TECHNISCHE UNIVERSITÄT WIEN
Vienna University of Technology

DIPLOMARBEIT

Poisson Approximation for Structure Floors

Ausgeführt am Institut für
Stochastik und Wirtschaftsmathematik
der Technischen Universität Wien

unter der Anleitung von
Privatdoz. Dipl.-Ing. Dr.techn. Stefan Gerhold

durch
Alexander Eder, BSc.

Simm. Haide 4/24/336
1110 Wien

Wien, 07.09.2015

Abstract

This thesis is about the approximation of the price for structure floors. The underlying structured note consists of an arbitrary number of double barrier options. For a small number of options, it's numerically shown by a Monte Carlo simulation that they fulfill a special dependency criterium. To approximate the distribution of the structured note's payoff, the Chen-Stein method is used. Using this approximation, bounds for the exact price of a structure floor are given. These results are implemented using the coding language Mathematica. With this implementation, several examples are given to illustrate the results.

Keywords: *Poisson approximation, Chen-Stein method, structured note, structure floor, coupling*

Contents

List of figures

List of tables

Chapter 1

Introduction

Structured notes as they are considered here, consist of an arbitrary number of coupons. The payoff of a coupon depends on an underlying stock. It only pays, if the underlying stays between two barriers on a given time interval. The payoff of the structure note is then the sum of the coupon's payoffs. To guarantee that a specified amount is paid, structure floors are used. If the structured note pays less than this amount, the structure floor pays the difference.
In the Black-Scholes model, the arbitrage-free price of such a structure floor is the discounted expected value of its payoff. Therefore the distribution of the sum of the coupon's payoffs is needed to price the structure floor. This distribution as well as an algorithm for the computation of the exact price are derived in [9].
The complexity of this algorithm increases, as the number of coupons increases. The computation of the distribution and therefore the computation of the exact price, has a high computational effort, even for a small number of coupons. Hence the distribution of the sum of the coupon's payoffs is approximated by the Poisson distribution here. Then the price can be computed easily. Of course the price is not exact anymore with this approximation. The approximation error is bounded and lower and upper bounds for the exact price are given. These bounds are derived using the Chen-Stein method. It's a well known method for Poisson approximation.

This paper has the following structure. Chapter 2 gives basic definitions and results of stochastic calculus. Furthermore, the Black-Scholes model is introduced and structure floors are formal defined. Theorems about the distribution of the sum of the coupon's payoffs and the exact price of a structure floor can be found there too.
Chapter 3 discusses the Chen-Stein method. Bounds for the total variation

distance of a Poisson distributed and another random variable are given there. The results of this chapter are used to approximate the point probabilities of a random variable by the point probabilities of a Poisson distributed random variable in chapter 4. The bounds given there depend on whether the coupon's payoffs fulfill some dependency criteria or not. Using the bounds given in chapter 4, a theorem for the approximation of the price for a structure floor is proved in chapter 5.
In chapter 6 the results of chapter 4 and 5 are applied to given problems. For a small number of coupons, a Monte Carlo simulation is used to show that the dependencies of their payoff's have a special structure. Several examples are given, which illustrate the results of the previous chapters. The last chapter contains the Mathematica code, which was used to obtain the numerical results in chapter 6.

Chapter 2

Mathematical Theory

In this chapter basic results of stochastic calculus are given first. These results refer to [7]. Next, the Black-Scholes model is introduced to define a framework, in which structured notes and especially structure floors can be defined. The references for the two subsections are [12] for the Black-Scholes model and [1] for the theory about structure floors.

First, a probability space must be defined. To do that, the terms σ-algebra and probability measure are needed.

Definition 2.1 *A σ-algebra $\mathcal{F}$ on a non-empty set Ω is a family of subsets of Ω fulfilling*

(1) $\emptyset \in \mathcal{F}$,

(2) $A \in \mathcal{F} \Rightarrow \Omega \backslash A \in \mathcal{F}$,

(3) $(A_i)_{i\in\mathbb{N}} \subseteq \mathcal{F} \Rightarrow \bigcup_{i\in\mathbb{N}} A_i \in \mathcal{F}$.

Remark: The set of natural numbers is defined here as

$$\mathbb{N} := \{1, 2, 3, \dots\},$$

while $\mathbb{N}_0$ denotes $\mathbb{N} \cup \{0\}$.

Definition 2.2 *A function $\mathbb{P} : \mathcal{F} \to [0, 1]$ from a σ-algebra on a non-empty set Ω to the interval $[0, 1]$ is called a probability measure, if*

(1) $\mathbb{P}(\Omega) = 1$,

(2) $(A_i)_{i\in\mathbb{N}} \subseteq \mathcal{F}$ *and* $A_i \cap A_j = \emptyset,\ \forall i, j \in \mathbb{N}$ *with* $i \neq j$

$$\Rightarrow \mathbb{P}(\textstyle\bigcup_{i\in\mathbb{N}} A_i) = \sum_{i=1}^{n} \mathbb{P}(A_i).$$

The probability space defined as follows is used throughout this whole paper.

Definition 2.3 *A probability space is a triple (Ω, $\mathcal{F}$, $\mathbb{P}$), where Ω is a non-empty set, $\mathcal{F}$ is a σ-algebra on Ω and $\mathbb{P}$ is a probability measure on $\mathcal{F}$.*

On this probability space, random variables and other instruments of stochastic modeling can be defined. A few basic definitions are given next.

Definition 2.4 *A pair (S, $\mathcal{S}$), where S is a non-empty set and $\mathcal{S}$ is a σ-algebra on S is called a measurable space. A function $f : (\Omega, \mathcal{F}, \mathbb{P}) \to (S, \mathcal{S})$ from a probability space to a measurable space is called measurable, if*

$$f^{-1}(A) := \{\omega \in \Omega : f(\omega) \in A\} \in \mathcal{F}, \quad \forall A \in \mathcal{S}.$$

Definition 2.5 *A measurable function $X : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, where $\mathcal{B}(\mathbb{R})$ denotes the family of Borel sets, is called a random variable.*

Remark: The family of Borel sets $\mathcal{B}(\mathbb{R})$ is the smallest σ-algebra containing all intervals in $\mathbb{R}$.

To model the available information at time t, filtrations are used.

Definition 2.6 *A family of σ-algebras $(\mathcal{F}_t)_{t\geq 0}$ on a probability space with*

(1) $\mathcal{F}_t \subseteq \mathcal{F}$, $\forall t \geq 0$,

(2) $\mathcal{F}_s \subseteq \mathcal{F}_t$, $\forall s, t \geq 0$ *with* $s \leq t$,

is called filtration.

With this definition, the expected value of a random variable at a specific moment in time t is the conditional expectation of the random variable given the filtration at t.

Definition 2.7 *The conditional expectation of an integrable random variable X, given a σ-algebra $\mathcal{G}$ is a random variable, denoted by $\mathbb{E}[X|\mathcal{G}]$, fulfilling*

(1) $\mathbb{E}[X|\mathcal{G}]$ *is measurable,*

(2) $\mathbb{E}[\mathbb{1}_A(\mathbb{E}[X|\mathcal{G}])] = \mathbb{E}[\mathbb{1}_A(X)]$, $\forall A \in \mathcal{G}$.

Remarks: A random variable X is called integrable, if

$$\mathbb{E}[X] < \infty.$$

The indicator function for a set A is defined as

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Random variables that vary in time are described by stochastic processes.

Definition 2.8 *If for each $0 \le t < \infty$, X_t is a random variable, the collection*

$$X := \{X_t : 0 \le t < \infty\}$$

is called a stochastic process. For each $\omega \in \Omega$ the function

$$\omega \to (X_t(\omega))_{t \ge 0}$$

is called a sample path of the stochastic process X.

A special stochastic process is the Brownian motion, also called Wiener process. It's defined as follows.

Definition 2.9 *A stochastic process $W := \{W_t : 0 \le t < \infty\}$ is called Brownian motion, if it fulfills*

(1) $W_0 = 0$ a.s.,

(2) the sample paths of W are continuous a.s.,

(3) for $0 < t_1 < t_2 < \cdots < t_k < \infty$, the increments
$$W_{t_2} - W_{t_1}, W_{t_3} - W_{t_2}, \ldots, W_{t_k} - W_{t_{k-1}}$$
are independent,

(4) for $s < t, W_t - W_s \sim \mathcal{N}(0, t - s)$.

Remark: The normal distribution with expected value μ and variance σ is denoted by $\mathcal{N}(\mu, \sigma)$.

Now the Itô stochastic integral is introduced. It is used to integrate with respect to a Brownian motion. The definition takes place in two steps. First, the integral is defined for random step processes as integrands. In the second step, this definition is expanded to a larger class of stochastic processes.

Definition 2.10 *A random step process X is a stochastic process that can be represented as*

$$X_t = \sum_{j=1}^{n} \eta_i \mathbb{1}_{(t_{j-1}, t_j]}(t), \tag{2.1}$$

with $0 = t_0 < t_1 < \cdots < t_n < \infty$ and random variables $(\eta_j)_{j=1}^n$ taking values in $\mathbb{N}_0$, where η_j is measurable with respect to $\mathcal{F}_{t_j}$ and $\mathbb{E}[\eta_j^2] < \infty$, for all $j \in \{1, \ldots, n\}$.

Now the stochastic integral can be defined for the class of random step processes.

Definition 2.11 *For a random step process X of the form (2.1), the stochastic integral with respect to a Brownian motion W is defined by*

$$I(X) := \sum_{j=1}^{n} \eta_j (W_{t_j} - W_{t_{j-1}}).$$

Using this definition, the integral can be defined for all stochastic processes that can be approximated by random step processes.

Definition 2.12 *Let X be a stochastic process with $\mathbb{E}[X^2] < \infty$, for which a sequence $(X^{(n)})_{n\in\mathbb{N}}$ of random step processes exists, such that*

$$\lim_{n\to\infty} \mathbb{E}\left[\int_0^\infty |X_t - X_t^{(n)}|^2 dt\right] = 0. \tag{2.2}$$

Then $I(X)$ is called the Itô stochastic integral, if

$$\lim_{n\to\infty} \mathbb{E}\left[|I(X) - I(X^{(n)})|^2\right] = 0.$$

For a clearer notation write

$$\int_0^\infty X_t dW_t$$

instead of $I(X)$. For the integration over an interval $[0,T]$ define

$$\int_0^T X_t dW_t := \int_0^\infty \mathbb{1}_{[0,T]}(t) X_t dW_t.$$

For this definition of the Itô stochastic integral holds the so called Itô formula, which is given in the following theorem. For a proof see [7], chapter 7.

Theorem 2.13 *Let W be a Brownian motion and $f(t,x)$ be a real valued function with continuous partial derivatives $f_t(t,x)$, $f_x(t,x)$ and $f_{xx}(t,x)$, for all $t \geq 0$, $x \in \mathbb{R}$. Also assume that the process $\mathbb{1}_{[0,T]}(t) f_x(t, W_t)$ can be approximated by random step processes in the sense of (2.2), for all $T \geq 0$. Then*

$$\begin{aligned} &f(T, W_T) - f(0, W_0) \\ &\quad = \int_0^T f_t(t, W_t) dt + \frac{1}{2}\int_0^T f_{xx}(t, W_t) dt + \int_0^T f_x(t, W_t) dW_t \end{aligned} \tag{2.3}$$

holds almost sure.

With the definition of the Itô stochastic integral it's also possible to define stochastic differential equations.

Definition 2.14 *Let f and g be real valued functions. A differential equation of the form*

$$dX_t = f(X_t)dt + g(X_t)dW_t, \tag{2.4}$$

where X is a stochastic process and W is a Brownian motion, is called stochastic differential equation. Combined with an initial condition

$$X_0 = x_0 \in \mathbb{R},$$

it's called an initial value problem.

With these basics of stochastic calculus, the Black-Scholes model can be defined in the following section.

2.1 The Black-Scholes model

The Black-Scholes model goes back to Fischer Black and Myron Scholes (see [6]). A risk free bank account as well as a stock are modeled through stochastic processes, which are defined as the solutions of differential equations. It makes use of the following assumptions about the market:

(1) the market of the stock, options and cash is perfectly liquid, i.e. it's possible to buy and sell resp. borrow and lend at any time any amount of stocks and options resp. cash and there are no margin requirements,

(2) the interest rates of the bank account are known and constant,

(3) interest rates for borrowing and lending cash are the same,

(4) the volatility of the stock price is known and constant and

(5) there are no transaction costs or taxes.

The bank account is modeled by a stochastic process B. It continuously increases with an interest rate $r > 0$. By convention, $B_0 = 1$. Therefore, for all $t \geq 0$, B can be defined through

$$dB_t = rB_t dt$$

$$B_0 = 1.$$

It's easy to see that the solution of this ordinary differential equation is given by

$$B_t = e^{rt}, \quad \forall t \geq 0.$$

The dynamics of the stock price S are given by the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad \forall t > 0, \tag{2.5}$$

where μ and σ are real valued constants. They are called drift resp. volatility. With the initial condition $S_0 = s_0 > 0$, (2.5) is an initial value problem. It has a solution, which is given in the following theorem.

Theorem 2.15 *The stochastic process given by*

$$S_t = S_0 exp\left(\sigma W_t + \left(\mu - \frac{1}{2}\sigma^2\right) t\right), \quad \forall t > 0, \tag{2.6}$$

with a Brownian motion W and $\mu, \sigma \in \mathbb{R}$ is a solution of the stochastic differential equation (2.5).

Proof: To show that (2.6) is a solution of (2.5), Ito's formula from theorem 2.13 can be used. Define a stochastic process X by

$$X_t = \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t, \quad \forall t > 0.$$

Then

$$S_t = g(X_t), \quad \forall t > 0,$$

follows, where the function g is defined by

$$g : \mathbb{R} \to \mathbb{R} : x \to S_0 e^x.$$

With Ito's formula follows

$$\begin{aligned} dS_t &= dg(X_t) \\ &= g'(X_t)\left(\mu - \frac{1}{2}\sigma^2\right) dt + \frac{1}{2}g''(X_t)\sigma^2 dt + g'(X_t)\sigma dW_t \\ &= g(X_t)(\mu dt + \sigma dW_t) \\ &= S_t(\mu dt + \sigma dW_t) \\ &= \mu S_t dt + \sigma S_t dW_t, \end{aligned}$$

since g(x)=g'(x)=g"(x). Therefore S_t given by (2.6) is a solution of (2.5).

□

Remarks: It's also possible to show that this solution is unique (for a proof see [12], section 3.1). A stochastic process as in (2.6) is called geometric Brownian motion.

The following section uses this results to define structure floors and price them. In the Black-Scholes model the arbitrage free prices are used. These are the discounted expected returns of the considered financial instruments.

2.2 Structure floors

In this section a structured floor consisting of an arbitrary number of coupons n is considered. The coupons pay 1 in case the underlying stays between two barriers during a specified time interval at the end of this interval and 0 otherwise. Let $0 < T_0 < T_1 < \cdots < T_n$ with $T_k = T_{k-1} + P$, for all $k \in \{1 \ldots, n\}$ and $P \in \mathbb{R}$. The value P defines the length of the time intervals. The payoff of the coupons can be written as

$$C_i = \mathbb{1}_{\{B_{low} < S_t < B_{up}, t \in [T_{i-1}, T_i]\}}, \quad \forall i \in \{1, \ldots, n\}, \tag{2.7}$$

where S is the stock price of the underlying as defined in (2.6) and B_{low} resp. B_{up} are the lower resp. upper barriers. These coupons can be priced using the following theorem. The proof is omitted, it can be found in [1], section 3.

Theorem 2.16 *The discounted expected value of the product of various coupon's payoffs, defined as in (2.7), at $t = 0$ is given by*

$$\begin{aligned} BD(S_0, (T_i)_{i \in J}, P, B_{low}, B_{up}, \sigma, r) :&= e^{-rT_n} \mathbb{E}\left[\prod_{i \in J} C_i\right] \\ &= e^{\alpha x + \beta \tau} U(x, \tau), \quad \forall J \subseteq \{1, \ldots, n\}, \end{aligned}$$

with the following definitions.

$$j := |J|, \qquad \alpha := -\frac{1}{2}\left(\frac{2r}{\sigma^2} - 1\right),$$

$$\tilde{T} := (T_i)_{i \in J}, \qquad \beta := -\frac{2r}{\sigma^2} - \alpha^2,$$

$$\tau := \frac{\sigma^2}{2}\tilde{T}_j, \qquad x := \log\left(\frac{S_0}{B_{low}}\right),$$

$$p := \frac{\sigma^2}{2}P, \qquad L := \log\left(\frac{B_{up}}{B_{low}}\right),$$

$$\tau_i := \frac{\sigma^2}{2}(\tilde{T}_{j-1} - \tilde{T}_{i-1}) \textit{ and}$$

$$U(x,\tau) := \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_0^L \cdots \int_0^L \sum_{k_1=0}^{\infty} \cdots \sum_{k_j=0}^{\infty} \tag{2.8}$$

$$h_{j-1}(k_1,\ldots,k_j;x_1,\ldots,x_j;y_1,\ldots,y_j;x,\tau)dx_1\ldots dx_j dy_1\ldots dy_j.$$

The function h is given by

$$h_i(k_1,\ldots,k_{i+1};x_1,\ldots,x_{i+1};y_1,\ldots,y_{i+1};x,\tau)$$

$$:= \sqrt{\frac{e^{-y_{i+1}^2}}{2\pi}}\mathbb{1}_{[-x,L-x]}\left(y_{i+1}\sqrt{2(\tau-(\tau_{j-i}+p))}\right)$$

$$\cdot g_i(k_1,\ldots,k_{i+1};x_1,\ldots,x_{i+1};y_1,\ldots,y_i;x+y_{i+1}\sqrt{2(\tau-(\tau_{j-i}+p))},\tau_{j-i}+p) \tag{2.9}$$

with

$$g_i(k_1,\ldots,k_{i+1};x_1,\ldots,x_{i+1};y_1,\ldots,y_i;x,\tau)$$

$$:= \frac{2}{L}\sin\frac{k_{i+1}\pi x_{i+1}}{L}\sin\frac{k_{i+1}\pi x}{L}e^{-(k_{i+1}\pi/L)^2(\tau-\tau_{j-i})}$$

$$\cdot\, h_{i-1}(k_1,\ldots,k_i;x_1,\ldots,x_i;y_1,\ldots,y_i;x_{i+1},\tau_{j-i})$$

and

$$g_0(k_1;x_1;;x;\tau) := \frac{2}{L}e^{-\alpha x_1}\sin\frac{k_1\pi x_1}{L}\sin\frac{k_1\pi x}{L}e^{-(k_1\pi/L)^2\tau}.$$

Remarks: Because this theorem is only used for $t=0$ here, some parts of the original theorem in [1] were left out. Also the indicator function in h_i was changed. Originally it was

$$\mathbb{1}_{\left[-\frac{x}{\sqrt{2(\tau-(\tau_{j-i}+p))}},\frac{L-x}{\sqrt{2(\tau-(\tau_{j-i}+p))}}\right]}(y_{i+1}),$$

but since the square roots are possibly 0, this expression is not defined in some cases.

The payoff of the structured note defined as above is given by

$$W := \sum_{i=1}^{n} C_i.$$

To guarantee a minimum payout, structure floors can be used. Its payoff is given by

$$(x - W)^+, \tag{2.10}$$

where $x > 0$ is the level of the structure floor. By combining a structured note with a structure floor, the minimum payoff is always x. The question about the arbitrage-free price of such a structure floor is answered by the next theorem.

Theorem 2.17 *The arbitrage-free price of a structure floor defined as in (2.10) at $t = 0$ is given by*

$$SF(x) := e^{-rT_n}\mathbb{E}\left[(x - W)^+\right] = e^{-rT_n} \sum_{i=0}^{n \wedge \lfloor x \rfloor} (x - i)\mathbb{P}(W = i) \tag{2.11}$$

with

$$\mathbb{P}(W = n) = BD(S_0, 0; (T_0), T_n - T_0, B_{low}, B_{up}, 0). \tag{2.12}$$

The other point probabilities $\mathbb{P}(W = i)$, for all $i \in \{0, \ldots, n-1\}$, can be obtained by solving the system of equations

$$\begin{gathered} \sum_{i=0}^{n} \mathbb{P}(W = i) = 1 \\ \sum_{i=0}^{n} i^\nu \mathbb{P}(W = i) = \sum_{J \subseteq \{1,\ldots,n\}} c(\nu, J) BD(S_0, 0; (T_j)_{j \in J}, P, B_{low}, B_{up}, 0), \end{gathered} \tag{2.13}$$

for all $\nu \in \{1, \ldots, n\}$. The coefficient function c is given by

$$c(\nu, J) := \sum_{\substack{0 \le i_1, \ldots, i_n \le \nu \\ supp(i_1, \ldots, i_n) = J}} \binom{\nu}{i_1, \ldots, i_n},$$

where $supp(i_1, \ldots, i_n) = J$ means that $i_k \neq 0$, for all $k \in J$.

Proof: Equation (2.11) holds by definition of the expected value. $\mathbb{P}(W = n)$ is the probability that all coupons pay 1. This means that the underlying has to stay between the barriers for all intervals $[T_{i-1}, T_i]$, $i \in \{1, \dots, n\}$. Since

$$\bigcup_{i=1}^{n} [T_{i-1}, T_i] = [T_0, T_n],$$

the case $W = n$ can be considered as a coupon with only one barrier on the time interval $[T_0, T_n]$. Therefore (2.12) holds.

The last part is to show that the equalities in the system of equations (2.13) hold. The first equation is obvious. By definition of the k-th moment of a random variable X, taking values in $\{0, \dots, n\}$,

$$\mathbb{E}[X^k] = \sum_{i=0}^{n} i^k \mathbb{P}(X = i),$$

the left hand side of the second equality is the ν-th moment of W, $\mathbb{E}[W^\nu]$. Therefore the aim is to prove

$$\mathbb{E}[W^\nu] = \sum_{J \subseteq \{1,\dots,n\}} \left(\sum_{\substack{0 \le i_1,\dots,i_n \le \nu \\ supp(i_1,\dots,i_n) = J}} \binom{\nu}{i_1, \dots, i_n} \right) \mathbb{E}\left[\prod_{j \in J} C_j \right], \quad \forall \nu \in \{1, \dots, n\}.$$

It follows from

$$\begin{aligned}
\mathbb{E}[W^\nu] &= \mathbb{E}\left[\left(\sum_{i=1}^{n} C_i \right)^{\nu} \right] \\
&= \sum_{0 \le i_1,\dots,i_n \le \nu} \binom{\nu}{i_1, \dots, i_n} \mathbb{E}\left[C_1^{i_1} \dots C_n^{i_n} \right] \\
&= \sum_{0 \le i_1,\dots,i_n \le \nu} \binom{\nu}{i_1, \dots, i_n} \mathbb{E}\left[\prod_{\substack{j=1 \\ i_j > 0}}^{n} C_j \right] \\
&= \sum_{J \subseteq \{1,\dots,n\}} \left(\sum_{\substack{0 \le i_1,\dots,i_n \le \nu \\ supp(i_1,\dots,i_n) = J}} \binom{\nu}{i_1, \dots, i_n} \right) \mathbb{E}\left[\prod_{j \in J} C_j \right],
\end{aligned}$$

for all $\nu \in \{1, \dots, n\}$. □

Chapter 3

The Chen-Stein method

This chapter gives the main results of the Chen-Stein method, which is used for Poisson approximation. Although these results refer to several sources, they can all be found in [5].

Let $\lambda > 0$ and $(C_i)_{i=1}^n$ be indicator random variables with

$$\mathbb{P}(C_i = 1) = 1 - \mathbb{P}(C_i = 0) = \frac{\lambda}{n}, \quad \forall i \in \{1, \dots, n\}.$$

Poisson's limit theorem states that the distribution of

$$W := \sum_{i=1}^{n} C_i$$

converges to the Poisson distribution with parameter λ as $n \to \infty$, if the indicators $(C_i)_{i=1}^n$ are independent. Generalizing this to the case, where the indicator random variables $(C_i)_{i=1}^n$ are not identical distributed and

$$\mathbb{P}(C_i = 1) = 1 - \mathbb{P}(C_i = 0) = \mathbb{E}[C_i], \quad \forall i \in \{1, \dots, n\}$$

holds, the distribution of W can still be approximated by a Poisson distribution. The approximation error is measured by the total variation distance, defined as follows.

Definition 3.1 *Let X, Y be two random variables taking values in $\mathbb{N}_0$ and let $\mathcal{L}(X)$, $\mathcal{L}(Y)$ denote their distributions. Then the total variation distance of $\mathcal{L}(X)$ and $\mathcal{L}(Y)$ is defined by*

$$d_{TV}(\mathcal{L}(X), \mathcal{L}(Y)) := \sup_{A \subseteq \mathbb{N}_0} |\mathbb{P}(X \in A) - \mathbb{P}(Y \in A)|.$$

Le Cam proved in [11] that

$$d_{TV}(\mathcal{L}(W), Poi(\lambda)) \leq 2\sum_{i=1}^{n} \mathbb{E}[C_i]^2,$$

where $Poi(\lambda)$ denotes the Poisson distribution with parameter

$$\lambda := \sum_{i=1}^{n} \mathbb{E}[C_i].$$

Therefore a Poisson approximation is reasonable, if the expected values $(\mathbb{E}[C_i])_{i=1}^{n}$ are small. The Chen-Stein method generalizes this approximation to the case, where the indicators are not independent.

From now on let X denote a Poisson distributed random variable with parameter λ. The aim is to bound $d_{TV}(\mathcal{L}(X), \mathcal{L}(W))$. To do that (see [8]), define for each $A \subseteq \mathbb{N}_0$ a function through

$$w f_A(w) - \lambda f_A(w+1) = \mathbb{1}_A(w) - \mathbb{P}(X \in A), \quad \forall w \in \mathbb{N}_0. \tag{3.1}$$

This function is unique except for $w = 0$. It is explicitly given by

$$f_A(w) := \frac{(w-1)!}{\lambda^w} \sum_{i=0}^{w-1} \left(\mathbb{P}(X \in A) - \mathbb{1}_A(i)\right) \frac{\lambda^i}{i!}, \quad \forall w \in \mathbb{N}. \tag{3.2}$$

Since $f_A(0)$ has no effect on the following calculations, set $f_A(0) = 0$. Taking expectations of (3.1) at W leads to

$$\begin{aligned} \mathbb{E}[W f_A(W) - \lambda f_A(W+1)] &= \mathbb{E}\left[\mathbb{1}_A(W)\right] - \mathbb{P}(X \in A) \\ &= \mathbb{P}(W \in A) - \mathbb{P}(X \in A). \end{aligned}$$

Although the following method to bound the left hand side was used before, Stein was the first who referred to it as a method of coupling. It is described in [13], pp. 92-93. For the error term holds

$$\begin{aligned} \mathbb{P}(W \in A) - \mathbb{P}(X \in A) &= \mathbb{E}[W f_A(W) - \lambda f_A(W+1)] \\ &= \sum_{i=1}^{n} \left(\mathbb{E}\left[C_i f_A(W)\right] - \mathbb{E}[C_i]\mathbb{E}\left[f_A(W+1)\right]\right) \\ &= \sum_{i=1}^{n} \left(\mathbb{E}[C_i]\mathbb{E}\left[f_A(W) | C_i = 1\right] - \mathbb{E}[C_i]\mathbb{E}\left[f_A(W+1)\right]\right) \\ &= \sum_{i=1}^{n} \mathbb{E}[C_i]\left(\mathbb{E}\left[f_A(W) | C_i = 1\right] - \mathbb{E}\left[f_A(W+1)\right]\right) \end{aligned}$$

Now define random variables $(V_i)_{i=1}^n$ with

$$(V_i + 1) \overset{(d)}{=} (W | C_i = 1), \quad \forall i \in \{1, \ldots, n\}. \tag{3.3}$$

From above follows

$$\begin{aligned} |\mathbb{P}(W \in A) - \mathbb{P}(X \in A)| &= \left| \sum_{i=1}^{n} \mathbb{E}[C_i] \mathbb{E}\left[f_A(V_i + 1) - f_A(W + 1) \right] \right| \\ &\leq \sum_{i=1}^{n} \mathbb{E}[C_i] \mathbb{E}\left[|f_A(W + 1) - f_A(V_i + 1)| \right] \end{aligned} \tag{3.4}$$

One way to construct $(V_i)_{i=1}^n$ is described in [4]. For every $i \in \{1, \ldots, n\}$ set $\Gamma_i := \{1, \ldots, n\} \setminus \{i\}$ and define indicator random variables $(J_{ik})_{k \in \Gamma_i}$ with

$$(J_{ik}, k \in \Gamma_i) \overset{(d)}{=} (C_k, k \in \Gamma_i | C_i = 1). \tag{3.5}$$

Setting

$$V_i := \sum_{k \in \Gamma_i} J_{ik}, \tag{3.6}$$

V_i fulfills (3.3), for all $i \in \{1, \ldots, n\}$. The sequence $(V_i)_{i=1}^n$ as well as $\{(J_{ik})_{k \in \Gamma_i} | i \in \{1, \ldots, n\}\}$ are referred to as couplings.

Now for the right hand side of (3.4) holds

$$\begin{aligned} &\sum_{i=1}^{n} \mathbb{E}[C_i] \mathbb{E}[|f_A(W + 1) - f_A(V_i + 1)|] \\ &\qquad \leq \|\Delta f_A\| \sum_{i=1}^{n} \mathbb{E}[C_i] \mathbb{E}\left[|W - V_i| \right] \\ &\qquad = \|\Delta f_A\| \sum_{i=1}^{n} \mathbb{E}[C_i] \mathbb{E}\left[\left| C_i + \sum_{k \in \Gamma_i} C_k - J_{ik} \right| \right] \\ &\qquad \leq \|\Delta f_A\| \sum_{i=1}^{n} \mathbb{E}[C_i] \mathbb{E}\left[C_i + \sum_{k \in \Gamma_i} |C_k - J_{ik}| \right] \\ &\qquad = \|\Delta f_A\| \sum_{i=1}^{n} \left(\mathbb{E}[C_i]^2 + \sum_{k \in \Gamma_i} \mathbb{E}[C_i] \mathbb{E}\left[|C_k - J_{ik}| \right] \right), \end{aligned} \tag{3.7}$$

with

$$\Delta f(k) := f(k+1) - f(k), \quad \forall k \in \mathbb{N},$$

and

$$\|\Delta f_A\| := \sup_{k \in \mathbb{N}} |f(k) - f(k+1)|.$$

The following estimate for $\|\Delta f_A\|$ was proved by Barbour and Holst (see the appendix in [3]).

Lemma 3.2 *Let f_A be defined as in (3.2) with $A \subseteq \mathbb{N}_0$. Then*

$$\|\Delta f_A\| \leq \frac{1 - e^{-\lambda}}{\lambda}. \tag{3.8}$$

Proof: The function f_A defined as in (3.2) for $A = \{j\}$ is given by

$$f_{\{j\}}(k) = \begin{cases} 0 & \text{if } k = 0 \\ \frac{(k-1)!}{\lambda^k} \frac{\lambda^j}{j!} \left(\sum_{i=0}^{k-1} \frac{\lambda^i}{i!} e^{-\lambda} \right) & \text{if } k \leq j \\ \frac{(k-1)!}{\lambda^k} \frac{\lambda^j}{j!} \left(\sum_{i=0}^{k-1} \frac{\lambda^i}{i!} e^{-\lambda} - 1 \right) & \text{if } k > j \end{cases}.$$

Since

$$\sum_{i=0}^{k-1} \frac{\lambda^i}{i!} e^{-\lambda} = \mathbb{P}(X \leq k-1),$$

for a Poisson distributed random variable X with parameter λ, $f_{\{j\}}(k)$ is positive and increasing for $k \leq j$ and negative and increasing for $k > j$. Hence the only positive increment is

$$\begin{aligned} f_{\{j\}}(j) - f_{\{j\}}(j+1) &= e^{-\lambda} \left(\frac{1}{j} \sum_{i=0}^{j-1} \frac{\lambda^i}{i!} + \frac{1}{\lambda} \sum_{i=j+1}^{\infty} \frac{\lambda^i}{i!} \right) \\ &= e^{-\lambda} \left(\frac{1}{j} \sum_{i=1}^{j} \frac{\lambda^{i-1}}{(i-1)!} + \frac{1}{\lambda} \sum_{i=j+1}^{\infty} \frac{\lambda^i}{i!} \right) \\ &= \frac{e^{-\lambda}}{\lambda} \left(\sum_{i=1}^{j} \frac{i}{j} \frac{\lambda^i}{i!} + \sum_{i=j+1}^{\infty} \frac{\lambda^i}{i!} \right) \\ &\leq \frac{e^{-\lambda}}{\lambda} \left(e^{\lambda} - 1 \right) \\ &= \frac{1 - e^{-\lambda}}{\lambda}. \end{aligned}$$

Because of

$$\mathbb{1}_A(\omega) - \mathbb{P}(X \in A) = \sum_{j \in A} \left(\mathbb{1}_{\{j\}}(\omega) - \mathbb{P}(X = j) \right)$$

in the definition of f_A, the function can be expressed as

$$f_A(\omega) = \sum_{j \in A} f_{\{j\}}(\omega).$$

For the increments of f_A with $A \subseteq \mathbb{N}_0$ holds

$$\begin{aligned} f_A(m) - f_A(m+1) = & \mathbb{1}_A(m) \left(f_{\{m\}}(m) - f_{\{m\}}(m+1) \right) \\ & + \sum_{\substack{j \in A \\ j \neq m}} \left(f_{\{j\}}(m) - f_{\{j\}}(m+1) \right), \quad \forall m \in \mathbb{N}. \end{aligned} \tag{3.9}$$

Because of the properties of $f_{\{j\}}$ above, this expression is positive if $m \in A$. If $m \notin A$

$$\begin{aligned} f_A(m) - f_A(m+1) &= f_{\mathbb{N}_0 \setminus A}(m+1) - f_{\mathbb{N}_0 \setminus A}(m) \\ &= -\left(f_{\{m\}}(m) - f_{\{m\}}(m+1) \right) \\ &\quad - \sum_{\substack{j \in \mathbb{N}_0 \setminus A \\ j \neq m}} \left(f_{\{j\}}(m) - f_{\{j\}}(m+1) \right) \end{aligned} \tag{3.10}$$

holds, because

$$f_A(k) = -f_{\mathbb{N}_0 \setminus A}(k), \quad \forall k \in \mathbb{N}_0.$$

In conclusion, the absolute value of an increment $\Delta f(m)$ takes the maximum, if A only contains m. Then the sums in (3.9) and (3.10) are 0. The lemma follows now from

$$\begin{aligned} \|\Delta f_A\| &= \sup_{k \in \mathbb{N}} |f_A(k) - f_A(k+1)| \\ &\leq \sup_{k \in \mathbb{N}} \max_{M \subseteq \mathbb{N}_0} |f_M(k) - f_M(k+1)| \\ &= \sup_{k \in \mathbb{N}} |f_{\{k\}}(k) - f_{\{k\}}(k+1)| \\ &\leq \frac{1 - e^{-\lambda}}{\lambda}. \end{aligned}$$

for any set $A \subseteq \mathbb{N}_0$. □

Combining (3.4), (3.7) and lemma 3.2 leads to

$$|\mathbb{P}(W \in A) - \mathbb{P}(X \in A)| \leq \frac{1-e^{-\lambda}}{\lambda} \sum_{i=1}^{n} \left(\mathbb{E}[C_i]^2 + \sum_{k \in \Gamma_i} \mathbb{E}[C_i] \mathbb{E}\left[|C_k - J_{ik}|\right] \right)$$

Since the right hand side doesn't depend on the set A, the next theorem follows.

Theorem 3.3 *With the definitions above*

$$\begin{aligned} d_{TV}(\mathcal{L}(W), Poi(\lambda)) &= \sup_{A \subseteq \mathbb{N}_0} |\mathbb{P}(X \in A) - \mathbb{P}(W \in A)| \\ &\leq \frac{1-e^{-\lambda}}{\lambda} \left(\sum_{i=1}^{n} \mathbb{E}[C_i]^2 + \sum_{i=1}^{n} \sum_{k \in \Gamma_i} \mathbb{E}[C_i] \mathbb{E}\left[|C_k - J_{ik}|\right] \right), \end{aligned} \tag{3.11}$$

where $Poi(\lambda)$ denotes the Poisson distribution with parameter λ. □

This bound can be significantly simplified, if $\{(J_{ik})_{k \in \Gamma_i} : i \in \{1, \ldots, n\}\}$ is monotone in the sense of

$$J_{ik} \leq C_i, \quad \forall k \in \Gamma_i, i \in \{1, \ldots, n\} \tag{3.12}$$

or

$$J_{ik} \geq C_i, \quad \forall k \in \Gamma_i, i \in \{1, \ldots, n\}. \tag{3.13}$$

3.1 Monotone couplings

Monotone couplings were introduced by Barbour and Holst in [4]. The terms positive and negative relation are defined through monotone couplings. The results of this subsection, especially the next definition, refer to [10].

Definition 3.4 *The random variables $(C_i)_{i=1}^n$ are said to be negatively related, if a coupling $\{(J_{ik})_{k \in \Gamma_i} : i \in \{1, \ldots, n\}\}$ exists, fulfilling (3.12). They are said to be positively related, if a coupling that fulfills (3.13) exists.*

The following two theorems are extensions of theorem 3.3. They give bounds in case the indicators $(C_i)_{i=1}^n$ are positively resp. negatively related.

Theorem 3.5 *If the indicator random variables $(C_i)_{i=1}^n$ are positively related,*

$$d_{TV}(\mathcal{L}(W), Poi(\lambda)) \leq \frac{1-e^{-\lambda}}{\lambda} \left(2 \sum_{i=1}^{n} \mathbb{E}[C_i]^2 + Var(W) - \lambda \right)$$

holds.

Proof: From (3.5) follows

$$\mathbb{P}(J_{ik} = 1) = \mathbb{P}(C_k = 1 | C_i = 1), \quad \forall k \in \Gamma_i, i \in \{1, \dots, n\}.$$

Therefore

$$\begin{aligned}
\mathbb{E}[C_i]\mathbb{E}[J_{ik}] &= \mathbb{P}(C_i = 1)\mathbb{P}(J_{ik} = 1) \\
&= \mathbb{P}(C_i = 1)\mathbb{P}(C_k = 1 | C_i = 1) \\
&= \mathbb{P}(C_i = 1)\frac{\mathbb{P}(C_k = 1, C_i = 1)}{\mathbb{P}(C_i = 1)} \\
&= \mathbb{P}(C_k = 1, C_i = 1) \\
&= \mathbb{E}[C_i C_k], \quad \forall k \in \Gamma_i, i \in \{1, \dots, n\}.
\end{aligned}$$

For the expected values $\mathbb{E}[C_i]\mathbb{E}[|C_k - J_{ik}|]$ on the right hand side of (3.11) follows from above

$$\begin{aligned}
\mathbb{E}[C_i]\mathbb{E}[|C_k - J_{ik}|] &= \mathbb{E}[C_i]\mathbb{E}[J_{ik} - C_k] \\
&= \mathbb{E}[C_i]\mathbb{E}[J_{ik}] - \mathbb{E}[C_i]\mathbb{E}[C_k] \\
&= \mathbb{E}[C_i C_k] - \mathbb{E}[C_i]\mathbb{E}[C_k] \\
&= Cov(C_i, C_k),
\end{aligned}$$

for all $k \in \Gamma_i$ and all $i \in \{1, \dots, n\}$. The first equality holds, because the indicators $(C_i)_{i=1}^n$ are positively related.

Using this, the double sum in (3.11) can be simplified by

$$\begin{aligned}
\sum_{i=1}^n \sum_{k \in \Gamma_i} \mathbb{E}[C_i]\mathbb{E}[|C_k - J_{ik}|] &= \sum_{i=1}^n \sum_{k \in \Gamma_i} Cov(C_i, C_k) \\
&= \sum_{i=1}^n \sum_{k=1}^n Cov(C_i, C_k) - \sum_{i=1}^n Var(C_i) \\
&= Var(W) - \sum_{i=1}^n \left(\mathbb{E}[C_i^2] - \mathbb{E}[C_i]^2\right) \\
&= Var(W) - \sum_{i=1}^n \mathbb{E}[C_i] + \sum_{i=1}^n \mathbb{E}[C_i]^2 \\
&= Var(W) - \lambda + \sum_{i=1}^n \mathbb{E}[C_i]^2.
\end{aligned}$$

This proves the theorem.

□

Theorem 3.6 *If the indicator random variables* $(C_i)_{i=1}^n$ *are negatively related,*

$$d_{TV}(\mathcal{L}(W), Poi(\lambda)) \leq \frac{1-e^{-\lambda}}{\lambda}(\lambda - Var(W))$$

holds.

Proof: The only difference to the proof of theorem 3.5 is that

$$\begin{aligned}\mathbb{E}[C_i]\mathbb{E}\left[|C_k - J_{ik}|\right] &= \mathbb{E}[C_i]\mathbb{E}\left[C_k - J_{ik}\right] \\ &= -\mathbb{E}[C_i]\mathbb{E}\left[J_{ik} - C_k\right] \\ &= -Cov(C_i, C_k), \quad \forall k \in \Gamma_i, i \in \{1, \dots, n\},\end{aligned}$$

because the random variables are negatively related. Therefore

$$\begin{aligned}\sum_{i=1}^{n}\sum_{k\in\Gamma_i}\mathbb{E}[C_i]\mathbb{E}\left[|C_k - J_{ik}|\right] &= -\sum_{i=1}^{n}\sum_{k\in\Gamma_i} Cov(C_i, C_k) \\ &= -\left(Var(W) - \lambda + \sum_{i=1}^{n}\mathbb{E}[C_i]^2\right) \\ &= \lambda - Var(W) - \sum_{i=1}^{n}\mathbb{E}[C_i]^2.\end{aligned}$$

Using this in (3.11) proves the theorem. □

For the bounds given in theorem 3.5 and theorem 3.6 it's not necessary to explicitly know a monotone coupling. The existence of such a coupling is sufficient. The next theorem uses Strassen's theorem (see [14]) to obtain a criterium for this existence. For a proof see [2].

Theorem 3.7 *The indicator random variables* $(C_i)_{i=1}^n$ *are positively (negatively) related if and only if*

$$Cov(\phi(C_1, \dots, C_{k-1}, C_{k+1}, \dots, C_n), C_k) \geq (\leq)\ 0, \quad \forall k \in \{1, \dots n\},$$

for every increasing indicator function $\phi : \{0,1\}^{n-1} \to \{0,1\}$.

Remark: A function $\phi : \{0,1\}^{n-1} \to \{0,1\}$ is increasing, if $\phi(x) \leq \phi(y)$ for all $x, y \in \{0,1\}^{n-1}$ with $x \leq y$. Here the natural partial order

$$x \leq y \Leftrightarrow x_i \leq y_i \quad \forall i \in \{1, \dots, n-1\}, \tag{3.14}$$

where $x = (x_1, \dots, x_{n-1})$ and $y = (y_1, \dots, y_{n-1})$, is used.

Chapter 4

Approximation of point probabilities

Approximation of point probabilities using the Chen-Stein method is already discussed in [5], section 2.4. Since the results there are not very convenient for direct calculations, some simpler considerations are used in this chapter.

The first section gives obvious bounds for the point probabilities. These bounds are the worst ones possible. They are only used, if the bounds given in the second section are not applicable, because they are too inaccurate.

In the second section the Chen-Stein method is used to obtain bounds for the approximation error, which are easy to calculate. It contains three subsections. In the first and second subsection bounds are given that hold, if the random variables fulfill some special dependencies. These dependencies are positive and negative relation as in definition 3.4. The third subsection is addressed to the point probability of the point 0. A bound, which only holds for the approximation error of this point probability, is given there.

4.1 Trivial bounds

The bounds in the following theorem use the property that probabilities are always greater or equal 0 and less or equal 1. They can be seen as a maximum and minimum for the bounds in the next section.

Theorem 4.1 *Let X and W be arbitrary random variables taking values in $\mathbb{N}_0$. Then for all $k \in \mathbb{N}_0$ holds*

$$\mathbb{P}(X = k) + \varepsilon_-(k) \leq \mathbb{P}(W = k) \leq \mathbb{P}(X = k) + \varepsilon_+(k),$$

where

$$\varepsilon_-(k) = -\mathbb{P}(X = k)$$

and

$$\varepsilon_+(k) = 1 - \mathbb{P}(X = k).$$

Proof: Because of

$$\mathbb{P}(W = k) \geq 0 = \mathbb{P}(X = k) - \mathbb{P}(X = k)$$

and

$$\mathbb{P}(W = k) \leq 1 = \mathbb{P}(X = k) + (1 - \mathbb{P}(X = k)), \quad \forall k \in \mathbb{N}_0,$$

the theorem follows. □

4.2 The Chen-Stein method for point probabilities

Throughout this section let

$$W := \sum_{i=1}^{n} C_i,$$

$$\lambda := \mathbb{E}[W] = \sum_{i=1}^{n} \mathbb{E}[C_i] > 0,$$

$$X \sim Poi(\lambda),$$

where $(C_i)_{i=1}^n$ are indicator random variables. To obtain bounds for the point probabilities using the Chen-Stein method, the same starting point is used as in [5]. The Chen-Stein method is usually used to bound the total variation distance as in (3.11). To do this, the estimate (3.4) is used. For the point probability $\mathbb{P}(W = j)$, with $j \in \mathbb{N}_0$, A can be set to $\{j\}$ in (3.4). Let f_j denote f_A defined as in (3.2), with $A = \{j\}$. Then f_j is explicitly given by

$$f_j(k) = \begin{cases} 0 & \text{if } k = 0 \\ \frac{(k-1)!}{\lambda^k} \frac{\lambda^j}{j!} \left(\sum_{i=0}^{k-1} \frac{\lambda^i}{i!} e^{-\lambda} - \mathbb{1}_{\mathbb{N}_0 \setminus \{0,\ldots,k-1\}}(j) \right) & \text{if } k \geq 1 \end{cases}. \tag{4.1}$$

The bound given in theorem 3.3 can now be improved, by finding a better estimate for $||\Delta f_j||$ than (3.8), using the special structure of f_j. The following lemma lists some useful, basic properties of f_j.

Lemma 4.2 *Let f_j be given by (4.1) for $j \in \mathbb{N}_0$, $\lambda > 0$. Then f_j has the following properties:*

(p1) $f_j(k) > 0, \quad \forall k \leq j$

(p2) $f_j(k) < 0, \quad \forall k \geq j+1$

(p3) $f_j(k+1) - f_j(k) > 0, \quad \forall k \neq j$

(p4) $\Delta f_j(k) - \Delta f_j(k+1) \geq 0, \quad \forall k \geq j+1$

Remark: It can also be shown that

$$\Delta f_j(k) - \Delta f_j(k+1) \leq 0, \quad \forall k \leq j$$

holds. But since this property is not used here, the proof is omitted.

Proof of Lemma 4.2: The properties (1)-(3) follow from the proof of lemma 3.2. Property (4) is equivalent to

$$2f_j(k+1) - f_j(k) - f_j(k+2) \geq 0, \quad \forall k \geq j+1. \tag{4.2}$$

Since for $k \geq j+1$, f_j can be written as

$$\begin{aligned} f_j(k) &= \frac{(k-1)!}{\lambda^k} \frac{\lambda^j}{j!} \left(\sum_{i=0}^{k-1} \frac{\lambda^i}{i!} \exp^{-\lambda} - 1 \right) \\ &= -\frac{(k-1)!}{\lambda^k} \frac{\lambda^j}{j!} \sum_{i=k}^{\infty} \frac{\lambda^i}{i!} \exp^{-\lambda}, \end{aligned}$$

the inequality (4.2) is equivalent to

$$\begin{aligned} &\frac{(k-1)!}{\lambda^k} \sum_{i=k}^{\infty} \frac{\lambda^i}{i!} + \frac{(k+1)!}{\lambda^{k+2}} \sum_{i=k+2}^{\infty} \frac{\lambda^i}{i!} - 2\frac{(k)!}{\lambda^{k+1}} \sum_{i=k+1}^{\infty} \frac{\lambda^i}{i!} \geq 0 \\ \Leftrightarrow\ &\frac{\lambda}{k} \sum_{i=k}^{\infty} \frac{\lambda^i}{i!} + \frac{k+1}{\lambda} \sum_{i=k+2}^{\infty} \frac{\lambda^i}{i!} - 2 \sum_{i=k+1}^{\infty} \frac{\lambda^i}{i!} \geq 0 \\ \Leftrightarrow\ &\sum_{i=k}^{\infty} \frac{1}{k} \frac{\lambda^{i+1}}{i!} + \sum_{i=k+1}^{\infty} \frac{k+1}{i+1} \frac{\lambda^i}{i!} - 2 \sum_{i=k+1}^{\infty} \frac{\lambda^i}{i!} \geq 0, \quad \forall k \geq j+1. \end{aligned}$$

The left hand side is 0 for $\lambda = 0$. It holds that it is increasing in λ, if the first derivative in λ is non-negative. The first derivative of the left hand side is given by

$$\sum_{i=k}^{\infty} \left(\frac{i+1}{k} + \frac{k+1}{i+2} - 2 \right) \frac{\lambda^i}{i!}. \tag{4.3}$$

Since $\lambda > 0$, (4.3) is non-negative, if the coefficients fulfill

$$\frac{i+1}{k} + \frac{k+1}{i+2} - 2 \geq 0,$$

for all $i \geq k$. Multiplying this inequality with $k(i+2)$ leads to

$$\begin{aligned} (i+1)(i+2) + k(k+1) - 2k(i+2) &\geq 0 \\ \Leftrightarrow i^2 + 3i + 2 + k^2 - 2ki - 3k &\geq 0 \\ \Leftrightarrow (i-k)^2 + 3(i-k) + 2 &\geq 0. \end{aligned}$$

This is true for all $i \geq k$. □

The next theorem gives a bound for the approximation error, by improving the estimate (3.8) for $A = \{j\}$.

Theorem 4.3 *Let*

$$W = \sum_{i=1}^{n} C_i, \qquad \lambda = \mathbb{E}[W] = \sum_{i=1}^{n} \mathbb{E}[C_i] > 0,$$

where $(C_i)_{i=1}^n$ *are indicator variables and* f_j *be given by (4.1) for* $j \in \mathbb{N}_0$. *For each* $i \in \{1, \ldots, n\}$ *set* $\Gamma_i := \{1, \ldots, n\} \setminus \{i\}$ *and let the random variables* $\{C_k : k \in \{1, \ldots, n\}\}$ *and* $\{J_{ik} : k \in \Gamma_i\}$ *be defined on the same probability space with*

$$(J_{ik}, k \in \Gamma_i) \overset{(d)}{=} (C_k, k \in \Gamma_i | C_i = 1).$$

Then for all $j \in \mathbb{N}_0$

$$\begin{aligned} &|\mathbb{P}(W = j) - \mathbb{P}(X = j)| \\ &\quad \leq |\Delta f_j(j)| \sum_{i=1}^{n} \left(\mathbb{E}[C_i]^2 + \sum_{k \in \Gamma_i} \mathbb{E}[C_i] \, \mathbb{E}[|C_k - J_{ik}|] \right), \end{aligned} \tag{4.4}$$

where X *is a Poisson distributed random variable with parameter* λ.

Proof: From property (1), (2) and (3) of f_j in lemma 4.2 follows

$$\begin{aligned} f_j(k) &\leq f_j(j) \qquad \textit{and} \\ f_j(k) &\geq f_j(j+1), \quad \forall k \in \mathbb{N}. \end{aligned}$$

Therefore

$$\|\Delta f_j\| = |f_j(j) - f_j(j+1)| = |\Delta f_j(j)|. \tag{4.5}$$

Setting $A = \{j\}$ in (3.11) and using (4.5), proves the theorem. □

Remark: Since $f_j(j+1) \leq f_j(j)$, $|\Delta f_j(j)| = f_j(j) - f_j(j+1)$, for all $j \in \mathbb{N}_0$.

A bound that is even easier to calculate is given in the next theorem.

Theorem 4.4 *Let*

$$W = \sum_{i=1}^{n} C_i, \qquad \lambda = \mathbb{E}\left[W\right] = \sum_{i=1}^{n} \mathbb{E}\left[C_i\right] > 0,$$

where $(C_i)_{i=1}^n$ *are indicator variables and* f_j *be given by (4.1) for* $j \in \mathbb{N}_0$. *Then*

$$|\mathbb{P}(W = j) - \mathbb{P}(X = j)| \leq \lambda |\Delta f_j(j)|, \tag{4.6}$$

where X *is a Poisson distributed random variable with parameter* λ.

Proof: As in the proof of theorem 4.3, it holds that

$$\begin{aligned} f_j(k) &\leq f_j(j) \qquad \textit{and} \\ f_j(k) &\geq f_j(j+1), \quad \forall k \in \mathbb{N}. \end{aligned}$$

Setting $A = \{j\}$ in (3.4) and using this estimates for f_j leads to

$$\begin{aligned} |\mathbb{P}(W = j) - \mathbb{P}(X = j)| &\leq \sum_{i=1}^{n} \mathbb{E}[C_i]\mathbb{E}[|f_j(W+1) - f_j(V_i+1)|] \\ &\leq \sum_{i=1}^{n} \mathbb{E}[C_i]\mathbb{E}[|f_j(j) - f_j(j+1)|] \\ &= |\Delta f_j(j)| \sum_{i=1}^{n} \mathbb{E}[C_i] \\ &= |\Delta f_j(j)|\lambda, \end{aligned}$$

where $\{V_i : 1 \le i \le n\}$ are random variables defined on the same probability space as W with

$$V_i \overset{(d)}{=} (W|C_i = 1), \quad \forall i \in \{1, \dots, n\}.$$

□

The proof of theorem 4.4 uses

$$\mathbb{E}[|f_j(W+1) - f_j(V_i+1)|] \le |\Delta f_j(j)|, \tag{4.7}$$

while the proof of theorem 4.3 uses (3.7) with $A = \{j\}$ and (4.5) to bound the left hand side of (4.7). Since both estimates hold, it is reasonable to take the minimum of them. The following corollary combines theorem 4.3 and theorem 4.4.

Corollary 4.5 *Let*

$$W = \sum_{i=1}^{n} C_i, \qquad \lambda = \mathbb{E}[W] = \sum_{i=1}^{n} \mathbb{E}[C_i] > 0,$$

where $(C_i)_{i=1}^n$ are indicator variables and f_j be given by (4.1) for $j \in \mathbb{N}_0$. For each $i \in \{1, \dots, n\}$ set $\Gamma_i := \{1, , \dots, n\} \setminus \{i\}$ and let the random variables $\{C_k : k \in \{1, \dots, n\}\}$ and $\{J_{ik} : k \in \Gamma_i\}$ be defined on the same probability space with

$$(J_{ik}, k \in \Gamma_i) \overset{(d)}{=} (C_k, k \in \Gamma_i | C_i = 1).$$

Then for all $j \in \mathbb{N}_0$

$$\begin{aligned} &|\mathbb{P}(W = j) - \mathbb{P}(X = j)| \\ &\quad \le |\Delta f_j(j)| \sum_{i=1}^{n} \min\left(\mathbb{E}[C_i], \mathbb{E}[C_i]^2 + \sum_{k \in \Gamma_i} \mathbb{E}[C_i]\, \mathbb{E}[|C_k - J_{ik}|] \right), \end{aligned} \tag{4.8}$$

where X is a Poisson distributed random variable with parameter λ. □

Note that the bound given in corollary 4.5 is not just the minimum of the bounds given in theorem 4.3 and theorem 4.4. The minimum is taken over each summand. Therefore this estimate may be better than both of the other two bounds.

4.2.1 Positively related random variables

The next theorem gives a bound for the approximation errors

$$|\mathbb{P}(W = j) - \mathbb{P}(X = j)|, \quad \forall j \in \mathbb{N}_0, \tag{4.9}$$

if the random variables $(C_i)_{i=1}^n$ are positively related, in the sense of definition 3.4. Note that this bound is just an extension of the bound given in corollary 4.5.

Theorem 4.6 *Under the assumptions of corollary 4.5*

$$\begin{aligned} &|\mathbb{P}(W = j) - \mathbb{P}(X = j)| \\ &\leq |\Delta f(j)| \sum_{i=1}^{n} \min\left(\mathbb{E}[C_i], \mathbb{E}\left[C_i\right]^2 + \sum_{k\in\Gamma_i} Cov(C_i, C_k)\right), \end{aligned} \tag{4.10}$$

for all $j \in \mathbb{N}_0$, if the indicators $(C_i)_{i=1}^n$ are positively related.

Proof: By definition of positive relation there exist random variables $\{J_{ik} : i \in \{1, \ldots, n\}, k \in \Gamma_i\}$, which fulfill the assumptions, with

$$J_{ik} \geq C_k, \quad \forall k \in \Gamma_i, i \in \{1, \ldots, n\}.$$

Therefore

$$\mathbb{E}[|C_k - J_{ik}|] = \mathbb{E}[J_{ik} - C_k].$$

In the proof of theorem 3.5 it's shown that

$$\mathbb{E}[C_i]\mathbb{E}[J_{ik} - C_k] = Cov(C_i, C_k), \quad \forall k \in \Gamma_i, i \in \{1, \ldots, n\}.$$

Using this in (4.8) proves the theorem. □

4.2.2 Negatively related random variables

In this section let the random variables $(C_i)_{i=1}^n$ be negatively related in the sense of definition 3.4, instead of positively related. The bound given in the next theorem, is once more an extension of corollary 4.5.

Theorem 4.7 *Under the assumptions of corollary 4.5*

$$\begin{aligned} &|\mathbb{P}(W = j) - \mathbb{P}(X = j)| \\ &\leq |\Delta f(j)| \sum_{i=1}^{n} \min\left(\mathbb{E}[C_i], \mathbb{E}\left[C_i\right]^2 - \sum_{k\in\Gamma_i} Cov(C_i, C_k)\right), \end{aligned} \tag{4.11}$$

for all $j \in \mathbb{N}_0$, if the indicators $(C_i)_{i=1}^n$ are negatively related.

Proof: For negatively related random variables $(C_i)_{n=1}^n$ exist random variables $\{J_{ik} : i \in \{1, \ldots, n\}, k \in \Gamma_i\}$ fulfilling the assumptions with

$$J_{ik} \leq C_k, \quad \forall k \in \Gamma_i, i \in \{1, \ldots, n\}.$$

It follows

$$\mathbb{E}[|C_k - J_{ik}|] = \mathbb{E}[C_k - J_{ik}] = -\mathbb{E}[J_{ik} - C_k].$$

As in the proof of theorem 4.6, (4.8) follows from

$$\mathbb{E}[C_i](-\mathbb{E}[J_{ik} - C_k]) = -Cov(C_i, C_k), \quad \forall k \in \Gamma_i, i \in \{1, \ldots, n\}.$$

□

4.2.3 Special case for the point 0

For the approximation of the point probability $\mathbb{P}(W = 0)$ the function f_0, defined by (4.1) with $A = \{0\}$ is explicitly given by

$$f_0(k) = \begin{cases} 0 & \text{if } k = 0 \\ \frac{(k-1)!}{\lambda^k} \left(\sum_{i=0}^{k-1} \frac{\lambda^i}{i!} e^{-\lambda} - 1 \right) & \text{if } k \geq 1 \end{cases}. \tag{4.12}$$

Now define a function by

$$\tilde{f}_0(k) := \max_{i \in \{1, \ldots, n-k+1\}} |f_0(i) - f_0(i+k)|, \quad k \in \{0, \ldots, n\}, \tag{4.13}$$

where f_0 is defined as in (4.12). Then for the approximation error follows

$$\begin{aligned} |\mathbb{P}(W = 0) - \mathbb{P}(X = 0)| &\leq \sum_{i=1}^{n} \mathbb{E}[C_i]\mathbb{E}[|f_0(W+1) - f_0(V_i+1)|] \\ &\leq \sum_{i=1}^{n} \mathbb{E}[C_i]\mathbb{E}[\tilde{f}_0(|W - V_i|)] \end{aligned} \tag{4.14}$$

from (3.4), where $(V_i)_{i=1}^n$ is defined as in (3.6). The following lemma gives two properties of the function $\tilde{f}_0$.

Lemma 4.8 *Let f_0 and $\tilde{f}_0$ be defined as in (4.12) and (4.13), $\lambda > 0$. Then $\tilde{f}_0$ is increasing and*

$$\tilde{f}_0(k) = f_0(1+k) - f_0(1), \quad \forall k \in \{0, \ldots, n\}.$$

Proof: From property (3) and (4) in theorem 4.2 follows

(1) $f_0(k+1) - f_0(k) > 0, \quad \forall k \geq 1$

(2) $\Delta f_0(k) \geq \Delta f_0(k+1), \quad \forall k \geq 1$

Therefore

$$\begin{aligned}
\tilde{f}_0(k) &= \max_{i \in \{1,\ldots,n-k+1\}} |f_0(i) - f_0(i+k)| \\
&= \max_{i \in \{1,\ldots,n-k+1\}} (f_0(i+k) - f_0(i)) \\
&= \max_{i \in \{1,\ldots,n-k+1\}} \sum_{m=0}^{k-1} \Delta f_0(i+m) \\
&= \sum_{m=0}^{k-1} \Delta f_0(1+m) \\
&= f_0(1+k) - f_0(1), \qquad \forall k \in \{0,\ldots,n\}\,.
\end{aligned}$$

The second equality holds because of (1) and the fourth equality holds because of (2). Using this representation, it is easy to see that $\tilde{f}_0$ is increasing, since (1) holds for $f_0(k)$. □

For the continuous function $\hat{f}_0$ defined by

$$\hat{f}_0(x) = (x - \lfloor x \rfloor)\tilde{f}_0(\lfloor x \rfloor) + (1 - (x - \lfloor x \rfloor))\tilde{f}_0(\lceil x \rceil), \quad \forall x \in [0,n]\,, \tag{4.15}$$

holds

(1) $\hat{f}_0(k) = \tilde{f}_0(k), \quad \forall k \in \{0,\ldots,n\}$,

(2) $\hat{f}$ is linear on $[k, k+1]\,, \quad \forall k \in \{0,\ldots,n-1\}$.

The next lemma gives other properties of $\hat{f}$.

Lemma 4.9 *The function $\hat{f}_0$, defined as in (4.15), is concave and increasing.*

Proof: The first derivative of $\hat{f}$ can be interpreted as the slope of $\hat{f}$. Since $\hat{f}$ is linear on $[k, k+1], \forall k \in \{0,\ldots,n-1\}$, the slope of $\hat{f}$ is given by

$$\begin{aligned}
\hat{f}_0'(x) &= \hat{f}_0(\lceil x \rceil) - \hat{f}_0(\lfloor x \rfloor) \\
&= \tilde{f}_0(\lceil x \rceil) - \tilde{f}_0(\lfloor x \rfloor) \\
&= f_0(1 + \lceil x \rceil) - f_0(1 + \lfloor x \rfloor) \\
&= \Delta f_0(\lfloor x \rfloor + 1), \qquad \forall x \in (k, k+1)
\end{aligned}$$

Because the increments of f_0 are decreasing, it follows for all $x \in (k, k+1)$, $y \in (\tilde{k}, \tilde{k}+1)$, $x \leq y$, with $k, \tilde{k} \in \{0, \dots, n-1\}$

$$\hat{f}_0'(x) \geq \hat{f}_0'(y). \tag{4.16}$$

Now pick two arbitrary points $x, y \in [0, n]$. If they are both in the same interval $[k, k+1]$ for an arbitrary $k \in \{0, \dots, n-1\}$, $\hat{f}_0$ is linear between them and therefore concave. If they are not in the same interval, draw a straight line g from $\hat{f}_0(x)$ to $\hat{f}_0(y)$.

Following $\hat{f}_0$ from x to y the slope of $\hat{f}_0$ is greater than that of g in the beginning. Here the points $\{0, \dots, n\}$ are excepted, since the derivative doesn't exist at these points. Going on, the slope of $\hat{f}_0$ decreases because of (4.16) until g crosses $\hat{f}_0$ at point p. Because the slope of $\hat{f}_0$ is smaller than that of g after they hit, there are no more points of intersection. This means that $p = \hat{f}_0(y)$.

Hence $\hat{f}_0$ is greater than g on $[x, y]$. This is also true for the points $\{0, \dots, n\}$, since $\hat{f}_0$ is continuous. Therefore $\hat{f}_0$ fulfills

$$\hat{f}_0(tx + (1-t)y) \geq t\hat{f}_0(x) + (1-t)\hat{f}_0(y), \quad \forall t \in [0, 1],$$

for all $x, y \in [0, n]$. Since this is the definition of concaveness, $\hat{f}_0$ is concave.

Because of lemma 4.8, $\tilde{f}$ is increasing. From the properties above follows that $\hat{f}$ is increasing too. □

The function $\hat{f}_0$ can now be used to obtain a bound for the point probability $\mathbb{P}(W = 0)$.

Theorem 4.10 *Let*

$$W = \sum_{i=1}^{n} C_i, \qquad \lambda = \mathbb{E}[W] = \sum_{i=1}^{n} \mathbb{E}[C_i] > 0,$$

where $(C_i)_{i=1}^n$ *are indicator variables and* $\hat{f}_0$ *be defined as in (4.15). For each* $i \in \{1, \dots, n\}$ *set* $\Gamma_i := \{1, , \dots, n\} \setminus \{i\}$ *and let the random variables* $\{C_k : k \in \{1, \dots, n\}\}$ *and* $\{J_{ik} : k \in \Gamma_i\}$ *be defined on the same probability space with*

$$(J_{ik}, k \in \Gamma_i) \stackrel{(d)}{=} (C_k, k \in \Gamma_i | C_i = 1).$$

Then

$$|\mathbb{P}(W = 0) - \mathbb{P}(X = 0)| \leq \sum_{i=1}^{n} \mathbb{E}[C_i] \hat{f}_0 \left(\mathbb{E}\left[C_i + \sum_{k \in \Gamma_i} |C_k - J_{ik}| \right] \right), \tag{4.17}$$

where X is a Poisson distributed random variable with parameter λ.

Proof: As in (4.14) the approximation error can be estimated by

$$|\mathbb{P}(W=0) - \mathbb{P}(X=0)| \leq \sum_{i=1}^{n} \mathbb{E}[C_i]\mathbb{E}[\tilde{f}_0(|W-V_i|)],$$

with $(V_i)_{i=1}^n$ defined as in (3.6). By definition of $\hat{f}_0$,

$$\mathbb{E}[\tilde{f}_0(|W-V_i|)] = \mathbb{E}[\hat{f}_0(|W-V_i|)]$$

holds. Because of lemma 4.9, $\hat{f}_0$ is concave. Therefore Jensen's inequality

$$\mathbb{E}[\hat{f}_0(|W-V_i|)] \leq \hat{f}_0\left(\mathbb{E}[|W-V_i|]\right)$$

can be applied and

$$\begin{aligned}|\mathbb{P}(W=0) - \mathbb{P}(X=0)| &\leq \sum_{i=1}^{n} \mathbb{E}[C_i]\hat{f}_0\left(\mathbb{E}[|W-V_i|]\right) \\ &\leq \sum_{i=1}^{n} \mathbb{E}[C_i]\hat{f}_0\left(\mathbb{E}\left[C_i + \sum_{k\in\Gamma_i}|C_k - J_{ik}|\right]\right),\end{aligned}$$

where the second inequality follows from the definition of $(V_i)_{i=1}^n$ and because $\hat{f}_0$ is increasing. □

The following two corollaries can be obtained from theorem 4.10 the same way as theorem 4.6 and theorem 4.7 are obtained from corollary 4.5.

Corollary 4.11 *With the same assumptions as in theorem 4.10,*

$$|\mathbb{P}(W=0) - \mathbb{P}(X=0)| \leq \sum_{i=1}^{n} \mathbb{E}[C_i]\hat{f}_0\left(\mathbb{E}\left[C_i\right] + \sum_{k\in\Gamma_i}\frac{Cov(C_i,C_k)}{\mathbb{E}[C_i]}\right).$$

if the random variables $(C_i)_{i=1}^n$ are positively related. □

Corollary 4.12 *With the same assumptions as in theorem 4.10,*

$$|\mathbb{P}(W=0) - \mathbb{P}(X=0)| \leq \sum_{i=1}^{n} \mathbb{E}[C_i]\hat{f}_0\left(\mathbb{E}\left[C_i\right] - \sum_{k\in\Gamma_i}\frac{Cov(C_i,C_k)}{\mathbb{E}[C_i]}\right).$$

if the random variables $(C_i)_{i=1}^n$ are negatively related. □

Chapter 5

Price approximation for structure floors

In this chapter bounds for the price of a structure floor are given, where the point probabilities of the payoff of the underlying structured note are approximated. The following theorem is the result of this chapter.

Theorem 5.1 *Let W be the payoff of a structured note taking values in $\{0,1,...,n\}$, n be the number of coupons in the structured note, x be the level of a structure floor, X be a Poisson distributed random variable and f be given by*

$$f(k) := x - k, \quad \forall k \in \{0,\dots,n \wedge \lfloor x \rfloor\}.$$

If for sequences $(\varepsilon_-(k))_{k=0}^n$ and $(\varepsilon_+(k))_{k=0}^n$

$$\mathbb{P}(X = k) + \varepsilon_-(k) \leq \mathbb{P}(W = k) \leq \mathbb{P}(X = k) + \varepsilon_+(k)$$

holds, then for the price SF of the structure floor holds

$$\begin{aligned} e^{-rT_n}\left(\mathbb{E}[f(X)] + \sum_{k=0}^{n} f(k)\hat{\varepsilon}_-(k)\right) &\leq SF(x) \\ &\leq e^{-rT_n}\left(\mathbb{E}[f(X)] + \sum_{k=0}^{n} f(k)\hat{\varepsilon}_+(k)\right), \end{aligned} \tag{5.1}$$

with r and T_n as described in section 2.2. In (5.1), $(\hat{\varepsilon}_-(k))_{k=0}^n$ and $(\hat{\varepsilon}_+(k))_{k=0}^n$ are given by

$$\hat{\varepsilon}_-(k) = \begin{cases} \varepsilon_-(k) & \text{if } k = 0,\dots,j-1 \\ 1 - \sum_{i=0}^{n}\mathbb{P}(X=i) - \sum_{i=0}^{j-1}\varepsilon_-(i) - \sum_{i=j+1}^{n}\varepsilon_+(i) & \text{if } k = j \\ \varepsilon_+(k) & \text{if } k = j+1,\dots,n \end{cases}$$

where j fulfills

$$\left(1-\sum_{i=0}^{n}\mathbb{P}(X=i)-\sum_{i=0}^{j-1}\varepsilon_-(i)-\sum_{i=j+1}^{n}\varepsilon_+(i)\right)\in[\varepsilon_-(j),\varepsilon_+(j)],$$

and

$$\hat{\varepsilon}_+(k)=\begin{cases}\varepsilon_+(k) & \text{if } k=0,\dots,j-1\\ 1-\sum_{i=0}^{n}\mathbb{P}(X=i)-\sum_{i=0}^{j-1}\varepsilon_+(i)-\sum_{i=j+1}^{n}\varepsilon_-(i) & \text{if } k=j\\ \varepsilon_-(k) & \text{if } k=j+1,\dots,n\end{cases}$$

where j fulfills

$$\left(1-\sum_{i=0}^{n}\mathbb{P}(X=i)-\sum_{i=0}^{j-1}\varepsilon_+(i)-\sum_{i=j+1}^{n}\varepsilon_-(i)\right)\in[\varepsilon_-(j),\varepsilon_+(j)].$$

Proof: If

$$\varepsilon(k):=\mathbb{P}(W=k)-\mathbb{P}(X=k),\quad \forall k\in\{0,\dots,n\}$$

denotes the true error, then

$$\varepsilon(k)\in[\varepsilon_-(k),\varepsilon_+(k)],\quad \forall k\in\{0,\dots,n\}\,. \tag{5.2}$$

Another condition for $(\varepsilon(k))_{k=0}^{n}$ can be obtained by observing that W can only take values from 0 to n. Therefore

$$1=\sum_{k=0}^{n}\mathbb{P}(W=k)=\sum_{k=0}^{n}(\mathbb{P}(X=k)+\varepsilon(k)),$$

what implies

$$\sum_{k=0}^{n}\varepsilon(k)=1-\sum_{k=0}^{n}\mathbb{P}(X=k). \tag{5.3}$$

The bounds for the expectation can now be written as

$$\sum_{k=0}^{n}f(k)\mathbb{P}(X=k)+\inf A\le\mathbb{E}[f(W)]\le\sum_{k=0}^{n}f(k)\mathbb{P}(X=k)+\sup A, \tag{5.4}$$

where

$$A:=\left\{\sum_{k=0}^{n}f(k)\varepsilon(k):\varepsilon(i)\in[\varepsilon_-(i),\varepsilon_+(i)]\quad\forall i\in\{0,\dots,n\},\right.$$
$$\left.\sum_{k=0}^{n}\varepsilon(k)=1-\sum_{k=0}^{n}\mathbb{P}(X=k)\right\}.$$

By setting

$$\hat{\varepsilon}_+(k) := \varepsilon_+(k), \qquad \forall k \in \{0, \dots, n\}$$

and

$$E := \sum_{k=0}^{n} f(k)\hat{\varepsilon}_+(k),$$

E is the greatest possible error without the additional condition (5.3). Obviously it holds that

$$\sum_{k=0}^{n} \hat{\varepsilon}_+(k) \geq 1 - \sum_{k=0}^{n} \mathbb{P}(X = k). \tag{5.5}$$

Since f is positive, E will decrease if $\hat{\varepsilon}_+(k)$ is reduced, for any k. Because f is decreasing, the least change of E is achieved by reducing $\hat{\varepsilon}_+(n)$. To obtain the supremum in (5.4), reduce $\hat{\varepsilon}_+(n)$ until equality in (5.5) holds or $\hat{\varepsilon}_+(n) = \varepsilon_-(k)$. In the latter case, $\hat{\varepsilon}_+(n)$ can't be reduced anymore. Otherwise the condition

$$\hat{\varepsilon}(n) \in [\varepsilon_-(n), \varepsilon_+(n)]$$

wouldn't be fulfilled. Now the least change of E is achieved by reducing $\hat{\varepsilon}_+(n-1)$. Repeating this steps until equality in (5.5) holds, leads to

$$\hat{\varepsilon}_+(k) = \begin{cases} \varepsilon_+(k) & \text{if } k = 0, \dots, j-1 \\ 1 - \sum_{i=0}^{n} \mathbb{P}(X = i) - \sum_{i=0}^{j-1} \varepsilon_+(i) - \sum_{i=j+1}^{n} \varepsilon_-(i) & \text{if } k = j \\ \varepsilon_-(k) & \text{if } k = j+1, \dots, n \end{cases}$$

where j fulfills

$$\left(1 - \sum_{k=0}^{n} \mathbb{P}(X = k) - \sum_{k=0}^{j-1} \varepsilon_+(k) - \sum_{k=j+1}^{n} \varepsilon_-(k)\right) \in [\varepsilon_-(j), \varepsilon_+(j)].$$

It holds that

$$E = \sup A.$$

For the infimum in (5.4) the same procedure can be used. Just set

$$\hat{\varepsilon}_-(k) := \varepsilon_-(k), \qquad \forall k \in \{0, \dots, n\}$$

and increase some of these $\hat{\varepsilon}_-(k)$ as described above. Then $\hat{\varepsilon}_-(k)$ is given by

$$\hat{\varepsilon}_-(k) = \begin{cases} \varepsilon_-(k) & \text{if } k = 0, \dots, j-1 \\ 1 - \sum_{i=0}^{n} \mathbb{P}(X = i) - \sum_{i=0}^{j-1} \varepsilon_-(i) + \sum_{i=j+1}^{n} \varepsilon_+(i) & \text{if } k = j \\ \varepsilon_+(k) & \text{if } k = j+1, \dots, n \end{cases}$$

where j fulfills

$$\left(1 - \sum_{k=0}^{n} \mathbb{P}(X = k) - \sum_{k=0}^{j-1} \varepsilon_-(k) - \sum_{k=j+1}^{n} \varepsilon_+(k)\right) \in [\varepsilon_-(j), \varepsilon_+(j)].$$

From the definition of the price for a structure floor (2.11) at level x follows

$$SF(x) = e^{-rT_n}\mathbb{E}\left[(x - W)^+\right] = e^{-rT_n}\mathbb{E}[f(W)].$$

Hence multiplying (5.4) with e^{-rT_n} proves the theorem. □

Remark: Here the assumption that the approximating random variable is Poisson distributed is made. This is not necessary. The considerations in this chapter are also true for an arbitrary random variable.

Chapter 6

Numerical results

This chapter is addressed to the numerical calculation of the bounds given in chapter 4 and 5. The aim is to approximate the price of a structure floor, since the computational effort for the calculation of the exact price given in theorem 2.17 is very high.

Throughout the whole chapter, let a structure note with payoff W be defined as in section 2.2 consisting of coupons $(C_i)_{i=1}^n$ defined as in (2.7) and

$$SF(x) := e^{-rT_n}\mathbb{E}\left[(x-W)^+\right] \quad \forall x \in [0,n] \tag{6.1}$$

denotes the exact price of a structure floor at level x, as given by theorem 2.17. Furthermore let SF_X be the price of the structure floor (6.1), where W is substituted by a Poisson distributed random variable X with parameter $\mathbb{E}[W]$. The expected values

$$BD(S_0,(T_i)_{i\in I},P,B_{low},B_{up},\sigma,0) = \mathbb{E}\left[\prod_{i\in I} C_i\right], \quad I \subseteq \{0,\dots,n\} \tag{6.2}$$

can be computed using theorem 2.16.

Remark: For the calculation of the values (6.2), the corrected Mathematica-function `BDMult` from [9] is used. In the original function, wrong integration bounds are used. The corrected Mathematica function can be found in the appendix.

The computational effort of these values increases significantly, as the number of elements in I increases. To calculate the bounds from the previous two chapters, only the values $(\mathbb{E}[C_i])_{i=1}^n$ and $(\mathbb{E}[C_iC_j])_{i,j=1}^n)$ are needed. Therefore the computational effort for the approximation is much less than that for the exact price. Since

$$\mathbb{E}[C_iC_j] = \mathbb{E}[C_jC_i], \quad \forall i, j \in \{1, \dots, n\}$$

and

$$\mathbb{E}[C_iC_i] = \mathbb{E}[C_i], \quad \forall i \in \{1, \dots, n\}$$

the n values $(\mathbb{E}[C_i])_{i=1}^n$ and $\frac{n^2-n}{2}$ values $(\mathbb{E}[C_iC_j])_{j=1}^{i-1}$ for all $i \in \{1, \dots, n\}$ must be computed. Hence the computational effort for the approximation is high too, for large n. Since theorem 4.4 only uses $\sum_{i=1}^n \mathbb{E}[C_i]$, the bounds given there can be used in cases where n is large. Then only the n values $(\mathbb{E}[C_i])_{i=1}^n$ must be computed.

An improvement for the approximation in all cases can be made, by noting that

$$\mathbb{E}\left[\prod_{i=1}^n C_i\right] = \mathbb{E}\left[\tilde{C}_1\right],$$

where $\tilde{C}_1$ is defined as C_1 with barrier length nP. Therefore

$$\begin{aligned}\mathbb{E}\left[\prod_{i=1}^n C_i\right] &= BD(S_0, (T_i)_{i=0}^{n-1}, P, B_{low}, B_{up}, \sigma, 0)\\ &= BD(S_0, (T_0), nP, B_{low}, B_{up}, \sigma, 0),\end{aligned}$$

as also used in theorem 2.17.

6.1 A general coupling

The bounds given in theorem 4.3 and corollary 4.5 use random variables $\{J_{ik} : i \in \{1, \dots, n\}, k \in \{1, \dots, n\} \setminus \{i\}\}$, which are defined on the same probability space as $\{C_i : i \in \{1, \dots, n\}\}$ and fulfill

$$(J_{ik}, k \in \Gamma_i) \stackrel{(d)}{=} (C_k, k \in \Gamma_i | C_i = 1), \tag{6.3}$$

where $\Gamma_i := \{1, \dots, n\} \setminus \{i\}$ for all $i \in \{1, \dots, n\}$.

One way to construct such indicator random variables is to simply define their joint distribution by (6.3) and let them be independent from the random variables $(C_i)_{i=1}^n$. Then for all $k \in \Gamma_i, i \in \{1, \dots, n\}$ follows

$$\begin{aligned}
\mathbb{P}(J_{ik} = 1) = 1 - \mathbb{P}(J_{ik} = 0) &= \mathbb{P}(J_{ik} = 1, J_{il} \leq 1, l \in \Gamma_i \backslash \{k\}) \\
&= \mathbb{P}(C_k = 1, C_l \leq 1, l \in \Gamma_i \backslash \{k\} | C_i = 1) \\
&= \mathbb{P}(C_k = 1 | C_i = 1) \\
&= \frac{\mathbb{P}(C_k = 1, C_i = 1)}{\mathbb{P}(C_i = 1)} \\
&= \frac{\mathbb{E}[C_k C_i]}{\mathbb{E}[C_i]}.
\end{aligned}$$

The expected values $\mathbb{E}[|C_k - J_{ik}|]$, $i \in \{1, \dots, n\}$, $k \in \Gamma_i$, which are used in the bounds of theorem 4.3 and corollary 4.5, are then given by

$$\begin{aligned}
\mathbb{E}[|C_k - J_{ik}|] &= \mathbb{P}(|C_k - J_{ik}| = 1) \\
&= \mathbb{P}(C_k = 1, J_{ik} = 0) + \mathbb{P}(C_k = 0, J_{ik} = 1) \\
&= \mathbb{P}(C_k = 1)\mathbb{P}(J_{ik} = 0) + \mathbb{P}(C_k = 0)\mathbb{P}(J_{ik} = 1) \\
&= \mathbb{E}[C_k] \left(1 - \frac{\mathbb{E}[C_k C_i]}{\mathbb{E}[C_i]}\right) + (1 - \mathbb{E}[C_k]) \frac{\mathbb{E}[C_k C_i]}{\mathbb{E}[C_i]} \\
&= \mathbb{E}[C_k] + \frac{\mathbb{E}[C_k C_i]}{\mathbb{E}[C_i]} (1 - 2\mathbb{E}[C_k])
\end{aligned}$$

The third equality holds, because of the assumption of independence.

This construction can be used for any parameters of $(C_i)_{i=1}^n$ and is easy to calculate.

6.2 A general example

In this section an example is given, to show how the results of chapter 4 and 5 can be applied to a given problem. Set $n = 7$, $r = 0.02$ and let the parameters of the coupons $(C_i)_{i=1}^7$ be given by

$$\begin{aligned}
S_0 &= 100, & T_0 &= 1, \\
P &= 1, & B_{low} &= 85, \\
B_{up} &= 115, & \sigma &= 0.18.
\end{aligned} \tag{6.4}$$

For all $i \in \{1, \dots, 7\}, j \in \Gamma_i := \{1, \dots, 7\} \setminus \{i\}$, the first and second moments of the coupons are given by

$$\mathbb{E}[C_i] = BD(100, (T_i), 1, 85, 115, 0.18, 0), \quad \textit{and}$$

$$\mathbb{E}[C_i C_j] = BD(100, (T_i, T_j), 1, 85, 115, 0.18, 0),$$

where BD is defined as in theorem 2.16.

The aim is to approximate the expected value on the right hand side of (6.1). To do this, it's necessary to approximate the point probabilities of

$$W := \sum_{i=1}^{7} C_i$$

first. The following table gives the expected values $\mathbb{E}[C_i C_j]$, for all $i, j \in \{1, \dots, 7\}$.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
C_1	0.0882	0.0153	0.0073	0.0055	0.0045	0.0039	0.0035
C_2	0.0153	0.0641	0.0111	0.0053	0.0040	0.0033	0.0029
C_3	0.0073	0.0111	0.0527	0.0091	0.0044	0.0033	0.0027
C_4	0.0055	0.0053	0.0091	0.0458	0.0079	0.0038	0.0028
C_5	0.0045	0.0040	0.0044	0.0079	0.0409	0.0071	0.0034
C_6	0.0039	0.0033	0.0033	0.0038	0.0071	0.0373	0.0065
C_7	0.0035	0.0029	0.0027	0.0028	0.0034	0.0065	0.0344

Table 1: Expected values $\mathbb{E}[C_i C_j]$

Since $(C_i)_{i=1}^7$ are indicator variables,

$$\mathbb{E}[C_i C_i] = \mathbb{E}[C_i], \quad \forall i \in \{1, \dots, 7\}$$

holds. Therefore the diagonal elements of table 1 are the expected values of $(C_i)_{i=1}^7$. Using the general coupling from the previous section, table 2 gives the expected values $\mathbb{E}[|C_k - J_{ik}|]$, for all $i \in \{1, \dots, 7\}$ and $k \in \Gamma_i$.

	1	2	3	4	5	6	7
1	$---$	0.2150	0.1270	0.1020	0.0881	0.0787	0.0718
2	0.2842	$---$	0.2076	0.1211	0.0977	0.0848	0.0822
3	0.2025	0.2477	$---$	0.2030	0.1171	0.0945	0.0027
4	0.1864	0.1656	0.2313	$---$	0.1998	0.1141	0.0920
5	0.1794	0.1488	0.1485	0.2217	$---$	0.1975	0.1117
6	0.1755	0.1412	0.1311	0.1383	0.2153	$---$	0.1956
7	0.1729	0.1369	0.1232	0.1205	0.1314	0.2107	$---$

Table 2: Expected values $\mathbb{E}[|C_k - J_{ik}|]$

These values can now be used to apply theorem 4.3, theorem 4.4 and corollary 4.5. Let $(\varepsilon(k))_{k=0}^{7}$ denote the approximation errors,

$$\varepsilon(k) := \mathbb{P}(W = k) - \mathbb{P}(X = k), \quad \forall k \in \{0, \dots, 7\},$$

where X is a Poisson distributed random variable with parameter

$$\lambda := \sum_{i=1}^{7} \mathbb{E}[C_i] = 0.363411.$$

Then

$$\begin{aligned}\mathbb{P}(W = k) &= \mathbb{P}(X = k) + (\mathbb{P}(W = k) - \mathbb{P}(X = k)) \\ &= \mathbb{P}(X = k) + \varepsilon(k), \quad \forall k \in \{0, \dots, 7\}.\end{aligned}$$

The three general bounds from chapter 4 are given by

	theorem 4.3	theorem 4.4	corollary 4.5
$\lvert\varepsilon(0)\rvert$	0.286325	0.3047	0.282254
$\lvert\varepsilon(1)\rvert$	0.286325	0.3047	0.282254
$\lvert\varepsilon(2)\rvert$	0.167604	0.17836	0.165221
$\lvert\varepsilon(3)\rvert$	0.113649	0.120942	0.112033
$\lvert\varepsilon(4)\rvert$	0.0853644	0.0908424	0.0841505
$\lvert\varepsilon(5)\rvert$	0.0682989	0.0726818	0.0673276
$\lvert\varepsilon(6)\rvert$	0.0569161	0.0605685	0.0561067
$\lvert\varepsilon(7)\rvert$	0.0487852	0.0519159	0.0480915

Table 3: General bounds from chapter 4 with $\sigma = 0.18$

In this example the first bound is better than the second one. The third bound is even better than the first one. This is because the third bound is not just the minimum of the first and second one, as described in chapter 4. But there are also many cases (if not most) in which the third bound turns out to be the minimum of the first two ones.

Setting $\sigma = 0.14$ in (6.4) gives the following bounds.

	theorem 4.3	theorem 4.4	corollary 4.5
$\lvert\varepsilon(0)\rvert$	1.18326	0.602055	0.602055
$\lvert\varepsilon(1)\rvert$	1.18326	0.602055	0.602055
$\lvert\varepsilon(2)\rvert$	0.822923	0.418713	0.418713
$\lvert\varepsilon(3)\rvert$	0.592137	0.301287	0.301287
$\lvert\varepsilon(4)\rvert$	0.451249	0.229601	0.229601
$\lvert\varepsilon(5)\rvert$	0.362018	0.184199	0.184199
$\lvert\varepsilon(6)\rvert$	0.301809	0.153564	0.153564
$\lvert\varepsilon(7)\rvert$	0.258707	0.131634	0.131634

Table 4: General bounds from chapter 4 with $\sigma = 0.14$

Now the second bound is better than the first one and the third estimate is the minimum of the first and second one. So it depends on the parameters of the coupons, if the bound given in corollary 4.5 is just the minimum of the other two bounds or not. It also depends on the parameters of the coupons, if the bound given in theorem 4.3 is better than the bound given in theorem 4.4 or vice versa.

Going on with $\sigma = 0.18$, table 5 lists the trivial bounds for the approximation error, given by theorem 4.1.

	lower bound	upper bound
$\varepsilon(0)$	-0.69530044	0.30469956
$\varepsilon(1)$	-0.25267999	0.74732001
$\varepsilon(2)$	-0.04591337	0.95408663
$\varepsilon(3)$	-0.00556181	0.99443819
$\varepsilon(4)$	-0.00050531	0.99949469
$\varepsilon(5)$	-0.00003673	0.99996327
$\varepsilon(6)$	$-2.22449 \cdot 10^{-6}$	0.99999778
$\varepsilon(7)$	$-1.15486 \cdot 10^{-7}$	0.99999988

Table 5: Trivial bounds for the approximation errors

For all lower bounds except the first one, the trivial bounds are better than the ones given in table 3. Therefore it is reasonable to take the smallest values from the tables above.

Table 6 gives the best lower and upper bounds for the approximation errors of the point probabilities, using the best values of table 3 and table 5.

	lower bound	upper bound
$\varepsilon(0)$	-0.282254	0.282254
$\varepsilon(1)$	-0.25267999	0.282254
$\varepsilon(2)$	-0.04591337	0.165221
$\varepsilon(3)$	-0.00556181	0.112033
$\varepsilon(4)$	-0.00050531	0.0841505
$\varepsilon(5)$	-0.00003673	0.0673276
$\varepsilon(6)$	$-2.22449 \cdot 10^{-6}$	0.0561067
$\varepsilon(7)$	$-1.15486 \cdot 10^{-7}$	0.0480915

Table 6: Best bounds for the approximation errors

Some of these bounds can still be improved. The next step is to tighten the bounds for the approximation error $\varepsilon(0)$ by using theorem 4.10. In section 4.2.3 the functions $\tilde{f}_0$ and $\hat{f}_0$ are defined, which are used in the proof of this theorem.

Figure 1 and 2 show these functions, defined as in (4.13) and (4.15).

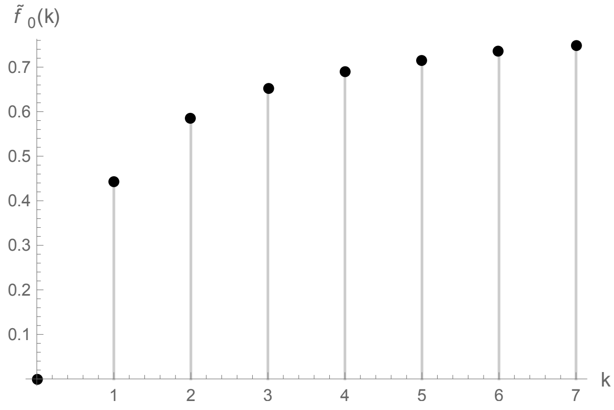

Figure 1: Function $\tilde{f}_0$ as defined in (4.13)

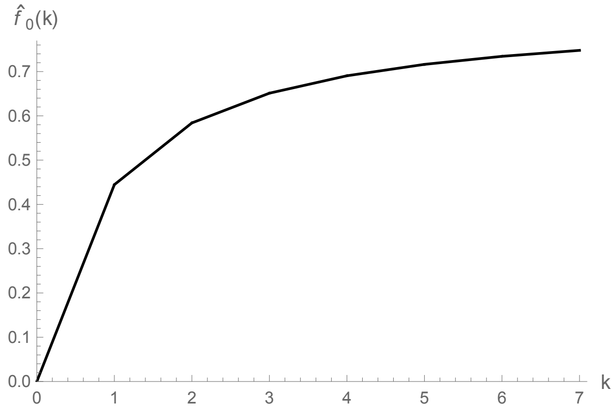

Figure 2: Function $\hat{f}_0$ as defined in (4.15)

It's easy to see that $\hat{f}$ is concave, as proved in lemma 4.9. Therefore Jensen's inequality can be applied in the proof of theorem 4.10. Using the general coupling from the previous section and (4.17) gives

$$|\mathbb{P}(W = 0) - \mathbb{P}(X = 0)| \leq 0.150333$$

as a bound for the approximation error of the point 0.

The last step for the approximation of the point probabilities is to calculate the exact value of $\mathbb{P}(W = 7)$ as described above. It is given by

$$\begin{aligned}\mathbb{P}(W = 7) &= \mathbb{E}\left(\prod_{i=1}^{7} C_i\right) \\ &= BD(100, (1), 7, 85, 115, 0.18, 0) \\ &= 2.37285 \cdot 10^{-6}.\end{aligned}$$

To include this in the setting above, just take

$$2.37285 \cdot 10^{-6} - \mathbb{P}(X = 7)$$

as the lower and upper bound for the approximation error of $\mathbb{P}(W = 7)$. Then

$$2.37285 \cdot 10^{-6} \leq \mathbb{P}(W = 7) \leq 2.37285 \cdot 10^{-6}$$

holds.

The final bounds for the approximation errors are given in table 7.

	lower bound	upper bound
$\varepsilon(0)$	-0.150333	0.150333
$\varepsilon(1)$	-0.25267999	0.282254
$\varepsilon(2)$	-0.04591337	0.165221
$\varepsilon(3)$	-0.00556181	0.112033
$\varepsilon(4)$	-0.00050531	0.0841505
$\varepsilon(5)$	-0.00003673	0.0673276
$\varepsilon(6)$	$-2.22449 \cdot 10^{-6}$	0.0561067
$\varepsilon(7)$	$2.25736 \cdot 10^{-6}$	$2.25736 \cdot 10^{-6}$

Table 7: Final bounds for the approximation errors

Figure 3 shows the point probabilities of a Poisson distributed random variable with parameter λ (black dots) and the lower respectively upper bounds for the point probabilities of W (gray dots).

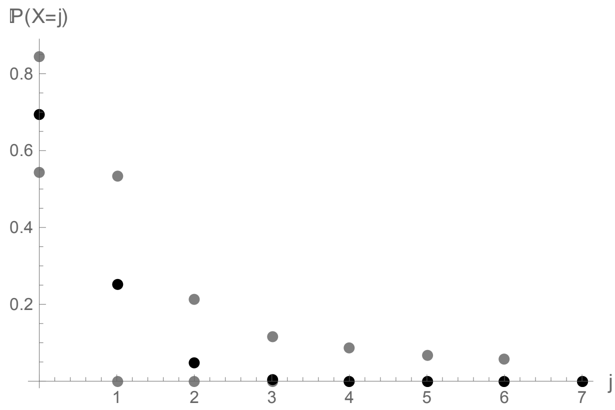

Figure 3: Point probabilities of X and bounds

This approximations of the point probabilities can now be used in theorem 5.1 to approximate SF. Table 8 gives the values of $SF_X(x)$ for $x \in \{0, \ldots, 7\}$ and bounds for the real price SF as defined in (6.1).

x	lower bound for $SF(x)$	$SF_X(x)$	upper bound for $SF(x)$
0	0.00	0.00	0.00
1	0.464391	0.592496	0.720601
2	0.928781	1.40031	1.57274
3	1.50336	2.24725	2.42488
4	2.17815	3.09893	3.27703
5	2.92507	3.95104	4.12917
6	3.7294	4.80318	4.98131
7	4.58154	5.65533	5.83345

Table 8: Approximated price and bounds for the real price

Note that SF_X and SF are continuous functions. Therefore table 7 only gives the values of the functions at a few points. Figure 4 shows the approximated price SF_X (black line) and the lower and upper bounds for the real price SF (gray lines).

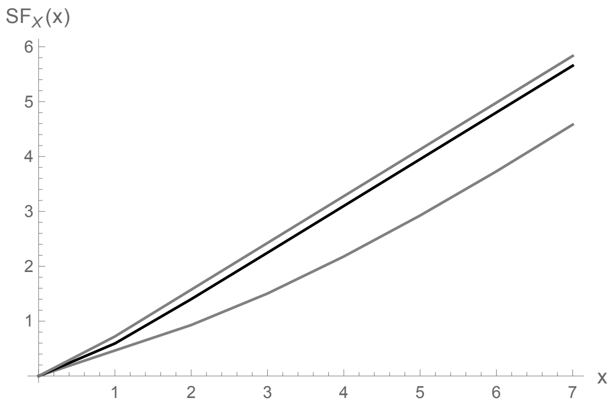

Figure 4: Approximated price SF_X and bounds for the real price SF

6.3 Positive relation

Theorem 3.7 states that the random variables $(C_i)_{i=1}^n$ are positively related, if

$$Cov(\phi(C_1, \dots, C_{k-1}, C_{k+1}, \dots, C_n), C_k) \geq 0, \quad \forall k \in \{1, \dots, n\}, \tag{6.5}$$

for every increasing function $\phi : \{0,1\}^{n-1} \to \{0,1\}$. Every increasing ϕ is clearly determined by a set of $(n-1)$-tuples

$$I := \{i = (i_1, ..., i_{n-1}) : \phi(i) = 1, \phi(l) = 0, \forall l < i\},$$

where $<$ is the natural partial order, given by (3.14). Then

$$J := \left\{j \in \{0,1\}^{n-1} : \exists i \in I \text{ with } i \leq j\right\}$$

is the index set of all points j with $\phi(j) = 1$. By definition of the partial order $<$,

$$\tilde{j} \geq j \Leftrightarrow \tilde{j}_i = 1, \forall i \in \{l : j_l = 1\}, \quad \forall j, \tilde{j} \in \{0,1\}^{n-1}$$

holds. Let $k \in \{1, \dots, n\}$ and $\Gamma_k := \{1, \dots, n\} \setminus \{k\}$. Then

$$(C_l, l \in \Gamma_k) \geq i \Leftrightarrow C_l = 1, \forall l \in L_{ki}, \quad \forall i \in \{0,1\}^{n-1}, \tag{6.6}$$

where L_{ki} is defined by

$$L_{ki} := \{l \in \Gamma_k : (l = j \text{ with } j < k \wedge i_j = 1) \vee (l = j + 1 \text{ with } j \geq k \wedge i_j = 1)\}.$$

The definition of L_{ki} takes into account that k is not in the index set of the coupon's payoffs. Now define an index set for all points of I using (6.6) by

$$\tilde{I}_k := \{\{j \in \Gamma_k : j \in L_{ki}\} : i \in I\}. \tag{6.7}$$

Then the left hand side of condition (6.5) is equivalent to

$$\begin{aligned}
&\mathbb{E}[\phi(C_i, i \in \Gamma_k)C_k] - \mathbb{E}[\phi(C_i, i \in \Gamma_k)]\mathbb{E}[C_k] \\
&\quad = \mathbb{P}(\phi(C_i, i \in \Gamma_k) = 1, C_k = 1) - \mathbb{P}(\phi(C_i, i \in \Gamma_k) = 1)\mathbb{P}(C_k = 1) \\
&\quad = \mathbb{P}\left(\bigcup_{j \in J_k} \{(C_i, i \in \Gamma_k) = j\} \cap \{C_k = 1\}\right) \\
&\qquad\qquad - \mathbb{P}\left(\bigcup_{j \in J_k} \{C_i, i \in \Gamma_k) = j\}\right)\mathbb{P}(C_k = 1) \qquad (6.8) \\
&\quad = \mathbb{P}\left(\bigcup_{I \in \tilde{I}_k}\bigcap_{i \in I}\{C_i = 1\} \cap \{C_k = 1\}\right) \\
&\qquad\qquad - \mathbb{P}\left(\bigcup_{I \in \tilde{I}_k}\bigcap_{i \in I}\{C_i = 1\}\right)\mathbb{P}(C_k = 1).
\end{aligned}$$

The right hand side can now be approximated by a Monte Carlo simulation. To approximate a probability $\mathbb{P}(C_i = 1)$, $i \in \Gamma_k$, a large number of paths of S_t (defined as in (2.6)) are determined. Every path that fulfills $C_i = 1$ is counted as a valid path.

Example of a valid path
Set $n = 5$, $k = 2$ and let ϕ be clearly determined by the set

$$\{(1, 1, 0, 0), (0, 0, 0, 1)\}.$$

Then $\tilde{I}_2$ as defined in (6.7) is given by

$$\tilde{I}_2 := \{\{1, 3\}, \{5\}\}.$$

Approximating

$$\begin{aligned}
\mathbb{E}[\phi(C_1, C_3, C_4, C_5)] &= \mathbb{P}\left(\bigcup_{I \in \tilde{I}_2}\bigcap_{i \in I}\{C_i = 1\}\right) \\
&= \mathbb{P}\left((\{C_1 = 1\} \cap \{C_3 = 1\}) \cup \{C_5 = 1\}\right),
\end{aligned}$$

a path is valid if C_1 and C_3 equal 1 or $C_5 = 1$. Figure 5 gives an example for a valid path.

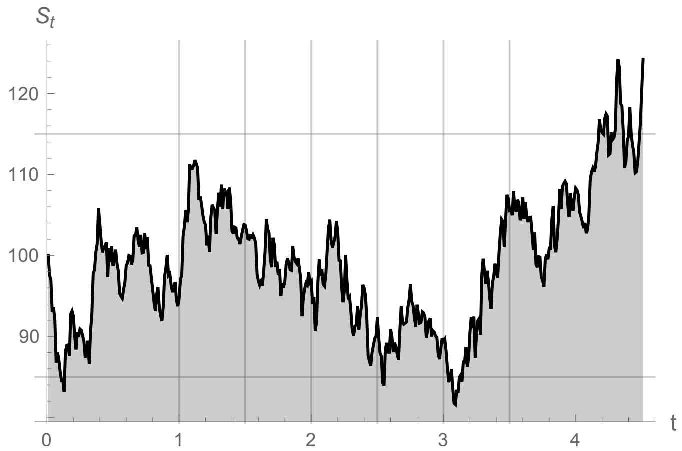

Figure 3: Example of a valid path

The number of valid paths divided by the number of all paths gives an approximation of the probability. Since the number of functions ϕ increases, as n increases, the computational effort is high for large n.

Let the indicator variables $(C_i)_{i=1}^n$ be defined through the parameters

$$\begin{array}{rr} S_0 = 100, & T_0 = 1, \\ P = 0.5, & B_{low} = 85, \\ B_{up} = 115, & \sigma = 0.2. \end{array} \tag{6.9}$$

The next table gives the results of the Monte Carlo simulations for n up to 5. For the computations a modification of the Mathematica function `BDMC` in [9] was used. The function `BDMC` approximates the function BD, as defined in theorem 2.16, by a Monte Carlo simulation. The code of the modified function can be found in chapter 7.

n	number of functions ϕ	number of functions ϕ fulfilling (6.5)
2	1	1
3	4	4
4	18	18
5	159	159

Table 9: Results of the Monte Carlo simulation

Since (6.5) is fulfilled for all increasing indicator functions ϕ, it can be assumed that the random variables $(C_i)_{i=1}^n$ with parameters given in (6.9) are positively related, for $n \in \{2, \ldots, 5\}$.

6.4 Several Examples

In this section three examples for the approximation of SF are given. For all examples set $r = 0.02$. The first example uses the assumption that the random variables $(C_i)_{i=1}^n$ are positively related. The second example compares bounds, if the assumption of positive relation is made or not. If n is large, the only bound that can be easily calculated is the one given in theorem 4.4. This is what the third example is addressed to.

Example 6.1 *Set $n = 5$ and let the parameters of the coupons be defined as in (6.9). Then*

$$\lambda = \sum_{n=1}^{5} \mathbb{E}[C_i] = 0.595488.$$

As shown in the previous section it is reasonable to assume that the random variables $(C_i)_{i=1}^5$ are positively related. Therefore theorem 4.6 can be applied. Table 10 gives the final bounds for the approximation errors of the point probabilities. Theorem 4.1, theorem 4.10 and the exact value for $\mathbb{P}(W = 5)$ were used as described in section 6.2.

	lower bound	*upper bound*
$\varepsilon(0)$	-0.171094	0.171094
$\varepsilon(1)$	-0.311458	0.311458
$\varepsilon(2)$	-0.097746	0.197521
$\varepsilon(3)$	-0.0194022	0.136926
$\varepsilon(4)$	-0.00288844	0.103262
$\varepsilon(5)$	0.00172288	0.00172288

Table 10: Final bounds for the approximation errors

Now theorem 5.1 can be used to approximate the price of the structure floor. The results are illustrated in figure 6. The black line represents the approximated price of the structure floor SF_X. The gray lines are the bounds for the real price.

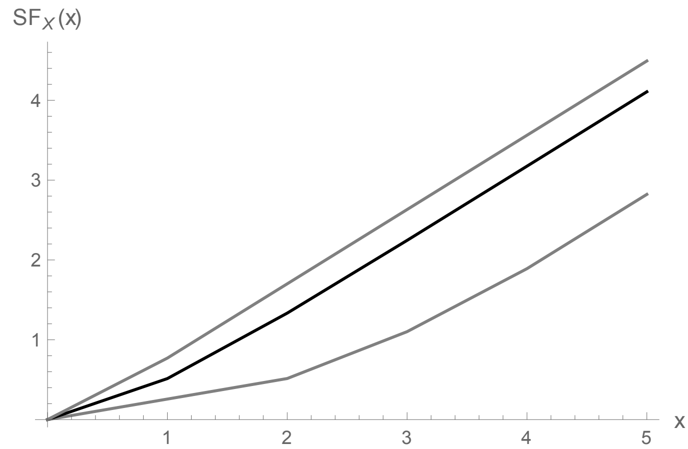

Figure 6: Approximated price and bounds for the real price with $n = 5$

Remark: Although the assumption of positive relation is made in this example, the bounds in section 6.2 are tighter than the ones here. This is because the parameters and the number of coupons are different.

The next example compares two approximations with the same parameters for the coupons. One approximation uses the assumption of positive relation, while the other one does not.

Example 6.2 *In this example set $n = 20$ and let the parameters for the coupons again be given by (6.9). If the assumption of positive relation is made, theorem 4.6 can be used instead of corollary 4.5. Figure 7 shows the upper bounds $(\varepsilon_+(k))_{k=0}^n$ for the approximation errors of the point probabilities, if theorem 4.6 is used (gray dots) and if corollary 4.5 is used (black dots).*

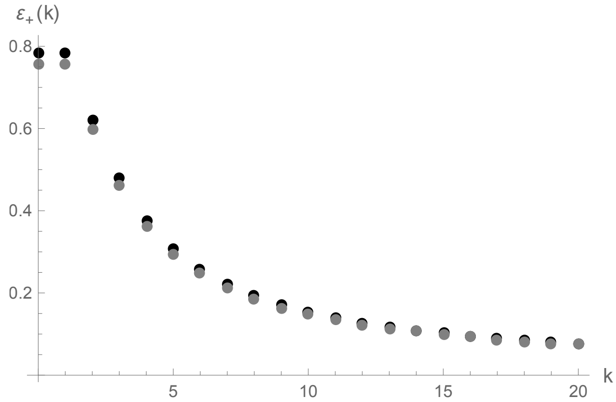

Figure 7: Upper bounds for the approximation errors with $n = 20$

In this example, the differences are pretty small. This is because n is quite large. For comparison, figure 8 shows the upper bounds for $n = 5$, as in example 6.1.

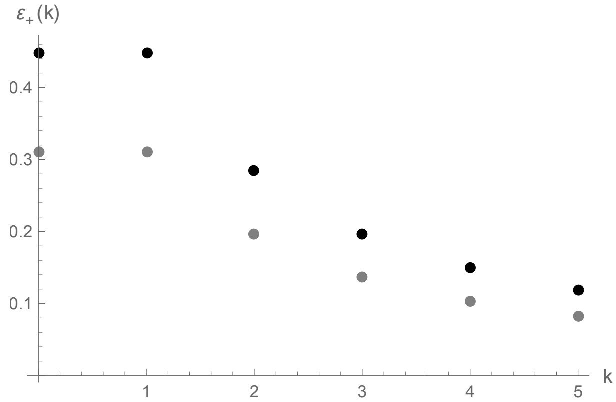

Figure 8: Upper bounds for the approximation errors with $n = 5$

For small n it makes a big difference, if the coupons are positively related or not.

Going on with $n = 20$, the lower bounds are the same in both cases. The best lower bounds are the trivial bounds given by theorem 4.1. By proceeding as described in section 6.2, the lower and upper bounds can be improved.

For the approximation of the structure floor's price, theorem 5.1 can be applied. Figure 9 and 10 show the approximated prices SF_X (black lines) and the bounds for the real prices SF (gray lines). Note that SF_X is the same in both cases, only the bounds vary.

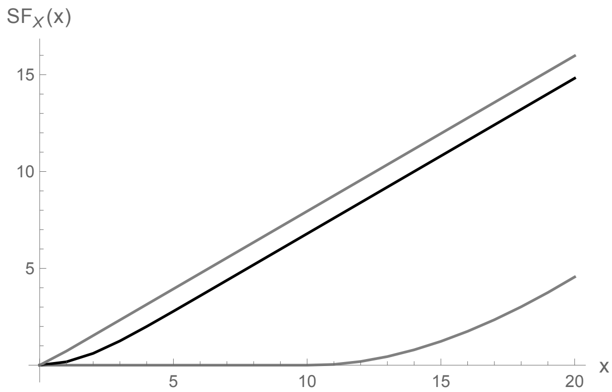

Figure 9: Approximation results without assumption of positive relation

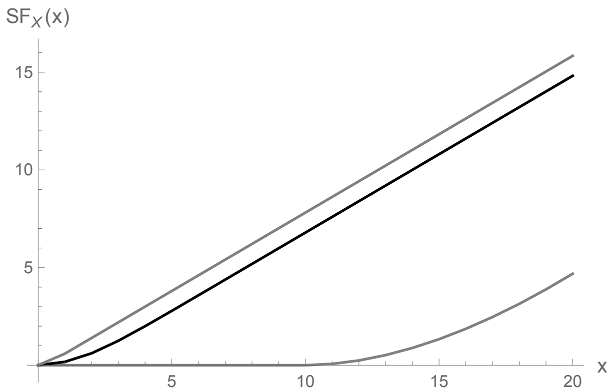

Figure 10: Approximation results with assumption of positive relation

It is easy to see that the approximations are almost the same. Note that the bounds in example 6.1, with n small, are much better.

In conclusion, if n is large, there's not a big difference, if the assumption of positive relation is made or not. If n is small, the difference is much bigger. From section 6.3 follows that it is reasonable to make this assumption, if n is small.

Since the computational effort for the calculation of the expected values $\mathbb{E}[|C_k - J_{ik}|]$ is very high as n grows, the bound given in corollary 4.5 can't be used for large n. Theorem 4.4 can be used instead. For increasing n, the number of summands in (4.4) increase. Therefore the larger n, the looser the bound given in theorem 4.3. It follows that the bound given in corollary 4.5 equals the bound given in theorem 4.4 for sufficiently large n.

The following example shows the results for large n.

Example 6.3 *Let now be $n = 60$. By using the bound given in theorem 4.4, only the expected values $\mathbb{E}[C_i]$, for all $i \in \{1, \dots, 60\}$ must be computed.*

Remark: The improvement for the point 0 is also not used here, as it would need further calculations with high computational efforts.

The trivial bounds given in theorem 4.1 and the exact point probability $\mathbb{P}(W = 60)$ can be used as described in section 6.2. Figure 11 shows the point probabilities of a Poisson distributed random variable with parameter

$$\lambda = \sum_{n=1}^{60} \mathbb{E}[C_i] = 2.901277,$$

as well as the lower and upper bounds for the point probabilities $\mathbb{P}(W = j)$, for all $j \in \{0, \dots, 60\}$.

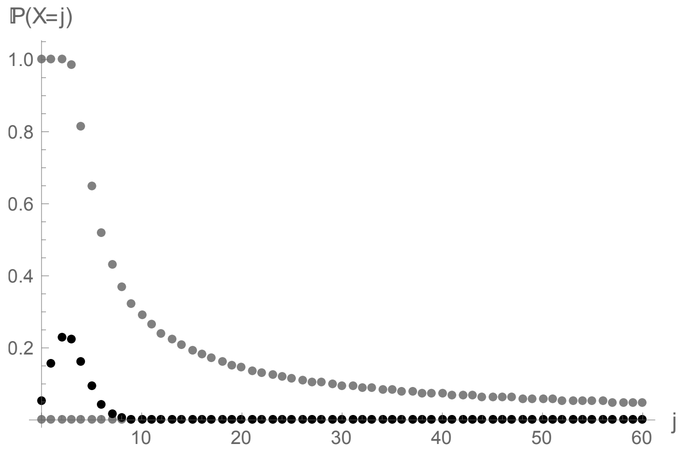

Figure 11: Point probabilities of X and bounds

The results after applying theorem 5.1 to approximate the price of the structure floor are illustrated in figure 12.

As in the previous examples, the approximated price SF_X is represented by the black line, while the upper resp. lower bounds of the real price SF are represented by the gray lines.

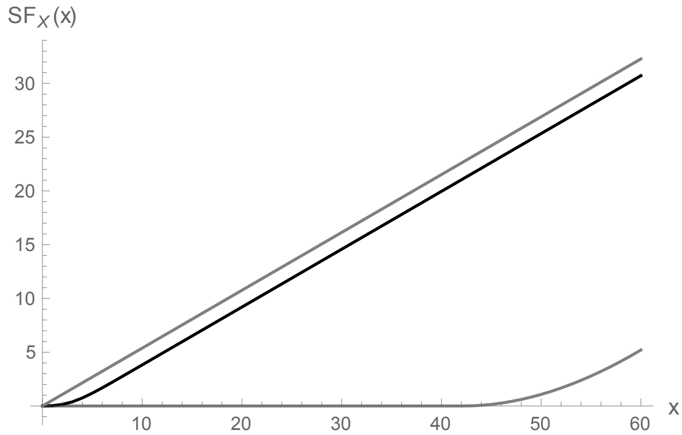

Figure 12: Approximated price and bounds for the real price, with $n = 60$

Chapter 7

Implementation in Mathematica

7.1 Functions f_j, $\tilde{f}_j$ and $\hat{f}_j$

The function `fj` computes the values of f_j defined as in (4.1). The first input parameter λ is the sum of the expectations of the coupons $(C_i)_{i=1}^n$ as described in section 4.2. The second parameter `j` is the point of which the point probability should be approximated. The third and last parameter `k` is the point at which the function f_j is evaluated. The output is $f_j(k)$ with parameter λ.

```
fj[λ_, j_, k_] := If[k == 0, 0,
   (k - 1)! / λ^k  λ^j / j! (Sum[λ^i / i! Exp[-λ], {i, 0, k - 1}] - Boole[j ≤ (k - 1)])];
```

The function $\tilde{f}_j$ defined by (4.13) is evaluated by the function `ftilj`. The input parameters λ, `j` and `k` are the same as for the function `fj`. The parameter `n` is the number of coupons. Its output is $\tilde{f}_j(k)$ with parameter λ.

```
ftilj[λ_, j_, k_, n_] := If[k == 0, 0,
   Max[Table[Abs[fj[λ, j, i] - fj[λ, j, i + k]], {i, 1, n - k + 1}]]];
```

The values of the third function $\hat{f}_j$ defined as in (4.15), are computed by the function `fhatj`. The point at which the function should be evaluated is given by the input parameter `x`. In contrast to function f_j and $\tilde{f}_j$, $\hat{f}_j$ is continuous. The other parameters λ, `j` and `n` are the same as for function `ftilj`. The output is $\hat{f}_j(x)$ with parameter λ.

```
fhatj[λ_, j_, x_, n_] := (x - Floor[x]) ftilj[λ, j, Ceiling[x], n] +
   (1 - (x - Floor[x])) ftilj[λ, j, Floor[x], n];
```

7.2 Error bounds for the approximation of point probabilities

The following functions compute the bounds given in chapter 4. They basically all use the same input parameters. The parameter `j` is the point for which the approximation error bounds should be calculated, `n` is the number of coupons and `ECiCk` is a two dimensional list with the expected values $\mathbb{E}[C_i C_k]$, for all $i, k \in \{1, \ldots n\}$, of the coupon's payoffs $(C_i)_{i=1}^n$. The diagonal elements of this list are the elements of the list `ECi`. The input parameter λ is the sum of the elements in `ECi`. The last parameter that is used is `ECkmJik`. It is a two dimensional list with the expected values $\mathbb{E}[C_k - J_{ik}]$, for all $i, k \in \{1, \ldots n\}$, where the random variables J_{ik} are defined as in (3.5). Since the expected values for $i = k$ are not defined (and not used), they are set to 0 in `ECkmJik`.

The output of the function `TrivialBounds` is a list with the trivial bounds given by theorem 4.1 as elements.

```
TrivialBounds[λ_, j_] := {PDF[PoissonDistribution[λ], j],
   1 - PDF[PoissonDistribution[λ], j]};
```

The function `FirstGeneralBound` computes the bound given in theorem 4.3.

```
FirstGeneralBound[λ_, j_, ECi_, ECkmJik_, n_] :=
  (fj[λ, j, j] - fj[λ, j, j + 1])
   Sum[ECi[[i]]^2 + Sum[ECi[[i]] ECkmJik[[i]][[k]],
      {k, Delete[Table[m, {m, 1, n}], i]}], {i, 1, n}];
```

The second bound in chapter 4, given in theorem 4.4, is evaluated by the function `SecondGeneralBound`.

```
SecondGeneralBound[λ_, j_] := λ (fj[λ, j, j] - fj[λ, j, j + 1]);
```

The function `GeneralBound` has the bound given in corollary 4.5 as its output.

```
GeneralBound[λ_, j_, ECi_, ECkmJik_, n_] :=
  (fj[λ, j, j] - fj[λ, j, j + 1]) Sum[Min[ECi[[i]],
     ECi[[i]]^2 + Sum[ECi[[i]] ECkmJik[[i]][[k]],
       {k, Delete[Table[m, {m, 1, n}], i]}]], {i, 1, n}];
```

The following two functions, PosRelBound and NegRelBound, compute the bounds given in theorem 4.6 and theorem 4.7, if the coupon's payoffs are positively or negatively related.

```
PosRelBound[λ_, j_, ECi_, ECiCk_, n_] :=
  (fj[λ, j, j] - fj[λ, j, j + 1]) Sum[Min[ECi[[i]],
     ECi[[i]]^2 + Sum[ECiCk[[i]][[k]] - ECi[[i]] ECi[[k]],
       {k, Delete[Table[m, {m, 1, n}], i]}]], {i, 1, n}];
NegRelBound[λ_, j_, ECi_, ECiCk_, n_] :=
  (fj[λ, j, j] - fj[λ, j, j + 1]) Sum[Min[ECi[[i]],
     ECi[[i]]^2 - Sum[ECiCk[[i]][[k]] - ECi[[i]] ECi[[k]],
       {k, Delete[Table[m, {m, 1, n}], i]}]], {i, 1, n}];
```

The bounds for the point 0 are calculated by the next three functions. The first one, ZeroBound, has the bound given in theorem 4.10 as its output. The second and third function, NegRelZeroBound and PosRelZeroBound, compute bounds, if the coupon's payoffs are negatively resp. positively related.

```
ZeroBound[λ_, ECi_, ECkmJik_, n_] := Sum[ECi[[i]] fhat[λ, 0,
     (ECi[[i]] + Sum[ECkmJik[[i]][[k]],
         {k, Delete[Table[m, {m, 1, n}], i]}])[[1]], n], {i, 1, n}];

PosRelZeroBound[λ_, ECi_, ECiCk_, n_] := Sum[ECi[[i]] fhat[λ, 0,
     (ECi[[i]] + Sum[(ECiCk[[i]][[k]] - ECi[[i]] ECi[[k]]) / ECi[[i]],
         {k, Delete[Table[m, {m, 1, n}], i]}])[[1]], n], {i, 1, n}];
```

```
NegRelZeroBound[λ_, ECi_, ECiCk_, n_] := Sum[ECi[[i]] fhat[λ, 0,
    (ECi[[i]] - Sum[(ECiCk[[i]][[k]] - ECi[[i]] ECi[[k]]) / ECi[[i]],
        {k, Delete[Table[m, {m, 1, n}], i]}])[[1]], n], {i, 1, n}];
```

7.3 Bounds for the price of a structure floor

The function `PriceBounds` calculates the error terms in (5.1). According to theorem 5.1, j is chosen for the lower and the upper bound before the sequences $(\hat{\varepsilon}_-(k))_{k=0}^n$ and $(\hat{\varepsilon}_+(k))_{k=0}^n$ are defined. The input parameters are the values of f as a list `f`, the sum of the point probabilities

$$\sum_{k=0}^{n} \mathbb{P}(X = k)$$

of the arbitrary random variable `sopp`, the lower and upper error bounds for the point probabilities as a two dimensional list ϵ and `n`, the number of coupons. The output is a list with the lower error term as the first and the upper error term as the second element.

```
PriceBounds[f_, sopp_, ϵ_, n_] :=
 Module[{j, ϵhatminus = Table[0, {n + 1}],
   ϵhatplus = Table[0, {n + 1}]},
  Do[If[ϵ[[1]][[jt + 1]] ≤
      (1 - sopp - Sum[ϵ[[1]][[i + 1]], {i, 0, jt - 1}] -
        Sum[ϵ[[2]][[i + 1]], {i, jt + 1, n}]) ≤ ϵ[[2]][[jt + 1]], j = jt],
   {jt, 0, n}] ;
  Do[ϵhatminus[[l + 1]] = If[l < j, ϵ[[1]][[l + 1]],
     If[l > j, ϵ[[2]][[l + 1]],
      1 - sopp - Sum[ϵ[[1]][[i + 1]], {i, 0, j - 1}] -
        Sum[ϵ[[2]][[i + 1]], {i, j + 1, n}] 1]], {l, 0, n}];
  Do[If[ϵ[[1]][[jt + 1]] ≤
      (1 - sopp - Sum[ϵ[[2]][[i + 1]], {i, 0, jt - 1}] -
        Sum[ϵ[[1]][[i + 1]], {i, jt + 1, n}]) ≤ ϵ[[2]][[jt + 1]], j = jt],
   {jt, 0, n}] ;
```

```
Do[ϵhatplus[[l + 1]] = If[l < j, ϵ[[2]][[l + 1]],
      If[l > j, ϵ[[1]][[l + 1]],
       1 - sopp - Sum[ϵ[[2]][[i + 1]], {i, 0, j - 1}] -
        Sum[ϵ[[1]][[i + 1]], {i, j + 1, n}]]], {l, 0, n}];
 Sum[f[[i + 1]] {ϵhatminus[[i + 1]], ϵhatplus[[i + 1]]}, {i, 0, n}]]
```

7.4 Approximation of the price for a structure floor

The function `PriceApproximation` uses the functions from the previous sections and `BDMult` from [9] to compute a lower and upper bound for the price of the strucure floor. It proceeds as described in section 6.2.

The input parameters are the number of coupons `n` and the parameters of the coupons `S0`, `T0`, `P`, `Blow`, `Bup`, σ as well as the interest rate `r`. Furthermore is `x` the level of the structure floor and `posRel` a Boolean value, which should be set to `true` if the assumption of positive relation is made. Otherwise it should be set to `false`. In the latter case, the general coupling from section 6.1 is used for the calculations. The input parameter `lim` is used by the function `BDMult`. For a detailed description of `lim` see [9].

The output is a list with three elements. The first resp. third element is the lower resp. upper bound for the exact price SF. The second element is the approximated price SF_X.

```
PriceApproximation[S0_, T0_, P_, Blow_, Bup_, σ_, r_, n_, x_,
  posRel_, lim_] :=
 Module[{secMomTmp, ECiCk, ECi, λ, gBound, ECkmJik, zBound,
   lowerBounds, upperBounds, ppn, fval, sopp, SFX},
  secMomTmp = Table[Table[
     If[j ≤ i, If[j == i, BDMult[S0, 0, {T0 + i * P}, P, Blow, Bup,
        0, σ, lim], BDMult[S0, 0, {T0 + j * P, T0 + i * P}, P,
        Blow, Bup, 0, σ, lim]], 0], {j, 0, n - 1}], {i, 0, n - 1}];
  ECiCk = Table[Table[secMomTmp[[Max[i, j]]][[Min[i, j]]], {j, 1, n}],
    {i, 1, n}];
  ECi = Table[ECiCk[[i]][[i]], {i, 1, n}];
  λ = Sum[ECi[[i]], {i, 1, n}];
```

```
If[posRel,
  gBound = Table[PosRelBound[λ, j, ECi, ECiCk, n] , {j, 0, n}];
  zBound = PosRelZeroBound[λ, ECi, ECiCk, n][[1]],
  ECkmJik =
   Table[
    Table[If[i == k, 0, ECi[[k]] + (ECiCk[[i]][[k]] / ECi[[i]]) (1 - 2 ECi[[k]])],
     {k, 1, n}], {i, 1, n}];
  gBound = Table[GeneralBound[λ, j, ECi, ECkmJik, n] , {j, 0, n}];
  zBound = ZeroBound[λ, ECi, ECkmJik, n][[1]]];
 lowerBounds =
  Table[-Min[TrivialBounds[λ[[1]], j][[1]], gBound[[j + 1]]],
   {j, 0, n}];
 upperBounds =
  Table[Min[TrivialBounds[λ[[1]], j][[2]], gBound[[j + 1]]],
   {j, 0, n}];
 lowerBounds[[1]] = Max[lowerBounds[[1]], -zBound];
 upperBounds[[1]] = Min[upperBounds[[1]], zBound];
 ppn = BDMult[S0, 0, {T0}, n * P, Blow, Bup, 0, σ, lim];
 lowerBounds[[n + 1]] =
  (ppn - PDF[PoissonDistribution[λ[[1]]], n])[[1]];
 upperBounds[[n + 1]] = (ppn - PDF[PoissonDistribution[λ[[1]]], n])[[
   1]];
 fval = Table[If[i ≥ x, 0, x - i], {i, 0, n}];
 sopp = Sum[PDF[PoissonDistribution[λ[[1]]], j], {j, 0, n}];
 SFX = Sum[fval[[j + 1]] PDF[PoissonDistribution[λ[[1]]], j],
   {j, 0, n}];
 Exp[-r * (T0 + n * P)]
  {SFX + PriceBounds[fval, sopp, {lowerBounds, upperBounds}, n][[1]],
   SFX, SFX + PriceBounds[fval, sopp, {lowerBounds, upperBounds}, n]
     [[2]]}]
```

7.5 Monte Carlo simulation

This section contains four functions, GBMPathCompiled is from [9]. The functions ECkMC and PathTest are modified versions of the functions BDMultMC and PartialBarrierTest in [9].

ECkMC approximates the probabilities in (6.8). If the parameter k is set to 0, the second probability on the right hand side is calculated. Otherwise the function computes the first probability, with k as in the input parameter k. The other parameters are the parameters of the coupons S0, T0, P, Blow, Bup and σ, the number of paths that should be evaluated pathNum, the number of points that are evaluated per path dt and fct. The last parameter is a list with the elements of $\tilde{I}_k$, defined as in (6.7).

The function pathTest is used by ECkMC to evaluate, if a path is counted as a valid path or not. The output is a Boolean value with true for a valid path and false otherwise.

The fourth function PosRelTest uses the three functions above to test, whether a sequence of payoffs of coupons is positively related or not. To do that, it generates all possible functions ϕ by generating sets as defined in (6.7). The input parameters are again the number of coupons n, the parameters of the coupons S0, T0, P, Blow, Bup, σ the number of paths that should be evaluated pathNum, the number of points that are evaluated per path dt and the expected values of the coupon's payoffs as a list ECi. The output is a list with the number of functions ϕ as the first element and the number of this functions that fulfill condition (6.5) as the second element.

```
GBMPathCompiled =
  Compile[{{S0, _Real}, {drift, _Real}, {diff, _Real},
    {nSteps, _Integer}}, FoldList[(#1 drift Exp[diff #2]) &,
    S0, RandomVariate[NormalDistribution[0, 1], nSteps]]];

ECkMC[S0_, T0_, P_, Blow_, Bup_, σ_, n_, pathNum_, dt_, fct_, k_] :=
 Module[{T, drift, diff, paths, τ, remainingPaths, value, j},
  T = T0 + n * P;
  {drift, diff} = {Exp[(-σ^2 / 2) dt], diff = σ Sqrt[dt]};
  paths = Table[GBMPathCompiled[S0, drift, diff, (T / dt)], {pathNum}];
```

```
τ = Table[{Floor[(T0 + (i - 1) * P) / dt], Floor[(T0 + i * P) / dt]},
    {i, 1, n}];
  remainingPaths = Select[paths, PathTest[#, τ, Blow, Bup, fct, k] &];
  value = N[Length[remainingPaths] / pathNum]];

PathTest[path_, τ_, L_, U_, fct_, k_] :=
  Module[{pathIndex = False, index, min, max, tmpFct},
   Do[
    index = True;
    If[k > 0, tmpFct = Append[fct[[j]], k], tmpFct = fct[[j]]];
    Do[
     min =
      Min[path[[τ[[tmpFct[[i]]]][[1]] ;; τ[[tmpFct[[i]]]][[2]]]]];
     max =
      Max[path[[τ[[tmpFct[[i]]]][[1]] ;; τ[[tmpFct[[i]]]][[2]]]]];
     If[Or[min < L, max > U], index = False];
     , {i, 1, Length[tmpFct]}];
    If[index, pathIndex = True];
    , {j, 1, Length[fct]}];
   pathIndex
  ];

PosRelTest[S0_, T0_, P_, Blow_, Bup_, σ_, ECi_, n_, pathNum_, dt_] :=
 Module[{index, points, permutations, functions, results, ϕ,
   functionResults},
  index = Table[i, {i, 1, n - 1}];
  points = Drop[DeleteDuplicates[Permutations[index, n - 1],
     Union[#1, #2] == #1 &], 1];
  permutations =
   Drop[DeleteDuplicates[Permutations[points, n - 1],
     Union[#1, #2] == #1 &], 1];
```

```
functions =
  Table[If[i > n - 1, DeleteDuplicates[permutations[[i]],
     Intersection[#1, #2] == #1 &], permutations[[i]]],
   {i, 1, Length[permutations]}];
 functions = DeleteDuplicates[functions];
 results = Table[Table[0, {i, 1, Length[functions]}], {k, 1, n}];
 Do[
  ϕ = functions + Ceiling[Floor[functions/k], n - 1]/(n - 1);
  results[[k]] =
   Table[
    Boole[
     (ECkMC[S0, T0, P, Blow, Bup, σ, n, pathNum, dt, ϕ[[i]], k] -
         ECkMC[S0, T0, P, Blow, Bup, σ, n, pathNum, dt, ϕ[[i]], 0]
          ECi[[k]])[[1]] ≥ 0], {i, 1, Length[ϕ]}];
  , {k, 1, n}];
 functionResults = Sum[results[[k]], {k, 1, n}];
 {Length[functions], Sum[Boole[functionResults[[ϕ]] == n],
  {ϕ, 1, Length[functions]}]}]
```

Appendix A

Appendix

A.1 Corrected function BDMult

The corrected function BDMult from [9] for the calculation of BD as described in theorem 2.16 is given here. Originally the indicator function in (2.9) was left out and the integration boundaries of y in (2.8) were changed.

Since theorem 2.16 uses only parts of this code, the parts that are not used are left out. Also the auxiliary functions, which weren't changed (namely τj, g0, gj) are omitted. They can be found in the appendix of [9].

The following two functions h0 and hj are the modified auxiliary functions, which are used by BDMult.

```
hj[kj__, xj__, yj__, x_, τ_, τj_, p_, j_, α_, L_, n_] :=
  If[j == 0, h0[kj, xj, yj, x, τ, τj, p, j, n, α, L],
   hj[kj, xj, yj, x, τ, τj, p, j, α, L, n] =
    Module[{yjp1 = yj⟦j + 1⟧, τjnmj = τj⟦n - j⟧},
     1/Sqrt[2 π] Exp[-yjp1^2/2] *
      Boole[-x ≤ yjp1 Sqrt[2 (τ - (τjnmj + p))] ≤ L - x] *
      gj[kj, xj, Most[yj], x + yjp1 Sqrt[2 (τ - (τjnmj + p))],
       τjnmj + p, τj, p, j, α, L, n]]];
```

```
h0[k1_, x1_, y1_, x_, τ_, τj_, p_, 0, n_, α_, L_] :=
  1/Sqrt[2 π] Exp[-y1^2/2] *
   Boole[-x ≤ y1[[1]] Sqrt[2 (τ - (τj[[n - 0]] + p))] ≤ L - x] *
   g0[k1, x1, 0, x + y1 Sqrt[2 (τ - (τj[[n - 0]] + p))], τj[[n - 0]] + p,
    τ, p, 0, α, L];
```

Next, the modified function BDMult is given.

```
BDMult[S_, t_, Ti_, P_, Blow_, Bup_, r_, σ_, lim_] :=
 Module[{n, x, τ, α, β, p, L, tj, gxvars, gxrange, gyvars,
   gyrange, gkvars, gkrange, hxvars, hxrange, hyvars, hyrange,
   hkvars, hkrange, expression0, expression1, j, value, i, k},
  n = Length[Ti];
  x = N[Log[S/Blow]]; τ = 1./2. σ^2 (Ti[[n]] + P - t);
  α = -1./2. (2. r/σ^2 - 1); β = -2. r/σ^2 - α^2; p = σ^2 P/2.;
  L = N[Log[Bup/Blow]]; tj = 1. / 2. * σ^2 * τj[Ti];
  If[Length[Pick[#, tj[[n - (# - 1)]] + p < τ < If[# == n, Infinity,
             tj[[n - (#)]]] & /@ #] &[Range[Length[tj]]] - 1] != 0,
   j = Pick[#, tj[[n - (# - 1)]] + p < τ < If[# == n, Infinity,
            tj[[n - (#)]]] & /@ #] &[Range[Length[tj]]] - 1,
   j = Pick[#, tj[[n - (#)]] <= τ ≤ tj[[n - (#)]] + p & /@ #] &[
     Range[0, Length[tj] - 1]]];
  j = j[[1]];
  hxvars = Table[Symbol["x" <> ToString[i]], {i, 1, j + 1}];
  hxrange = Table[{hxvars[[i]], 0, L}, {i, 1, j + 1}];
  hyvars = Table[Symbol["y" <> ToString[i]], {i, 1, j + 1}];
```

```
hyrange =
  Append[Table[If[(tj〚n - k〛 - (tj〚n - k + 1〛 + p)) == 0,
     {hyvars[[k]], -∞, ∞},
     {hyvars[[k]], - L / Sqrt[2 (tj〚n - k〛 - (tj〚n - k + 1〛 + p))],
      L / Sqrt[2 (tj〚n - k〛 - (tj〚n - k + 1〛 + p))]}], {k, 1, j}],
   {hyvars[[j + 1]], - x / (σ Sqrt[Ti[[n - j]]]), (L - x) / (σ Sqrt[Ti[[n - j]]])}];
 hkvars = Table[Symbol["k" <> ToString[i]], {i, 1, j + 1}];
 hkrange = Table[{hkvars[[i]], 0, lim}, {i, 1, j + 1}];
 If[tj[[n - j]] ≤ τ && tj[[n - j]] + p >= τ,
  If[j == 0,
   expression0 =
    Sum[gj[gkvars, gxvars, gyvars, x, τ, tj, p, j, α, L, n],
     Evaluate[Sequence @@ gkrange]];
   expression1 = Integrate[expression0, Sequence @@ gxrange];
   value = Re[Exp[α * x + β * τ] * expression1],
   expression0 =
    Sum[gj[gkvars, gxvars, gyvars, x, τ, tj, p, j, α, L, n],
     Evaluate[Sequence @@ gkrange]];
   expression1 = Integrate[expression0, Sequence @@ gxrange];
   value = Re[Exp[α * x + β * τ] * NIntegrate[expression1,
       Evaluate[Sequence @@ gyrange],
       Method → {Automatic, "SymbolicProcessing" → 0}]]],
  expression0 =
   Sum[hj[hkvars, hxvars, hyvars, x, τ, tj, p, j, α, L, n],
    Evaluate[Sequence @@ hkrange]];
  expression1 = Integrate[expression0,
    Evaluate[Sequence @@ hxrange]];
  value = Re[Exp[α * x + β * τ] * NIntegrate[expression1,
      Evaluate[Sequence @@ hyrange],
      Method → {Automatic, "SymbolicProcessing" → 0}]]]]
```

References

[1] S. Altay, S. Gerhold, R. Haidinger, and K. Hirhager. Digital double barrier options: Several barrier periods and structure floors. *International Journal of Theoretical and Applied Finance*, 16(08):1350044, 2013.

[2] A.D. Barbour and L.H.Y. Chen. *An introduction to Stein's method*, volume 4. World Scientific, 2005.

[3] A.D. Barbour and G.K. Eagleson. Poisson approximation for some statistics based on exchangeable trials. *Advances in applied probability*, 15(03):585–600, 1983.

[4] A.D. Barbour and L. Holst. Some applications of the Stein-Chen method for proving Poisson convergence. *Advances in Applied Probability*, 21(01):74–90, 1989.

[5] A.D. Barbour, L. Holst, and S. Janson. *Poisson approximation*. Clarendon Press Oxford, 1992.

[6] F. Black and M. Scholes. The pricing of options and corporate liabilities. *The journal of political economy*, 81(03):637–654, 1973.

[7] Z. Brzezniak and T. Zastawniak. *Basic stochastic processes: a course through exercises*. Springer Science & Business Media, 1999.

[8] L.H.Y. Chen. Poisson approximation for dependent trials. *The Annals of Probability*, 03(03):534–545, 1975.

[9] R. Haidinger. Barrier options and their application to structure floors. Master's thesis, Vienna University of Technology, 2014.

[10] S. Janson. Coupling and Poisson approximation. *Acta Applicandae Mathematica*, 34(1-2):7–15, 1994.

[11] L. Le Cam et al. An approximation theorem for the Poisson binomial distribution. *Pacific J. Math*, 10(4):1181–1197, 1960.

[12] M. Musiela and M. Rutkowski. *Martingale methods in financial modelling*, volume 36. Springer Science & Business Media, 2006.

[13] C. Stein. Approximate computation of expectations. *Lecture Notes-Monograph Series*, 7:i–164, 1986.

[14] V. Strassen. The existence of probability measures with given marginals. *The Annals of Mathematical Statistics*, 36:423–439, 1965.