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Parameter estimation of stochastic volatility models

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Affidavit

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hereby declare

that I am the sole author of the present Master's Thesis,

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26 pages, bound, and that I have not used any source or tool other than those referenced or any other illicit aid or tool, and that I have not prior to this date submitted this Master's Thesis as an examination paper in any form in Austria or abroad.

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ABSTRACT

In the current paper the exponential Ornstein-Uhlenbeck stochastic volatility model and its parameter estimation are discussed. As for the latter, the assessment of quasi-maximum likelihood estimation based on the Kalman filter for this class of models is the key aspect of the current study. Thus, the description of the main methodological steps in order to get the state space formulation of the model is given. Moreover, precision of the estimation method for different parameters' scenarios is evaluated on simulated data. The obtained result suggests that for two settings the quasi-maximum likelihood estimation based on the Kalman filter fails to get precise estimates: first, if parameters are relatively high; second, if the noise term in the volatility driven stochastic process is too low. In all the others the estimation performs better. The same method was applied for the real data as well. Estimated parameters for the real data are not similar to the most problematic scenario; however, if one considers extension to the multifactor models, the scenarios for which estimates are the most inaccurate could be plausible.

I. INTRODUCTION

Stochastic volatility models are among the main approaches to describe asset price behavior that would be consistent with "stylized facts" of financial markets. It is concluded by many studies that financial time-series data often exhibits volatility clustering, fat-tails distribution and leverage effects.

Volatility clustering means that high fluctuations of asset returns are usually followed by further high fluctuations. That is, it is typical to observe grouping of the volatility fragments (Comte and Renault (1998)). An important conclusion of the empirical analysis of financial data is that the volatility of the asset price shows autocorrelation for periods exceeding one year. Another fact that justifies the statement about time-variant volatility is that empirical results provide evidence of the leverage effect, that is to say negative correlation between asset returns and future volatility (Masoliver et al. (2008)). Thus, empirical analyses show that volatility is not constant over time and needs to be included in the modeling of asset prices. Additionally, the distribution of the asset returns is not normally distributed because of the fatter tails providing evidence of a leptokurtic distribution (Taylor (1982)).

Described above stylized facts motivate the great widespread of stochastic volatility models in the analysis of asset returns. Continuous time models of this class are two-dimensional or higher-dimensional diffusion processes in which both the price dynamics and corresponding volatility term are represented as stochastic processes. Among related models the most prominent are: Hull and White (1987), Heston (1993), Barndorff-Nielsen and Shephard (2001), Scott (1987) and Stein and Stein (1991), etc.

A relatively new model, where the latent volatility is driven by the exponential Ornstein-Uhlenbeck (OU) process, has been advocated (Masoliver et al. (2008)). The authors derive the European call option price, which depends on the parameters of the underlying model. Hence, the estimation of the parameters is a crucial task for the later usage in the option price theory.

The main approaches for estimation of parameters of these models are generalized method of moments (GMM) (Melino and Turnbull (1990), Andersen and Sørensen (1996), Hoffmann (2002)), Gaussian quasi-maximum likelihood estimation for the state-space representation of the discrete time stochastic volatility model

(Harvey et al. (1994), Nelson (1988)) and the simulation based inference, specifically, Monte Carlo Markov Chain based on Bayesian framework (Kim et al.(1998), Broto and Ruiz (2004), Elerian et al. (2001) etc.). The second presented method is of great interest to the current study as it could be applied for the parameter estimation of the exponential OU process.

The key idea of this method proposed independently by Harvey et al. (1994) and Nelson (1988) is to obtain the state-space representation of a discrete stochastic volatility model in order to apply the Kalman filter recursion. Estimation is then based on maximum likelihood function. The state space representation can be obtained by linearizing initial model, namely, taking squares and then logarithm of asset return equation. However, the obtained linear system is different from the standard Gaussian state space form, because this transformation changes the error term, which is no longer distributed as normal random variable. Therefore, the quasi-maximum estimation is used (QML).

In the current study a similar approach is used for the model with the exponential OU process. Its evaluation is one of the main tasks of the paper. Therefore, the estimation will be done for simulated data of different parameters. The last step is to estimate the parameters for real data –stock index and exchange rates.

The paper consists of five sections. In section II the model specification is presented, including statistical properties of the exponential OU process. In section III the method of the parameter estimation is explained. It includes the discrete approximation of the continuous time model described in section II, and quasi-maximum likelihood estimation using the Kalman filter for state space models. In the section IV the empirical analysis, precisely, the assessment of the method for different possible realizations of parameters based on simulated data and parameter estimation for real data is performed. The fifth section presents conclusion.

II. MODEL SPECIFICATION

In the current study the exponential OU stochastic volatility model will be considered. The model specification is presented in Masoliver et al. (2008).

The mathematical formulation is:

$$\frac{dS(t)}{S(t)} = \mu dt + me^{Y(t)} dW_1(t), \tag{1a}$$

$$dY(t) = -\alpha Y(t)dt + kdW_2(t), \tag{1b}$$

where $S(t)$ is a asset price or exchange rate; μ, m, α, k – nonrandom real parameters; $dW_i(t)$ for $i = 1,2$ are Wiener processes. Parameters m, α, k are assumed to be positive. In the current study the Wiener processes are assumed to be uncorrelated for simplicity, which is equivalent to the no leverage effect condition. For the further analysis the key statistical properties of the model are presented.

As for the second equation, applying stochastic calculus and setting $Y(0) = Y_0$ leads to:

$$Y(t) = Y_0 e^{-\alpha(t-t_o)} + k \int_{t_0}^{t} e^{-\alpha(t-s)}\, dW_2(s). \tag{2}$$

Calculating the first and second moments for the Gaussian OU processes $Y(t)$ conditional on realization Y_0 results in:

$$E[Y(t)|Y_0] = \ Y_0 e^{-\alpha(t-t_o)}, \tag{3}$$

and

$$Var[Y(t)|Y_0] = \frac{k^2}{2\alpha}\left(1 - e^{-2\alpha(t-t_o)}\right). \tag{4}$$

If $Y(0)$ arises form the stationary process and $\alpha > 0$, the process Y(t) is stationary. In the following we assume that this is the case. For $(t - t_o) \to \infty$, the first two moments are:

$$E(Y(t)) = 0, \tag{5}$$

$$Var\big(Y(t)\big) = \frac{k^2}{2\alpha}\ . \tag{6}$$

The instantaneous volatility for the process S(t) is given as:

$$\sigma(t) = me^{Y(t)}. \tag{7}$$

Thus, the distribution of the volatility term can be characterized by presented below probability density function:

$$p(\sigma, t|\sigma_0, 0) = \frac{1}{\sigma\sqrt{2\pi \frac{k^2}{2\alpha} * (1 - e^{-2\alpha t})}} \exp\left(-\frac{\left[\ln\left(\frac{\sigma}{m}\right) - e^{-\alpha t}\ln\left(\frac{\sigma_0}{m}\right)\right]^2}{2\frac{k^2}{2\alpha} * (1 - e^{-2\alpha t})}\right). \tag{8}$$

Among four parameters of the model, three are of a great interest for current study, specifically, m, α, k. The reason of such a decision is their importance for option price theory based on the exponential OU model.

For instance, Perello J., Sircar R., Masoliver J. derive the approximate solution to the European call option price (Masoliver et al. (2008)) which is defined by parameters m, α, k in the notation of the model (1). That is why, it is important to have efficient method for their estimation. The approximation for the European call option price for exponential OU stochastic volatility is presented below:

$$C(S, T, z_o) = SN(d_1) - Ke^{-rT}N(d_2) + \left(\vartheta + \kappa + \frac{\vartheta^2}{2}\right)SN(d_1) + \frac{Ke^{-rT}}{\sqrt{\breve{m}^2 T}} \times$$

$$\times N'(d_2)\left[\frac{\kappa + \frac{\vartheta^2}{2}}{2\breve{m}^2 T} H_2\left(\frac{d_2}{\sqrt{2}}\right) - \frac{\kappa + \frac{\vartheta^2}{2}}{\sqrt{2\breve{m}^2 T}} H_2\left(\frac{d_2}{\sqrt{2}}\right) + \vartheta + \kappa + \frac{\vartheta^2}{2}\right]. \tag{9}$$

where r stands for constant risk-free interest rate (the model transformed to equivalent martingale measure), T is expiration day of the option; K is the strike price, $N(d)$ is the cumulative distribution function for a standard normal distribution and $N'(x) = dN(x)/dx$, $H_n(x)$ are the Hermite polynomials.

New variables introduced in the equation (9) are defined as following:

$$z_o = Y_o + \frac{k\Lambda_o}{\alpha + \Lambda_1 k}, \text{ and } \breve{m} = m * \exp\left(\frac{-k * \Lambda_o}{\alpha + \Lambda_1 k}\right), \tag{10, 11}$$

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\breve{m}^2}{2}\right)T}{\sqrt{\breve{m}^2 T}}, \text{ and } d_2 = \frac{\ln\left(\frac{S}{K}\right) + \left(r - \frac{\breve{m}^2}{2}\right)T}{\sqrt{\breve{m}^2 T}}, \tag{12}$$

where Λ_o and Λ_1 are the parameters of the function $\Lambda(Y)$, the market price of risk.

The expressions for ϑ, κ are:

$$\vartheta = \frac{z_o}{\lambda^2} * \left(1 - e^{-\breve{\alpha}t}\right), \tag{13}$$

$$\kappa = \frac{1}{2\lambda^4\nu^3}\left[\breve{\alpha}\, t + 0.5\left(1 - e^{-2\breve{\alpha}t}\right) - 2\left(1 - e^{-\breve{\alpha}t}\right)\right]\frac{\rho^2}{2\lambda^4\nu^3}\left(\breve{\alpha}t - 2\left(1 - e^{-\breve{\alpha}t}\right) + \breve{\alpha}te^{-\breve{\alpha}t}\right), \tag{14}$$

where $\lambda = k/\breve{m}$, $\breve{\alpha} = (\alpha + k\Lambda_1)$ and $\nu = \breve{\alpha}/k^2$ (Masoliver et al. (2008)).

The presented formula for the option price includes first two terms that are classic Black-Scholes option price. The difference between $C(S, T, z_o)$ and Black-Scholes price could be positive or negative depending on the moneyness region (S/K). For example, in the case of leverage effect for $S/K < 1$ the option is cheaper than Black-Scholes one and for $S/K > 1$ $C(S, T, z_o)$ is higher than Blach Sholes option price.

Thus, the presented formula for the option price indicates that the parameters m, α, k determine the option price of the underlying asset when it is generated by the exponential OU process. The next section includes the description of the method for their estimation used in the current study.

IV. Parameter estimation

In order to be able to implement method of quasi-maximum likelihood estimation based on the Kalman filter mentioned in section I, a discrete time approximation of the continuous time exponential OU model should be derived.

To get a discrete time approximation of the continuous stochastic process two approaches are considered: either to use the approximation directly for the differential stochastic equation or to discretize its solution (Lamberton and Lapeyre (1996)). It has been chosen to use the first one, where the differential equation for the asset prices (equation 1a) and stochastic process Y(t) (1b) will be approximated. The Euler scheme will be used in the paper as a simple and relatively reasonable approximation.

The general form for the stochastic differential equation with initial value $X(0)$ is defined as:

$$dX(t) = \alpha\big(t, X(t)\big)dt + \sigma\big(t, X(t)\big)dW(t), \tag{15}$$

where W is standard Brownian motion, α and σ are functions of time and underlying process X(t), respectively its drift and standard deviation. The simple Euler discretization for the general form is given by (Bruti-Liberati and Platen (2007)):

$$X_{n+1} = X_n + \alpha(t_n, X_n)(t_{n+1} - t_n) + \sigma(t_n, X_n)\Delta W_n. \quad (16)$$

The convergence in probability property is proven for such a process, its limit is continuous – time stochastic differential equation, the sketch of the proof could be found in Lindner (2009).

Applying this approximation to the continuous time exponential OU process yields:

$$\frac{S_{n+1} - S_n}{S_n} = \mu\Delta t_n + mexp(Y_n)\Delta W_{1_n}, \quad (17)$$

$$Y_{n+1} = Y_n - \alpha Y_n \Delta t_n + k\Delta W_{2_n}, \quad (18)$$

where $\Delta t_n = t_{n+1} - t_n$ is the step width with $n = 0,1,2,\dots,N$ and $\Delta W_n = W(t_{n+1}) - W(t_n)$ stands for difference of Wiener processes. The definition of Wiener process is that its increments distributed independently and normally with magnitude of order $\Delta t_n{}^{1/2}$; therefore, $\Delta W_n \sim N(0, \Delta t_n)$. An efficient way to evaluate the increments of the Wiener process ΔW_n is to consider $\Delta W_n = \sqrt{\Delta t_n} I_n$, where $I_n \sim N(0,1)$. Equations (17) and (18) are discrete approximations of the continuous time stochastic differential equations.

Applying parameterization as $\beta = \mu\Delta t_n$, $\psi = (1 - \alpha\Delta t_n)$ and $\varepsilon_1 = \Delta W_{1_n}$, $\varepsilon_2 = \Delta W_{2_n}$ the transformed equations are following:

$$\frac{S_{n+1} - S_n}{S_n} = \beta + m * exp(Y_n)\varepsilon_1, \quad (19)$$

$$Y_{n+1} = \psi Y_n + k\varepsilon_2. \quad (20)$$

where ε_1 and ε_2 are white noises that are distributed as $N(0, \Delta t_n)$.

In order to be able to estimate the parameters by using standard machinery for the Kalman filter, the system (19)-(20) should be transformed into state-space form. It could be obtained by taking logarithm of the squared equation (19). Setting $H_n = \frac{S_{n+1} - S_n}{S_n} - \beta$, equation (19) becomes:

$$\log(H_n^2) = \log(m^2) + 2Y_{n-1} + \log({\varepsilon_1}_n^2). \tag{21}$$

The noise component in (21) is no longer distributed according to normal distribution but as $\log(\chi^2)$. The mean and variances of this random variable are -1.270 and 4.934 (Abramowitz and Stegun (1970)). Adding expectation of $\log({\varepsilon_1}_n^2)$ to (21), the model is presented in linear form:

$$\log(H_t^2) = \log(m^2) - 1.27 + 2Y_{n-1} + \zeta_t, \tag{22}$$

$$Y_{n+1} = \psi Y_n + \eta_t. \tag{23}$$

The noise term ζ_t after adding the expectation to the observation equation has zero mean. The noise term is distributed as normal random variable with zero mean and variance Δk^2.

The linear version of the discretized continuous time exponential OU process could be formulated in the state space form; therefore, it makes it possible to apply the Kalman filter. The parameters could be estimated by maximum-likelihood estimation.

However, instead of considering the true distribution of the noise term in (22), we will focus on its approximation according to a Gaussian distribution. The method is known as quasi-maximum likelihood estimation.

Its main idea is to approximate one distribution with another. Assuming that the underlying true distribution has finite first four moments and parameters are not on the boundaries of parameter space, the method yields asymptotically normal and consistent estimators; however, they are no longer efficient (Harvey and Shephard (1996)). Therefore, despite the fact that the asymptotic result holds, the estimator could have poor small sample properties depending on how well chosen distribution approximates the true one (Kim et al. (1998)).

Denote the parameter vector as θ, if both assumptions described above hold, then the asymptotic result is formulated as:

$$\sqrt{T}\left(\theta - \hat{\theta}\right) \xrightarrow{d} N(0, E^{-1}DE^{-1}), \tag{24}$$

where $\frac{1}{T} * \frac{\partial^2 \log L}{\partial\theta\,\partial\theta'} \xrightarrow{p} E$, and $\frac{1}{T} * \frac{\partial \log L}{\partial\theta} \xrightarrow{d} N(0, D)$; L denote likelihood function (Harvey and Shephard (1996)). Information matrix (the inverse of the variance-covariance matrix) will be approximated numerically in the current analysis.

All first four moments of the noise term ζ_t are finite, the parameters values are assumed to belong to the interior of restricted parameter set; therefore, the estimators of the parameters in the discrete time exponential OU process are consistent and asymptotically normal.

The discussion of this method for the parameter estimation of a stochastic volatility model is still the topic of concern for many researchers; specifically, the accuracy of this method is controversial. For example, there is evidence that performance of quasi-maximum likelihood estimation for stochastic volatility models gets more biased in the case where the variance of stochastic process is small (Jacquier et al. (1994)). Thus, analysis of the precision of the method for exponential OU process is the task for the next sections.

In the section V the estimation of the parameters for the simulated and real data will be presented based on which the efficiency of the method will be discussed.

V. Empirical analysis

In order to resolve opened questions of the previous section the Kalman recursion is done for the state space representation of the model (22-23) and based on it maximum likelihood estimation is performed. The important part of the current research is to investigate for which parameters' realizations of the model the method works relatively imprecisely and for which possible realizations its performance improves. Thus, the estimation is applied to the simulated data. In the final step of analysis the estimates of the parameters for four real data time-series (exchange rates and stock index) are obtained.

The state-space representation of the derived in subchapter IV discrete-time linear model is:

$$\log(H_n) = \log(\widetilde{m}^2) - 1.27 + [0 \;\; 2] * \begin{bmatrix} Y_n \\ Y_{n-1} \end{bmatrix} + \zeta_t, \tag{25}$$

$$\begin{bmatrix} Y_{n+1} \\ Y_n \end{bmatrix} = \begin{bmatrix} \tilde{\psi} & 0 \\ 1 & 0 \end{bmatrix} * \begin{bmatrix} Y_n \\ Y_{n-1} \end{bmatrix} + \begin{bmatrix} \eta_t \\ 0 \end{bmatrix}. \quad (26)$$

The first equation will be referred to as observation equation and the second one as state. The variance-covariance matrix for the noise term in state equation is given as:

$$E\left(\begin{bmatrix} \eta_t \\ 0 \end{bmatrix} * \begin{bmatrix} \eta_t & 0 \end{bmatrix}\right) = \begin{pmatrix} \tilde{k}^2 & 0 \\ 0 & 0 \end{pmatrix}. \quad (27)$$

The parameters presented above are the initial parameters scaled by the step width. In the notation of the continuous-time model (1) the transformed parameters are $\tilde{\psi} = 1 - \alpha * \Delta \mathrm{t_n}$; $\tilde{k} = k * \sqrt{\Delta \mathrm{t_n}}$; $\tilde{m} = m * \sqrt{\Delta \mathrm{t_n}}$.

The step width ($\Delta \mathrm{t_n}$) is chosen to be 0.004 (which is equal to 1 divided by 250 - one year over the average number of trading days in a year).

The presented state-space model includes a singular matrix in the state equation; therefore, it is impossible to derive analytically the initial value of mean squared errors in the Kalman recursion. The common way to resolve this problem is to use the positive definite matrix (diagonal) as the confidence of the researcher in her guess of initial state vector. Then a diagonal element summarizes how big error of initial guess is: big value means less confidence about the guess (Hamilton (1994)). In the current research the initial value of the state vector is chosen to be a zero vector. The initial matrix of mean squared errors is set to be a diagonal matrix with entries of 10.

The conclusion about performance of the method will be made for three possible scenarios– low, middle and high parameters' values. As the equivalent numerical realization, we choose the values 0.1, 0.5 and 0.9.

Thus, for different combinations of possible scenarios for the model (19-20) 100 simulations have been done for time span equals to 1000 observations. The estimate for the variable H_n in the notation of the equation (22) is demeaned returns. This step is common in the literature, which includes the linearization of the volatility term (Harvey (1994), Machieu (1998)).

As for the numerical aspects of the analysis, the algorithm coded in Matlab relies on the constraint optimization, because the parameters $\tilde{m}, \tilde{k}$ are assumed to be positive in initial model and absolute value of $\tilde{\psi}$ should be smaller than 1 as the process $Y(t)$ is stationary if parameter α is positive. The initial starting values for the optimization routine are chosen to be true parameters.

Obtained result is summarized in Tables 1-3 in which true parameters, mean of the estimates for 100 simulations and standard deviation from the true value of parameter are presented.

The obtained estimates indicate that the method works quite precisely except for some scenarios. The worst estimate is the estimate for parameter $\tilde{m}$ in the case where all three parameters are relatively high. Its deviation for the mean from the true one is 1.9948 and the standard deviation is around 3.3 (see Table 1, scenario 1). Therefore, in the case where all three parameters are relatively high (scenario 1-2, Table1) estimate for $\tilde{m}$ is inaccurate. Its value is more that three times bigger that the underlying true one. In the other settings estimated value for $\tilde{m}$ is close to the true one with the smallest standard deviation among all parameters (see Table 2, Table 3).

Also the least precise estimate for $\tilde{k}$ corresponds to the case with all high values of the parameters (scenario 1-2, Table1) as the standard deviation is of almost the same value as the mean of estimates. However, precision is relatively high for the parameter $\tilde{\psi}$.

Table 1. Result for relatively less accurate parameter estimation

Scenario	Parameters	True parameters	Estimated parameters	Standard deviation
1	$\tilde{m}$	0.9	2.8948	3.3038
	$\tilde{\psi}$	0.9	0.9038	0.0254
	$\tilde{k}$	0.9	0.4710	0.4725
2	$\tilde{m}$	0.5	1.6082	1.8355
	$\tilde{\psi}$	0.9	0.9038	0.0254
	$\tilde{k}$	0.9	0.4710	0.4725
3	$\tilde{m}$	0.9	0.9019	0.0331
	$\tilde{\psi}$	0.1	-0.0718	0.4593
	$\tilde{k}$	0.1	0.1291	0.1568
4	$\tilde{m}$	0.5	0.5010	0.0184
	$\tilde{\psi}$	0.1	-0.0718	0.4593
	$\tilde{k}$	0.1	0.1291	0.1568
5	$\tilde{m}$	0.1	0.1002	0.0037
	$\tilde{\psi}$	0.1	-0.0796	0.4541

	$\tilde{k}$	0.1	0.1278	0.1571

Thus, high values of all three parameters are most problematic for estimation. In order to find the possible explanation of such result, the possibility of higher than in other cases computational imprecision in optimization routine should be checked. Changing initial values and using different algorithm have not influenced on the estimated values. Hence, the weak identification is the most reasonable explanation of the biases in the estimation. Indeed, if compare the numerically approximated empirical information matrix for the estimates when the true vector of parameters equal to (0.9,0.9,0.9) with one for the case of relatively accurate estimation (see scenario 1, Table 3), second derivative with respect to parameter $\tilde{m}$ is 46,7 in comparison with 52234.

As for the last three scenarios in the Table 1, it coincides with the found empirical evidence of the drawbacks of quasi-maximum estimation for stochastic volatility models with the small variance of noise term for volatility process (Jacquier et al. (1994)). For the exponential OU stochastic volatility model in this setting where parameter $\tilde{k}$ is low, the estimated parameter $\tilde{\psi}$ is not just much lower than the true one but it also flips the sign. Additional information about the empirical analysis of the parameter estimation in the cases with $\tilde{k} = 0.1$ is summarized in Table 2.

Table 2. Parameter estimation for scenarios with $\tilde{k} = 0.1$.

Scenario	Parameters	True parameters	Estimated parameters	Standard deviation
1	$\tilde{m}$	0.9	0.8996	0.0451
	$\tilde{\psi}$	0.9	0.6683	0.4453
	$\tilde{k}$	0.1	0.1410	0.1258
2	$\tilde{m}$	0.1	0.0999	0.0050
	$\tilde{\psi}$	0.9	0.6558	0.4526
	$\tilde{k}$	0.1	0.1410	0.1258
3	$\tilde{m}$	0.5	0.4997	0.0248
	$\tilde{\psi}$	0.9	0.6076	0.5191
	$\tilde{k}$	0.1	0.1424	0.1291
4	$\tilde{m}$	0.9	0.9019	0.0337
	$\tilde{\psi}$	0.5	0.0687	0.5997
	$\tilde{k}$	0.1	0.1379	0.1577

5	$\tilde{m}$	0.1	0.1002	0.0037
	$\tilde{\psi}$	0.5	0.0762	0.5927
	$\tilde{k}$	0.1	0.1369	0.1579

Except for result presented in Table 1-2, the method performs relatively well (see Table 3).

Table 3. Estimation for the set of parameters for which method performs the most precisely:

Scenario	Parameters	True parameters	Estimated parameters	Standard deviation
1	$\tilde{m}$	0.1	0.1011	0.0050
	$\tilde{\psi}$	0.1	0.0882	0.0950
	$\tilde{k}$	0.9	0.8740	0.0716
2	$\tilde{m}$	0.5	0.4996	0.0246
	$\tilde{\psi}$	0.5	0.4637	0.1589
	$\tilde{k}$	0.5	0.5053	0.1188
3	$\tilde{m}$	0.1	0.0999	0.0049
	$\tilde{\psi}$	0.5	0.4637	0.1589
	$\tilde{k}$	0.5	0.5053	0.1188
4	$\tilde{m}$	0.5	0.5053	0.0250
	$\tilde{\psi}$	0.1	0.0882	0.0950
	$\tilde{k}$	0.9	0.8740	0.0716
5	$\tilde{m}$	0.9	0.9341	0.1564
	$\tilde{\psi}$	0.9	0.8878	0.0298
	$\tilde{k}$	0.5	0.4799	0.0647
6	$\tilde{m}$	0.1	0.1038	0.0174
	$\tilde{\psi}$	0.9	0.8878	0.0298
	$\tilde{k}$	0.5	0.4799	0.0647

Table 3 indicates that the precision of the method is significantly better for the presented scenarios. The best estimation is for the case with high value of noise term and relatively low autoregressive coefficient $\tilde{\psi}$ and scaling factor of the asset price volatility $\tilde{m}$.

To conclude, the worst two scenarios that cause problems in identifying parameters are: if all parameters have high values and if the noise term for the state equation is low. The last one corresponds to the low variance for the process that governs asset price volatility. However, despite the enumerated cases the evaluation of the quasi-maximum likelihood for exponential OU stochastic volatility model is ambiguous due to the clear empirical evidence of relatively good precision for some of parameter scenarios (presented in Table 3).

The final step of the empirical part of the current study is to estimate parameters of the exponential OU process for the real data. Three exchange rate returns and one stock index were chosen. The data is taken from Bloomberg stream: daily price for Euro to USD, JPY to USD, GBP to USD and S&P500 in the period from 1990 till 2011 (for Euro from 2000). Summary statistics and plots are presented below.

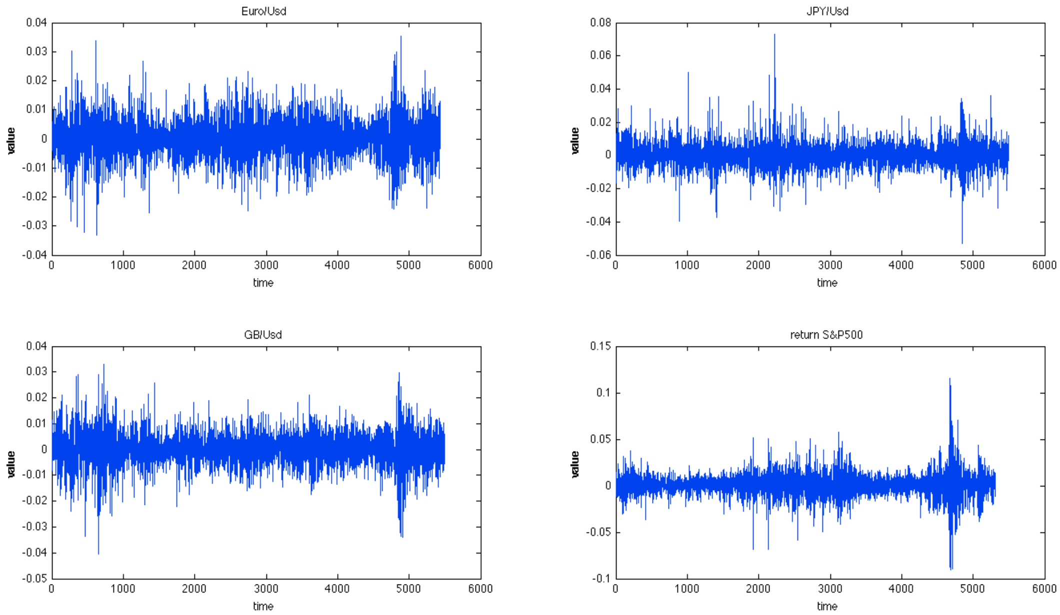

Figure 1. The daily return of exchange rates EURO/USD, JPY/USD, GB/USD and stock index S&P 500 from 1990-2011.

To investigate the volatility clustering of the data, the autocorrelation function of absolute value and squared returns might be used (Figure 2 and Figure 3).

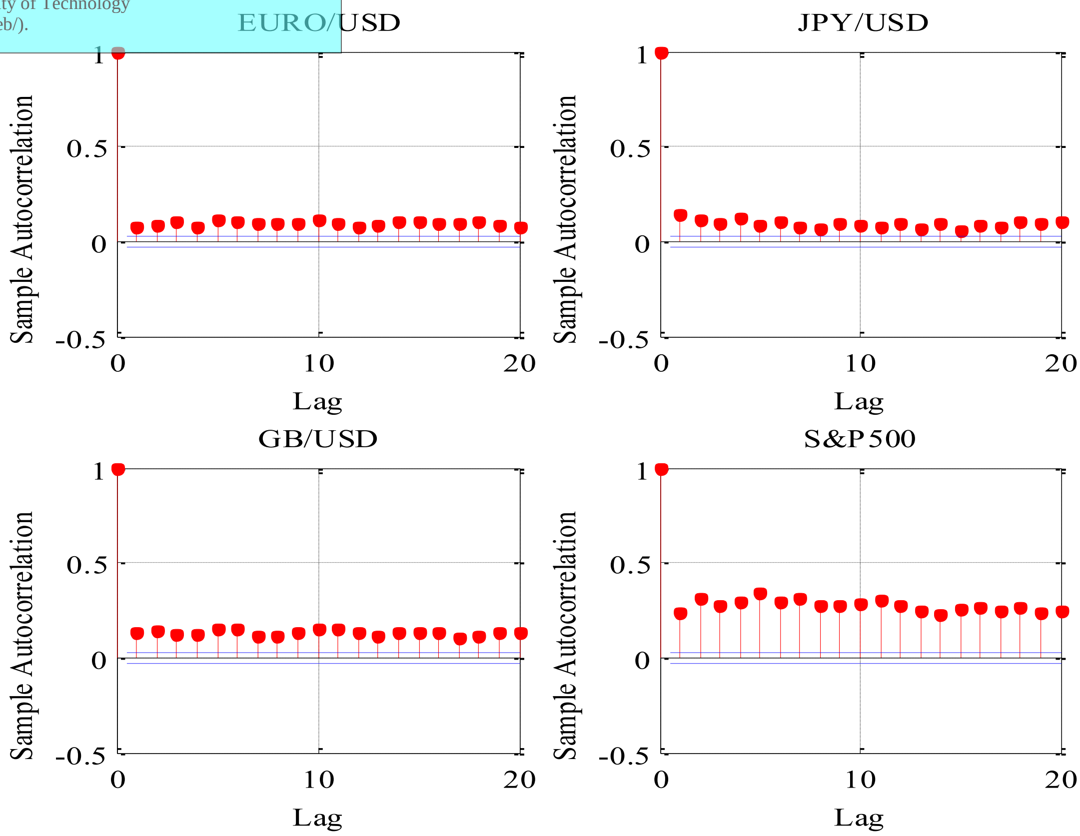

Figure 2. Autocorrelation function of absolute value of return for empirical data.

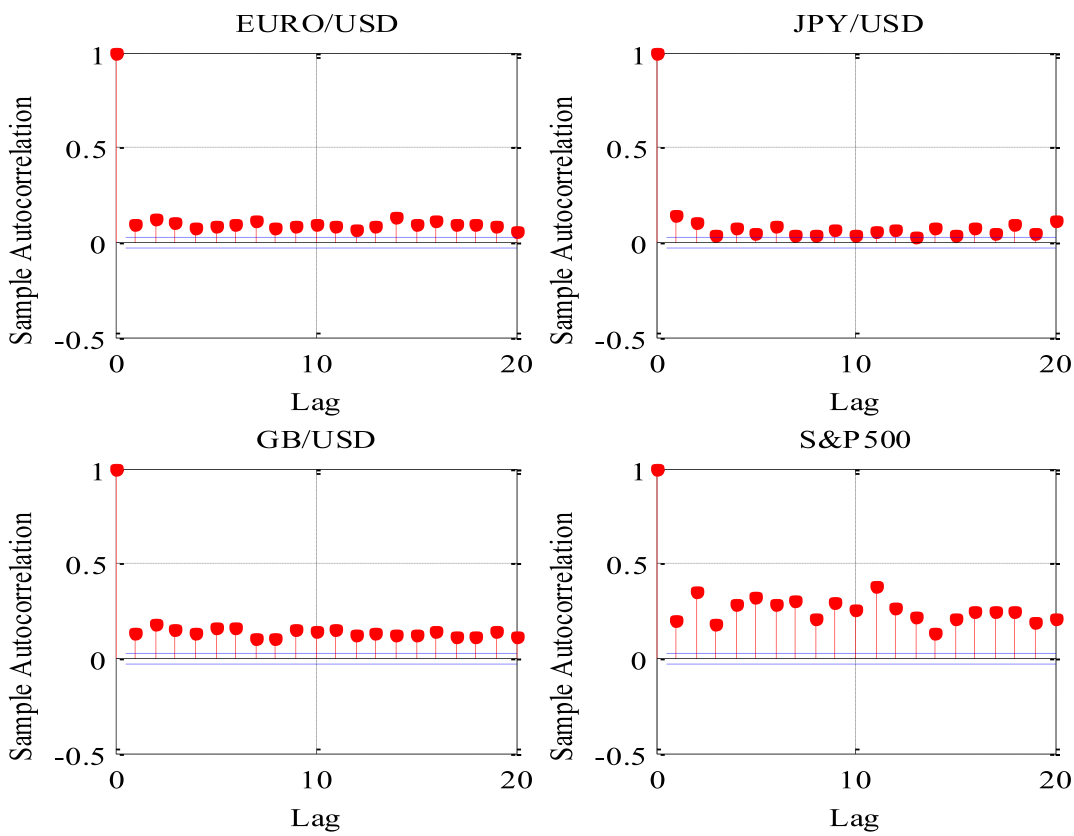

Figure 2. Autocorrelation function of squared return for empirical data.

Figure 2 and Figure 3 indicate the evidence of volatility clustering, especially for the S&P 500 index.

Descriptive statistics are presented below for transformed return data into H_n.

Table 4. Descriptive statistic for daily log squared demeaned return for the exchange rates EURO/USD, JPY/USD, GB/USD and asset stock index S&P 500.

Statistic	Euro/USD	JPY/USD	GBP/USD	S&P500
mean	-11.6507	-11.5672	-11.8645	-10.8468
variance	5.5518	5.7630	5.7634	6.3654
skewness	-1.1283	-1.2665	-1.1241	-1.1642
kurtosis	4.7639	5.7392	4.8215	6.1039
min	-23.0148	-25.4514	-21.2267	-29.0901
max	-6.6930	-5.2351	-6.4128	-4.3174

The sample statistics indicate negative skewness and kurtosis that are higher than normal, which means that distributions of the underlying transformed asset returns are leptokurtic. Higher kurtosis indicates heavier tails of the distribution; more share of variance is due to infrequent extreme deviations. The reason for negative skewness is logarithm transformation of the data. The sample variance for all time-serieses is higher that for the log of chi squared (4.934), which could indicate violation of the error term being normally distributed in the initial nonlinear discrete time model.

The parameters of the exponential OU process were estimated for the presented four time series and given in Table 5. The estimates are:

Table 5. The estimated parameters of the exponential OU stochastic volatility model for the real data:

	Euro/USD	JPY/USD	GBP/USD	S&P500
$\tilde{m}$	0.0056	0.0058	0.0050	0.0083
$\tilde{\psi}$	0.9920	0.9751	0.9923	0.9949
$\tilde{k}$	0.0366	0.0668	0.0391	0.0492

The smoothed estimate for volatility of each time series presented below (Figure 4, Figure 5, Figure 6 and Figure 7).

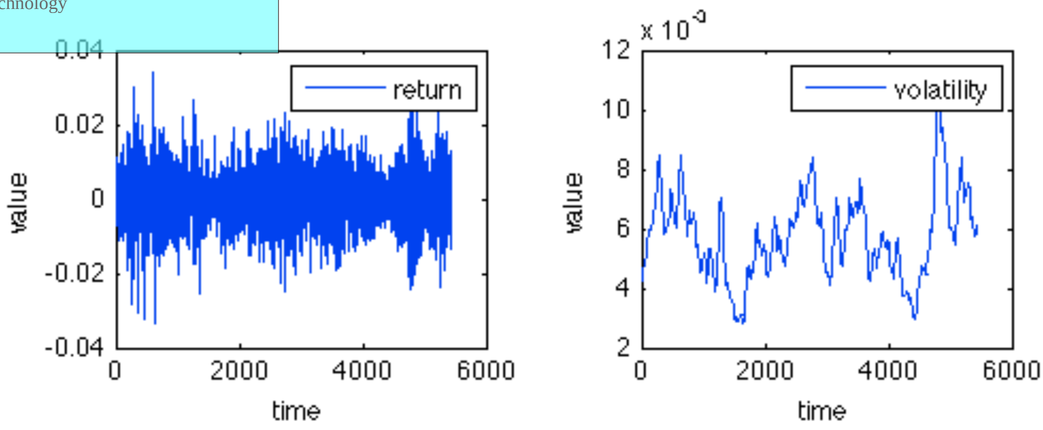

Figure 4. Volatility and return of EURO/USD exchange rate.

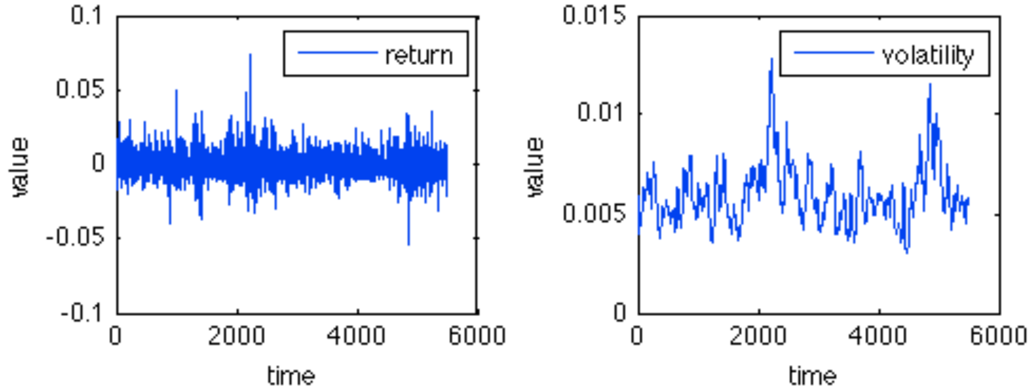

Figure 5. Volatility and return of JPY/USD exchange rate.

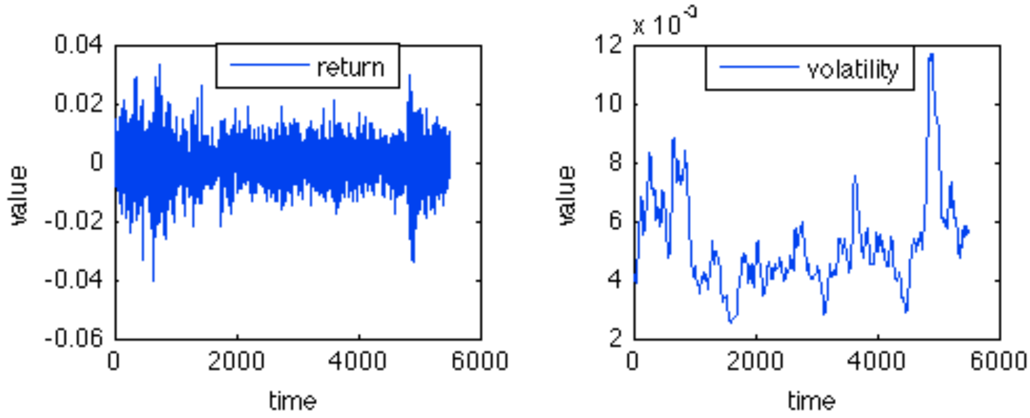

Figure 6. Volatility and return of GB/USD exchange rate.

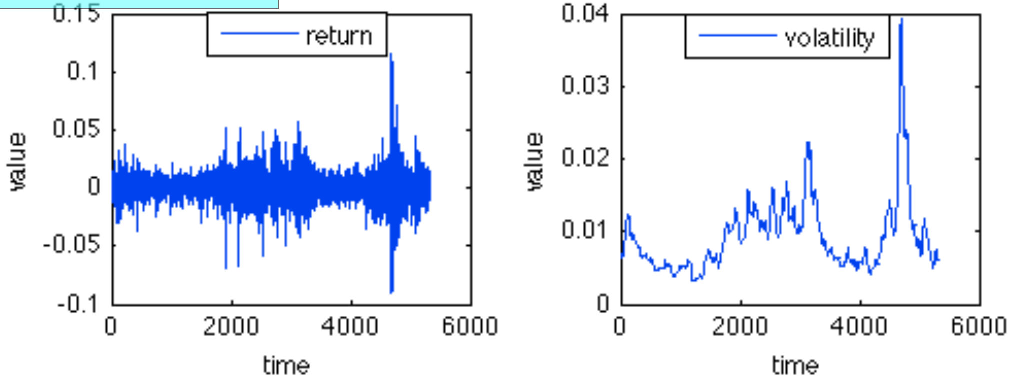

Figure 7. Volatility and return of S&P 500.

The most volatile is the return of stock index. It has large jumps, which is visually confirmed by the volatility fluctuation as well. The second most volatile process is return of EURO/USD. The highest pick of the volatility corresponds to the biggest jump of its return. For the other exchange rates, similar conclusion could be inferred: the bigger the outliers in the asset price return the higher volatility fluctuation.

Thus, estimated parameters suggest that it is more likely to have small values for parameters $\tilde{m}$ and $\tilde{k}$ for the asset price and close to unit root value of parameter $\tilde{\psi}$. The obtained result is consistent with many empirical studies that conclude that the most realistic parameters for stochastic volaility model are those with small noise term and high persistency of the stochastic voaltility driven process and low scaling factor of asset price volatility (Masoliver et al. (2008), Kim et al. (1996), Machieu (1998)). Obtained parameters do not belong to the case where estimator was mostly biased. Despite this fact, for other model specification the most problematic scenarios could be considered as realistic ones; hence, the quasi-maximum likelihood estimation will give inaccurate estimate.

V. Conclusions

The main goal of the current study was to apply the quasi-maximum likelihood estimation based on the Kalman filtering for the continuous time exponential OU process and evaluation of its performance on different parameters of the underlying model. The current study focuses on the estimation of three parameters of the asset price volatiltiy as they define the option price formula.

Therefore, the model is discretized by the Euler scheme and linearized for state-space representation, afterwards the estimation is done for the simulated and real data.

On simulated data the performance of the method for three different parameter scenarios with high, middle and low value is investigated. Obtained result indicates that the method exhibits the worst precision in the case where all three parameters are relatively high and in the case where the noise component of the volatility driven stochastic process is relatively small. However, in all the other settings the method performs significantly better.

The final step is to estimate the parameters of the exponential OU process for the real data. The founding is consistent with the relevant literature – the realistic parameters are small noise term and high persistency of the stochastic voaltility driven process, low scaling factor of asset price volatility. This setting is not identical to one for which the method gives the most imprecise estimate. However, considering extension of the model to the multifactor one, the problematic scenarios for the parameters could be plausible.

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