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Institut für Höhere Studien
Institute for Advanced Studies
Vienna

TU Wien
Continuing Education Center

MSc Economics

"Optimal Strategies for Apartment Hunting"

A Master's Thesis submitted for the degree of
"Master of Science"

supervised by

Martin Meier

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0426322

Vienna, 14 June 2011

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Affidavit

I, Stefan Schneeberger

hereby declare

that I am the sole author of the present Master's Thesis,

"Optimal Strategies for Apartment Hunting"

60 pages, bound, and that I have not used any source or tool other than those referenced or any other illicit aid or tool, and that I have not prior to this date submitted this Master's Thesis as an examination paper in any form in Austria or abroad.

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Abstract

I study the problem of a decision maker with rank-based preferences who wants to rent an apartment. The objective is either to maximize the probability of chosing the best apartment (best choice problem) or to minimize the expected rank of the chosen apartment (expected rank problem). Information can be obtained by sequentially inspecting the available options. A renting decision has to be made immediately after inspection, and there is no recall. Options are modeled by an i.i.d. sequence $X_1, \dots, X_n$ with $X_i \sim \mathcal{U}[0, 1]$, where n is known. There are two informational regimes: either, the decision maker knows the distribution of the X_i and observes them directly (full information); or she does not know their distribition and can only observe the relative ranks of the X_i (no information).

I study an extension where the number of options n is random, with a uniform distibution on $\{1, \dots, K\}$, where K is known. I also consider an extension where the landlord refuses to rent an inspected apartment with a certain probability.

The expected-rank problem under full information is an open problem and is known as Robbin's problem. The solution can be approximated (see Bruss (2005)) by the optimal stratgy for maximizing $\mathrm{E}[X_N]$, where X_N denotes the chosen option. Maximizing $\mathrm{E}[X_N]$ was studied by Moser (1956).

For the best-choice problem under no information the solution is a threshold strategy where the first $k-1$ apartments are inspected but rejected, and the first relatively best apartment thereafter is selected. Asymptotically, $k = \frac{n}{e}$ and the success probablility equals $\frac{1}{e} \approx 0.37$. For the best-choice problem under full information the asymptotic success probability improves to ≈ 0.58, where the optimal strategy is to stop with the first relatively best apartment X_i that lies above a certain threshold d_i that only depends on the number of remaining options $n-i$. These results are well known (see Gilbert and Mosteller (1966)).

For the expected-rank problem under no information the solution is to stop with the first option whose relative rank r_i lies below a certain threshold rank s_i that depends on n and i. Asymptotically, the expected rank equals ≈ 3.87. This finding is due to Chow *et. al* (1964). The approximative solution for the expected-rank problem under full information is to stop with the first option X_i exceeding a certain threshold s_i that only depends on the number of remaining options $n-i$. I use simulations to show that the resulting expected rank is in the order of 2.

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1 A Motivating Example

Suppose you want to rent an apartment. There are three apartments available among which you can choose. Your objective is to choose the apartment that you like best. A priori you know nothing about the apartments. To obtain information, you can sequentially inspect the apartments in whatever order you like. Inspecting an apartment only allows you to rank it relatively to the other options inspected so far, i.e., the only information obtained from inspection are the relative ranks of the inspected options. After ranking you have to decide whether you want to rent the apartment or not. There is no possibility of recall, i.e., if you decide to reject the apartment there is no option of renting it at a later time, after inspecting other options.

What strategy will maximize the probability that you rent the best among the three availabe apartments? One strategy would be not to inspect any apartment and simply choose one of them. This strategy ensures you a probability of $\frac{1}{3}$ that you rent the best apartment. If you want to have full information on all apartments you have to inspect all of them. In this case you also have a probability of $\frac{1}{3}$ of ending up with the best option. This is because you do not rent the first two apartments that you inspect, and only with probability $\frac{1}{3}$ will the third and last option be the one that you like best. So both no and complete inspection of the available options will guarantee you a success probability of $\frac{1}{3}$. But can you do better? And the answer is yes! Consider the following strategy:

1. You select one of the apartments and inspect it, but you do not rent it. You use this apartment as benchmark.

2. You then inspect a second apartment and take it if you like it better than the benchmark apartment. Otherwise you rent the remaining third apartment.

What success probability does this strategy ensure you? With probability $\frac{1}{3}$ you pick the second-best apartment as your benchmark, in which case you will for sure rent the best option. Additionally, with probability $\frac{1}{6}$ you chose the third-best option as your benchmark and chose the best option as second apartment that you inspect, in which case you also rent the best one. In total, this strategy ensures you a probability of $\frac{1}{2}$ of

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renting the apartment that you like best. And, as can be shown, this strategy is optimal in the sense that it provides the highest success probability among all strategies that only use the relative ranks of the inspected apartments as available information (see Remark 10).

2 Introduction

This thesis is concerned with problems of the type described in Section 1. I study the problem of a decision maker who wants to rent an apartment and can obtain information by sequentially inspecting the available options. After inspecting an apartment the decision maker immediately has to decide whether she accepts or rejects an option, without the possibility of recall. The decision maker has rank-based preferences, i.e., she derives utility only from the ranking of the chosen apartment relative to the other available options. Her objective is either to maximize the probability of chosing the best apartment, or to minimize the expected rank of the chosen apartment, where the best apartment gets assigned rank 1. The informational regime is either such that she can only rank the inspected apartments, or that she can express her evaluation of an option by a subjective quality level and that she knows the probability distribution of these levels across the population from which the options are drawn. The first regime is referred to as no-information case, and the second as full-information case.

In the basic model the number of available apartments is fixed and ex-ante known to the decision maker. In an extension I relax this assumption and study a model where the number of options is random with a uniform distibution over a known range. An interesting and counterintuitive observation in this setting is the following: if the maximal number of options is small ($1 < n < 6$), then the uncertainty about the number of options leads to an increased success probability. In another extension I consider the possibility that a landlord might refuse to rent an inspected apartment, with an ex-ante known probability.

The above described apartment hunting problems can be formulated as optimal stopping problems. Such problems arise frequently in economics, e.g., in search theory, option pricing or decision theory, and there is a large literature on them (see, e.g.,

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Ferguson (2011)). The prime example of an optimal stopping problem is the well-known (Classical) Secretary Problem (see Gardener (1969)): An employer interviews sequentially a known number of applicants for a secretary post and wants to hire the best among the applicants, with the constraint that he has to accept or reject a candidate immediately after interviewing, without recall. The problem of maximizing the probability of renting the best apartment is just an instance thereof.

The problem of minimizing the expected rank under knowledge of the quality levels is a version of Robbin's Problem (see Bruss (2005)). A complete solution is still unknown. I discuss what is special about this problem and the resulting difficulties in finding a solution. The solution of the problem can be approximated by the optimal strategy for the related problem of maximizing the (observable) quality levels directly, which was studied by Moser (1956). Intuitively, one might think that these problems are equivalent, but it turns out that this is not the case. I use simulations to assess the quality of the approximative solution in the original expected rank problem.

For the remaining problems I present the optimal strategies and the resulting success probabilities and expected ranks, including their asymptotic behavior for a large number of available apartments. The solutions for the best-choice problems be found in Gilbert and Mosteller (1966). The solution for the expected-rank problem under no information can be found in Chow *et. al* (1964).

The above described informational regimes can be seen as boundary cases. I consider a partial-information setting in which the quality levels are observable, but the underlying distribution is unknown. I use a simply parametric setting in which inference can be used to approximate the underlying distribution. The decision maker, who wants to choose the best option, can then use this approximation along with the optimal strategy for the full-information case. Intuitively, the resulting success probability should lie between the success probabilities of the optimal strategies of the no- and the full-information setting. However, as it turns out the above described procedure does not necessarily improve the success probability in comparison to the no-information strategy.

The structure of the thesis is as follows: Section 3 incroduces the notion of an optimal stopping problem and provides some relevant concepts and results from probability.

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The proofs are relegated to Appendix A. Section 4 specifies the model. Section 5 considers the apartment hunting problem under no-information. It contains the extensions of the model to a stochastic number of options, and the possible refusal of a landlord. It also gives a digression on Bruss' (2000) Odds-Theorem, which provides a general solution for a class of optimal stopping problems that, for instance, contains the Classical Secretary Problen. Section 6 presents the apartment hunting problem under the full-information regime. In Section 7 I study the above mentioned partial information setting. In Section 8 I provide an outlook of possible extensions.

3 Preliminaries

All considerations in this thesis are based on a fixed complete probability space $(\Omega, \mathcal{F}, \mathrm{P})$. All appearing random variables are defined on this space and are, if not explicitely stated otherwise, univariate.

3.1 Optimal Stopping Problems

This section reviews the notion of an optimal stopping problem and presents a formally concise description of it. The notation follows Ferguson (2011).

An optimal stopping problem consists of two components:

1. A description of the *available information* at each point in time.
2. A description of the *reward for stopping* at each point in time.

The decision maker who is facing the optimal stopping problem can at each point in time decide, given the available information, whether to stop or not. The reward for stopping at any point in time is a function of the available information at that point, where the functional form is ex-ante known to her; the reward can, conditional on the information, either be deterministic or stochastic. In the latter case she ex-ante knows the functional form of the corresponding conditional probability distribution of the reward. If she decides to stop, she will obtain the (potentially random) reward. The

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objective of the decision maker is to find a strategy for stopping that maximizes her expected reward and that only depends on the available information. Such a strategy is called an optimal stopping rule.

This can be modeled in the following way. Fix an index set of the form $T \subseteq \mathbb{N}_0$ or $T = \mathbb{N}_0 \cup \{\infty\}$. With the usual linear order structure the set T may be interpreted as time. Let $(\mathcal{F}_t)_{t\in T}$ be a filtration of $\mathcal{F}$, with $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_\infty = \sigma\left(\bigcup_{t\in T\setminus\{\infty\}} \mathcal{F}_t\right)$[1] if $0, \infty \in T$. The weakly increasing sequence of σ-algebras $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_\infty \subseteq \mathcal{F}$ models the evolution of available information over time. The collection of events $\mathcal{F}_t$ represents the information available at time $t \in T$, where information means that you can tell of every event in $\mathcal{F}_t$ whether it has occured or not. The interpretation of $t = 0$ is the opportunity of the decision maker to stop right at the beginning with the trivial information $\{\emptyset, \Omega\}$. The interpretation of $t = \infty$ is the opportunity of the decision maker to never stop, in which case her available information is described by $\mathcal{F}_\infty$. Let $\mathcal{N} = \{N : (\Omega, \mathcal{F}) \to (T, 2^T) \,|\, \{N = t\} \in \mathcal{F}_t,\, t \in T\}$ be the set of all stopping times w.r.t. $(\mathcal{F}_t)_{t\in T}$. The event $\{N = t\}$ can be interpreted as stopping at time $t \in T$ under the rule N. The event $\{N = \infty\}$ can be interpreted as never stopping. The $\mathcal{F}_t$-measurability of $\{N = t\}$ captures the idea that the decision whether to stop or not at time $t \in T$ has to be based on the available information at that time point. Let $Y = (Y_t)_{t\in T}$ be a sequence of random variables that is uniformly bounded from above by an L^1 random variable (i.e., with existing and finite expected value). This assumption ensures the existence of $\mathrm{E}\,[\,Y_t\,] \in \mathbb{R} \cup \{-\infty\}$ for all $t \in T$. For $N \in \mathcal{N}$ and $(Y_t)_{t\in T}$ define the random variable[2]

$$Y_N := \sum_{t\in T} Y_t \,\mathbf{1}_{\{N=t\}}. \tag{1}$$

The random variable Y_t models the (random) reward for stopping at time $t \in T$. Y_∞ models the (random) reward for never stopping. Y_N may be interpreted as (random) reward under the stopping rule N.

[1]The definition of the *terminal* σ-algebra $\mathcal{F}_\infty$ is relevant because $\bigcup_{t\in T} \mathcal{F}_t$ is a σ-algebra only if $(\mathcal{F}_t)_{t\in T\setminus\{\infty\}}$ is constant from some index onward (cf. Broughton and Huff (1977)).

[2]The fact that T is countable ensures the $\mathcal{F}$-measurability of Y_N for any $N \in \mathcal{N}$. Additional conditions to ensure the measurability of Y_N in a more general setting (e.g., in continuous time with $T \subseteq \mathbb{R}_{\geq 0}$), like the progressive measurability of $(Y_t)_{t\in T}$, are not required here. In such cases also an alternative definition of Y_N, other than (1), is required.

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An optimal stopping problem w.r.t. the available information $(\mathcal{F}_t)_{t\in T}$ and the rewards Y then is defined as finding, if it exists, the maximizer $N^* \in \mathcal{N}$ of $\mathrm{E}[\,Y_N\,]$, i.e., finding the stopping rule that satisfies

$$N^* = \arg\max_{N\in\mathcal{N}} \mathrm{E}[\,Y_N\,]\,.$$

Under suitable conditions that allow for interchanging integration and summation[3],

$$\begin{aligned}
\mathrm{E}[\,Y_N\,] &= \sum_{t\in T} \mathrm{E}\left[\,Y_t\,\mathbf{1}_{\{N=t\}}\,\right] \\
&= \sum_{t\in T} \mathrm{E}\left[\,\mathrm{E}\left[\,Y_t\,\mathbf{1}_{\{N=t\}}\middle|\,\mathcal{F}_t\,\right]\,\right] \\
&= \sum_{t\in T} \mathrm{E}\left[\,\mathrm{E}[\,Y_t|\,\mathcal{F}_t\,]\;\mathbf{1}_{\{N=t\}}\,\right] \\
&= \sum_{t\in T} \mathrm{E}\left[\,y_t\,\mathbf{1}_{\{N=t\}}\,\right] = \mathrm{E}[\,y_N\,] \qquad (2)
\end{aligned}$$

with $y_t := \mathrm{E}[\,Y_t \mid \mathcal{F}_t\,]$. Equation (2) shows that (under the mentioned conditions) Y_t may be replaced by its conditional expectation w.r.t. $\mathcal{F}_t$ without changing the problem. The optimal stopping rule (if it exists) is the same. Thus, we may w.l.o.g. assume that Y_t is adapted to $(\mathcal{F}_t)_{t\in T}$ (by replacing Y_t with y_t if necessary). This resembles the availability of the information contained in $\mathcal{F}_t$ when deciding whether or not to stop at time $t \in T$.

Remark 1. In some cases it might be convenient to minimize $\mathrm{E}[\,Y_N\,]$ instead of maximizing it. Both approaches are equivalent, since $-\max_{N\in\mathcal{N}} \mathrm{E}[\,Y_N\,] = \min_{N\in\mathcal{N}} \mathrm{E}[\,-Y_N\,]$.

Remark 2. For $T = \{1, \ldots, n\}$ the problem is called a finite horizon problem with horizon n. Such problems may be solved, for instance, by backward induction. To do so, define $V_n := y_n$ and then recursively for $1 \le i \le n-1$

$$V_i := \max\{y_i\,,\, \mathrm{E}[\,V_{i+1} \mid \mathcal{F}_i\,]\}.$$

[3]Such conditions are, for instance,

1. T is finite, or
2. $Y_t \ge 0$ P-a.s. $\forall t \in T$ (facilitating Lebesgue's monotone convergence theorem), or
3. $\exists$ a random variable Z s.t. $|Y_t| \le Z$ P-a.s. $\forall t \in T$ and $\mathrm{E}[\,Z\,] < \infty$ (facilitating Lebesgue's dominated convergence theorem).

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It is optimal to stop at stage $1 \leq i \leq n-1$ iff $y_i \geq \mathrm{E}\,[\, V_{i+1} \mid \mathcal{F}_i \,]$, which is also referred to as principle of optimality.

Remark 3. The stopping problem may be based on the sequential observation of a sequence of random variables $X = (X_t)_{t \in T}$ (the observables) and a sequence of rewards $Y = (Y_t)_{t \in T}$ where every Y_t may depend on the whole vector X. The distribution of X and Y as well as their joint distribution are assumed to be known. The filtration $(\mathcal{F}_t)_{t \in T}$ is then generated by X, i.e., $\mathcal{F}_t = \sigma\,(X_i \mid i \leq t)$ for $t \in T$. At time $t \in T$, the available information on which the stopping decision may be based are the observations $X_1 = x_1, \ldots, X_t = x_t$. Given these observations, the reward of stopping, Y_t, might still be a random variable. As discussed above, under suitable conditions the rewards (Y_t) may be replaced by their conditional expectations w.r.t. $(\mathcal{F}_t)$. Hence, when deciding whether to stop or to continue at time $t \in T$ the reward for stopping equals w.l.o.g. $y_t(x_1, \ldots, x_t) = \mathrm{E}\,[\, Y_t \mid X_1 = x_1, \ldots, X_t = x_t \,]$, which has the advantage of being deterministic. This might come in quite handy in solving for the optimal stropping rule.

3.2 Absolute and Relative Ranks of Random Variables

The following two rank concepts for a (finite) sequence of random variables are relevant in this thesis:

Definition 1. *Let $X_1, \ldots, X_n$ be a finite sequence of random variables. The* absolute rank *R_k of X_k, $1 \leq k \leq n$, is defined by*

$$R_k := R_k(X_1, \ldots, X_n) := \sum_{i=1}^{n} \mathbf{1}_{\{X_k \leq X_i\}}. \tag{3}$$

The relative rank *r_k of X_k, $1 \leq k \leq n$, is defined by*

$$r_k := r_k(X_1, \ldots, X_k) := \sum_{i=1}^{k} \mathbf{1}_{\{X_k \leq X_i\}}. \tag{4}$$

The absolute and relative ranks of $X_1, \ldots, X_n$ are themselves random variables. For $1 \leq k \leq n$, the relative rank r_k is $\sigma\,(X_1, \ldots, X_l)$-measurable for $k \leq l \leq n$, whereas R_k is only $\sigma\,(X_1, \ldots, X_l)$-measurable for $l = n$.

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For $1 \leq k \leq n$ and $\omega \in \Omega$, the realization $R_k(\omega)$ of the absolute rank is a number in $\{1, \ldots, n\}$ and indicates the rank of the realization $X_k(\omega)$ among the realizations $X_1(\omega), \ldots, X_n(\omega)$ when ordered from the largest to the smallest value. In contrast, for $1 \leq k \leq n$ and $\omega \in \Omega$, the realization $r_k(\omega)$ of the relative rank is a number in $\{1, \ldots, k\}$ and indicates the rank of the realization $X_k(\omega)$ only among the first k realizations $X_1(\omega), \ldots, X_k(\omega)$, again ordered from the largest to the smallest value.

The following fact about random variables will be used later on:

Lemma 1. *Let X, Y be independent random variables with continuous c.d.f. F_X and F_Y, respectively. Then*

$$\mathrm{P}\left(\{X = Y\}\right) = 0. \tag{5}$$

Proof. See Appendix A. □

Remark 4. The fact that a random variable has a continuous c.d.f. does not imply that it has a density. On the other hand, every random variable that has a density has a continuous c.d.f..

Remark 5. Lemma 1 obviously holds for the case of two i.i.d. random variables X_1, X_2 with continuous (univariate) c.d.f. F. It implies that ties between X_1 and X_2 occur only with probability 0. This can be generalized to the case of $n > 2$ i.i.d. random variables $X_1, \ldots, X_n$ with continuous c.d.f. F by noting that

$$\mathrm{P}\left(\exists i \neq j : X_i = X_j\right) = \mathrm{P}\left(\bigcup_{i \neq j} \{X_i = X_j\}\right) \leq \sum_{i \neq j} \mathrm{P}\left(\{X_i = X_j\}\right) = 0.$$

The next theorem provides some facts about relative ranks that will be of central importance in the later parts of this thesis. It is attributed to Alfred Rényi (cf. Lemmas 1 and 2 in Rényi (1962)).

Theorem 1 (Rényi (1962))**.** *Let $X_1, \ldots, X_n$ be i.i.d. random variables with continuous (univariate) c.d.f. F. Then,*

(a) the corresponding absolute ranks R_i, $1 \leq i \leq n$, are identically distributed and have a discrete uniform distribution on $\{1, \ldots, n\}$, and all events of the form

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$\{R_1 = i_1, \ldots, R_n = i_n\}$ *with* $i_1, \ldots, i_n \in \{1, \ldots, n\}$ *and* $i_j \neq i_k$ *for* $j \neq k$ *have the same probability*

$$\mathrm{P}\left(\{R_1 = i_1, \ldots, R_n = i_n\}\right) = \frac{1}{n!}.$$

(b) the corresponding relative ranks $r_1, \ldots, r_n$ *are stochastically independent, and*

$$\mathrm{P}\left(\{r_k = i\}\right) = \frac{1}{k}, \quad 1 \leq k \leq n,\, 1 \leq i \leq k,$$

i.e., r_k *has a discrete uniform distribution on* $\{1, \ldots, k\}$*.*

Proof. See Appendix A. □

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4. The Model

The assumptions for the appartment hunting problem are:

(1) A decision maker wants to rent an apartment. There are $n \in \mathbb{N}$ apartments (options) that she can choose from, and she has to choose one of the options. The number of options is ex-ante known to her.

(2) Every option $1 \leq i \leq n$ corresponds to an i.i.d. draw X_i from a continuous c.d.f. F, where X_i represents a (subjective) quality level that describes how much the decision maker likes option i.

(3) Ex-ante the decision maker has no information about an apartment, but she can obtain information through inspecting the apartments sequentially.

(4) The informational regime is either:

   (a) F is not known to the decision maker, but she has knowledge of i.i.d.-ness of the draws. Inspection of option $1 \leq i \leq n$ does not reveal X_i, but only the relative rank r_i of option i among the first i options.

   (b) F and i.i.d.-ness of the draws is known to the decision maker. Inspection of option $1 \leq i \leq n$ reveals X_i.

(5) Immediately after inspecting an apartment the decision maker has to decide whether she wants to rent or to reject it. There is no opportunity of recall, i.e., an apartment that has once been rejected cannot be rented at a later point in time, after inspecting other options.

(6) The objective of the decision maker is either

   (a) to maximize the probability of choosing the best apartment (the option with absolute rank 1) among all available options, or

   (b) to minimize the expected absolute rank of the chosen option.

By identifying the decision to rent the i-th apartment (after inspection) with stopping at time i, this can be formulated as optimal stopping problem.

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Remark 6. If the random variable X has a continuous c.d.f. F, then the random variable $F(X) \sim \mathcal{U}[0,1]$. The transformation $x \mapsto F(x)$ is monotonic. Hence, for two independent random variables $X_1, X_2 \sim F$, it holds that

$$X_1(\omega) \leq X_2(\omega) \Longleftrightarrow F(X_1(\omega)) \leq F(X_2(\omega))$$

for P-a.a. $\omega \in \Omega$, since, by continuity of the c.d.f. of $F(X)$, ties occur only with probability zero. Working with $(X_i)_{i=1}^n$ or $(F(X_i))_{i=1}^n$ is for a rank-based context equivalent.

A suitable model for the information structure satisfying the above assumptions is a sequence of n i.i.d. random variables $X_1, \dots, X_n$ with $X_i \sim \mathcal{U}[0,1]$, i.e., with a continuous uniform distribution on the interval $[0,1]$. Using this distribution is justified by Remark 6.

Under assumption (4a) the observables are the relative ranks, i.e., the decision maker sequentially observes the relative ranks r_i , $1 \leq i \leq n$, of $X_1, \dots, X_n$. Under assumption (4b) the observables are the quality levels, i.e., the decision maker sequentially observes $X_1, \dots, X_n$. The filtration $(\mathcal{F}_t)_{t=1}^n$ generated by the observables yields the information structure for the problem.

The payoffs $(Y_t)_{t=1}^n$ depend on the objective of the decision maker. For asssumption (6a), only renting the best apartment counts. A suitable payoff structure for this is a payoff of 1 if the best option (with absolute rank 1) is selected, and 0 else. For assumption (6b), also options with absolute rank larger than 1 generate a positive "payoff". A suitable payoff structure is a "payoff" of j if the option with absolute rank j is selected, and the decision maker's objective is to minimize the expected "payoff" (which could be interpreted as expected "loss" in this case). For both objectives, the payoff for stopping at time t, $1 \leq t \leq n-1$, is a random variable at that time point, since the absolute ranks of the options are not observable for $t \leq n-1$. As discussed in Remark 3, the random payoffs $(Y_t)_{t=1}^n$ may be replaced by their conditional expectations w.r.t. the avaiable information $\mathcal{F}_t$, denoted by $(y_t)_{t=1}^n$.

Under assumption (6a) the problem is called the *best choice* problem; under assumption (6b) the problem is called the *expected rank* problem.

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The literature (see Bruss (2005)) calls the informational regime (4a) for problems of the above type, where F is unknown and the observables are only the relative ranks of the X_i, the *no-information* case. The informational regime (4b) where F is known and the observables are the X_i is called the *full-information* case.

Remark 7. By assumption (2) and Lemma 1, ties between the X_i occur only with probability zero. Both objectives (6a) and (6b) are not affected by such null-events. Hence, w.l.o.g. the decision maker is always able to rank the inspected options without ties.

Remark 8. Another suitable, and more natural, model for the no-information case is a Laplace probability space with $\Omega = \{\omega \mid \omega \text{ is a permutation of } \{1, \ldots, n\}\}$, $\mathcal{F} = 2^\Omega$ and $\mathrm{P}(\omega) = \dfrac{1}{n!}$ for all $\omega \in \Omega$. Each $\omega = (\omega_1, \ldots, \omega_n)$ corresponds to a realization of the absolute ranks of the options, and each such realization is equally likely (by Theorem 1). The observables are the relative ranks of the ω_i, i.e., at time $1 \leq i \leq n$ the decision maker observes $r_i(\omega_1, \ldots, \omega_n)$. The advantage of the model proposed above is its generality; it is suitable for both informational regimes by simply choosing the observables appropriately.

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5 Apartment Hunting under No-Information

This section studies the apartment hunting problem under assumption (4a), i.e., the decision maker sequentially observes only the relative ranks $(r_i)_{i=1}^n$ of the inspected apartments.

5.1 The Best-Choice Problem under No-Information

Remark 9. This problem is a version of the classical secretary problem (CSP), reformulated to the apartment hunting setting. The solution, first given by Lindley (1961), together with its properties are well known and there is an extensive literature on it.

The objective of the decision maker in the best-choice problem is to maximize the probability of choosing the best apartment (i.e., assumption (6a)).

The payoff for renting apartment i , $1 \leq i \leq n$, after inspection is

$$Y_i = \mathbf{1}_{\{R_i=1\}} = \begin{cases} 1, \text{ if } R_i = 1, \\ 0, \text{ else.} \end{cases}$$

The expected payoff conditional on the observations $\mathcal{I}_i := \{r_1 = \widehat{r}_1, \ldots, r_i = \widehat{r}_i\}$ is

$$y_i = y_i(\widehat{r}_1, \ldots, \widehat{r}_i) = \mathrm{E}\left[\, Y_i \mid \mathcal{I}_i \,\right] = \mathrm{P}\left(\{R_i = 1\} \mid \mathcal{I}_i\right) = \begin{cases} \dfrac{i}{n}, \text{ if } \widehat{r}_i = 1, \\ 0, \text{ else.} \end{cases} \tag{6}$$

The last identity in (6) follows from the fact that an option that has relative rank 1 among the first i options has a probability of $\dfrac{i}{n}$ to have rank 1 among all options.

The objective function is

$$\max_{N \in \mathcal{N}} \mathrm{P}\left(\{R_N = 1\}\right) = \max_{N \in \mathcal{N}} \sum_{i=1}^{n} \mathrm{E}\left[\, y_i \, \mathbf{1}_{\{N=i\}} \,\right] , \tag{7}$$

where $\mathcal{N}$ denotes the set of all stopping times w.r.t. the filtration generated by the relative ranks.

Using (6) and (7), the problem can be solved with backward induction. However, Ferguson (2011) proposes an alternative way that draws on the structure of the optimal strategy, and this approach is pursued in the following.

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The first step is to determine the structure of the optimal strategy. Let W_i , $0 \leq i \leq n-1$, denote the probability of choosing the best apartment under the restriction that the first i options are rejected after inspection. The relation $W_i \geq W_{i+1}$ holds because the optimal strategy rejecting the first $i+1$ options, resulting in the success probability W_{i+1}, is available among the rules that reject the first i objects. (6) implies that it is only optimal to stop with a candidate that has relative rank 1. The decision to stop with option i if it has relative rank 1 is optimal iff $\frac{i}{n} \geq W_i$. This means that, if option i has not relative rank 1 but $\frac{i}{n} \geq W_i$, it is optimal to stop with option $i+1$ if it has relative rank 1, since

$$\frac{i+1}{n} > \frac{i}{n} \geq W_i \geq W_{i+1}.$$

This implies that the optimal strategy is a threshold strategy, i.e., to reject the first $r-1$ options for some $1 \leq r \leq n$ and then rent the first apartment that has relative rank 1 (among all apartments inspected, including the first $r-1$ options), if any such option exists. The number r is called the threshold of the strategy. Thus, finding the optimal strategy for the apartment hunting problem amounts to finding the optimal threshold r^*.

The success probability $P(r)$ of choosing the best apartment using the r-threshold rule N_r, for $2 \leq r \leq n$, is

$$\begin{aligned}
P(r) &= \mathrm{P}\left(\{R_{N_r} = 1\}\right) \\
&= \mathrm{P}\left(\text{"best apartment is selected"}\right) \\
&= \sum_{k=r}^{n} \mathrm{P}\left(\text{"apartment } k \text{ is best and is selected"}\right) \\
&= \sum_{k=r}^{n} \mathrm{P}\left(\text{"apartment } k \text{ is best"}\right) \mathrm{P}\left(\text{"apartment } k \text{ is selected"} \mid \text{"it is best"}\right) \\
&= \sum_{k=r}^{n} \frac{1}{n} \mathrm{P}\left(\text{"best among first } k-1 \text{ options appears before option } r\text{"}\right) \\
&= \sum_{k=r}^{n} \frac{1}{n} \frac{r-1}{k-1} = \frac{r-1}{n} \sum_{k=r}^{n} \frac{1}{k-1}. \qquad (8)
\end{aligned}$$

The success probability of the 1-threshold rule (i.e., renting the first option) is

$$P(1) = \frac{1}{n}. \qquad (9)$$

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The optimal threshold r^* is the maximizer of $P(r)$ for $1 \leq r \leq n$. The inequalities

$$\begin{aligned} P(r+1) &\leq P(r) \\ \iff \frac{r}{n}\sum_{k=r+1}^{n}\frac{1}{k-1} &\leq \frac{r-1}{n}\sum_{k=r}^{n}\frac{1}{k-1} \\ \iff \sum_{k=r+1}^{n}\frac{1}{k-1} &\leq 1 \end{aligned} \tag{10}$$

imply that r^* is given by

$$r^* = \min\left\{1 \leq r \leq n \;\middle|\; \sum_{k=r+1}^{n}\frac{1}{k-1} \leq 1\right\}. \tag{11}$$

Theorem 2. *The optimal strategy N^*_{BC-NI} for the best-choice apartment hunting problem under no-information, i.e., under assumptions (1), (2), (3), (4), (4a), (5) and (6a), is the r^*-threshold rule with r^* given by (11). That is, it is optimal to reject the first r^*-1 options and to rent the first apartment thereafter that has relative rank 1. Formally,*

$$N^*_{BC-NI} = \min\left\{r^* \leq i \leq n \mid \hat{r}_i = 1\right\}.$$

The corresponding success probability is given by (8) for $r^* > 1$, and by (9) for $r^* = 1$.

Proof. See above considerations. □

Table 1 contains the optimal threshold r^* and the corresponding success probability $P(r^*)$ for different values of the number of options n. Figure 1 depicts the optimal threshold and the success probability for $1 \leq n \leq 100$.

Table 1: Optimal thresholds r^* and corresponding success probabilities $P(r^*)$ for the best-choice problem under no information.

n	1	2	3	4	5	6	7	8	9	10	20	30
r^*	1	1	2	2	3	3	3	4	4	4	8	12
$P(r^*)$	1	.500	.500	.458	.433	.428	.414	.410	.406	.399	.384	.379

Remark 10. The values in Table 1 for $n = 3$ confirm the claim at the end of Section 1 that the optimal strategy for three available apartments is the threshold strategy with $r^* = 2$, with a success probability of 0.5.

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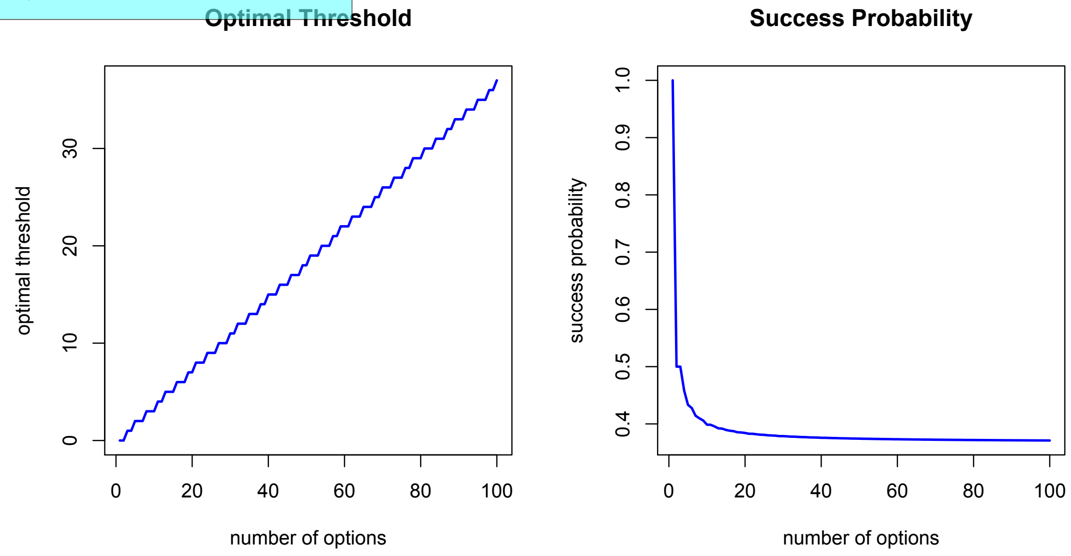

Figure 1: Optimal thresholds r^* and corresponding success probabilities $P(r^*)$ for the best-choice problem under no information.

The optimal threshold and the success probability exhibit a certain regularity. The optimal threshold seems to increase linearly in n. The success probability seems to stabilize at a value of ≈ 0.37 as n increases. To analyze the asymptotic properties of r^* and $P(r^*)$, Ferguson (2011) interprets the defining sum of r^* in (11) as (upper) Rieman sum of $x \mapsto x^{-1}$ for $x \in [r\,,\, n]$. This leads, for large n, to the approximation[4]

$$1 \approx \sum_{k=r^*+1}^{n} \frac{1}{k-1} \approx \ln\left(\frac{n}{r^*}\right),$$

which yields

$$r^* \approx \frac{n}{e}. \tag{12}$$

Relation (12) implies that the optimal threshold rule, for large n, is to skip approximately a fraction of $\exp(-1) \approx 0.368$ of the available options. This relationship is reflected by the linearity of the optimal threshold in Figure 1. I have run an OLS regression of the optimal threshold values on the number of options for $1 \leq n \leq 100$ which yields a slope coefficient of 0.369, with an adjusted R^2 of 0.999. Repeating the regression for $1 \leq n \leq 10$ yields a slope coefficient of 0.370, with an adjusted R^2 of 0.923. This indicates that Relation (12) is also a good approximation for small n.

[4]This approximation makes sense due to the decreasing curvature of $x \mapsto x^{-1}$ for large x.

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To get an approximation of the success probability $P(r^*)$ for large n, Ferguson (2011) uses the approximation (12) of r^* and plugs it into the (exact) formula (8) for $P(r^*)$ to get

$$P(r^*) \approx \frac{\frac{n}{e}}{n} \underbrace{\sum_{k=r^*+1}^{n} \frac{1}{k-1}}_{\approx 1 \text{ by def. of } r^*} \approx \frac{1}{e} \approx 0.368. \tag{13}$$

The asymptotic relation (13) is reflected by the aforementioned stabilizing behavior of the success probability in Figure 1.

To verify the analytical results I have simulated the success probabilities for the optimal strategy for $1 \leq n \leq 30$. As a benchmark I have also simulated the success probability for the naive strategy of choosing randomly one of the available options. The exact success probability for this strategy for the problem of n available options equals $\frac{1}{n}$. The results of these simulations are displayed in Figure 2. They confirm the analytical results.

The higher variation of the simulated success probabilities around the exact values for the optimal strategy, in comparison to the results for the naive strategy, might be explained by the higher (exact) success probability for the optimal strategy. For both the optimal and the naive strategy each simulation run corresponds to a Bernoulli trial with the corresponding exact success probability. The higher success probability of the optimal strategy implies a higher variance of the corresponding Bernoulli trials. This higher variance might be reflected in the higher variation of the average simulated success probability of the optimal strategy around its theoretical value.

The increased congruence of the simulated and the exact success probablities in the right plot, compared to the left plot, might be explained by the Law of Large Numbers.

5.1.1 A Digression on Bruss' Odds-Theorem

Bruss (2000) presents a theorem on the solution of a class of optimal stopping problems involving indicator functions of independent events. This theorem can be used to derive the solution to the CSP, as well as to the best-choice apartment hunting problem under no information. It also shows that the structure of the solution to the apartment hunting problem presented in Theorem 2 is the same for a whole class of optimal stopping

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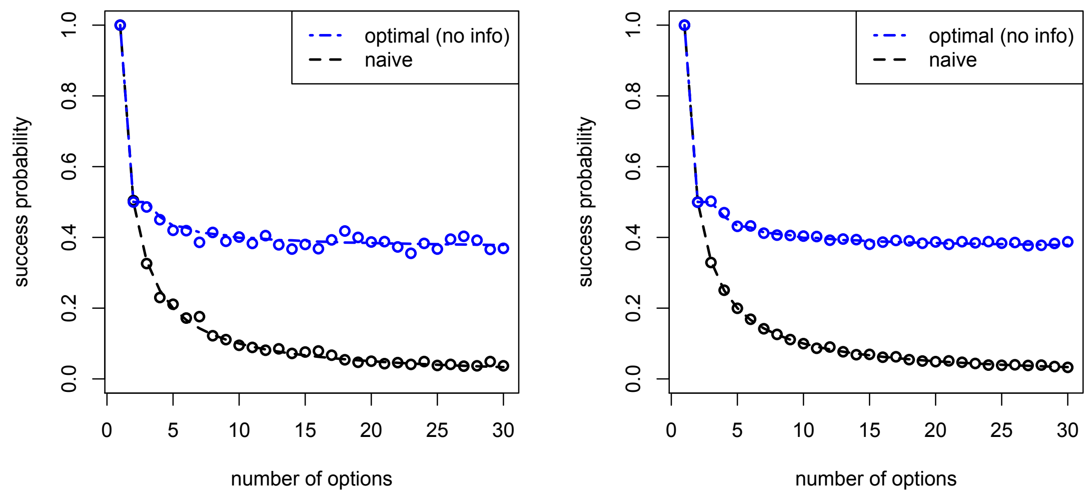

Figure 2: Simulated and exact success probabilities for the optimal and the naive strategy in the best-choice problem under no information. (Number of simulation runs $r = 10^3$ (left) and $r = 10^4$ (right); dots = simulated, dashed lines = exact)

problems. The optimal stopping rule is always a threshold rule, where the optimal threshold is determined by summing up the odds of the defining events of the indicators until this sum is larger or equal to 1.

The problem in Bruss (2000) is the following. Let $A_1, \ldots, A_n$ be a sequence of independent events and $\mathbf{1}_1, \ldots, \mathbf{1}_n$ be the sequence of corresponding indicator functions (i.e., $\mathbf{1}_i := \mathbf{1}_{A_i}$). A decision maker observes the indicators $\mathbf{1}_i$ sequentially. The indicators are the only information that she has, and she may stop after each observation. There is no recall of preceeding indicators. An event $\{\mathbf{1}_i = 1\}$ is called a success. The decision maker gets a payoff of 1 if she stops at the last success in the sequence of observations, and she gets a payoff of 0 else. That is, her objective is to find (if it exists) a stopping rule N^* that maximizes the expected payoff

$$P := \mathrm{P}\left(\{\mathbf{1}_N = 1\} \cap \{\mathbf{1}_{N+1} = \cdots = \mathbf{1}_n = 0\}\right)$$

among all stopping rules N w.r.t. the filtration generated by the indicators $\mathbf{1}_1, \ldots, \mathbf{1}_n$.

The next theorem provides the solution to this problem.

Theorem 3 (Odds-Theorem, Bruss (2000))**.** *Let* $\mathbf{1}_1, \ldots, \mathbf{1}_n$ *be the indicator functions of a sequence of independent events* $A_1, \ldots, A_n$*. Let* $p_i = \mathrm{P}(A_i)$*,* $q_i = 1 - p_i$ *and*

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$o_j = p_j/q_j$ (i.e., o_i denotes the odds of event A_i). Then an optimal rule N^* for stopping on the last success exists and is to stop on the first index (if any) k with $\mathbf{1}_k = 1$ and $k \geq s$, where

$$s := \sup\left\{1, \sup\left\{1 \leq k \leq n \;\middle|\; \sum_{j=k}^{n} o_j \geq 1\right\}\right\}, \tag{14}$$

with $\sup\{\emptyset\} := -\infty$. Thus, the optimal strategy is the s-threshold rule.

The expected payoff (the success probability) under the optimal strategy is given by

$$P = \left(\prod_{j=s}^{n} q_j\right)\left(\sum_{j=s}^{n} o_j\right).$$

Proof. See Bruss (2000) □

The odds-theorem 3 immediately provides the solution for the best-choice apartment hunting problem under no-information. The corresponding events are $A_i := \{r_i = 1\}$, and these are independent by Theorem 1. Using Theorem 1, $p_i = \frac{1}{i}$, hence $r_i = \frac{1}{i-1}$. The odds-theorem implies that the optimal strategy is the s-threshold rule with threshold

$$s = \sup\left\{1, \sup\left\{1 \leq k \leq n \;\middle|\; \sum_{j=k}^{n} \frac{1}{j-1} \geq 1\right\}\right\}.$$

Some algebra shows that indeed $s = r^*$, the optimal threshold for the problem determined in (11). The success probability according to the odds-theorem equals

$$P = \prod_{j=s}^{n} \frac{j-1}{j} \sum_{j=s}^{n} \frac{1}{j-1} = \frac{s-1}{n} \sum_{j=s}^{n} \frac{1}{j-1}. \tag{15}$$

For $r = s$, (15) coincides with the previously determined success probability (8).

Remark 11. A priori it is not clear (at least for $n > 2$) that the odds-theorem will yield the same optimal strategy as the approach in section 5.1. The odds-theorem maximizes the success probability over the set $\mathcal{N}_1$ of all stopping rules w.r.t. to the filtration generated by the indicator functions of the events $\{r_i = 1\}$, $1 \leq i \leq n$; in section 5.1 the objective was to maximize the success probability over the set $\mathcal{N}_2$ of all stopping rules w.r.t. the filtration generated by the relative ranks $r_1, \ldots, r_n$. The former set of stopping rules is for $n > 2$ strictly smaller than the latter, since $\sigma(\mathbf{1}_1, \ldots, \mathbf{1}_m)$ is a strict subset of $\sigma(r_1, \ldots, r_m)$ for $2 < m \leq n$. It is not clear whether the maximizer over $\mathcal{N}_2$ is contained in $\mathcal{N}_1$. This problem vanishes once it is known that the optimal

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strategy over $\mathcal{N}_2$ is a threshold strategy. The maximizer over $\mathcal{N}_2$ is then contained in $\mathcal{N}_1$, since it only requires information that is contained in the filtration generated by the indicators, namely whether the option at hand has relative rank 1 (and whether the number of the option is larger than the threshold).

5.1.2 Uncertain Availability of Options

In practice it might happen that the landlord refuses to rent an apartment to an interested applicant. In this case the decision maker has to continue with her search. The possibility of refusal is not contained in the model so far. To study the effect of the uncertain availability of apartments the model (i.e., assumptions (1), (2), (3), (4), (5), (6a)) is extended by assumption

(7) If the decision maker decides to rent an option (with or without inspection), the landlord may refuse to rent the apartment to her. This happens for each apartment with a fixed and ex-ante known probability $1-p$, with $p \in (0,1)$. If the apartment is not avaiable, the search goes on. The decision of the landlord becomes known to her after her renting decision. Whether apartment $1 \leq i \leq n$ is avaiable is independent of the ranking of the apartments, and is also independent of whether the other apartments $j \neq i$ are avaiable upon request.

Remark 12. This extension draws on Smith (1975) who considers a modification of the CSP in which each applicant declines an offer of employment with a certain probability. This modification is also discussed in Ferguson (2011).

The solution follows the steps of Ferguson (2011). By the same reasoning as for the case without assumption (7), the optimal strategy for this problem is a threshold strategy. The solution thus goes along the same lines as before. The success probability of the r-threshold rule N_r, for $1 \leq r \leq n$, is given by

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$$
\begin{aligned}
P(r) &= \mathrm{P}\left(\{R_{N_r} = 1\}\right) \\
&= \mathrm{P}\left(\text{"best apartment is selected"}\right) \\
&= \sum_{k=r}^{n} \mathrm{P}\left(\text{"apartment } k \text{ is available, is selected and is best"}\right) \\
&= \sum_{k=r}^{n} \mathrm{P}\left(\text{"}k\text{ available and best"}\right) \mathrm{P}\left(\text{"}k\text{ selected"} \mid \text{"}k\text{ available and best"}\right) \\
&= \sum_{k=r}^{n} \frac{p}{n} \mathrm{P}\left(\text{"option } k \text{ is first available with rank 1 after option } r-1\text{"}\right) \\
&= \sum_{k=r}^{n} \frac{p}{n} \underbrace{\left(\prod_{i=r}^{k-1}\left(1-\frac{p}{i}\right)\right)}_{=\frac{\Gamma(r)\Gamma(k-p)}{\Gamma(k)\Gamma(r-p)}} = \frac{p}{n}\sum_{k=r}^{n}\frac{\Gamma(r)\Gamma(k-p)}{\Gamma(k)\Gamma(r-p)}\,, \qquad (16)
\end{aligned}
$$

where $\Gamma(.)$ denotes the Gamma function. The optimal threshold r^* is the maximizer of $P(r)$ for $1 \leq r \leq n$. Some algebra reveals that

$$
\begin{aligned}
P(r+1) &\leq P(r) \\
\iff \quad \frac{p\Gamma(r)}{\Gamma(r-p+1)} \sum_{k=r+1}^{n} \frac{\Gamma(k-p)}{\Gamma(k)} &\leq 1. \qquad (17)
\end{aligned}
$$

Inequality (17) yields that r^* is given by

$$
r^* = \min\left\{1 \leq r \leq n \;\middle|\; \frac{p\Gamma(r)}{\Gamma(r-p+1)} \sum_{k=r+1}^{n} \frac{\Gamma(k-p)}{\Gamma(k)} \leq 1\right\}. \qquad (18)
$$

Thus, the optimal strategy for the problem with uncertain availability of options is the r^*-threshold rule with r^* given by (18). The success probability $P(r^*)$ is given by (16).

Figure 3 depicts the optimal threshold and the success probability for the problem of uncertain availability of options for $1 \leq n \leq 100$ and $p \in \{.1, .3, .5, .7, .9, 1\}$.

Figure 3 reveals the qualitative influence of the parameter p on the optimal strategy. An increase of p yields to lower optimal thresholds (i.e., it is optimal to accept apartments earlier on), and to lower success probabilities (i.e., there is a cost in terms of optimality). Both effects are intuitive:

- If the probability that options are not available increases, it seems reasonable to start accepting apartments at an earlier stage (thus the lower threshold r^*).

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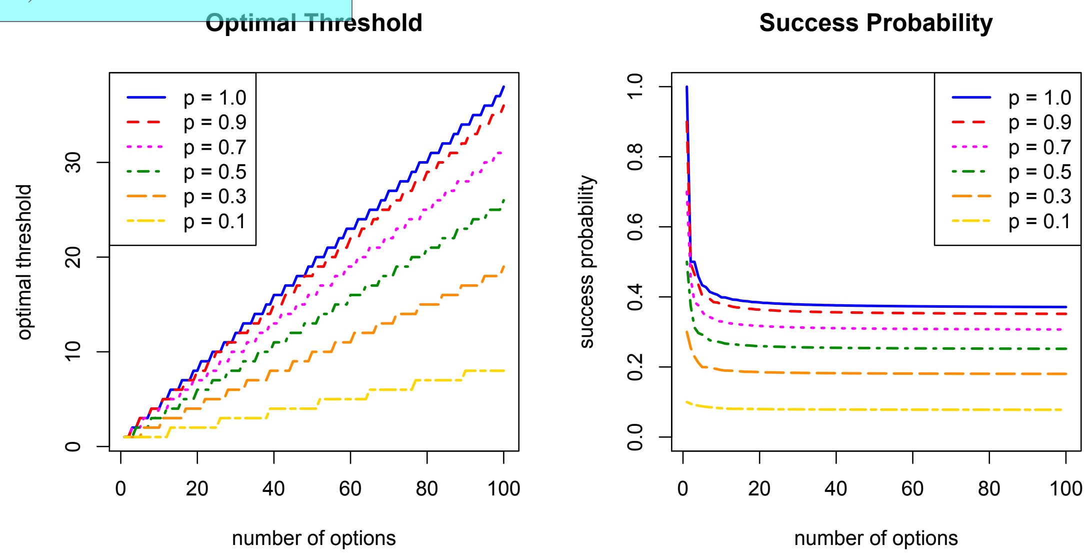

Figure 3: Optimal thresholds r^* and corresponding success probabilities $P(r^*)$ for the best-choice problem under no information with uncertain avaiability of options (p denotes the probability that an option is available).

- Earlier stopping leads to a lower benchmark (the best among the first $r^* - 1$ options), and additionally apartments that the decision maker wants to rent might not be available (thus the lower success probability).

For fixed $p \in (0, 1)$, the qualitative behavior of the optimal threshold (linear growth in n) and the corresponding success probability (stabilizing behavior as $n \to \infty$) is the same as for the case without uncertain availability of options (i.e., for $p = 1$). Ferguson (2011) uses the convergence result

$$k^p \frac{\Gamma(k-p)}{\Gamma(k)} \xrightarrow[k \to \infty]{} 1$$

to approximate the defining sum in (18) by an integral in order to derive that for large n the optimal threshold $r^* = r^*(p)$ satisfies

$$r^*(p) \approx n\, p^{\frac{1}{1-p}}. \tag{19}$$

Relation (19) is reflected by the linear growth of r^*. Using the rule of L'Hospital shows that the r.h.s. of (19) converges to $\dfrac{n}{e}$ for $p \to 1$, which is in line with Relation (12) for the case without uncertain availability of options.

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To get an approximation for $P(r^*)$ for large n I use the exact formula (16) and approximation (19) for r^* to obtain

$$P(r^*) = \underbrace{\frac{p}{n}\sum_{k=r}^{n}\frac{\Gamma(r)\Gamma(k-p)}{\Gamma(k)\Gamma(r-p)}}_{\approx\frac{r^*-p}{p}\text{ by def. of } r^*} \approx \frac{r^*-p}{n} \approx p^{\frac{1}{1-p}}. \tag{20}$$

Relation (20) is reflected by the stabilizing behavior of $P(r^*)$ for $n \to \infty$ in Figure 3. Using again the rule of L'Hospital shows that the last expression in (20) converges to $\frac{1}{e}$ for $p \to 1$, which is in line with Relation (13) for the case without uncertain availability of options.

Remark 13. Tamaki (2001) provides a generalization of Bruss' Odds-Theorem (see Theorem 3) to the case of uncertain availability of options. His result can be used to derive the optimal strategy for the problem studied in this section.

5.1.3 An Unknown Number of Options

A crucial component of assumption (1) is the ex-ante knowledge of the number of options $n \in \mathbb{N}$. In practice this number might not be known exactly. But the decision maker might have a subjective belief on what the number of options is. To study such a situation assumption (1) is replaced by

(1*) A decision maker wants to rent an apartment. There are $n \in \mathbb{N}$ apartments (options) that she can choose from, where n is a random variable that follows a discrete uniform distribution on $\{1, \ldots, N\}$, with some fixed $N > 0$. The number N and the distribution of n are ex-ante known to the decision maker.

The remaining assumptions for the model in this section are (2), (3), (4), (5) and (6a).

Remark 14. This modification draws on Presman and Sonin (1972) who introduced the idea of an unknown number of applicants to the CSP. This modification is also discussed in Ferguson (2011).

The solution follows Ferguson (2011). Suppose the decision maker has just inspected apartment $1 \leq j \leq N$. Conditional on the event $\{n \geq j\}$ the distribution of n is

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uniform on the set $\{j, \dots, N\}$. If option j has relative rank 1 it is the best option overall if the remaining $k - j$ options, with $j \leq k \leq N$, all have relative rank larger than 1. Thus, given the conditional distribution of n on $\{n \geq j\}$ and the distribution of the relative ranks, the probability p_j that option j is the best option overall is

$$\begin{aligned} p_j &= \frac{1}{N-j+1} \sum_{k=j}^{N} \mathrm{P}\left(\{r_{j+1} > 1, \dots, r_k > 1\}\right) \\ &= \frac{1}{N-j+1} \sum_{k=j}^{N} \frac{j}{j+1} \frac{j+1}{j+2} \cdots \frac{k-1}{k} = \frac{j}{N-j+1} \sum_{k=j}^{N} \frac{1}{k}. \end{aligned} \tag{21}$$

The sequence $(p_j)_{j=1}^N$ is strictly increasing, i.e., $p_{j+1} > p_j$ for $1 \leq j \leq N-1$. This follows because the difference

$$p_{j+1} - p_j = \left(\frac{N+1}{(N-j+1)(N-j)} \sum_{k=j+1}^{N} \frac{1}{k} \right) - \frac{1}{N-j+1}$$

is strictly positive iff

$$\frac{1}{N-j} \sum_{k=j+1}^{N} \frac{1}{k} > \frac{1}{N+1}\,, \tag{22}$$

which holds because the l.h.s. of (22) is the mean of numbers which are all strictly larger than the r.h.s. of (22).

A reasoning analogous to the case with assumption (1) instead of (1*) shows that the optimal strategy for this problem is a threshold strategy. Denote with W_j the success probability if option j is reached but rejected. Since rejecting option $j+1$ is possible if option j is rejected, it holds that $W_j \geq W_{j+1}$. It is optimal to rent apartment j iff it has relative rank 1 and $p_j \geq W_j$. The strict monotonicity of $(p_j)_{j=1}^N$ then implies that

$$p_{j+1} > p_j \geq W_j \geq W_{j+1}.$$

Hence, if it is optimal to rent apartment j if it has relative rank 1, then it is optimal to rent apartment $j+1$ if this apartment has relative rank 1 but the previous has not. This shows that the optimal strategy is a threshold strategy N_r with some threshold $1 \leq r \leq N$.

The success probability using N_r conditional on the event $\{n < N\}$ is zero (since there are less apartments than the threshold). The success probability using N_r conditional

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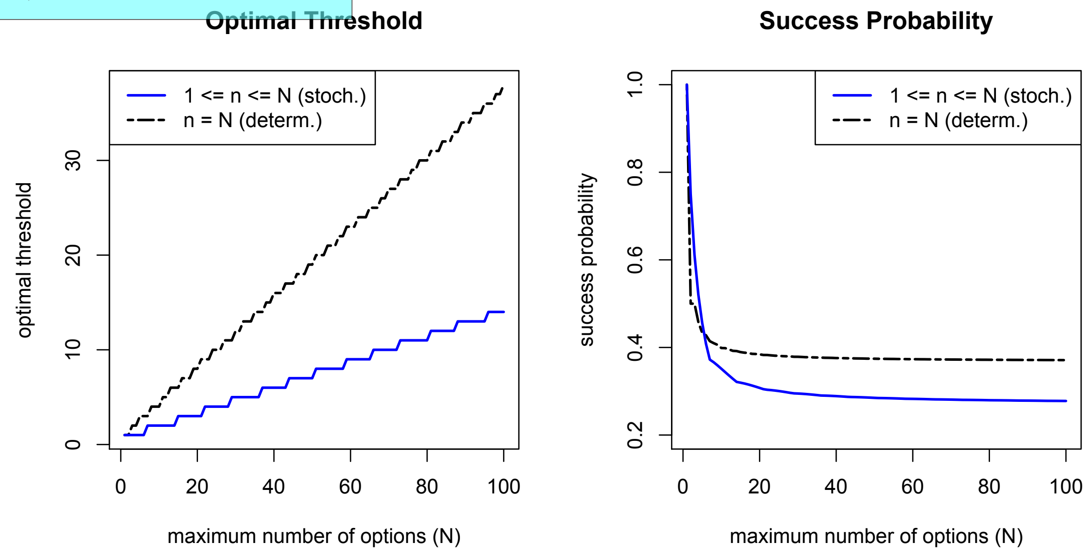

Figure 4: Optimal thresholds r^* and corresponding success probabilities $P(r^*)$ for the best-choice problem under no information with stochastic ($n \sim \mathcal{U}(\{1,\dots,N\})$) and deterministic ($n = N$) number of options.

on the event $\{n \geq N\}$ is the same as if the number of options n was fixed and known ex-ante, i.e. (see Equation (8)),

$$\mathrm{P}\left(\{R_{N_r} = 1\} \mid \{n \geq N\}\right) = \begin{cases} \dfrac{r-1}{n} \displaystyle\sum_{i=r}^{n} \dfrac{1}{i-1}, & \text{for } 2 \leq r \leq N, \\ \dfrac{1}{n}, & \text{for } r = 1. \end{cases} \tag{23}$$

Since the distribution of n is uniform on $\{1,\dots,N\}$, the (unconditional) success probability using N_r is given by

$$P(r) = \begin{cases} \dfrac{1}{n} \displaystyle\sum_{n=r}^{N} \dfrac{r-1}{n} \sum_{i=r}^{n} \dfrac{1}{i-1}, & \text{for } 2 \leq r \leq N, \\ \dfrac{1}{N} \displaystyle\sum_{n=1}^{N} \dfrac{1}{n}, & \text{for } r = 1. \end{cases} \tag{24}$$

The optimal threshold r^* is the maximizer of $P(r)$ for $1 \leq r \leq N$ and can be computed numerically. Figure 4 depicts the optimal threshold and the corresponding success probability for $1 \leq n \leq 100$. For large N (and thus $r^* > 2$) the defining sum for $P(r)$

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in (24) can be approximated by an integral (see Ferguson (2011)) as

$$\begin{aligned} P(r) = \frac{1}{n}\sum_{n=r}^{N}\frac{r-1}{n}\sum_{i=r}^{n}\frac{1}{i-1} \quad &\approx \quad x\int_x^1 \frac{1}{y}\int_x^y \frac{1}{z}\,\mathrm{d}z\,\mathrm{d}y \\ &= \quad \frac{1}{2}x(\log(x))^2, \end{aligned} \tag{25}$$

if $x \approx \dfrac{r}{n}$. The last expression in (25) has a unique maximum at $x^* = \exp(-2) \approx 0.135$. Thus, for large n, the optimal threshold r^* satisfies

$$r^* \approx n\exp(-2). \tag{26}$$

Plugging (26) into (25) yields that for large n

$$P(r^*) \approx 2\exp(-2) \approx 0.271.$$

These findings are also reflected by the linear growth of r^* and the stabilizing behavior of $P(r^*)$ in Figure 4.

An interesting observation is the following. One might intuitively expect that the uncertainty about n leads to a cost in terms of optimality compared to the case where $n = N$ with probability one (i.e., when the number of options is known ex-ante). Surprisingly this is not true for small numbers of options: for $2 \leq N \leq 5$ the probability of success under the optimal strategy is strictly larger in the case with uncertainty than in the deterministic case (for $N = 1$ the success probabilities trivially coincide). Figure 5 depicts the optimal threshold and the success probability for both cases for $1 \leq N \leq 6$. To investigate this counterintuitive result consider the case of $N = 2$. For the deterministic case, where N is known ex-ante then, both the 1- and the 2-threshold rule yield a success probability of $\frac{1}{2}$. In the stochastic case, where N is uniformly distributed on $\{1, 2\}$, the success probability of the 1-threshold rule equals $\frac{3}{4}$; this is because with probability $\frac{1}{2}$ the number of options is 1, in which case the decision maker for sure chooses the best option, and with probability $\frac{1}{2}$ the number of options is 2, in which case the option that she chooses (the first one) is best with probability $\frac{1}{2}$. The 2-threshold rule in this case yields a success probability of only $\frac{1}{4}$. The optimal strategy is thus the 1-threshold strategy for both the deterministic and the stochastic case. For the case with uncertainty this strategy is optimal on the events $\{n = 1\}$ and

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$\{n = 2\}$. The conditional success probabilities, both weighted by $\frac{1}{2}$, yield the unconditional success probability. The success probability for the certainty case is identical to the conditional success probability on the event $\{n = 2\}$ in the uncertainty case, weighted by 1. The higher weight of 1 of this conditional success probability in the unconditional success probability for the certainty case does not outweigh the contribution of the conditional success probability on the event $\{n = 1\}$ in the unconditional success probability for the case with uncertainty. Figure 5 shows that for $N = 3$ the optimal strategy in the uncertainty case is still the 1-threshold rule. This rule is only optimal on the events $\{n = 1\}$ and $\{n = 2\}$, but not on $\{n = 3\}$. The success probability in the uncertainty case is thus a linear combination of the optimal success probabilities on the events $\{n = 1\}$ and $\{n = 2\}$ and a suboptimal success probability on the event $\{n = 2\}$, each weighted by $\frac{1}{3}$. The optimal rule for the certainty case is the 2-threshold rule, and this is also the optimal rule on the event $\{n = 3\}$. The success probability for the certainty case is thus the optimal success probability on the event $\{n = 3\}$, weighted by 1. The higher optimal success probabilities on the events $\{n = 1\}$ and $\{n = 2\}$, weighted by $\frac{1}{3}$, still outweigh the optimal success probability on the event $\{n = 3\}$ for the certainty case. Continuing with these considerations a pattern becomes visible: as N increases, the fraction of events on which the threshold rule for the uncertainty case is optimal decreases (see Figure 6), and the weights of those events, especially for small values of n, where this strategy is optimal are decreasing as well. This constitutes the cost in terms of optimality caused by the uncertainty about n for N getting larger. From $N = 6$ onwards the optimality loss in terms of a decreased weighted conditional success probability on the individual events $\{n = i\}$ for $1 \leq i \leq N$ dominates the (overall) unconditional success probability.

5.2 The Expected-Rank Problem under No-Information

Remark 15. The underlying optimal stopping problem was introduced by Lindley (1961). A complete discussion, including the asymptotics of the problem, is given by Chow *et al.* (1964)).

The objective of the decision maker in the expected-rank problem is to minimize the

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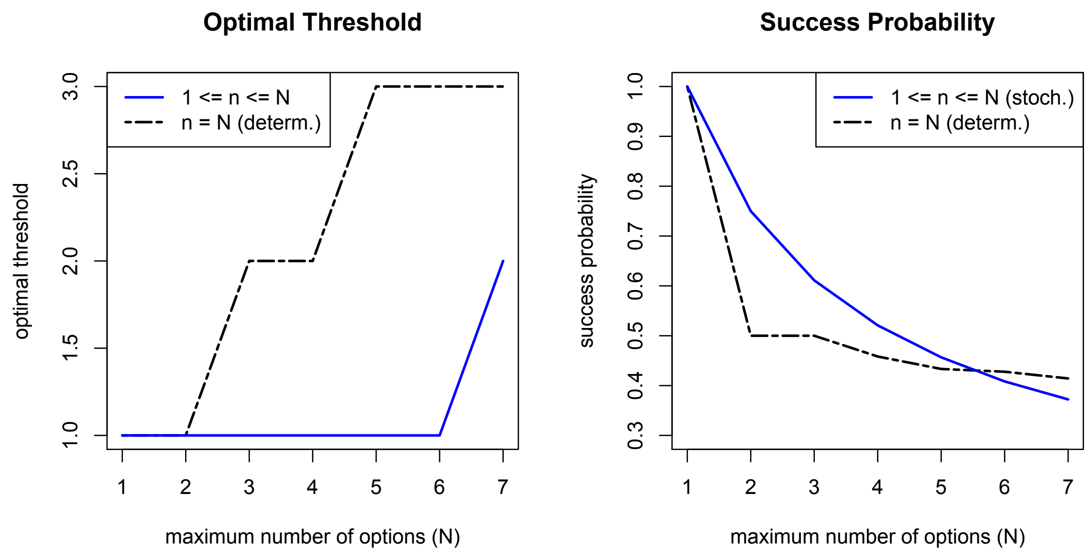

Figure 5: Optimal thresholds r^* and corresponding success probabilities $P(r^*)$ for the best-choice problem under no information with stochastic ($n \sim \mathcal{U}(\{1, \ldots, N\})$) and deterministic ($n = N$) number of options.

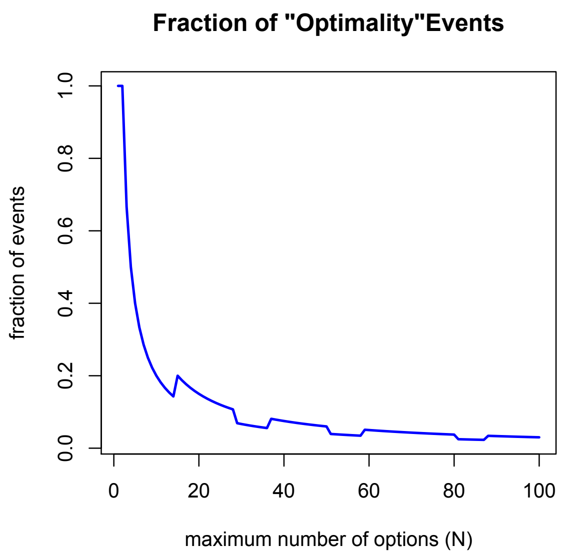

Figure 6: Fraction of "optimality" events, i.e., events of the form $\{n = 1\}$, $1 \leq i \leq N$, on which the optimal strategy for the case of an uncertain number of options is (conditionally) optimal.

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expected absolute rank of the chosen apartment (i.e., assumption (6b)).

The payoff for renting apartment i, $1 \le i \le n$, after inspection is

$$Y_i = R_i. \tag{27}$$

The probability of the event $\{R_i = k\}$, $1 \le k \le n$, conditional on the observations $\mathcal{I}_i := \{r_1 = \widehat{r}_1, \ldots, r_i = \widehat{r}_i\}$ is

$$\mathrm{P}\left(\{R_i = k\} \mid \mathcal{I}_i\right) = \mathrm{P}\left(\{R_i = k\} \mid \{r_i = \widehat{r}_i\}\right) = \binom{n}{i}^{-1} \binom{k-1}{\widehat{r}_i - 1} \binom{n-k}{i - \widehat{r}_i}, \tag{28}$$

i.e., the distribution of R_i conditional on $\mathcal{I}_i$ is a negative hypergeometric distribution with parameters n, i and $\hat{r}_i$. Equation (28) holds because, by Theorem 1, all permutations of the abolute ranks are equally likely, and the probability that option i with absolute rank $\widehat{r}_i$ among the first i options has absolute rank k among all options is the same as the probability that an option with absolute rank k is found in a random sample of size i and has absolute rank $\widehat{r}_i$ there.

Using (28), the expected payoff for renting apartment i conditional on $\mathcal{I}_i$ is

$$\begin{aligned} y_i = y_i(\widehat{r}_1, \ldots, \widehat{r}_i) = \mathrm{E}\left[\, R_i \mid \mathcal{I}_i \,\right] &= \sum_{k=1}^{n} k \, \mathrm{P}\left(\{R_i = k\} \mid \{r_i = \widehat{r}_i\}\right) \\ &= \binom{n}{i}^{-1} \sum_{k=1}^{n} k \binom{k-1}{\widehat{r}_i - 1} \binom{n-k}{i - \widehat{r}_i} \\ &= \binom{n}{i}^{-1} \widehat{r}_i \underbrace{\sum_{k=1}^{n} \binom{k}{\widehat{r}_i} \binom{n-k}{i - \widehat{r}_i}}_{= \binom{n+1}{i+1}} = \frac{n+1}{i+1} \widehat{r}_i. \end{aligned} \tag{29}$$

The objective function is

$$\min_{N \in \mathcal{N}} \mathrm{E}\left[\, R_N \,\right] = \min_{N \in \mathcal{N}} \sum_{i=1}^{n} \mathrm{E}\left[\, y_i \, \mathbf{1}_{\{N=i\}} \,\right], \tag{30}$$

where $\mathcal{N}$ denotes the set of all stopping times w.r.t. the filtration generated by the relative ranks $(r_i)_{i=1}^{n}$.

Following Chow *et al.* (1964), backward induction is used to solve the problem. Let W_i, $0 \le i \le n-1$, denote the expected rank of the chosen apartment under the optimal strategy among all strategies that satisfy the restriction that the first i options

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are rejected after inspection. The optimal strategy that corresponds to W_0 gives the solution to the expected-rank apartment hunting problem.

The initial condition for the backward induction is

$$W_{n-1} = \mathrm{E}\left[\, y_n(r_n) \mid \mathcal{I}_{i-1}\,\right] = \mathrm{E}\left[\frac{n+1}{n+1}\, r_n \;\middle|\; \mathcal{I}_{n-1}\right] = \mathrm{E}\left[\, r_n\,\right] = \frac{1}{n}\sum_{j=1}^{n} j = \frac{n+1}{2}. \tag{31}$$

The third equality in (31) is due to the independence of the r_i. The backward induction recursion for $1 \le i \le n-1$ is

$$\begin{aligned} W_{i-1} = \mathrm{E}\left[\, \min\left(y_i(r_i)\,,\, W_i\right) \mid \mathcal{I}_{i-1}\,\right] \;&=\; \mathrm{E}\left[\min\left(\frac{n+1}{i+1}\, r_i\,,\, W_i\right)\right] \\ &=\; \frac{1}{i}\sum_{j=1}^{i} \min\left(\frac{n+1}{i+1} j\,,\, W_i\right). \end{aligned} \tag{32}$$

Expression (32) for W_{i-1} can be simplified by using the floor function $\lfloor . \rfloor$, where $\lfloor x \rfloor = \max\{y \in \mathbb{Z} \mid y \le x\}$, $x \in \mathbb{R}$. With

$$s_i := \left\lfloor \frac{i+1}{n+1}\, W_i \right\rfloor, \quad 1 \le i \le n-1, \tag{33}$$

the r.h.s. of (32) becomes

$$\begin{aligned} W_{i-1} \;&=\; \frac{1}{i}\left(\frac{n+1}{i+1}(1+2+\cdots+s_i) + (i-s_i)\, W_i\right) \\ &=\; \frac{1}{i}\left(\frac{n+1}{i+1}\, \frac{s_i(s_i+1)}{2} + (i-s_i)\, W_i\right). \end{aligned} \tag{34}$$

Equations (31) and (32) imply

$$W_0 \le W_1 \le \cdots \le W_{n-1} = \frac{n+1}{2}.$$

Equation (33) implies

$$s_1 \le s_2 \le \cdots \le s_{n-1} = \left\lfloor \frac{n}{2} \right\rfloor.$$

The sequence $W_1, \ldots, W_{n-1}$ implicitly characterizes the optimal strategy: Equation (32) shows that it is optimal to stop with option i after inspection if it is the first option that satisfies

$$\begin{aligned} \widehat{r}_i \;&\le\; W_i\, \frac{i+1}{n+1} \\ \iff \widehat{r}_i \;&\le\; s_i. \end{aligned} \tag{35}$$

Using (35), the optimal strategy can be described by the vector $s := (s_1, \ldots, s_{n-1})$.

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Theorem 4. *The optimal strategy N^*_{ER-NI} for the expected-rank apartment hunting problem under no-information, i.e., under assumptions (1), (2), (3), (4), (4a), (5) and (6b), is to stop with the first option i that satisfies $\widehat{r}_i \leq s_i$, with s_i defined by (33). Formally,*

$$N^*_{ER-NI} = \min\left\{1 \leq i \leq n \mid \hat{r}_i \leq s_i\right\}.$$

The expected absolute rank of the chosen apartment under this strategy equals W_0, recursively defined by (31) and (32).

Proof. See above considerations. □

Table 2 contains the optimal strategies s and the corresponding expected absolute ranks for different values of the number of options n. Figure 7 depicts the expected absolute rank under the optimal strategy for $1 \leq n \leq 100$.

Table 2: Optimal strategies s and corresponding expected absolute ranks W_0 for the expected-rank problem under no information.

n	W_0	$s = (s_1, \ldots, s_{n-1})$
1	1.00	-
2	1.50	(1)
3	1.67	$(0, 1)$
4	1.88	$(0, 1, 2)$
5	2.05	$(0, 1, 1, 2)$
6	2.22	$(0, 0, 1, 2, 3)$
7	2.28	$(0, 0, 1, 1, 2, 3)$
8	2.40	$(0, 0, 1, 1, 2, 2, 4)$
9	2.50	$(0, 0, 0, 1, 1, 2, 3, 4)$
10	2.56	$(0, 0, 0, 1, 1, 2, 2, 3, 5)$
20	3.00	$(0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 2, 2, 2, 3, 3, 4, 5, 7, 10)$
30	3.20	$(0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 3, 3, 4, 4, 5, 6, 7, 8, 11, 15)$

Figure 7 indicates a stabilizing behavior of the expected absolute rank W_n for $n \to \infty$ between 3.5 and 4. Chow *et al.* (1964) show that $W_0 = W_0(n)$ is strictly monotonically

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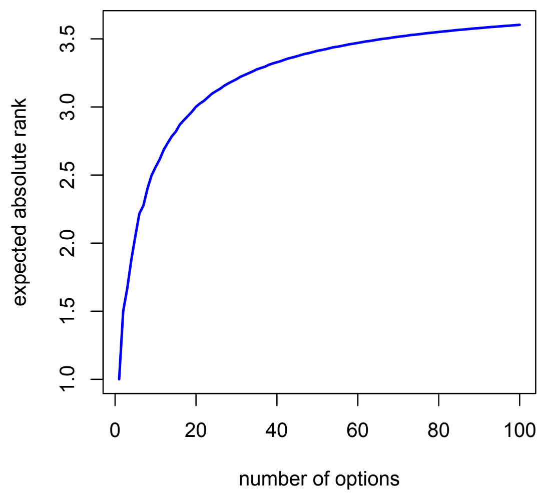

Figure 7: Expected absolute rank under the optimal strategy for the expected-rank problem under no information.

increasing in n and that

$$W_0 \xrightarrow[n\to\infty]{} \prod_{j=1}^{\infty} \left(\frac{j+2}{j}\right)^{\frac{1}{j+1}} \approx 3.8695. \tag{36}$$

Given the monotonicity of W_0, the r.h.s. of (36) provides an upper bound for the expected absolute rank for an arbitrary number of options n. For large n, the expected absolute rank under the optimal strategy is approximately equal to this upper bound.

Remark 16. Bruss (2005) provides a neat non-technical argument for the (weak) monotonicity of $W_0(n)$ for the full-information case. The same idea works for the no-information case as well. Suppose there is a prophet who can at the beginning foresee the index $1 \leq j \leq n$ of the option with absolute rank $R_j = n$ (i.e., the worst option). Apart from this he has no further prophetic abilities. His expected rank under optimal behavior, $W_0^P(n)$ say, has to satisfy $W_0^P(n) \leq W_0(n)$, since he cannot do worse than under the optimal strategy without the additional information of the index j. Optimal behavior forces the prophet to reject option j. This implies that $W_0^P(n) = W_0(n-1)$, since the prophet has to solve the problem of choosing the best apartment among the remaining $n-1$ options. Hence, $W_0(n-1) \leq W_0(n)$ for $n \geq 2$. To extend this result and see that this equality is strict note that, by Theorem 1, the

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event $\{R_1 < R_2 < \cdots < R_n\}$ has strictly positive probability $\frac{1}{n!}$. On this event the worst option is the last one, i.e., $j = n$. The prophet would thus skip option n, whereas the optimal strategy without the prophetic ability would be to stop with the last option (since $s_i < i$ for all $1 \leq i \leq n-1$ and $n \geq 2$). Thus, on this event the prophet would do strictly better. Hence $W_0(n-1) < W_0(n)$ for $n \geq 2$.

To verify the analytical results I have simulated the expected absolute rank for the optimal strategy for $1 \leq n \leq 30$. As a benchmark I have simulated the expected absolute rank for the naive strategy of choosing randomly one of the available options. The exact expected absolute rank for this strategy for the problem of n available options equals $\frac{1}{n}\sum_{i=1}^{n} i = \frac{n+1}{2}$. The results of these simulations are displayed in Figure 8. They confirm the analytical results.

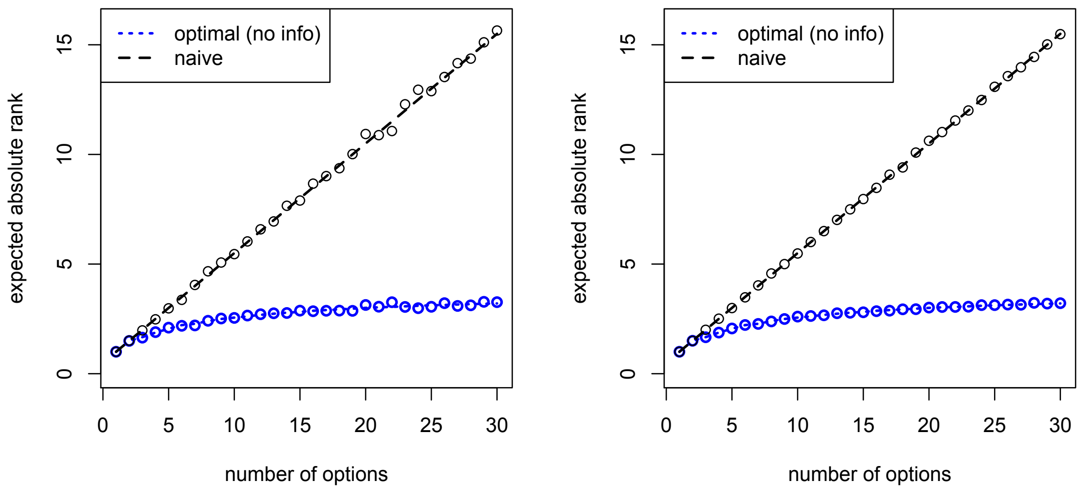

Figure 8: Simulated and exact expected absolute ranks for the optimal and the naive strategy in the expected-rank problem under no information. (Number of simulation runs $r = 10^3$ (left) and $r = 10^4$ (right); dots = simulated, dashed lines = exact)

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6 Apartment Hunting under Full-Information

This section studies the apartment hunting problem under assumption (4b), i.e., the decision maker sequentially observes the quality levels $(X_i)_{i=1}^n$ of the inspected apartments and knows the underlying distribution of these levels.

6.1 The Best-Choice Problem under Full-Information

Remark 17. The underlying optimal stopping problem was, according to Bruss (2005), first solved by Gilbert and Mosteller (1966) in the context of a game.

The objective of the decision maker in the best-choice problem is to maximize the probability of choosing the best apartment (i.e., assumption (6a)).

The payoff for renting apartment i, $1 \leq i \leq n$, after inspection is

$$Y_i = \mathbf{1}_{\{R_i=1\}} = \begin{cases} 1, & \text{if } R_i = 1, \\ 0, & \text{else.} \end{cases}$$

The expected payoff conditional on the observations $\mathcal{I}_i := \{X_i = x_1, \ldots, X_i = x_i\}$ is

$$y_i = y_i(x_1, \ldots, x_i) = \mathrm{E}\left[\, Y_i \mid \mathcal{I}_i \,\right] = \mathrm{P}\left(\{R_i = 1\} \mid \mathcal{I}_i\right) = \begin{cases} x_i^{n-i}, & \text{if } \hat{r}_i = 1, \\ 0, & \text{else.} \end{cases} \tag{37}$$

The last identity in (37) follows from the fact that an option that has relative rank 1 among the first i options has a probability of x_i^{n-i} to have rank 1 among all options. This is because the $(X_j)_{j=1}^n$ are i.i.d., with $\mathrm{P}\left(\{X_j \leq x_i\}\right) = x_i$, and there are $n - i$ options left.

The objective function is

$$\max_{N \in \mathcal{N}} \mathrm{P}\left(\{R_N = 1\}\right) = \max_{N \in \mathcal{N}} \sum_{i=1}^{n} \mathrm{E}\left[\, y_i \, \mathbf{1}_{\{N=i\}} \,\right] , \tag{38}$$

where $\mathcal{N}$ denotes the set of all stopping times w.r.t. the filtration generated by the quality levels $(X_j)_{j=1}^n$.

The solution follows Gilbert and Mosteller (1966). The first step is to determine the optimal strategy. Let $W_i \,|\, \mathcal{I}_i$, $0 \leq i \leq (n-1)$, denote the optimal probability of choosing the best apartment conditional on the observations $\mathcal{I}_i := \{X_i = x_1, \ldots, X_i = x_i\}$

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and under the restriction that the first $i \leq n-1$ options are rejected. By an analogous reasoning as in Section 5.1, the relation $W_i \,|\, \mathcal{I}_i \geq W_{i+1} \,|\, \mathcal{I}_{i+1}$ holds. Equation (37) implies that it is only optimal to stop with an option that has relative rank 1. By the principle of optimality, the decision to stop with option i if it has relative rank 1 is optimal iff $x_i^{n-i} \geq W_i \,|\, \mathcal{I}_i$. If

$$x_i^{n-i} = W_i \,|\, \mathcal{I}_i \,, \tag{39}$$

the decision maker is indifferent between renting apartment i and continuing with her search. To find the optimal strategy, the values of the x_i that solve equation (39) for every $0 \leq i \leq (n-1)$ have to be found, i.e., the value of x_i that makes the success probability for stopping with x_i the same as for rejecting x_i. If option i has relative rank 1 and $x_i^{n-1} = W_i \,|\, \mathcal{I}_i$, but the decision maker decides to continue, it is optimal to stop with the first option $l > i$ that satisfies $x_l \geq x_i$, since

$$x_l^{n-l} > x_i^{n-i} = W_i \,|\, \mathcal{I}_i \geq W_l \,|\, \mathcal{I}_l. \tag{40}$$

Let $j := n - i$, i.e., j denotes the number of remaining options. If there is exactly one option among the last j options that has a quality at least as high as x_i, the decision maker will choose it (since it is only optimal to stop with options that have relative rank 1). This happens with probability

$$\binom{j}{1} x_i^{j-1}(1 - x_i).$$

If there are exactly two such options, there is a probability of 0.5 that the decision maker will stop with the best (since the X_i are independent). The success probability is then

$$\frac{1}{2}\binom{j}{2} x_i^{j-2}(1 - x_i)^2.$$

Repeating this argument and summing up shows that the probability of success for rejecting option i if it has relative rank 1 and quality level x_i equals

$$\sum_{k=1}^{j} \frac{1}{k}\binom{j}{k} x_i^{j-k}(1 - x_i)^k. \tag{41}$$

Equating (41) with x_i^j (the success probability for stopping with option i) gives an

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equation whose root is the desired indifference quality level:

$$
\begin{aligned}
x^j &= \sum_{k=1}^{j} \frac{1}{k}\binom{j}{k} x^{j-k}(1-x)^k \\
\Longleftrightarrow 1 &= \sum_{k=1}^{j} \binom{j}{k}\frac{z^j}{k} \quad \text{with } z := \frac{1-x}{x}.
\end{aligned} \tag{42}
$$

In line with Gilbert and Mosteller (1966) the solution of (42) is denoted as b_j, where j denotes the number of remaining options, and the decision numbers d_i are defined by

$$
d_i := \begin{cases} 0\,, & \text{for } i = 0, \\ b_{n-i}\,, & \text{for } 1 \le i \le n-1. \end{cases}
$$

The numbers b_j only depend on j, the number of remaining options, and not on n, the total number of options. Relation (40) implies that the b_j are strictly monotonically increasing, and that the decision numbers d_i are strictly monotonically decreasing. Table 6.1 contains the values of b_j for $1 \le j \le 30$. The optimal strategy can be described by the vector $d := (d_1, \dots, d_n)$.

Theorem 5. *The optimal strategy N^*_{BC-FI} for the best-choice apartment hunting problem under full information, i.e. under assumptions (1), (2), (3), (4), (4b), (5) and (6a), is to stop with the first option $i\,, 1 \le i \le (n-1)$, that has relative rank 1 and that satisfies $x_i \ge d_i = b_{n-i}$, with b_{n-i} defined as solution of equation (42) for $j = n-i$. Formally,*

$$
N^*_{BC-FI} = \min\{1 \le i \le n \mid X_i = \max(X_1, \dots, X_i) \wedge X_i \ge d_i\}
$$

Proof. See above considerations. □

The next step is to determine the success probability under the optimal strategy. Let $1 \le i \le k \le n-1$. The probability of the event that option i has absolute rank 1 among the first k options, i.e., $X_i = \max(X_1, \dots, X_k)$, and that $X_i < d_i$ equals $\frac{d_i^r}{r}$. The probability of the event that option i has absolute rank 1 among all options, i.e., $X_i = \max(X_1, \dots, X_n)$, and that $X_i < d_i$ equals $\frac{d_i^n}{n}$. The difference

$$
\frac{d_i^r}{r} - \frac{d_i^n}{n} \tag{43}
$$

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Table 3: Optimal decision numbers b_j for the best-choice problem under full-information.

j	b_j	j	b_j	j	b_j	j	b_j	j	b_j	j	b_j
1	0.5000	6	0.8778	11	0.9305	16	0.9515	21	0.9627	26	0.9697
2	0.6899	7	0.8939	12	0.9361	17	0.9542	22	0.9644	27	0.9708
3	0.7758	8	0.9063	13	0.9408	18	0.9567	23	0.9659	28	0.9719
4	0.8246	9	0.9160	14	0.9448	19	0.9589	24	0.9673	29	0.9728
5	0.8559	10	0.9240	15	0.9484	20	0.9609	25	0.9686	30	0.9737

is the probability of the event that option i is the best option among the first k options, that $X_i < d_i$, and option i is not the best option among all choices. Since $X_i < d_i$, the decision maker would not choose option i under the optimal strategy on this event. Since the d_i are monotonically decreasing, she would neither choose one of the options before option i. Summing (43) over $1 \leq i \leq r$ thus yields the probability of the event that no choice is made among the first k options and that the best option is among the remaining $n-k$ options. Conditional on this event, the probability that option $k+1$ is the best option equals $\frac{1}{n-k}$. The probability that no choice is made among the first k options and that option $k+1$ is best overall equals

$$\frac{1}{n-r}\sum_{i=1}^{r}\left(\frac{d_i^r}{r}-\frac{d_i^n}{n}\right). \tag{44}$$

The probability that option $k+1$ is best overall and that the decision maker rejects it under the optimal strategy equals $\frac{d_{k+1}^n}{n}$. Hence, the probability under the optimal strategy that the decision maker chooses option $k+1$ and that this option is best overall equals

$$P(k+1) := \left(\frac{1}{n-r}\sum_{i=1}^{r}\left(\frac{d_i^r}{r}-\frac{d_i^n}{n}\right)\right)-\frac{d_{k+1}^n}{n}, \quad 1 \leq k \leq n-1. \tag{45}$$

The probability that the first option has absolute rank 1 equals $\frac{1}{n}$. The probability that the first option has absolute rank 1 and that it is rejected under the optimal strategy equals $\frac{d_1^n}{n}$. Hence, the probability that the decision maker chooses option 1 when it is the best overall equals

$$P(1) := \frac{1}{n}-\frac{d_1^n}{n}=\frac{1-d_1^n}{n}. \tag{46}$$

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The success probability under the optimal strategy N^*_{BC-FI} for the best-choice apartment hunting problem under full information is thus given by

$$\begin{aligned} \mathrm{P}\left(\{R_{N^*} = 1\}\right) &= \mathrm{P}\left(\text{"best apartment is chosen"}\right) \\ &= \sum_{k=0}^{n-1} \mathrm{P}\left(\text{"apartment } k+1 \text{ is chosen and it is best"}\right) \\ &= \sum_{k=0}^{n-1} P(k+1), \end{aligned} \tag{47}$$

where $P(k+1)$ is given by (45) and (46). Figure 9 shows the success probablity for $1 \leq n \leq 50$.

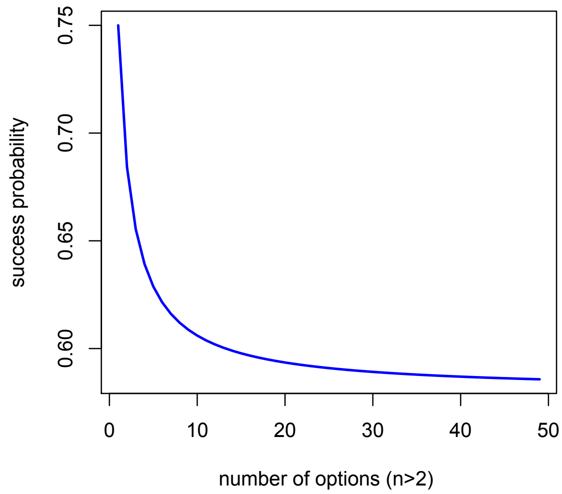

Figure 9: Success probability under the optimal strategy for the best-choice problem under full information.

By replacing sums with integrals in (45) and (47) and passing to the limit, Gilbert and Mosteller (1966) show that

$$\mathrm{P}\left(\left\{R_{N^*_{BC-FI}} = 1\right\}\right) \xrightarrow[n\to\infty]{} 0.580164\ldots. \tag{48}$$

This is also reflected in Figure 9 and Table 4.

Thus, asymptotically the success probability from using the optimal strategy improves from approx. $\frac{1}{e} \approx 0.368$ in the no-information case (see Equation (13)) to ≈ 0.580 in the full-information case, which is a relative improvement of approx. 58%.

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Table 4: Success probability under the optimal strategy $N^* = N^*_{BC-FI}$ for the best-choice problem under full information.

n	$\mathrm{P}(\{R_{N^*} = 1\})$	n	$\mathrm{P}(\{R_{N^*} = 1\})$	n	$\mathrm{P}(\{R_{N^*} = 1\})$
1	1.000	6	0.629	20	0.594
2	0.750	7	0.622	30	0.589
3	0.684	8	0.616	40	0.587
4	0.655	9	0.612	50	0.586
5	0.639	10	0.609	100	0.583

To verify the analytical results I have simulated the success probability for the optimal strategy under full information for $1 \leq n \leq 30$. As benchmarks I have simulated the success probabilities for the optimal strategy under no information and the naive strategy of choosing randomly one of the available options. The results are displayed in Figure 10.

Remark 18. Under the optimal strategy, the decision whether to stop or not with option $1 \leq i \leq n-1$ depends on the number of remaining options $n-i$, the quality level x_i of option i, and the relative rank of X_i. That is, $r_i = r_i(X_1, \ldots, X_i)$ is a sufficient summary statistic that contains all the relevant information about the previously rejected options (the history of the problem). Apart from the relative ranking, the quality levels $x_1, \ldots, x_{i-1}$ itself play no role for the optimal decision at stage i.

6.2 The Expected-Rank Problem under Full Information

Remark 19. The underlying optimal stopping problem is known as Robbin's problem (see Bruss (2005)). A general solution for it is not known. For a detailed discussion and an account of its history see Bruss (2005) and Bruss and Ferguson (1993).

The objective of the decision maker in the expected-rank problem is to minimize the expected absolute rank of the chosen apartment (i.e., assumption (6b)), with ranks defined as in Section 3.2. With this definition, the largest X_i gets assigned the smallest

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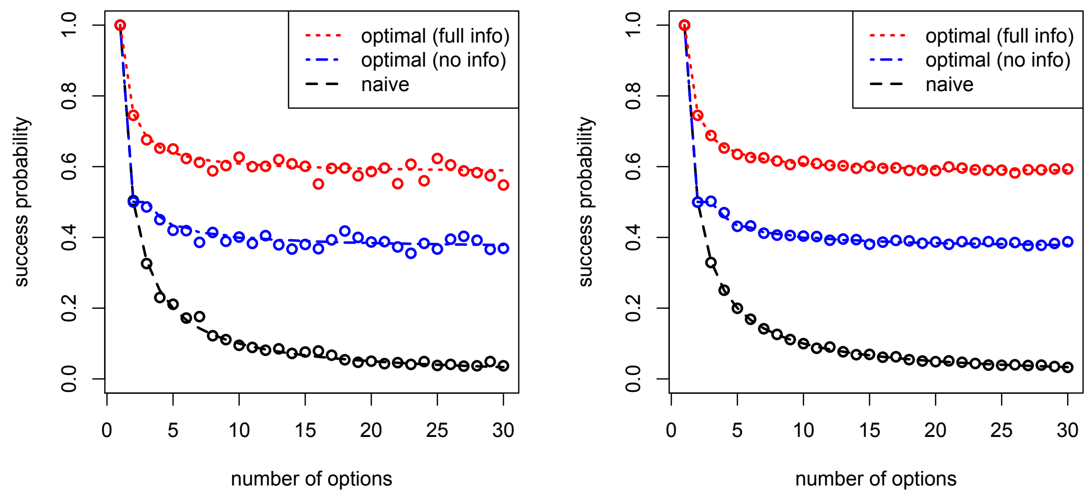

Figure 10: Simulated and exact success probabilities for the best-choice problem under the optimal strategies in the full- and no-information regime and under the naive strategy. (Number of simulation runs $r = 10^3$ (left) and $r = 10^4$ (right); dots = simulated, dashed lines = exact)

absolute rank $R_i = 1$. However, in this section the 'inverse' definitions of ranks, i.e.,

$$\widetilde{R}_k := \widetilde{R}_k(X_1, \ldots, X_n) := \sum_{i=1}^{n} \mathbf{1}_{\{X_i \leq X_k\}} \tag{49}$$

and

$$\widetilde{r}_k := \widetilde{r}_k(X_1, \ldots, X_k) := \sum_{i=1}^{k} \mathbf{1}_{\{X_i \leq X_k\}}, \tag{50}$$

where the largest X_i gets assigned the largest absolute rank $\widetilde{R}_i = n$, is more convenient (e.g., because of Relation (61)). This is also the definition of ranks used in the relevant literature (Bruss and Ferguson (1993), Assaf and Samuel-Cahn (1996)), and this convention will be adopted for this section. With absolute ranks defined by (49), the objective of the decision maker is to maximize the expected rank of the chosen apartment.

The payoff for renting apartment i, $1 \leq i \leq n$, after inspection is

$$Y_i = \widetilde{R}_i. \tag{51}$$

The expected payoff for renting apartment i conditional on the observations

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$\mathcal{I}_i := \{X_i = x_1, \ldots, X_i = x_i\}$ is

$$y_i = y_i(x_1, \ldots, x_i) = \mathrm{E}\left[\, \widetilde{R}_i \,\middle|\, \mathcal{I}_i \right] = \mathrm{E}\left[\widetilde{r}_i + \sum_{j=i+1}^{n} \mathbf{1}_{\{X_j \leq X_i\}} \,\middle|\, \mathcal{I}_i \right] = \widehat{\widetilde{r}}_i + (n-i)\, x_i. \tag{52}$$

The objective function is

$$\max_{N \in \mathcal{N}} \mathrm{E}\left[\widetilde{R}_N \right] = \max_{N \in \mathcal{N}} \sum_{i=1}^{n} \mathrm{E}\left[y_i \, \mathbf{1}_{\{N=i\}} \right], \tag{53}$$

where $\mathcal{N}$ denotes the set of all stopping times w.r.t. the filtration generated by the quality levels $(X_j)_{j=1}^n$.

As usual, using (51), (52) and (53), the problem may be solved by backward induction. However, this method turns out to be cumbersome and involved, even for small numbers of n. Using "a considerable amount of arithmetic", Assaf and Samuel-Cahn (1996) give the solution via backward induction for $n = 3$:

$$N^*_{ER-FI} = \min\{1 \leq k \leq 3 \mid 1 - X_k \leq p_k^{(3)}(X_1, \ldots, X_{k-1})\} \tag{54}$$

with

$$\begin{aligned} p_1^{(3)} &= \frac{1}{4}\left(5 - \sqrt{13}\right) \approx 0.3486. \\ p_2^{(3)}(x_1) &= \begin{cases} x_1, & \text{if } p_1^{(3)} \leq 1 - x_1 \leq \frac{2}{3}, \\ 1 - \frac{x_1}{2}, & \text{if } \frac{2}{3} \leq 1 - x_1 \leq 1. \end{cases} \qquad (55) \\ p_3^{(3)}(x_1, x_2) &\equiv 1. \qquad (56) \end{aligned}$$

The expected rank under this strategy for $n = 3$ equals ≈ 1.392.

The method of backward induction does not yield much intuition on a general solution for an arbitrary number of options n. A general solution would be an algorithm, that can be implemented numerically, that only takes the number n as input and returns the corresponding optimal strategy for the problem. For the previous problems in sections 5.1, 5.2 and 6.1, as shown there, such algorithms exists. For the problem at hand in this section no such algorithm is known yet.

Although a general solution is not available, the structure of the optimal strategy for arbitrary n is known (and is analogous to the structure of (54)).

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Theorem 6 (Proposition 7 in Bruss and Ferguson (1993))**.** *For all n, the optimal rule is a stepwise-monotone-increasing threshold function rule defined by the stopping time*

$$N^* := \min\{1 \leq k \leq n \mid 1 - X_k \leq p_k^{(n)}(X_1, \ldots, X_{k-1})\}, \tag{57}$$

where the threshold functions $p_k^{(n)}(.)$ satisfy

$$0 \leq p_k^{(n)}(X_1, \ldots, X_{k-1}) \leq p_{k+1}^{(n)}(X_1, \ldots, X_k) \quad a.s., \tag{58}$$

and

$$p_{n-1}^{(n)}(X_1, \ldots, X_{n-1}) < 1 \equiv p_n^{(n)}(.) \quad a.s.. \tag{59}$$

Proof. See Bruss and Ferguson (1993). □

Thus, under the optimal strategy, if it is optimal to stop with X_k at stage k, then it would also be optimal to stop with a larger value $\widetilde{X}_k > X_k$ at stage k. For each history $X_1, \ldots, X_{k-1}$ for which it is optimal to stop with X_k at stage k, it is optimal to stop with $X_{k+1} \geq X_k$ at stage $k+1$.

A general solution would require the knowledge of all threshold functions $p_k^{(n)}(.)$ for $1 \leq k \leq (n-1)$ and all $n \geq 1$. As Assaf and Samuel-Cahn (1996) put it: "Finding explicit expressions for $p_k^{(n)}(.)$ when $n > 3$ is a formidable task, and seems to lend little insight to the asymptotic value [of the problem]."

In the expected-rank problem under no-information (see Section 5.2), the optimal strategy, described by the vector $s = (s_1, \ldots, s_{n-1})$, is to stop with the first option i that satisfied $\widehat{r}_i \leq s_i$. The threshold ranks s_i only depend on i and n, but not on the history of previous observations $\widehat{r}_1, \ldots, \widehat{r}_{i-1}$. As discussed in Remark 18, all that is needed for deciding whether to stop is the relative rank $\widehat{r}_i$ of the option at hand, i.e., $\widehat{r}_i$ is a sufficient summary statistic of the history of the problem at stage i. A source of complexity in the full-information case is the fact that the threshold functions $p_k^{(n)}(.)$ depend on the entire history of the problem, i.e., on $X_1, \ldots, X_{k-1}$. As Bruss (2005) states: "Full history dependence is a strong result, actually proving that the problem is complex". This is made precise in the following theorem.

Theorem 7 (Proposition 8 in Bruss (2005))**.** *The optimal stopping time N^*, given by (57), which yields the optimal expected rank $\mathrm{E}\,[\,R_{N^*}\,]$, is fully history dependent in the*

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sense that, at each step $2 \leq k \leq n-1$*, the decision to accept or refuse* X_k *depends on all values of* $X_1, \ldots, X_k$*, and no nontrivial summary satistic of* $X_1, \ldots, X_k$ *is sufficient to achieve* $\mathrm{E}\left[\, R_{N^*} \,\right]$.

Proof. See Bruss and Ferguson (1993), Section 4.2. □

The proof of Theorem 7 is technical in nature, but Bruss (2005) summarizes the underlying ideas in a simpler style:

Suppose there are two decision makers. The first (male) has no prophetical abilities and behaves optimally (i.e., he follows N^*). The second (female) is a 'half-prophet' in the following sense: if she "enters" the problem at stage $1 \leq k \leq n-1$, which means that she observes the realizations of $X_1, \ldots, X_k$ but cannot choose one of the options 1 to $k-1$, then she can foresee the future realizations of $X_{k+1}, \ldots, X_n$ *conditional* on the decision that she rejects option k and goes on to stage $k+1$. Thus, both decision makers face the same expected absolute rank of stopping with option k, given by $y_k = y_k(x_1, \ldots, x_k) = \widehat{\widehat{r}}_k + (n-k)\, x_k$ (see (52)), since both have the same available information $\mathcal{I}_k = \{X_1 = x_1, \ldots, X_k = x_k\}$. However, to decide whether to stop or not with option k, *he* has to compute the expected payoff (the expected absolute rank) for continuing, W_k say. That is, he has to solve Robbin's problem at stage k. *She*, on the other hand, only has to compute the expected absolute rank of $\max(X_{k+1}, \ldots, X_n)$ conditional on $\mathcal{I}_k$, denoted by W_k^{HP}, since, if she decides to continue, her prophetic abilities reveal the future realization of $\max(X_{k+1}, \ldots, X_n)$, and she can stop precisely at the corresponding observation. Her expected conditional payoff is given by

$$W_k^{HP} = \int_0^1 \left(n - k + \sum_{j=1}^{k} \mathbf{1}_{\{x_j \leq s\}} \right) \mathrm{d}F^{(n-k)}(s) = 1 + \sum_{j=1}^{k} x_j^{(n-k)}, \qquad (60)$$

where $F^{(n-k)}$ is the c.d.f. of the maximum of $(n-k)$ i.i.d. $\mathcal{U}[0,1]$ random variables with c.d.f. F. By the principle of optimality, she stops if $y_k \geq W_k^{HP}$. Equation (60) shows that all values $x_1, \ldots, x_k$ are required to compute W_k^{HP}. For instance, to compute W_{k+1}^{HP} in the next stage, she would need again all values $x_1, \ldots, x_k$ plus the value x_{k+1}. Knowledge of just W_k^{HP} and x_{k+1} would not be enough, since W_{k+1}^{HP} cannot be computed from these two numbers. Thus, she always needs the entire previous history of observations for her optimal decisions.

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Given *her* prophetic abilities, *he* must stop whenever she stops under her optimal strategy, since he cannot do better than her (i.e., $W_k^{HP} \geq W_k \; \forall \, k$). The contrary is not true; even if *she* does not stop, *his* optimal strategy might force him to stop.

Where does now the full history dependence become visible? The trick is to look from stage k, $1 \leq k < n-1$, ahead to stage $n-1$. By the principle of optimality, for deciding whether to stop at stage k or not, *he* has to compute the expected absolute rank of continuing under optimal behavior at stages $k+1, \ldots, n-1$. He has to do so for all stages, since all of them are reached with a strictly positive probability (this follows from (58) and (59)), hence the behavior at every stage has a strictly positive weight in the expected absolute rank of continuing. So he has to think about the optimal behavior at stage $n-1$. At this stage, *he* has the same power as *her* if she entered the problem at stage $n-1$: there is only one option left, and if *he* decides to continue, he can also be sure to end up with this best remaining option. At stage $n-1$ the optimal strategies of her and him, if she enters the problem at this stage, coincide. Since she needs all the previous information from stages 1 to $n-2$ to make her decision at stage $n-1$, so does he. Thus, when *he*, at stage k, forms his exepectation conditional on $\mathcal{I}_k$ about his payoff at stage $n-1$ under optimal behavior, this payoff will depend on the values $x_1, \ldots, x_k$. Hence, his decision at stage k depends on these values, and the optimal strategy N^* is fully history dependent.

6.2.1 Approximative Solutions of Robbin's Problem

If an exact solution is not available, it is natural to try to approximate the problem under consideration by a similar problem whose solution is known. Trivially, the no-information case discussed in Section 5.2 yields an upper bound for the expected rank under optimal behavior via

$$\mathrm{E}\left[\widetilde{R}_{N^*}\right] = n + 1 - \mathrm{E}\left[R_{N^*}\right]$$

and an applicable strategy to attain this value, since knowledge of the quality levels X_i implies knowledge of the relative ranks r_i.

Another idea to obtain an approximation is to use the relation between X_i and $\widetilde{R}_i$, $1 \leq i \leq n$; indeed, these random variables are highly correlated for large values of n

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(see Bruss and Ferguson (1993)):

$$\operatorname{corr}\left(X_i\,,\,\widetilde{R}_i\right) = \sqrt{\frac{n-1}{n+1}} \xrightarrow[n\to\infty]{} 1. \tag{61}$$

Thus, at least for large values of n, the optimal strategy N_M^* that maximizes $\mathrm{E}\left[\,X_N\,\right]$ should also yield a good approximation for the problem at hand of maximizing $\mathrm{E}\left[\,\widetilde{R}_N\,\right]$, since by (61) there is approximately an affine relationship between X_N and $\widetilde{R}_N$ if n is large, for any stopping rule N. The problem of maximizing $\mathrm{E}\left[\,X_N\,\right]$ is known as Moser's problem (see Section 6.3), and the optimal strategy is a memoryless threshold rule of the form

$$N^* = \min\{1 \leq i \leq n \mid X_i \geq s_{n-i}\},$$

with constant thresholds $s_{n-i} \in \mathbb{R}$ that depend only on the number of remaining draws $n-i$, but not on the history of previous observations. Thus, although the maximizer of $\mathrm{E}\left[\,X_N\,\right]$ might be a good approximation for the maximizer of $\mathrm{E}\left[\,\widetilde{R}_N\,\right]$, they cannot coincide since the latter maximizer is history dependent by Theorem 7.

To assess the quality of approximating the expected-rank problem by Moser's problem I have run some simulations. The results are presented in Figure 11 and Figure 12. Figure 11 depicts the expected absolute rank $\mathrm{E}\left[\,R_{N_M^*}\,\right] = n+1-\mathrm{E}\left[\,\widetilde{R}_{N_M^*}\,\right]$ under the optimal strategy for Moser's problem (henceforth: the Moser strategy), and, as benchmark, the expected absolute rank $\mathrm{E}\left[\,R_{N_{ER-NI}^*}\,\right]$ under the optimal strategy for the expected-rank problem under no information. Figure 12 shows the relative improvement for the two strategies in terms of expected absolute rank. For n around 30, the Moser strategy yields an expected absolute rank of ≈ 2, which is approximately 35% below the expected abolute rank in the no-information case.

The attained absolute rank under the Moser strategy seems to stabilize around 2 for n getting larger. An intuitive explanation for this observation is the fact that

$$\mathrm{E}\left[\,X_{N_M^*}\,\right] \approx 1 - \frac{2}{n}. \tag{62}$$

for large n (see Equation (67)), and this is (see Bruss (2005)) in the order of the expectation of the second-largest order statistic[5] $X_{(n-1)}$ of $X_1, \ldots, X_n$, which is

$$\mathrm{E}\left[\,X_{(n-1)}\,\right] = \frac{n-1}{n+1} = 1 - \frac{2}{n+1}. \tag{63}$$

[5] The i-th order statistic $X_{(i)}$, $1 \leq i \leq n$, of an i.i.d. sample $X_i, \ldots, X_n$ with $X_1 \sim \mathcal{U}[0,1]$ has expectation $\mathrm{E}\left[\,X_{(i)}\,\right] = \frac{i}{n+1}$.

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To assess the situation for large n I have simulated the expected absolute ranks for the value $n = 100$ (5000 simulation runs). The Moser strategy yields an average absolute rank of 2.2178, which is also in line with (62) and (63) above. The optimal strategy for the no-information case yields an average absolute rank of 3.6888, which is in line with the corresponding asymptotically attained rank of 3.8695 (see the limit (36)). The gain in optimality of using the Moser strategy is thus $\approx 40\%$.

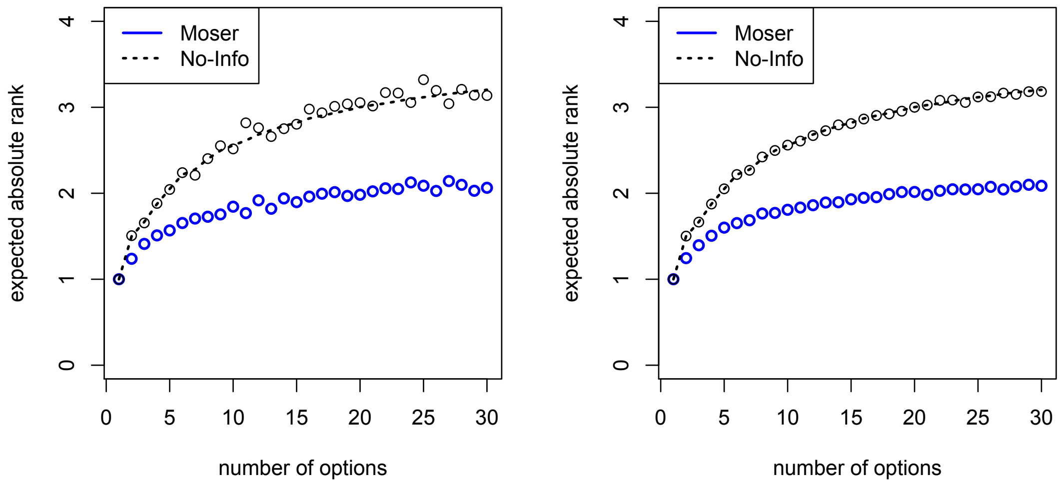

Figure 11: Simulated expected absolute ranks under the Moser strategy in the full-information case, and simulated and expected absolute ranks under the optimal strategy in the no-information case. (number of simulation runs $r = 10^3$ (left) and $r = 10^4$ (right); dots = simulated, dashed lines = exact)

6.3 A Digression on Moser's Problem

Moser (1956) studies the problem of finding

$$N_M^* = \arg\max_{N \in \mathcal{N}} \mathrm{E}\left[\, X_N \,\right],$$

where $X_i, \ldots, X_n$ are i.i.d. with $X_1 \sim \mathcal{U}[0,1]$ and $\mathcal{N}$ denotes the set of all finite stopping times w.r.t. the filtration $(\mathcal{F}_i)_{i=1}^n$ generated by $(X_i)_{i=1}^n$.

The solution can be found by backward induction (see Ferguson (2011)). With the notation of Remark 2 , $V_n = X_n$ and $V_i = \max\{X_i\,,\, \mathrm{E}\left[\, V_{i+1} \mid \mathcal{F}_i \,\right]\}$ for $1 \le i \le n-1$.

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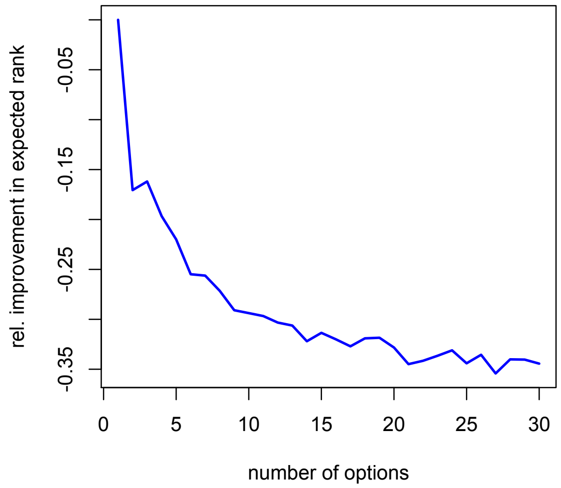

Figure 12: Relative improvement in absolute rank under the Moser strategy in the full-information case, compared to the optimal strategy for the no-information case.

The independence of $(X_i)_{i=1}^n$ implies that V_i, for $1 \le i \le n$, only depends on X_i, and that $s_{n-i} := \mathrm{E}\,[\,V_{i+1} \mid \mathcal{F}_i\,]$, for $1 \le i \le n-1$, is a constant that depends only on the number of remaining draws $n-i$. It is optimal to stop at stage $1 \le i \le n$ iff $X_i \ge s_{n-i}$. Hence, the optimal stopping rule is given by

$$N_M^* = \min\{1 \le i \le n \mid X_i \ge s_{n-i}\}, \tag{64}$$

where the thresholds s_i are given by $s_0 := 0$ and, recursively, for $0 \le i \le n-1$ by

$$\begin{aligned} s_{i+1} = \mathrm{E}\,[\,\max\{X_{n-i}\,,\,A_i\}\,] \;&=\; \int_0^{A_i} A_i\,\mathrm{d}x + \int_{A_i}^1 x\,\mathrm{d}x \\ &=\; \frac{1}{2}\left(A_i^2+1\right). \end{aligned} \tag{65}$$

The optimal strategy yields a payoff $\mathrm{E}\left[\,X_{N_M^*}\,\right] = s_n$. Since the optimal strategy only depends on the value of X_i at hand and the number of remaining draws, but not on the history $X_1, \ldots, X_{i-1}$, it is called a *memoryless* threshold rule.

Table contains the threshold values s_i for $1 \le i \le 30$. Figure 13 displays the expected payoff $\mathrm{E}\left[\,X_{N_M^*}\,\right]$ under the optimal strategy for $1 \le n \le 100$.

Moser (1956) shows that for large n

$$s_n \approx 1 - \frac{2}{n + \log(n) + c} \tag{66}$$

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for some constant $c > 0$. Gilbert and Mosteller (1966) state that $c \approx 1.768$. Thus, for large values of n, the expected payoff $\mathrm{E}\left[X_{N_M^*}\right]$ under the optimal strategy N^* satisfies

$$\mathrm{E}\left[X_{N_M^*}\right] \approx 1 - \frac{2}{n}. \tag{67}$$

This is also reflected in Figure 13.

Table 5: Optimal thresholds s_i for Moser's problem.

j	s_j	j	s_j	j	s_j	j	s_j	j	s_j	j	s_j
1	0.5000	6	0.8003	11	0.8707	16	0.9037	21	0.9230	26	0.9358
2	0.6250	7	0.8203	12	0.8790	17	0.9083	22	0.9260	27	0.9379
3	0.6953	8	0.8364	13	0.8864	18	0.9125	23	0.9287	28	0.9398
4	0.7417	9	0.8498	14	0.8928	19	0.9163	24	0.9313	29	0.9416
5	0.7750	10	0.8610	15	0.8985	20	0.9198	25	0.9336	30	0.9433

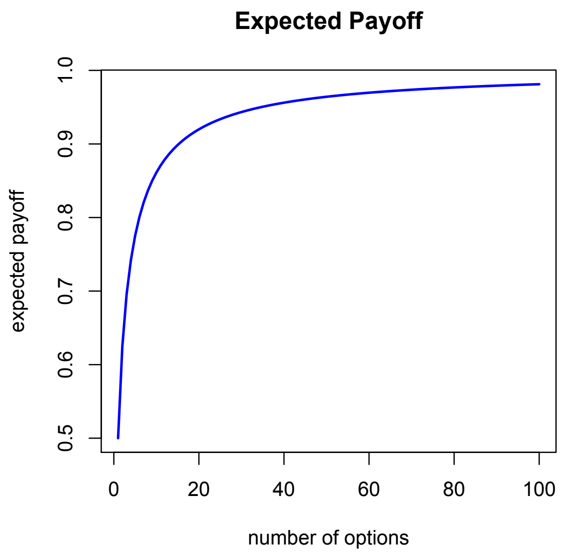

Figure 13: Expected payoff $\mathrm{E}\left[X_{N_M^*}\right]$ under the optimal strategy for Moser's problem.

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7 A Partial Information Setting

Gilbert and Mosteller (1966) note that in many statistical decision problems some but not all information about the underlying distribution is assumed to be known. For instance, the distribution might only be known up to an unknown parameter, which can be estimated by statistical inference.

In the apartment hunting problem it could be that the quality levels $(X_i)_{i=1}^n$ are observable, but the distribution $F = F(\theta)$ is only known up to a parameter θ. In this respect, the informational regimes describe by assumptions (4a), i.e., no information, and (4b), i.e., full information, can be interpreted as boundary cases of this partial information setting. Observing the X_i implies knowledge of the relative ranks, thus the optimal no-information strategy is applicable. The optimal full-information strategy on the other hand is not applicable, since F is unknown. But the X_i can be used to obtain an estimate $\widehat{\theta}$ of θ. The approximation $F(\widehat{\theta})$ can then be used along with the optimal full-information strategy. Intuitively, the outcomes associated with the optimal strategies for the no- and the full-information case should provide lower and upper bounds for the outcome of this procedure.

To study a simple partial information setting of the above type I use the following setup: Suppose $X_1, \ldots, X_n$ are i.i.d. with $X_i \sim \mathrm{N}(\mu, \sigma^2)$ and $\theta := (\mu, \sigma^2) = (0, 1)$, i.e., X_i has a standard Normal distribution. The decision maker can observe the quality levels X_i and knows that they come from a Normal distribution, but she does not know θ. Her objective is to maximize the probability of choosing the best option. A viable approach for her is to inspect and reject a number of options, k say, and use the observations of $X_1, \ldots, X_k$ to estimate θ (by the sample mean $\widehat{\mu}$ and the sample variance $\widehat{\sigma}^2$ of these k observations). For the remaining $n - k$ options she uses the optimal strategy for the full-information setting, based on $\widehat{\mu}$ and $\widehat{\sigma}^2$.

I have simulated this situation for $30 \leq n \leq 50$ and $k = \lfloor nf \rfloor$ for $f = \exp(-1)$ and $f = 0.2$, and for $r = 2500$ simulation runs. The choice $f = \exp(-1) \approx 0.37$ stems from the fact that this is the fraction of options that would asymptotically be discarded under the optimal strategy in the no-information case. On the other hand, in the full-information case the very first option might be chosen under the optimal strategy,

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which suggests to also try a value of f which is smaller than $\exp(-1)$, like $f = 0.2$. Figure 14 presents the results, together with the success probabilities of the optimal strategies for the no- and the full-information case as benchmarks. Surprisingly, the outcome for both choices of f is in the exact same range as for the optimal strategy under no information. This indicates that this strategy is a sensible choice even if the X_i, and not only their relative ranks, are observable but F is unknown. This also shows that the crucial component of assumption (4b), i.e., the full-information regime, is not the observability of the quality levels X_i, but knowledge of their underlying distribution F.

I have also simulated the above situation under the assumption that the decision maker does not use a fraction f of the available n options for inference; instead, she has access to another sample of size k, from the same underlying distribution, from which she cannot choose and which is only used for inference. For simplicity, I have chosen $k = n$, with $30 \leq n \leq 50$ and $r = 2500$ simulation runs. Figure 15 shows the results.

The resulting success probability in this setting has increased by ≈ 0.15, which again indicates the importance of an accurate knowledge about the underlying c.d.f. F for the optimality of the full-information strategy.

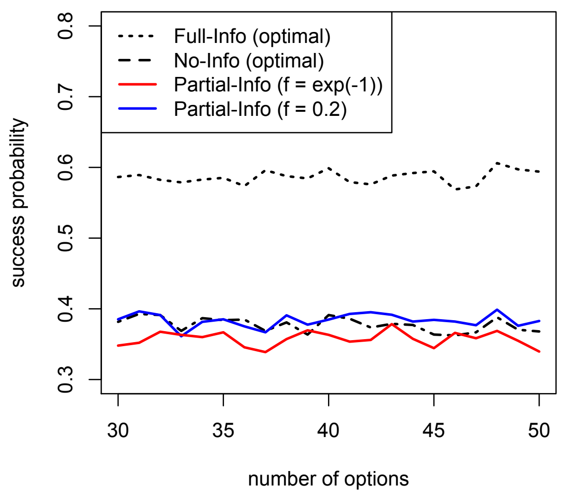

Figure 14: Simulated success probabilities for the partial information setting.

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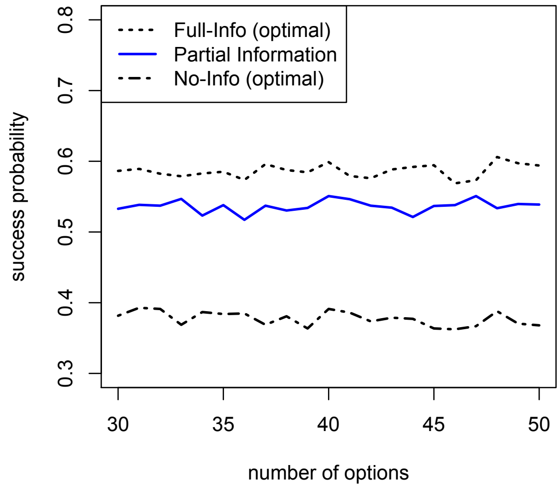

Figure 15: Simulated success probabilities for the partial information setting with an extra sample of size $k = n$ for parameter estimation.

8 Outlook

A crucial assumption in the model is that there is no recall of rejected options (i.e., assumption (5)). This might not hold in real-world situations. A potential extension of the model would be to introduce the possibility that the decision maker can go back, from stage k say, a certain number of stages, l say, and choose among the already inspected options $X_{k-l}, \ldots, X_k$; the current version of the model corresponds to $l = 0$. A variation of the CSP with a possibility of this kind was studied in Yang (1974).

It would also be interesting to study the optimal strategy under different distributions of a stochastic number of options, as studied in Section 5.1.2 for the case of a discrete uniform distribution.

A further analysis of partial-information settings, as indicated in Section 7, could also yield interesting results. One thing to do would be to analyse a non-parametric setting, in which F is completely unknown. A first attempt might be to use a fraction of the inspected options to estimate the empirical c.d.f. and use the optimal strategy for the full-information setting based on this estimate, analogously to the procedure employed in Section 7. It would be interesting to see whether the optimal strategy for the no-information case yields again a good performance.

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References

[1] Assaf David, Samuel-Cahn Ester (1996): The secretary problem: Minimizing the expected rank with i.i.d. random variables. Advanced in Applied Probability, Vol. 18, pp. 828-852.

[2] Bauer Heinz (2001): Measure and Integration Theory. 1st. Ed., Gruyter, Berlin.

[3] Bauer Heinz (1995): Probability Theory. 1st. Ed., Gruyter, Berlin.

[4] Broughton Allen, Huff B. W. (1977): A comment on unions of sigma-fields, American Mathematical Monthly, Vol. 84, No. 7, pp. 553-554.

[5] Bruss Franz T (2005): What is known about Robbin's Problem? Journal of Applied Probability, Vol. 42, pp. 108-120.

[6] Bruss Franz T. (2000): Sum the Odds to One and Stop. Annals of Probability, Vol. 28, No. 3, pp. 1384-1391.

[7] Bruss Franz T., Ferguson Thomas S. (1993): Minimizing the expected rank with full information. Journal of Applied Probability, Vol. 30, pp. 616-626.

[8] Chow Yuan-Shih, Moriguti S., Robbins H., Samuels S.M. (1964): Optimal selection based on relative rank (The "secretary problem"). Israel Journal of Mathematics, Vol. 2, pp. 81-90.

[9] Ferguson Thomas (2011): Optimal Stopping and Applications. http://www.math.ucla.edu/ tom/Stopping/Contents.html - accessed on: 21 March 2011.

[10] Gardener Martin (1969): Mathematical Games. Scientific American, Vol. 3, pp. 202-203.

[11] Gilbert John P., Mosteller Frederick (1966): Recognizing the maximum of a sequence. Journal of the American Statistical Association, Vol. 61, No. 313, pp. 35-73.

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[12] Lindley Dennis V. (1961): Dynamic programming and decision theory: Applied Statistcs, Vol. 10, pp. 39-51.

[13] Moser L. (1956): On a problem of Cayley. Scripta Mathematica, Vol. 22, pp. 289-292.

[14] Presman Ernst L., Sonin Isaac M. (1972): The best choice problem for a random number of objects. Theory of Probability and its Applications, Vol. 17, pp. 657-668.

[15] Rényi Alfred (1962): Théorie des éléments saillants d'une suite d'observations. Annales scientifiques de l'Université de Clermont-Ferrand 2, tome 8, série Mathématiques, No. 2, pp. 7-13.

[16] Smith M. H. (1975): A secretary problem with uncertain employment. Journal of Applied Probability, Vol. 12, pp. 620-624.

[17] Tamaki Mitsushi (2001): A note on odds-theorem. Departmental Bulletin Paper, Department of Business Administration, Aichi University, Vol. 1241, pp. 166-170.

[18] Yang M. C. K. (1974): Recognizing the maximum of a random sequence based on relative rank with backward solicitation. Journal of Applied Probability, Vol. 11, pp. 504-512.

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Appendix

A Proofs

Proof of Lemma 1. $X = Y \Leftrightarrow X - Y = 0$, hence

$$\{X = Y\} = \{X - Y \in \{0\}\}. \tag{68}$$

Using the definition of the image measure (cf. Definition 7.6 in Bauer (2001) and the product measure (cf. Definition 23.4 in Bauer (2001)), and the fact that independence of X and Y implies that the image measure $\mathrm{P}^{(X,Y)}$ of the vector (X, Y) is the product of the marginals, i.e., $\mathrm{P}^{(X,Y)} = \mathrm{P}^{X \otimes Y}$ (cf. Theorem 7.5 in Bauer (1995)), it holds for an arbitrary Borel set $B \in \mathcal{B}$ that

$$\begin{aligned} \mathrm{P}(\{X - Y \in B\}) &= \mathrm{P}^{X\otimes Y}\left(\{(x, y) \in \mathbb{R}^2 | x - y \in B\}\right) \\ &= \int_{\mathbb{R}} \mathrm{P}^Y(x - B)\, \mathrm{dP}^X(x). \end{aligned} \tag{69}$$

Using (68) and (69) (with $B = \{0\}$) yields

$$\mathrm{P}(\{X = Y\}) = \int_{\mathbb{R}} \mathrm{P}^Y(\{x\})\, \mathrm{dP}^X(x) = 0, \tag{70}$$

where the last identity follows from the continuity of F_Y, which implies that $\mathrm{P}^Y(\{x\}) = F(x) - F^-(x) = 0, \forall x \in \mathbb{R}$, where F^- denotes the left-side limit of F. □

Remark 20. The proof of lemma 1 only uses the continuity of F_Y (in the last step (70)), but not the continuity of F_X. Since the situation in the lemma is symmetric in X and Y, the continuity of only one of the c.d.f.s suffices for the lemma to hold.

Proof of Theorem 1. Fix $k \in \{1, \ldots, n\}$. Denote with $S(k)$ the set of all permutations of $\{1, \ldots, k\}$. It is well known that $S(k)$ has $k!$ elements. Let $\sigma \in S(k)$ and define the associated random vector $\mathbf{X}_\sigma := \left(X_{\sigma(1)}, \ldots, X_{\sigma(k)}\right)$, i.e., the components of $\mathbf{X}_\sigma$ are the random variables $X_1, \ldots, X_k$ ordered according to σ. Due to i.i.d-ness, for any $\sigma, \tau \in S(k)$ the vectors $\mathbf{X}_\sigma$ and $\mathbf{X}_\tau$ have the same distribution, i.e., the multivariate c.d.f.s F_σ and F_τ of $\mathbf{X}_\sigma$ and $\mathbf{X}_\tau$, respectively, coincide $\forall\, \sigma, \tau \in S(k)$. For $\sigma \in S(k)$ define the event $A_\sigma := \left\{X_{\sigma(1)} < X_{\sigma(2)} < \cdots < X_{\sigma(k)}\right\}$. Then $(A_\sigma)_{\sigma \in S(k)}$

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is a family of disjoint events, i.e., $A_\sigma \cap A_\tau = \emptyset$ for $\sigma \neq \tau$. Define furthermore the event $A_0 := \{\exists i \neq j : X_i = X_j\}$, and note that A_0 is disjoint from any $A_\sigma, \sigma \in S(k)$. Thus, the family $(A_\sigma)_{\sigma \in S(k)}$ together with A_0 is a partition of the sample space Ω. This implies that $\mathrm{P}(A_0) + \sum_{\sigma \in S(k)} \mathrm{P}(A_\sigma) = \mathrm{P}(\Omega) = 1$. Lemma 1, together with remark 5, implies that $\mathrm{P}(A_0) = 0$. Hence

$$\sum_{\sigma \in S(k)} \mathrm{P}(A_\sigma) = 1. \tag{71}$$

The equality of the c.d.f.s F_σ and F_τ for $\sigma, \tau \in S(n)$ implies (using the notation $\mathbf{z} := (z_1, \ldots, z_k) \in \mathbb{R}^k$)

$$\mathrm{P}(A_\sigma) = \int_{\mathbb{R}} \mathbf{1}_{\{z_1 < \cdots < z_k\}}(\mathbf{z}) \mathrm{d}F_\sigma(\mathbf{z}) = \mathrm{P}(A_\tau), \quad \forall \sigma, \tau \in S(k). \tag{72}$$

This symmetry, together with (71) and the fact that the cardinality of $S(k)$ equals $k!$, implies

$$\mathrm{P}(A_\sigma) = \frac{1}{k!}, \quad \forall \sigma \in S(n), \tag{73}$$

which proves part (a) of the theorem. Every event of the form $\{r_1 = \widehat{r}_1, \ldots, r_k = \widehat{r}_k\}$, with $1 \leq \widehat{r}_i \leq i$ for $1 \leq i \leq k$, coincides with an event A_σ for some $\sigma \in S(k)$. Therefore, using (73),

$$\mathrm{P}(\{r_1 = \widehat{r}_1, \ldots, r_k = \widehat{r}_k\}) = \frac{1}{k!} \tag{74}$$

for $1 \leq \widehat{r}_i \leq i$ for $1 \leq i \leq k$. The disjointness of the events A_σ implies the disjointness of the events $\{r_1 = \widehat{r}_1, \ldots, r_k = \widehat{r}_k\}$. Together with (74) this yields, for $1 \leq \widehat{r}_k \leq k$,

$$\mathrm{P}(\{r_k = \widehat{r}_k\}) = \sum_{\widehat{r}_1, \ldots, \widehat{r}_{k-1}} \mathrm{P}(\{r_1 = \widehat{r}_1, \ldots, r_k = \widehat{r}_k\}) = \frac{1}{k}, \tag{75}$$

where the summation is over all combinations of $\widehat{r}_1, \ldots, \widehat{r}_{k-1}$ satisfying $1 \leq \widehat{r}_i \leq i$ for $1 \leq i \leq k-1$, and the last equality follows becaue there are $(k-1)!$ such combinations. This establishes the discrete distribution of r_k on $\{1, \ldots, k\}$ for $1 \leq k \leq n$.

The independence of $r_1, \ldots, r_n$ follows from (74), for $k = n$, and (75), for $1 \leq k \leq n$, by

$$\mathrm{P}(\{r_1 = \widehat{r}_1, \ldots, r_n = \hat{r}_n\}) = \frac{1}{n!} = \prod_{i=1}^{n} \frac{1}{i} = \prod_{i=1}^{n} \mathrm{P}(\{r_i = \widehat{r}_i\}).$$

□