 & -5 & 6 & -2 & 1 & 3 \end{pmatrix}$  and  $\pi = \{\{1, 2, 3\}, \{4, 5, 6, \}\}$ . Then we have  $p_\omega = p_2(1) \cdot p_4(-1)$ .  $\sigma$  consists of one cycle with length 2 and type  $-$ . We then have

$$\sigma = (\{1, 2, 3\} + \{4, 5, 6\} -) \text{ and } \tau = \omega^2 | \{\pm 4, \pm 5, \pm 6, \} = \begin{pmatrix} 4 & 5 & 6 \\ 5 & -4 & 6 \end{pmatrix}.$$

We have  $p_\tau = p_1(1) \cdot p_2(-1)$ . The lemma claims that  $p_\omega = p_{\tau*_3 -2} = p_{1.2}(1) \cdot p_{2.2}(-1)$  as  $1^1 \cdot (-1)^1 = -1$ , which is obviously true in this case.

For a graphical representation see figure 10.

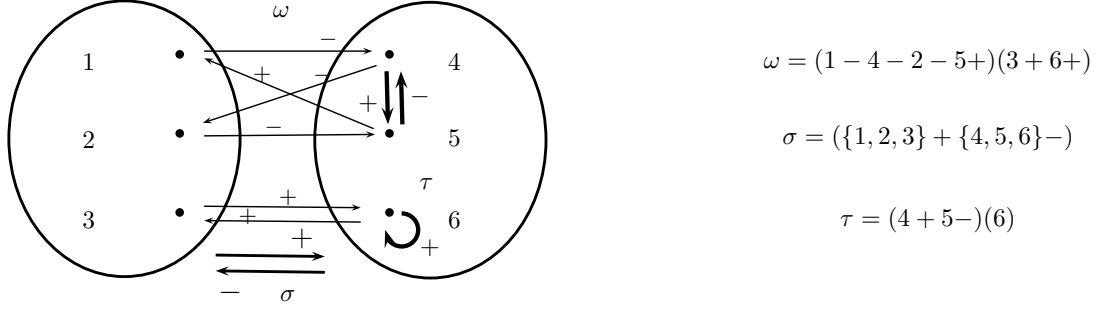


FIGURE 10. An example of a  $*_3$ -operation.

PROOF. If  $\sigma$  is induced by  $\omega$ , the condition  $\prod_{k,j} (\zeta^j)^{p_k(\zeta^j)} = \xi$ , for  $\xi$  being the type of a cycle in  $\sigma$ , and  $p$  being the symmetric function of  $\omega|_{b_c}^l$  is fulfilled, so all terms in this product are non-zero. Therefore, every  $*_3\xi c_l$  acts like  $*_1l_c$ , so the rest of the proof is exactly the same as the proof of lemma 4.3.18.  $\square$

DEFINITION 4.4.13. We define  $s_{i,l,\xi}$  as the number of cycles of  $\sigma$  with length  $l$ , type  $\xi$ , blocks of size  $u_i$ . Furthermore, we define  $t_{i,l,\xi,\tau}$  as the number of cycles with length  $l$ , type  $\xi$ , blocks of size  $u_i$ , and with  $\tau = (\omega^l)|_{b_c}$ , where  $b_c$  is one (arbitrary but fixed) block of the cycle.

LEMMA 4.4.14. *There are three identities following directly through definition 4.4.13:*

- (1)  $\sum_i s_{i,l,\xi} = \sigma_l(\xi)$
- (2)  $\sum_\tau t_{i,l,\xi,\tau} = s_{i,l,\xi}$
- (3)  $\sum s_{i,l,\xi} \cdot l$  is the number of blocks with size  $u_i$  ( $= j_i$ ).

REMARK 4.4.15. Note that for  $t_{i,l,\xi,\tau}$  with a given  $\tau$ ,  $i$  is also given, so we sometimes omit it (with  $\tau$  we also know the size of blocks of which  $\tau$  is a signed permutation).

LEMMA 4.4.16. *For fixed  $\sigma$  and  $\pi$  the number of different  $\omega$  which induce  $\sigma$  and the same multiset of  $(\omega^l)|_{b_c}$  for every cycle  $c$  (with length  $l_c$  and an arbitrary but fixed block  $b_c$  belonging to this cycle  $c$ ) where a signed permutation of this multiset can belong to any proper cycle  $c$  is*

$$\prod_{i,l,\xi} u_i!^{s_{i,l,\xi}(l-1)} r^{(u_i-1)s_{i,l,\xi}(l-1)} \cdot \prod_{i,l,\xi} \frac{s_{i,l,\xi}!}{\prod_\tau t_{i,l,\xi,\tau}!}$$

REMARK 4.4.17. For better understanding compare with type 1. (See lemma 4.3.32 and particularly remark 4.3.33!)

PROOF. For a given  $\omega|_{b_c}^l$ , we can construct  $\omega$  on the blocks  $\omega^m(b_c)$ ,  $m = 1, 2, \dots, l-1$ , in  $u_c!(l_c-1)r^{u_c(l_c-1)}$  different ways: We have  $u_c!r^{u_c-1}$  ways to define  $\omega|_{b_c} : b_c \rightarrow \sigma(b_c)$  (we can choose  $u_c - 1$  signs, but the last one is defined by  $\sigma$ , as the product of the signs need to be the sign of  $\sigma(b_c)$ ), another  $u_c!r^{u_c-1}$  ways to define  $\omega|_{\sigma b_c}$  and so on. Only for  $\omega|_{\sigma^{l-1}(b_c)}$  we are forced to define in a way that we get  $\omega|_{b_c}^l$ , as  $\omega|_{b_c}^l$  also defines the sign of the  $c$  in  $\sigma$  the right way ( $\sigma$  is induced by  $\omega$ , the cycle type can be defined as the product over all signs or as  $\xi$  with  $\xi x = \omega^l(x)$ ). All together, we have  $u_c!(l_c-1)r^{u_c(l_c-1)}$  for one cycle, considering all cycles, the factors multiply and give us the first part of our formula.

The second part is, once more, analogous to that of lemma 4.3.32.  $\square$

LEMMA 4.4.18. *The number of permutations  $\sigma$  with the same cycle type on  $\pi$  and the same term  $p_\omega$  (with notation as before) is:*

$$\frac{\prod_i j_i!}{\prod_{i,l,\xi} s_{i,l,\xi}! l^{s_{i,l,\xi}}} \cdot \prod_{i,l,\xi} r^{s_{i,l,\xi}(l-1)}.$$

PROOF. Note that here the cycle type is given so  $\sigma$  is a possible choice for being induced by an  $\omega$  with  $p_\omega$ . Therefore, the rest of the proof is analogous to the one of lemma 4.3.35.  $\square$

Now we will transform  $Z_{F \circ G}$  to get a formula that defines  $Z_F \circ Z_G$  for type 3 in a reasonable way:

$$\begin{aligned} Z_{F \circ G} &= \sum_{n, \omega \in \mathfrak{W}_{r,n}} \frac{1}{n! r^n} |\text{fix}(F \circ G)[\omega]| p_\omega \\ &= \sum_n \sum_{\pi \in \text{Par}[[n]]} \sum_{\sigma \in W_{r,|\pi|}} \sum_{\omega \text{ induces } \sigma \text{ on } \pi} \frac{1}{n! r^n} |\text{fix}_\pi(F \circ G)[\omega]| p_\omega \end{aligned}$$

Where  $\text{fix}_\pi(F \circ G)[\omega]$  denotes those fixed points that have  $\pi$  as their partition.

lemma 4.4.6 tells us the number of fixed points appearing in this term, due to lemma 4.4.10 we can transform  $p_\omega$  appropriately.  $b_c$  is here an arbitrary but fixed block belonging to the cycle  $c$  with length  $l_c$  and type  $\xi_c$  of  $\sigma$ .

Note that if we sum over all  $\omega$ , we sum over all  $\omega$  in  $\mathfrak{W}_{r,n}$  for all possible  $n$ , analogous for  $\pi$  and  $\sigma$ .  $n$  is then determined by  $\sigma, \pi$  and  $\omega$ :

$$= \sum_\pi \sum_\sigma \sum_\omega \frac{1}{n! r^n} \prod_{c \text{ cycle of } \sigma} |\text{fix } G[(\omega^l)_{|b_c|}] \cdot p_{(\omega^l)_{|p_c * 3 \xi_c l_c}} \cdot |\text{fix } F[\sigma]|$$

We use lemma 4.4.16 and consider, from now on, only  $\tau = (\omega^l)_{|b_c}$  and  $\sigma$ . For the manipulation of the next lines we use lemma 4.4.14:

$$\begin{aligned} &= \sum_{\pi, \sigma} |\text{fix } F[\sigma]| \frac{1}{n! r^n} \prod_{i,l,\xi} u_i!^{s_{i,l,\xi}(l-1)} r^{(u_i-1)s_{i,l,\xi}(l-1)} s_{i,l,\xi}! \prod_\tau (|\text{fix } G[\tau]| p_{\tau * 3 \xi l})^{t_{i,l,\xi,\tau}} \frac{1}{t_{i,\xi,l,\tau}!} \\ &= \sum_{\pi, \sigma} |\text{fix } F[\sigma]| \frac{1}{n! r^n} \prod_{i,l,\xi} \frac{u_i!^{s_{i,l,\xi} l r^{(u_i-1)s_{i,l,\xi} l}}}{u_i!^{\sum_\tau t_{i,l,\xi,\tau}} r^{(u_i-1)(\sum_\tau t_{i,l,\xi,\tau})}} s_{i,l,\xi}! \prod_\tau \frac{(|\text{fix } G[\tau]| p_{\tau * 3 \xi l})^{t_{i,\xi,l,\tau}}}{t_{i,l,\xi,\tau}!} \\ &= \sum_{\pi, \sigma} |\text{fix } F[\sigma]| \frac{1}{n! r^n} \prod_i u_i!^{j_i} r^{(u_i-1)j_i} \prod_{l,\xi} s_{i,l,\xi}! \prod_\tau \frac{(|\text{fix } G[\tau]| \frac{p_{\tau * 3 \xi l}}{u_i! r^{u_i-1}})^{t_{i,l,\xi,\tau}}}{t_{i,\xi,l,\tau}!} \end{aligned}$$

Next we use lemma 4.4.18, so from now on, we only consider the cycle type of  $\sigma$ :

$$\begin{aligned} &= \sum_{\sigma \vdash r, k} |\text{fix } F[\sigma]| \sum_\pi \frac{1}{n! r^n} \frac{\prod_i j_i! \cdot \prod_{i,l,\xi} r^{s_{i,l,\xi}(l-1)}}{\prod_{i,l,\xi} s_{i,l,\xi}! l^{s_{i,l,\xi}}} \prod_i u_i!^{j_i} r^{(u_i-1)j_i} \prod_{l,\xi} s_{i,l,\xi}! \prod_\tau \frac{(|\text{fix } G[\tau]| \frac{p_{\tau * 3 \xi l}}{u_i! r^{u_i}} r)^{t_{i,l,\xi,\tau}}}{t_{i,l,\xi,\tau}!} \\ &= \sum_{\sigma \vdash r, k} |\text{fix } F[\sigma]| \sum_\pi \frac{1}{n! r^n} \prod_i r^{(u_i-1)j_i} j_i! u_i!^{j_i} \prod_{l,\xi} \frac{r^{j_i}}{r^{s_{i,l,\xi} l^{s_{i,l,\xi}}}} \prod_\tau \frac{(|\text{fix } G[\tau]| \frac{p_{\tau * 3 \xi l}}{u_i! r^{u_i}} r)^{t_{i,l,\xi,\tau}}}{t_{i,\xi,l,\tau}!} \\ &= \sum_{\sigma \vdash r, k} |\text{fix } F[\sigma]| \sum_\pi \frac{1}{n! r^n} r^n \frac{1}{\prod_{l,\xi} r^{\sum_i s_{i,l,\xi}}} \prod_i j_i! u_i!^{j_i} \prod_{l,\xi} \frac{1}{l^{s_{i,l,\xi}}} \prod_\tau \frac{(|\text{fix } G[\tau]| \frac{p_{\tau * 3 \xi l}}{u_i! r^{u_i}} r)^{t_{i,\xi,l,\tau}}}{t_{i,l,\xi,\tau}!} \end{aligned}$$

Now we use lemma 4.3.25 and from now on just consider the sizes of the parts of  $\pi$ :

$$\begin{aligned}
&= \sum_{\sigma \vdash_r k} |\text{fix } F[\sigma]| \frac{1}{\prod_{l,\xi} r^{\sigma_l(\xi)}} \sum_{\pi \text{ with sizes } u_i} \prod_{i,l,\xi} \frac{1}{l^{s_{i,l,\xi}}} \prod_{\tau} \frac{(|\text{fix } G[\tau]| \frac{p_{\tau * 3 \xi l}}{u_i! r^{u_i}} r)^{t_{i,l,\tau}}}{t_{i,\xi,l,\tau}!} \\
&= \sum_{\sigma \vdash_r k} |\text{fix } F[\sigma]| \frac{1}{\prod_{l,\xi} r^{\sigma_l(\xi)} l^{\sum_i s_{i,l,\xi}}} \sum_{\pi \text{ with sizes } u_i} \prod_{l,\xi} \frac{\sigma_l(\xi)!}{\sigma_l(\xi)!} \prod_{l,\xi,i,\tau} \frac{(|\text{fix } G[\tau]| \frac{p_{\tau * 3 \xi l}}{u_i! r^{u_i}} r)^{t_{i,l,\xi,\tau}}}{t_{i,\xi,l,\tau}!} \\
&= \sum_{\sigma \vdash_r k} |\text{fix } F[\sigma]| \frac{1}{\prod_{l,\xi} \sigma_l(\xi)! r^{\sigma_l(\xi)} l^{\sigma_l(\xi)}} \sum_{\pi \text{ with sizes } u_i} \prod_{l,\xi} \frac{\sigma_l(\xi)!}{\prod_{i,\tau} t_{i,l,\xi,\tau}!} \prod_{i,\tau} (|\text{fix } G[\tau]| \frac{p_{\tau * 3 \xi l}}{u_i! r^{u_i}} r)^{t_{i,l,\xi,\tau}}
\end{aligned}$$

The sizes  $u_i$  and the number of their appearance are fully determined by the sums of the  $t_{i,\xi,l,\tau}$ :

$$= \sum_{\sigma \vdash_r k} |\text{fix } F[\sigma]| \frac{1}{\prod_{l,\xi} \sigma_l(\xi)! r^{\sigma_l(\xi)} l^{\sigma_l(\xi)}} \sum_{\forall l,\xi: \sum_{\tau} t_{l,\xi,\tau} = \sigma_l(\xi)} \prod_{l,\xi} \frac{\sigma_l(\xi)!}{\prod_{i,\tau} t_{i,l,\xi,\tau}!} \prod_{i,\tau} (|\text{fix } G[\tau]| \frac{p_{\tau * 3 \xi l}}{u_i! r^{u_i}} r)^{t_{i,l,\xi,\tau}}$$

Now we change the order of product and sum: (In the term above we have a sum over all possible products of terms, depending on  $l$  and  $\xi$ , and in the term below we have a product over sums of all possible terms, belonging to fix  $l$  and  $\xi$ .)

$$= \sum_{\sigma \vdash_r k} |\text{fix } F[\sigma]| \frac{1}{\prod_{l,\xi} \sigma_l(\xi)! r^{\sigma_l(\xi)} l^{\sigma_l(\xi)}} \prod_{l,\xi} \sum_{\sum_{\tau} t_{l,\xi,\tau} = \sigma_l(\xi)} \frac{\sigma_l(\xi)!}{\prod_{\tau} t_{l,\xi,\tau}!} \prod_{\tau} (|\text{fix } G[\tau]| \frac{p_{\tau * 3 \xi l}}{u_{i(\tau)}! r^{u_{i(\tau)}}} r)^{t_{l,\xi,\tau}}$$

As we defined in definition 4.4.8  $*_3 \xi l$  as zero for any  $\tau$  that does not have the right signs, we can take here the sum over every  $\tau$ . Moreover we use the multinomial theorem:

$$\begin{aligned}
&= \sum_{\sigma \vdash_r k} |\text{fix } F[\sigma]| \frac{1}{\prod_{l,\xi} \sigma_l(\xi)! r^{\sigma_l(\xi)} l^{\sigma_l(\xi)}} \prod_{l,\xi} \left( \sum_{\tau} (|\text{fix } G[\tau]| \frac{p_{\tau * 3 \xi l}}{u_{i(\tau)}! r^{u_{i(\tau)}}} r)^{\sigma_l(\xi)} \right) \\
&= \sum_{\sigma \vdash_r k} |\text{fix } F[\sigma]| \frac{1}{k! r^k} \frac{k! r^k}{\prod_{l,\xi} \sigma_l(\xi)! r^{\sigma_l(\xi)} l^{\sigma_l(\xi)}} \prod_{l,\xi} \left( \sum_{\tau} (|\text{fix } G[\tau]| \frac{p_{\tau * 3 \xi l}}{u_{i(\tau)}! r^{u_{i(\tau)}}} r)^{\sigma_l(\xi)} \right)
\end{aligned}$$

As we here have the number of  $\sigma$  with the same cycle type, we can go back to considering every different  $\sigma$ :

$$\begin{aligned}
&= \sum_{\sigma} |\text{fix } F[\sigma]| \frac{1}{k! r^k} \prod_{l,\xi} \left( \sum_{\tau} (|\text{fix } G[\tau]| \frac{p_{\tau * 3 \xi l}}{u_{i(\tau)}! r^{u_{i(\tau)}}} r)^{\sigma_l(\xi)} \right) \\
&= \sum_{\sigma} |\text{fix } F[\sigma]| \frac{1}{k! r^k} \prod_{l,\xi} \left( r \sum_{\tau} (|\text{fix } G[\tau]| p_{\tau * 3 \xi l} \frac{1}{u_{i(\tau)}! r^{u_{i(\tau)}}})^{\sigma_l(\xi)} \right)
\end{aligned}$$

This result leads to the following definition. Note that the factor  $r$  was more or less expected, as it is necessary, also in the substitution of the exponential generating series: compare with lemma 4.1.9. Furthermore, this definition is the best we could have expected as [Hen04] wrote that there is no way to define a plethysm  $\Lambda(r) \times \Lambda(r) \rightarrow \Lambda(r)$ , as at least three of the four homomorphism rules are fulfilled.

DEFINITION 4.4.19. We define  $f \circ g$  for type 3 uniquely as follows:

(1)  $\forall g \in \Lambda(r)$ , the map  $\Lambda(1) \rightarrow \Lambda(r) : f \mapsto f \circ g$  is a  $\mathbb{C}$ -algebra homomorphism.

( $\forall g \in \Lambda(r), f_1, f_2 \in \Lambda(1) : (f_1 + f_2) \circ g = f_1 \circ g + f_2 \circ g$  and  $(f_1 \cdot f_2) \circ g = (f_1 \circ g) \cdot (f_2 \circ g)$ )

(2)  $\forall i, j \in \mathbb{N}$ , the map  $\Lambda(r) \rightarrow \Lambda(r) : g \mapsto p_i \circ g$  is regarding to addition a homomorphism.

$$(\forall i \in \mathbb{N}, g_1, g_2 \in \Lambda(r) : p_i \circ (g_1 + g_2) = p_i \circ g_1 + p_i \circ g_2)$$

$$(3) \quad p_i(\xi) \circ \prod_{k,j} p_k(\zeta^j)^{\tau_{k,j}} = \begin{cases} r \prod_{k,j} p_{k \cdot i}(\zeta^j)^{\tau_{k,j}} & \prod_{k,j} (\zeta^j)^{\tau_{k,j}} = \xi \\ 0 & \text{otherwise} \end{cases}.$$

This can be defined in another way (compare [BLL98]):

LEMMA 4.4.20. *If  $\tilde{Z}_{G\xi}$  are the terms of  $Z_G$  where the products of the signs fulfill  $\prod_{k,j} (\zeta^j)^{p_k(\zeta^j)} = \xi$  it holds that:*

$$\begin{aligned} & Z_F(p_1(1), p_1(\zeta), p_1(\zeta^2), \dots, p_1(\zeta^{r-1}), p_2(1), \dots, p_2(\zeta^{r-1}), p_3(1), \dots) \\ & \text{circ} Z_G(p_1(1), p_1(\zeta), p_1(\zeta^2), \dots, p_1(\zeta^{r-1}), p_2(1), \dots, p_2(\zeta^{r-1}), p_3(1), \dots) \\ = & Z_F(r \tilde{Z}_{G1}(p_{1.1}(1), p_{1.1}(\zeta) \dots, p_{1.1}(\zeta^{r-1}), p_{2.1}(1), \dots), r \tilde{Z}_{G\zeta}(p_{1.1}(1), p_{1.1}(\zeta) \dots, p_{1.1}(\zeta^{r-1}), p_{2.1}(1), \dots) \\ & \dots r \tilde{Z}_{G\zeta^{r-1}}(p_{1.1}(1), p_{1.1}(\zeta) \dots, p_{1.1}(\zeta^{r-1}), p_{2.1}(1), \dots), r \tilde{Z}_{G1}(p_{1.2}(1), \dots, p_{1.2}(\zeta^{r-1}), p_{2.2}(1), \dots), \dots \\ & r \tilde{Z}_{G1}(p_{1.3}(1), \dots, p_{1.3}(\zeta^{r-1}), p_{2.3}(1), \dots), \dots) \end{aligned}$$

In other words we substitute every  $p_i(\xi)$  in  $Z_F$  by a modified  $Z_G$ , where we have substituted every  $p_\tau$  into a  $p_{\tau * 3\xi i}$ .

PROOF. The first part of the prove is analogous to that of type 1 (lemma 4.4.20).

When we get to the last part, where we consider  $p_i \circ (b_\mu p_\mu)$ , we now use  $p_i(\xi) \circ \prod_{k,j} p_k(\zeta^j)^{\tau_{k,j}} = \begin{cases} r \prod_{k,j} p_{k \cdot i}(\zeta^j)^{\tau_{k,j}} & \prod_{k,j} (\zeta^j)^{\tau_{k,j}} = \xi \\ 0 & \text{otherwise} \end{cases}$  where the condition  $\prod_{k,j} (\zeta^j)^{\tau_{j,k}}$  is fulfilled by the definition of  $\tilde{Z}_{G\xi}$ . Therefore, the lemma holds.  $\square$

With that we can proof the following theorem of type 3:

THEOREM 4.4.21 (Type 3). *Let  $F$  and  $G$  be  $r$ -species. Then  $Z_{F \circ G} = Z_F \circ Z_G$ .*

PROOF. We have already shown that

$$Z_{F \circ G} = \sum_{\sigma} |\text{fix } F[\sigma]| \frac{1}{k! r^k} \prod_{l, \xi} \left( r \sum_{\tau} (|\text{fix } G[\tau]| p_{\tau * 3\xi l} \frac{1}{u_{i(\tau)}! r^{u_{i(\tau)}}}) \right)^{\sigma_l(\xi)}.$$

Now, with the help of lemma 4.4.20 we see that this is nothing different as:

$$= \left( \sum_{n, \sigma \in \mathfrak{W}_{r,n}} |\text{fix } F[\sigma]| \frac{1}{r^n n!} p_\sigma \right) \circ \left( \sum_{n, \tau \in \mathfrak{W}_{r,n}} |\text{fix } G[\tau]| \frac{1}{n! r^n} p_\tau \right) = Z_F \circ Z_G.$$

$\square$

For examples see the following section( 4.5).



## 4.5. Examples

EXAMPLE 4.5.1 (Signed Permutations). We want to recalculate  $Z_{\mathcal{E} \circ \mathcal{C}^r} = Z_{\mathcal{E}} \circ Z_{\mathcal{C}^r}$  directly:

$$\begin{aligned} Z_{\mathcal{E}} \circ Z_{\mathcal{C}^r} &= \left( \exp \left( \sum_{k>0} \frac{p_k}{k} \right) \right) \circ \left( \sum_{\xi \in \mathbf{C}_r} \sum_{l,j>0} \frac{\phi(l)}{lj} p_l(\xi)^j \right) = \exp \left( \sum_{k>0} \frac{p_k}{k} \right)_{|p_k = \sum_{\xi \in \mathbf{C}_r} \sum_{l,j>0} \frac{\phi(l)}{lj} p_{l \cdot k}(\xi)^j} \\ &= \exp \left( \sum_{k>0} \frac{\sum_{\xi \in \mathbf{C}_r} \sum_{l,j>0} \frac{\phi(l)}{lj} p_{l \cdot k}(\xi)^j}{k} \right) = \exp \left( \sum_{k>0} \sum_{\xi \in \mathbf{C}_r} \sum_{l,j>0} \frac{\phi(l)}{jlk} p_{lk}(\xi)^j \right) \end{aligned}$$

We substitute  $kl$  by  $m$  and use  $\sum_{l|m} \phi(l) = m$ :

$$\begin{aligned} &= \exp \left( \sum_{\xi \in \mathbf{C}_r} \sum_{m,j>0} \frac{p_m(\xi)^j}{jm} \left( \sum_{l|m} \phi(l) \right) \right) = \exp \left( \sum_{\xi \in \mathbf{C}_r} \sum_{m,j>0} \frac{p_m(\xi)^j}{j} \right) \\ &= \exp \left( \sum_{\xi \in \mathbf{C}_r} \sum_{m>0} \log \frac{1}{1 - p_m(\xi)} \right) = \prod_{\xi \in \mathbf{C}_r} \prod_{m>0} \exp \left( \log \frac{1}{1 - p_m(\xi)} \right) \\ &= \prod_{\xi \in \mathbf{C}_r} \prod_{m>0} \frac{1}{1 - p_m(\xi)} = \prod_{\xi \in \mathbf{C}_r} \prod_{m>0} \sum_{k>0} p_m^k(\xi) \\ &= \sum_{n, \omega \vdash_r n} p_{\omega} = Z_{\mathcal{W}^r} = Z_{\mathcal{E} \circ \mathcal{C}^r} \end{aligned}$$

Furthermore, we want to show an equivalent way to build signed permutations by showing  $\mathcal{E} \circ \mathcal{C}^r \cong \mathcal{E}^r \circ \mathcal{C}^r$ : An  $\mathcal{E}^r \circ \mathcal{C}^r$ -structure is a tuple  $(\pi, f, (g_b)_{b \in \pi})$  where:

- (1)  $\pi$
- (2)  $f = \pi$
- (3)  $g_b = \mathcal{C}^r[b_{\mathbf{C}_r}]$

As there is a natural bijection to the  $\mathbf{C}_r$ -partitions (see lemma 3.4.6), and  $b_{\mathbf{C}_r} = b'$  for  $b \in \pi$  and  $b'$  the associated block in the associated  $\mathbf{C}_r$ -partition, we can easily build an isomorphism to  $\mathcal{E} \circ \mathcal{C}^r$ . We want to calculate  $Z_{\mathcal{E}^r} \circ Z_{\mathcal{C}^r}$  as another example of the plethysm of type 3:

$$\begin{aligned} Z_{\mathcal{E}^r} \circ Z_{\mathcal{C}^r} &= \left( \exp \left( \sum_{j=0}^{r-1} \sum_{k>0} \frac{p_k(\zeta^j)}{kr} \right) \right) \circ \left( \sum_{\xi \in \mathbf{C}_r} \sum_{l,j>0} \frac{\phi(l)}{lj} p_l(\xi)^j \right) \\ &= \left( \exp \left( \sum_{j=0}^{r-1} \sum_{k>0} \frac{p_k(\zeta^j)}{kr} \right) \right)_{|p_k(1) = \begin{cases} \sum_{\xi \in \mathbf{C}_r} \sum_{l,j>0} \frac{\phi(l)}{lj} p_{l \cdot k}(\xi)^j & \prod_{n=0}^{r-1} \zeta^{n \cdot j} = 1 \\ 0 & \text{otherwise} \end{cases}} \end{aligned}$$

As  $\prod_{n=0}^{r-1} \zeta^{n \cdot j} = \zeta^{j \sum_{n=0}^{r-1} n} = \zeta^{j1/2r(r-1)} = 1$  we get:

$$= \exp \left( r \sum_{k>0} \frac{\sum_{\xi \in \mathbf{C}_r} \sum_{l,j>0} \frac{\phi(l)}{lj} p_{l \cdot k}(\xi)^j}{rk} \right)$$

This is just the second line of the calculation above so we obtain:

$$= \sum_{n, \omega \vdash_r n} p_\omega = Z_{\mathcal{V}^r}$$

EXAMPLE 4.5.2 (Vertices of Cycles). We now calculate  $Z_{\mathcal{V}^r \circ \mathcal{C}}$  by using theorem 4.3.1:

$$\begin{aligned} Z_{\mathcal{V}^r \circ \mathcal{C}} &= Z_{\mathcal{V}^r} \circ Z_{\mathcal{C}} = \left( \exp \left( \sum_{k>0} \frac{p_k(1)}{k} \right) \right) \circ \left( \sum_{l,j>0} \frac{\phi(l)}{l^j} p_l^j \right) \\ &= \exp \left( \sum_{k>0} \frac{p_k(1)}{k} \right)_{|p_k(\xi) = \sum_{l,j>0} \frac{\phi(l)}{l^j} p_{l \cdot k}(\xi^l)^j} = \exp \left( \sum_{k>0} \frac{\sum_{l,j>0} \frac{\phi(l)}{l^j} p_{l \cdot k}(1^l)^j}{k} \right) \end{aligned}$$

Now we do the same calculations as in Example 4.5.1 and get:

$$= \sum_{n, \omega \vdash n} p_\omega(1)$$

EXAMPLE 4.5.3 (Vertices of Signed Cycles). We now calculate  $Z_{\mathcal{V}^r \circ \mathcal{C}^r}$  by using theorem 4.3.1:

$$\begin{aligned} Z_{\mathcal{V}^r \circ \mathcal{C}^r} &= Z_{\mathcal{V}^r} \circ Z_{\mathcal{C}^r} = \left( \exp \left( \sum_{k>0} \frac{p_k(1)}{k} \right) \right) \circ \left( \sum_{\xi \in \mathbf{C}_r} \sum_{l,j>0} \frac{\phi(l)}{l^j} p_l(\xi)^j \right) \\ &= \exp \left( \sum_{k>0} \frac{p_k(1)}{k} \right)_{|p_k(1) = \begin{cases} \sum_{\xi \in \mathbf{C}_r} \sum_{l,j>0} \frac{\phi(l)}{l^j} p_{l \cdot k}(\xi)^j & \prod_{n=0}^{r-1} \zeta^{n \cdot j} = 1 \\ 0 & \text{otherwise} \end{cases}} \end{aligned}$$

Again  $\prod_{n=0}^{r-1} \zeta^{n \cdot j} = \zeta^j \sum_{n=0}^{r-1} n = \zeta^{j^2/2r(r-1)} = 1$ , so we get:

$$= \exp \left( r \sum_{k>0} \frac{\sum_{\xi \in \mathbf{C}_r} \sum_{l,j>0} \frac{\phi(l)}{l^j} p_{l \cdot k}(\xi)^j}{k} \right) = \exp \left( \sum_{k>0} \sum_{\xi \in \mathbf{C}_r} \sum_{l,j>0} \frac{\phi(l)}{j l k} p_{l k}(\xi)^j \right)^r$$

Now we do the same calculations as in Example 4.5.1 and get:

$$= \left( \sum_{n, \omega \vdash_r n} p_\omega \right)^r$$

EXAMPLE 4.5.4 ( $r$ -Sets). We consider the species of  $r$ -sets. This species can also be seen as  $\mathcal{E} \circ \mathcal{E}_1^r$ :

$$\mathcal{E}^r \cong \mathcal{E} \circ \mathcal{E}_1^r[[\mathbf{C}_r, M]] = \sum_{\pi \in \text{Par}_{\mathbf{C}_r}[M]} (\mathcal{E} \times \prod_{N \in \pi} \mathcal{E}_1^r[N])$$

As  $\mathcal{E}_1^r[N] = \emptyset$  if  $|N| \neq r$  and  $\{N\}$  otherwise, we get the set  $\{\{\pi, f, g_b\}\}$  with:

- (1)  $\pi = \{\{|x|\}_{\mathbf{C}_r}\}$
- (2)  $f = \pi$
- (3)  $g_{\{|x|\}_{\mathbf{C}_r}} = \{|x|\}_{\mathbf{C}_r}$

This is isomorphic to  $\mathcal{E}^r$ .

Now we can calculate  $Z_{\mathcal{E}} \circ Z_{\mathcal{E}_1^r}$  directly:

$$Z_{\mathcal{E}} \circ Z_{\mathcal{E}_1^r} = \left( \exp \left( \sum_{k \geq n} \frac{p_k}{k} \right) \right) \circ \left( \sum_{j=0}^{r-1} p_1(\zeta^j) \right) = \exp \left( \sum_{k \geq n} \frac{\sum_{j=0}^{r-1} p_{1 \cdot k}(\zeta^j)}{k} \right) = \exp \left( \sum_{j=0}^{r-1} \sum_{k \geq n} \frac{p_k(\zeta^j)}{k \cdot r} \right)$$

EXAMPLE 4.5.5 (Vertices). Analogously, we can identify  $\mathcal{V}^r$  with  $\mathcal{E} \circ \mathcal{V}_1^r$ : We get the set  $\mathcal{E} \circ \mathcal{V}_1^r[M_{\mathbf{C}_r}] = \{\{\pi, f, g_b\}\}$  with:

- (1)  $\pi = \{\{|x|\}_{\mathbf{C}_r}\}$
- (2)  $f = \pi$
- (3)  $g_{\{|x|\}_{\mathbf{C}_r}} = \{x\}$

This is isomorphic to  $\mathcal{E}^r$ .

Now we can calculate  $Z_{\mathcal{E}} \circ Z_{\mathcal{V}_1^r}$  directly:

$$Z_{\mathcal{E}} \circ Z_{\mathcal{V}_1^r} = \left( \exp \left( \sum_{k \geq n} \frac{p_k}{k} \right) \right) \circ (p_1(1)) = \exp \left( \sum_{k \geq n} \frac{p_{1 \cdot k}(1)}{k} \right)$$

Furthermore, we can identify  $\Delta\mathcal{V}$  with  $\mathcal{E} \circ \{\Delta\mathcal{V}_1^r\}$ .

EXAMPLE 4.5.6 (Diagonals). We can identify  $\mathcal{D}$  with  $\mathcal{E}_1^r \circ \mathcal{E}$ :

$\mathcal{E}_1^r \circ \mathcal{E}[M_{\mathbf{C}_r}] = \{\{\pi, f, g_b\}\}$  with

- (1)  $\pi \in \text{Par}_{\mathbb{B}_r}[M_{\mathbf{C}_r}]$  with size 1
- (2)  $f = \pi$
- (3) ' $g_b = b$ '

Therefore, we can define  $r$ -diagonals as  $\mathcal{D}^r = \mathcal{E}_1^r \circ \mathcal{E}$ . We now calculate  $Z_{\mathcal{D}^r}$ :

$$\begin{aligned} Z_{\mathcal{D}^r} &= Z_{\mathcal{E}_1^r} \circ Z_{\mathcal{E}} = \left( \frac{1}{r} \sum_{j=0}^{r-1} p_1(\zeta^j) \right) \circ \left( \exp \left( \sum_{k \geq n} \frac{p_k}{k} \right) \right) \\ &= \frac{1}{r} \sum_{j=0}^{r-1} \exp \left( \sum_{k \geq n} \frac{p_{k \cdot 1}(\zeta^{j \cdot k})}{k} \right) \end{aligned}$$

EXAMPLE 4.5.7 (Set Partitions). We can identify the three kinds of set partitions as substitutions of species too:

- (1)  $\text{Par}[S] \cong \mathcal{E} \circ \mathcal{E}_+[S]$
- (2)  $\text{Par}_{\mathbf{C}_r}[S] \cong \mathcal{E} \circ \mathcal{E}_+^r[S]$
- (3)  $\text{Par}_{\mathbb{B}_r}[S] \cong \mathcal{E}^r \circ \mathcal{E}_+[S]$

Therefore we have:

$$\text{Par}_{\mathbb{B}_r}[S] \cong (\mathcal{E} \circ \mathcal{E}_1^r) \circ \mathcal{E}_+ = \mathcal{E} \circ (\mathcal{E}_1^r \circ \mathcal{E}_+) = \mathcal{E} \circ \mathcal{E}_1^r \circ \mathcal{E}_+$$

This holds, as for a tuple  $(\pi, f, g)$  in the substitution in all three cases  $\pi = f$  is exactly the partition, and  $g$  consists of the parts of this partition. With the help of this, we can easily calculate the cycle indicator series.

EXAMPLE 4.5.8. (Signed Cycles) We have seen that signed cycles have  $r$   $\mathfrak{W}_{r,n}$ -orbits: In every such orbit there are exactly the signed cycles with the same type. Therefore the signed cycles can be represented as the sum of  $r$ -species of signed cycles of type  $\xi \in \mathbf{C}_r$ . Unfortunately for types not equal 1 this is quite complicated.

However there is a way to represent the  $r$ -species of signed cycles with type 1 as they are  $\mathcal{E}_1^r \circ \mathcal{C}$ : Therefore we consider  $\mathcal{E}_1^r \circ \mathcal{C}$ -structures. Such a structure is a tuple  $(\pi, f, (g_{\mathcal{O}}))$  where:

- $\pi$  is a  $\mathbb{B}_r$ -partition,
- $f$  an  $\mathcal{E}_1^r$ -structures,
- $g$  a set of tuples of  $\mathcal{C}$ -structures.

As the only  $\mathcal{E}_1^r$ -structures come from  $r$ -sets of size  $r$ ,  $\pi$  can only have one  $\mathbf{C}_r$ -cycle.  $g$  then is a tuple of cycles on the elements of  $\pi$ . This tuple can be interpreted as the traditional cycle notation for signed cycles with type 1. (Every element is the only one with its absolute value in its cycle and is therefore mapped under  $c^l$  to itself.)

We have for example:

- $\pi = \{\{1, \zeta 2, \zeta^2 3, \zeta 4\}, \{\zeta 1, \zeta^2 2, 3, \zeta^2 4\}\{\zeta^2 1, 2, \zeta 3, 4\}\}$
- $f = \pi = \{\{1, \zeta 2, \zeta^2 3, \zeta 4\}, \{\zeta 1, \zeta^2 2, 3, \zeta^2 4\}\{\zeta^2 1, 2, \zeta 3, 4\}\}$
- $g = (1\zeta^2 3\zeta 4\zeta 2)(\zeta 1 3\zeta^2 4\zeta^2 2)((\zeta 1 \zeta 3 4 2) = (1\zeta^3 3\zeta^2 4\zeta^0 2\zeta^2)$

EXAMPLE 4.5.9. We can use the substitution also for calculating the cycle indicator series. We can consider, for example the different types of set compositions, decompositions and partitions defined in [Cho10], two kinds of partitions are exactly our  $\text{Par}_{\mathbf{C}_r}[M]$  and  $\text{Par}_{\mathbb{B}_r}[M]$ . We consider the others:

- We define a decomposition of a signed set  $S$  of length  $l$  as a sequence  $(S_1, S_2, \dots, S_l)$  of disjoint  $\mathcal{B}_r$ -subsets, whose union is  $S$ .

For example decompositions of  $S = [2]_{\mathbf{C}_2}$  of length 2 are:

$$(\emptyset, \{\pm 1, \pm 2\}), (\{\pm 1\}, \{\pm 2\}), (\emptyset, \{\pm 2, \pm 1\}), (\{\pm 1, \pm 2\}, \emptyset)$$

We can interpret decompositions of a signed set  $S$  of length  $l$  as lists of length  $l$  of  $r$ -sets and get  $\mathcal{L}_l \circ \mathcal{E}^r$ . The cycle indicator series for  $\mathcal{L}_l$  is well known (see for example in [BLL98]):  $Z_{\mathcal{L}_l} = p_1^l$ . Then the cycle indicator series for decompositions of a signed set  $S$  of length  $l$  is:

$$\begin{aligned} p_1^l \circ \exp \left( \sum_{j=0}^{r-1} \sum_{k>0} \frac{p_k(\zeta^j)}{k \cdot r} \right) &= \exp \left( l \sum_{j=0}^{r-1} \sum_{k>0} \frac{p_k(\zeta^j)}{k \cdot r} \right) \\ &= p_1^l \circ \sum_{k>0} h_k(x) = \left( \sum_{k>0} h_k(x) \right)^l \end{aligned}$$

- We define set compositions as sequences of nonempty disjoint  $\mathcal{B}_r$ -subsets, whose union is  $S$ .

For example set compositions of  $S = [2]_{\mathbf{C}_2}$  are:

$$(\{\pm 1, \pm 2\}), (\{\pm 1\}, \{\pm 2\}), (\{\pm 2\}, \{\pm 1\})$$

We can interpret them as lists of nonempty  $r$ -sets and get  $\mathcal{L}_l \circ \mathcal{E}_{n>0}^r$ . We also know the cycle indicator series  $\mathcal{L} = \frac{1}{1-p_1}$  and get as cycle indicator series for set compositions:

$$\begin{aligned} \frac{1}{1-p_1} \circ \left( \exp \left( \sum_{j=0}^{r-1} \sum_{k>0} \frac{p_k(\zeta^j)}{k \cdot r} \right) - 1 \right) &= \frac{1}{2 - \exp \left( \sum_{j=0}^{r-1} \sum_{k>0} \frac{p_k(\zeta^j)}{k \cdot r} \right)} \\ &= \frac{1}{1-p_1} \circ \sum_{n>0} h_n(x) = \sum_{k>0} \left( \sum_{n>0} h_n(x) \right)^k \end{aligned}$$

- We define a new kind of partition of  $M_{\mathbf{C}_r}$  in that way that we consider a partition of  $M$  and then chose signs for the elements in the sets.

For example those partitions of  $S = [2]_{\mathbf{C}_2}$  are:

$$\{\{1, 2\}\}, \{\{1, -2\}\}, \{\{-1, 2\}\}, \{\{-1, -2\}\}, \{\{1\}, \{2\}\}, \{\{1\}, \{-2\}\}, \{\{-1\}, \{2\}\}, \{\{-1\}, \{-2\}\}$$

We can interpret them as sets of vertices  $\mathcal{E} \circ \mathcal{V}_{n>0}^r$  and get as cycle indicator series:

$$\exp\left(\sum_{k>0} \frac{p_k}{k}\right) \circ \left(\exp\left(\sum_k \frac{p_k(1)}{k}\right) - 1\right)$$

- We define a new kind of set composition in analogy with the previous kind of set partition.
- For example those set compositions of  $S = [2]_{\mathbf{C}_2}$  are:

$$(\{1, 2\}), (\{1, -2\}), (\{-1, 2\}), (\{-1, -2\}), (\{1\}, \{2\}), (\{2\}, \{1\}), (\{1\}, \{-2\}), (\{-2\}, \{1\}),$$

$$(\{-1\}, \{2\}), (\{2\}, \{-1\}), (\{-1\}, \{-2\}), (\{-2\}, \{-1\})$$

We can interpret them as lists of nonempty vertices  $\mathcal{L} \circ \mathcal{V}_{n>0}^r$  and get as cycle indicator series:

$$\frac{1}{1-p_1} \circ \left(\exp\left(\sum_k \frac{p_k(1)}{k}\right) - 1\right) = \frac{1}{2 - \exp\left(\sum_k \frac{p_k(1)}{k}\right)}$$

$$= \frac{1}{1-p_1} \circ \sum_{n>0} h_n(x) = \sum_{k \geq 0} \left(\sum_{n>0} h_n(x)\right)^k$$



## Conclusion and Future Work

$r$ -Species are just like species a useful tool to analyze combinatorial objects. In the first part of this thesis we gave an introduction to the theory of  $r$ -species and considered a set of examples to work with.

The second part concentrated on different generalizations of the substitution of species and the associated operation the plethysm. We proved the relation between them ( $Z_{F \circ G} = Z_F \circ Z_G$ ) by means of computation and used similar methods for establishing a similar operation for the substitution of two  $r$ -species:

(1)  $\forall g \in \Lambda(r)$ , the map  $\Lambda(1) \rightarrow \Lambda(r) : f \mapsto f \circ g$  is a  $\mathbb{C}$ -algebra homomorphism.

(2)  $\forall i, j \in \mathbb{N}$ , the map  $\Lambda(r) \rightarrow \Lambda(r) : g \mapsto p_i \circ g$  is regarding to addition a homomorphism.

$$(3) p_i(\xi) \circ \prod_{k,j} p_k(\zeta^j)^{\tau_{k,j}} = \begin{cases} r \prod_{k,j} p_{k \cdot i}(\zeta^j)^{\tau_{k,j}} & \prod_{k,j} (\zeta^j)^{\tau_{k,j}} = \xi \\ 0 & \text{otherwise} \end{cases}.$$

However, the last condition is not very natural, so there is the question, whether there is a more sophisticated and more natural way to define a substitution of two  $r$ -species.

Furthermore, we stated examples for a better understanding of the substitution. For example, we proved that ‘signed permutations are sets of signed cycles’. Examples like this are important for getting an intuition for the substitution.

After all, species that are the substitution of two other species can be easier treated when they are considered as such, as we have seen in the end of the work. Associated series, that are often hard to calculate, can be combined for example, by plethysm.

Still, there are various open problems related with  $r$ -species, two examples are:

- For ordinary species there are a lot more operations, some of them are already considered for hyperoctahedral species for example, in [HLL98]. Generalizing and analyzing them, and the associated operations on the cycle indicator series, would strengthen the theory of  $r$ -species.
- When working with the substitutions, we sometimes get unexpected isomorphisms, for example, ‘ $r$ -sets of signed cycles’ are isomorphic to ‘sets of signed cycles’. It would be interesting to consider isomorphisms, finding rules when they occur, and analyzing the relations between the substitutions.





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