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Combining Supervaluation and Fuzzy Logic Based Theories of Vagueness

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Abstract

Vagueness is ubiquitous in our language and thinking. Reasoning with vague information therefore is a highly relevant task for information systems and artificial intelligence. In this thesis, we study non-classical logics for reasoning under vagueness.

Supervaluationism is one of several theories of vagueness that are discussed in analytic philosophy. The idea of supervaluationism is to consider all ways of making vague statements completely precise. In every such precisification, formulas are interpreted like in classical logic. Vague situations are modeled by precisification spaces which are sets of different precisifications. The supervaluationist's notion of truth is supertruth, which is defined as truth in all precisifications. Due to the similarity to Kripke semantics, the supervaluational approach results in a modal logic.

Fuzzy logics have a background in control engineering and are a well-studied class of many-valued logics. The fuzzy-logic approach has two main features: the unit interval is taken as the set of truth degrees and formulas are evaluated according to truth functions. We consider those fuzzy logics in which the truth function for conjunction is a continuous t-norm and the truth functions of the other connectives are also fully determined by the choice of the t-norm. Fuzzy logics are often extended by the $\triangle$-operator that indicates whether a formula has the truth value 1. The most important fuzzy logics for this thesis are Łukasiewicz logic and Gödel logic. We show that in some natural sense Gödel logic is the only "logic of comparison".

We combine supervaluationism and fuzzy logic to a hybrid logic by equipping every precisification space with a measure on its set of precisifications. We determine the truth value of each propositional variable by measuring the set of precisifications of the space in which it is true. The truth functions of the connectives are determined by a t-norm, like in fuzzy logic. In this way we obtain a hybrid logic for every continuous t-norm. We also add a modal operator S to the logic that indicates whether a formula is supertrue in the precisification space.

We obtain a normal form for the hybrid logic in which nestings of the S-operator are not necessary. Furthermore, we show that Gödel fuzzy logic with the $\triangle$-operator can be embedded into Łukasiewicz hybrid logic and that Łukasiewicz hybrid logic can be embedded into fuzzy Łukasiewicz logic with the $\triangle$-operator. We also consider certain natural restricted versions of precisification spaces and show the following: Łukasiewicz hybrid logic is the only hybrid logic in which truth in all precisification spaces is equivalent to truth in all precisification spaces with a measure of strictly positive range. In both Łukasiewicz hybrid logic and Gödel hybrid logic, truth in all precisification spaces with a measure of strictly positive range is equivalent to truth in all precisification spaces with a uniform measure.

Zusammenfassung

Vagheit ist ein allgegenwärtiges Phänomen unserer Sprache und unseres Denkens. Das Schlussfolgern aus vager Information ist daher äußerst relevant für Informationssysteme und für Künstliche Intelligenz. Diese Arbeit beschäftigt sich mit nichtklassischen Logiken für das Schlussfolgern in vagen Kontexten.

Supervaluationismus ist eine von mehreren Theorien der Vagheit, die in der analytischen Philosophie diskutiert werden. Die Idee dahinter ist, dass alle Möglichkeiten berücksichtigt werden sollen, eine vage Aussage vollständig präzise zu machen. In jeder Präzisierung werden Formeln wie in der klassischen Logik interpretiert. Vagheit wird durch Präzisierungsräume, d. h. durch Mengen verschiedener Präzisierungen, modelliert. Der supervaluationale Wahrheitsbegriff ist Superwahrheit, welche als Wahrheit in allen Präzisierungen definiert wird. Aufgrund der Ähnlichkeiten zur Kripke-Semantik, ergibt sich aus diesem Ansatz eine modale Logik.

Fuzzy-Logiken stammen ursprünglich aus der Kontrolltechnik und sind eine Klasse mehrwertiger Logiken. Fuzzy Logik hat zwei Hauptbestandteile: Zum einen wird das Einheitsintervall als die Menge der möglichen Wahrheitswerte verwendet und zum anderen ergibt sich der Wahrheitswert von Formeln durch Wahrheitsfunktionen. Wir betrachten jene Fuzzy-Logiken, bei denen die Wahrheitsfunktion der Konjunktion eine stetige T-Norm ist und auch die Wahrheitsfunktionen der restlichen Konnektive vollständig durch die Wahl der T-Norm bestimmt werden. Fuzzy-Logiken werden oft um den Operator $\vartriangle$ erweitert, der angibt, ob eine Formel den Wahrheitswert 1 erhält. Die wichtigsten Fuzzy-Logiken für diese Arbeit stellen die Łukasiewicz-Logik und die Gödel-Logik dar. Wir zeigen, dass die Gödel-Logik in einem gewissen, natürlichen Sinn die einzige "Logik des Vergleichs" ist.

Wir kombinieren Supervaluationismus und Fuzzy-Logik zu einer hybriden Logik, indem wir jeden Präzisierungsraum mit einem Maß auf seiner Menge von Präzisierungen ausstatten. Wir bestimmen den Wahrheitswert jeder propositionalen Variable durch das Maß jener Präzisierungen, in denen sie als wahr erachtet wird. Die Wahrheitsfunktionen der Konnektive werden, wie in der Fuzzy-Logik, durch eine T-Norm bestimmt. Auf diese Art und Weise erhalten wir eine hybride Logik für jede stetige T-Norm. Weiters fügen wir einen modalen Operator S hinzu, der angibt, ob eine Formel im Präzisierungsraum superwahr ist.

Wir erhalten eine Normalform für die hybride Logik, bei der S-Operatoren nicht geschachtelt werden müssen. Wir zeigen außerdem, dass die Gödel-Logik mit $\vartriangle$-Operator in die hybride Łukasiewicz-Logik eingebettet werden kann und dass die hybride Łukasiewicz-Logik in die Łukasiewicz-Logik mit $\vartriangle$-Operator eingebettet werden kann. Darüber hinaus betrachten wir bestimmte Einschränkungen für Präzisierungsräume und zeigen Folgendes: Die hybride Łukasiewicz-Logik ist die einzige hybride Logik, in der Wahrheit in allen Präzisierungsräumen äquivalent ist zu Wahrheit in allen Präzisierungsräumen, bei denen das Maß einen strikt positiven Wertebereich hat. Sowohl in der hybriden Łukasiewicz-Logik als auch in der hybriden Gödel-Logik ist Wahrheit in allen Präzisierungsräumen mit einem strikt positiven Maß äquivalent zu Wahrheit in allen Präzisierungsräumen mit einem uniformen Maß.

Erklärung zur Verfassung der Arbeit

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Contents

CHAPTER 1

Introduction

1.1 Overview

Consider the following two statements:

(i) A person with a height of 2 m or more is tall.

(ii) If a person with a height of x mm is tall, then also a person with a height of $x - 1$ mm is tall.

Although both premises seem to be intuitively correct, we arrive at a conclusion that is certainly not: Starting with the fact that a person with a height of 2 m is tall, we successively apply the second premise and conclude that even a person with a height of 40 cm is tall. This line of reasoning leads to the contradictory statement that even babies, the smallest among all people, are tall. A situation like this is called a sorites paradox. Sorites paradoxes were first formulated by the ancient Greek. Other variants of this argument address the property of being young and the age in seconds, the property of being bald and the number of hairs, or, in its classical formulation, the property of being a heap and the number of grains of sand.

Strictly speaking, there is of course no contradiction. Nothing is wrong with our well-known classical logic in this example: the second statement just cannot be true according to the classical notion of truth. However, this answer seems a bit unsatisfactory because, intuitively, the second statement is also not completely false, it is *somewhat* true. Even if we accept the second premise as *somewhat* true, the conclusion still seems completely unacceptable. The reason for this complication is that adjectives like "tall" are not completely precise in the sense that it is always possible to label every person as tall or small. We call such adjectives *vague*. As our example indicates, there seems to be a certain style of human reasoning in the presence of vague information that deviates from classical logic. Adequate formal models of reasoning under vagueness would be valuable for many applications. Therefore it is natural to ask: how should we deal with vagueness?

Computer science has an answer to that question: fuzzy logic or related degree-based approaches are the prevalent logical approach towards vagueness and alternatives are hardly ever discussed. In contrast to that, there is an ongoing debate about *theories of vagueness* in analytic philosophy. One of these theories is called *supervaluationism* and is diametrically opposed to fuzzy approaches. The goal of this thesis is to bring the two research fields closer together. Supervaluationism will be presented in a way that makes it accessible for logicians and computer scientists. Fuzzy logic will be presented in a way that makes a discussion of its appropriateness for vagueness possible. Finally, we consider a hybrid logic that combines fuzzy logic and supervaluationism in a specific way. Note that we do not just borrow ideas from philosophy as a source of inspiration, but also want to point out which consequences certain design choices for a theory of vagueness have. This, in turn, might be of particular interest for philosophers.

To overcome any scepticism concerning the usefulness of philosophical discussions for computer science in advance, we want to remind the readers that modal logic initially was developed out of purely philosophical concerns. However, nowadays modal logics are a central tool for the verification of hardware and software. Furthermore it will become apparent that a discussion of logics for vagueness includes many aspects that are technically challenging.

This thesis consists of two kinds of contributions: reviews of the key literature and theoretical results. The organisation is as follows:

- We start with a review of supervaluationism in Chapter 2. Here we consider Fine's classical article [40] as well as current research papers with an emphasis on technical results. Furthermore, we show that a certain variant of supervaluationism is not suitable for so-called higher-order vagueness.

- Fuzzy logics based on t-norms, as studied in Chapter 3, are the prevalent manifestation of degree-based reasoning. We give a review of the state of the art with a focus on aspects that are relevant for the discussion of vagueness.

- Chapter 4 is devoted to Gödel logic, one of the most important fuzzy logics. The central feature of Gödel logic is that its truth functions only compare truth values and do not involve any arithmetic operations. In principle there might be other logics with this behaviour. We prove that, under certain preconditions, Gödel logic is the only fuzzy logic of comparison.

- In Chapter 5 we consider a hybrid logic that combines supervaluational and fuzzy reasoning. We slightly generalize the concept introduced by Fermüller and Kosik [38] and investigate the resulting logics. Our results include a normal form for formulas and certain embeddings to or from fuzzy logics. Furthermore, we introduce natural restrictions of the corresponding semantics and show under which choices of the truth functions for logical connectives these restrictions affect the notion of validity.

In our presentation we assume that the reader is familiar with the basic notions of classical logic. We remark that consistency of notation was one of the goals of this thesis. This entails that cited results are often formulated in a different manner than in the original paper.

1.2 Theories of vagueness

Before we dig deeper into the subject it seems useful to give a short overview on the vagueness landscape. We follow the survey by Fermüller [36] for presenting the main aspects needed to provide some background.

Vagueness is a highly discussed topic in analytic philosophy that also is of potential interest to logicians. Several monographs have been published, each one arguing in favor of one of the existing approaches on vagueness (see for example [72, 93, 94, 101]). Unsurprisingly, there is no consensus on how vagueness is to be defined. However, it seems to be clear that one has to deal with the following phenomena of vagueness [36, 72]:

- **Borderline cases**: Vague predicates usually admit borderline cases. This means that there are objects for which it is "unclear whether or not the predicate applies" [72]. Note that the existence of borderline cases is not due to a lack of precision. Even if we knew the height of every person accurate to the nanometer, it would still be hard to decide for some people whether they should be counted as tall or not tall.

- **No sharp boundaries**: The classical conception says that the extension of a predicate consists of all objects for which the predicate applies. If an object is not in the extension, then it is clear that it is not subsumed by the predicate. Vague predicates do not have such sharp boundaries in the form of well-defined extensions. The boundary between the extension and the "anti-extension" of a vague predicate is "fuzzy". Note that this concept is in a way very similar to the existence of borderline cases and it is sometimes argued that both concepts coincide.

- **Sorites paradoxes**: A good theory of vagueness should help us to resolve the sorites paradox described in the introduction: In a line of persons in which every person is 1 mm smaller than her left neighbor, it seems intuitive to argue as follows: if a person in the line is considered tall, the right neighbor should be considered tall, too. If we assume that the first person in the line is tall, repeated applications of modus ponens lead to the paradoxical conclusion that there can never be a person in the line that is not tall.

- **Higher-order vagueness**: "Vague" itself is a vague predicate. Just as there are borderline cases of vague predicates, one can also think of objects for which one cannot definitely determine whether they are borderline cases or not. Thus, there might be borderline cases of borderline cases. This idea can be formulated up to arbitrarily high levels.

There are many suggestions how to deal with these phenomena. Among the most important theories of vagueness proposed by philosophers are the following:

- **Gap theories**: Two views are subsumed under this notion. The first one is that in the presence of vagueness there is no way of doing proper reasoning at all. The second one admits that there are statements that are neither true nor false. Technically, this can be modeled with an additional truth value and leads to a three-valued logic.

- **Epistemicism**: Epistemic theories consider vagueness as a lack of knowledge. The main idea is that in principle all predicates, also the vague ones, have exact extensions. However, the exact boundaries of some predicates are not known to us and therefore appear to be vague.

- **Supervaluationism**: The baseline of supervaluationism is that a vague statement is definitely true if it is true for all ways of making it completely precise. The different precisifications correspond to fully classical interpretations of a vague predicate. A borderline statement then is true in some but not all precisifications. Obviously, there is a connection between supervaluationism and Kripke-style semantics for modal logics.

- **Degree-based theories**: Borderline statements receive truth values between absolute truth and absolute falsehood. Fuzzy logics are one well-known implementation of this approach. However, there are also other degree-based approaches. In a more recent contribution, Smith proposes a degree-based approach where he takes fuzzy interpretations—as opposed to classical interpretations—as precisifications [94].

- **Pragmatic theories**: Some philosophers consider our use of language as the source of vagueness. Under this view, a language is always completely precise but its meaning depends on the context. Since it is not always determined which language a community of speakers uses, the phenomenon of vagueness arises.

- **Contextualism**: Contextualism assumes that over time or with context the meaning of vague terms might change. Such shifts in context might occur quickly, even during the course of a conversation. In particular, contextualism considers the possibility that in certain situations "a competent speaker of the language can go either way in the borderline area of a vague predicate" [93].

Philosophers usually end up in defending exactly one theory of vagueness. However, Fermüller argues that it is adequate to pursue competing approaches because vagueness is a "complex and multi-faceted phenomenon" [34]. Different applications may justify different means of dealing with vagueness. This thesis follows this point of view. We focus on supervaluationism and fuzzy logics as an instantiation of degree-based reasoning because the two concepts are complementary to each other and are both well-developed in their respective communities.

CHAPTER 2

Supervaluationism

The baseline of supervaluationism is that a vague statement should be considered true if it is true for all ways of making it completely precise. Although similar ideas have already been expressed by others, Fine's seminal article [40] is the standard reference on supervaluationism. In the following we try to present the canonical supervaluation theory based on Fine's article. In particular, we are interested in the logics arising from the supervaluational concepts. Higher-order vagueness and specification spaces based on partial interpretations are two topics that deserve special attention and are discussed separately. Finally, we also give an overview on some newer, more technical results that mostly concern different forms of entailment for supervaluational logic.

2.1 Fine's specification space approach

Fine motivates his approach by his view on vagueness: "I take it [vagueness] to be a semantic notion. Very roughly, it is deficiency of meaning" [40]. Hence, he ties vagueness to the existence of *truth-value gaps*. His central idea is the following: "A vague sentence can be made more precise; and this operation should preserve truth value" [40]. One possibility to deal with truth-value gaps is to simply introduce a third truth value for *neither-true-nor-false*. Fine dismisses truth-value approaches of that kind and presents an alternative framework for which the *specification space* is the central notion. As a simplification, he carries out his analysis for vague predicates only and does not consider vague names or vague quantifiers.

A special requirement that Fine imposes on a theory of vagueness is that it can deal with *penumbral connections*. He explains that a penumbral connection is a logical relation that holds among indefinite statements. "Truths that arise, wholly or in part, from penumbral connection [are called] truths on a penumbra or penumbral truths" [40]. His example is the following: Assume a blob whose color is on the border between red and pink. Although both statements "The blob is red" and "The blob is pink" are indefinite, the statement "The blob is red and pink" is clearly false because being pink and being red are contraries in this setting. On the other hand, if the blob is also a borderline case of small, the statement "The blob is small" is indefinite as well as the statement "The blob is red and small". Note that this example already indicates that the resulting logic for vagueness is not truth-functional.

Fine's approach can be summarized as follows: He considers a space of specification points that correspond to ways of making vague statements precise. At a specification point, a statement can be true, false, or neither-true-nor-false. Some specification points are complete, i.e., all vagueness is resolved. At these complete specification points, every statement is either (classically) true or (classically) false. At incomplete specification points, statements can also be neither-true-nor-false. Fine introduces a notion of truth suitable for specification spaces that is central to his approach: a vague statement is *supertrue* if it is true for all ways of making it completely precise. In the following we define Fine's abstract concepts more formally.

A specification space is a triple $\boldsymbol{S} = \langle \boldsymbol{P}, \geq, (\|\cdot\|_{s,\boldsymbol{S}})_{s\in\boldsymbol{P}} \rangle$ that consists of a nonempty set of specification points $\boldsymbol{P}$, a partial ordering[1] $\geq$ on $\boldsymbol{P}$ called *extension relation* and a function $(\|\cdot\|_{s,\boldsymbol{S}})_{s\in\boldsymbol{P}}$ that assigns to every $s \in \boldsymbol{P}$ a partial function $\|\cdot\|_{s,\boldsymbol{S}}$. The specification points "correspond to the different ways of making the language more precise" [40]. The intuition behind the expression $t \geq s$, which should be read as "t extends s", is that t is a precisification of s: it resolves some of the vagueness in t. For every $s \in \boldsymbol{P}$, the partial function $\|\cdot\|_{s,\boldsymbol{S}}$ assigns a truth value $\|\varphi\|_{s,\boldsymbol{S}} \in \{0, 1\}$ to some statements φ. The function might be undefined for some statements because their truth value might not be settled at the specification point s.

Fine imposes the following constraints on every specification space $\boldsymbol{S}$:

- **Admissibility**: A specification space should be admissible which means that the truth value assignment at each specification point is "in accordance with the intuitively understood meanings of the predicates" [40]. In particular, admissibility guarantees that penumbral connections are not violated.

- **Base point**: There is a *base point* $b \in \boldsymbol{P}$ such that $s \geq b$ for each $s \in \boldsymbol{P}$.

- **Completeability**: "Any point can be extended to a complete point within the same space" [40]. For every $s \in \boldsymbol{P}$ there is a complete point $t \in \boldsymbol{P}$ such that $t \geq s$. Fine does not explicitly define what it means for a specification point to be complete, but it seems reasonable to call a specification point $s \in \boldsymbol{P}$ *complete* if there is no $t \in \boldsymbol{P}$ such that $t \neq s$ and $t \geq s$.

- **Fidelity**: "The truth values at a complete point are classical" [40]. If $s \in \boldsymbol{P}$ is complete then there is a classical interpretation $\boldsymbol{M}_s$ such that, for every statement φ, $\|\varphi\|_{s,\boldsymbol{S}} = \|\varphi\|_{\boldsymbol{M}_s}$ (where the right hand side denotes the classical interpretation function). Therefore the function $\|\cdot\|_{s,\boldsymbol{S}}$ is not partial when s is a complete specification point.

[1] A partial ordering is a reflexive, transitive and antisymmetric relation.

- **Stability**: "Truth values are preserved under extension of points" [40]. If for a statement φ we have $\|\varphi\|_{s,\boldsymbol{S}} \in \{0,1\}$ and $t \geq s$, then $\|\varphi\|_{t,\boldsymbol{S}} = \|\varphi\|_{s,\boldsymbol{S}}$.

Fine also imposes two further constraints that we will not state formally because they are rather technical. In essence, these two conditions state that every specification point can be identified with the nonempty set of its complete extensions and vice versa. This means that the function that assigns to every specification point the set of its complete extensions is a bijective mapping between the set of specification points and the powerset of the set of complete specification points. The base point, for example, can be identified with the set of all complete specification points.

Fine's abstract approach allows several possibilities to define the function $\|\cdot\|_{s,\boldsymbol{S}}$ for every $s \in \boldsymbol{P}$. Due to the fidelity condition, the only freedom in defining $\|\cdot\|_{s,\boldsymbol{S}}$ for a complete specification point s is to choose a classical interpretation that fully determines $\|\cdot\|_{s,\boldsymbol{S}}$. Thus, the interesting case concerns the incomplete specification points. We present Fine's standard approach in the following.[2]

Fine says that the truth value of a statement φ at an incomplete specification point $s \in \boldsymbol{P}$ depends on the classical interpretations of the statement at all complete extensions of s in the following way:

$$\|\varphi\|_{s,\boldsymbol{S}} = \begin{cases} 1 & \text{if } \|\varphi\|_{t,\boldsymbol{S}} = 1 \text{ for every complete point } t \in \boldsymbol{P} \text{ with } t \geq s \\ 0 & \text{if } \|\varphi\|_{t,\boldsymbol{S}} = 0 \text{ for every complete point } t \in \boldsymbol{P} \text{ with } t \geq s\,. \end{cases}$$

Note that this definition makes $\|\cdot\|_{s,\boldsymbol{S}}$ a partial function because it might be the case that there are two complete extensions $t_1, t_2 \in \boldsymbol{P}$ of s such that $\|\varphi\|_{t_1,\boldsymbol{S}} = 1$ and $\|\varphi\|_{t_2,\boldsymbol{S}} = 0$.

Besides this local notion of truth, Fine also introduces a concept of global truth in a specification space $\boldsymbol{S}$. A statement φ is *supertrue* iff it is true at the base point b, i.e., if $\|\varphi\|_{b,\boldsymbol{S}} = 1$. With Fine's local notion of truth, truth at the base point means truth at all of its complete extensions and since the base point is extended by *every* specification point, supertruth can be identified with truth at all complete specification points. For the remainder of this chapter it is a very important observation that this notion of supertruth in a specification space only depends on the complete specification points of the space, and not on any incomplete specification point.

Another important aspect of Fine's paper is the introduction of a "definitely" operator D. Fine's definition is that a statement φ is definitely true iff it is true at the base point which is equivalent to being supertrue, i.e., $\|\mathrm{D}\varphi\|_{t,\boldsymbol{S}} = \|\varphi\|_{b,\boldsymbol{S}}$. Furthermore, Fine defines an indefinitely operator that indicates that a statement is "borderline" true. It is neither definitely true nor definitely false: the statement $\mathrm{I}\varphi$ is an abbreviation for $\neg\mathrm{D}\varphi \wedge \neg\mathrm{D}\neg\varphi$. Under this definition, the D-operator is not suitable for higher-order vagueness because it does not admit nontrivial iterations. We discuss Fine's approach on higher-order vagueness in Section 2.4.

[2] As an alternative, Fine offers a "bastard intuitionistic account" [40] that however has not gained the same perception as Fine's standard approach. On the bastard intuitionistic approach the truth condition for implication for example is the following: $\|\varphi \supset \psi\|_{s,\boldsymbol{S}} = 1$ if and only if for every $t \in \boldsymbol{S}$ such that $t \geq s$, if $\|\varphi\|_{t,\boldsymbol{S}} = 1$, then $\|\psi\|_{t,\boldsymbol{S}} = 1$.

It is interesting to note that Fine modifies the stability condition in presence of the D-operator. We can associate with each specification point s in a specification space $\boldsymbol{S}$ a subspace $\boldsymbol{S}_s$ with s as its base point. The set of complete precisifications of $\boldsymbol{S}_s$ then is a subset of the complete precisifications of $\boldsymbol{S}$. The "proper form of stability" [40] then is:

$$\text{If } \|\varphi\|_{s,\boldsymbol{S}} = 1 \text{ and } t \geq s, \text{ then } \|\varphi\|_{t,\boldsymbol{S}_t} = 1.$$

The reason why the original stability condition might be violated is that the D-operator "ignores any improvement in specification that may have taken place" [40]. Consider a specification space $\boldsymbol{S}$ in which $t \geq s$ and a statement φ such that $\|\varphi\|_{s,\boldsymbol{S}}$ is undefined and $\|\varphi\|_{t,\boldsymbol{S}} = 1$. Then, by the definition of the D-operator, $\|\mathrm{D}\varphi\|_{s,\boldsymbol{S}} = 0$ and $\|\mathrm{D}\varphi\|_{t,\boldsymbol{S}} = 1$ which would violate the original stability condition.

2.2 Supervaluational logic

We now review some properties of the logic emerging from Fine's concepts. The language of supervaluational logic is that of classical first-order logic together with the unary D-operator. Our language has predicate symbols, but no constant symbols, no function symbols and no identity sign. The reason for this choice is that supervaluationists are mainly interested in vague predicates. Sometimes only D-free formulas, in the language of pure classical first-order logic, will be considered. In this case it will be explicitly mentioned.

2.2.1 The standard approach

From a formal point of view all relevant questions concerning a specification space can be answered by knowing its set of complete specifications points and their corresponding classical interpretations.

Definition 2.2.1. A *precisification space* $\boldsymbol{S}$ is a triple $\boldsymbol{S} = \langle \boldsymbol{P}, \boldsymbol{D}, (\boldsymbol{M}_s)_{s\in\boldsymbol{P}} \rangle$ that consists of a nonempty set $\boldsymbol{P}$ of *precisifications*, a nonempty set $\boldsymbol{D}$, the *domain* of $\boldsymbol{S}$, and a function $(\boldsymbol{M}_s)_{s\in\boldsymbol{P}}$ that assigns a classical first-order interpretation $\boldsymbol{M}_s$ with domain $\boldsymbol{D}$ to every precisification $s \in \boldsymbol{P}$. As a simplification, we may write $s \in \boldsymbol{S}$ instead of $s \in \boldsymbol{P}$.

Note that we require that all first-order interpretations have the same domain. The vagueness then comes from different interpretations of predicates in that domain. This definition corresponds to the definition of interpretation structures for a version of modal predicate logic, but without an accessibility relation [69]. The set of possible worlds in modal logic corresponds to the set of precisifications in supervaluational logic.

The terms specification space and precisification space are usually used synonymously. We will stick to the following convention: a specification space refers to Fine's approach of modeling vague situations by supervaluation and the precisification space is the notion of an interpretation structure for supervaluational logic.

We now define how formulas are interpreted in a precisification space. Apart from the D-operator, this is straightforward. Just like in classical first-order logic, the interpretation function is defined modulo a variable assignment.

Definition 2.2.2. Let $\boldsymbol{S}$ be a precisification space with domain $\boldsymbol{D}$. A function $\boldsymbol{v}$ that assigns to every object variable an element of $\boldsymbol{D}$ is called *variable assignment* of $\boldsymbol{S}$.

Definition 2.2.3. For every precisification $s \in \boldsymbol{S}$ of a precisification space $\boldsymbol{S}$ and every variable assignment $\boldsymbol{v}$ of $\boldsymbol{S}$, the *interpretation of formulas* is defined inductively. For atomic formulas, the defining clause is

$$\|\varphi\|_{s,\boldsymbol{v},\boldsymbol{S}} = \|\varphi\|_{\boldsymbol{M}_s,\boldsymbol{v}}$$

where $\|\varphi\|_{\boldsymbol{M}_s,\boldsymbol{v}}$ is the classical interpretation of φ in $\boldsymbol{M}_s$ under the variable assignment $\boldsymbol{v}$. The classical connectives are defined in their standard way and the interpretation of the D-operator is given by the clause

$$\|\mathrm{D}\varphi\|_{s,\boldsymbol{v},\boldsymbol{S}} = \begin{cases} 1 & \text{if } \|\varphi\|_{t,\boldsymbol{v},\boldsymbol{S}} = 1 \text{ for every } t \in \boldsymbol{S} \\ 0 & \text{otherwise}. \end{cases}$$

We will at several occasions consider a different semantics of the D-operator, in particular in our discussion of higher-order vagueness (see Section 2.4). There, the formula Dφ will be true if φ is true at all precisifications that are accessible according to a binary relation on the precisifications. To avoid confusion, we will always mention if we deviate from this original definition. Note that some authors use the term "determinately" to describe the D-operator.

We now define the standard logical notions for supervaluational logics, i.e., we define what truth, validity and logical consequence mean in supervaluational logic. To define these notions we only consider closed formulas where every occurrence of an object variable is bound by a quantifier. Therefore we omit the index for the variable assignment in the interpretation function. First we define the supervaluational notion of truth, namely supertruth.

Definition 2.2.4. A formula φ is *supertrue* in a precisification space $\boldsymbol{S}$ iff φ is true at each precisification, i.e., $\|\varphi\|_{s,\boldsymbol{S}} = 1$ for every $s \in \boldsymbol{S}$.

One can also call formulas that are false at every precisification *superfalse*. A formula then is superfalse if and only if its negation is supertrue. As these definitions indicate there might be formulas that are neither supertrue nor superfalse in a given precisification space. Note that with our above semantics of the D-operator, a formula Dφ is true at some precisification if and only if φ is supertrue.

Validity in our supervaluational logic is then naturally defined from supertruth.

Definition 2.2.5. A formula φ is *valid* iff φ is supertrue in every precisification space.

Besides validity, which is a property of formulas, we also want a notion of logical consequence, which is a relation between formulas. Fine takes logical consequence as preservation of supertruth.

Definition 2.2.6. A set of formulas Γ *globally entails* a formula ψ (written $\Gamma \vDash_g \varphi$) iff for every precisification space $\boldsymbol{S}$ the following condition holds: if every premiss $\varphi \in \Gamma$ is supertrue in $\boldsymbol{S}$, then the conclusion ψ is supertrue in $\boldsymbol{S}$.

It will become clear later on, why it is called global entailment. Later in this chapter, we will also learn about other entailment relations for supervaluational logic.

We now investigate some simple properties of this logic that have already been pointed out by Fine [40]. As usual, validity amounts to being a logical consequence of the empty set of premisses.

Proposition 2.2.7. *A formula φ is valid if and only if $\emptyset \vDash_g \varphi$.*

As Fine remarks, there is a strong connection to classical logic.

Proposition 2.2.8. *Let Γ be a set of formulas and φ a formula, both in the language of classical first-order logic (i.e., without the* D*-operator). Then $\Gamma \vDash_g \varphi$ if and only if $\Gamma \vDash_{CL} \varphi$, where $\vDash_{CL}$ is the classical entailment relation.*

This means that the classical notions of validity and logical consequence are retained in supervaluational logic. Fine comments that "the supertruth theory makes a difference to truth, but not to logic" [40]. This fact is seen as one of the main advantages of supervaluationism by its defenders—Keefe [72] for example shares this point of view.

Note that supervaluational logic *does* deviate from classical logic for multiple-conclusion entailment. In this form of logical consequence an argument is valid if in the case that all premisses are true then at least one of the conclusions is true. For example, the argument $p \vee q \vDash \{p, q\}$, where p and q are propositional variables, classically holds, but does not hold in our supervaluational logic. This deviation from classical logic is sometimes part of the criticism of supervaluationism [99]. Entailment relations with multiple conclusions are considered by Kremer and Kremer [77] (see Section 2.5.2), and Varzi [99] (see Section 2.5.1).

At the level of validity an even stronger connection, namely to modal logic, holds.

Proposition 2.2.9. *A formula φ is valid in supervaluational logic if and only if $\varphi^{\Box}$ is valid in* S5 *where $\varphi^{\Box}$ denotes the result of replacing every occurrence of the* D*-operator in φ by the necessitation operator $\Box$.*

Remark. We obtain an axiomatization of the valid formulas of supervaluational logic by adding to a Hilbert-style system for classical first-order logic the S5-axioms

(K) $\mathrm{D}(\varphi \supset \psi) \supset (\mathrm{D}\varphi \supset \mathrm{D}\psi)$

(T) $\mathrm{D}\varphi \supset \varphi$

(5) $\neg \mathrm{D}\varphi \supset \mathrm{D}\neg \mathrm{D}\varphi$

and the necessitation rule “From φ, infer $\mathrm{D}\varphi$” [69].

The global entailment relation has a central property: it is always possible to eliminate or introduce the D-operator [101].

Proposition 2.2.10. *For every formula φ, we have $\mathrm{D}\varphi \vDash_g \varphi$ and $\varphi \vDash_g \mathrm{D}\varphi$.*

But this does not mean that the D-operator is redundant because, for example, the formula $p \supset \mathrm{D}p$, where p is a propositional variable, is not valid.

One of Williamson's objections against supervaluationism is that the following deductive principles, which are well-known for classical logic, do not hold in supervaluational logic in presence of the D-operator [101]:

- **Contraposition**: From $\Gamma, \varphi \vDash \psi$, infer $\Gamma, \neg\psi \vDash \neg\varphi$.

  Counterexample: $p \vDash_g \mathrm{D}p$, but $\neg\mathrm{D}p \nvDash_g \neg p$

- **Conditional proof**: From $\Gamma, \varphi \vDash \psi$, infer $\Gamma \vDash \varphi \supset \psi$.

  Counterexample: $p \vDash_g \mathrm{D}p$, but $\nvDash_g p \supset \mathrm{D}p$

- **Argument by cases**: From $\Gamma, \varphi \vDash \chi$ and $\Gamma, \psi \vDash \chi$, infer $\Gamma, \varphi \vee \psi \vDash \chi$.

  Counterexample: $p \vDash_g \mathrm{D}p \vee \mathrm{D}\neg p$ and $\neg p \vDash_g \mathrm{D}p \vee \mathrm{D}\neg p$, but $p \vee \neg p \nvDash_g \mathrm{D}p \vee \mathrm{D}\neg p$

- **Reductio ad absurdum**: From $\Gamma, \varphi \vDash \psi$ and $\Gamma, \varphi \vDash \neg\psi$, infer $\Gamma \vDash \neg\varphi$.

  Counterexample: $p \wedge \neg\mathrm{D}p \vDash_g \mathrm{D}p$ and $p \wedge \neg\mathrm{D}p \vDash_g \neg\mathrm{D}p$, but $\nvDash_g \neg(p \wedge \neg\mathrm{D}p)$

The failure of the principle of conditional proof means that the deduction theorem does not hold.

As a response to Williamson, Keefe suggests to modify these deductive principles [72]. In presence of the D-operator, the following inference patterns,[3] which hold in supervaluational logic, should be considered instead of their classical counterparts.

- **Contraposition**: From $\Gamma, \varphi \vDash \psi$, infer $\Gamma, \neg\psi \vDash \neg\mathrm{D}\varphi$.

- **Conditional proof**: From $\Gamma, \varphi \vDash \psi$, infer $\Gamma \vDash \mathrm{D}\varphi \supset \psi$.

- **Argument by cases**: From $\Gamma, \varphi \vDash \chi$ and $\Gamma, \psi \vDash \chi$, infer $\Gamma, \mathrm{D}\varphi \vee \mathrm{D}\psi \vDash \chi$.

- **Reductio ad absurdum**: From $\Gamma, \varphi \vDash \psi$ and $\Gamma, \varphi \vDash \neg\psi$, infer $\Gamma \vDash \neg\mathrm{D}\varphi$.

[3]Keefe's rules have been slightly adapted to match Williamson's original rules.

2.2.2 Local entailment

As seen above, supervaluational logic has many similarities to modal logic. Just as in modal logic, one can also consider the principle of preservation of truth at every world in the supervaluation framework. The corresponding consequence relation is usually called local entailment and it seems that Williamson was the first one who discussed it in the context of supervaluationism [101].

Definition 2.2.11. A set of formulas Γ *locally entails* a formula ψ (written $\Gamma \vDash_l \varphi$) iff for every precisification space $\boldsymbol{S}$ and every precisification $s \in \boldsymbol{S}$ the following condition holds: if $\|\varphi\|_{s,\boldsymbol{S}} = 1$ for every $\varphi \in \Gamma$, then $\|\psi\|_{s,\boldsymbol{S}} = 1$.[4]

Note that in modal logics both concepts, global and local entailment, are well-known. However, the local entailment relation is usually taken as the standard one in modal logics [9].

By its definition, local entailment does not rely on the notion of supertruth. Therefore, Williamson argues that global entailment is *the* consequence relation of supervaluationism. However, other authors (see for example Varzi [99], Section 2.5.1, or Asher, Dever, and Pappas [1], Section 2.5.4) argue that local entailment should indeed be considered as a suitable consequence relation for supervaluationism.[5] Furthermore, due to strong connections between global and local entailment, it might be useful to develop proof systems for global entailment from proof systems for local entailment that are very well-studied because local entailment is the standard in modal logics (see for example Cobreros [23], Section 2.5.3). Therefore we review some of the properties of local entailment that Williamson mentions [101].

Just like in the case of global entailment, validity means being entailed from the empty set of premisses.

Proposition 2.2.12. *A formula φ is valid if and only if $\emptyset \vDash_l \varphi$.*

Thus, at the level of validity both, the global and the local view, coincide. Also the connection to classical logic for D-free formulas carries over to local entailment, which makes both entailment relations coincide with the classical one.

Proposition 2.2.13. *Let Γ be a set of formulas and φ a formula, both in the language of classical first-order logic (i.e., without the* D*-operator). Then $\Gamma \vDash_l \varphi$ if and only if $\Gamma \vDash_{\mathrm{CL}} \varphi$ where $\vDash_{\mathrm{CL}}$ is the classical entailment relation.*

In general, local entailment is the stronger notion because it implies global entailment.

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[4]At this point we could introduce *pointed precisification spaces* as pairs $\langle \boldsymbol{S}, s \rangle$ where $\boldsymbol{S}$ is a precisification space and s is a precisification of $\boldsymbol{S}$. Then we could take pointed precisification spaces as our base notion for interpretation structures and could formulate the local entailment relation as preservation of truth in pointed precisification spaces.

[5]Some care has to be taken because some authors speak of validity when they mean entailment.

Proposition 2.2.14. *Let φ be a formula and Γ a set of formulas. Then $\Gamma \models_1 \varphi$ implies $\Gamma \models_g \varphi$.*

The converse direction does not hold because $p \models_g \mathrm{D}p$ by Proposition 2.2.10, but $p \not\models_1 \mathrm{D}p$.

2.3 Partial interpretations for specification spaces

As seen above, only the complete specification points are relevant for Fine's notion of supertruth. There are other approaches that also take into account the incomplete specification points. Such approaches are usually based on the following idea: Just as we assign a classical interpretation to every complete specification point, we can also assign a partial interpretation to every incomplete specification point. The idea behind partial interpretations is to modify classical interpretations in a way that they leave some things unspecified. The most prominent approach for partial interpretations is the following one [1, 71, 77, 93].

Definition 2.3.1. A *partial (first-order) interpretation* is a triple $\boldsymbol{M} = \langle \boldsymbol{D}, \boldsymbol{I}^+, \boldsymbol{I}^- \rangle$ where $\boldsymbol{D}$ is a nonempty set, the domain of discourse, and $\boldsymbol{I}^+$ and $\boldsymbol{I}^-$ are functions that assign to each n-ary predicate symbol a subset of $\boldsymbol{D}^n$. For every predicate symbol Q, $\boldsymbol{I}^+(Q)$ is called the *extension* of Q and $\boldsymbol{I}^-(Q)$ is called the *anti-extension* of Q. We require that the extension and the anti-extension of Q are disjoint, i.e., $\boldsymbol{I}^+(Q) \cap \boldsymbol{I}^-(Q) = \emptyset$. A partial interpretation is *complete* iff $\boldsymbol{I}^+(Q) \cup \boldsymbol{I}^-(Q) = \boldsymbol{D}^n$ for every predicate symbol Q that has some arity n.

For this definition of partial interpretations there is a very natural way of "making more precise".

Definition 2.3.2. Let $\boldsymbol{M}_1 = \langle \boldsymbol{D}_1, \boldsymbol{I}_1^+, \boldsymbol{I}_2^- \rangle$ and $\boldsymbol{M}_2 = \langle \boldsymbol{D}_2, \boldsymbol{I}_2^+, \boldsymbol{I}_2^- \rangle$ be two partial interpretations. Then $\boldsymbol{M}_2$ *extends* $\boldsymbol{M}_1$ (written $\boldsymbol{M}_2 \geq \boldsymbol{M}_1$) iff

(i) $\boldsymbol{D}_1 = \boldsymbol{D}_2$

(ii) $\boldsymbol{I}_1^+(Q) \subseteq \boldsymbol{I}_2^+(Q)$ for every predicate symbol Q

(iii) $\boldsymbol{I}_1^-(Q) \subseteq \boldsymbol{I}_2^-(Q)$ for every predicate symbol Q

The symbol $\geq$ denotes an extension relation and the symbol $\geq$ denotes the concrete extension relation on partial interpretations. Note that the definition of completeness for partial interpretations matches the definition of completeness for specification points given in Section 2.1 since a partial interpretation that is complete can be only extended by itself.

In this setting one can take the following view on specification spaces: the base point of the specification space is a partial interpretation. Other partial interpretations extend the base point by filling up the gap between extensions and anti-extensions in various ways, under consideration of the admissibility constraint. The hierarchy of extensions goes on until finally the complete specification points are reached, where there are no gaps left between the extensions and anti-extensions.

In his book [93], Shapiro essentially takes this approach, but without demanding that there is a complete extension for every incomplete specification point. He introduces new operators with sophisticated truth conditions that in particular take into account the extension relation. Since Shapiro's agenda mainly is to defend contextualism, we do not treat his approach here.

Note that there is a natural way to assign a partial interpretation to every specification point s in a specification space $\boldsymbol{S}$. We define the partial interpretation $\boldsymbol{M}_s = \langle \boldsymbol{D}, \boldsymbol{I}^+, \boldsymbol{I}^- \rangle$ by setting $\boldsymbol{D}$ to the domain of $\boldsymbol{S}$ and setting

- $(d_1, \dots, d_n) \in \boldsymbol{I}^+(Q)$ if and only if $\|Q(x_1, \dots, x_n)\|_{t,\boldsymbol{v}} = 1$ every $t \geq s$ such that t is complete

- $(d_1, \dots, d_n) \in \boldsymbol{I}^-(Q)$ if and only if $\|Q(x_1, \dots, x_n)\|_{t,\boldsymbol{v}} = 0$ every $t \geq s$ such that t is complete

where $\boldsymbol{Q}$ is a predicate symbol of arity $\boldsymbol{n}$ and $\boldsymbol{v}$ is a variable assignment on $\boldsymbol{S}$ such that $\boldsymbol{v}(x_i) = d_i$ for every $1 \leq i \leq n$.

2.3.1 Kleene semantics

Independent from the question of truth at a specification point in a specification space, for D-free formulas one might also be interested in a notion of truth in a partial interpretation isolated from the other specification points. This is usually done with a 3-valued Kleene semantics where i denotes the additional truth value 'indefinite'. There are two *Kleene schemes*, the *weak* and the *strong* Kleene scheme. Both Kleene schemes are truth-functional and their connectives behave classically for the classical truth values. The weak Kleene scheme returns the truth value i whenever some part of a statement is indefinite. Under the strong Kleene scheme, a truth function returns a definite truth value whenever every assignment of 0 or 1 instead of i would result in the same truth value. But note that this rule only applies at the level of truth functions and not at the level of whole formulas. Kleene introduced these 3-valued semantics in his book on metamathematics [73], but not with any application to vagueness in mind. Tye gave an example of a theory of vagueness revolving around Kleene's approach [98].

In the following, we describe the strong Kleene scheme because it is the more important one. It is used as a semantics for single specification points in a specification space by Shapiro [93] and in a suggestion of an extended notion of supertruth, truth at *all* specification points (including incomplete ones), by Asher, Dever, and Pappas [1].

For the classical connectives, the strong Kleene scheme is given by the truth functions of Table 2.1.

$\neg$	
0	1
i	i
1	0

$\wedge$	0	i	1
0	0	0	0
i	0	i	i
1	0	i	1

$\vee$	0	i	1
0	0	i	1
i	i	i	1
1	1	1	1

$\supset$	0	i	1
0	1	1	1
i	i	i	1
1	1	i	0

Table 2.1: Truth functions of the strong Kleene scheme

These principles can also be extended to the first-order case. The interpretation of a predicate symbol $\boldsymbol{Q}$ is given by

$$\|Q(x_1,\ldots,x_n)\|_{\boldsymbol{M},\boldsymbol{v}} = \begin{cases} 0 & \text{if } (\boldsymbol{v}(x_1),\ldots,\boldsymbol{v}(x_n)) \in \boldsymbol{I}^-(Q) \\ \mathrm{i} & \text{if } (\boldsymbol{v}(x_1),\ldots,\boldsymbol{v}(x_n)) \notin \boldsymbol{I}^-(Q) \cup \boldsymbol{I}^+(Q) \\ 1 & \text{if } (\boldsymbol{v}(x_1),\ldots,\boldsymbol{v}(x_n)) \in \boldsymbol{I}^+(Q) \end{cases}$$

where $\boldsymbol{M}$ is a partial interpretation in which $\boldsymbol{Q}$ has the extension $\boldsymbol{I}^+(\boldsymbol{Q})$ and the anti-extension $\boldsymbol{I}^-(\boldsymbol{Q})$ and $\boldsymbol{v}$ is a variable assignment.

The interpretation of quantified formulas is given by

$$\|\exists x\varphi\|_{\boldsymbol{M},\boldsymbol{v}} = \begin{cases} 0 & \text{if } \|\varphi\|_{\boldsymbol{M},\boldsymbol{v}\cup\{x\mapsto d\}} = 0 \text{ for every } d \in \boldsymbol{D} \\ 1 & \text{if there is some } d \in \boldsymbol{D} \text{ such that } \|\varphi\|_{\boldsymbol{M},\boldsymbol{v}\cup\{x\mapsto d\}} = 1 \\ \mathrm{i} & \text{otherwise} \end{cases}$$

$$\|\forall x\varphi\|_{\boldsymbol{M},\boldsymbol{v}} = \begin{cases} 0 & \text{if there is some } d \in \boldsymbol{D} \text{ such that } \|\varphi\|_{\boldsymbol{M},\boldsymbol{v}\cup\{x\mapsto d\}} = 0 \\ 1 & \text{if } \|\varphi\|_{\boldsymbol{M},\boldsymbol{v}\cup\{x\mapsto d\}} = 1 \text{ for every } d \in \boldsymbol{D} \\ \mathrm{i} & \text{otherwise} \end{cases}$$

where $\boldsymbol{M}$ is a partial interpretation with domain $\boldsymbol{D}$ and $\boldsymbol{v} \cup \{x \mapsto d\}$ is the variable assignment $\boldsymbol{v}'$ that is defined by

$$\boldsymbol{v}'(v) = \begin{cases} d & \text{if } v = x \\ \boldsymbol{v}(v) & \text{if } v \neq x\,. \end{cases}$$

A natural way to define validity is to say that a formula φ is valid iff φ is always true, where truth in the Kleene scheme amounts to the truth value 1. The formula $p \vee \neg p$ is not valid any of the two Kleene schemes, as opposed to classical logic and our supervaluational logic introduced in Section 2.2. Even more, it can easily be seen that the set of valid formulas is empty for both Kleene schemes. Asher, Dever, and Pappas therefore dismiss the idea of supertruth as truth in all specification points, including the incomplete ones [1].

Note that for both Kleene schemes the extension relation $\geq$ for partial interpretations satisfies the stability condition, i.e., the truth values 0 and 1 are preserved under extension.

2.3.2 Definite truth

For specification spaces built from partial interpretations, Asher, Dever, and Pappas [1] as well as Varzi [99] offer an alternative semantics of the D-operator. Under this semantics, the truth of the statement Dφ indicates that φ is true at the base specification point, which resembles Fine's concept of "truth at the base point". However, these ideas are very different from each other. As explained above, for Fine, truth at the base point is equivalent to truth at all complete specification points. The other approach however equates determinate truth with truth at the base partial interpretation according to a semantics for partial interpretations, as for example Kleene semantics. This would lead to the following definition of the D-operator:

$$\|\mathrm{D}\varphi\|_{s,\boldsymbol{v},\boldsymbol{S}} = \|\varphi\|_{\boldsymbol{M}_b,\boldsymbol{v}}$$

where $\|\cdot\|_{\boldsymbol{M}_b,\boldsymbol{v}}$ is the interpretation function of the partial interpretation $\boldsymbol{M}_b$ assigned to the base point under the variable assignment $\boldsymbol{v}$. This operator captures a notion of prior truth. It behaves as an actuality operator and leads to a logic at least as strong as S5 [99]. Asher, Dever, and Pappas as well as Varzi, do not further pursue this reading of the D-operator and instead go for a necessity-style operator as discussed in Section 2.4. Still, this idea seems worth mentioning.

2.4 Higher-order vagueness

The concept of higher-order vagueness arises from the observation that *vague* itself is a vague predicate [40]. Not only would one like to be able to express statements like "Kim is borderline tall" but also "Kim is a borderline case of a borderline case of tallness." For this purpose, an adequate semantics of the D-operator has to be found. This of course also affects its counterpart, the indefinitely operator I that is defined as follows: Iφ is an abbreviation for $\neg\mathrm{D}\varphi \wedge \neg\mathrm{D}\neg\varphi$, i.e., neither φ nor its negation are definitely true.

Up to now we defined definite truth as truth at all specification points, i.e., supertruth. Our first statement can be written as $\mathrm{I}T(k)$, where $\boldsymbol{T}$ is the predicate for "tall" and k stands for Kim, and we can adequately model the situation with the framework that we have presented so far. However, the second statement $\mathrm{II}T(k)$ would always turn out to be false because $\mathrm{I}T(k)$ is either true at all specification points or false at all specification points.

Similar examples all fail because the axioms (4) $\mathrm{D}\varphi\supset\mathrm{DD}\varphi$ and (5) $\neg\mathrm{D}\neg\varphi\supset\mathrm{D}\neg\mathrm{D}\neg\varphi$ are validated under the given semantics of the D-operator. As the formula $\mathrm{D}\varphi \vee \neg\mathrm{D}\varphi$ is valid, also all formulas of the form $\mathrm{D}^n\mathrm{D}\varphi \vee \mathrm{D}^n\neg\mathrm{D}\varphi$ are valid, where D^n denotes n iterations of the D-operator. According to Williamson, "[on] this semantics, it cannot be indefinite whether something is definite" [101]. Thus, our framework so far can deal with what is called "first-order vagueness" but not with "second-order vagueness" or any higher levels. But a standard requirement of a proper theory of vagueness is that it is possible to express n-th order vagueness for any n.

2.4.1 Relative admissibility

We will now discuss two approaches towards higher-order vagueness. The first one was given by Fine and the second one is an abstraction of Fine's ideas that has become the standard approach.

Fine's approach

Fine offers a treatment of higher-order vagueness of which he admits that it is rather complicated [40]. In essence, we follow Williamson's presentation [101] of this framework which simplifies some ideas of Fine's original article.

The main idea of the new framework is to generalize the concept of specification spaces. Instead of just picking a complete specification point in a specification space one also has to decide in which one of several possible specification spaces the specification point lies, and so on. We define a level-based hierarchy of interpretations where each interpretation at a certain level specifies which interpretations of those that lie one level below are admissible.

Fine inductively defines the hierarchy of interpretations as follows:

- A 0-th level interpretation is a classical interpretation of statements.
- An $n+1$-th level interpretation is a set of n-th level interpretations.
- An ω-interpretation is a sequence $s_0 s_1 s_2 \dots$ such that each s_i is a i-th level interpretation and $s_i \in s_{i+1}$, for $i \geq 0$.

The intuition behind this definition is that an $n+1$-th level interpretation considers exactly those n-th level interpretations admissible that it contains. Note that by our usual approach we could take a 0-th level interpretation as a complete specification point and a first level interpretation as the set of complete specification points of a specification space.

Fine connects these notions to the supervaluational framework by considering higher-order specification spaces, in which every complete specification point is an ω-interpretation. In order to obtain a semantics for the D-operator, he first defines an *admissibility relation* among ω-interpretations as follows: an ω-interpretation $s = s_0 s_1 \dots$ *admits* an ω-interpretation $t = t_0 t_1 \dots$ iff $t_i \in s_{i+1}$ for every $i \geq 0$. By $\boldsymbol{R}$ we denote the relation corresponding to admissibility: $s \boldsymbol{R} t$ iff s admits t. Note that by definition every ω-interpretation admits itself which means that $\boldsymbol{R}$ is reflexive.

We now show how vague statements are interpreted in a higher-order specification space $\boldsymbol{S}$ in which every complete specification point is an ω-interpretation. If φ is an atomic statement, then the interpretation at an ω-interpretation $s = s_0 s_1 \dots$ is given by[6]

$$\|\varphi\|_{s,\boldsymbol{S}} = \|\varphi\|_{s_0}$$

[6]Fine and Williamson do not explicitly define how atomic formulas should be evaluated in this framework, but the approach that is presented here should be the only one that makes sense.

where s_0 is the starting point of s and $\|\cdot\|_{s_0}$ denotes the classical 0-th level interpretation according to s_0. The new semantics of the D-operator takes into account the admissible specifications of the higher-order space:

$$\|\mathrm{D}\varphi\|_{s,\boldsymbol{S}} = \begin{cases} 1 & \text{if } \|\varphi\|_{t,\boldsymbol{S}} = 1 \text{ for every } t \in \boldsymbol{S} \text{ such that } s\, \boldsymbol{R}\, t \\ 0 & \text{otherwise}. \end{cases}$$

The other connectives are interpreted as usual. Note that the clause "truth at all admissible specification points" characterizes both, the old and the new semantics of the D-operator. The only difference is that in a setting with higher-order vagueness admissibility is not global.

In Fine's framework, the admissibility relation $\boldsymbol{R}$ of a higher-order space is given by the structure of the ω-interpretations. It determines for each complete specification point which other complete specification points are to be regarded as admissible. This concept is similar to the accessibility relation in Kripke semantics for modal logics. By its definition $\boldsymbol{R}$ is reflexive and the resulting logic is, according to Fine, the modal logic T. Reflexivity seems very natural because each specification point should regard itself as admissible. Fine remarks that one can also think of further restrictions on $\boldsymbol{R}$. If for example one would only be interested in at most n-th order vagueness for a concrete n, then one could demand that in every ω-interpretation $s = s_0 s_1 \dots$ one has $s_{i+1} = \{s_i\}$ for every $i \geq n$.

To illustrate these definitions, we now investigate a simple example that a theory of higher-order vagueness should be able to model.

Example. Let us assume a scenario where Kim is a borderline case of definite tallness. Let p denote the propositional variable standing for "Kim is tall" Then we want the formula I Dp to be true. Under a classical interpretation, "Kim is tall" is either true or false. We denote the classical interpretation in which "Kim is tall" is true by $[p]$ and the one in which "Kim is tall" is false by $[\neg p]$.

We model this situation with a higher-order specification space $\boldsymbol{S}$ that consists of the ω-interpretations given in Table 2.2. In this example, we maximally deal with second-order vagueness. Therefore, we "stop" all ω-interpretations after the second level, i.e., $s_{i+1} = \{s_i\}$, $t_{i+1} = \{t_i\}$ and $u_{i+1} = \{u_i\}$ for $i \geq 2$.

	level 0	level 1	level 2
s	$[p]$	$\{[p]\}$	$\{\{[p]\}, \{[p], [\neg p]\}\}$
t	$[p]$	$\{[p], [\neg p]\}$	$\{\{[p]\}, \{[p], [\neg p]\}\}$
u	$[\neg p]$	$\{[p], [\neg p]\}$	$\{\{[p]\}, \{[p], [\neg p]\}\}$

Table 2.2: Higher-order specification space with ω-interpretations

According to Fine's definitions, the admissibility relation $\boldsymbol{R}$ is the one given in Figure 2.1. We for example have $u\, \boldsymbol{R}\, s$ because the 0-th level interpretation $[p]$ of s is contained in the first level interpretation $\{[p], [\neg p]\}$ of u and the first level interpretation $\{[p]\}$ of s is contained in the second level interpretation $\{\{[p]\}, \{[p], [\neg p]\}\}$ of u.

We do *not* have $s\,\boldsymbol{R}\,u$ because the 0-th level interpretation $[\neg p]$ of u is not contained in the first level interpretation $\{[p]\}$ of s. Note that this admissibility relation is neither symmetric nor transitive.

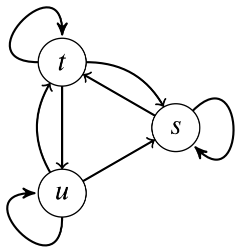

Figure 2.1: Admissibility relation

We now verify that we have found a model of our scenario:

$$\begin{array}{rrr}
\|p\|_{s,\mathcal{S}} = 1 & \|p\|_{t,\mathcal{S}} = 1 & \|p\|_{u,\mathcal{S}} = 0 \\
\|\mathrm{D}p\|_{s,\mathcal{S}} = 1 & \|\mathrm{D}p\|_{t,\mathcal{S}} = 0 & \|\mathrm{D}p\|_{u,\mathcal{S}} = 0 \\
\|\neg\mathrm{D}p\|_{s,\mathcal{S}} = 0 & \|\neg\mathrm{D}p\|_{t,\mathcal{S}} = 1 & \|\neg\mathrm{D}p\|_{u,\mathcal{S}} = 1 \\
\|\mathrm{DD}p\|_{s,\mathcal{S}} = 0 & \|\mathrm{DD}p\|_{t,\mathcal{S}} = 0 & \|\mathrm{DD}p\|_{u,\mathcal{S}} = 0 \\
\|\neg\mathrm{DD}p\|_{s,\mathcal{S}} = 1 & \|\neg\mathrm{DD}p\|_{t,\mathcal{S}} = 1 & \|\neg\mathrm{DD}p\|_{u,\mathcal{S}} = 1 \\
\|\mathrm{D}\neg\mathrm{D}p\|_{s,\mathcal{S}} = 0 & \|\mathrm{D}\neg\mathrm{D}p\|_{t,\mathcal{S}} = 0 & \|\mathrm{D}\neg\mathrm{D}p\|_{u,\mathcal{S}} = 0 \\
\|\neg\mathrm{D}\neg\mathrm{D}p\|_{s,\mathcal{S}} = 1 & \|\neg\mathrm{D}\neg\mathrm{D}p\|_{t,\mathcal{S}} = 1 & \|\neg\mathrm{D}\neg\mathrm{D}p\|_{u,\mathcal{S}} = 1
\end{array}$$

Since $\mathrm{ID}p$ is an abbreviation for $\neg\mathrm{DD}p \wedge \neg\mathrm{D}\neg\mathrm{D}p$, we get $\|\mathrm{ID}p\|_{\mathcal{S}} = 1$. Note that our ω-interpretations at the second level contain two first level interpretations. The first one is $\{[p]\}$, where Kim is definitely tall, and the second one is $\{[p], [\neg p]\}$ where Kim is a definite borderline case of tallness. We simply consider all ω-interpretations that contain this second level interpretation.

Abstract framework

The idea of modeling higher-order vagueness with the concept of relative admissibility is a very prominent one in the literature on supervaluation. In Fine's framework described above the admissibility relation is explicitly defined on ω-interpretations. In many cases, an abstraction of this framework is considered where one works with an admissibility relation that is at least reflexive without explicitly specifying it. This point of view might be taken because Fine stressed the fact that his account on the D-operator results in the modal logic T. Under the more abstract point of view the admissibility relation is usually taken directly on the complete specification points [1, 23, 99].

In the higher-order specification space of ω-interpretations we can assign to every complete specification point the classical interpretation that it starts with. Therefore, this abstraction is indeed a generalization of Fine's framework.

The abstract necessity-style D-operator is the standard in the literature on supervaluation and we therefore modify our definition of a precisification space.

Definition 2.4.1. A *precisification space* is a quadruple $\langle \boldsymbol{P}, \boldsymbol{R}, \boldsymbol{D}, (\boldsymbol{M}_s)_{s\in\boldsymbol{P}}\rangle$ that consists of a nonempty set $\boldsymbol{P}$ of *precisifications*, a binary *admissibility relation* $\boldsymbol{R}$ on $\boldsymbol{P}$, a nonempty set $\boldsymbol{D}$, the *domain* of $\boldsymbol{S}$, and a function $(\boldsymbol{M}_s)_{s\in\boldsymbol{P}}$ that assigns a classical first-order interpretation $\boldsymbol{M}_s$ with domain $\boldsymbol{D}$ to every precisification $s \in \boldsymbol{P}$.

The interpretation of formulas at precisifications in a precisification space $\boldsymbol{S}$ is standard, with exception of the D-operator:

$$\|\mathrm{D}\varphi\|_{s,\boldsymbol{v},\boldsymbol{S}} = \begin{cases} 1 & \text{if } \|\varphi\|_{t,\boldsymbol{v},\boldsymbol{S}} = 1 \text{ for every } t \in \boldsymbol{P} \text{ such that } s\ \boldsymbol{R}\ t \\ 0 & \text{otherwise} \end{cases}$$

where $s \in \boldsymbol{S}$ and $\boldsymbol{v}$ is a variable assignment of $\boldsymbol{S}$.

Let us now turn to the logic that results from the new semantics of the D-operator. The set of valid formulas is just the set of valid formulas of the modal logic K, where D is taken as the necessity operator $\Box$. Therefore we obtain a Hilbert-style proof system for this logic by enriching a proof system for classical first-order logic with the axioms

(K) $\mathrm{D}(\varphi \supset \psi) \supset (\mathrm{D}\varphi \supset \mathrm{D}\psi)$

(BF) $\forall x \mathrm{D}\varphi \supset \mathrm{D}\forall x\varphi$

and the rule of necessitation "From φ, infer $\mathrm{D}\varphi$" [1]. Axiom (BF) is the Barcan formula and combines modal and first-order reasoning [69]. This axiom is needed because we required that all classical first-order interpretations in the precisification space have the same domain. Hughes and Cresswell remark that in some modal systems, like for example S5, the Barcan formula does not have to be included because it can be derived from the other axioms [69].

It is often the case that one only considers certain classes of precisification spaces. Usually one demands that the admissibility relation has to fulfill a certain property. As explained above one might postulate that the admissibility relation be reflexive. In such cases, our logical notions are then to be understood as relative to the set of precisification spaces in which this property holds. A formula then for example is called valid iff it is supertrue in all precisification spaces in which the admissibility relation fulfills this property. It is well-known from modal logics [12] that in certain cases a property P on the admissibility relation can be characterized by an axiom A in the following sense: for every precisification space $\boldsymbol{S}$, A is supertrue in $\boldsymbol{S}$ if and only if $\boldsymbol{S}$ has the property P. Table 2.3 gives an overview over the most important characterizable properties.

[7] The symbols used in the description of these properties should be understood as mere abbreviations of informal language, e.g., $\forall s$ stands for "for every precisification s of the precisification space".

Axiom schema		Property of admissibility relation[7]	
(T)	$D\varphi \supset \varphi$	reflexive	$\forall s : s \boldsymbol{R}\, s$
(B)	$\varphi \supset D\neg D\neg\varphi$	symmetric	$\forall s \forall t : s \boldsymbol{R}\, t \Rightarrow t \boldsymbol{R}\, s$
(D)	$D\varphi \supset \neg D\neg\varphi$	serial	$\forall s \exists t : s \boldsymbol{R}\, t$
(4)	$D\varphi \supset DD\varphi$	transitive	$\forall s \forall t \forall u : s \boldsymbol{R}\, t \,\&\, t \boldsymbol{R}\, u \Rightarrow s \boldsymbol{R}\, u$
(5)	$\neg D\neg\varphi \supset D\neg D\neg\varphi$	euclidean	$\forall s \forall t \forall u : s \boldsymbol{R}\, t \,\&\, s \boldsymbol{R}\, u \Rightarrow t \boldsymbol{R}\, u$

Table 2.3: Correspondence between axioms and properties of the admissibility relation

One obtains a Hilbert-style proof system for a certain class of precisification spaces by adding the corresponding axioms to our proof system for the logic K. If for example we restrict ourselves to precisification spaces with reflexive admissibility relations, then the axiom (T) is added. If we demand reflexivity, symmetry and transitivity, then we arrive at the additional axioms (T), (B), and (4). It can be shown that these axioms characterize the logic S5 and therefore, at the level of validity, reflexivity, symmetry, and transitivity yield our previous semantics of the D-operator as defined in Section 2.2.

Concerning global entailment, we remark that the inference $\varphi \models_g D\varphi$ also holds for the new semantics of our D-operator. However, the converse inference from $D\varphi$ to φ is only possible if every precisification is admissible for some other precisification, a principle inverse to seriality. This is for example the case in the class of precisification spaces with reflexive admissibility relations.

We will at some occasions also consider an operator called D_u, where u stands for "universal access". We want this operator to indicate supertruth which is truth at all precisifications. Thus, this operator ignores the admissibility relation. This corresponds to Fine's first definition for a definitely operator that, as we have seen, is not suited for higher-order vagueness (see Definition 2.2.3).

Definition 2.4.2. Let $\boldsymbol{S}$ be a precisification space, $s \in \boldsymbol{S}$ a precisification and $\boldsymbol{v}$ a variable assignment of $\boldsymbol{S}$. Then the operator D_u is interpreted as follows:

$$\|D_u\varphi\|_{s,\boldsymbol{v},\boldsymbol{S}} = \begin{cases} 1 & \text{if } \|\varphi\|_{t,\boldsymbol{v},\boldsymbol{S}} = 1 \text{ for every } t \in \boldsymbol{S} \\ 0 & \text{otherwise}. \end{cases}$$

2.4.2 Inexpressibility of higher-order vagueness under certain admissibility relations

We now show that certain classes of admissibility relations are not suited for modeling higher-order vagueness. In particular, we consider formulas of the form $ID\varphi$ that express that it is indefinite whether φ is definitely true. This is Williamson's prototypical example of second-order vagueness that cannot be expressed with a universal semantics of the D-operator (see Definition 2.2.3). Williamson mentions that the admissibility relation must be allowed to be non-transitive for modeling higher-order vagueness. In the following, we extend Williamson's analysis by showing with simple arguments that formulas of the form IDφ are unsatisfiable in certain classes of precisification spaces.[8]

There are two types of satisfiability of a formula φ that one can consider. The first one is that there is precisification space in which φ is supertrue. The second one is that there is a precisification space and a precisification s in that space such that φ is true at s. Note that the first notion has a global flavor and the second notion has a local flavor. The first notion of satisfiability is in some sense stronger than the second one because it implies the second one. It seems more intuitive that the first notion of satisfiability should be considered for supervaluation. But, just as one can argue about global entailment versus local entailment, one could also discuss which notion of satisfiability is more suitable. Therefore we will in the following always be precise which form of (un)satisfiability we mean. What matters for our concerns is that a formula that is not satisfiable in the second sense is also not satisfiable in the first sense.

We start with a basic property of the I-operator.

Lemma 2.4.3. *If $||\mathrm{I}\varphi||_{s,\boldsymbol{S}} = 1$ for a precisification s in a precisification space $\boldsymbol{S}$ and a formula φ, then there are precisifications $u, t \in \boldsymbol{S}$ such that $s\ \boldsymbol{R}\ t$, $s\ \boldsymbol{R}\ u$, $||\varphi||_{t,\boldsymbol{S}} = 1$, $||\varphi||_{u,\boldsymbol{S}} = 0$ and $t \neq u$.*

Proof. Let $s \in \boldsymbol{S}$ be a precisification such that $||\mathrm{I}\varphi||_{s,\boldsymbol{S}} = 1$. Suppose that there is no precisification $u \in \boldsymbol{S}$ such that $s\ \boldsymbol{R}\ u$ and $||\varphi||_{u,\boldsymbol{S}} = 1$. Then $||\varphi||_{u,\boldsymbol{S}} = 0$ for every precisification u that is admissible for s. Therefore we get $||\mathrm{D}\neg\varphi||_{s,\boldsymbol{S}} = 1$ from which $||\neg\mathrm{D}\neg\varphi||_{s,\boldsymbol{S}} = 0$ and therefore, by definition of the I-operator, $||\mathrm{I}\varphi||_{s,\boldsymbol{S}} = 0$ follows. This contradicts our initial assumption and therefore we may conclude that there is some $u \in \boldsymbol{S}$ such that $s\ \boldsymbol{R}\ u$ and $||\varphi||_{u,\boldsymbol{S}} = 1$. With the same argument we get that there is some $t \in \boldsymbol{S}$ such that $s\ \boldsymbol{R}\ t$ and $||\varphi||_{t,\boldsymbol{S}} = 0$. Due to $||\varphi||_{u,\boldsymbol{S}} = 1 \neq 0 = ||\varphi||_{t,\boldsymbol{S}}$ we can conclude that $t \neq u$. □

Note that from this proposition, seriality follows for the admissibility relation if a formula of the form Iφ is true at every precisification. We now show unsatisfiability in the second sense for euclidean admissibility relations.

Proposition 2.4.4. *If the admissibility relation $\boldsymbol{R}$ is euclidean, then any formula of the form* IDφ *cannot be true at any precisification (i.e.* IDφ *is false at all precisifications).*

Proof. Let $\boldsymbol{S}$ be a precisification space with a euclidean admissibility relation $\boldsymbol{R}$ and suppose that there is a precisification $s \in \boldsymbol{S}$ such that $||\mathrm{ID}\varphi||_{s,\boldsymbol{S}} = 1$. Then by Lemma 2.4.3 there are two precisifications $t, u \in \boldsymbol{S}$ such that $s\ \boldsymbol{R}\ t$, $s\ \boldsymbol{R}\ u$, $||\mathrm{D}\varphi||_{t,\boldsymbol{S}} = 1$ and $||\mathrm{D}\varphi||_{u,\boldsymbol{S}} = 0$. Due to $||\mathrm{D}\varphi||_{u,\boldsymbol{S}} = 0$ there must be a precisification $v \in \boldsymbol{S}$ such that $||\varphi||_{v,\boldsymbol{S}} = 0$. Since $\boldsymbol{R}$ is euclidean, we know that $u\ \boldsymbol{R}\ t$ and therefore also $t\ \boldsymbol{R}\ v$. But then $||\varphi||_{v,\boldsymbol{S}} = 1$ is a contradiction to $||\mathrm{D}\varphi||_{t,\boldsymbol{S}} = 1$. □

[8] We remark that very similar results can be obtained for formulas of the form IIφ that express the property of being a borderline case of a borderline case. However, the arguments then get longer.

Note that we could have also argued by assuming that axiom (5) holds, which is the axiom corresponding to a euclidean admissibility relation. Our result can easily be extended to admissibility relations that are symmetric and transitive.

Proposition 2.4.5. *If the admissibility relation* $\boldsymbol{R}$ *is symmetric and transitive, then any formula of the form* $\mathrm{ID}\varphi$ *cannot be true at any precisification (i.e.* $\mathrm{ID}\varphi$ *is false at all precisifications).*

Proof. The claim follows from the following property: if a binary relation $\boldsymbol{R}$ on a set $\boldsymbol{S}$ is symmetric and transitive, then it is euclidean. Let $s, t, u \in \boldsymbol{S}$ such that $s\ \boldsymbol{R}\ t$ and $s\ \boldsymbol{R}\ u$. Since $\boldsymbol{R}$ is symmetric we also have $t\ \boldsymbol{R}\ s$. Due to transitivity, from $t\ \boldsymbol{R}\ s$ and $s\ \boldsymbol{R}\ u$, we conclude $t\ \boldsymbol{R}\ u$. □

We now show that the first notion of unsatisfiability is already given if the admissibility relation is symmetric *or* transitive.

Proposition 2.4.6. *If the admissibility relation* $\boldsymbol{R}$ *is symmetric, then any formula of the form* $\mathrm{ID}\varphi$ *cannot be true at all precisification (i.e.* $\mathrm{ID}\varphi$ *is false at some precisification).*

Proof. Let $\boldsymbol{S}$ be a precisification space with a symmetric admissibility relation $\boldsymbol{R}$ and suppose that $\|\mathrm{ID}\varphi\|_{s,\boldsymbol{S}} = 1$ for every precisification $s \in \boldsymbol{S}$. Let $s \in \boldsymbol{S}$ be an arbitrary precisification. By assumption $\|\mathrm{ID}\varphi\|_{s,\boldsymbol{S}} = 1$ holds and therefore by Lemma 2.4.3 there is a precisification $t \in \boldsymbol{S}$ such that $\|\mathrm{D}\varphi\|_{t,\boldsymbol{S}} = 0$. By the definition of D-operator this means that there is a precisification $u \in \boldsymbol{S}$ such that $\|\varphi\|_{u,\boldsymbol{S}} = 0$. Since by assumption $\|\mathrm{ID}\varphi\|_{u,\boldsymbol{S}} = 1$ holds, we know by Lemma 2.4.3 that there is a precisification $v \in \boldsymbol{S}$ such that $u\ \boldsymbol{R}\ v$ and $\|\mathrm{D}\varphi\|_{v,\boldsymbol{S}} = 1$. Because $\boldsymbol{R}$ is symmetric, also $v\ \boldsymbol{R}\ u$ holds. But then $\|\varphi\|_{u,\boldsymbol{S}} = 0$ is a contradiction to $\|\mathrm{D}\varphi\|_{v,\boldsymbol{S}} = 1$. □

Proposition 2.4.7. *If the admissibility relation* $\boldsymbol{R}$ *is transitive, then any formula of the form* $\mathrm{ID}\varphi$ *cannot be true at all precisifications (i.e.* $\mathrm{ID}\varphi$ *is false at some precisification).*

Proof. Let $\boldsymbol{S}$ be a precisification space with a transitive admissibility relation $\boldsymbol{R}$ and suppose that $\|\mathrm{ID}\varphi\|_{s,\boldsymbol{S}} = 1$ for all precisifications $s \in \boldsymbol{S}$. Let $s \in \boldsymbol{S}$ be an arbitrary precisification. Since by assumption $\|\mathrm{ID}\varphi\|_{s,\boldsymbol{S}} = 1$ holds, we know by Lemma 2.4.3 that there is a precisification $t \in \boldsymbol{S}$ such that $\|\mathrm{D}\varphi\|_{t,\boldsymbol{S}} = 1$. Again, by assumption $\|\mathrm{ID}\varphi\|_{t,\boldsymbol{S}} = 1$ holds and by Lemma 2.4.3 there is a precisification $u \in \boldsymbol{S}$ such that $t\ \boldsymbol{R}\ u$ and $\|\mathrm{D}\varphi\|_{u,\boldsymbol{S}} = 0$. By the definition of the D-operator there must be a precisification $v \in \boldsymbol{S}$ such that $\|\varphi\|_{v,\boldsymbol{S}} = 0$ and $u\ \boldsymbol{R}\ v$. Since $\boldsymbol{R}$ transitive we conclude $t\ \boldsymbol{R}\ v$. But then $\|\varphi\|_{v,\boldsymbol{S}} = 0$ is a contradiction to $\|\mathrm{D}\varphi\|_{t,\boldsymbol{S}} = 1$. □

We remark that the reflexivity of the admissibility was no ingredient in any of our proofs.

This investigation shows that no precisification space with a symmetric or transitive admissibility relation can model a situation of higher-order vagueness. This indicates that the question of the nature of the admissibility relation should be taken seriously. The possibility of constraints like symmetry and transitivity on the admissibility relation have been mentioned by Asher, Dever, and Pappas [1] as well as Cobreros [23]. However, the motivation of introducing a necessity-style D-operator was the deficiency of the D_u-operator to express higher-order vagueness. Therefore it can be questioned whether a necessity-style D-operator based on admissibility relations with constraints like symmetry or transitivity is then necessary at all.

2.4.3 Fara's argument

In her 2003 paper [33], Fara raises an objection against theories of vagueness that admit a gap between definite cases and definite non-cases of a predicate in the context of higher-order vagueness. In supervaluationism this is the case because only one of the following conditions can hold: $D\varphi$, $\neg D\varphi \vee \neg D\neg\varphi$, or $D\neg\varphi$.

First of all, Fara argues that the inference rule of D-introduction $\varphi \vDash D\varphi$ should be accepted. She gives the reason that in the presence of truth-value gaps the D-operator means something like "it is true that" and it seems impossible that a statement is true whereas the statement stating that it is true is not true. The global entailment relation with a necessity-style D-operator indeed verifies this rule, and Cobreros sees this argument as an argument in favor of the global entailment relation [23].

Fara assumes the following scenario: Consider a finite sorites series for a vague predicate Q and objects $x_1, \ldots x_m$ in which the first object x_1 is "definitely Q", the last object x_m is "definitely not Q" and x_{i+1} is the successor of x_i, for $1 \leq i \leq i-1$. Since Q has borderline cases, there is a gap between those objects that are "definitely Q" and those that are "definitely not Q". We arrive at the *gap principle*: if a member of the series is "definitely Q", then its successor in the series is "not definitely not Q". If the successor were "definitely not Q", then an object that is "definitely Q" would be followed by an object that is "definitely not Q" which would mean that there is no gap in our sorites series. Thus the implication $DQ(x_i) \supset \neg D\neg Q(x_{i+1})$ should hold for every $1 \leq i \leq m-1$. In the sorites series there are also borderline cases of being "definitely Q" and therefore there is also a gap between "definitely definitely Q" and "definitely not definitely Q". The same idea can be iterated which leads to the *generalized gap principle*: The formula

$$DD^n Q(x_i) \supset \neg D\neg D^n Q(x_{i+1})$$

should hold for every $1 \leq i \leq m-1$ and every $n \geq 0$ where D^n is a concatenation of n D-operators. Fara's argument uses an equivalent formulation of the generalized gap principle. the formula

$$D\neg D^n Q(x_i) \supset \neg DD^n Q(x_{i-1})$$

should hold for every $2 \leq i \leq m$ and every $n \geq 0$.

Now it is clear that, by first applying D-introduction and then the generalized gap principle in its second formulation, we get that the inference

$$\neg D^i Q(x_{m-i}) \vDash \neg D^{i+1} Q(x_{m-(i+1)})$$

holds for every $0 \leq i \leq m-2$. Thus, by iterating this derivation k times, we conclude that the inference

$$\neg D^i Q(x_{m-i}) \vDash \neg D^{i+k} Q(x_{m-(i+k)})$$

holds for $i+k \leq m-1$. Finally, we set $i=0$ and $k=m-1$ and get that

$$\neg Q(x_m) \vDash \neg D^{m-1} Q(x_1)$$

holds.

On the other hand we get the inference

$$Q(x_1) \vDash D^{m-1} Q(x_1)$$

by $m-1$ applications of the D-introduction rule. Fara concludes that the D-introduction rule and the generalized gap principle are inconsistent. As she explains, the main reason for this fact is that there cannot be a dense linear order on a finite set. Varzi derives from this argument the objection that supervaluationism cannot deal with unrestricted higher-order vagueness [99].

2.5 The discussion on local entailment and miscellaneous results

In the following we take a rather non-systematic look at some interesting recent results on supervaluational logic. Most of them deal with the question of suitable entailment relations.

2.5.1 Varzi

The main goal of Varzi's paper [99] is to argue in favor of local entailment as the standard notion of logical consequence for supervaluationism because it has two advantages in comparison to global entailment. First, he shows that supervaluational logic with local entailment does not suffer from the most common objections against supervaluationism. Second, he emphasizes that global entailment is definable from local entailment.

Entailment relations galore

Varzi discusses entailment relations with multiple conclusions that follow variations of a scheme that, in classical logic, can be formulated as follows: "if all of the premisses on the left hand side are true, then at least one conclusion on the right hand side is true". Remember that formulas in a precisification space are one of the following:

- supertrue: true at all precisifications
- superfalse: false at all precisifications

- neither supertrue nor superfalse: true at some precisifications and false at others

Thus, in a certain sense, supervaluational logic has three truth values. Varzi considers a necessity-style D-operator following Definition 2.4.1 with an admissibility relation that is at least reflexive.

Definition 2.5.1. Four *global* entailment relations are defined as follows.

- $\Gamma \vDash_A \Sigma$ iff for every precisification space $\boldsymbol{S}$, if every premiss $\varphi \in \Gamma$ is supertrue in $\boldsymbol{S}$, then some conclusion $\psi \in \Sigma$ is supertrue in $\boldsymbol{S}$.
- $\Gamma \vDash_B \Sigma$ iff for every precisification space $\boldsymbol{S}$, if every conclusion $\psi \in \Sigma$ is superfalse in $\boldsymbol{S}$, then some premiss $\varphi \in \Gamma$ is superfalse in $\boldsymbol{S}$.
- $\Gamma \vDash_C \Sigma$ iff for every precisification space $\boldsymbol{S}$, if every premiss $\varphi \in \Gamma$ is supertrue in $\boldsymbol{S}$, then some conclusion $\psi \in \Sigma$ is not superfalse in $\boldsymbol{S}$.
- $\Gamma \vDash_D \Sigma$ iff for every precisification space $\boldsymbol{S}$, if every conclusion $\psi \in \Sigma$ is not supertrue in $\boldsymbol{S}$, then some premiss $\varphi \in \Gamma$ is superfalse in $\boldsymbol{S}$.

The counterparts of these global entailment relations in classical logic are all equivalent due to the principle of bivalence: true and false are the only possible truth values. Varzi gives easy examples that prove that none of these four entailment relations coincide in the case of supervaluation. For single conclusion arguments, A-entailment and C-entailment as well as B-entailment and D-entailment coincide in absence of the D-operator. In the usual notation of this chapter, A-entailment refers to the entailment relation $\vDash_g$.

The same scheme can now be applied to local entailment.

Definition 2.5.2. Four *local* entailment relations are defined as follows.

- $\Gamma \vDash_\alpha \Sigma$ iff for every precisification space $\boldsymbol{S}$ and every $s \in \boldsymbol{S}$, if $\|\varphi\|_{s,\boldsymbol{S}} = 1$ for every premiss $\varphi \in \Gamma$, then $\|\psi\|_{s,\boldsymbol{S}} = 1$ for some conclusion $\psi \in \Sigma$.
- $\Gamma \vDash_\beta \Sigma$ iff for every precisification space $\boldsymbol{S}$ and every $s \in \boldsymbol{S}$, if $\|\psi\|_{s,\boldsymbol{S}} = 0$ for every conclusion $\psi \in \Sigma$, then $\|\varphi\|_{s,\boldsymbol{S}} = 0$ for some premiss $\varphi \in \Gamma$.
- $\Gamma \vDash_\gamma \Sigma$ iff for every precisification space $\boldsymbol{S}$ and every $s \in \boldsymbol{S}$, if $\|\varphi\|_{s,\boldsymbol{S}} = 1$ for every premiss $\varphi \in \Gamma$, then $\|\psi\|_{s,\boldsymbol{S}} \neq 0$ for some conclusion $\psi \in \Sigma$.
- $\Gamma \vDash_\delta \Sigma$ iff for every precisification space $\boldsymbol{S}$ and every $s \in \boldsymbol{S}$, if $\|\psi\|_{s,\boldsymbol{S}} \neq 1$ for every conclusion $\psi \in \Sigma$, then $\|\varphi\|_{s,\boldsymbol{S}} = 0$ for some premiss $\varphi \in \Gamma$.

Due to bivalence of classical interpretations, α-, β-, γ- and δ-entailment are all equivalent. Therefore it is sufficient to restrict the discussion to α-entailment, which, in the usual notation of this chapter, refers to the entailment relation $\vDash_l$. For D-free, single conclusion arguments, α-entailment is equivalent to A- and C-entailment. But in general, α-entailment is different from A-, B-, C- and D-entailment.

Varzi presents another possibility how multiple-conclusion entailment could be understood in a supervaluational logic.

Definition 2.5.3. Four *collective* entailment relations are defined as follows.

- $\Gamma \vDash_X \Sigma$ iff for every precisification space $\boldsymbol{S}$, if for every $s \in \boldsymbol{S}$ and every premiss $\varphi \in \Gamma$, $\|\varphi\|_{s,\boldsymbol{S}} = 1$ holds, then for every $s \in \boldsymbol{S}$ there is some conclusion $\psi \in \Sigma$ such that $\|\psi\|_{s,\boldsymbol{S}} = 1$.
- $\Gamma \vDash_Y \Sigma$ iff for every precisification space $\boldsymbol{S}$, if for every $s \in \boldsymbol{S}$ and every conclusion $\psi \in \Sigma$, $\|\psi\|_{s,\boldsymbol{S}} = 0$ holds, then for every $s \in \boldsymbol{S}$ there is some premiss $\varphi \in \Gamma$ such that $\|\varphi\|_{s,\boldsymbol{S}} = 0$.
- $\Gamma \vDash_Z \Sigma$ iff for every precisification space $\boldsymbol{S}$, if for every $s \in \boldsymbol{S}$ and every premiss $\varphi \in \Gamma$, $\|\varphi\|_{s,\boldsymbol{S}} = 1$ holds, then for every $s \in \boldsymbol{S}$ there is some conclusion $\psi \in \Sigma$ such that $\|\psi\|_{s,\boldsymbol{S}} \neq 0$.
- $\Gamma \vDash_W \Sigma$ iff for every precisification space $\boldsymbol{S}$, if for every $s \in \boldsymbol{S}$ and every conclusion $\psi \in \Sigma$, $\|\psi\|_{s,\boldsymbol{S}} \neq 1$ holds, then for every $s \in \boldsymbol{S}$ there is some premiss $\varphi \in \Gamma$ such that $\|\psi\|_{s,\boldsymbol{S}} = 0$.

The intuition behind the collective entailment relations is to relate the conjunction of the premisses to the disjunction of the conclusions. Varzi observes that Z-entailment is equivalent to X-entailment and W-entailment is equivalent to Y-entailment. Furthermore, X-entailment and Y-entailment are not equivalent to each other and are also not equivalent to any other entailment relation considered by Varzi so far.

In summary, only the following types of entailment are all pairwise distinct: A, B, C, D, α, X, Y. The following relations hold:

- A-entailment implies C-entailment.
- D-entailment implies B-entailment.
- α-entailment implies X-entailment.
- α-entailment implies Y-entailment.
- For an empty set of premisses $\Gamma = \varnothing$ and a singleton set of conclusions $\Sigma = \{\psi\}$, all entailment relations coincide.
- For an empty set of conclusions $\Sigma = \varnothing$ and a singleton set of premisses $\Gamma = \{\varphi\}$, all entailment relations coincide.

Varzi discusses three popular objections[9] against the theory of supervaluation and analyzes for which entailment relations the objections apply. The result of this analysis is that α- and Y-entailment are the only entailment relations for which none of the objections apply. Varzi's judgement concerning A-entailment, which is seen as the

[9]Varzi considers the following objections: (1) Non-classicality of multiple-conclusion entailment (see Section 2.2.1), (2) Williamson's objections (see Section 2.2.1, (3) Fara's argument (see Section 2.4.3). Varzi emphasizes that he does not necessarily share these objections, but only compiled this list of important objections against supervaluationism.

standard approach towards supervaluational entailment, is that it "yields a logic that is both far from classical and far from acceptable in the presence of higher-order vagueness" [99]. Moreover, Varzi argues that supervaluationists should prefer α-entailment over Y-entailment because Y-entailment is merely preservation of non-falsehood and not preservation of truth.

Reduction to local entailment

In another part of his paper, Varzi introduces two operators T and F that amount to being supertrue and being superfalse.

Definition 2.5.4. Let $\boldsymbol{S}$ be a precisification space, $s \in \boldsymbol{S}$ a precisification, and $\boldsymbol{v}$ a variable assignment of $\boldsymbol{S}$. The *interpretation of formulas* is extended as follows:

$$\|\mathrm{T}\varphi\|_{s,\boldsymbol{v},\boldsymbol{S}} = \begin{cases} 1 & \text{if } \|\varphi\|_{t,\boldsymbol{v},\boldsymbol{S}} = 1 \text{ for every } t \in \boldsymbol{S} \\ 0 & \text{otherwise} \end{cases}$$

$$\|\mathrm{F}\varphi\|_{s,\boldsymbol{v},\boldsymbol{S}} = \begin{cases} 1 & \text{if } \|\varphi\|_{t,\boldsymbol{v},\boldsymbol{S}} = 0 \text{ for every } t \in \boldsymbol{S} \\ 0 & \text{otherwise}\,. \end{cases}$$

Note that the T-operator has the same semantics as the D_u-operator of Definition 2.4.2. Varzi shows how these operators can be used to reduce the global and the collective entailment relations to the local one:

- $\Gamma \vDash_{\mathrm{A}} \Sigma$ if and only if $\{\mathrm{T}\varphi \mid \varphi \in \Gamma\} \vDash_{\alpha} \{\mathrm{T}\psi \mid \psi \in \Sigma\}$
- $\Gamma \vDash_{\mathrm{B}} \Sigma$ if and only if $\{\mathrm{F}\psi \mid \psi \in \Sigma\} \vDash_{\alpha} \{\mathrm{F}\varphi \mid \varphi \in \Gamma\}$
- $\Gamma \vDash_{\mathrm{C}} \Sigma$ if and only if $\{\mathrm{T}\varphi \mid \varphi \in \Gamma\} \vDash_{\alpha} \{\neg\mathrm{F}\psi \mid \psi \in \Sigma\}$
- $\Gamma \vDash_{\mathrm{D}} \Sigma$ if and only if $\{\neg\mathrm{T}\psi \mid \psi \in \Sigma\} \vDash_{\alpha} \{\mathrm{F}\varphi \mid \varphi \in \Gamma\}$
- $\Gamma \vDash_{\mathrm{X}} \Sigma$ if and only if $\{\mathrm{T}(\varphi_1 \wedge \ldots \wedge \varphi_n) \mid n > 0, \varphi_1, \ldots, \varphi_n \in \Gamma\} \vDash_{\alpha} \{\mathrm{T}(\psi_1 \vee \ldots \vee \psi_n) \mid n > 0, \psi_1, \ldots, \psi_n \in \Sigma\}$
- $\Gamma \vDash_{\mathrm{Y}} \Sigma$ if and only if $\{\mathrm{F}(\psi_1 \vee \ldots \vee \psi_n) \mid n > 0, \psi_1, \ldots, \psi_n \in \Sigma\} \vDash_{\alpha} \{\mathrm{F}(\varphi_1 \wedge \ldots \wedge \varphi_n) \mid n > 0, \varphi_1, \ldots, \varphi_n \in \Gamma\}$

The fact that all other entailment relations become definable from α-entailment in presence of the two operators T and F is, according to Varzi, a major advantage of α-entailment.

2.5.2 Kremer and Kremer

As mentioned above, supervaluational logic deviates from classical logic for a multiple-conclusion entailment relation also without the D-operator (see Section 2.2.1). In their work [77], Kremer and Kremer study supervaluational entailment relations with multiple conclusions for a fully classical language, i.e., without the D-operator. They make two basic distinctions:

- Preservation of supertruth, preservation of superfalsehood and both
- Arbitrary specification spaces and specification spaces without admissibility constraint

Kremer and Kremer consider a full first-order language with predicate symbols, function symbols, constants and the identity sign.

Arbitrary precisification spaces.

Throughout their paper, Kremer and Kremer consider three types of entailment relations. Using our terminology from above, we can say that their entailment relations have a "global" flavor.

Definition 2.5.5. Let Γ and Σ be sets of formulas.

- $\Gamma \vDash_1 \Sigma$ iff for every precisification space $\boldsymbol{S}$ the following holds: if every $\varphi \in \Gamma$ is supertrue in $\boldsymbol{S}$, then some $\psi \in \Sigma$ is supertrue in $\boldsymbol{S}$.
- $\Gamma \vDash_0 \Sigma$ iff for every precisification space $\boldsymbol{S}$ the following holds: if every $\psi \in \Sigma$ is superfalse in $\boldsymbol{S}$, then some $\varphi \in \Gamma$ is superfalse in $\boldsymbol{S}$.
- $\Gamma \vDash_b \Sigma$ iff $\Gamma \vDash_0 \Sigma$ and $\Gamma \vDash_1 \Sigma$.

Kremer and Kremer are especially interested in the axiomatizability of their entailment relations. An entailment relation is axiomatizable iff the relation that it defines between finite sets of formulas on its left and right hand side is recursively enumerable. If an entailment relation is not axiomatizable, then it is not possible to find a sound and complete proof system. They remark, that for a language without the identity sign, there is a rather trivial axiomatization by connecting the entailment relations to classical logic. Thus, a proof system can be easily obtained for the entailment relations.

Theorem 2.5.6 (Axiomatizability). *Let Γ and Σ be sets of formulas. If the identity sign does not occur in any formula of $\Gamma \cup \Sigma$ then*

- *$\Gamma \vDash_1 \Sigma$ if and only if either $\Gamma \vDash_{CL} \psi$ for some $\psi \in \Sigma$ or Γ is classically inconsistent.*
- *$\Gamma \vDash_0 \Sigma$ if and only if either $\varnothing \vDash_{CL} \Sigma$ or $\varphi \vDash_{CL} \Sigma$ for some $\varphi \in \Gamma$.*
- *$\Gamma \vDash_b \Sigma$ if and only if both $\Gamma \vDash_{CL} \psi$ for some $\psi \in \Sigma$ and $\varphi \vDash_{CL} \Sigma$ for some $\varphi \in \Gamma$.*

The symbol $\vDash_{CL}$ denotes the classical multiple-conclusion entailment.

Kremer and Kremer also provide sound and complete proof systems for $\vDash_1$ and $\vDash_0$ in the case that the identity sign *does* occur in the premisses or the conclusions. These systems can then be combined to a proof system for $\vDash_b$. The proof system for $\vDash_1$ for example consists of the following axioms:

(i) $\Gamma \vdash_1 \varnothing$ if Γ is classically inconsistent

(ii) $\Gamma \vdash_1 \psi$ if $\Gamma \vDash_{\mathrm{CL}} \psi$

(iii) $\varnothing \vdash_1 \psi, \neg\psi$

Its rules are weakening, cut, and negation introduction. Kremer and Kremer also extend this proof system to what they call *non-classical relations*. A relation is non-classical iff its interpretation is the same in every precisification of the space.

As usual, compactness follows from the axiomatization.

Theorem 2.5.7 (Compactness). *The entailment relations $\vDash_1$, $\vDash_0$ and $\vDash_{\mathrm{b}}$ are compact: Let $\vDash_\bigstar$ be on of the three consequence relations*[10] *and let Γ and Σ be sets of formulas such that $\Gamma \vDash_\bigstar \Sigma$. Then there are finite subsets $\Gamma' \subseteq \Gamma$ and $\Sigma' \subseteq \Sigma$ such that $\Gamma' \vDash_\bigstar \Sigma'$.*

Precisification spaces generated from partial interpretations.

Kremer and Kremer also study their consequence relations when they are restricted to a certain class of precisification spaces. The idea is to consider a specification space with a base point and all complete extensions of the base point but without any admissibility constraint. Technically, this idea is implemented by assuming that the base point has a partial interpretation that at all complete specifications has to be extended faithfully. We use the concepts of a partial interpretation and the extension relation $\geq$ on partial interpretation as introduced in Section 2.3.

Definition 2.5.8. The precisification space $\boldsymbol{S_M}$ *generated from a partial interpretation* $\boldsymbol{M}$ has the set of precisifications given by

$$\boldsymbol{P_M} = \left\{ \boldsymbol{M'} \mid \boldsymbol{M'} \geq \boldsymbol{M} \text{ and } \boldsymbol{M'} \text{ is a classical interpretation} \right\} .$$

and the same domain as $\boldsymbol{M}$. We say that a precisification space is *generated* iff it is generated from some partial interpretation.

For the rest of this short review of the Kremer and Kremer paper, we adapt the definitions of the entailment relations as follows.

Definition 2.5.9. Let Γ and Σ be sets of formulas.

- $\Gamma \vDash_1^{\mathrm{g}} \Sigma$ iff for every *generated* precisification space $\boldsymbol{S}$ the following holds: if every $\varphi \in \Gamma$ is supertrue in $\boldsymbol{S}$, then some $\psi \in \Sigma$ is supertrue in $\boldsymbol{S}$.
- $\Gamma \vDash_0^{\mathrm{g}} \Sigma$ iff for every *generated* precisification space $\boldsymbol{S}$ the following holds: if every $\psi \in \Sigma$ is superfalse in $\boldsymbol{S}$, then some $\varphi \in \Gamma$ is superfalse in $\boldsymbol{S}$.
- $\Gamma \vDash_{\mathrm{b}}^{\mathrm{g}} \Sigma$ iff $\Gamma \vDash_0^{\mathrm{g}} \Sigma$ and $\Gamma \vDash_1^{\mathrm{g}} \Sigma$.

[10] The symbol $\bigstar$ is to be understood as a placeholder.

As already mentioned in Section 2.2, the global entailment relation has very strong connections to classical logic that can also be seen for these three entailment relations of Kremer and Kremer.

Proposition 2.5.10. *Let Γ and Σ be sets of formulas and let φ and ψ be formulas. Then the following relations the classical multiple-conclusion entailment relation $\vDash_{\mathrm{CL}}$ hold:*

- $\Gamma \vDash^{\mathrm{g}}_{1} \psi$ *if and only if* $\Gamma \vDash_{\mathrm{CL}} \psi$
- $\varphi \vDash^{\mathrm{g}}_{0} \Sigma$ *if and only if* $\varphi \vDash_{\mathrm{CL}} \Sigma$
- $\varphi \vDash^{\mathrm{g}}_{\mathrm{b}} \psi$ *if and only if* $\varphi \vDash_{\mathrm{CL}} \psi$

Kremer and Kremer also discuss the question of axiomatizations for these three entailment relations. In the cases of the previous proposition, the entailment relations are axiomatizable because their classical counterparts are. In all other relevant cases, Kremer and Kremer show that an axiomatization is not possible.

Theorem 2.5.11 (Non-axiomatizability). *If the language contains two two-ary relations symbols, then the following sets are not recursively enumerable:*

- $\{\langle \Gamma, \psi\rangle \mid \Gamma$ *is a finite set of formulas,* ψ *is a formula and* $\Gamma \vDash^{\mathrm{g}}_{0} \psi\}$
- $\{\langle \Gamma, \psi\rangle \mid \Gamma$ *is a finite set of formulas,* ψ *is a formula and* $\Gamma \vDash^{\mathrm{g}}_{b} \psi\}$
- $\{\langle \varphi, \Sigma\rangle \mid \varphi$ *is a formula,* Σ *is a finite set of formulas and* $\varphi \vDash^{\mathrm{g}}_{1} \Sigma\}$
- $\{\langle \varphi, \Sigma\rangle \mid \varphi$ *is a formula,* Σ *is a finite set of formulas and* $\varphi \vDash^{\mathrm{g}}_{b} \Sigma\}$
- $\{\langle \Gamma, \Sigma\rangle \mid \Gamma$ *and* Σ *are finite sets of formulas and* $\Gamma \vDash^{\mathrm{g}}_{1} \Sigma\}$
- $\{\langle \Gamma, \Sigma\rangle \mid \Gamma$ *and* Σ *are finite sets of formulas and* $\Gamma \vDash^{\mathrm{g}}_{0} \Sigma\}$
- $\{\langle \Gamma, \Sigma\rangle \mid \Gamma$ *and* Σ *are finite sets of formulas and* $\Gamma \vDash^{\mathrm{g}}_{b} \Sigma\}$.

Kremer and Kremer remark that the nonaxiomatizability theorem can be strengthened to languages containing only one two-ary predicate symbol, but the proof gets more difficult.

Besides axiomatizability, Kremer and Kremer are again also interested in the compactness of their entailment relations. They show that compactness fails for the class of generated precisification spaces.

Theorem 2.5.12 (Non-compactness). *The entailment relations $\vDash^{\mathrm{g}}_{1}$, $\vDash^{\mathrm{g}}_{0}$ and $\vDash^{\mathrm{g}}_{\mathrm{b}}$ are not compact, i.e., for each $\vDash^{\mathrm{g}}_{\star}$ of the three entailment relations, there are sets of formulas Γ and Σ such that $\Gamma \vDash^{\mathrm{g}}_{\star} \Sigma$ but $\Gamma' \nvDash^{\mathrm{g}}_{\star} \Sigma'$ for all finite subsets $\Gamma' \subseteq \Gamma$ and $\Sigma' \subseteq \Sigma$.*

Note the divergence between both meta-theorems: although all three multiple-conclusion entailment relations can be axiomatized and are compact for arbitrary precisification spaces, neither of the two properties holds for the narrower class of generated precisification spaces.

2.5.3 Cobreros

Cobreros' paper [23] also discusses global and local entailment for supervaluationism. His main contributions are a tableaux-style proof system for supervaluational logic and a new entailment relation called *regional entailment*. Cobreros' D-operator is the necessity-style operator of Definition 2.4.1 whose evaluation depends on an admissibility relation among the specification points where in principle no constraints are put on the admissibility relation. At some parts he also analyzes the special cases of reflexive or transitive relations. He also considers the operator D_u as given by Definition 2.4.2.

Proof systems

First, Cobreros points out a connection between global and local entailment. Using the D_u-operator, Cobreros reduces global entailment to local entailment. Varzi made the same observation with his operator T [99] (see Section 2.5.1).

Proposition 2.5.13. *Let Γ be a set of formulas and φ a formula. Then*

$$\Gamma \vDash_{\mathrm{g}} \varphi \text{ if and only if } \left\{\mathrm{D}_u\psi \mid \psi \in \Gamma\right\} \vDash_{\mathrm{l}} \varphi\,.$$

Cobreros presents a tableaux system for local entailment that considers both operators, D and D_u. Such systems are well-known for related modal logics (see for example [90]). Due to the just mentioned reduction of global entailment to local entailment, this tableaux system can then be used as a proof system for supervaluational logic with global entailment.

In a related paper [22] uses another well-known relation between global and local entailment [9].

Proposition 2.5.14. *Let Γ be a set of formulas and φ a formula. Then*

$$\Gamma \vDash_{\mathrm{g}} \varphi \text{ if and only if } \{\mathrm{D}^n\psi \mid \psi \in \Gamma, n \in \mathbb{N}\} \vDash_{\mathrm{l}} \varphi\,.$$

Cobreros uses this result to develop a proof system for global entailment in which he incorporates Keefe's suggestion of modified rules for certain deduction principles (see Section 2.2.1).

Regional entailment

In the second part of his paper, Cobreros introduces a new notion of logical consequence: regional entailment is preservation of truth at precisifications that are relatively admissible.

Definition 2.5.15. A set of formulas Γ *regionally entails* a formula φ (written $\Gamma \vDash_r \varphi$) iff for every precisification space $\boldsymbol{S}$ with admissibility relation $\boldsymbol{R}$ and every precisification $s \in \boldsymbol{S}$ the following condition holds: if for every premiss $\psi \in \Gamma$ and every $t \in \boldsymbol{S}$ such that $s\ \boldsymbol{R}\ t$, $\|\psi\|_{t,\boldsymbol{S}} = 1$, then for every $t \in \boldsymbol{S}$ such that $s\ \boldsymbol{R}\ t$, $\|\varphi\|_{t,\boldsymbol{S}} = 1$.

Cobreros explains that "'definitely' is introduced in the object language in order to express the supervaluationist notion of truth" [23]. He argues that this notion of truth in relatively admissible precisifications for a necessity-style D-operator should also be preserved by logical consequence.

By its definition, regional entailment can be reduced to local entailment.

Proposition 2.5.16. *Let Γ be a set of formulas and φ a formula. Then*

$$\Gamma \vDash_r \varphi \text{ if and only if } \{\mathrm{D}\psi \mid \psi \in \Gamma\} \vDash_l \mathrm{D}\varphi\,.$$

Cobreros' tableaux system for local entailment therefore also gives a proof system for regional entailment.

Cobreros also provides some analysis of the regional entailment relation.[11] He explains that regional entailment "oscillates between global and local [entailment]" [23]. Regional entailment is stronger than local entailment and weaker than global entailment. But the concrete strength of the regional entailment relation depends on the constraints that are put on the admissibility relation. One might for example only want to consider the class of precisification spaces with reflexive admissibility relations. If we want to consider entailment modulo a certain class of precisification spaces, we adapt the definition of entailment by not quantifying over all precisification spaces but only over the precisification spaces in that class.

Theorem 2.5.17. *Let Γ be a set of formulas and φ a formula.*

- *In any class of precisification spaces, we have:*

$$\Gamma \vDash_l \varphi \text{ implies } \Gamma \vDash_r \varphi$$
$$\Gamma \vDash_r \varphi \text{ implies } \Gamma \vDash_g \varphi\,.$$

- *In the class of all precisification spaces, we have*

$$\Gamma \vDash_r \varphi \text{ implies } \Gamma \vDash_l \varphi\,.$$

- *In the class of precisification spaces with reflexive and transitive admissibility relations, we have*

$$\Gamma \vDash_g \varphi \text{ implies } \Gamma \vDash_r \varphi\,.$$

- *In the class of precisification spaces with reflexive admissibility relations, there is a set of formulas Γ and a formula φ such that $\Gamma \vDash_r \varphi$, but $\Gamma \nvDash_l \varphi$.*

- *In the class of all precisification spaces, there is a set of formulas Γ and a formula φ such that $\Gamma \vDash_g \varphi$, but $\Gamma \nvDash_r \varphi$.*

[11]The full analysis can only be found in a draft version of his paper that was available on his website: Pablo Cobreros. Supervaluations and Logical Consequence: Retrieving the Local Perspective, October 2006.

Cobreros emphasizes that the rule of D-introduction

$$\varphi \vDash \mathrm{D}\varphi$$

does not hold for regional entailment. However, the related inference rule

$$\{\varphi, \neg\mathrm{D}\varphi\} \vDash_r \bot$$

holds and can be seen as a weak form of D-introduction. He argues that the failure of D-introduction means that Fara's argument (see Section 2.4.3) does not apply. However, just as with global entailment, some classically valid inference patterns do not hold for regional entailment with a necessity-style D-operator (see also Williamson's objections in Section 2.2.1).

Comparing global, local and regional entailment, Cobreros judgement is the following: Local entailment should be dismissed because it preserves truth at every precisification, and therefore does not allow failures of bivalence, i.e., that a statement is neither true nor false. "If we want to explain the semantic indeterminacy characteristic of vagueness in terms of truth-value gaps, we cannot be committed to the local notion" [23]. He also argues that regional entailment is better suited for supervaluationism than global entailment, because on the one hand Fara's argument does not apply to regional entailment and on the other hand the rule of D-introduction is also available for regional entailment in its weaker form.

2.5.4 Asher, Dever, and Pappas

In their article [1], Asher, Dever, and Pappas criticize the supervaluation theory as proposed by Fine. They say that "the central insight of supervaluation theory is that vagueness is a modal phenomenon" [1]. For this reason they disagree with Fine's treatment of supertruth as a notion of "simple truth" [1]. Instead it should be treated as a "mode of truth" [1]. Therefore, logical consequence should not be preservation of supertruth as considered by Fine, but instead should be based on a "world-indexed notion on truth" [1].

First approach

Based on an analysis of the frameworks proposed by Fine and others, Asher, Dever, and Pappas show which approach they prefer. They consider two operators for "definitely". The first one is the operator D_u of Definition 2.4.2. The second one, called D, is a necessity-style operator like in Definition 2.4.1 with the constraint that the admissibility relation is always assumed to be reflexive. First, they compare the logics arising from global and local entailment by comparing which principles from modal logic hold in these logics. They distinguish between the "axiomatic logic" as the set of valid formulas and the "inferential logic" as properties of the entailment relation. These two notions might come apart when the deduction theorem fails. Tables 2.4 and 2.5 show which axioms and inference rules hold in both notions of entailment. Note that in these tables, the symbol $\Box$ is either replaced by D_u or by D.

	axiom	global $\models_g$ D_u	global $\models_g$ D	local $\models_l$ D_u	local $\models_l$ D
(K)	$\Box(\varphi \supset \psi) \supset (\Box\varphi \supset \Box\psi)$	✓	✓	✓	✓
(T)	$\Box\varphi \supset \varphi$	✓	✓	✓	✓
(5)	$\neg\Box\varphi \supset \Box\neg\Box\varphi$	✓	×	✓	×
(T_c)	$\varphi \supset \Box\varphi$	×	×	×	×

Table 2.4: Comparison of axioms for global and local entailment

	inference rule	global $\models_g$ D_u	global $\models_g$ D	local $\models_l$ D_u	local $\models_l$ D
(K)	$\Box(\varphi \supset \psi) \models (\Box\varphi \supset \Box\psi)$	✓	✓	✓	✓
(T)	$\Box\varphi \models \varphi$	✓	✓	✓	✓
(5)	$\neg\Box\varphi \models \Box\neg\Box\varphi$	✓	✓	✓	×
(T_c)	$\varphi \models \Box\varphi$	✓	✓	×	×

Table 2.5: Comparison of inference rules for global and local entailment

The right sides of both tables are obvious because local entailment is the standard notion of logical consequence in modal logic, which has the deduction theorem [9]. In the case of D, the admissibility relation is reflexive and therefore the corresponding modal logic is KT. In the case of D_u, the corresponding modal logic is S5. Furthermore, as already seen before, the set of valid formulas coincides for global and local entailment. Therefore, in Table 2.4, the columns of D and D_u coincide for global and local entailment. In Table 2.5, the crucial observation for global entailment is that the reflexivity of the admissibility relation makes the inference $D\varphi \models_g \varphi$ possible. The "triviality" of global entailment then comes from the fact that both $\varphi \models_g D\varphi$ and $D\varphi \models_g \varphi$ hold.

Asher, Dever, and Pappas also consider the case that a partial interpretation is assigned to every incomplete specification point where partial interpretations have a Kleene-semantics (as introduced in Section 2.3). However they dismiss this approach because it does not preserve classical logical truths which they take as one of the main features of supervaluationism.

Since they want to avoid the inference rule (T_c) (usually also called D-introduction), their preferred approach for supervaluational logic is local entailment. Due to the correspondence with modal logic, the axiomatization of supervaluational logic with local entailment depends on the constraints on the admissibility relation. They remark that one could check whether principles like reflexivity, symmetry, and transitivity make sense for specification spaces. However, they want to keep an abstract perspective and avoid talking about the nature of specification points.[12]. Therefore they want to keep their constraints on the admissibility relation minimal. However, they do impose reflexivity because they "take the move from $D\varphi$ to φ to be partly constitutive of the

[12]They suggest such considerations about the nature of specification points as an "optional extra in constructing supervaluation theory" [1]

concept of determinacy" [1]. We discuss constraints on the admissibility relation in a setting with higher-order vagueness in Section 2.4.2.

A major advantage that they see is that Fara's argument concerning higher-order vagueness [33] (see Section 2.4.3) does not apply because their logic does not support D-introduction. In fact, they manage to give an easy example of a precisification space in which all of the "gap principles" are satisfied.

Refined approach

Supertruth does not play any role in the definition of the local entailment relation. The final suggestion of Asher, Dever, and Pappas is to reintroduce supertruth by adding an actuality-style operator S that indicates supertruth. The truth function of this operator is based on a subset of precisifications in the precisification space, the *base* precisifications, "supposed to specify the D-free facts" [1]. In the following we redefine our central notions according to this new framework.

Definition 2.5.18. A *precisification space* is a quintuple $\boldsymbol{S} = \langle \boldsymbol{P}, \boldsymbol{A}, \boldsymbol{R}, \boldsymbol{D}, (\boldsymbol{M}_s)_{s \in \boldsymbol{P}} \rangle$ that consists of a nonempty set $\boldsymbol{P}$ of *precisifications*, a nonempty subset $\boldsymbol{A} \subseteq \boldsymbol{P}$ of *actual precisifications*, a binary *admissibility relation* $\boldsymbol{R}$ on $\boldsymbol{P}$, a nonempty set $\boldsymbol{D}$, the *domain* of $\boldsymbol{S}$, and a function $(\boldsymbol{M}_s)_{s \in \boldsymbol{P}}$ that assigns a classical first-order interpretation $\boldsymbol{M}_s$ with domain $\boldsymbol{D}$ to every precisification $s \in \boldsymbol{P}$.

The semantics of the classical connectives and the predicates is standard. The interpretation of the operators D and S is given by

$$\|\mathrm{D}\varphi\|_{s,\boldsymbol{v},\boldsymbol{S}} = \begin{cases} 1 & \text{if } \|\varphi\|_{t,\boldsymbol{v},\boldsymbol{S}} = 1 \text{ for all } t \in \boldsymbol{P} \text{ with } s\ \boldsymbol{R}\ t \\ 0 & \text{otherwise} \end{cases}$$

$$\|\mathrm{S}\varphi\|_{s,\boldsymbol{v},\boldsymbol{S}} = \begin{cases} 1 & \text{if } \|\varphi\|_{t,\boldsymbol{v},\boldsymbol{S}} = 1 \text{ for all } t \in \boldsymbol{A} \\ 0 & \text{otherwise} \end{cases}$$

where $\boldsymbol{S}$ is a precisification space, $s \in \boldsymbol{S}$ and $\boldsymbol{v}$ is a variable assignment of $\boldsymbol{S}$.

Note that a formula $\mathrm{S}\varphi$ is true in a precisification if and only if it is true in all precisification. In this case one can therefore also think of $\mathrm{S}\varphi$ being true in the specification space. Thus, a formula is defined to be supertrue iff it is true in all base precisifications.

Definition 2.5.19. A formula φ is *supertrue* in a precisification space $\boldsymbol{S}$ with actual precisifications $\boldsymbol{A}$ iff $\|\varphi\|_{s\boldsymbol{S}} = 1$ for every $s \in \boldsymbol{A}$.

An equivalent formulation is that φ is supertrue if $\mathrm{S}\varphi$ is true at any precisification in $\boldsymbol{A}$.

Local entailment in this framework is defined in the standard way. The global entailment relation is just defined from the local one, which is similar to the connection between global and local entailment in presence of Varzi's T-operator [99] (see Section 2.5.1).

Definition 2.5.20. A set of formulas $\{\varphi_1, \ldots, \varphi_n\}$ *globally entails* a formula ψ iff $S\varphi_1, \ldots, S\varphi_n \vDash_1 S\psi$.[13]

In this logic, the following connections hold between the S- and the D-operator:

- $S\varphi \supset DS\varphi$
- $\neg S\varphi \supset D\neg S\varphi$

The following relation is also desirable for a supervaluational logic:

$$\text{(SD) } S\varphi \supset D\varphi, \text{ for D-free } \varphi.$$

For this principle to hold, the following constraint has to be fulfilled: if a precisification $s \in \boldsymbol{S}$ is an admissible precisification, i.e., there is a $t \in \boldsymbol{S}$ such that $t \,\boldsymbol{R}\, s$, then there is an actual precisification $u \in \boldsymbol{A}$ such that their classical interpretations are the same, i.e., $\boldsymbol{M}_s = \boldsymbol{M}_u$.

The advantage of this logic (including the constraint for the (SD)-principle), as seen by Asher, Dever, and Pappas, is that Williamson's examples of failures of classical inference patterns (compare Section 2.2.1) can easily be fixed.

[13]Asher, Dever, and Pappas give a definition for finite sets, but the extension of the definition to infinite sets is unproblematic.

CHAPTER 3

Fuzzy logics based on t-norms

This section consists of a review of relevant aspects of mathematical fuzzy logic. We follow Hájek's monograph [56] for introducing the basic theory of t-norm based fuzzy logics. Unless noted otherwise, all results in this section are taken from this book. Another valuable source is Gottwald's monograph on many-valued logics [44] that gives a systematic overview of the knowledge on many-valued logics, including fuzzy logics. The most important facts can also be found in several survey papers by Hájek and Gottwald [45–47, 61]. In a recent survey [20], Cintula and Hájek sum up the knowledge on t-norm based predicate fuzzy logics as of 2009. Another recent paper [19] summarizes the completeness results for various kinds of semantics.

Concerning proof theory we only present axiomatizations in the form of Hilbert-style calculi. For a systematic treatment of the proof theory of fuzzy logic, with an emphasis on Gentzen-style systems, we refer to the book by Metcalfe, Olivetti, and Gabbay [82].

3.1 Motivation of fuzzy logic

A set S can be identified with its characteristic function

$$\chi_S(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S \end{cases}$$

that defines for each object x if it is a member of S or not. Zadeh introduced fuzzy sets [104] by allowing the full unit interval $[0, 1]$ as the range of characteristic functions. Intuitively speaking, fuzzy sets allow infinitely many membership degrees. Zadeh demonstrated suitable functions on fuzzy sets that generalize set operations like union, intersection and complement.

The relation of fuzzy sets to logic arises from the usual semantics of logic that interprets predicates as sets. For example, the set T of tall individuals could be the meaning of the predicate *tall*. Classically, the statement "Kim is tall" is true if Kim belongs to T and is false otherwise. If T is a fuzzy set, the statement might be true to a certain degree. Fuzziness is introduced to logic by taking [0, 1] as the set of truth values together with suitable connectives that are inspired by operations on fuzzy sets. Set intersection, for example, corresponds to conjunction in logic.

The success of fuzzy logic is due to the application of its principles in control engineering. In a nutshell, a fuzzy controller approximates functions with input and output variables about which partial, imprecise knowledge is given by *IF-THEN rules* [51]. Zadeh's idea of linguistic variables enables formulations of rules close to natural language, like for example

IF temperature is high THEN heating is low.

The controller associates each input and output value with a degree of membership to a linguistic variable. A value of 19°C for example could be a "high temperature" to degree 0.7. A process called *fuzzy inference* computes the membership degrees of the output values from the membership degrees of the input values.

Fuzzy control is little concerned with mathematical logic. Zadeh therefore suggests to distinguish between fuzzy logic in a *wide sense* as a toolkit for approximate reasoning, and fuzzy logic in a *narrow sense* as a formal system of many-valued logic in which traditional logical questions arise [105]. One of the main motivations of fuzzy logic in a narrow sense is to offer a logical foundation for fuzzy control. A recent development is to view mathematical fuzzy logic as a degree-theoretic approach for reasoning under vagueness [36].

As stated in the introduction we follow Hájek's approach towards mathematical fuzzy logic because it is the most influential one. Before we present any of his results, it seems useful to examine Hájek's "design choices" on formalizing fuzzy logic.

- **Truth values**: The unit interval [0, 1] is the set of truth values with 1 meaning absolute truth and 0 meaning absolute falsehood. The usual linear order $\leq$ on the real numbers imposes a *comparative notion of truth.*

- **Truth-functionality**: The truth value of a compound formula should only depend on the truth values of its parts. For a two-ary connective $\circ$, the truth value of $\varphi \circ \psi$ is $f_\circ(x, y)$ where $f_\circ$ is a function, x is the truth value of φ and y is the truth value of ψ.

- **Generalization of classical logic**: The truth functions of fuzzy logic behave classically for the truth values 1 and 0. A conjunction, for example, receives the truth value 0 if one of its conjuncts does.

- **T-norm conjunction**: The truth function of conjunction is a continuous t-norm. T-norm conjunctions are called *strong conjunctions.* Gottwald explains that in the 1980s the fuzzy community reached the consensus that set intersection, the equivalent to conjunction, should be defined from a t-norm [46].

- **Other connectives**: The choice of a t-norm determines the whole logic. Other connectives are defined from the t-norm: implication, negation, disjunction and also a second connective for conjunction called *weak conjunction.*

Besides *standard* semantics with truth functions on the unit interval that are defined from a t-norm, Hájek also studies *general* semantics with generalized truth functions on abstract truth degree structures. We can compare this situation to classical logic where Boolean algebras provide a lattice semantics. In the following, we focus on the standard semantics because they are more relevant for fuzzy logic as a degree-based theory of vagueness.

3.2 Truth functions

In the following, we introduce the connectives that will be used for fuzzy logic. We also discuss some alternatives to the truth functions proposed by Hájek. For an in-depth "exploration" of the space of possible truth functions we recommend Gottwald's monograph on many-valued logics [44].

3.2.1 Conjunction

The main requirement on a truth function of conjunction is formulated by Hájek as follows: "A large truth degree of [the conjunction "φ and ψ"] should indicate that both the truth degree of φ and the truth degree of ψ is large, without any preference between φ and ψ" [56]. By formalizing this requirement, Hájek arrives at the class of t-norms as good candidates for truth functions of conjunction.

Definition 3.2.1. A binary operation $*$ on the real unit interval $[0, 1]$ is a *triangular norm* (short: t-norm) iff it satisfies the following conditions for all $x, y, z \in [0, 1]$:

(T1) $*$ is associative:

$$x * (y * z) = (x * y) * z$$

(T2) $*$ is commutative:

$$x * y = y * x$$

(T3) $*$ is non-decreasing in both arguments:

$$x \leq y \text{ implies } x * z \leq y * z$$
$$x \leq y \text{ implies } z * x \leq z * y$$

(T4) $*$ has 1 as its neutral element and 0 as its zero element:

$$1 * x = x$$
$$0 * x = 0$$

Note that in condition (T3) it would be sufficient to demand non-decreasingness in one argument and in (T4) it would be sufficient to demand that 1 is the neutral element. Hájek argues that these conditions are quite natural to demand from a truth function of conjunction and thus the concept of a t-norm offers a very wide generalization of adequate truth functions of conjunction. Indeed, conditions (T1), (T2) and (T4) seem to be very intuitive for a truth function of conjunction. Condition (T3) can be explained as follows: If the truth degree of a conjunct is increased, the truth degree of the conjunction should not decrease. Consider for example the conjunction "φ and ψ" and the statement χ. Furthermore assume that χ has a higher truth value than ψ. Then the fuzzy logician's intuition is that the truth value of the conjunction "φ and χ" should not be lower than the truth value of "φ and ψ".

Following Hájek's monograph [56], we will restrict ourselves to t-norms that are continuous. Informally speaking, continuity means that small changes in the arguments of a function lead only to small changes in the result of the function. This condition is formalized with the usual ε-δ-criterion in the two-dimensional space $[0, 1]^2$. For t-norms, continuity is equivalent to continuity in each argument [44].

Proposition 3.2.2. *A t-norm $*$ is a continuous binary function if and only if for every $a \in [0, 1]$ the unary function f_a characterized by the equation*

$$f_a(x) = x * a$$

is a continuous (unary) function.

Example. The most important continuous t-norms are the following:

(i) Łukasiewicz t-norm: $x *_{Ł} y = \max(x + y - 1, 0)$

(ii) Gödel t-norm: $x *_{G} y = \min(x, y)$

(iii) Product t-norm: $x *_{P} y = x \cdot y$

These three t-norms are important from a historical point of view. The Łukasiewicz t-norm generalizes the conjunction used in Łukasiewicz' system of many-valued logic [79, 80]. Gödel, in his proof that intuitionistic logic is not a finite-valued logic [41], defined a finitely-valued non-classical logic with minimum conjunction that Dummett extended to infinitely many truth values [26]. Goguen, in his paper on fuzzy logic [43], suggested that multiplication and division are adequate truth functions of conjunction and implication in the unit interval. The concept of a continuous t-norm can be seen as a generalization of these three truth functions of conjunction.

The second importance of these these three t-norms arises from the fact that each continuous t-norm is a combination of these three fundamental t-norms. It can be shown that for any continuous t-norm $*$ the unit square $[0, 1]^2$ can be decomposed into a partition of disjoint sets $(X_i)_{i \in I}$ such that, for every $i \in I$, $*$ restricted to X_i is either Łukasiewicz, Gödel or the product t-norm. For a precise formulation of this statement we have to introduce the concepts of an order isomorphism and a generalized ordinal sum [44].

Definition 3.2.3. Let $[a_1, b_1] \subseteq [0, 1]$ and $[a_2, b_2] \subseteq [0, 1]$ be subintervals of the unit interval. An *order isomorphism* between $[a_1, b_1]$ and $[a_2, b_2]$ is a bijective function $f : [a_1, b_1] \to [a_2, b_2]$ such that

$$x < y \text{ if and only if } f(x) < f(y) .$$

Theorem 3.2.4 (Generalized ordinal sum representation). *For every continuous t-norm $*$ there is a countable family $([a_i, b_i], f_i, *_i)_{i \in I}$ with the following properties:*

- *For every $i \in I$, $[a_i, b_i]$ is a subinterval of $[0, 1]$ that is not a singleton.*
- *For all $i, j \in I$ such that $i \neq j$, the intersection $[a_i, b_i] \cap [a_j, b_j]$ is either empty or a singleton.*
- *For every $i \in I$, f_i is an order isomorphism from $[a_i, b_i]$ onto $[0, 1]$.*
- *For every $i \in I$, the t-norm $*_i$ is either equal to the Łukasiewicz t-norm or to the product t-norm.*
- *The t-norm $*$ can be characterized as follows:*

$$x * y = \begin{cases} f_k^{-1}\left(f_k(x) *_k f_k(y)\right) & \text{if } x, y \in \left[a_k, b_k\right] \text{ for some } k \in I \\ \min(x, y) & \text{otherwise} \end{cases}$$

Note that the index set I is allowed to be empty

Another important notion that we will need several times are idempotents.

Definition 3.2.5. An element $x \in [0, 1]$ is an *idempotent* of a continuous t-norm $*$ iff $x * x = x$.

In the ordinal sum representation of a continuous t-norm, the idempotents are exactly those points that are end-points of a subinterval or not contained in any subinterval at all.

Due to their central role in fuzzy logic and fuzzy control, t-norms themselves have become a field of study and many properties about them are known. A standard reference is the monograph by Klement, Mesiar and Pap [74].

3.2.2 Implication

The next preliminary that we have to settle is the truth function of implication. The standard approach is to take the residuum of a continuous t-norm.

Definition 3.2.6. Let $*$ be a continuous t-norm. Then the *residuum* $\Rightarrow_*$ of $*$ is defined by

$$x \Rightarrow_* y = \max\{z \in [0, 1] \mid x * z \leq y\} .$$

Hájek spends some time in his book to justify this choice. First he observes that the residuum has some properties that should be demanded from a truth function of implication. It behaves classically for 0 and 1 and it is non-increasing in the first argument, the antecedent, and non-decreasing in the second argument, the succedent. Intuitively, an increase in the truth value of the antecedent should not yield an increase in the truth value of the implication because it gets more difficult for the implication to be applicable. Conversely, an increase in the truth value of the succedent should make it easier for the implication to be applicable which means that the truth value of the implication should not decrease. As Hájek puts it: "A large truth value of [the implication "φ implies ψ"] should indicate that the truth value of φ is *not too much larger* than the truth value of ψ" [56].

He then shows that the residuum is not only suitable but also the best choice. His main argument is that the truth function of implication should allow a form of *fuzzy modus ponens* which, as he shows, imposes a characterizing condition. The argument goes as follows: Let $\Rightarrow$ denote a truth function of implication. Now consider the following situation for the implication "φ implies ψ", its antecedent φ, and its succedent ψ. Let a be a lower bound of the truth value x of φ and let b be a lower bound of the truth value $x \Rightarrow y$ of "φ implies ψ". Then we want to find out how true ψ is, i.e., we want to compute a lower bound c of the truth value y of ψ.

Now the question is how the lower bound c should be computed from the lower bounds a and b. This means that we are searching for a binary operation $\circ$ such that $c = a \circ b$. Hájek argues that a natural requirement on $\circ$ is that it is non-decreasing in both arguments: The higher a is, the more we know that φ is true and the more the implication "φ implies ψ" should apply which makes ψ more true and thus also increases c. The higher b is, the more we know that the implication "φ implies ψ" is true and the more true it should be to conclude ψ from φ which again increases c. As Hájek notes it is difficult to find arguments why the operation $\circ$ should be associative and commutative. Nevertheless he suggests that it would be useful to take a t-norm for this operation.

Thus we get the following condition on $\Rightarrow$:

$$\text{If } a \leq x \text{ and } b \leq (x \Rightarrow y)\text{, then } a * b = c \leq y\,.$$

By setting $a = x$ and writing z instead of b we get

$$\text{If } z \leq (x \Rightarrow y)\text{, then } x * z \leq y\,. \tag{3.1}$$

A second constraint emerges by the wish to make the implication as powerful as possible: Hájek demands that whenever in condition (3.1) the conclusion holds also the premiss should hold. Therefore, whenever $x * z \leq y$ holds, z is a possible candidate for the truth value $x \Rightarrow y$. Thus we get a stronger condition:

$$z \leq (x \Rightarrow y) \text{ if and only if } x * z \leq y\,. \tag{3.2}$$

Hájek then shows that for every continuous t-norm $*$, the residuum, which computes the maximal z satisfying $x * z$, is the unique operation that satisfies condition (3.2).

Proposition 3.2.7. *Let $*$ be a continuous t-norm and $x, y, z \in [0, 1]$. Then*

$$(x * z) \leq y \text{ if and only if } z \leq (x \Rightarrow_* y)\,.$$

The residuum $\Rightarrow_*$ is the only operation that has this property.

Gottwald explains that due to this property, residua generalize the characterization of intuitionistic implication connectives in Heyting algebras [46]. As the discussion above shows, the principle of fuzzy modus ponens holds for the residuum. This is the content of the next result.

Proposition 3.2.8. *Let $*$ be a continuous t-norm and $x, y \in [0, 1]$. Then*

$$x * (x \Rightarrow_* y) \leq y\,.$$

The inequality in this proposition can be made precise: For every continuous t-norm, the left term in the inequality defines the function that computes the minimum of two values, which is exactly the Gödel t-norm. In the fuzzy logics that we will work with, one therefore usually considers two conjunction connectives. The first one is a *strong conjunction* given by a continuous t-norm and the second one is a *weak conjunction* given by the Gödel t-norm.

Proposition 3.2.9. *For every continuous t-norm and all $x, y \in [0, 1]$ we have*

$$\min(x, y) = x * (x \Rightarrow_* y)\,.$$

The main reason why the continuity of the t-norm was demanded is to ensure that the residuum is well-defined. However this seems to be a rather shallow justification compared to Hájek's other design choices. Alternatively one can relax this constraint to left-continuity and the definition of the residuum would still work, but left-continuity seems even harder to justify. And still it would leave the problem why one demands (left)-continuity for the truth function of conjunction but not for the truth function of implication. Gottwald argues that the class of all t-norms is not yet well understood and therefore the restriction to continuous or left-continuous t-norms is a reasonable simplification [46].

We close our discussion of residua by stating two simple but important properties.

Proposition 3.2.10. *Let $*$ be a continuous t-norm. Then we have the following properties:*

(i) $x \leq y$ *if and only if* $(x \Rightarrow_* y) = 1$.

(ii) $(1 \Rightarrow_* y) = y$.

According to Gottwald [46], an alternative to residua is to introduce implication using either disjunction and negation or using conjunction and negation. For example one could read the implication "φ implies ψ" merely as an abbreviation for "not φ or ψ" if the truth functions of negation and disjunction have already been defined independently from the truth function of implication. Such truth functions of implication are called *S-implications* in the fuzzy community whereas residua are called *R-implications*. Gottwald argues that this approach has some difficulties with the main disadvantage being the following: In the residuum-based approach we get a natural lower bound on the truth degree of the succedent of an implication due to fuzzy modus ponens, as seen above. A similar property in general fails for S-implications in many cases. A survey by Klement and Navara [75] mentions as an advantage of the S-implication based approach that the truth function of implication is continuous if the other truth functions are continuous, too, whereas a continuous t-norm does not guarantee a continuous residuum. However, they explain, the residuum-based approach allows deeper logical results. We will not explicitly deal with S-implications in the following with the exception of Łukasiewicz logic where the S-implication and the residuum coincide. For results on S-fuzzy logics, in particular concerning axiomatizations, compactness, and comparisons to the residuum-based approach we refer to the papers [10, 68, 75].

3.2.3 Biimplication

As usual, two statements φ and ψ are considered equivalent iff both "φ implies ψ" and "ψ implies φ" hold

Definition 3.2.11. The *truth function of biimplication* $\Leftrightarrow_*$ of a continuous t-norm $*$ is given by the relation

$$x \Leftrightarrow_* y = \left(x \Rightarrow_* y\right) * \left(y \Rightarrow_* x\right)$$

One might wonder if it would be better to define the biimplication with the weak minimum conjunction instead of the strong conjunction. It turns out that it does not matter because both approaches are equivalent.

Proposition 3.2.12. *For every continuous t-norm $*$ and all $x, y \in [0, 1]$ we have*

$$x \Leftrightarrow_* y = \min\left(\left(x \Rightarrow_* y\right), \left(y \Rightarrow_* x\right)\right) .$$

With Proposition 3.2.10 it is easy to check that the biimplication evaluates to 1 only if the left hand side and the right hand side are equal and thus it is really suited to capture equivalence of formulas.

Proposition 3.2.13. *For every continuous t-norm $*$ and all $x, y \in [0, 1]$ we have*

$$x \Leftrightarrow_* y = 1 \text{ if and only if } x = y .$$

3.2.4 Negation

Hájek suggests an "implies falsum" negation like in intuitionistic logic, which means that its truth function is defined from the residuum and the truth value 0.

Definition 3.2.14. For every continuous t-norm $*$, the *precomplement* $-_*$ *of* $*$ is the unary relation on the real unit interval $[0,1]$ given by

$$-_*(x) = \left(x \Rightarrow_* 0\right) .$$

Example. The precomplements of Łukasiewicz and the Gödel t-norm are two important examples of truth functions of negation.

- Łukasiewicz negation:

$$-_Ł(x) = 1 - x$$

- Gödel negation:

$$-_G(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

Note that Gödel negation only returns the "crisp" values 0 and 1. This property could be seen as a disadvantage in applications [75]. It can be shown that several other t-norms have Gödel negation as its precomplement.

Proposition 3.2.15. *If the continuous t-norm* $*$ *fulfills the condition*[14]

$$x * y = 0 \textit{ implies } x = 0 \textit{ or } y = 0$$

for all $x, y \in [0,1]$, *then its precomplement* $-_*$ *is equal to Gödel negation* $-_G$.

Due to Proposition 3.2.10 only the negation of falsehood returns truth.

Proposition 3.2.16. *Let* $*$ *be a continuous t-norm. Then* $-_*(x) = 1$ *if and only if* $x = 0$.

Note that this relation does not hold when the roles of 1 and 0 are exchanged: Gödel negation is a counterexample.

It would also be possible to allow negation functions based on other considerations. *Strong* negation functions seem to be the most important class of alternative negation functions [44].

Definition 3.2.17. A unary relation n on the unit interval $[0,1]$ is called *strong negation function* iff it satisfies the following conditions for all $x \in [0,1]$:

(N1) n is classical negation on $\{0,1\}$:

$$n(0) = 1$$
$$n(1) = 0$$

[14]In algebraic terms, this condition states that $*$ has no *nontrivial zero divisors*.

(N2) n is strictly decreasing:

$$x < y \text{ implies } n(x) > n(y)$$

(N3) n is continuous

(N4) n is involutive:

$$n(n(x)) = x$$

Esteva, Godo, Hájek, and Navara remark that condition (N3) is redundant [30].

Remark. The Łukasiewicz negation is a strong negation function. Gödel negation is not a strong negation function.

3.2.5 Disjunction

One possibility to define a truth function of disjunction would be to make similar considerations as for the truth function of conjunction. Dually to t-norms, one then arrives at t-conorms as good candidates for such truth functions.

Definition 3.2.18. A binary operation $\oplus$ on the real unit interval $[0, 1]$ is a *triangular conorm* (short: t-conorm) iff it satisfies the following conditions for all $x, y, z \in [0, 1]$:

(S1) $\oplus$ is associative:

$$x \oplus (y \oplus z) = (x \oplus y) \oplus z$$

(S2) $\oplus$ is commutative:

$$x \oplus y = y \oplus x$$

(S3) $\oplus$ is non-decreasing in both arguments:

$$x \leq y \text{ implies } x \oplus z \leq y \oplus z$$
$$x \leq y \text{ implies } z \oplus x \leq z \oplus y$$

(S4) $\oplus$ has 0 as its neutral element and 1 as its zero element:

$$0 \oplus x = x$$
$$1 \oplus x = 1$$

Note that the conditions (S1)–(S3) for t-conorms are exactly the same as (T1)–(T3) for t-norms.

In the presence of a strong negation function it is possible to define a t-conorm from a t-norm and vice versa using DeMorgan's laws [44].

Proposition 3.2.19. *Let n be a strong negation function. For every t-norm $*$, the function given by*

$$x \oplus y = n(n(x) * n(y))$$

is a t-conorm. For every t-conorm $\oplus$, the function given by

$$x * y = n(n(x) \oplus n(y))$$

is a t-norm.

Although t-conorms seem to be a natural choice for truth functions of disjunction, and are also used in fuzzy control, Hájek's approach mostly disregards them. Instead, only a weak disjunction connective corresponding to the maximum operator is introduced. This maximum disjunction is definable from the continuous t-norm and its residuum.

Proposition 3.2.20. *For every continuous t-norm and all $x, y \in [0, 1]$ we have*

$$\max(x, y) = \min\left(\left(x \Rightarrow_* y\right) \Rightarrow_* y, \left(y \Rightarrow_* x\right) \Rightarrow_* x\right) .$$

3.3 Logical notions for fuzzy logics

Before we proceed with presenting concrete fuzzy logics, we discuss how the central logical notions satisfiability, validity and entailment can be defined in a logic based on the truth value set $[0, 1]$ with the usual order $\leq$ on the truth values. We compare different definitions for these concepts as they occur in the literature. Relatedly, we also discuss some aspects of provability.

In the following, we want to define logical notions for a fuzzy logic X. We assume that there is a set of formulas and a set of possible interpretations $\boldsymbol{I}$. Each interpretation $\boldsymbol{M} \in \boldsymbol{I}$ determines an interpretation function $\|\cdot\|_{\boldsymbol{M}}^{\mathsf{X}}$ that maps formulas to truth values in the unit interval $[0, 1]$. The concrete syntax and semantics of our degree-theoretic logic will be left unspecified. Instead, an abstract framework will be presented with the idea being that a concrete syntax and semantics can easily be "plugged in" to define a concrete logic.

Note that we only consider "crisp" approaches towards validity, entailment and provability. For example we assume that a formula can be either valid or not, but we do not consider that a formula might be valid to a certain degree. It is of course also possible to transfer the concept of fuzziness to the outer logical level and to work with graded notions of validity, entailment and provability. We shortly describe one such approach called *Pavelka logic* in Section 3.7.

3.3.1 Designated truth values

The crucial question for a many-valued logic is which truth value(s) should be regarded as designated [44]. The usual assumption is that 1 should be a designated truth value in any case and that 0 should not. The following options for the designated truth values appear to be quite natural:

(i) 1 is the only designated truth value.

(ii) Every value in $(0, 1]$ is designated, i.e., 0 is the only truth value that is not designated.

(iii) Every value in $[\varepsilon, 1]$ is designated for a concrete $\varepsilon \in [0, 1]$.

(iv) Every value in $[\varepsilon, 1]$ is designated, but we abstract from a concrete choice of ε. Every $\varepsilon \in [0, 1]$ should be considered.

Option (i) is the prevalent way of defining satisfiability, validity and entailment. In the following definitions we will speak of 1-models, 1-satisfiability, 1-validity and 1-entailment, but later we will occasionally omit the 1-prefix because it is standard to do so. Option (ii) is usually considered only in the scope of the complexity of satisfiability and validity in fuzzy logics. Option (iii) can be made for every many-valued logic with a comparative notion of truth [44]: as soon as a certain $\varepsilon \in [0, 1]$ is considered designated, also every $\delta \geq \varepsilon$ has to be considered designated. The intrinsic problem seems to be how a concrete choice of ε can be justified and—to the best of our knowledge—this approach has only little relevance for mathematical fuzzy logics. A further possibility would be to consider the possibility of an arbitrary subset of $[0, 1]$ being designated, as it is done by Cintula and Navara who studied different forms of compactness in fuzzy logics [21]. However, this approach does not seem to fit together with the idea of a comparative notion of truth which is why we do not pursue it here. Option (iv) is advocated by Priest [90] and only makes a difference to option (i) when it comes to entailment relations.

Similar to satisfiability, validity and entailment with regard to truth, one could also be explicitly interested in the analogous relations for falsehood. In this case it would be necessary not only to look at the designated truth values, but also at the *anti-designated* ones [44], where 0 is usually anti-designated 1 is usually not anti-designated. However, we omit this discussion because in t-norm based fuzzy logics the relations for falsehood should in principle be dual to the relations for truth and therefore no fundamentally new insights can be expected. Furthermore, due to our choice of truth functions and Proposition 3.2.16, we know that a formula evaluates to 0 if and only if its negation evaluates to 1. Therefore most interesting relations concerning falsehood should be transformable to their counterparts for truth.

3.3.2 Satisfiability

First, we define a satisfaction relation between interpretations and formulas, or sets of formulas respectively. This relation denotes whether a formula is true in an interpretation, where the notion of truth depends on the set of designated truth values.

Definition 3.3.1. Let $\boldsymbol{M}$ be an interpretation, φ a set of formulas and $\varepsilon \in [0, 1]$.

- $\boldsymbol{M}$ is a 1-*model of* φ ($\boldsymbol{M} \models_1^{\mathsf{X}} \varphi$) iff $\|\varphi\|_{\boldsymbol{M}}^{\mathsf{X}} = 1$.
- $\boldsymbol{M}$ is a *(>0)-model of* φ ($\boldsymbol{M} \models_{>0}^{\mathsf{X}} \varphi$) iff $\|\varphi\|_{\boldsymbol{M}}^{\mathsf{X}} > 0$.
- $\boldsymbol{M}$ is a *(≥ε)-model of* φ ($\boldsymbol{M} \models_{\geq\varepsilon}^{\mathsf{X}} \varphi$) iff $\|\varphi\|_{\boldsymbol{M}}^{\mathsf{X}} \geq \varepsilon$.
- $\boldsymbol{M}$ is a α-*model of* φ ($\boldsymbol{M} \models_{\alpha}^{\mathsf{X}} \varphi$) iff $\boldsymbol{M} \models_{\geq\delta}^{\mathsf{X}}$ for every $\delta \in [0, 1]$.

Let ★ be one of the following:[15] 1, (>0), (≥ε) or α. We say that φ is ★*-satisfiable* iff it has a ★-model. Let Γ be a set of formulas. Then $\boldsymbol{M} \models_{\bigstar}^{\mathsf{X}} \Gamma$ iff $\boldsymbol{M} \models_{\bigstar}^{\mathsf{X}} \psi$ for every $\psi \in \Gamma$.

Note that α-satisfiability and 1-satisfiability trivially coincide. In the literature, 1-satisfiability and (>0)-satisfiability are the dominant concepts, especially when the complexity of satisfiability in a propositional fuzzy logic is studied. Therefore it is convenient to define the following sets of satisfiable propositional formulas.

Definition 3.3.2. For every fuzzy logic X we define the following sets:

$$\mathbf{SAT}_1^{\mathsf{X}} = \{\varphi \mid \varphi \text{ has a 1-model of } \mathsf{X}\}$$
$$\mathbf{SAT}_{>0}^{\mathsf{X}} = \{\varphi \mid \varphi \text{ has a (>0)-model of } \mathsf{X}\}\,.$$

3.3.3 Validity

As usual we say that a formula is valid iff it is true for every interpretation where the meaning of truth depends on our notion of satisfaction. This means that a formula φ is valid iff every interpretation is a model of φ.

Definition 3.3.3. Let φ be a formula and $\varepsilon \in [0, 1]$.

- φ is 1-*valid* iff $\boldsymbol{M} \models_1^{\mathsf{X}} \varphi$ for every interpretation $\boldsymbol{M}$.
- φ is *(>0)-valid* iff $\boldsymbol{M} \models_{>0}^{\mathsf{X}} \varphi$ for every interpretation $\boldsymbol{M}$.
- φ is *(≥ε)-valid* iff $\boldsymbol{M} \models_{\geq\varepsilon}^{\mathsf{X}} \varphi$ for every interpretation $\boldsymbol{M}$.
- φ is α-*valid* iff φ is (≥δ)-valid for every $\delta \in [0, 1]$.

Note that α-validity and 1-validity trivially coincide. Again the most important concepts are 1-validity and (>0)-validity which is why we make an additional definition for the corresponding sets of propositional formulas.

Definition 3.3.4. For every fuzzy logic X we define the following sets:

$$\mathbf{TAUT}_1^{\mathsf{X}} = \{\varphi \mid \varphi \text{ is 1-valid in } \mathsf{X}\}$$
$$\mathbf{TAUT}_{>0}^{\mathsf{X}} = \{\varphi \mid \varphi \text{ is (>0)-valid in } \mathsf{X}\}$$

[15] The symbol ★ is to be understood as a placeholder.

3.3.4 Entailment

Finally, we take a look at the entailment relation. The entailment relation denotes whether a formula φ, the conclusion, is a logical consequence of a set of formulas Γ, the premisses. In this case we also say that Γ *entails* φ. An entailment relation should always be truth-preserving which means that the conclusion should always be at least as true as the premisses. We define different entailment relations according to our four options for the set of designated truth values.

Definition 3.3.5. Let Γ be a (possibly infinite) set of formulas and φ a formula.

- $\Gamma \models_1^{\mathsf{X}} \varphi$ iff $\boldsymbol{M} \models_1^{\mathsf{X}} \Gamma$ implies $\boldsymbol{M} \models_1^{\mathsf{X}} \Gamma$ for every interpretation $\boldsymbol{M}$
- $\Gamma \models_{>0}^{\mathsf{X}} \varphi$ iff $\boldsymbol{M} \models_{>0}^{\mathsf{X}} \Gamma$ implies $\boldsymbol{M} \models_{>0}^{\mathsf{X}} \Gamma$ for every interpretation $\boldsymbol{M}$
- $\Gamma \models_{\varepsilon}^{\mathsf{X}} \varphi$ iff $\boldsymbol{M} \models_{\varepsilon}^{\mathsf{X}} \Gamma$ implies $\boldsymbol{M} \models_{\varepsilon}^{\mathsf{X}} \Gamma$ for every interpretation $\boldsymbol{M}$
- $\Gamma \models_{\alpha}^{\mathsf{X}} \varphi$ iff $\Gamma \models_{\delta}^{\mathsf{X}} \varphi$ for every $\delta \in [0, 1]$

As usual, validity amounts to being a logical consequence of the empty set of premisses.

Proposition 3.3.6. *Let φ be a formula and $\bigstar$ one of the following: 1, (>0), (≥ε), or α. Then*

$$\varphi \text{ is } \bigstar\text{-valid if and only if } \varnothing \models_{\bigstar} \varphi\,.$$

As Priest points out, there is an easy characterization of the fourth consequence relation [90].

Proposition 3.3.7. *For every formula φ, $\Gamma \models_{\alpha}^{\mathsf{X}} \varphi$ if and only if*

$$\inf\left(\left\{\|\psi\|_{\boldsymbol{M}}^{\mathsf{X}} \mid \psi \in \Gamma\right\}\right) \leq \|\varphi\|_{\boldsymbol{M}}^{\mathsf{X}} \text{ for every interpretation } \boldsymbol{M}.$$

The infimum $\inf(S)$ *of a set S is its greatest lower bound.*

Following the idea of truth preservation, it would of course be legitimate to define an entailment relation according to this characterization in the first place.

Priest also mentions the following reduction to 1-validity. In a finite set, the infimum is equal to the supremum. If we consider a fuzzy logic $\mathsf{FL}(*)$ based on a continuous t-norm $*$, a notion that we make precise in Section 3.4, and a finite set of formulas Γ, then $\Gamma \models_{\alpha} \varphi$ if and only if $\min(\|\psi_1\|_{\boldsymbol{M}}^{\mathsf{X}}, \ldots, \|\psi_n\|_{\boldsymbol{M}}^{\mathsf{X}}) \leq \|\varphi\|_{\boldsymbol{M}}^{\mathsf{X}}$ for every interpretation $\boldsymbol{M}$. Define the formula χ as $(\psi_1 \wedge \ldots \wedge \psi_n) \supset \varphi$. Then, our truth functions of fuzzy logic guarantee that for every interpretation $\boldsymbol{M}$, we have $\|\chi\|_{\boldsymbol{M}}^{\mathsf{FL}(*)}$ if and only if $\min(\|\psi_1\|_{\boldsymbol{M}}^{\mathsf{X}}, \ldots, \|\psi_n\|_{\boldsymbol{M}}^{\mathsf{X}}) \leq \|\varphi\|_{\boldsymbol{M}}^{\mathsf{X}}$. Thus we obtain the following result.

Proposition 3.3.8. *Let $\Gamma = \{\psi_1, \ldots, \psi_n\}$ be a finite set of formulas and φ a formula. Then $\Gamma \models_{\alpha}^{\mathsf{FL}(*)} \varphi$ if and only if the formula $(\psi_1 \wedge \ldots \wedge \psi_n) \supset \varphi$ is 1-valid in $\mathsf{FL}(*)$.*

According to Priest, this connection justifies focusing on 1-validity as the central notion for studying mathematical fuzzy logic [90]. To the best of our knowledge, there are no approaches towards fuzzy logic that are based on (>0)-entailment or ($\geq\varepsilon$)-entailment.

3.3.5 Provability

Usually one is not only interested in semantic definitions of a logic but also in syntactic characterizations in the form of proof systems. A proof system, also called calculus, determines a set of provable formulas and a provability relation for which we will use the symbol $\vdash$, usually together with an index that indicates for which logic the proof system is intended. If Γ is a set of formulas and φ a formula, then $\Gamma \vdash \varphi$ means that φ is provable from Γ and $\vdash\varphi$ means that φ is provable from the empty set of formulas. We will concentrate on Hilbert-style proof systems that consist of a set of axiom schemata and a set of rules. Furthermore we will identify a proof system with its provability relation.

When a proof system for a logic is developed, the natural question to ask is whether it is an adequate syntactic characterization of the logic under consideration. In such a case we say that the logic can be axiomatized or has an axiomatization. The first condition one imposes on such a proof system is that its proofs, as syntactic objects, have finite length. The second condition is that the set of proofs must be recursively enumerable, i.e., the question whether an arbitrary syntactic object is a legal proof in the given proof system is decidable. The two remaining conditions are called *soundness* and *completeness* that we define in the following. Both refer to the set of valid formulas of a logic and its entailment relation. Precisely speaking we will, unless noted otherwise, always mean 1-validity and 1-entailment as discussed in section 3.3.

Definition 3.3.9. Let $\vdash$ be a provability relation and X a logic (with a notion of validity and an entailment relation $\models^{\mathsf{X}}$).

- $\vdash$ *is sound for* X iff for every formula φ

$$\vdash\varphi \text{ implies } \varphi \text{ is valid in } \mathsf{X}.$$

- $\vdash$ *is strongly sound for* X iff for every set of formulas Γ and every formula φ

$$\Gamma \vdash \varphi \text{ implies } \Gamma \models^{\mathsf{X}} \varphi\,.$$

- $\vdash$ *is complete for* X iff for every formula φ

$$\varphi \text{ is valid in } \mathsf{X} \text{ implies } \vdash \varphi\,.$$

- $\vdash$ *is strongly complete for* X iff for every set of formulas Γ and every formula φ

$$\Gamma \models^{\mathsf{X}} \varphi \text{ implies } \Gamma \vdash \varphi\,.$$

- $\vdash$ *is finitely strongly complete for* X iff for every *finite* set of formulas Γ and every formula φ

$$\Gamma \models^{X} \varphi \text{ implies } \Gamma \vdash \varphi\,.$$

Remark. For the logics that we consider, the set of valid formulas is always equal to the set of formulas that are entailed by the empty set of formulas. Therefore we have the following relations:

- Strong soundness implies soundness.
- Strong completeness implies finite strong completeness.
- Finite strong completeness implies completeness.

We will only explicitly mention the strongest result known for every logic that we present.

Soundness proofs are usually a routine matter whereas completeness proofs often demand mathematical depth. Usually it makes little sense to talk about proof systems that are not sound.

For fuzzy logics, a further distinction has to be made between *general* completeness and *standard* completeness. Standard completeness refers to completeness with regard to standard semantics with their t-norm based truth functions on the unit interval $[0, 1]$ as defined in Section 3.4. General completeness refers to completeness with regard to algebraic semantics that generalize truth functions and truth degree structures. In this thesis, we always state standard completeness results because the standard semantics are more relevant for a logic of vagueness.

Sometimes it is not possible to obtain strong completeness, but only finite strong completeness. This situation usually occurs when the entailment relation is not compact [90].[16]

Definition 3.3.10. An entailment relation $\models$ is *compact* iff whenever $\Gamma \models \varphi$ for a set of formulas Γ and a formula φ there is some finite subset $\Gamma' \subseteq \Gamma$ such that $\Gamma' \models \varphi$.

Remark. If an entailment relation $\models$ is not compact, then there is no strongly sound and complete proof system for $\models$. Every proof can only use finitely many premisses. If we had a strongly sound and complete proof system we would therefore obtain compactness.

Furthermore we will see cases where completeness is not possible for any adequate proof system. The reason then usually is that the set of valid formulas is not recursively enumerable. Once we know that completeness is not possible, we also know that strong completeness and finite strong completeness are not possible.

[16]Attention: In most other contexts, *compactness* refers to compactness of the satisfaction relation which we do not discuss in this thesis. Cintula and Navara study this form of compactness [21].

Remark. There is no sound and complete proof system with a recursively enumerable set of legal proofs for a logic where the set of valid formulas is not recursively enumerable. If we had such a proof system it would give us a recursive enumeration of the set of valid formulas

3.4 Propositional fuzzy logics

After having settled all preliminaries, we now define some propositional t-norm based fuzzy logics and present the most important results about them.

3.4.1 Syntax and semantics

First, we define the set of legal formulas.

Definition 3.4.1. The *language of fuzzy logic* is generated by a set of propositional variables, the truth constant $\bar{0}$, and the two-ary connectives & and $\supset$. We use the following abbreviations:

$\bar{1}$	abbreviates	$\neg\bar{0}$
$\varphi \wedge \psi$	abbreviates	$\varphi \,\&\, (\varphi \supset \psi)$
$\varphi \vee \psi$	abbreviates	$((\varphi \supset \psi) \supset \psi) \wedge ((\psi \supset \varphi) \supset \varphi)$
$\neg\varphi$	abbreviates	$\varphi \supset \bar{0}$
$\varphi \equiv \psi$	abbreviates	$(\varphi \supset \psi) \,\&\, (\psi \supset \varphi)$

The connective & is called *strong conjunction* and the connective $\wedge$ is called *weak conjunction.*

These abbreviations guarantee that the corresponding connectives will receive the intended semantics as discussed in Section 3.2. While the abbreviations for negation and biimplication are fairly standard, it would be less clear how the abbreviations for weak conjunction and weak disjunctions could be justified without the knowledge that they will be interpreted as the minimum and the maximum operator for every continuous t-norm.

We now define how formulas of propositional fuzzy logic are interpreted. The idea is that the truth value of a formula depends on the truth values of the propositional variables and on the semantics of the connectives, which is fully determined by the choice of a continuous t-norm.

Definition 3.4.2. An *evaluation* of propositional variables is a function $\boldsymbol{e}$ that assigns to each propositional variable a truth value in $[0, 1]$.

Definition 3.4.3. Let $*$ be a continuous t-norm and $\boldsymbol{e}$ an evaluation of propositional variables. Then the *interpretation of formulas* is inductively defined as follows:

$$\begin{aligned} \|p\|_{\boldsymbol{e}}^{*} &= \boldsymbol{e}(p) \\ \|\bar{0}\|_{\boldsymbol{e}}^{*} &= 0 \\ \|\varphi \mathbin{\&} \psi\|_{\boldsymbol{e}}^{*} &= \|\varphi\|_{\boldsymbol{e}}^{*} * \|\psi\|_{\boldsymbol{e}}^{*} \\ \|\varphi \supset \psi\|_{\boldsymbol{e}}^{*} &= \|\varphi\|_{\boldsymbol{e}}^{*} \Rightarrow_{*} \|\psi\|_{\boldsymbol{e}}^{*} \end{aligned}$$

It can now easily be checked that the abbreviated connectives receive their intended semantics (see the discussion in Section 3.2).

Proposition 3.4.4. *Let $*$ be a continuous t-norm and $\boldsymbol{e}$ an evaluation of propositional variables. Then for all formulas φ and ψ*

$$\begin{aligned} \|\neg\varphi\|_{\boldsymbol{e}}^{*} &= -_{*}\left(\|\varphi\|_{\boldsymbol{e}}^{*}\right) \\ \|\varphi \wedge \psi\|_{\boldsymbol{e}}^{*} &= \min\left(\|\varphi\|_{\boldsymbol{e}}^{*}, \|\psi\|_{\boldsymbol{e}}^{*}\right) \\ \|\varphi \vee \psi\|_{\boldsymbol{e}}^{*} &= \max\left(\|\varphi\|_{\boldsymbol{e}}^{*}, \|\psi\|_{\boldsymbol{e}}^{*}\right) \\ \|\varphi \equiv \psi\|_{\boldsymbol{e}}^{*} &= \|\varphi\|_{\boldsymbol{e}}^{*} \Leftrightarrow_{*} \|\psi\|_{\boldsymbol{e}}^{*} \end{aligned}$$

The semantics of a t-norm based fuzzy logic is now defined in the obvious way: We fix a continuous t-norm $*$ and take all evaluations as possible interpretations. Satisfiability, validity and entailment are then defined as outlined in Section 3.3. The propositional t-norm based logic defined in this way will be called $\mathsf{FL}(*)$. We use the following abbreviations:

- Łukasiewicz logic is $\text{Ł} = \mathsf{FL}(*_{\text{Ł}})$ where $*_{\text{Ł}}$ is the Łukasiewicz t-norm.
- Gödel logic is $\mathsf{G} = \mathsf{FL}(*_{\mathrm{G}})$ where $*_{\mathrm{G}}$ is the Gödel t-norm.
- Product logic is $\mathsf{P} = \mathsf{FL}(*_{\mathrm{P}})$ where $*_{\mathrm{P}}$ is the product t-norm.

It is this notion of a t-norm based logic that we are mainly interested in. However it is also of interest to know which relations hold for all continuous t-norms. For this purpose we also define a logic called *basic logic* BL.

Definition 3.4.5. A BL-*interpretation* is a pair $\langle *, \boldsymbol{e}\rangle$ that consists of a continuous t-norm $*$ and an evaluation $\boldsymbol{e}$. Every BL-interpretation $\boldsymbol{M} = \langle *, \boldsymbol{e}\rangle$ determines an interpretation function $\|\cdot\|_{\boldsymbol{M}}^{\mathsf{BL}}$ given by

$$\|\varphi\|_{\boldsymbol{M}}^{\mathsf{BL}} = \|\varphi\|_{\boldsymbol{e}}^{*}.$$

Basic logic BL is the logic emerging from all BL-interpretations together with the usual definitions of 1-satisfaction, 1-validity and 1-entailment.[17]

[17]The usual approach to define basic logic is the following: A $*$-tautology is a formula that always evaluates to 1 for a continuous t-norm $*$. A t-tautology is a formula that is a $*$-tautology for every

3.4.2 Basic logic

Strictly speaking, basic logic is not a t-norm based logic because it was not defined from a single t-norm, but the class of all continuous t-norms. Despite this special status, basic logic is very important as a common theoretical ground for logics based on continuous t-norms. In particular, we obtain proof systems for our three basic propositional logics by extending a proof system for basic logic. We define this Hilbert-style proof system in the following.

Definition 3.4.6. The proof system $\vdash^{\mathsf{BL}}$ is given by the axiom schemata

(A1) $(\varphi \supset \psi) \supset ((\psi \supset \chi) \supset (\varphi \supset \chi))$

(A4) $(\varphi \wedge \psi) \supset (\psi \wedge \varphi)$

(A5) $(\varphi \supset (\psi \supset \chi)) \equiv ((\varphi \,\&\, \psi) \supset \chi)$

(A6) $((\varphi \supset \psi) \supset \chi) \supset (((\psi \supset \varphi) \supset \chi) \supset \chi)$

(A7) $\bar{0} \supset \varphi$

and the deduction rule *modus ponens* "From φ and $\varphi \supset \psi$, infer ψ".

We remark that after the publication of Hájek's monograph [56] two of Hájek's original axioms, called (A2) and (A3), have been shown to be redundant, whereas the remaining axioms are independent [14, 18]. For historical reasons we keep the original names of the axioms. Axiom schema (A1) expresses the transitivity of implication. (A4) is the commutativity of weak conjunction. (A5) expresses residuation and deresiduation corresponding to Proposition 3.2.7. (A6) is a variant of the proof by cases. (A7) says that $\bar{0}$ is the bottom element, it implies everything (*ex falso quodlibet*).

It can easily be seen that this deduction system is sound by checking that every instance of an axiom schema is valid and that if φ and $\varphi \supset \psi$ are valid, also ψ is valid. This soundness does not only hold for basic logic but also for every propositional fuzzy logic $\mathsf{FL}(*)$ based on a continuous t-norm $*$.

The completeness of this proof system with regard to basic logic was not yet known when Hájek's monograph [56] was written. A proof of this fact was given by Cignoli, Esteva, Goda, and Torrens [16], building on Hájek's subsequent work [48]. Montagna mentions in the introduction of one of his papers [83] that the proof given in [16] also establishes finite strong completeness.

continuous t-norm and basic logic consists of all t-tautologies. But it makes sense to explicitly define the interpretation structure for basic logic as above because in our framework also 1-satisfiability and 1-entailment then correspond to satisfiability and entailment as they are usually understood for basic logic. However, one could be interested in alternative notions of satisfiability for a formula φ in basic logic: One could for example demand that for every continuous t-norm there is a 1-evaluation for φ, or even more, demand that there exists *one* 1-evaluation that satisfies φ for every continuous t-norm

Theorem 3.4.7 (Finite strong completeness). *For every* finite *set of formulas Γ and every formula φ we have*

$$\Gamma \vdash^{\mathsf{BL}} \varphi \text{ if and only if } \Gamma \models_1^{\mathsf{BL}} \varphi\,.$$

Finally, we want to mention that basic logic enjoys a variant of the deduction theorem.

Theorem 3.4.8. *Let φ and ψ be formulas and let Γ be a set of formulas. Then $\Gamma \cup \{\varphi\} \models_1^{\mathsf{BL}} \psi$ if and only if there is an n such that $\Gamma \models_1^{\mathsf{BL}} \varphi^n \supset \psi$ where φ^n denotes the formula $\varphi \mathbin{\&} \ldots \mathbin{\&} \varphi$ with n factors.*

3.4.3 Łukasiewicz logic

Historically, Łukasiewicz logic is the first fuzzy logic. It is also a very important one because it has many pleasing properties. We start with an overview on its truth functions that are have already been implicitly defined by our previous definitions.

Proposition 3.4.9. *The truth functions of Łukasiewicz logic are given by*

$$\begin{aligned}
\|\varphi \mathbin{\&} \psi\|_e^{\text{Ł}} &= \max\left(\|\varphi\|_e^{\text{Ł}} + \|\psi\|_e^{\text{Ł}} - 1, 0\right)\\
\|\varphi \supset \psi\|_e^{\text{Ł}} &= \min\left(1 - \|\varphi\|_e^{\text{Ł}} + \|\psi\|_e^{\text{Ł}}, 1\right)\\
\|\varphi \equiv \psi\|_e^{\text{Ł}} &= 1 - \left|\|\varphi\|_e^{\text{Ł}} - \|\psi\|_e^{\text{Ł}}\right|\\
\|\neg\varphi\|_e^{\text{Ł}} &= 1 - \|\varphi\|_e^{\text{Ł}}
\end{aligned}$$

where φ and ψ are arbitrary formulas and $\boldsymbol{e}$ is an evaluation.

It can be seen that the law of double negation $\neg\neg\varphi \supset \varphi$ holds in Łukasiewicz logic because the truth function of negation is a strong one and therefore it is involutive (see Section 3.2.4). The following completeness result shows that this condition characterizes Łukasiewicz logic.

Definition 3.4.10. The Hilbert-style proof system $\vdash^{\text{Ł}}$ consists of all axiom schemata and rules of $\vdash^{\mathsf{BL}}$ together with the additional axiom schema of involution

$$\text{(INV) } \neg\neg\varphi \supset \varphi\,.$$

Theorem 3.4.11 (Finite strong completeness). *For every* finite *set of formulas Γ and every formula φ we have*

$$\Gamma \vdash^{\text{Ł}} \varphi \text{ if and only if } \Gamma \models_1^{\text{Ł}} \varphi\,.$$

Strong completeness is not possible for Łukasiewicz logic because the entailment relation $\models_1^{\text{Ł}}$ is not compact.

Ignoring the relation to basic logic, it is also possible to give a shorter axiomatization of Łukasiewicz logic. We present a list of four axioms that, together with an additional fifth axiom, was initially conjectured by Łukasiewicz [80]. Rose and Rosser then proved the completeness of Łukasiewicz' conjectured axioms [91]. The fifth axiom was then shown to be derivable from the others [13, 81].

Definition 3.4.12. The Hilbert-style proof system $\vdash^{Ł'}$ is given by the axiom schemata

(Ł1) $\varphi \supset (\psi \supset \varphi)$

(Ł2) $(\varphi \supset \psi) \supset ((\psi \supset \chi) \supset (\varphi \supset \chi))$

(Ł3) $(\neg\varphi \supset \neg\psi) \supset (\psi \supset \varphi)$

(Ł4) $((\varphi \supset \psi) \supset \psi) \supset ((\psi \supset \varphi) \supset \varphi)$

and the deduction rule *modus ponens*: “From φ and $\varphi \supset \psi$, infer ψ”.

Proposition 3.4.13. *For every set of formulas Γ and every formula φ we have*

$$\Gamma \vdash^{Ł} \varphi \text{ if and only if } \Gamma \vdash^{Ł'} \varphi .$$

In the following we look at the truth functions of Łukasiewicz logic in more detail. In Łukasiewicz logic, strong conjunction can be defined from implication and negation which means that every formula is equivalent to a formula containing only implication and $\bar{0}$. Furthermore a shorter definition of minimum conjunction is possible.

Proposition 3.4.14. *For all formulas φ and ψ and every evaluation e we have*

$$\begin{aligned} \|\varphi \,\&\, \psi\|^{Ł}_{e} &= \|\neg(\varphi \supset \neg\psi)\|^{Ł}_{e} \\ \|\varphi \vee \psi\|^{Ł}_{e} &= \|(\varphi \supset \psi) \supset \psi\|^{Ł}_{e} . \end{aligned}$$

One usually introduces a second conjunction connective to Łukasiewicz logic.

Definition 3.4.15. The two-ary connective $\veebar$ is called *strong disjunction*. For all formulas φ and ψ we take $\varphi \veebar \psi$ as an abbreviation for $\neg\varphi \supset \psi$.

Strong disjunction in Łukasiewicz logic is interesting because it allows the addition of truth values. Furthermore, DeMorgan’s laws hold for strong conjunction, strong disjunction and negation. Together with Proposition 3.2.19, this shows that the truth function of strong disjunction is a t-conorm. Note that, following Hájek’s approach, Łukasiewicz logic is the only one of the important fuzzy logics where a t-conorm is relevant. It would also be possible to define the implication from strong disjunction and negation which means that Łukasiewicz implication can be seen as both, an R-implication and an S-implication.

Proposition 3.4.16. *For all formulas φ and ψ and every evaluation e we have*

$$\begin{aligned} \|\varphi \veebar \psi\|^{Ł}_{e} &= \min\left(\|\varphi\|^{Ł}_{e} + \|\psi\|^{Ł}_{e}, 1\right) \\ \|\varphi \veebar \psi\|^{Ł}_{e} &= \|\neg(\neg\varphi \,\&\, \neg\psi)\|^{Ł}_{e} \\ \|\varphi \,\&\, \psi\|^{Ł}_{e} &= \|\neg(\neg\varphi \veebar \neg\psi)\|^{Ł}_{e} \\ \|\varphi \supset \psi\|^{Ł}_{e} &= \|\neg\varphi \veebar \psi\|^{Ł}_{e} . \end{aligned}$$

Another peculiarity of Łukasiewicz logic is that the residuum resulting from the Łukasiewicz t-norm is continuous. This fact is also characteristic for Łukasiewicz logic as has been pointed out, for example, by Fermüller and Kosik [38].

Proposition 3.4.17. *The residuum $\Rightarrow_*$ of a continuous t-norm $*$ is continuous if and only if $*$ is order isomorphic to the Łukasiewicz t-norm $*_{Ł}$, i.e., there is an order isomorphism f such that $x * y = f^{-1}(f(x) *_{Ł} f(y))$ for all $x, y \in [0,1]$.*

It must be noted that in our framework order isomorphic t-norms yield the same logics. Thus Łukasiewicz logic is the only fuzzy logic based on a continuous t-norm that has a continuous truth function of implication.[18]

3.4.4 Gödel logic

Gödel logic was introduced by Gödel as a by-product of his proof that intuitionistic logic is not a many-valued logic [41] and later extended to an infinite set of truth values by Dummett [26].[19] We will see that it has some nice properties and in a way is "nearer" to classical logic than other fuzzy logics.

We first give an overview on the truth functions of Gödel logic. In particular, weak and strong conjunction coincide and it would be possible to omit one of these connectives.

Proposition 3.4.18. *The truth functions of Gödel logic are given by*

$$\|\varphi \mathbin{\&} \psi\|_e^G = \|\varphi \wedge \psi\|_e^G = \min\left(\|\varphi\|_e^G, \|\psi\|_e^G\right)$$

$$\|\varphi \supset \psi\|_e^G = \begin{cases} 1 & \textit{if } \|\varphi\|_e^G \leq \|\psi\|_e^G \\ \|\psi\|_e^G & \textit{otherwise} \end{cases}$$

$$\|\varphi \equiv \psi\|_e^G = \begin{cases} 1 & \textit{if } \|\varphi\|_e^G = \|\psi\|_e^G \\ \|\varphi\|_e^G & \textit{if } \|\varphi\|_e^G < \|\psi\|_e^G \\ \|\psi\|_e^G & \textit{if } \|\varphi\|_e^G > \|\psi\|_e^G \end{cases}$$

$$\|\neg\varphi\|_e^G = \begin{cases} 1 & \textit{if } \|\varphi\|_e^G = 0 \\ 0 & \textit{otherwise} \end{cases}$$

for all formulas φ and ψ and every evaluation $\boldsymbol{e}$.

Note that the truth functions in Gödel logic only take into account the relative order of the truth values and not their concrete values. In fact, Gödel logic is characterized by this property as we show in Chapter 4. It seems therefore suitable to call

[18] The class of Yager t-norms provides examples of continuous t-norms with continuous residua that are not equal to the Łukasiewicz t-norm. For every $p > 0$, the function $*$ given by $x * y = \max(0, 1 - ((1-x)^p + (1-y)^p)^{1/p})$ is in the class of Yager t-norms and isomorphic to the Łukasiewicz t-norm [74].

[19] Dummett considered the truth value set $\{1/n \mid n \geq 1\} \cup \{0\}$ instead of $[0,1]$. Due to Theorem 4.3.7 this difference does not affect the sets of valid formulas of Gödel logic.

Gödel logic "the logic of order" [82]. This relationship is important if one considers reasoning under vagueness [35]: Degree-based approaches are often criticized because in many cases it seems not clear how the assignment of particular numbers in the unit interval as truth values can be justified. Gödel logic avoids this criticism because only an order on the truth value has to be settled which seems to be a weaker commitment.

The distinguishing feature of the Gödel t-norm is that every $x \in [0, 1]$ is idempotent due to $x *_G x = \min(x, x) = x$. The characterizing axiom of Gödel logic expresses this fact.

Definition 3.4.19. The Hilbert-style proof system $\vdash^G$ consists of all axiom schemata and rules of $\vdash^{BL}$ together with the additional axiom schema of contraction

$$\text{(C) } \varphi \supset \varphi \,\&\, \varphi.$$

In Gödel logic, we get a completeness theorem where the finiteness of the set of premisses is not necessary.

Theorem 3.4.20 (Strong completeness). *For every set of formulas Γ and every formula φ we have*

$$\Gamma \vdash^G \varphi \text{ if and only if } \Gamma \models^G_1 \varphi\,.$$

In contrast to other propositional fuzzy logics, Gödel logic has a classical deduction theorem.

Theorem 3.4.21 (Deduction theorem). *Gödel logic is the unique fuzzy logic that has the deduction theorem:*

(i) For every set of formulas Γ and all formulas φ and ψ we have

$$\Gamma \cup \{\varphi\} \models^G_1 \psi \text{ if and only if } \Gamma \models^G_1 (\varphi \supset \psi)\,.$$

(ii) If a logic $\mathsf{FL}(*)$ *based on a continuous t-norm has the above classical deduction theorem, then $*$ is the Gödel t-norm.*

In Gödel logic, another entailment relation than the standard one is also relevant, namely α-entailment as defined in Section 3.3.5.

Proposition 3.4.22. *For every set of formulas Γ and every formula φ we have*

$$\Gamma \models^G_1 \varphi \text{ if and only if } \Gamma \models^G_\alpha \varphi\,.$$

As an interesting side note, we remark that propositional Gödel logic, as a set of valid formulas, is an intermediate logic, i.e., it is a logic that includes intuitionistic logic and is included in classical logic. A sound and complete Hilbert-style calculus for Gödel logic is obtained by adding the linearity axiom $(\varphi \supset \psi) \vee (\psi \supset \varphi)$ to a Hilbert-style calculus for intuitionistic logic [26]. Furthermore, the law of excluded middle $\varphi \vee \neg\varphi$, which separates intuitionistic and classical logic, is not valid in Gödel logic.

3.4.5 Product logic

In contrast to Łukasiewicz and Gödel logic, product logic was introduced by Hájek, Godo, and Esteva not long ago [62], using truth functions of conjunction and implication that can be traced back to Goguen [43]. We first give its truth functions.

Proposition 3.4.23. *The truth functions of product logic are given by*

$$\|\varphi \mathbin{\&} \psi\|_e^{\mathsf{P}} = \|\varphi\|_e^{\mathsf{P}} \cdot \|\psi\|_e^{\mathsf{P}}$$
$$\|\varphi \supset \psi\|_e^{\mathsf{P}} = \begin{cases} 1 & \textit{if } \|\varphi\|_e^{\mathsf{P}} \leq \|\psi\|_e^{\mathsf{P}} \\ \|\psi\|_e^{\mathsf{P}} / \|\varphi\|_e^{\mathsf{P}} & \textit{otherwise} \end{cases}$$
$$\|\varphi \equiv \psi\|_e^{\mathsf{P}} = \begin{cases} \|\varphi\|_e^{\mathsf{P}} / \|\psi\|_e^{\mathsf{P}} & \textit{if } \|\varphi\|_e^{\mathsf{P}} \leq \|\psi\|_{\mathsf{P}} \\ \|\psi\|_e^{\mathsf{P}} / \|\varphi\|_e^{\mathsf{P}} & \textit{otherwise} \end{cases}$$
$$\|\neg\varphi\|_e^{\mathsf{P}} = \begin{cases} 1 & \textit{if } \|\varphi\|_e^{\mathsf{P}} = 0 \\ 0 & \textit{otherwise} \end{cases}$$

for all formulas φ and ψ and every evaluation $\boldsymbol{e}$.

The truth function of implication is also called *Goguen implication.* Note that the truth function of negation is Gödel negation.

Also in product logic we get the desired completeness result.

Definition 3.4.24. The Hilbert-style proof system $\vdash^{\mathsf{P}}$ consists of all axiom schemata and rules of $\vdash^{\mathsf{BL}}$ together with the additional axiom schema

$$\text{(A) } \neg\neg\varphi \supset ((\varphi \supset \varphi \mathbin{\&} \psi) \supset (\psi \mathbin{\&} \neg\neg\psi))\,.$$

Theorem 3.4.25 (Finite strong completeness). *For every* finite *set of formulas Γ and every formula φ we have*

$$\Gamma \vdash^{\mathsf{P}} \varphi \textit{ if and only if } \Gamma \models_1^{\mathsf{P}} \varphi\,.$$

The result that only one additional axiom schema is necessary is due to Cintula [17]. Furthermore he showed that it is not possible to find an according axiom schema with only one variable. In contrast to Łukasiewicz and Gödel logic, it seems to be hard to find an intuitive interpretation of this additional axiom, or the original two [56], that does not refer to certain properties of t-norms.

As a side note we remark that Łukasiewicz logic can be embedded into product logic [4], which means that many properties of Łukasiewicz logic also hold in product logic, e.g., the non-compactness of the entailment relation.

Theorem 3.4.26. *For every formula φ we can construct a formula φ' such that φ is valid in Łukasiewicz logic if and only if φ' is valid in product logic.*

3.4.6 Relations between logics

In the following, we present some properties that concern relations between the propositional logics that we have introduced or that hold for all propositional fuzzy logics. It should mainly be seen as a reference for interesting results.

Combining results from [49, 56, 67] we get some connections between different sets of satisfiable and valid formulas. Note that these relations imply several results on the complexity of satisfiability and validity checking. The logic CL is classical propositional logic.

Theorem 3.4.27. *The following holds for sets of satisfiable formulas:*

$$\mathbf{SAT}_1^{\mathsf{BL}} = \bigcap_* \mathbf{SAT}_1^{\mathsf{FL}(*)}$$

$$\mathbf{SAT}_{>0}^{\mathsf{BL}} = \bigcap_* \mathbf{SAT}_{>0}^{\mathsf{FL}(*)}$$

$$\begin{aligned}\mathbf{SAT}_1^{\mathsf{G}} = \mathbf{SAT}_{>0}^{\mathsf{G}} = \mathbf{SAT}_1^{\mathsf{P}} = \mathbf{SAT}_{>0}^{\mathsf{P}} \\ = \mathbf{SAT}^{\mathsf{CL}} \subset \mathbf{SAT}_1^{\text{Ł}} = \mathbf{SAT}_1^{\mathsf{BL}} \subset \mathbf{SAT}_{>0}^{\text{Ł}} = \mathbf{SAT}_{>0}^{\mathsf{BL}}\end{aligned}$$

All inclusions are strict.

Theorem 3.4.28. *The following holds for sets of valid formulas:*

$$\mathbf{TAUT}_1^{\mathsf{BL}} = \bigcap_* \mathbf{TAUT}_1^{\mathsf{FL}(*)}$$

$$\mathbf{TAUT}_{>0}^{\mathsf{BL}} = \bigcap_* \mathbf{TAUT}_{>0}^{\mathsf{FL}(*)}$$

$$\mathbf{TAUT}_1^{\mathsf{G}} \neq \mathbf{TAUT}_1^{\mathsf{P}}$$

$$\mathbf{TAUT}_1^{\mathsf{G}} \subset \mathbf{TAUT}_1^{\mathsf{CL}}$$

$$\mathbf{TAUT}_1^{\mathsf{P}} \neq \mathbf{TAUT}_1^{\mathsf{CL}}$$

$$\mathbf{TAUT}_1^{\mathsf{BL}} \subset (\mathbf{TAUT}_1^{\text{Ł}} \cap \mathbf{TAUT}_1^{\mathsf{G}} \cap \mathbf{TAUT}_1^{\mathsf{P}})$$

$$\mathbf{TAUT}_1^{\text{Ł}} \subset \mathbf{TAUT}_{>0}^{\text{Ł}} = \mathbf{TAUT}_{>0}^{\mathsf{BL}} \subset \mathbf{TAUT}^{\mathsf{CL}} = \mathbf{TAUT}_{>0}^{\mathsf{G}} = \mathbf{TAUT}_{>0}^{\mathsf{P}}$$

All inclusions are strict.

3.5 First-order fuzzy logics

The extension of our propositional systems to the first-order case is straightforward in Hájek's framework. Instead of fuzzy propositions we now consider fuzzy relations. The only crucial question is how the quantifiers should be interpreted. The interpretation of formulas gets a bit more complicated, which makes some definitions necessary that are mostly like in classical predicate logic.

Definition 3.5.1. The *language of first-order fuzzy logic* consists of a set of *object variables*, a set of *predicate symbols*, and the usual connectives. Formulas are built in the usual manner.

Definition 3.5.2. A *fuzzy predicate* on a nonempty set $\boldsymbol{S}$ of arity n is a function from the set $\boldsymbol{S}^n$ to the unit interval $[0, 1]$.

Definition 3.5.3. A *variable assignment* on a set $\boldsymbol{S}$ is a function $\boldsymbol{v}$ that assigns to each object variable x an element $\boldsymbol{v}(x)$ in $\boldsymbol{S}$.

Definition 3.5.4. A *first-order fuzzy interpretation* is a pair $\boldsymbol{M} = \langle \boldsymbol{D}, \boldsymbol{I} \rangle$ where the *domain* $\boldsymbol{D}$ is a nonempty set, the universe of discourse, and $\boldsymbol{I}$ is a function that assigns a fuzzy predicate $\boldsymbol{I}(Q)$ on $\boldsymbol{D}$ of suiting arity to each predicate symbol Q.

We now define the interpretation of formulas in a first-order fuzzy interpretation.

Definition 3.5.5. Let $*$ be a continuous t-norm, $\boldsymbol{M} = \langle \boldsymbol{D}, \boldsymbol{I} \rangle$ a first-order fuzzy interpretation and $\boldsymbol{v}$ a variable assignment on $\boldsymbol{D}$. The *interpretation of formulas* is defined inductively like in the propositional case with the following additions:

$$\begin{aligned}
\|x\|^*_{\boldsymbol{M},\boldsymbol{v}} &= \boldsymbol{v}(x) \text{ for every object variable } x \\
\|Q(x_1, \dots, x_n)\|^*_{\boldsymbol{M},\boldsymbol{v}} &= \boldsymbol{I}(Q)\left(\|x_1\|^*_{\boldsymbol{M},\boldsymbol{v}}, \dots, \|x_n\|^*_{\boldsymbol{M},\boldsymbol{v}}\right) \text{ for every predicate symbol } Q \\
\|\forall x\varphi\|^*_{\boldsymbol{M},\boldsymbol{v}} &= \inf\left\{\|\varphi\|^*_{\boldsymbol{M},\boldsymbol{v}\cup\{x\mapsto d\}} \mid d \in \boldsymbol{D}\right\} \\
\|\exists x\varphi\|^*_{\boldsymbol{M},\boldsymbol{v}} &= \sup\left\{\|\varphi\|^*_{\boldsymbol{M},\boldsymbol{v}\cup\{x\mapsto d\}} \mid d \in \boldsymbol{D}\right\}
\end{aligned}$$

where $\boldsymbol{v} \cup \{x \mapsto d\}$ is the variable assignment $\boldsymbol{v}'$ that is defined by

$$\boldsymbol{v}'(v) = \begin{cases} d & \text{if } v = x \\ \boldsymbol{v}(v) & \text{if } v \neq x \,. \end{cases}$$

This means that for a universally quantified formula $\forall x\varphi$ we compute the infimum of all interpretations with a concrete value of x and for an existential quantification we compute the supremum. Note that these definitions of the quantifiers behave classically for the truth values 0 and 1. In classical predicate logic, universal quantification can be seen as a “possibly infinite conjunction” and existential quantification as a “possibly infinite disjunction”. This point of view can also be taken in fuzzy logic because the truth function of weak conjunction is the minimum operator which generalizes to the infimum for infinite sets and the truth function of weak conjunction is the maximum operator which generalizes to the supremum.

As in the propositional case, our definitions give us a first-order fuzzy logic $\mathsf{FL}\forall(*)$ for every continuous t-norm $*$ with the usual notions of 1-satisfaction, 1-validity and 1-entailment where we only consider closed formulas.

By giving a suitable definition of its interpretations we also get a first-order variant of basic logic.

Definition 3.5.6. A BL∀-*interpretation* is a pair $\langle *, \boldsymbol{M} \rangle$ that consists of a continuous t-norm $*$ and a first-order fuzzy interpretation $\boldsymbol{M}$.

Unlike in the propositional case, there are no completeness result for the standard semantics of first-order fuzzy logics in most cases. This situation can occur if the set of valid formulas is not recursively enumerable which makes an axiomatization impossible (see section 3.3.5).

Theorem 3.5.7. *The following sets are not recursively enumerable:*

- *The set of formulas valid in first-order basic logic*
- *The set of formulas valid in first-order Łukasiewicz logic*
- *The set of formulas valid in first-order product logic*

Hájek gave a proof of the first proposition after his monograph appeared [54]. The second proposition was proved by Scarpellini in the early 1960s [92]. The third proposition was shown by Baaz, Hájek, Švejda, and Krajíček by finding an embedding of predicate product logic into predicate Łukasiewicz logic [4]. As this theorem indicates, an axiomatization exists for Gödel logic.

We present a Hilbert-style proof system for predicate basic logic that although not complete can at least be shown to be sound.

Definition 3.5.8. The Hilbert-style proof system $\vdash^{\mathrm{BL}\forall}$ consists of all axiom schemata and rules of $\vdash^{\mathrm{BL}}$ together with the additional axiom schemata

(∀1) $(\forall x \varphi(x)) \supset \varphi(t)$

(∃1) $\varphi(t) \supset (\exists x \varphi(x))$

(∀2) $(\forall x(\psi \supset \varphi)) \supset (\psi \supset \forall x \varphi)$

(∃2) $(\forall x(\varphi \supset \psi)) \supset ((\exists x \varphi) \supset \psi)$

(∀3) $(\forall x(\varphi \vee \psi)) \supset ((\forall x \varphi) \vee \psi)$

where x is not free in ψ and the additional deduction rule of *generalization*: "From φ, infer $\forall x \varphi$".

Definition 3.5.9. The Hilbert-style proof system $\vdash^{\mathrm{G}\forall}$ consists of all axiom schemata and rules of $\vdash^{\mathrm{BL}\forall}$ together with the additional axiom schema of contraction

$$\text{(C) } \varphi \supset \varphi \mathbin{\&} \varphi\,.$$

Theorem 3.5.10 (Strong completeness). *For every set of formulas Γ and every formula φ we have*

$$\Gamma \vdash^{\mathrm{G}\forall} \varphi \text{ if and only if } \Gamma \models_1^{\mathrm{G}\forall} \varphi$$

where G∀ *refers to first-order Gödel logic.*

3.6 Modalities

Since linguistic modalities are highly relevant for the purpose of reasoning under vagueness we show some examples of such modalities in fuzzy logic. We consider both, modalities that are truth-functional and modalities that are not.

3.6.1 Projection modalities

We first introduce two projection modalities that were first studied in the context of Gödel logic [2]. Their semantics is defined as follows.

Definition 3.6.1. Let $*$ be a continuous t-norm. We extend the language of fuzzy logic by the two unary connectives $\vartriangle$ and $\triangledown$, called *projection modalities.* The interpretation of formulas is extended by

$$\|\vartriangle\varphi\|_{e}^{*} = \lfloor \|\varphi\|_{e}^{*} \rfloor = \begin{cases} 1 & \text{if } \|\varphi\|_{e}^{*} = 1 \\ 0 & \text{if } \|\varphi\|_{e}^{*} \neq 1 \end{cases}$$

$$\|\triangledown\varphi\|_{e}^{*} = \lceil \|\varphi\|_{e}^{*} \rceil = \begin{cases} 1 & \text{if } \|\varphi\|_{e}^{*} \neq 0 \\ 0 & \text{if } \|\varphi\|_{e}^{*} = 0 \end{cases}$$

where $\lfloor\cdot\rfloor$ is the floor and $\lceil\cdot\rceil$ is the ceiling function. The operator $\vartriangle$ is also called *globalization.*

Thus $\vartriangle\varphi$ indicates that the formula φ is absolutely true and $\triangledown\varphi$ indicates that φ is not absolutely false. Therefore the role of the $\vartriangle$-operator for fuzzy logic is similar to the role of the D-operator for supervaluational logic (see Section 2.2).

We only have to discuss $\vartriangle$ because $\triangledown$ is definable from $\vartriangle$ as an abbreviating formula. Furthermore it can be seen that $\vartriangle\neg\varphi$ corresponds to Gödel negation for every choice of the continuous t-norm and thus $\triangledown$ corresponds to double Gödel negation. Also note that for each of the operators only the most inner application is relevant.

Proposition 3.6.2. *For every formula φ, every continuous t-norm $*$ and every evaluation e we have*

$$\|\vartriangle\neg\varphi\|_{e}^{*} = -_{G}\left(\|\varphi\|_{e}^{*}\right)$$
$$\|\triangledown\varphi\|_{e}^{*} = -_{G}\left(-_{G}\left(\|\varphi\|_{e}^{*}\right)\right) = \|\vartriangle\neg\vartriangle(\neg\varphi)\|_{e}^{*}$$

and

$$\|\vartriangle\vartriangle\varphi\|_{e}^{*} = \|\vartriangle\varphi\|_{e}^{*} \qquad \|\triangledown\vartriangle\varphi\|_{e}^{*} = \|\vartriangle\varphi\|_{e}^{*}$$
$$\|\vartriangle\triangledown\varphi\|_{e}^{*} = \|\triangledown\varphi\|_{e}^{*} \qquad \|\triangledown\triangledown\varphi\|_{e}^{*} = \|\triangledown\varphi\|_{e}^{*}.$$

Baaz obtained an axiomatization of the $\vartriangle$-operator [2].

Definition 3.6.3. We denote by $\vdash_{\vartriangle}$ the extension of a Hilbert-style calculus $\vdash$ with the axiom schemata

(△1) $\triangle\varphi \supset \varphi$

(△2) $\triangle\varphi \supset \triangle\triangle\varphi$

(△3) $\triangle(\varphi \supset \psi) \supset (\triangle\varphi \supset \triangle\psi)$

(△4) $\triangle\varphi \vee \neg\triangle\varphi$

(△5) $\triangle(\varphi \vee \psi) \supset (\triangle\varphi \vee \triangle\psi)$

and the △-rule "From φ infer $\triangle\varphi$".

Note that the axiom schemata (△1), (△2), (△3) and the △-rule correspond to the modal axioms of the logic S4 and its necessitation rule. Hájek comments that in the axiom schema (△4) the △-operator rather behaves as possibility than necessity.

All the completeness results that we encountered so far also hold in presence of the △-operator [20].

Theorem 3.6.4. *The completeness results for* $\vdash^{\mathsf{BL}}$, $\vdash^{\text{Ł}}$, $\vdash^{\mathsf{G}}$, $\vdash^{\mathsf{G}\forall}$ *and* $\vdash^{\mathsf{P}}$ *extend to* $\vdash^{\mathsf{BL}}_{\triangle}$, $\vdash^{\text{Ł}}_{\triangle}$, $\vdash^{\mathsf{G}}_{\triangle}$, $\vdash^{\mathsf{G}\forall}_{\triangle}$ *and* $\vdash^{\mathsf{P}}_{\triangle}$.

As a side remark we mention a connection between 1-entailment and 1-validity as well as between (>0)-entailment and 1-entailment in the presence of the projections (see Section 3.3.4 for the difference between these notions).

Remark. For every finite set of formulas $\Gamma = \{\psi_1, \dots, \psi_n\}$, every formula φ and every continuous t-norm $*$ we have

$$\Gamma \models^*_1 \varphi \text{ if and only if } \left(\triangle\psi_1 \wedge \dots \wedge \triangle\psi_n\right) \supset \triangle\varphi \text{ is 1-valid in } \mathsf{FL}(*)\,.$$

Remark. For every finite set of formulas $\Gamma = \{\psi_1, \dots, \psi_n\}$, every formula φ and every continuous t-norm $*$ we have

$$\Gamma \models^*_{>0} \varphi \text{ if and only if } \triangledown\Gamma \models^*_1 \triangledown\varphi$$

where $\triangledown\Gamma = \{\triangledown\psi \mid \psi \in \Gamma\}$.

3.6.2 Hedges

An early idea of Zadeh was that fuzzy logic should make reasoning with vague modifiers like *very true* or *quite true* possible [102, 103]. Lakoff discusses modifiers of that kind from a linguistic point of view [78]. In the following we shortly discuss some modalities for this purpose. In the context of fuzzy logics, such modalities are called hedges and are usually assumed to be *truth-functional.* Hedges can, according to this understanding, also be seen as *fuzzy truth values* [59].

Definition 3.6.5. A *hedge* is a function from the real unit interval [0, 1] to itself.

In particular, the globalization operator $\triangle$ is a hedge. We discussed it separately from other hedges because of its prominence in the literature.

Hájek introduced a modality to fuzzy logic that can be interpreted as "very true" by defining a class of truth-stressing hedges [59, 61].

Definition 3.6.6. A hedge v is $*$-*truth-stressing* (or: a $*$-*truth stresser*) for a continuous t-norm $*$ iff it fulfills the following conditions for all $x, y \in [0, 1]$:

$$
\begin{aligned}
v(1) &= 1 && \\
v(x) &\leq x && \text{(sub-diagonal)} \\
v(x \Rightarrow_* y) &\leq v(x) \Rightarrow_* v(y) && \text{($*$-regular)}
\end{aligned}
$$

Example. The following are some natural examples of truth stressers:

- The identity function $v(x) = x$ is a trivial example of a truth stresser for every continuous t-norm.
- The function $v(x) = \lfloor x \rfloor$ is the truth function of the globalization operator $\triangle$ and a truth stresser for every continuous t-norm, as can be seen from the axiomatization of $\triangle$.
- The function $v(x) = x * x$ is a truth stresser for every continuous t-norm $*$. A corresponding operator vt can be syntactically defined by taking vtφ as an abbreviation for $\varphi \mathbin{\&} \varphi$.
- The function $v(x) = x^2$, which corresponds to $x *_{\mathrm{P}} x$ for the product t-norm, is also a truth stresser for the Łukasiewicz and the Gödel t-norm.
- The function $v(x) = \max(2x - 1, 0)$, which corresponds to $x *_{\text{Ł}} x$ for the Łukasiewicz t-norm, is also a truth stresser for the Gödel t-norm but *not* for the product t-norm.

The extension of the usual semantics to truth stressers is straightforward. We define our logical notions with regard to the class of all truth stressers which creates a situation similar to basic logic where we quantify over all continuous t-norms.

Definition 3.6.7. Let $*$ be a continuous t-norm. A vt-*interpretation* for $*$ is a pair $\langle \boldsymbol{e}, v \rangle$ where $\boldsymbol{e}$ is an evaluation and v is a truth stresser for $*$.

We extend the language of fuzzy logic by the unary connective vt. The interpretation of formulas is defined in the usual way with exception of the vt-operator where the defining clause is

$$\|\mathrm{vt}\varphi\|^*_{\langle e, v \rangle} = v\left(\|\varphi\|^*_{\langle e, v \rangle}\right) .$$

The logical consequence relation $\models^*_{\mathrm{vt}}$ is defined as usual by 1-truth preservation (see Section 3.3.4).

Hájek also suggests an axiomatization of the vt-operator.

Definition 3.6.8. We denote by $\vdash_{\mathrm{vt}}$ the extension of a Hilbert-style calculus $\vdash$ with the axiom schemata

(VE1) $\mathrm{vt}\varphi \supset \varphi$

(VE2) $\mathrm{vt}(\varphi \supset \psi) \supset (\mathrm{vt}\varphi \supset \mathrm{vt}\psi)$

(VE3) $\mathrm{vt}(\varphi \vee \psi) \supset (\mathrm{vt}\varphi \vee \mathrm{vt}\psi)$

and the deduction rule of *truth confirmation*: "From φ infer $\mathrm{vt}\varphi$."

The axiom schema (VE1) says that a statement that is very true is also true. The meaning of (VE2) is the following: if both φ and $\varphi \supset \psi$ are very true, then also ψ is very true. The schema (VE3) states that if a disjunction is very true, then one of its disjuncts is very true.

The calculi introduced for propositional and first-order basic, Gödel, Łukasiewicz, and product logic can be shown to be sound for the extension with the vt-operator. Hájek also proves general strong completeness for these logics. However, it seems that to date the standard completeness with regard to the class of all truth stressers is still an open problem for t-norm based fuzzy logics, with exception to Gödel logic.

Theorem 3.6.9 (Strong standard completeness). *Let Γ be a set of formulas and φ a formula. Then $\Gamma \vdash^{G\forall}_{\mathrm{vt}} \varphi$ if and only if $\Gamma \models^{G\forall}_{\mathrm{vt}} \varphi$.*

In a related investigation, Vychodil studies truth-depressing hedges [100]. Dual to the interpretation of truth-stressers as "very true"-operators, truth-depressing hedges can be interpreted as operators for "slightly true". Vychodil obtains two completeness results with regard to algebraic semantics: First, he provides an axiomatization of basic logic together with Hájek's "very true"-operator and his "slightly true"-operator. From this result, axiomatizations of the corresponding variants of Łukasiewicz, Gödel and product logic immediately follow. Second, an axiomatization of the slightly true-operator for Łukasiewicz, Gödel and product logic is given without adding the very true-operator.

Besides general truth stressers, Hájek also studied two concrete hedges [60]. For a truth value x, the two hedges are defined as

- the (unique) greatest idempotent $\leq x$ (called $l(x)$) and
- the (unique) least idempotent $\geq x$ (called $u(x)$).

The corresponding connectives are denoted by L and U and may be interpreted as "very true" and "more or less true", respectively. Hájek provides an axiomatization of basic logic together with $\triangle$, L and U with regard to the standard semantics. Note that the hedge l is a truth stresser for every continuous t-norm [7].[20] Hájek's definition of l corresponds to the storage operator which was introduced by Montagna [84]. Montagna provides an axiomatization with finite strong standard completeness for propositional basic, Łukasiewicz, Gödel and product logic with storage operator.

Related work for hedges in fuzzy logic is relatively sparse. A hedge with the meaning "more or less true" was presented for Gödel logic by Hájek and Harmoncová [63]. In a discussion of the sorites paradox, a connective for "almost true" is introduced and several examples for a possible semantics are given [64]. Truth stressers for extensions of MTL, the logic of left-continuous t-norms, are studied by Ciabattoni, Metcalfe, and Montagna [15], but without explicitly covering basic, Łukasiewicz and product logic. Esteva, Godo, and Noguera propose definitions of truth-stressers and truth-depressers that are different from those of Hájek [59] and Vychodil [100] with the advantage that they can show that the corresponding axiomatizations preserve the standard completeness of the original logic.

3.6.3 Probably and many

In his monograph [56], Hájek presents a framework in which a modal operator with the meaning "probably" can be introduced. We summarize the main ideas at this point because they are related to the hybrid logic for reasoning under vagueness that will be discussed in Chapter 5.

Hájek suggests to determine the probability of an event by its number of occurrences in a Kripke-style set of classical worlds. Events are described by classical propositional formulas and a modal operator for probability extracts their probability. Then the usual fuzzy logic machinery is applied to the extracted truth values.

Definition 3.6.10. The *language of fuzzy probability logic* consists of formulas of the following form:

- *Atomic formulas* are formulas of the type $\mathrm{P}\varphi$ where P is a unary connective and φ is a classical propositional formula.
- All formulas are built from atomic formulas using the usual propositional fuzzy connectives.

Hájek defines an interpretation of this logic as a set of worlds, each with a classical interpretation of the propositional variables, and a function that measures sets of worlds. The truth value of the statement "φ is probable" is then just the measure of the set of worlds in which φ is true.

Definition 3.6.11. An *interpretation of fuzzy probability logic* PŁ is a triple $\boldsymbol{\Pi} = \langle \boldsymbol{W}, (\boldsymbol{M}_s)_{s\in\boldsymbol{W}}, \mu\rangle$ where $\boldsymbol{W}$ is a set of *worlds*, $(\boldsymbol{M}_s)_{s\in\boldsymbol{W}}$ is a function that assigns a classical propositional interpretation $\boldsymbol{M}_w$ to every world $w \in \boldsymbol{W}$ and μ is a finitely

[20]This can be shown using the following property: for a continuous t-norm $*$, the greatest idempotent which is less or or equal to x is the infimum of the set $\{x, x * x, x * x * x, \dots\}$.

additive *probability measure* on a field of subsets of $\boldsymbol{W}$ such that for each propositional variable p the set $\{w \in \boldsymbol{W} \mid \|p\|_{\boldsymbol{M}_w} = 1\}$ is measurable, i.e., in the domain of μ.

The interpretation of formulas has the usual inductive definition with $\mathrm{P}\varphi$ for an atomic formula φ being interpreted as

$$\|\mathrm{P}\varphi\|^*_{\Pi} = \mu\left(\left\{ w \in \boldsymbol{W} \mid \|\varphi\|_{\boldsymbol{M}_w} = 1 \right\}\right) .$$

The logical consequence relation $\models^{\mathsf{PŁ}}$ is defined as usual by 1-truth preservation (see Section 3.3.4).

Hájek also defines another semantics for which the P-operator has the meaning "For many worlds φ is true".

Definition 3.6.12. An *interpretation of the fuzzy logic of "many"* MŁ is a pair $\boldsymbol{\Pi} = \langle \boldsymbol{W}, (\boldsymbol{M}_s)_{s\in\boldsymbol{W}} \rangle$ where $\boldsymbol{W}$ is a *finite* set of *worlds* and $(\boldsymbol{M}_s)_{s\in\boldsymbol{W}}$ is a function that assigns a classical propositional interpretation $\boldsymbol{M}_w$ to every world $w \in \boldsymbol{W}$.

The interpretation of formulas has the usual inductive definition with $\mathrm{P}\varphi$ for an atomic formula φ being interpreted as

$$\|\mathrm{P}\varphi\|^*_{\Pi} = \frac{\left|\left\{ w \in \boldsymbol{W} \mid \|\varphi\|_{\boldsymbol{M}_w} = 1 \right\}\right|}{|\boldsymbol{W}|}$$

where $|\boldsymbol{W}|$ denotes the cardinality of $\boldsymbol{W}$.

The logical consequence relation $\models^{\mathsf{MŁ}}$ is defined as usual by 1-truth preservation (see Section 3.3.4).

Technically, the fuzzy logic of "many" arises from setting the probability measure μ to the uniform distribution.

Despite the generality of these definitions, Hájek suggests to use the truth functions determined by the Łukasiewicz t-norm for the "outer" connectives because then the truth functions are well suited to capture some properties of probability. For this case he gives an axiomatization.[21] His proof system combines classical reasoning for the classical propositional formulas inside the modal operator and fuzzy reasoning in Łukasiewicz logic.

Definition 3.6.13. The Hilbert-style proof system $\vdash^{\mathsf{PŁ}}$ consists of all axiom schemata and rules of $\vdash^{\mathsf{Ł}}$ together with the additional axiom schemata

(P1) $\mathrm{P}\varphi \supset (\mathrm{P}(\varphi \supset \psi) \supset \mathrm{P}\psi)$

(P2) $\mathrm{P}(\neg\varphi) \equiv \neg\mathrm{P}\varphi$

(P3) $\mathrm{P}(\varphi \vee \psi) \equiv ((\mathrm{P}\varphi \supset \mathrm{P}(\varphi \wedge \psi)) \supset \mathrm{P}\psi)$

[21] Precisely speaking, Hájek gives an axiomatization of rational Pavelka logic extended by the P-operator. Completeness for pure Łukasiewicz logic then immediately follows.

and the deduction rule of *necessitation*: "If φ is classically provable, then deduce $P\varphi$."[22]

Theorem 3.6.14 (Finite strong completeness). *Let Γ be a finite set of formulas and φ a formula. Then*

$$\Gamma \vdash^{\mathsf{P}Ł} \varphi \textit{ if and only if } \Gamma \models^{\mathsf{P}Ł} \varphi$$

and

$$\Gamma \vdash^{\mathsf{P}Ł} \varphi \textit{ if and only if } \Gamma \models^{\mathsf{M}Ł} \varphi\,.$$

Note that the completeness result also shows that the two logics coincide for finite sets of premisses.

Further refinements of this framework include conditional probabilities that can be treated by introducing product conjunction in addition to Łukasiewicz conjunction. A related approach is the introduction of a belief operator using Dempster-Shafer belief functions for which Godo, Hájek, and Esteva present an axiomatization [42].

3.7 Other topics

Besides the material presented in this chapter many other topics have been explored. In the following we give a short overview on the most important developments.

We presented an axiomatization of Łukasiewicz, Gödel and product logic with regard to their standard semantics. This can be generalized: Esteva, Godo, and Montagna gave an algorithm that computes an axiomatization of any fuzzy logic based on a continuous t-norm and its residuum [31].

Besides the standard semantics one might also be interested in rational semantics, i.e., allowing only rational numbers between 0 and 1 as the set of truth values. These semantics could be of particular importance for applications due to the necessity of discretization. Concerning the previously introduced logics, we can summarize the results of the two central papers on this topic [19, 27] as follows:

- Propositional basic, Łukasiewicz, Gödel and product logic enjoy strong completeness for rational semantics.
- First-order Łukasiewicz, Gödel and product logic enjoy strong completeness for rational semantics.
- First-order basic logic enjoys strong completeness for rational semantics when one axiom is added.

[22] We could also be more explicit by taking all formulas $P\varphi$ as axioms where φ is an axiom of a Hilbert-style calculus for classical propositional logic, together with an adapted version of classical modus ponens.

An idea that goes back to one of the first treatments of mathematical fuzzy logic by Pavelka [86–88] is to consider fuzzy sets of formulas for logical deductions instead of crisp sets where the goal is to "prove *partially* true conclusions from *partially* true premisses" [56]. In this way, one consequently arrives at graded notions of validity and entailment. A syntactic correspondence to the graded consequence relation in the form of a suitable calculus for a graded provability relation can be gained by adding truth value constants to the language: for every real $r \in [0,1]$ a constant $\overline{r}$ is defined and, in every evaluation e, $\overline{r}$ is interpreted by itself, i.e., $e(\overline{r}) = r$. This concept can be defined in any fuzzy logic, but in Łukasiewicz logic it has some essential well-behaving properties due to the characteristic continuous implication connective. A logic enjoys *strong Pavelka-style completeness* iff the degree of provability and the degree of entailment always coincide for every fuzzy set of formulas. This is the case for propositional and first-order Pavelka Łukasiewicz logic. Hájek's analysis showed that this result also holds for the relaxed version of the concept where only rational truth value constants for $r \in \mathbb{Q} \cap [0,1]$ are considered, which keeps the language countable [52, 53]. The propositional version of this rational Pavelka Łukasiewicz logic enjoys finite strong standard completeness [56]. Furthermore, for formulas without truth value constants validity in usual Łukasiewicz logic and (rational) Pavelka Łukasiewicz logic coincide [66]. For a more detailed introduction, Gottwald's papers [44, 45] can be recommended.[23] The standard reference on Pavelka-style fuzzy logic is the monograph by Novák, Perfilieva, and Močkoř [85].

Another important direction is to consider additional connectives with truth functions obtained from another t-norm. Esteva, Godo, Hájek, and Navara have enriched t-norm based logics with Gödel negation by a strong negation connective (see Definition 3.2.17) [30]. They remark that in the presence of a strong negation $\sim$ the following connectives can be defined as abbreviating formulas:

- A *strong disjunction* $\varphi \veebar \psi$ can be defined as $\sim(\sim\varphi \,\&\, \sim\psi)$ because the function defined by $n(n(x) * n(y))$ is always a t-conorm for a t-norm $*$ and a strong negation function n (see Proposition 3.2.19).

- A *contrapositive implication* (also called *strong implication*) $\varphi \hookrightarrow \psi$ can be defined as $\sim\varphi \veebar \psi$.

- In the presence of both, an involutive negation $\sim$ and Gödel negation $\neg$, the globalization $\triangle\varphi$ can be defined as $\neg\sim\varphi$.

For propositional and first-order Gödel logic together with the connective $\sim$ interpreted as Łukasiewicz negation, they obtain standard completeness. For propositional

[23] As an interesting side note we remark that Gottwald distinguishes between *t-norm based logics* and *fuzzy logic*. With the former he refers to the usual approach as presented in this thesis where only crisp versions of validity, entailment and provability are considered. With the latter he refers to Pavelka-style versions of t-norm based logics. Relatedly, Hájek writes that Novák considers Pavelka logic, with graded consequences and truth value constants, as *the* fuzzy logic and that he (Hájek) does not share this opinion [61].

product logic together with the connective $\sim$ they get semi-standard completeness meaning that a formula is provable if and only if it is a tautology for all involutive negation functions of $\sim$.

A related approach is pursued with the logic ŁP that combines Łukasiewicz and product logic and was introduced by Esteva, Godo, and Montagna [32]. It consists of Łukasiewicz implication, product conjunction, product implication and the truth constant $\bar{0}$. For the logic ŁP½ the additional truth value constant $\overline{½}$, which is always interpreted as ½, is defined. For both logics, an axiomatization with finite strong standard completeness can be obtained. Furthermore, the following connectives can be defined as abbreviating formulas: Łukasiewicz negation, Gödel negation, $\triangle$-operator, Łukasiewicz conjunction, strong Łukasiewicz disjunction, weak minimum conjunction, weak maximum disjunction and Gödel implication. Furthermore the rational numbers are definable in ŁP½. With this in mind it can be shown that most of the important concrete[24] fuzzy logics can be embedded into ŁP½.

There are several ways to generalize the presented approach towards fuzzy logic. For example, recall our definition of the residuum $\Rightarrow_*$ of a t-norm $*$. It is the unique function that fulfills the condition

$$x \Rightarrow_* y = \max\{z \in [0,1] \mid x * z \leq y\}\,.$$

For the residuum to be well-defined, the t-norm $*$ has to be left-continuous and not necessarily continuous as we have demanded. Similar to basic logic, it is possible to define *monoidal t-norm logic* MTL [28] which is the logic of left-continuous t-norms [70]. An example of a t-norm that is left-continuous but not continuous is the nilpotent minimum t-norm defined by

$$x * y = \begin{cases} \min(x,y) & \text{if } x + y > 1 \\ 0 & \text{otherwise}\,. \end{cases}$$

Another generalizations arises when the constant $\bar{0}$ is removed from the language which allows only a falsehood-free set of truth values, e.g., the half-closed interval $(0,1]$. In this way one arrives at *hoop* logics [29]. Or one could give up the commutativity of conjunction which makes sense in certain contexts, e.g., some conjunction connectives in natural language. T-norms without commutativity are called *pseudo t-norms* and the corresponding logic is called *pseudo basic logic* [55, 57]. If all three generalizations are considered, i.e. non-commutative left-continuous pseudo t-norms without falsehood, one gets Hájek's flea logic [50], which is a very weak fuzzy logic.

Another generalization of fuzzy logic is to consider intervals of truth-values. Instead of assigning a concrete truth value in $[0,1]$ to a proposition one just gives a lower bound a and an upper bound b of its truth value, hence saying that the truth value is in $[a,b] \subseteq [0,1]$. This framework adds a flavor of imprecision to fuzzy logic. An overview on interval-valued fuzzy logic is given by Cornelis, Deschrijver, and

[24]By a *concrete* fuzzy logic we mean a fuzzy logic that is based on a single t-norm and not a class of t-norms. Thus, Gödel logic with $\triangle$ would be a concrete fuzzy logic and basic logic would not.

Kerre [24]. A further analysis and some philosophical motivation of this approach was provided by Fermüller [37].

CHAPTER 4

Gödel logic as the fuzzy logic of comparison

One of the main criticism of degree-based theories of vagueness is the concern that the choice of a concrete real number, like for example $1/\sqrt{2}$, as the exact truth value of a vague proposition can usually hardly be justified and might be artificially precise [65, 72, 95]. One can address this criticism by proposing that only the order of the truth values matters and not the concrete values [35].

Gödel logic as introduced in Section 3.4.4 is such a logic in which truth values are only compared and no real arithmetic, as for example in Łukasiewicz and product logic, has to be carried out. As we see in this chapter even more holds: Gödel logic, with the globalization operator, is in some very natural sense the *only* logic of comparison over a set of linearly ordered truth values. To prove this statement we need to settle our notion of a "logic of comparison".[25] We will understand such a logic as specified over the theory of a set of linearly ordered truth values. The overall result might not be completely new (see [5]), but nevertheless a detailed proof seems to be valuable.

4.1 Specified logics

In the following, we define the class of specified logics. Our definitions are slight generalizations of those for projective logics given by Baaz and Fermüller [3].

We distinguish between the syntax and semantics of an underlying theory and the logic that is specified by using terms and formulas of this theory. We allow arbitrary, classical first-order theories with their usual syntax and semantics.

[25] Our concept of logics of comparison should not be confused with Casari's comparative logics [11]. Casari introduces his logics in order to formalize certain aspects of comparative reasoning in natural language. The goal of this chapter is a different one: we explore the space of possible logics when the truth functions of the connectives are defined by conditions on the order of truth values.

Definition 4.1.1. A *theory* is a set of first-order formulas.

We assume that a theory implicitly determines a first-order language. The theory is meant to describe the truth values of the specified logic. Therefore the domain of a model of the theory is taken as the set of truth values of the specified logic.

The language of the specified logic consists of a set of variables, a set of truth constants, and a set of connectives. The set of propositional variables of the specified logic is equal to the set of object variables of the theory with regard to which the logic is specified. Also the set of constants is the same in both languages. The semantics of the specified logic will mainly be given by special rules that we call *specifications*.

Definition 4.1.2. A $\boldsymbol{S}$-evaluation $\boldsymbol{e}$ for a nonempty set $\boldsymbol{S}$ is a function that assigns to each propositional variable an element in $\boldsymbol{S}$. For a first-order interpretation $\boldsymbol{M}$ with domain $\boldsymbol{D}$ we call a $\boldsymbol{D}$-evaluation also $\boldsymbol{M}$-evaluation.

Note that we take evaluations as both, assignments of object variables in the underlying theory and evaluations of propositional variables in the specified logic.

In this section we use the notation $\varphi[x_1, \dots, x_n]$ to indicate that the free variables of the formula φ are among the set $\{x_1, \dots, x_n\}$. The notation $\varphi[\psi_1, \dots, \psi_n]$ should indicate that each occurrence of x_i has been replaced by ψ_i for every $1 \leq i \leq n$.

Definition 4.1.3. A *specification* of an n-ary connective $\square$ with respect to a theory $\mathcal{T}$ is a rule of the form

$$\square(x_1, \dots, x_n) = \begin{cases} t_1[x_1, \dots, x_n] & \text{if } A_1[x_1, \dots, x_n] \\ \vdots & \vdots \\ t_m[x_1, \dots, x_n] & \text{if } A_m[x_1, \dots, x_n] \end{cases}$$

where each x_i is an object variable, each t_j is a term and each condition A_j is a *quantifier-free* formula of $\mathcal{T}$. The free variables of each t_j and each A_j are among the set $\{x_1, \dots, x_n\}$.

The conditions of the specification have to satisfy the following two properties.

- *Totality.* At least one condition must hold: $\mathcal{T} \vDash \forall x_1 \dots \forall x_n \bigvee_{i=1}^{m} A_i$.
- *Functionality.* If two conditions hold simultaneously, then they must yield the same result. For all models $\boldsymbol{M}$ of $\mathcal{T}$ and every $\boldsymbol{M}$-evaluation $\boldsymbol{e}$: if $\boldsymbol{M}, \boldsymbol{e} \vDash A_i$ and $\boldsymbol{M}, \boldsymbol{e} \vDash A_j$, then $\|t_i\|_{\boldsymbol{M},\boldsymbol{e}} = \|t_j\|_{\boldsymbol{M},\boldsymbol{e}}$.

The specification is called *projective* iff each $t_j[x_1, \dots, x_n]$ is either a constant or one of the variables in the set $\{x_1, \dots, x_n\}$.

Note that we only allow quantifier-free conditions. In Section 4.5 we investigate what happens if this condition is relaxed. Besides the specifications of connectives we also need a mechanism that specifies the designated truth values and thus the valid formulas.

Definition 4.1.4. A *designating predicate* is a first-order formula $\boldsymbol{D}[x]$ with exactly one free variable x.

We will assume that the variables $x_1, ...x_n$ of a specification and the variable x of a designating predicate are always fresh. This means that these variables occur nowhere else. Note that specifications and designating predicates are merely syntactic objects that should capture a natural way of defining the semantics of a logic. We use the specifications to define a semantics for a propositional logic by defining the truth function corresponding to a connective in the obvious way.

Definition 4.1.5. Let $\Box$ be a connective that has a specification with respect to a theory T as in Definition 4.1.3 and $\boldsymbol{M}$ a model of T with domain $\boldsymbol{D}$. Then the *truth function* $\widetilde{\Box}_{\boldsymbol{M}}$ of $\Box$ is defined by

$$\widetilde{\Box}_{\boldsymbol{M}}\left(d_1, \dots, d_n\right) = \begin{cases} \|t_1[x_1, \dots, x_n]\|_{\boldsymbol{M},\boldsymbol{e}} & \text{if } \boldsymbol{M}, \boldsymbol{e} \vDash A_1[x_1, \dots, x_n] \\ \vdots & \vdots \\ \|t_m[x_1, \dots, x_n]\|_{\boldsymbol{M},\boldsymbol{e}} & \text{if } \boldsymbol{M}, \boldsymbol{e} \vDash A_m[x_1, \dots, x_n] \end{cases}$$

for all $d_i \in \boldsymbol{D}$ and every $\boldsymbol{D}$-evaluation $\boldsymbol{e}$ such that $\boldsymbol{e}(x_i) = d_i$, for $1 \leq i \leq n$. This means that $\boldsymbol{e}$ assigns to each variable of the specification its corresponding argument of the truth function.

Remark. Due to the two requirements of totality and functionality in Definition 4.1.3, the truth function $\widetilde{\Box}_{\boldsymbol{M}}$ is always well-defined.

Definition 4.1.6. Let $\{\Box_1, \dots, \Box_k\}$ be a finite set of connectives, each with a specification with regard to a theory T as in Definition 4.1.3, $\boldsymbol{M}$ a model of T, and $\boldsymbol{e}$ an $\boldsymbol{M}$-evaluation. Then the *interpretation of formulas* in the specified logic S is inductively defined as follows:

$$\|\bar{c}\|^{\mathrm{S}}_{\boldsymbol{M},\boldsymbol{e}} = \|\bar{c}\|_{\boldsymbol{M}} \text{ for every truth constant } \bar{c}$$
$$\|x\|^{\mathrm{S}}_{\boldsymbol{M},\boldsymbol{e}} = \boldsymbol{e}(x) \text{ for every variable } x$$
$$\|\Box_i(\varphi_1, \dots, \varphi_n)\|^{\mathrm{S}}_{\boldsymbol{M},\boldsymbol{e}} = \widetilde{\Box}_{i,\boldsymbol{M}}\left(\|\varphi_1\|^{\mathrm{S}}_{\boldsymbol{M},\boldsymbol{e}}, \dots, \|\varphi_n\|^{\mathrm{S}}_{\boldsymbol{M},\boldsymbol{e}}\right) \text{ for every connective } \Box_i\,.$$

Now, together with the designating predicate we can define a logic.

Definition 4.1.7. Let $\{\Box_1, \dots, \Box_k\}$ be a finite set of connectives, each with a specification with regard to a theory T as in Definition 4.1.3, $\boldsymbol{D}[x]$ a designating predicate, and $\boldsymbol{M}$ a model of T. A formula φ is *valid* in the *specified logic* S of $\boldsymbol{M}$ iff $\boldsymbol{M}, \boldsymbol{e}' \vDash \boldsymbol{D}[x]$, where $\boldsymbol{e}$' is an $\boldsymbol{M}$-evaluation with $\boldsymbol{e}'(x) = \|\varphi\|^{\mathrm{S}}_{\boldsymbol{M},\boldsymbol{e}}$, for every $\boldsymbol{M}$-evaluation $\boldsymbol{e}$.

4.2 Linear orders with endpoints

We have to make precise what we mean by comparison of truth values without losing the generality of the idea of "logics of comparison." The abstract concept of comparison that seems adequate for our situation is the linear order. In a linear order we can always impose an order on two values that are not equal. As the intended domain of our theory are truth values, we also include a maximal element for truth and a minimal element for falsehood.

We consider a language with a two-ary relation symbol $<$ and the constants $\bar{1}$ and $\bar{0}$. Since this language has no function symbols, every term is either a variable or a constant.

Definition 4.2.1. The theory LOE of *linear orders with endpoints* is the deductive closure of the following formulas.

(LOE1) $\forall x(\neg(x < x))$ (irreflexive)

(LOE2) $\forall x \forall y \forall z(x < y \wedge y < z) \supset x < z$ (transitive)

(LOE3) $\forall x \forall y \forall z(x = y \vee x < y \vee y < x)$ (linear)

(LOE4) $\forall x(x = \bar{0} \vee \bar{0} < x)$ (minimal element)

(LOE5) $\forall x(x = \bar{1} \vee x < \bar{1})$ (maximal element)

(LOE6) $\neg(\bar{0} = \bar{1})$ (distinct)

We use the abbreviation $x \leq y$ for $x < y \vee x = y$.

Now we show some minor results that are important for our discussion of Gödel logic, which we will define as a certain specified logic over the theory LOE.

Lemma 4.2.2. *The following formulas are equivalent in the theory* LOE.

- $\neg(x \leq y)$ *is equivalent to* $y < x$.
- $\neg(x = \bar{0})$ *is equivalent to* $\bar{0} < x$.
- $\neg(x = \bar{1})$ *is equivalent to* $x < \bar{1}$.

Proof. Let $\boldsymbol{M}$ be a model of LOE and $\boldsymbol{e}$ an $\boldsymbol{M}$-evaluation. Assume that $\boldsymbol{M}, \boldsymbol{e} \models \neg(x \leq y)$. Since $x \leq y$ is an abbreviation for $x < y \vee x = y$ we then know that $\boldsymbol{M}, \boldsymbol{e} \not\models x < y$ and $\boldsymbol{M}, \boldsymbol{e} \not\models x = y$. Then, by the axiom of linearity (LOE3), we know that $\boldsymbol{M}, \boldsymbol{e} \models y < x$.

Now assume that $\boldsymbol{M}, \boldsymbol{e} \models y < x$. Suppose that $\boldsymbol{M}, \boldsymbol{e} \models x = y$. Then we get $\boldsymbol{M}, \boldsymbol{e} \models x < x$ which is not possible due to the axiom of irreflexivity (LOE1). Therefore the opposite holds and we know that $\boldsymbol{M}, \boldsymbol{e} \models \neg(x = y)$. Suppose that $\boldsymbol{M}, \boldsymbol{e} \models x < y$. Then, by the axiom of transitivity (LOE2), we also get the contradiction $\boldsymbol{M}, \boldsymbol{e} \models x < x$. Therefore the opposite holds and we know that $\boldsymbol{M}, \boldsymbol{e} \models \neg(x < y)$. Putting together both of these results we get $\boldsymbol{M}, \boldsymbol{e} \models \neg(x \leq y)$.

The other two equivalencies directly follow from (LOE4) and (LOE5) respectively (for the left-to-right direction), and the irreflexivity axiom (LOE1) (for the right-to-left direction). □

Lemma 4.2.3. *The following formulas are theorems of the theory* LOE*.*

$$\bar{0} = \bar{0} \qquad \bar{0} \leq \bar{0}$$
$$\bar{1} = \bar{1} \qquad \bar{1} \leq \bar{1}$$
$$\neg(\bar{0} = \bar{1}) \qquad \bar{0} \leq \bar{1}$$
$$\neg(\bar{1} = \bar{0}) \qquad \neg(\bar{1} \leq \bar{0})$$

Proof. The formulas $\bar{0} = \bar{0}$ and $\bar{1} = \bar{1}$ are theorems due to the definition of identity in classical first-order logic. By the definition of $\leq$ then also $\bar{0} \leq \bar{0}$ and $\bar{1} \leq \bar{1}$ are theorems.

The formula $\neg(\bar{0} = \bar{1})$ is axiom (LOE6) of the theory. Since identity, by definition, is symmetric, the formula $\neg(\bar{1} = \bar{0})$ is also a theorem of the theory.

Since $\bar{1}$ is the maximal element by axiom (LOE5), we get the theorem $\bar{0} = \bar{1} \vee \bar{0} < \bar{1}$. Because $\neg(\bar{0} = \bar{1})$ is axiom (LOE6), we conclude that $\bar{0} < \bar{1}$ is a theorem. By the definition of $\leq$ this means that that $\bar{0} \leq \bar{1}$ is a theorem. And, by Lemma 4.2.2, $\bar{0} < \bar{1}$ is equivalent to $\neg(\bar{1} \leq \bar{0})$ which is therefore also a theorem. □

We will see that for Gödel logic only the cardinality of the truth-degree structure is relevant. For finite cardinalities this also holds for models of LOE. We will see that two models with cardinality n are always equivalent up to renaming. To make this idea precise, we need the notion of an isomorphism.

Definition 4.2.4. Let $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ be two models of LOE with domains $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$, respectively. A function $f : \boldsymbol{D}_1 \to \boldsymbol{D}_2$ is a *homomorphism* $f : \boldsymbol{M}_1 \to \boldsymbol{M}_2$ iff the following structure-preserving properties hold:[26]

- $f\left(\|\bar{0}\|_{\boldsymbol{M}_1}\right) = \|\bar{0}\|_{\boldsymbol{M}_2}$ and $f\left(\|\bar{1}\|_{\boldsymbol{M}_1}\right) = \|\bar{1}\|_{\boldsymbol{M}_2}$.
- $\boldsymbol{M}_1, \boldsymbol{e} \vDash x < y$ if and only if $\boldsymbol{M}_2, f \circ \boldsymbol{e} \vDash x < y$ for every $\boldsymbol{D}_1$-evaluation $\boldsymbol{e}$.
- $\boldsymbol{M}_1, \boldsymbol{e} \vDash x = y$ if and only if $\boldsymbol{M}_2, f \circ \boldsymbol{e} \vDash x = y$ for every $\boldsymbol{D}_1$-evaluation $\boldsymbol{e}$.

The symbol $\circ$ denotes the operator for the composition of functions, i.e., $(f \circ \boldsymbol{e})(x) = f(\boldsymbol{e}(x))$ for every variable x. A bijective homomorphism is called *isomorphism.* Two models are called *isomorphic* iff there is an isomorphism between them.

We remark that, due to the inclusion of the identity sign, the third condition implies that the homomorphisms that we consider are always injective.

Lemma 4.2.5. *Every model* $\boldsymbol{M}$ *of* LOE *with finite cardinality* n *is isomorphic to the model* $\boldsymbol{M}^{(n)}$ *defined as follows:*

- *The domain of* $\boldsymbol{M}^{(n)}$ *is* $\boldsymbol{D}^{(n)} = \{0, \ldots, n-1\}$.
- $\|\bar{0}\|_{\boldsymbol{M}^{(n)}} = 0$ *and* $\|\bar{1}\|_{\boldsymbol{M}^{(n)}} = n - 1$.

[26]Precisely speaking, these conditions define *strict* homomorphisms. However, since we never work with non-strict homomorphisms, we omit this distinction.

- *For every $\boldsymbol{D}^{(n)}$-evaluation $\boldsymbol{e}$, $\boldsymbol{M}^{(n)}, \boldsymbol{e} \models x < y$ iff $\boldsymbol{e}(x) < \boldsymbol{e}(y)$ where $<$ is the usual smaller-than relation on the natural numbers.*

Proof. First of all it is clear that $\boldsymbol{M}^{(n)}$ is indeed a model of LOE because natural numbers are a prototypical example of a linear order.

Let $\boldsymbol{D}$ denote the domain of $\boldsymbol{M}$. We proof our claim by induction on n. Due to axiom (LOE6) a model of LOE must have at least two distinct elements. Therefore we do not have to care about the case $n = 1$. Thus, consider the case $n = 2$. Then it is clear that f defined by $f(0) = \|\bar{0}\|_{\boldsymbol{M}}$ and $f(1) = \|\bar{1}\|_{\boldsymbol{M}}$ is an isomorphism between $\boldsymbol{M}^{(n)}$ and $\boldsymbol{M}$.

Now consider the case $n > 2$. We want to define a restriction of $\boldsymbol{M}$ to the domain $\boldsymbol{D}' = \boldsymbol{D} \setminus \{\|\bar{1}\|_{\boldsymbol{M}}\}$. To do this, we just need to find a maximal element for the constant $\bar{1}$ in the set $\boldsymbol{D}'$, all other axioms of LOE also hold in $\boldsymbol{D}'$. Suppose that the following formula is true in $\boldsymbol{M}$:

$$\forall x(\neg(x = \bar{1}) \supset \exists y(\neg(y = \bar{1}) \wedge x < y)) .$$

Then clearly $\boldsymbol{M}$ cannot have finite cardinality, because, starting from the minimal element, we always find a strictly greater element. Therefore the formula is false in $\boldsymbol{M}$ and its negation is true which means that we find a maximal element in the set $\boldsymbol{D}'$. Since $\boldsymbol{D}'$ has cardinality at least 2, the maximal element is distinct from the minimal element. Therefore the restriction $\boldsymbol{M}'$ of $\boldsymbol{M}$ to the domain $\boldsymbol{D}'$ is a model of LOE.

Because $\boldsymbol{M}'$ has cardinality $n-1$, we know by our induction hypothesis that there is an isomorphism $f' : \boldsymbol{M}^{(n-1)} \to \boldsymbol{M}'$. Now define the function $f : \boldsymbol{D}^{(n)} \to \boldsymbol{D}$ by

$$f(i) = \begin{cases} \|\bar{1}\|_{\boldsymbol{M}} & \text{if } i = n-1 \\ f'(i) & \text{otherwise} . \end{cases}$$

Clearly, f is bijective because f' is bijective. It is easy to see that f is structure-preserving. We only have to consider the cases that are not already handled by f'. By our definition we have $f(\|\bar{1}\|_{\boldsymbol{M}^{(n)}}) = f(n-1) = \|\bar{1}\|_{\boldsymbol{M}}$. It is clear that $\boldsymbol{M}^{(n)}, \boldsymbol{e} \models x = \bar{1}$ if and only if $\boldsymbol{M}, f \circ \boldsymbol{e} \models x = \bar{1}$ for every $\boldsymbol{M}^{(n)}$-evaluation $\boldsymbol{e}$. Since both $\boldsymbol{M}^{(n)}$ and $\boldsymbol{M}$ are models of LOE, we have $\boldsymbol{M}^{(n)}, \boldsymbol{e} \not\models \bar{1} < x$ and $\boldsymbol{M}, f \circ \boldsymbol{e} \not\models \bar{1} < x$ for every $\boldsymbol{M}^{(n)}$-evaluation $\boldsymbol{e}$. For the same reason we have $\boldsymbol{M}^{(n)}, \boldsymbol{e} \models x < \bar{1} \wedge \neg(x = \bar{1})$ and $\boldsymbol{M}, f \circ \boldsymbol{e} \models x < \bar{1} \wedge \neg(x = \bar{1})$ for every $\boldsymbol{M}^{(n)}$-evaluation $\boldsymbol{e}$. Therefore $\boldsymbol{M}^{(n)}, \boldsymbol{e} \models x < y$ if and only if $\boldsymbol{M}, f \circ \boldsymbol{e} \models x < y$ for every $\boldsymbol{M}^{(n)}$-evaluation $\boldsymbol{e}$ with $\boldsymbol{e}(x) = \|\bar{1}\|_{\boldsymbol{M}^{(n)}}$ or $\boldsymbol{e}(y) = \|\bar{1}\|_{\boldsymbol{M}^{(n)}}$.

□

Since all models with cardinality n are isomorphic to a certain model and the property of being isomorphic is transitive, it follows as a corollary that all models with a fixed finite cardinality are isomorphic.

Corollary 4.2.6. *For every natural number n, all models of* LOE *with cardinality n are isomorphic.*

4.3 Gödel logic

In Section 3.4.4 we defined Gödel logic as a fuzzy logic over the truth-value interval $[0, 1]$. Now we define Gödel logics for more general truth-degree structures.

Definition 4.3.1. For every model $\boldsymbol{M}$ of the theory LOE, the *Gödel logic* G of $\boldsymbol{M}$ with globalization is given by the following specifications:

$$x \wedge y = \begin{cases} x & \text{if } x \leq y \\ y & \text{if } \neg(x \leq y) \end{cases} \qquad x \vee y = \begin{cases} x & \text{if } y \leq x \\ y & \text{if } \neg(y \leq x) \end{cases}$$

$$x \supset y = \begin{cases} \bar{1} & \text{if } x \leq y \\ y & \text{if } \neg(x \leq y) \end{cases} \qquad \neg x = \begin{cases} \bar{1} & \text{if } x = \bar{0} \\ \bar{0} & \text{if } \neg(x = \bar{0}) \end{cases}$$

$$\triangle x = \begin{cases} \bar{1} & \text{if } x = \bar{1} \\ \bar{0} & \text{if } \neg(x = \bar{1}) \end{cases}$$

Its designating predicate is $D_{\mathrm{G}}[x] = 1 \leq x$.

In Gödel logic, disjunction is implemented by the maximum operator and conjunction is implemented by the minimum operator. An implication is true if and only if the left argument is smaller than or equal to the right argument. Negation is the implication of falsehood and the globalization connective $\triangle$ has a definition inverse to negation. In the following, we always speak of Gödel logic when we mean Gödel logic with globalization.

Remark. It is obvious that the specifications for Gödel logic fulfill the requirements of totality and functionality because each specification consists of two contrary conditions. Furthermore, the negated conditions have their intuitive meanings (compare Lemma 4.2.2).

Lemma 4.3.2. *For every model $\boldsymbol{M}$ of* LOE*, the connectives of Gödel logic behave classically for the truth values $\|\bar{1}\|_{\boldsymbol{M}}$ and $\|\bar{0}\|_{\boldsymbol{M}}$.*

$$\begin{aligned} \|\bar{0} \wedge \bar{0}\|^{\mathrm{G}}_{\boldsymbol{M}} &= \|\bar{0}\|_{\boldsymbol{M}} & \qquad \|\bar{0} \vee \bar{0}\|^{\mathrm{G}}_{\boldsymbol{M}} &= \|\bar{0}\|_{\boldsymbol{M}} \\ \|\bar{0} \wedge \bar{1}\|^{\mathrm{G}}_{\boldsymbol{M}} &= \|\bar{0}\|_{\boldsymbol{M}} & \|\bar{0} \vee \bar{1}\|^{\mathrm{G}}_{\boldsymbol{M}} &= \|\bar{1}\|_{\boldsymbol{M}} \\ \|\bar{1} \wedge \bar{0}\|^{\mathrm{G}}_{\boldsymbol{M}} &= \|\bar{0}\|_{\boldsymbol{M}} & \|\bar{1} \vee \bar{0}\|^{\mathrm{G}}_{\boldsymbol{M}} &= \|\bar{1}\|_{\boldsymbol{M}} \\ \|\bar{1} \wedge \bar{1}\|^{\mathrm{G}}_{\boldsymbol{M}} &= \|\bar{1}\|_{\boldsymbol{M}} & \|\bar{1} \vee \bar{1}\|^{\mathrm{G}}_{\boldsymbol{M}} &= \|\bar{1}\|_{\boldsymbol{M}} \end{aligned}$$

$$\begin{aligned} \|\bar{0} \supset \bar{0}\|^{\mathrm{G}}_{\boldsymbol{M}} &= \|\bar{1}\|_{\boldsymbol{M}} & \qquad \|\neg\bar{0}\|^{\mathrm{G}}_{\boldsymbol{M}} &= \|\bar{1}\|_{\boldsymbol{M}} \\ \|\bar{0} \supset \bar{1}\|^{\mathrm{G}}_{\boldsymbol{M}} &= \|\bar{1}\|_{\boldsymbol{M}} & \|\neg\bar{1}\|^{\mathrm{G}}_{\boldsymbol{M}} &= \|\bar{0}\|_{\boldsymbol{M}} \\ \|\bar{1} \supset \bar{0}\|^{\mathrm{G}}_{\boldsymbol{M}} &= \|\bar{0}\|_{\boldsymbol{M}} & \|\triangle\bar{0}\|^{\mathrm{G}}_{\boldsymbol{M}} &= \|\bar{0}\|_{\boldsymbol{M}} \\ \|\bar{1} \supset \bar{1}\|^{\mathrm{G}}_{\boldsymbol{M}} &= \|\bar{1}\|_{\boldsymbol{M}} & \|\triangle\bar{1}\|^{\mathrm{G}}_{\boldsymbol{M}} &= \|\bar{1}\|_{\boldsymbol{M}} \end{aligned}$$

Proof. Let $\boldsymbol{e}$ be an evaluation such that $\boldsymbol{e}(x) = \|\bar{c}_1\|_{\boldsymbol{M}}$ and $\boldsymbol{e}(y) = \|\bar{c}_2\|_{\boldsymbol{M}}$ where $\bar{c}_1$ and $\bar{c}_2$ are two truth constants from the set $\{\bar{0}, \bar{1}\}$ and let $A[x, y]$ be a formula with at most two free variables. Then $\boldsymbol{M}, \boldsymbol{e} \vDash A[x, y]$ if and only if $\boldsymbol{M}, \boldsymbol{e} \vDash A[\bar{c}_1, \bar{c}_2]$.

Therefore it can easily be verified that the truth functions of Gödel logic behave classically by checking, in the list of Lemma 4.2.3, for each condition whether it applies. □

We now start an investigation with the aim of showing that for Gödel logics only the cardinality of the truth-degree structure is relevant. All Gödel logics of models with a fixed finite cardinality coincide. For models with infinite cardinality, even more holds: their sets of valid formulas all coincide. This result is well-known but to make the material in this chapter self-contained we include our elementary proofs.

We first formulate our central technical lemma that states that for Gödel logics it does not matter whether we first apply a homomorphism and evaluate afterwards or if we first evaluate and then apply the homomorphism.

Lemma 4.3.3. *Let $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ be two models of* LOE *such that there is a homomorphism $f : \boldsymbol{M}_1 \to \boldsymbol{M}_2$. Let φ be a formula of Gödel logic and $\boldsymbol{e}$ an $\boldsymbol{M}_1$-evaluation. Then*

$$f\left(\|\varphi\|^{\mathrm{G}}_{\boldsymbol{M}_1,\boldsymbol{e}}\right) = \|\varphi\|^{\mathrm{G}}_{\boldsymbol{M}_2,f\circ\boldsymbol{e}}\,.$$

Proof. The proof claim is by induction on the complexity of φ.

- $\varphi = x$ for a variable x: Trivially, $f(\|x\|^{\mathrm{G}}_{\boldsymbol{M}_1,\boldsymbol{e}}) = f(\boldsymbol{e}(x)) = (f \circ \boldsymbol{e})(x)$.
- $\varphi = \bar{c}$ where $\bar{c}$ is one of the truth constants $\bar{0}$ or $\bar{1}$: Since f is a homomorphism, we have $f(\|\bar{c}\|^{\mathrm{G}}_{\boldsymbol{M}_1,\boldsymbol{e}}) = f(\|\bar{c}\|_{\boldsymbol{M}_1}) = \|\bar{c}\|_{\boldsymbol{M}_2} = \|\bar{c}\|^{\mathrm{G}}_{\boldsymbol{M}_2,f\circ\boldsymbol{e}}$.
- $\varphi = \psi \wedge \chi$: First, define the $\boldsymbol{M}_1$-evaluation $\boldsymbol{e}_1$ and the $\boldsymbol{M}_2$-evaluation $\boldsymbol{e}_2$ as follows:

$$\boldsymbol{e}_1(v) = \begin{cases} \|\psi\|^{\mathrm{G}}_{\boldsymbol{M}_1,\boldsymbol{e}} & \text{if } v = x \\ \|\chi\|^{\mathrm{G}}_{\boldsymbol{M}_1,\boldsymbol{e}} & \text{if } v = y \\ \boldsymbol{e}(v) & \text{otherwise} \end{cases} \qquad \boldsymbol{e}_2(v) = \begin{cases} \|\psi\|^{\mathrm{G}}_{\boldsymbol{M}_2,f\circ\boldsymbol{e}} & \text{if } v = x \\ \|\chi\|^{\mathrm{G}}_{\boldsymbol{M}_2,f\circ\boldsymbol{e}} & \text{if } v = y \\ f \circ \boldsymbol{e}(v) & \text{otherwise}\,. \end{cases}$$

  The interpretation of the formula $\psi \wedge \varphi$, by the definition of conjunction in Gödel logic, depends on $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ in the following way:

$$\|\psi \wedge \chi\|^{\mathrm{G}}_{\boldsymbol{M}_1,\boldsymbol{e}} = \begin{cases} \|\psi\|^{\mathrm{G}}_{\boldsymbol{M}_1,\boldsymbol{e}} & \text{if } \boldsymbol{M}_1, \boldsymbol{e}_1 \vDash x \leq y \\ \|\chi\|^{\mathrm{G}}_{\boldsymbol{M}_1,\boldsymbol{e}} & \text{if } \boldsymbol{M}_1, \boldsymbol{e}_1 \nvDash x \leq y \end{cases}$$

$$\|\psi \wedge \chi\|^{\mathrm{G}}_{\boldsymbol{M}_2,f\circ\boldsymbol{e}} = \begin{cases} \|\psi\|^{\mathrm{G}}_{\boldsymbol{M}_2,f\circ\boldsymbol{e}} & \text{if } \boldsymbol{M}_2, \boldsymbol{e}_2 \vDash x \leq y \\ \|\chi\|^{\mathrm{G}}_{\boldsymbol{M}_2,f\circ\boldsymbol{e}} & \text{if } \boldsymbol{M}_2, \boldsymbol{e}_2 \nvDash x \leq y\,. \end{cases}$$

By the induction hypothesis we get

$$e_2(v) = \begin{cases} f\left(\|\psi\|^{\mathrm{G}}_{M_1,e}\right) & \text{if } v = x \\ f\left(\|\chi\|^{\mathrm{G}}_{M_1,e}\right) & \text{if } v = y \\ f \circ e(v) & \text{otherwise}. \end{cases}$$

and therefore $e_2 = f \circ e_1$.

Since f is a homomorphism, the following relations hold:

$$\begin{aligned} M_1, e_1 \models x < y &\text{ if and only if } M_2, f \circ e_1 \models x < y \\ &\text{ if and only if } M_2, e_2 \models x < y \\ M_1, e_1 \models x = y &\text{ if and only if } M_2, f \circ e_1 \models x = y \\ &\text{ if and only if } M_2, e_2 \models x = y. \end{aligned}$$

Since $x \leq y$ is an abbreviation for $x < y \vee x = y$, we then know that $M_1, e_1 \models x \leq y$ if and only if $M_2, e_2 \models x \leq y$.

First, consider the case that $M_1, e_1 \models x \leq y$. Then also $M_2, e_2 \models x \leq y$. By the definition of conjunction in Gödel logic and the induction hypothesis we then get

$$f\left(\|\psi \wedge \chi\|^{\mathrm{G}}_{M_1,e}\right) = f\left(\|\psi\|^{\mathrm{G}}_{M_1,e}\right) = \|\psi\|^{\mathrm{G}}_{M_2,f\circ e} = \|\psi \wedge \chi\|^{\mathrm{G}}_{M_2,f\circ e}.$$

Second, consider the case that $M_1, e_1 \not\models x \leq y$. Then also $M_2, e_2 \not\models x \leq y$. By the definition of conjunction in Gödel logic and the induction hypothesis we then get

$$f\left(\|\psi \wedge \chi\|^{\mathrm{G}}_{M_1,e}\right) = f\left(\|\chi\|^{\mathrm{G}}_{M_1,e}\right) = \|\chi\|^{\mathrm{G}}_{M_2,f\circ e} = \|\psi \wedge \chi\|^{\mathrm{G}}_{M_2,f\circ e}$$

In both cases, we get $f(\|\psi \wedge \chi\|^{\mathrm{G}}_{M_1,e}) = \|\psi \wedge \chi\|^{\mathrm{G}}_{M_2,f\circ e}$.

- The proofs for the other connectives work in the same manner as for $\wedge$.

□

Now we build upon our results for the theory LOE to show that all Gödel logics of truth-degree structures with a fixed finite cardinality coincide.

Lemma 4.3.4. *Let M_1 and M_2 be models of* LOE *such there is an injective homomorphism $f : M_1 \to M_2$. Then every formula that is valid in the Gödel logic of M_2 is also valid in the Gödel logic of M_1.*

Proof. By Lemma 4.3.3 we know that $f(\|\varphi\|^{G}_{\boldsymbol{M}_1,\boldsymbol{e}}) = \|\varphi\|^{G}_{\boldsymbol{M}_2,f\circ\boldsymbol{e}}$. Since φ is valid in the Gödel logic of $\boldsymbol{M}_2$ we have $\|\varphi\|^{G}_{\boldsymbol{M}_2,f\circ\boldsymbol{e}} = \|\bar{1}\|_{\boldsymbol{M}_2}$. Because f is a homomorphism we also have $f(\|\bar{1}\|_{\boldsymbol{M}_1}) = \|\bar{1}\|_{\boldsymbol{M}_2}$. Thus, we get $f(\|\varphi\|^{G}_{\boldsymbol{M}_1,\boldsymbol{e}}) = f(\|\bar{1}\|_{\boldsymbol{M}_1})$ and since f is injective this means that $\|\varphi\|^{G}_{\boldsymbol{M}_1,\boldsymbol{e}} = \|\bar{1}\|_{\boldsymbol{M}_1}$. Since $\boldsymbol{e}$ was an arbitrary $\boldsymbol{M}_1$-evaluation, we may conclude that φ is valid in the Gödel logic of $\boldsymbol{M}_1$. □

Since an isomorphism gives an injective homomorphism in both directions, we get an easy corollary.

Corollary 4.3.5. *Let $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ be two isomorphic models of* LOE. *Then a formula φ is valid in the Gödel logic of $\boldsymbol{M}_1$ if and only if φ is valid in the Gödel logic of $\boldsymbol{M}_2$.*

Since all models of LOE with a fixed finite cardinality are isomorphic (see Corollary 4.2.6), we already get our desired result for finite-valued Gödel logics.

Corollary 4.3.6. *For every natural number n, the set of valid formulas is the same for all Gödel logics based on a model of* LOE *with cardinality n.*

Remark. The consequences of Lemma 4.3.3 for isomorphic models $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ of LOE are even stronger, namely that φ has a designated truth value in $\boldsymbol{M}_1$ if and only if it has a designated truth value in $\boldsymbol{M}_2$. Therefore the two Gödel logics also coincide at the level of logical consequence. However, we do not consider entailment in this chapter because the overall result we aim at only holds for validity and not for entailment.

Finally, we show that for infinite cardinalities, no distinction can be made between different Gödel logics in terms of valid formulas. The theorem and its proof idea are taken from Gottwald's monograph [44].

Theorem 4.3.7. *For every formula φ of Gödel logic the following statements are equivalent:*

(i) For every model $\boldsymbol{M}$ of LOE *with infinite cardinality, φ is valid in the Gödel logic of $\boldsymbol{M}$.*

(ii) For some model $\boldsymbol{M}$ of LOE *with infinite cardinality, φ is valid in the Gödel logic of $\boldsymbol{M}$.*

(iii) For every model $\boldsymbol{M}$ of LOE *with finite cardinality, φ is valid in the Gödel logic of $\boldsymbol{M}$.*

We are mainly interested in the equivalence between (i) and (ii). The equivalence to (iii) is just an interesting by-product that is however essential for the proof.

Proof. The step from (i) to (ii) is trivial because a model of LOE with infinite cardinality exists.

For the step from (ii) to (iii) we first show that for every model $\boldsymbol{M}_1$ of LOE with finite domain $\boldsymbol{D}_1$ and every model $\boldsymbol{M}_2$ of LOE with infinite domain $\boldsymbol{D}_2$, there is an injective homomorphism from $\boldsymbol{M}_1$ to $\boldsymbol{M}_2$. Since $\boldsymbol{D}_2$ is infinite, there is a subset $A \subseteq \boldsymbol{D}_2$

such that $|A| = |\boldsymbol{D}_1|$. Clearly, we also find such a set A such that $\|\bar{0}\|_{\boldsymbol{M}_2} \in A$ and $\|\bar{1}\|_{\boldsymbol{M}_2} \in A$. Now we define $\boldsymbol{M}'$ to be the restriction of $\boldsymbol{M}_2$ to the domain A. By restriction we mean that $\|\bar{0}\|_{\boldsymbol{M}'} = \|\bar{0}\|_{\boldsymbol{M}_2} \in A$, $\|\bar{1}\|_{\boldsymbol{M}'} = \|\bar{1}\|_{\boldsymbol{M}_2} \in A$ and $\boldsymbol{M}', \boldsymbol{e} \vDash x < y$ iff $\boldsymbol{M}_2, \boldsymbol{e} \vDash x < y$ for every $\boldsymbol{M}'$-evaluation $\boldsymbol{e}$.[27] Clearly, $\boldsymbol{M}'$ is a model of LOE.

Since both $\boldsymbol{M}_1$ and $\boldsymbol{M}'$ have the same finite cardinality, there is an isomorphism f between $\boldsymbol{M}_1$ and $\boldsymbol{M}'$ due to Corollary 4.2.6. The function f is also an injective homomorphism from $\boldsymbol{M}_1$ to $\boldsymbol{M}_2$. Now let $\boldsymbol{e}$ be an arbitrary $\boldsymbol{M}_1$-evaluation. Then by Lemma 4.3.3 we have $f(\|\varphi\|^{\mathrm{G}}_{\boldsymbol{M}_1,\boldsymbol{e}}) = \|\varphi\|^{\mathrm{G}}_{\boldsymbol{M}_2,f\circ\boldsymbol{e}}$. Since φ is valid in the Gödel logic of $\boldsymbol{M}_2$ we know that $\|\varphi\|^{\mathrm{G}}_{\boldsymbol{M}_2,f\circ\boldsymbol{e}} = \|\bar{1}\|_{\boldsymbol{M}_2} = f(\|\bar{1}\|_{\boldsymbol{M}_1})$. Because f is injective, the identity $\|\varphi\|^{\mathrm{G}}_{\boldsymbol{M}_1,\boldsymbol{e}} = \|\bar{1}\|_{\boldsymbol{M}_1}$ follows. Since $\boldsymbol{e}$ was an arbitrary evaluation, φ is valid in the Gödel logic of $\boldsymbol{M}_1$.

We now show the step from (iii) to (i). Let $x_1, \ldots x_n$ denote the n variables occurring in φ. Suppose that that there is a model $\boldsymbol{M}$ of LOE with infinite cardinality such that φ is not valid. This means that there is an $\boldsymbol{M}$-evaluation $\boldsymbol{e}$ such that $\|\varphi\|^{\mathrm{G}}_{\boldsymbol{M},\boldsymbol{e}} \neq \|\bar{1}\|_{\boldsymbol{M}}$. Now we define $\boldsymbol{M}'$ to be the restriction of $\boldsymbol{M}$ to the domain $\{\|\bar{0}\|_{\boldsymbol{M}}, \|\bar{1}\|_{\boldsymbol{M}}, \boldsymbol{e}(x_1), \ldots, \boldsymbol{e}(x_n)\}$, i.e., $\|\bar{0}\|_{\boldsymbol{M}'} = \|\bar{0}\|_{\boldsymbol{M}}$, $\|\bar{1}\|_{\boldsymbol{M}'} = \|\bar{1}\|_{\boldsymbol{M}}$ and $\boldsymbol{M}', \boldsymbol{e}' \vDash x < y$ iff $\boldsymbol{M}, \boldsymbol{e}' \vDash x < y$ for every $\boldsymbol{M}'$-evaluation $\boldsymbol{e}'$. Obviously, $\boldsymbol{M}'$ is also a model of LOE and has finite cardinality.

Consider the following $\boldsymbol{M}'$-evaluation $\boldsymbol{e}'$:

$$\boldsymbol{e}'(v) = \begin{cases} \boldsymbol{e}(v) & \text{if } v \in \{x_1, \ldots, x_n\} \\ \|\bar{0}\|_{\boldsymbol{M}'} & \text{otherwise}. \end{cases}$$

It can easily be seen that $\|\varphi\|^{\mathrm{G}}_{\boldsymbol{M},\boldsymbol{e}} = \|\varphi\|^{\mathrm{G}}_{\boldsymbol{M}',\boldsymbol{e}'}$. But $\|\varphi\|^{\mathrm{G}}_{\boldsymbol{M},\boldsymbol{e}} \neq \|\bar{1}\|_{\boldsymbol{M}} = \|\bar{1}\|_{\boldsymbol{M}'}$ then is a contradiction to the assumption that φ is valid in every model of LOE with finite cardinality. □

Due to this result it makes sense to speak of *the* n-valued Gödel logic and *the* infinite-valued Gödel logic. Note that n-valued Gödel logic and infinite-valued Gödel logic do not coincide. For example, the formula $x \vee \neg x$ is valid in 2-valued Gödel logic, which is just classical propositional logic, but not in any n-valued Gödel logic for $n > 2$ nor in infinite-valued Gödel logic.

Another question that one might ask is whether our presentation of Gödel logic is optimal. Some of our connectives could be defined syntactically from other connectives:

- $\neg\varphi$ is equivalent $\varphi \supset \bar{0}$
- $\varphi \vee \psi$ is equivalent to $((\varphi \supset \psi) \supset \psi) \wedge ((\psi \supset \varphi) \supset \varphi)$
- $\bar{1}$ is equivalent to $x \supset x$
- $\bar{0}$ is equivalent to $\neg\bar{1}$

[27]Note that every $\boldsymbol{M}'$-evaluation is also a $\boldsymbol{M}_2$-evaluation

For the other connectives, such syntactic definitions are not possible:

- Conjunction is not definable from implication, disjunction and negation [8].
- Implication is not definable from conjunction, disjunction and negation [96].
- Globalization cannot be defined from the remaining connectives [89].

In the remainder of this chapter we will need the globalization operator $\triangle$ very often. It is therefore interesting to know that this operator is really needed for the full expressibility of Gödel logic.

4.4 Logics of comparison

In the following, we show that any logic that is specified over the theory of linear orders is already contained in a Gödel logic. We show that this statement holds in a very strong sense. Every connective that can be specified with respect to LOE is syntactically definable from the connectives of Gödel logic.

Note that the theory LOE is purely relational. Therefore every specification over LOE is projective.

Lemma 4.4.1. *Let $\varphi[x_1, \dots, x_n]$ be a formula of* LOE *with free variables among the set $\{x_1, \dots, x_n\}$. Then we can construct a formula $F_\varphi[x_1, \dots, x_n]$ of Gödel logic such that for every model $\boldsymbol{M}$ of* LOE *and every $\boldsymbol{M}$-evaluation $\boldsymbol{e}$:*

- *If $\boldsymbol{M}, \boldsymbol{e} \models \varphi$, then $\|F_\varphi\|^{\mathrm{G}}_{\boldsymbol{M},\boldsymbol{e}} = \|\bar{1}\|_{\boldsymbol{M}}$.*
- *If $\boldsymbol{M}, \boldsymbol{e} \not\models \varphi$, then $\|F_\varphi\|^{\mathrm{G}}_{\boldsymbol{M},\boldsymbol{e}} = \|\bar{0}\|_{\boldsymbol{M}}$.*

Proof. The proof is by induction on the structure of φ.

- If φ is a classical truth constant we simply define $F_\varphi = \bar{1}$ if $\varphi = \top$ and $F_\varphi = \bar{0}$ if $\varphi = \bot$. This is correct because $\boldsymbol{M}, \boldsymbol{e} \models \top$ and $\boldsymbol{M}, \boldsymbol{e} \not\models \bot$ in any case.
- If φ is of the form $t_1 < t_2$ for two terms t_1 and t_2, then we define F_φ as $\triangle(t_1 \supset t_2) \wedge \neg\triangle(t_2 \supset t_1)$. We then arrive at the following evaluations for the left and the right part of F_φ:

$$\|\triangle(t_1 \supset t_2)\|^{\mathrm{G}}_{\boldsymbol{M},\boldsymbol{e}} = \begin{cases} \|\triangle\bar{1}\|^{\mathrm{G}}_{\boldsymbol{M},\boldsymbol{e}} & \text{if } \boldsymbol{M}, \boldsymbol{e} \models t_1 \leq t_2 \\ \|\triangle t_2\|^{\mathrm{G}}_{\boldsymbol{M},\boldsymbol{e}} & \text{if } \boldsymbol{M}, \boldsymbol{e} \models \neg(t_1 \leq t_2) \end{cases}$$

$$\|\neg\triangle(t_2 \supset t_1)\|^{\mathrm{G}}_{\boldsymbol{M},\boldsymbol{e}} = \begin{cases} \|\neg\triangle\bar{1}\|^{\mathrm{G}}_{\boldsymbol{M},\boldsymbol{e}} & \text{if } \boldsymbol{M}, \boldsymbol{e} \models t_2 \leq t_1 \\ \|\neg\triangle t_1\|^{\mathrm{G}}_{\boldsymbol{M},\boldsymbol{e}} & \text{if } \boldsymbol{M}, \boldsymbol{e} \models \neg(t_2 \leq t_1)\,. \end{cases}$$

  By the specifications of $\triangle$ and $\neg$ in Gödel logic, we get $\|\triangle\bar{1}\|^{\mathrm{G}}_{\boldsymbol{M},\boldsymbol{e}} = \|\bar{1}\|_{\boldsymbol{M}}$ and $\|\neg\triangle\bar{1}\|^{\mathrm{G}}_{\boldsymbol{M},\boldsymbol{e}} = \|\bar{0}\|_{\boldsymbol{M}}$. Now we investigate the second branches of the truth functions. Suppose that $\boldsymbol{M}, \boldsymbol{e} \models \neg(t_1 \leq t_2)$ and that $\|t_2\|_{\boldsymbol{M},\boldsymbol{e}} = \|\bar{1}\|_{\boldsymbol{M}}$. Since $\bar{1}$ is the maximal element by axiom (LOE5) we know that $\boldsymbol{M}, \boldsymbol{e} \vDash t_1 \leq t_2$ which contradicts our first assumption. Therefore we get $\|t_2\|_{\boldsymbol{M},\boldsymbol{e}} \neq \|\bar{1}\|_{\boldsymbol{M}}$ which means that $\|\triangle t_2\|^{\mathrm{G}}_{\boldsymbol{M},\boldsymbol{e}} = \|\bar{0}\|_{\boldsymbol{M}}$. With the same argument, under the assumption that $\boldsymbol{M}, \boldsymbol{e} \vDash \neg(t_2 \leq t_1)$, we get that $\|t_1\|_{\boldsymbol{M},\boldsymbol{e}} \neq \|\bar{1}\|_{\boldsymbol{M}}$ which means that $\|\neg\triangle t_1\|^{\mathrm{G}}_{\boldsymbol{M},\boldsymbol{e}} = \|\bar{1}\|_{\boldsymbol{M}}$.

Finally, we end up with the following truth functions:

$$\|\triangle(t_1 \supset t_2)\|^{\mathrm{G}}_{\boldsymbol{M},\boldsymbol{e}} = \begin{cases} \|\bar{1}\|_{\boldsymbol{M}} & \text{if } \boldsymbol{M}, \boldsymbol{e} \vDash t_1 \leq t_2 \\ \|\bar{0}\|_{\boldsymbol{M}} & \text{if } \boldsymbol{M}, \boldsymbol{e} \vDash t_2 < t_1 \end{cases}$$

$$\|\triangle\neg(t_2 \supset t_1)\|^{\mathrm{G}}_{\boldsymbol{M},\boldsymbol{e}} = \begin{cases} \|\bar{0}\|_{\boldsymbol{M}} & \text{if } \boldsymbol{M}, \boldsymbol{e} \vDash t_2 \leq t_1 \\ \|\bar{1}\|_{\boldsymbol{M}} & \text{if } \boldsymbol{M}, \boldsymbol{e} \vDash t_1 < t_2 . \end{cases}$$

Now it is clear that if $\boldsymbol{M}, \boldsymbol{e} \vDash t_1 < t_2$, then $\|F_\varphi\|^{\mathrm{G}}_{\boldsymbol{M},\boldsymbol{e}} = \|\bar{1}\|_{\boldsymbol{M}}$. If however $\boldsymbol{M}, \boldsymbol{e} \nvDash t_1 < t_2$, then by Lemma 4.2.2 $\boldsymbol{M}, \boldsymbol{e} \vDash t_2 \leq t_1$ and therefore we get $\|F_\varphi\|^{\mathrm{G}}_{\boldsymbol{M},\boldsymbol{e}} = \|\bar{0}\|_{\boldsymbol{M}}$.

- If φ is of the form $t_1 = t_2$ for two terms t_1 and t_2, then we define F_φ as $\triangle(t_1 \supset t_2) \wedge \triangle(t_2 \supset t_1)$. We omit the proof because it is very similar to proof of the previous case.

- If φ is a composed formula, we apply Lemma 4.3.2. If φ is a negation $\neg\psi$ then we know by the induction hypothesis that there is a formula F_ψ with the desired property and we define F_φ as $\neg F_\psi$.

  - If $\boldsymbol{M}, \boldsymbol{e} \vDash \neg\psi$ then $\boldsymbol{M}, \boldsymbol{e} \nvDash \psi$. The induction hypothesis gives us $\|F_\psi\|^{\mathrm{G}}_{\boldsymbol{M},\boldsymbol{e}} = \|\bar{0}\|_{\boldsymbol{M}}$. Therefore by Lemma 4.3.2 we get $\|F_\varphi\|^{\mathrm{G}}_{\boldsymbol{M},\boldsymbol{e}} = \|\neg F_\psi\|^{\mathrm{G}}_{\boldsymbol{M},\boldsymbol{e}} = \|\bar{1}\|_{\boldsymbol{M}}$.
  - If $\boldsymbol{M}, \boldsymbol{e} \nvDash \neg\psi$ then $\boldsymbol{M}, \boldsymbol{e} \vDash \psi$. By the induction hypothesis we get $\|F_\psi\|^{\mathrm{G}}_{\boldsymbol{M},\boldsymbol{e}} = \|\bar{1}\|_{\boldsymbol{M}}$. Therefore by Lemma 4.3.2 we get $\|F_\varphi\|^{\mathrm{G}}_{\boldsymbol{M},\boldsymbol{e}} = \|\neg F_\psi\|^{\mathrm{G}}_{\boldsymbol{M},\boldsymbol{e}} = \|\bar{0}\|_{\boldsymbol{M}}$.

  If φ is a conjunction $\psi \wedge \chi$ then we know by the induction hypothesis that there are formulas F_ψ and F_χ with the desired property and we define F_φ as $F_\psi \wedge F_\chi$.

  - If $\boldsymbol{M}, \boldsymbol{e} \vDash \psi \wedge \chi$ then $\boldsymbol{M}, \boldsymbol{e} \vDash \psi$ and $\boldsymbol{M}, \boldsymbol{e} \vDash \chi$. The induction hypothesis gives us $\|F_\psi\|^{\mathrm{G}}_{\boldsymbol{M},\boldsymbol{e}} = \|\bar{1}\|_{\boldsymbol{M}}$ and $\|F_\chi\|^{\mathrm{G}}_{\boldsymbol{M},\boldsymbol{e}} = \|\bar{1}\|_{\boldsymbol{M}}$. Therefore by Lemma 4.3.2 we get $\|F_\varphi\|^{\mathrm{G}}_{\boldsymbol{M},\boldsymbol{e}} = \|F_\psi \wedge F_\chi\|^{\mathrm{G}}_{\boldsymbol{M},\boldsymbol{e}} = \|\bar{1}\|_{\boldsymbol{M}}$.
  - If $\boldsymbol{M}, \boldsymbol{e} \nvDash \psi \wedge \chi$ then one of the following three cases applies:
    * If $\boldsymbol{M}, \boldsymbol{e} \nvDash \psi$ and $\boldsymbol{M}, \boldsymbol{e} \nvDash \chi$, then, by the induction hypothesis, $\|F_\psi\|^{\mathrm{G}}_{\boldsymbol{M},\boldsymbol{e}} = \|\bar{0}\|_{\boldsymbol{M}}$ and $\|F_\chi\|^{\mathrm{G}}_{\boldsymbol{M},\boldsymbol{e}} = \|\bar{0}\|_{\boldsymbol{M}}$.
    * If $\boldsymbol{M}, \boldsymbol{e} \vDash \psi$ and $\boldsymbol{M}, \boldsymbol{e} \nvDash \chi$, then, by the induction hypothesis, $\|F_\psi\|^{\mathrm{G}}_{\boldsymbol{M},\boldsymbol{e}} = \|\bar{1}\|_{\boldsymbol{M}}$ and $\|F_\chi\|^{\mathrm{G}}_{\boldsymbol{M},\boldsymbol{e}} = \|\bar{0}\|_{\boldsymbol{M}}$.
    * If $\boldsymbol{M}, \boldsymbol{e} \nvDash \psi$ and $\boldsymbol{M}, \boldsymbol{e} \vDash \chi$, then, by the induction hypothesis, $\|F_\psi\|^{\mathrm{G}}_{\boldsymbol{M},\boldsymbol{e}} = \|\bar{0}\|_{\boldsymbol{M}}$ and $\|F_\chi\|^{\mathrm{G}}_{\boldsymbol{M},\boldsymbol{e}} = \|\bar{1}\|_{\boldsymbol{M}}$.

In either case we get $\|F_\varphi\|^{\mathrm{G}}_{\boldsymbol{M},\boldsymbol{e}} = \|F_\psi \wedge F_\chi\|^{\mathrm{G}}_{\boldsymbol{M},\boldsymbol{e}} = \|\bar{0}\|_{\boldsymbol{M}}$ by Lemma 4.3.2.

The two remaining cases can be handled analogously. If φ is a disjunction $\psi \wedge \chi$, then we define F_φ as $F_\psi \wedge F_\chi$ and if φ is an implication $\psi \supset \chi$, then we define F_φ as $F_\psi \supset F_\chi$.

□

Lemma 4.4.2. *Let* $\Box$ *be a connective of a logic* S *that is specified over* LOE *with a truth function defined by a specification of the form*

$$\Box(x_1, \dots, x_n) = \begin{cases} t_1[x_1, \dots, x_n] & \text{if } A_1[x_1, \dots, x_n] \\ \vdots & \vdots \\ t_m[x_1, \dots, x_n] & \text{if } A_m[x_1, \dots, x_n] \,. \end{cases}$$

Then we can construct a formula $F_\Box[x_1, \dots, x_n]$ *of Gödel logic such that*

$$\|\Box(x_1, \dots, x_n)\|^{\mathrm{S}}_{\boldsymbol{M},\boldsymbol{e}} = \|F_\Box[x_1, \dots, x_n]\|^{\mathrm{G}}_{\boldsymbol{M},\boldsymbol{e}}$$

for every model $\boldsymbol{M}$ *of* LOE *and every* $\boldsymbol{M}$*-evaluation* $\boldsymbol{e}$*.*

Proof. We define the formula $F_\Box[x_1, \dots, x_n]$ as

$$(F_{A_1} \wedge t_1[x_1, \dots, x_n]) \vee \dots \vee (F_{A_m} \wedge t_m[x_1, \dots, x_n])\,.$$

where each F_{A_j} $(1 \leq j \leq m)$ is determined by Lemma 4.4.1.

Note that by Lemma 4.4.1 the following holds for every $1 \leq j \leq m$:

- If $\boldsymbol{M}, \boldsymbol{e} \vDash A_j$, then $\|F_{A_j}\|^{\mathrm{G}}_{\boldsymbol{M},\boldsymbol{e}} = \|\bar{1}\|_{\boldsymbol{M}}$ and therefore $\|F_{A_j} \wedge t_j\|^{\mathrm{G}}_{\boldsymbol{M},\boldsymbol{e}} = \|t_j\|_{\boldsymbol{M},\boldsymbol{e}}$.
- If $\boldsymbol{M}, \boldsymbol{e} \nvDash A_j$, then $\|F_{A_j}\|^{\mathrm{G}}_{\boldsymbol{M},\boldsymbol{e}} = \|\bar{0}\|_{\boldsymbol{M}}$ and therefore $\|F_{A_j} \wedge t_j\|^{\mathrm{G}}_{\boldsymbol{M},\boldsymbol{e}} = \|\bar{0}\|_{\boldsymbol{M}}$.

Now assume that $\|\Box(x_1, \dots, x_n)\|^{\mathrm{S}}_{\boldsymbol{M},\boldsymbol{e}} = \|t_{j_1}\|_{\boldsymbol{M},\boldsymbol{e}}$ for some $1 \leq j_1 \leq m$. Then $\boldsymbol{M}, \boldsymbol{e} \vDash A_{j_1}$. Let $1 \leq j_2 \leq m$ such that $\|F_{A_{j_2}}\|^{\mathrm{G}}_{\boldsymbol{M},\boldsymbol{e}} \neq \bar{0}$. Then, as seen above, it must be the case that $\boldsymbol{M}, \boldsymbol{e} \vDash A_{j_2}$. Due to functionality we get that $\|t_{j_1}\|_{\boldsymbol{M},\boldsymbol{e}} = \|t_{j_2}\|_{\boldsymbol{M},\boldsymbol{e}}$.

Therefore, for every $1 \leq j \leq m$, we have $\|F_{A_j} \wedge t_j\|^{\mathrm{G}}_{\boldsymbol{M},\boldsymbol{e}} = \|t_{j_1}\|_{\boldsymbol{M},\boldsymbol{e}}$ or $\|F_{A_j} \wedge t_j\|^{\mathrm{G}}_{\boldsymbol{M},\boldsymbol{e}} = \|\bar{0}\|_{\boldsymbol{M}}$. Thus, by the definition of Gödel disjunction, $\|F_\Box\|^{\mathrm{G}}_{\boldsymbol{M},\boldsymbol{e}} = \|t_{j_1}\|_{\boldsymbol{M},\boldsymbol{e}} = \|\Box(x_1, \dots, x_n)\|^{\mathrm{S}}_{\boldsymbol{M},\boldsymbol{e}}$. □

Lemma 4.4.3. *For every formula* φ *of a logic* S *that is specified over the theory* LOE *we can construct a formula* F_φ *of Gödel logic such that* $\|\varphi\|^{\mathrm{S}}_{\boldsymbol{M},\boldsymbol{e}} = \|F_\varphi\|^{\mathrm{G}}_{\boldsymbol{M},\boldsymbol{e}}$ *for every model* $\boldsymbol{M}$ *of* LOE *and every* $\boldsymbol{M}$*-evaluation* $\boldsymbol{e}$*.*

Proof. The proof is by induction on the structure of φ.

- If $\varphi = \bar{0}$, $\varphi = \bar{1}$ or $\varphi = x$ for a variable x, we take $F_\varphi = \varphi$ and the identity clearly holds.

- If $\varphi = \Box_i(\psi_1, \dots, \psi_n)$ for one of the connectives $\Box_i$, then we know by the induction hypothesis that, for every $1 \le j \le n$, there is a formula F_{ψ_j} such that $\|\psi_j\|^{\mathrm{S}}_{\boldsymbol{M},\boldsymbol{e}} = \|F_{\psi_j}\|^{\mathrm{G}}_{\boldsymbol{M},\boldsymbol{e}}$. Now we take some fresh variables x_1, ..., x_n and define the evaluation $\boldsymbol{e}'$ by

$$\boldsymbol{e}'(v) = \begin{cases} \|\psi_j\|^{\mathrm{S}}_{\boldsymbol{M},\boldsymbol{e}} & \text{if } v = x_j \text{ for some } 1 \le j \le n \\ v & \text{otherwise}. \end{cases}$$

  Due to truth-functionality we get

$$\|\varphi\|^{\mathrm{S}}_{\boldsymbol{M},\boldsymbol{e}} = \|\Box_i(\psi_1, \dots, \psi_n)\|^{\mathrm{S}}_{\boldsymbol{M},\boldsymbol{e}} = \|\Box_i(x_1, \dots, x_n)\|^{\mathrm{S}}_{\boldsymbol{M},\boldsymbol{e}'} .$$

  Now we apply Lemma 4.4.2 and conclude

$$\|\Box_i(x_1, \dots, x_n)\|^{\mathrm{S}}_{\boldsymbol{M},\boldsymbol{e}'} = \left\|F_{\Box_i}[x_1, \dots, x_n]\right\|^{\mathrm{G}}_{\boldsymbol{M},\boldsymbol{e}'} = \left\|F_{\Box_i}[F_{\psi_1}, \dots, F_{\psi_n}]\right\|^{\mathrm{G}}_{\boldsymbol{M},\boldsymbol{e}} .$$

  Thus, $F_{\Box_i}[F_{\psi_1}, \dots, F_{\psi_n}]$ is the desired formula F_φ.

□

In the last step we also have to take into account that the status of validity of the transformed formula depends on the designating predicate of the specified logic.

Theorem 4.4.4. *For every formula φ of a logic* S *that is specified over the theory* LOE *we can construct a formula φ' such that for every model $\boldsymbol{M}$ of* LOE*, φ is valid in the specified logic* S *of $\boldsymbol{M}$ if and only if φ' is valid in the Gödel logic of $\boldsymbol{M}$.*

Proof. Let φ be a formula of the specified logic S. By $D_{\mathrm{S}}[x]$ we denote the designating predicate of S and by $D_{\mathrm{G}}[x]$ we denote the designating predicate of Gödel logic.

Now by Lemma 4.4.1 we know that there is a formula $F_{D_{\mathrm{S}}}[x]$ such that

$$\boldsymbol{M}, \boldsymbol{e} \vDash D_{\mathrm{S}}[x] \text{ if and only if } \left\|F_{D_{\mathrm{S}}}[x]\right\|^{\mathrm{G}}_{\boldsymbol{M},\boldsymbol{e}} = \|\bar{1}\|_{\boldsymbol{M}} . \tag{4.1}$$

for every model $\boldsymbol{M}$ of LOE and every $\boldsymbol{M}$-evaluation $\boldsymbol{e}$.

By Lemma 4.4.3 there is a formula F_φ such that $\|\varphi\|^{\mathrm{S}}_{\boldsymbol{M},\boldsymbol{e}} = \|F_\varphi\|^{\mathrm{G}}_{\boldsymbol{M},\boldsymbol{e}}$ for every model $\boldsymbol{M}$ of LOE and every $\boldsymbol{M}$-evaluation $\boldsymbol{e}$.

We denote by $F_{D_{\mathrm{S}}}[F_\varphi]$ the formula where each occurrence of the free variable x in the formula $F_{D_{\mathrm{S}}}[x]$ is replaced by F_φ. We want to take $F_{D_{\mathrm{S}}}[F_\varphi]$ as our φ'.

Let $\boldsymbol{e}$ be an arbitrary evaluation. Now we define the following evaluations:

$$\boldsymbol{e}'(v) = \begin{cases} \|\varphi\|^{\mathrm{S}}_{\boldsymbol{M},\boldsymbol{e}} & \text{if } v = x \\ \boldsymbol{e}(v) & \text{if } v \neq x \end{cases}$$

$$\boldsymbol{e}^*(v) = \begin{cases} \left\|F_{D_{\mathrm{S}}}[F_\varphi]\right\|^{\mathrm{G}}_{\boldsymbol{M},\boldsymbol{e}} & \text{if } v = x \\ \boldsymbol{e}(v) & \text{if } v \neq x \end{cases}$$

$$\boldsymbol{e}^\times(v) = \begin{cases} \left\|F_\varphi\right\|^{\mathrm{G}}_{\boldsymbol{M},\boldsymbol{e}} & \text{if } v = x \\ \boldsymbol{e}(v) & \text{if } v \neq x \end{cases}$$

Since $\|\varphi\|^{\mathsf{S}}_{\boldsymbol{M},\boldsymbol{e}} = \|F_\varphi\|^{\mathsf{G}}_{\boldsymbol{M},\boldsymbol{e}}$, the evaluations $\boldsymbol{e}'$ and $\boldsymbol{e}^\times$ are equal. Now we get the following chain of equivalences:

$$\boldsymbol{M}, \boldsymbol{e}^* \vDash D_{\mathsf{G}}[x]$$

is equivalent to

$$\left\|F_{D_{\mathsf{S}}}[F_\varphi]\right\|^{\mathsf{G}}_{\boldsymbol{M},\boldsymbol{e}} = \|\bar{1}\|_{\boldsymbol{M}}$$

because the designating predicate of Gödel logic is $x = \bar{1}$. The last statement is equivalent to

$$\left\|F_{D_{\mathsf{S}}}[x]\right\|^{\mathsf{G}}_{\boldsymbol{M},\boldsymbol{e}^\times} = \|\bar{1}\|_{\boldsymbol{M}}$$

because the connectives are necessarily truth-functional. Due to the equality of $\boldsymbol{e}'$ and $\boldsymbol{e}^\times$ this statement is equivalent to

$$\left\|F_{D_{\mathsf{S}}}[x]\right\|^{\mathsf{G}}_{\boldsymbol{M},\boldsymbol{e}'} = \|\bar{1}\|_{\boldsymbol{M}} .$$

Finally, we may apply (4.1) from above and get that the last statement is equivalent to

$$\boldsymbol{M}, \boldsymbol{e}' \vDash D_{\mathsf{S}}[x] .$$

Now since $\boldsymbol{e}$ was an arbitrary evaluation and $\boldsymbol{e}'$ and $\boldsymbol{e}^*$ were suitably defined we may conclude that φ is valid in the specified logic of $\boldsymbol{M}$ if and only if φ' is valid in the Gödel logic of $\boldsymbol{M}$.

□

This shows that every logic that is specified over a model of LOE can be embedded into the corresponding Gödel logic. Note that our proof is fully constructive which means that we can extract an explicit procedure that tells us how the formula φ' can be constructed.

4.5 Dense linear orders with endpoints

So far, the conditions of the specifications were required to be quantifier-free (compare Definition 4.1.3). The natural question to ask is whether this requirement is necessary for obtaining the preceding result.

We can easily show that this is indeed the case by introducing the concept of density. A linear order is dense if between two non-identical points we always find a third one that lies in between.

Definition 4.5.1. The theory DLOE of *dense linear orders with endpoints* is the deductive closure of the theory LOE together with the additional axiom of density

$$\text{(D)}\ \forall x \forall y (x < y \supset \exists z (x < z \wedge z < y)) .$$

Clearly, DLOE only has models with infinite cardinality. A natural example of a dense, linearly ordered set is the real unit interval $[0, 1]$ with the usual smaller-than relation on the real numbers. We call this model $\boldsymbol{M}^{[0,1]}$. An example of a model of LOE in which the density axiom does not hold is the set $\mathbb{N} \cup \{\infty\}$ which is the set of natural numbers with the usual smaller-than relation together with an artificial maximal element ∞. We call this model $\boldsymbol{M}^{(\infty)}$.

We can now specify a connective that simply checks whether the density property holds for two elements.

$$\boldsymbol{D}(x, y) = \begin{cases} \bar{1} & \text{if } x < y \supset \exists z(x < z \wedge z < x) \\ \bar{0} & \text{otherwise}. \end{cases}$$

Consider now the specified logic D that only consists of the connective $\boldsymbol{D}$ and has the designating predicate $x = \bar{1}$. Although this logic is rather meaningless it serves as a simple counterexample that our theorem can fail for specifications allowing quantifiers. It is clear that the formula $\boldsymbol{D}(x, y)$ is valid in the specified logic D of a model $\boldsymbol{M}$ of LOE if and only if $\boldsymbol{M}$ fulfills the density axiom.

Suppose, for the sake of contradiction, that Theorem 4.4.4 still holds. The formula $\boldsymbol{D}(x, y)$ is valid in the logic D of the LOE-model $\boldsymbol{M}^{[0,1]}$ because the set $[0, 1]$ is dense. By our theorem we therefore know that there is a formula φ' such that φ' is valid in the Gödel logic of $\boldsymbol{M}^{[0,1]}$. Since all infinite-valued Gödel logics coincide, we can apply Theorem 4.3.7 and get that φ' is valid in the Gödel logic of $\boldsymbol{M}^{(\infty)}$. Now we go back again with Theorem 4.4.4 and conclude that the formula $\boldsymbol{D}(x, y)$ is valid in the logic D of the model $\boldsymbol{M}^{(\infty)}$. But this cannot be the case because the set $\mathbb{N} \cup \{\infty\}$ is not dense. Therefore the connective $\boldsymbol{D}$ cannot be defined as an abbreviating formula with connectives of Gödel logic.

However, there might still be special cases in which the full generality of our result is possible. One important case in fact is the theory DLOE. It is well-known that DLOE admits quantifier elimination (see for example the classic book by Kreisel and Krivine [76])

Theorem 4.5.2. *For every formula φ of* DLOE *there is an equivalent quantifier-free formula φ' such that* DLOE $\models \varphi$ *if and only if* DLOE $\models \varphi'$.

Note that the usual proof gives an explicit procedure that puts a formula in its quantifier-free form. Therefore every specification over DLOE can be transformed into a quantifier-free specification and thus our result can still be applied. Since DLOE only has models with infinite cardinality, every specified logic over DLOE is subsumed by infinite-valued Gödel logic.

CHAPTER 5

A hybrid logic

In this chapter, we consider, for every continuous t-norm $*$, a logic that we call S$*$. We follow the definitions of [38] where the logic SŁ, obtained from the Łukasiewicz t-norm, was discussed. We restrict ourselves to the propositional case.

The main idea in this approach is to measure the amount of truth of propositional variables in a precisification space which can be attributed to Kamp [71]. This measure determines the truth values of propositional variables. Complex formulas are interpreted according to the truth functions given by a continuous t-norm. Furthermore, a supertruth operator is added that expresses truth in all precisifications. This approach is particularly interesting because it addresses the question how the truth values of atomic statements should be determined.

Hájek's "probably" modality [56] has some similarities to our system (compare Section 3.6.3). The main differences are that Hájek does not include something like the supertruth modality and does not only measure propositional variables but arbitrary formulas. Furthermore, the questions studied in the following are of a different nature than the ones answered by Hájek.

There is also another way of "plugging together" fuzzy logic and supervaluationism: An approach complementary to ours is Hájek's generalization of Shapiro's machinery [93] to interval-based fuzzy logics [58]. In that framework, the interpretation of a formula at a precisification is not classical, but based on a t-norm, and every propositional variable receives a set of possible truth values $[a, b]$. Precisifying then means to reduce the set of truth values to a subinterval $[c, d] \subseteq [a, b]$. In a completely sharp precisification all truth value intervals of propositional variables collapse to single truth values.

5.1 Definitions and basic properties

As explained above, the basic idea of our hybrid logic is to measure the "amount of truth" in a precisification space. For this purpose we have to make precise what we mean by measuring. We define the measure in a way that allows us to obtain truth values for propositional variables from it. The concept that is needed here is that of a probability measure. As a simplification, we restrict ourselves to precisification spaces with only countably many precisifications.

Definition 5.1.1. A *probability measure* on a countable set S is a function μ from S to the unit interval $[0,1]$ such that $\sum_{s\in S}\mu(s)=1$. To simplify notation we extend μ to subsets of S as follows: $\mu(T)=\sum_{s\in T}\mu(s)$ for every $T\subseteq S$.

There are some basic properties of probability measures that we will need quite often.

Proposition 5.1.2. *Let μ be a probability measure on a set S. Then the following holds:*

- *If $T\subseteq S$ and $U\subseteq S$ are two disjoint subsets of S, i.e., $T\cap U=\emptyset$, then $\mu(T\cup U)=\mu(T)+\mu(U)$.*
- *If $T\subseteq U\subseteq S$, then $\mu(T)\leq\mu(U)$.*
- *If $T\subseteq U\subseteq S$, then $\mu(U\setminus T)=\mu(U)-\mu(T)$.*

Proof. The first property follows directly from the definition. Assume that $T\cap U=\emptyset$. Then we have

$$\mu(T\cup U)=\sum_{s\in T\cup U}\mu(s)=\sum_{s\in T}\mu(s)+\sum_{s\in U}\mu(s)=\mu(T)+\mu(U).$$

If $T\subseteq U$, we have

$$\mu(U)=\sum_{s\in U}\mu(s)=\sum_{s\in T}\mu(s)+\sum_{s\in U\setminus T}\mu(s)=\mu(T)+\sum_{s\in U\setminus T}\mu(s).$$

Since $\mu(s)\geq 0$ for every $s\in S$ we get $\sum_{s\in U\setminus T}\mu(s)\geq 0$ and therefore $\mu(U)\geq\mu(T)$. Furthermore we get

$$\mu(U\setminus T)=\sum_{s\in U\setminus T}\mu(s)=\mu(U)-\mu(T).$$

□

We now consider precisification spaces that are equipped with a probability measure on the set of precisifications and give appropriate definitions for the truth values of formulas in such a structure. We will only work in the simplest possible setting: we restrict ourselves to propositional variables and do not impose an admissibility relation on the precisifications. As will be seen in the following, already in this simple setting many interesting questions can be asked.

Definition 5.1.3. A *precisification space* $\boldsymbol{S}$ is a triple $\boldsymbol{S} = \langle \boldsymbol{P}, (\boldsymbol{M}_s)_{s\in\boldsymbol{P}}, \mu\rangle$ that consists of a nonempty, countable set $\boldsymbol{P}$ of *precisifications*, a function $(\boldsymbol{M}_s)_{s\in\boldsymbol{P}}$ that assigns a classical propositional interpretation $\boldsymbol{M}_s$ to every precisification $s \in \boldsymbol{P}$, and a probability measure μ on $\boldsymbol{P}$. As a simplification, we may write $s \in \boldsymbol{S}$ instead of $s \in \boldsymbol{P}$. Furthermore, we define the *interpretation of formulas* in a precisification space with an associated continuous t-norm $*$.

The *local* truth value $\|\varphi\|_{s,\boldsymbol{S}}$ of a formula φ at a precisification $s \in \boldsymbol{S}$ in a precisification space $\boldsymbol{S}$ is inductively defined by:

$$\begin{aligned}
\|\bar{0}\|_{s,\boldsymbol{S}} &= 0 \\
\|p\|_{s,\boldsymbol{S}} &= \begin{cases} 1 & \text{if } \|p\|_{\boldsymbol{M}_s} = 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{for atomic } p \\
\|\varphi \,\&\, \psi\|_{s,\boldsymbol{S}} &= \begin{cases} 1 & \text{if } \|\varphi\|_{s,\boldsymbol{S}} = 1 \text{ and } \|\psi\|_{s,\boldsymbol{S}} = 1 \\ 0 & \text{otherwise} \end{cases} \\
\|\varphi \supset \psi\|_{s,\boldsymbol{S}} &= \begin{cases} 0 & \text{if } \|\varphi\|_{s,\boldsymbol{S}} = 1 \text{ and } \|\psi\|_{s,\boldsymbol{S}} = 0 \\ 1 & \text{otherwise} \end{cases} \\
\|\mathrm{S}\varphi\|_{s,\boldsymbol{S}} &= \begin{cases} 1 & \text{if } \|\varphi\|_{t,\boldsymbol{S}} = 1 \text{ for every } t \in \boldsymbol{S} \\ 0 & \text{otherwise}\,. \end{cases}
\end{aligned}$$

The *global* truth value $\|\varphi\|^*_{\boldsymbol{S}}$ of a formula φ for a continuous t-norm $*$ and its residuum $\Rightarrow_*$ is inductively defined as follows:

$$\begin{aligned}
\|\bar{0}\|^*_{\boldsymbol{S}} &= 0 \\
\|p\|^*_{\boldsymbol{S}} &= \mu\left(\left\{s \in \boldsymbol{S} \mid \|p\|_{s,\boldsymbol{S}} = 1\right\}\right) \text{ for atomic } p \\
\|\varphi \,\&\, \psi\|^*_{\boldsymbol{S}} &= \|\varphi\|^*_{\boldsymbol{S}} * \|\psi\|^*_{\boldsymbol{S}} \\
\|\varphi \supset \psi\|^*_{\boldsymbol{S}} &= \|\varphi\|^*_{\boldsymbol{S}} \Rightarrow_* \|\psi\|^*_{\boldsymbol{S}} \\
\|\mathrm{S}\varphi\|^*_{\boldsymbol{S}} &= \begin{cases} 1 & \text{if } \|\varphi\|_{s,\boldsymbol{S}} = 1 \text{ for every } s \in \boldsymbol{S} \\ 0 & \text{otherwise}\,. \end{cases}
\end{aligned}$$

We sometimes omit the superscript in $\|\cdot\|$ if the result does not depend on the t-norm. If a conjunction is in the scope of an S-operator, we usually write $\varphi \wedge \psi$ instead of $\varphi \,\&\, \psi$ for convenience. If a conjunction is not in the scope of an S-operator, then $\varphi \,\&\, \psi$ refers to the strong conjunction connective of fuzzy logic and $\varphi \wedge \psi$ refers to the weak conjunction connective of fuzzy logic and is an abbreviation for $\varphi \,\&\, (\varphi \supset \psi)$ (see Sections 3.2.1 and 3.4).

Using precisification spaces as interpretation structures of formulas, we obtain, for every continuous t-norm $*$, a logic that we call S$*$. The notions of truth and validity in this logic are defined in the standard way.

Definition 5.1.4. Let $*$ be a continuous t-norm. A formula φ is *true* for $*$ in a precisification space $\boldsymbol{S}$ iff $\|\varphi\|^*_{\boldsymbol{S}} = 1$. A formula φ is *valid* in S$*$ iff φ is true for $*$ in every precisification space $\boldsymbol{S}$.

Furthermore, we introduce some simplifying notation.

Definition 5.1.5. Let $\boldsymbol{S}$ be a precisification space with probability measure μ. For a formula φ, the *extension* of φ is

$$[\varphi]_{\boldsymbol{S}} = \left\{ s \in \boldsymbol{S} \mid \|\varphi\|_{s,\boldsymbol{S}} = 1 \right\}$$

and the *measure* of φ is

$$[\![\varphi]\!]_{\boldsymbol{S}} = \mu\left([\varphi]_{\boldsymbol{S}}\right) .$$

Note that by definition $\|p\|_{\boldsymbol{S}} = [\![p]\!]_{\boldsymbol{S}}$ for every propositional variable p in a precisification space $\boldsymbol{S}$.

Remark. For every formula φ and every precisification space $\boldsymbol{S}$ with a set of precisifications $\boldsymbol{P}$, the following statements are equivalent:

(i) $\|\mathrm{S}\varphi\|_{\boldsymbol{S}} = 1$

(ii) $\|\mathrm{S}\varphi\|_{s,\boldsymbol{S}} = 1$ for some $s \in \boldsymbol{P}$

(iii) $\|\mathrm{S}\varphi\|_{s,\boldsymbol{S}} = 1$ for all $s \in \boldsymbol{P}$

(iv) $[\varphi]_{\boldsymbol{S}} = \boldsymbol{P}$

(v) $[\mathrm{S}\varphi]_{\boldsymbol{S}} = \boldsymbol{P}$

Example. Consider a precisification space $\boldsymbol{S}$ with probability measure μ consisting of two precisifications s_1 and s_2. The local truth values for a propositional variable p are given by $\|p\|_{s_1,\boldsymbol{S}} = 1$ and $\|p\|_{s_2,\boldsymbol{S}} = 0$ and μ is defined by $\mu(s_1) = 1$ and $\mu(s_2) = 0$. Then we get $\|p\|_{\boldsymbol{S}} = 1$ and $\|\mathrm{S}p\|_{\boldsymbol{S}} = 0$. Thus, in that precisification space supertruth and 1-truth come apart for propositional variables.

One possibility to overcome the issue of this example is to forbid precisifications with measure 0, which leads to the concept of positive precisification spaces.

Definition 5.1.6. A precisification space $\boldsymbol{S}$ with probability measure μ is *positive* iff $\mu(s) > 0$ for every $s \in \boldsymbol{S}$. In such a case, μ is called a *positive* probability measure.

One could argue that it makes no sense to give the measure 0 to any precisification because in this case the precisification should not be included in the precisification space anyway. Following this argument, all precisification spaces should be positive. In positive precisification spaces the notions of truth and falsehood in terms of truth values and in terms of supertruth and superfalsehood coincide for propositional variables.

Proposition 5.1.7. *For every positive precisification $\boldsymbol{S}$ and every p a propositional variable the following holds:*

- $\|p\|_{\boldsymbol{S}} = 1$ *if and only if* $\|\mathrm{S}p\|_{\boldsymbol{S}} = 1$
- $\|p\|_{\boldsymbol{S}} = 0$ *if and only if* $\|\mathrm{S}\neg p\|_{\boldsymbol{S}} = 1$

Proof. In both cases, the direction from right to left holds for all precisification spaces and directly follows from the definitions.

Assume that $\|p\|_{\boldsymbol{S}} = 1$ and suppose that there is an $s \in \boldsymbol{S}$ such that $\|p\|_{s,\boldsymbol{S}} = 0$. Then the sets $[p]_{\boldsymbol{S}}$ and $\{s\}$ are disjoint and by Proposition 5.1.2 we have

$$1 \geq \mu\left([p]_{\boldsymbol{S}} \cup \{s\}\right) = \mu\left([p]_{\boldsymbol{S}}\right) + \mu(s) = \|p\|_{\boldsymbol{S}} + \mu(s) = 1 + \mu(s) > 1 + 0 = 1$$

which is a contradictory statement. Therefore $\|p\|_{s,\boldsymbol{S}} = 1$ for all $s \in \boldsymbol{S}$ which means that $\|\mathrm{S}p\|_{\boldsymbol{S}} = 1$.

Assume that $\|p\|_{\boldsymbol{S}} = 0$ and suppose that there is an $s \in \boldsymbol{S}$ such that $\|\neg p\|_{s,\boldsymbol{S}} = 0$. Then $\|p\|_{s,\boldsymbol{S}} = 1$ and therefore $\{s\} \subseteq [p]_{\boldsymbol{S}}$. By Proposition 5.1.2 we then get

$$0 = \|p\|_{\boldsymbol{S}} = \mu\left([p]_{\boldsymbol{S}}\right) \geq \mu(\{s\}) = \mu(s) > 0$$

which is a contradictory statement. Therefore $\|\neg p\|_{s,\boldsymbol{S}} = 0$ for all $s \in \boldsymbol{S}$ which means that $\|\mathrm{S}\neg p\|_{\boldsymbol{S}} = 1$. □

Another basic question that we answer now is to give a simple condition under which precisification spaces agree on the truth values of formulas.

Proposition 5.1.8. *Let $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ be two precisification spaces, $*$ a continuous t-norm, and $\mathcal{P}$ a set of propositional variables. Then $\|\varphi\|^*_{\boldsymbol{S}_1} = \|\varphi\|^*_{\boldsymbol{S}_2}$ for every formula φ containing only propositional variables of $\mathcal{P}$ if and only if the following two conditions hold:*

(i) $\|p\|_{\boldsymbol{S}_1} = \|p\|_{\boldsymbol{S}_2}$ *for every propositional variable $p \in \mathcal{P}$.*

(ii) $\|\mathrm{S}\psi\|_{\boldsymbol{S}_1} = \|\mathrm{S}\psi\|_{\boldsymbol{S}_2}$ *for every formula φ.*

Proof. The direction from left to right is clear, we simply set $\varphi = p$ or $\varphi = \mathrm{S}\psi$ respectively. We prove the direction from right to left by induction on the structure of φ.

- $\varphi = \bar{0}$: $\|\bar{0}\|_{\boldsymbol{S}_1} = 0 = \|\bar{0}\|_{\boldsymbol{S}_2}$.
- $\varphi = p \in \mathcal{P}$: Then by condition (i) we have $\|p\|_{\boldsymbol{S}_1} = \|p\|_{\boldsymbol{S}_2}$.
- $\varphi = \mathrm{S}\psi$: Then by condition (ii) we have $\|\mathrm{S}\psi\|_{\boldsymbol{S}_1} = \|\mathrm{S}\psi\|_{\boldsymbol{S}_2}$.
- $\varphi = \psi \,\&\, \chi$: By the induction hypothesis we have $\|\psi\|^*_{\boldsymbol{S}_1} = \|\psi\|^*_{\boldsymbol{S}_2}$ and $\|\chi\|^*_{\boldsymbol{S}_1} = \|\chi\|^*_{\boldsymbol{S}_2}$. Therefore we get

$$\|\psi \,\&\, \chi\|^*_{\boldsymbol{S}_1} = \|\psi\|^*_{\boldsymbol{S}_1} * \|\chi\|^*_{\boldsymbol{S}_1} = \|\psi\|^*_{\boldsymbol{S}_2} * \|\chi\|^*_{\boldsymbol{S}_2} = \|\psi \,\&\, \chi\|^*_{\boldsymbol{S}_2}.$$

- $\varphi = \psi \supset \chi$: By the induction hypothesis we have $\|\psi\|^*_{S_1} = \|\psi\|^*_{S_2}$ and $\|\chi\|^*_{S_1} = \|\chi\|^*_{S_2}$. Therefore we get

$$\|\psi \supset \chi\|^*_{S_1} = \left(\|\psi\|^*_{S_1} \Rightarrow_* \|\chi\|^*_{S_1}\right) = \left(\|\psi\|^*_{S_2} \Rightarrow_* \|\chi\|^*_{S_2}\right) = \|\psi \supset \chi\|^*_{S_2}.$$

□

We can make the second condition in the previous proposition more precise.

Proposition 5.1.9. *Let S_1 and S_2 be two precisification spaces and $\mathcal{P}$ a set of propositional variables. Then $\|S\psi\|_{S_1} = \|S\psi\|_{S_2}$ for every formula φ containing only propositional variables in $\mathcal{P}$ if the following two conditions hold:*

(i) For every precisification $s_1 \in S_1$ there is a precisification $s_2 \in S_2$ such that $\|p\|_{s_1,S_1} = \|p\|_{s_2,S_2}$ for every propositional variable $p \in \mathcal{P}$.

(ii) For every precisification $s_2 \in S_2$ there is a precisification $s_1 \in S_1$ such that $\|p\|_{s_2,S_2} = \|p\|_{s_1,S_1}$ for every propositional variable $p \in \mathcal{P}$.

Proof. Assume that conditions (i) and (ii) hold. We first prove the following claim:

(i') For every precisification $s_1 \in S_1$ there is a precisification $s_2 \in S_2$ such that $\|\varphi\|_{s_1,S_1} = \|\varphi\|_{s_2,S_2}$ for every formula φ.

(ii') For every precisification $s_2 \in S_2$ there is a precisification $s_1 \in S_1$ such that $\|\varphi\|_{s_2,S_2} = \|\varphi\|_{s_1,S_1}$ for every formula φ.

The proof is by induction on the structure of φ. Let φ be a formula. We may assume that (i') and (ii') hold for all formulas that are "smaller" than φ. We show that condition (i') also holds for φ, the proof of condition (ii') uses the same arguments.

Let $s_1 \in S_1$. By condition (i) there is an $s_2 \in S$ such that $\|p\|_{s_1,S_1} = \|p\|_{s_2,S_2}$ for every propositional variable $p \in \mathcal{P}$. We have to show that $\|\varphi\|_{s_1,S_1} = \|\varphi\|_{s_2,S_2}$ for every formula φ.

- $\varphi = \bar{0}$: Then $\|\bar{0}\|_{s_1,S_1} = 0 = \|\bar{0}\|_{s_2,S_2}$.
- $\varphi = p \in \mathcal{P}$: There is nothing to show because $\|p\|_{s_1,S_1} = \|p\|_{s_2,S_2}$.
- $\varphi = S\psi$: We show that $\|S\psi\|_{s_1,S_1} = 1$ if and only if $\|S\psi\|_{s_2,S_2} = 1$. Since $\|S\psi\|_{s_1,S_1} \in \{0,1\}$ and $\|S\psi\|_{s_2,S_2} \in \{0,1\}$ this implies that $\|S\psi\|_{s_1,S_1} = \|S\psi\|_{s_2,S_2}$.

  Assume that $\|S\psi\|_{s_1,S_1} = 1$ and suppose that $\|S\psi\|_{s_2,S_2} = 0$. Then there is some $t_2 \in S_2$ such that $\|\psi\|_{t_2,S_2} = 0$. By the induction hypothesis, (ii') holds and there is some $t_1 \in S_1$ such that $\|\psi\|_{t_1,S_1}, \|\psi\|_{t_2,S_2} = 0$. Then $\|S\psi\|_{s_1,S_1} = 0$ which contradicts our assumption. Therefore $\|S\psi\|_{s_2,S_2} = 1$. The direction from right to left can be shown using the same argument.

- $\varphi = \psi \wedge \chi$: By the induction hypothesis we know that $\|\psi\|_{s_1,\boldsymbol{S}_1} = \|\psi\|_{s_2,\boldsymbol{S}_2}$ and $\|\psi\|_{s_1,\boldsymbol{S}_1} = \|\psi\|_{s_2,\boldsymbol{S}_2}$. Therefore also $\|\psi \wedge \chi\|_{s_1,\boldsymbol{S}_1} = \|\psi \wedge \chi\|_{s_1,\boldsymbol{S}_2}$.

For the remaining connectives the same argument as for conjunction applies.

We now show that $\|\mathrm{S}\psi\|_{\boldsymbol{S}_1} = 1$ if and only if $\|\mathrm{S}\psi\|_{\boldsymbol{S}_2} = 1$ which implies $\|\mathrm{S}\psi\|_{\boldsymbol{S}_1} = \|\mathrm{S}\psi\|_{\boldsymbol{S}_2} = 1$. Assume that $\|\mathrm{S}\psi\|_{\boldsymbol{S}_1}$. Then there is some $s_1 \in \boldsymbol{S}_1$ such that $\|\mathrm{S}\psi\|_{s_1,\boldsymbol{S}_1} = 1$. By the claim we just proved there is an $s_2 \in \boldsymbol{S}_2$ such that $\|\mathrm{S}\psi\|_{s_2,\boldsymbol{S}_2} = 1$. Then we know that $\|\mathrm{S}\psi\|_{\boldsymbol{S}_2} = 1$. The other direction can be shown using the same argument. □

We also have a certain finite model property, observed in [38] where it is formulated for the logic SŁ.

Proposition 5.1.10. *Let $\boldsymbol{S}$ be a (positive) precisification space and φ a formula that contains n different propositional variables. Then there is a (positive) precisification space $\boldsymbol{S}'$ with 2^n precisifications such that $\|\varphi\|_{\boldsymbol{S}} = \|\varphi\|_{\boldsymbol{S}'}$.*

Proof. We prove the proposition in two steps. First, we show that we can reduce the number of precisifications to some $m \leq 2^n$. In the next step, we show that in the case $m < 2^n$ we can also get an equivalent precisification space with 2^n precisifications. By $\mathcal{P}$ we denote the set of propositional variables contained in φ.

Let $\boldsymbol{S}$ be a precisification space with a set of precisifications $\boldsymbol{P}$ and a probability measure μ. For every subset $S \subseteq \mathcal{P}$ we define the formula φ_S by

$$\varphi_S = \bigwedge_{p \in S} p \wedge \bigwedge_{p \in \mathcal{P} \setminus S} \neg p\,.$$

We define a precisification space $\boldsymbol{S}'$ that has certain subsets of $\mathcal{P}$ as elements of its set of precisifications $\boldsymbol{P}'$ and a probability measure μ' by

$$\begin{aligned} \boldsymbol{P}' &= \left\{ S \subseteq \mathcal{P} \mid [\varphi_S]_{\boldsymbol{S}} \neq \varnothing \right\} \\ \mu(S) &= \mu\left([\varphi_S]_{\boldsymbol{S}}\right) \\ \|p\|_{S,\boldsymbol{S}'} &= \begin{cases} 1 & \text{if } p \in S \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

for all $S \in \boldsymbol{P}'$ and $p \in \mathcal{P}$.

First we show that $\boldsymbol{S}$ is well-defined. Due to $\|p\|_{s,\boldsymbol{S}} \in \{0,1\}$ for every $p \in \mathcal{P}$ and every $s \in \boldsymbol{P}$ it is clear that for every precisification $s \in \boldsymbol{P}$ there is exactly one $S \in \boldsymbol{P}'$ such that $\|\varphi_S\|_{s,\boldsymbol{S}} = 1$. Therefore we have a disjoint partition $\boldsymbol{P} = \bigcup_{S \in \boldsymbol{P}'} [\varphi_S]_{\boldsymbol{S}}$ and we get

$$1 = \mu(\boldsymbol{P}) = \mu\left(\bigcup_{S \subseteq \boldsymbol{P}'} [\varphi_S]_{\boldsymbol{S}}\right) = \sum_{S \in \boldsymbol{P}'} \mu\left([\varphi_S]_{\boldsymbol{S}}\right) = \sum_{S \in \boldsymbol{P}'} \mu'(S) = \mu'(\boldsymbol{P}')\,.$$

Now we show that $\|\varphi\|_{\boldsymbol{S}} = \|\varphi\|_{\boldsymbol{S}'}$ for every formula φ. Note that $\|p\|_{s,\boldsymbol{S}} = 1$ if and only if there is some $S \in \boldsymbol{P}'$ such that $\|\varphi_S\|_{s,\boldsymbol{S}} = 1$ and $p \in S$. Therefore we have a disjoint partition $[p]_{\boldsymbol{S}} = \bigcup_{S\in \boldsymbol{P}', p\in S} [\varphi_S]_{\boldsymbol{S}}$ and we get

$$\begin{aligned}\|p\|_{\boldsymbol{S}} = \mu\left([p]_{\boldsymbol{S}}\right) = \mu\left(\bigcup_{\substack{S\in \boldsymbol{P}'\\ p\in S}} [\varphi_S]_{\boldsymbol{S}}\right) &= \sum_{\substack{S\in \boldsymbol{P}'\\ p\in S}} \mu\left([\varphi_S]_{\boldsymbol{S}}\right)\\ &= \sum_{\substack{S\in \boldsymbol{P}'\\ p\in S}} \mu'(S) = \mu\left([p]_{\boldsymbol{S}'}\right) = \|p\|_{\boldsymbol{S}'}\end{aligned}$$

because $\|p\|_{S,\boldsymbol{S}} = 1$ if and only if $p \in S$.

For every $s \in \boldsymbol{S}$, there is some $s' \in \boldsymbol{S}'$ such that $\|p\|_{s,\boldsymbol{S}} = \|p\|_{s',\boldsymbol{S}'}$ for every $p \in \mathcal{P}$: we know that there is some $S \subseteq \boldsymbol{P}'$ such that $\|\varphi_S\|_{s,\boldsymbol{S}} = 1$ and therefore pick $s' = S$. Also the converse holds: for every $S \in \boldsymbol{S}'$, there is some $s \in \boldsymbol{S}$ such that $\|p\|_{S,\boldsymbol{S}'} = \|p\|_{s,\boldsymbol{S}}$ for every $p \in \mathcal{P}$. Since $S \subseteq \boldsymbol{P}'$, there is some $s \in \boldsymbol{S}$ such that $\|\varphi_S\|_{s,\boldsymbol{S}} = 1$.

By Proposition 5.1.8 and Proposition 5.1.9 we then know that $\|\varphi\|_{\boldsymbol{S}} = \|\varphi\|_{\boldsymbol{S}'}$ for every formula φ. Since the set $\mathcal{P}$ has $2^{|\mathcal{P}|} = 2^n$ subsets, the precisification space $\boldsymbol{S}'$ has at most 2^n precisifications. If $\boldsymbol{S}$ was a positive precisification space, then, by the definitions of μ' and $\boldsymbol{P}'$, $\boldsymbol{S}'$ clearly is a positive precisification space, too.

We now prove the second part. Assume that $\boldsymbol{S}$ is a precisification space with a set $\boldsymbol{P}$ of $m < 2^n$ precisifications and a probability measure μ. Let $t \in \boldsymbol{P} \neq \emptyset$ be one of the precisifications of $\boldsymbol{S}$. Then we consider a set S of $k = 2^n - m$ new precisifications. We define the precisification space $\boldsymbol{S}'$ with a set of precisifications $\boldsymbol{P}'$ and a probability measure μ by setting $\boldsymbol{P}' = \boldsymbol{P} \setminus \{t\} \cup S$ and

$$\mu'(s) = \begin{cases} \mu(s) & \text{if } s \in \boldsymbol{P} \setminus \{t\} \\ \frac{\mu(T)}{|S|} & \text{otherwise} \end{cases}$$

for every $s \in \boldsymbol{P}$.

Then we have $\mu(S) = \sum_{s\in S} \mu(T)/|S| = |S| \cdot \mu(T)/|S| = \mu(T)$. We now show that μ is well-defined:

$$\sum_{s\in \boldsymbol{P}'} \mu(s) = \sum_{s\in \boldsymbol{P}\setminus\{t\}} \mu(s) + \sum_{s\in S} \mu(s) = (1 - \mu(T)) + \mu(T) = 1$$

Now let p be a propositional variable. If $t \notin [p]_{\boldsymbol{S}}$, then $[p]_{\boldsymbol{S}'} = [p]_{\boldsymbol{S}} \subseteq \boldsymbol{P} \setminus \{t\}$ and therefore $\|p\|_{\boldsymbol{S}'} = \mu'([p]_{\boldsymbol{S}'}) = \mu([p]_{\boldsymbol{S}'}) = \mu([p]_{\boldsymbol{S}}) = \|p\|_{\boldsymbol{S}}$. If $t \in [p]_{\boldsymbol{S}}$, then $[p]_{\boldsymbol{S}'} = [p]_{\boldsymbol{S}} \setminus \{t\} \cup S$. Since $[p]_{\boldsymbol{S}} \setminus \{t\}$ and S are disjoint we get

$$\begin{aligned}\|p\|_{\boldsymbol{S}'} = \mu'\left([p]_{\boldsymbol{S}'}\right) &= \mu'\left([p]_{\boldsymbol{S}} \setminus \{t\}\right) + \mu'(S)\\ &= \mu\left([p]_{\boldsymbol{S}} \setminus \{t\}\right) + \mu(T)\\ &= \mu\left([p]_{\boldsymbol{S}}\right) - \mu(T) + \mu(T)\\ &= \mu\left([p]_{\boldsymbol{S}}\right) = \|p\|_{\boldsymbol{S}}\,.\end{aligned}$$

For every $s \in \boldsymbol{S}$, there is some $s' \in \boldsymbol{S}'$ such that $\|p\|_{s,\boldsymbol{S}} = \|p\|_{s',\boldsymbol{S}'}$ for every propositional variable p: if $s \neq t$, then pick $s' = s$ and if $s = t$, then pick any $s' \in S$. Also the converse holds: for every $s' \in \boldsymbol{S}'$, there is some $s \in \boldsymbol{S}$ such that $\|p\|_{s',\boldsymbol{S}} = \|p\|_{s,\boldsymbol{S}'}$ for every propositional variable p. If $s \notin S$, then pick $s' = s$ and if $s \in S$, then pick $s = t$.

By Proposition 5.1.8 and Proposition 5.1.9 we then know that $\|\varphi\|_{\boldsymbol{S}} = \|\varphi\|_{\boldsymbol{S}'}$ for every formula φ. If $\boldsymbol{S}$ was a positive precisification space, then, by the definition of μ', $\boldsymbol{S}'$ clearly is a positive precisification space, too. □

Corollary 5.1.11. *A formula φ is valid in* S∗ *if and only if $\|\varphi\|^*_{\boldsymbol{S}} = 1$ for every precisification spaces $\boldsymbol{S}$ with a finite set of precisifications.*

Note however that a finite precisification space is not necessarily a “finitary object” because its probability measure allows all the real numbers in the interval [0, 1].

As already discussed in Section 2.2, precisification spaces can also be seen as interpretation structures of the modal logic S5. This connection affects the notion of validity in S∗ in a certain way that has been pointed out by Fermüller and Kosik [38].

Proposition 5.1.12. *For every continuous t-norm $*$, a formula of the form* Sφ *is valid in* S∗*, if and only if $\varphi^\Box$ is valid in* S5 *where $\varphi^\Box$ denotes the following translation of φ into a formula of modal logic: every occurrence of the* S*-operator is replaced by the necessitation operator* □ *and every occurrence of the conjunction connective* & *is replaced by* ∧.

A similar question is how the two logics S∗ and FL(∗), the fuzzy logic based on the continuous t-norm ∗, relate.

Proposition 5.1.13. *For every continuous t-norm, every precisification space $\boldsymbol{S}$ and every evaluation $\boldsymbol{e}$ the following holds:*

- *There is an evaluation $\boldsymbol{e}_{\boldsymbol{S}}$ such that $\|\varphi\|^*_{\boldsymbol{S}} = \|\varphi\|^*_{\boldsymbol{e}_{\boldsymbol{S}}}$ for every* S*-free formula φ.*
- *There is a precisification space $\boldsymbol{S}_{\boldsymbol{e}}$ such that $\|\varphi\|^*_{\boldsymbol{e}} = \|\varphi\|^*_{\boldsymbol{S}_{\boldsymbol{e}}}$ for every* S*-free formula φ.*

Proof. We show the easy proof of the first claim. The second claim will be covered by Lemma 5.6.2 in Section 5.6. Define the evaluation $\boldsymbol{e}_{\boldsymbol{S}}$ by $\boldsymbol{e}_{\boldsymbol{S}}(p) = \|p\|_{\boldsymbol{S}}$ for every propositional variable p. Let φ be a formula that does not contain any S-operator. Then all connectives and truth constants occurring in φ. are defined in the same way for S∗ and FL(∗). The truth value of φ in both cases then only depends on the truth values of the propositional variables and therefore we get $\|\varphi\|^*_{\boldsymbol{S}} = \|\varphi\|^*_{\boldsymbol{e}_{\boldsymbol{S}}}$. □

A simple corollary of this observation allows us to relate validity in S∗ to validity in the corresponding fuzzy logic (as pointed out in [38]).

Corollary 5.1.14. *For every continuous t-norm $*$ and every* S*-free formula φ the following holds: φ is valid in* S∗ *if and only if φ is valid in* FL(∗).

Next, we show that a certain deduction rule, that is usually called necessitation rule, holds for our logic.

Proposition 5.1.15. *For every continuous t-norm $*$ and every formula φ the following holds: if φ is valid in* S$*$*, then also* Sφ *is valid in* S$*$.

Proof. Assume that φ is valid in S$*$ and let $\boldsymbol{S}$ be an arbitrary precisification space with a set of precisifications $\boldsymbol{P}$ and a probability measure μ. For every $s \in \boldsymbol{S}$ we define the precisification space $\boldsymbol{S}_s$ that has $\boldsymbol{P}$ as its set of precisifications, the same local truth values as $\boldsymbol{S}$, and a probability measure μ_s that is given by

$$\mu_s(t) = \begin{cases} 1 & \text{if } t = s \\ 0 & \text{otherwise} \end{cases}$$

for every $t \in \boldsymbol{S}_s$.

For every formula of the form $\mathrm{S}\psi$ we clearly have $\|\mathrm{S}\psi\|_{s,\boldsymbol{S}} = \|\mathrm{S}\psi\|_{\boldsymbol{S}} = \|\mathrm{S}\psi\|_{\boldsymbol{S}_s}$. Furthermore, $\|p\|_{s,\boldsymbol{S}} = \|p\|_{\boldsymbol{S}_s} \in \{0,1\}$ for every propositional variable p which becomes clear by a simple case distinction. If $\|p\|_{s,\boldsymbol{S}} = 1$, then also $\|p\|_{s,\boldsymbol{S}_s} = 1$ and we have $\|p\|_{\boldsymbol{S}_s} = \mu([p]_{\boldsymbol{S}_s}) \geq \mu_s(s) = 1$ which means that $\|p\|_{\boldsymbol{S}_s} = 1$. If $\|p\|_{s,\boldsymbol{S}} = 0$, then also $\|p\|_{s,\boldsymbol{S}_s} = 0$ and we have $\|p\|_{\boldsymbol{S}_s} = \mu([p]_{\boldsymbol{S}_s}) = 1 - \mu([\neg p]_{\boldsymbol{S}_s}) \leq 1 - \mu_s(s) = 1 - 1 = 0$ which means that $\|p\|_{\boldsymbol{S}_s} = 0$.

Since these two facts hold we have $\|\varphi\|_{s,\boldsymbol{S}} = \|\varphi\|^*_{\boldsymbol{S}_s}$ because the t-norm based connectives behave classically for the truth value set $\{0,1\}$. Furthermore we know that φ is valid and therefore $\|\varphi\|^*_{\boldsymbol{S}_s} = 1$. Hence, our argument shows that $\|\varphi\|_{s,\boldsymbol{S}} = 1$ for every $s \in \boldsymbol{S}$ and therefore $\|\mathrm{S}\varphi\|_{\boldsymbol{S}} = 1$. Since $\boldsymbol{S}$ was an arbitrary precisification space we get that Sφ is valid. □

Together with Proposition 5.1.12 we get the following corollary.

Corollary 5.1.16. *For a continuous t-norm $*$ and every formula φ the following holds: If φ is valid in* S$*$*, then $\varphi^\Box$ is valid in* S5 *where $\varphi^\Box$ denotes the following translation of φ into a formula of modal logic: every occurrence of the* S*-operator is replaced by the necessitation operator $\Box$ and every occurrence of the conjunction connective & is replaced by $\wedge$.*

One way of syntactically characterizing a logic is giving an axiomatization in the form a Hilbert-style proof system. Although we do not give a sound and complete proof system for our hybrid logic, we want to gain some intuition on what an axiomatization could look like. Due to the relations shown above we take the axiomatizations of S5 and $\triangle$FL($*$) as the starting point of our little investigation, where $\triangle$ is the globalization operator that was discussed in Section 3.6.1. We distinguish between two kinds of axioms: modal axioms (containing one of the operators S, $\Box$ or $\triangle$) and non-modal axioms.

Let us first take a look at the non-modal axioms. An axiomatization of $\triangle$FL($*$) can be obtained by adding to an axiomatization of FL($*$) a certain set of modal axioms. Due to Corollary 5.1.14 it is clear that every axiom of FL($*$) is also valid in S$*$. The logic S5 also has non-modal axioms which are exactly those of classical logic. Some of these axioms, in particular those relating to the law of excluded middle $\varphi \vee \neg\varphi$ do not hold in the fuzzy logic FL($*$). However, if φ is an axiom of S5, then Sφ is valid in S$*$ due to Proposition 5.1.12.

The interesting question now is what the axioms for the S-operator might be. In Tables 5.1 and 5.2 we have listed all modal axioms of S5 and $\triangle$FL($*$). For each modal axiom we have checked whether its "translation" to S$*$ is valid.

Axioms of S5	translation into S$*$	valid?
(K) $\Box(\varphi \supset \psi) \supset (\Box\varphi \supset \Box\psi)$	$S(\varphi \supset \psi) \supset (S\varphi \supset S\psi)$	✓
(T) $\Box\varphi \supset \varphi$	$S\varphi \supset \varphi$	×
(5) $\Diamond\varphi \supset \Box\Diamond\varphi$	$\neg S\varphi \supset S\neg S\varphi$	✓

Table 5.1: Validity of S5-axioms in S$*$

Axioms of $\triangle$FL($*$)	translation into S$*$	valid?
($\triangle$1) $\triangle\varphi \supset \varphi$	$S\varphi \supset \varphi$	×
($\triangle$2) $\triangle\varphi \supset \triangle\triangle\varphi$	$S\varphi \supset SS\varphi$	✓
($\triangle$3) $\triangle(\varphi \supset \psi) \supset (\triangle\varphi \supset \triangle\psi)$	$S(\varphi \supset \psi) \supset (S\varphi \supset S\psi)$	✓
($\triangle$4) $\triangle\varphi \vee \neg\triangle\varphi$	$S\varphi \vee \neg S\varphi$	✓
($\triangle$5) $\triangle(\varphi \vee \psi) \supset (\triangle\varphi \vee \triangle\psi)$	$S(\varphi \vee \psi) \supset (S\varphi \vee S\psi)$	×

Table 5.2: Validity of $\triangle$FL($*$)-axioms in S$*$

For the schemata that are not valid in S$*$ it might still be the case that there are certain instances of the schemata that are valid in S$*$. One can for example ask the question which instances of $S\varphi \supset \varphi$ are valid in SŁ. We do not give a complete characterization, but only hint at some formulas of that form that are valid in S$*$ for every continuous t-norm $*$ (p and q are propositional variables, φ is a formula):

- $Sp \supset p$
- $(S\neg p) \supset (\neg p)$
- $(S(p \supset q)) \supset (p \supset q))$
- $S(p \wedge q) \supset (p \wedge q)$
- $SS\varphi \supset S\varphi$.

Note that, as these considerations indicate, the set of valid formulas in S$*$ is not closed under substitution: $Sp \supset p$ is valid whereas $S(p \vee q) \supset (p \vee q)$ is not valid. Things get even more difficult because there are instances of the schema $S\varphi \supset \varphi$ that are only valid for particular continuous t-norms.[28]

[28]We give an example in Proposition 5.3.5.

5.2 Normal Form

In the following we show that formulas of S∗ can be put in a certain normal form. There are many motivations why normal forms are desired. We expect a simplified analysis of S∗ with normal forms. Our result will be that every formula of S∗ is equivalent to a formula with no nested S-operators. Note that the normal form does not depend on the t-norm. Therefore the results in this section hold for every choice of a continuous t-norm.

The normal form for S∗ is based on well-known normal forms for classical logic and the modal logic S5. The following definitions show what these normal forms look like.

Definition 5.2.1. A *classical literal* is a propositional variable or its negation. A *classical clause* is a disjunction of classical literals. A formula is in *classical conjunctive normal form* iff it is a conjunction of classical clauses.

A *modal literal* is an S-free formula or a formula of the form $\mathrm{S}\varphi$ or $\neg\mathrm{S}\varphi$ where φ is an S-free formula. A *modal clause* is a disjunction of modal literals. A formula is in *modal conjunctive normal form* iff it is a conjunction of modal clauses.

We will base our normal form for S∗ on the modal conjunctive normal form that is available for the modal logic S5 [69].

Theorem 5.2.2. *For very formula φ of* S5 *there is a formula φ' in modal conjunctive normal form such that $\varphi \equiv \varphi'$ is valid in* S5.

Proposition 5.2.3. *For every formula of the form $\mathrm{S}\varphi$ there is a formula $\mathrm{S}\varphi'$ where φ' is in modal conjunctive normal form such that $\|\mathrm{S}\varphi\|_{\boldsymbol{S}} = \|\mathrm{S}\varphi'\|_{\boldsymbol{S}}$ for every precisification space $\boldsymbol{S}$.*

Proof. We ignore the syntactic difference between the modalities S and □ and use S instead of the necessity operator □ in formulas of S5. Let φ be a formula and φ' the equivalent formula in modal conjunctive normal form of Theorem 5.2.2. Since $\varphi \equiv \varphi'$ is valid in S5 we know by Proposition 5.1.12 that $\mathrm{S}(\varphi \equiv \varphi')$ is valid in S∗. Therefore we get $\|\mathrm{S}(\varphi \equiv \varphi')\|_{\boldsymbol{S}} = 1$ and thus $\|\varphi \equiv \varphi'\|_{s,\boldsymbol{S}} = 1$ for every $s \in \boldsymbol{S}$. From this we conclude that $\|\varphi\|_{s,\boldsymbol{S}} = \|\varphi'\|_{s,\boldsymbol{S}} = 1$ for every $s \in \boldsymbol{S}$.

Assume that $\|\mathrm{S}\varphi\|_{\boldsymbol{S}} = 1$ and let $s \in \boldsymbol{S}$ be an arbitrary precisification. Then we know that $\|\varphi\|_{s,\boldsymbol{S}} = 1$ and therefore also $\|\varphi'\|_{s,\boldsymbol{S}} = 1 = \|\mathrm{S}\varphi\|_{\boldsymbol{S}}$. Since s was arbitrary we then get $\|\mathrm{S}\varphi'\|_{\boldsymbol{S}} = 1$.

Now assume that $\|\mathrm{S}\varphi\|_{\boldsymbol{S}} \neq 1$. Then we have $\|\mathrm{S}\varphi\|_{\boldsymbol{S}} = 0$ and therefore there is an $s \in \boldsymbol{S}$ such that $\|\varphi\|_{s,\boldsymbol{S}} = 0$. Therefore also $\|\varphi'\|_{s,\boldsymbol{S}} = 0$ and we get $\|\mathrm{S}\varphi'\|_{\boldsymbol{S}} = 0 = \|\mathrm{S}\varphi\|_{\boldsymbol{S}}$. □

With this proposition we know now that every formula is equivalent to a formula in which every nesting of S-operators has at most depth two. The further strategy now is to show that we can get rid of the remaining inner S-operators. For this purpose some equivalences will be very helpful.

Lemma 5.2.4. *The following equivalences hold for every precisification space $\boldsymbol{S}$:*

$$\|\mathrm{S}(\varphi \wedge \psi)\|_{\boldsymbol{S}} = \|\mathrm{S}\varphi \wedge \mathrm{S}\psi\|_{\boldsymbol{S}}$$
$$\|\mathrm{S}(\mathrm{S}\varphi \vee \psi)\|_{\boldsymbol{S}} = \|\mathrm{S}\varphi \vee \mathrm{S}\psi\|_{\boldsymbol{S}}$$
$$\|\mathrm{S}(\neg\mathrm{S}\varphi \vee \psi)\|_{\boldsymbol{S}} = \|\neg\mathrm{S}\varphi \vee \mathrm{S}\psi\|_{\boldsymbol{S}}\,.$$

Proof. The left hand side and right hand side formulas can only have the truth values 0 or 1. Thus, it is sufficient to prove that the truth value of the left hand side formula is 1 if and only if the truth value of the right hand side formula is 1. We do this by showing both directions for every formula. Remember that the truth functions of $\wedge$ and $\vee$ behave classically for the truth value set $\{0, 1\}$.

- $\|\mathrm{S}(\varphi \wedge \psi)\|_{\boldsymbol{S}} = \|\mathrm{S}\varphi \wedge \mathrm{S}\psi\|_{\boldsymbol{S}}$:

  Assume that $\|\mathrm{S}(\varphi\wedge\psi)\|_{\boldsymbol{S}} = 1$ and let $s \in \boldsymbol{S}$. Then $\|\varphi\wedge\psi\|_{s,\boldsymbol{S}} = 1$ and therefore $\|\varphi\|_{s,\boldsymbol{S}} = 1$ and $\|\psi\|_{s,\boldsymbol{S}} = 1$. Since s was an arbitrary precisification we get $\|\mathrm{S}\varphi\|_{\boldsymbol{S}} = 1$ and $\|\mathrm{S}\psi\|_{\boldsymbol{S}} = 1$. Therefore we may conclude $\|\mathrm{S}\varphi \wedge \mathrm{S}\psi\|_{\boldsymbol{S}} = 1$.

  Now assume that $\|\mathrm{S}\varphi \wedge \mathrm{S}\psi\|_{\boldsymbol{S}} = 1$. Then $\|\mathrm{S}\varphi\|_{\boldsymbol{S}} = 1$ and $\|\mathrm{S}\psi\|_{\boldsymbol{S}} = 1$. Let $s \in \boldsymbol{S}$. Then we get $\|\varphi\|_{s,\boldsymbol{S}} = 1$ and $\|\psi\|_{s,\boldsymbol{S}} = 1$. Therefore we may conclude $\|\varphi\wedge\psi\|_{s,\boldsymbol{S}} = 1$. Since s was an arbitrary precisification we get $\|\mathrm{S}(\varphi\wedge\psi)\|_{\boldsymbol{S}} = 1$.

- $\|\mathrm{S}(\mathrm{S}\varphi \vee \psi)\|_{\boldsymbol{S}} = \|\mathrm{S}\varphi \vee \mathrm{S}\psi\|_{\boldsymbol{S}}$:

  Assume that $\|\mathrm{S}(\mathrm{S}\varphi \vee \psi)\|_{\boldsymbol{S}} = 1$. If $\|\mathrm{S}\varphi\|_{\boldsymbol{S}} = 1$, then trivially $\|\mathrm{S}\varphi \vee \mathrm{S}\psi\|_{\boldsymbol{S}} = 1$. Now assume that $\|\mathrm{S}\varphi\|_{\boldsymbol{S}} = 0$ and let $s \in \boldsymbol{S}$. Then clearly $\|\mathrm{S}\varphi\|_{s,\boldsymbol{S}} = 0$. Since $\|\mathrm{S}(\mathrm{S}\varphi \vee \psi)\|_{\boldsymbol{S}} = 1$, we have $\|\mathrm{S}\varphi \vee \psi\|_{s,\boldsymbol{S}} = 1$. Therefore $\|\psi\|_{s,\boldsymbol{S}} = 1$ must hold. Since s was an arbitrary precisification we get $\|\mathrm{S}\psi\|_{\boldsymbol{S}} = 1$ and therefore $\|\mathrm{S}\varphi \vee \mathrm{S}\psi\|_{\boldsymbol{S}} = 1$.

  Assume that $\|\mathrm{S}\varphi \vee \mathrm{S}\psi\|_{\boldsymbol{S}} = 1$. If $\|\mathrm{S}\varphi\|_{\boldsymbol{S}} = 1$, then $\|\mathrm{S}\varphi\|_{s,\boldsymbol{S}} = 1$ for every $s \in \boldsymbol{S}$. Therefore also $\|\mathrm{S}\varphi \vee \psi\|_{s,\boldsymbol{S}} = 1$ for every $s \in \boldsymbol{S}$. This means that $\|\mathrm{S}(\mathrm{S}\varphi \vee \psi)\|_{\boldsymbol{S}} = 1$. If $\|\mathrm{S}\varphi\|_{\boldsymbol{S}} = 0$, then $\|\mathrm{S}\psi\|_{\boldsymbol{S}} = 1$ must hold because of $\|\mathrm{S}\varphi \vee \mathrm{S}\psi\|_{\boldsymbol{S}} = 1$. Thus, $\|\psi\|_{s,\boldsymbol{S}} = 1$ for every $s \in \boldsymbol{S}$ and consequently $\|\mathrm{S}\varphi \vee \psi\|_{s,\boldsymbol{S}} = 1$ for every $s \in \boldsymbol{S}$. Therefore we also get $\|\mathrm{S}(\mathrm{S}\varphi \vee \psi)\|_{\boldsymbol{S}} = 1$ in this case.

- $\|\mathrm{S}(\neg\mathrm{S}\varphi \vee \psi)\|_{\boldsymbol{S}} = \|\neg\mathrm{S}\varphi \vee \mathrm{S}\psi\|_{\boldsymbol{S}}$:

  Assume that $\|\mathrm{S}(\neg\mathrm{S}\varphi \vee \psi)\|_{\boldsymbol{S}} = 1$. If $\|\mathrm{S}\varphi\|_{\boldsymbol{S}} = 0$, then $\|\neg\mathrm{S}\varphi\|_{\boldsymbol{S}} = 1$ and therefore also $\|\neg\mathrm{S}\varphi \vee \mathrm{S}\psi\|_{\boldsymbol{S}} = 1$. Consider now the case $\|\mathrm{S}\varphi\|_{\boldsymbol{S}} = 1$ and let $s \in \boldsymbol{S}$. Then $\|\mathrm{S}\varphi\|_{s,\boldsymbol{S}} = 1$ which means that $\|\neg\mathrm{S}\varphi\|_{s,\boldsymbol{S}} = 0$. Because of $\|\mathrm{S}(\neg\mathrm{S}\varphi \vee \psi)\|_{\boldsymbol{S}} = 1$ we have $\|\neg\mathrm{S}\varphi \vee \psi\|_{s,\boldsymbol{S}} = 1$. Therefore $\|\psi\|_{s,\boldsymbol{S}} = 1$ must hold. Since s was an arbitrary precisification, we get $\|\mathrm{S}\psi\|_{\boldsymbol{S}} = 1$ and thus $\|\neg\mathrm{S}\varphi \vee \mathrm{S}\psi\|_{\boldsymbol{S}} = 1$.

  Assume that $\|\neg\mathrm{S}\varphi \vee \mathrm{S}\psi\|_{\boldsymbol{S}} = 1$. If $\|\mathrm{S}\psi\|_{\boldsymbol{S}} = 1$, we get $\|\psi\|_{s,\boldsymbol{S}} = 1$ for every $s \in \boldsymbol{S}$. Then also $\|\neg\mathrm{S}\varphi\vee\psi\|_{s,\boldsymbol{S}} = 1$ for every $s \in \boldsymbol{S}$ which means that $\|\mathrm{S}(\neg\mathrm{S}\varphi\vee \psi)\|_{\boldsymbol{S}} = 1$. If $\|\mathrm{S}\psi\|_{\boldsymbol{S}} = 0$, we have $\|\neg\mathrm{S}\varphi\|_{\boldsymbol{S}} = 1$ due to $\|\neg\mathrm{S}\varphi\vee\mathrm{S}\psi\|_{\boldsymbol{S}} = 1$. From this we get $\|S\varphi\|_{\boldsymbol{S}} = 0$. This means that there is a precisification $t \in \boldsymbol{S}$ such that $\|\varphi\|_t = 0$. Now let $s \in \boldsymbol{S}$. Then, due to the existence of t, we have $\|S\varphi\|_{s,\boldsymbol{S}} = 0$ which means that $\|\neg S\varphi\|_{s,\boldsymbol{S}} = 1$. Then we may conclude $\|\neg S\varphi \vee S\psi\|_{s,\boldsymbol{S}} = 1$. Since s was an arbitrary precisification, we get $\|S(\neg S\varphi \vee \psi)\|_{\boldsymbol{S}} = 1$.

□

Theorem 5.2.5. *For every formula* $S\varphi$ *there is a formula* φ' *that does not contain nested* S*-operators such that* $\|\varphi\|_{\boldsymbol{S}} = \|\varphi'\|_{\boldsymbol{S}}$ *for every precisification space* $\boldsymbol{S}$*. Furthermore, there is such a* φ' *such that if* φ' *contains a subformula* $S\psi$*, then* ψ *is a disjunction of classical literals.*

Proof. We start with the proof of the first part of the theorem. If φ does not contain an S-operator, there is nothing to prove. Otherwise, let $S\psi$ be a subformula of φ. Again, if ψ does not contain an S-operator, there is nothing to prove. Otherwise, we know by Proposition 5.2.3 that there is formula ψ' in modal conjunctive normal form such that $\|S\psi\|_{\boldsymbol{S}} = \|S\psi'\|_{\boldsymbol{S}}$. The formula ψ is a conjunction of modal clauses, i.e., $\psi = \bigwedge_{i=1}^{n} C_i$. By iterated application of the equivalence for conjunction of Lemma 5.2.4 we get

$$\|S\psi\|_{\boldsymbol{S}} = \left\| S\left(\bigwedge_{i=1}^{n} C_i\right)\right\|_{\boldsymbol{S}} = \left\|\bigwedge_{i=1}^{n} SC_i\right\|_{\boldsymbol{S}} .$$

Now let C_i be one of the modal clauses that contains an S-operator and let L be one of the modal literals that contains an S-operator. Then C_i is equivalent to a formula $L \vee \chi$, i.e. $\|SC_i\|_{\boldsymbol{S}} = \|S(L \vee \chi)\|_{\boldsymbol{S}}$, because classical commutativity and associativity hold and therefore $\|C_i\|_{s,\boldsymbol{S}} = \|L \vee \chi\|_{s,\boldsymbol{S}}$ for every $s \in \boldsymbol{S}$. Note that in the case $C_i = L$ we can set $\chi = \bar{0}$. Now we proceed with a case distinction on the shape of the modal literal L:

- Case 1: $L = S\alpha$ for an S-free formula α

  Due to Lemma 5.2.4 we get

  $$\|SC_i\|_{\boldsymbol{S}} = \|S(L \vee \chi)\|_{\boldsymbol{S}} = \|S(S\alpha \vee \chi)\|_{\boldsymbol{S}} = \|S\alpha \vee S\chi\|_{\boldsymbol{S}} .$$

- Case 2: $L = \neg S\alpha$ for an S-free formula α

  Due to Lemma 5.2.4 we get

  $$\|SC_i\|_{\boldsymbol{S}} = \|S(L \vee \chi)\|_{\boldsymbol{S}} = \|S(\neg S\alpha \vee \chi)\|_{\boldsymbol{S}} = \|\neg S\alpha \vee S\chi\|_{\boldsymbol{S}} .$$

The same procedure can now be applied to the smaller formula χ which is again a disjunction of modal literals.

So far we have showed that for every formula φ there is an equivalent formula φ' such that φ' does not contain nested S-operators. The second part of the theorem is easy to show. Let $S\psi$ be a subformula of φ'. Then there is a formula ψ' in classical conjunctive normal form such that $\|\psi\|_{s,\boldsymbol{S}} = \|\psi'\|_{s,\boldsymbol{S}}$ for every $s \in \boldsymbol{S}$ and therefore also $\|\mathrm{S}\psi\|_{\boldsymbol{S}} = \|\mathrm{S}\psi'\|_{\boldsymbol{S}}$. The formula ψ' is a conjunction of classical clauses, i.e., $\psi' = \bigwedge_{i=1}^{n} C_i$. Again we iteratedly apply the equivalence for conjunction of Lemma 5.2.4 and get

$$\|\mathrm{S}\psi'\|_{\boldsymbol{S}} = \left\| \mathrm{S}\left(\bigwedge_{i=1}^{n} C_i\right)\right\|_{\boldsymbol{S}} = \left\|\bigwedge_{i=1}^{n} \mathrm{S}C_i\right\|_{\boldsymbol{S}}$$

where each C_i is a disjunction of classical literals. □

5.3 Bounding measures and truth degrees

In this section, we want to show some relations for the extension $[\varphi]$, the measure $[\![\varphi]\!]$ and the truth value $\|\varphi\|$ of a formula φ.

Proposition 5.3.1. *For every precisification space $\boldsymbol{S}$ with a set of precisifications $\boldsymbol{P}$ and all formulas φ and ψ the following holds:*

$$\begin{aligned}
\emptyset \subseteq [\varphi]_{\boldsymbol{S}} &\subseteq \boldsymbol{P} \\
[\neg\varphi]_{\boldsymbol{S}} &= \boldsymbol{P} \setminus [\varphi]_{\boldsymbol{S}} \\
[\varphi \wedge \psi]_{\boldsymbol{S}} &= [\varphi]_{\boldsymbol{S}} \cap [\psi]_{\boldsymbol{S}} \\
[\varphi \vee \psi]_{\boldsymbol{S}} &= [\varphi]_{\boldsymbol{S}} \cup [\psi]_{\boldsymbol{S}} \\
[\varphi \supset \psi]_{\boldsymbol{S}} &= \boldsymbol{P} \setminus [\varphi]_{\boldsymbol{S}} \cup [\psi]_{\boldsymbol{S}} \\
[\mathrm{S}\varphi]_{\boldsymbol{S}} &= \begin{cases} \boldsymbol{P} & \text{if } [\varphi]_{\boldsymbol{S}} = \boldsymbol{P} \\ \emptyset & \text{otherwise}. \end{cases}
\end{aligned}$$

Proof. The relation $\emptyset \subseteq [\varphi]_{\boldsymbol{S}} \subseteq \boldsymbol{P}$ trivially holds because $[\varphi]_{\boldsymbol{S}}$ by its definition is a subset of $\boldsymbol{P}$. Furthermore, we have:

$$\begin{aligned}
[\neg\varphi]_{\boldsymbol{S}} &= \left\{s \in \boldsymbol{P} \mid \|\neg\varphi\|_{s,\boldsymbol{S}} = 1\right\} \\
&= \left\{s \in \boldsymbol{P} \mid \|\varphi\|_{s,\boldsymbol{S}} = 0\right\} \\
&= \left\{s \in \boldsymbol{P} \mid \|\varphi\|_{s,\boldsymbol{S}} \neq 1\right\} \\
&= \boldsymbol{P} \setminus \left\{s \in \boldsymbol{P} \mid \|\varphi\|_{s,\boldsymbol{S}} = 1\right\} \\
&= \boldsymbol{P} \setminus [\varphi]_{\boldsymbol{S}} \\
[\varphi \wedge \psi]_{\boldsymbol{S}} &= \left\{s \in \boldsymbol{P} \mid \|\varphi \wedge \psi\|_{s,\boldsymbol{S}} = 1\right\} \\
&= \left\{s \in \boldsymbol{P} \mid \|\varphi\|_{s,\boldsymbol{S}} \text{ and } \|\psi\|_{s,\boldsymbol{S}} = 1\right\} \\
&= \left\{s \in \boldsymbol{P} \mid \|\varphi\|_{s,\boldsymbol{S}}\right\} \cap \left\{s \in \boldsymbol{P} \mid \|\psi\|_{s,\boldsymbol{S}} = 1\right\} \\
&= [\varphi]_{\boldsymbol{S}} \cap [\psi]_{\boldsymbol{S}} \\
[\varphi \vee \psi]_{\boldsymbol{S}} &= \left\{s \in \boldsymbol{P} \mid \|\varphi \vee \psi\|_{s,\boldsymbol{S}} = 1\right\} \\
&= \left\{s \in \boldsymbol{P} \mid \|\varphi\|_{s,\boldsymbol{S}} \text{ or } \|\psi\|_{s,\boldsymbol{S}} = 1\right\} \\
&= \left\{s \in \boldsymbol{P} \mid \|\varphi\|_{s,\boldsymbol{S}}\right\} \cup \left\{s \in \boldsymbol{P} \mid \|\psi\|_{s,\boldsymbol{S}} = 1\right\} \\
&= [\varphi]_{\boldsymbol{S}} \cup [\psi]_{\boldsymbol{S}} \\
[\varphi \supset \psi]_{\boldsymbol{S}} &= [\neg\varphi \vee \psi]_{\boldsymbol{S}} = [\neg\varphi] \cup [\psi]_{\boldsymbol{S}} = \boldsymbol{P} \setminus [\varphi]_{\boldsymbol{S}} \cup [\psi]_{\boldsymbol{S}}\,.
\end{aligned}$$

We prove the last relation by a case distinction. If $[\varphi]_{\boldsymbol{S}} = \boldsymbol{P}$, then $\{s \in \boldsymbol{P} \mid \|\varphi\|_{s,\boldsymbol{S}} = 1\} = \boldsymbol{P}$. Therefore $\|\varphi\|_{t,\boldsymbol{S}} = 1$ for every $t \in \boldsymbol{P}$. Then it follows that $\|\mathrm{S}\varphi\|_{s,\boldsymbol{S}} = 1$ for every $s \in \boldsymbol{P}$ and we get $[\mathrm{S}\varphi]_{\boldsymbol{S}} = \{s \in \boldsymbol{P} \mid \|\mathrm{S}\varphi\|_{s,\boldsymbol{S}} = 1\} = \boldsymbol{P}$.

Now assume that $[\varphi]_{\boldsymbol{S}} \neq \boldsymbol{P}$. Then we know that $[\varphi]_{\boldsymbol{S}} \subset \boldsymbol{P}$ which means there is a $t \in \boldsymbol{P}$ such that $t \notin [\varphi]_{\boldsymbol{S}}$. This means that $\|\varphi\|_{t,\boldsymbol{S}} \neq 1$ and therefore $\|\mathrm{S}\varphi\|_{s,\boldsymbol{S}} \neq 1$ for every $s \in \boldsymbol{P}$. Thus we get $[\mathrm{S}\varphi]_{\boldsymbol{S}} = \{s \in \boldsymbol{P} \mid \|\mathrm{S}\varphi\|_{s,\boldsymbol{S}} = 1\} = \emptyset$. □

We give some useful bounds on the measure of a formula that partly have been pointed out by Fermüller and Roschger [39].

Proposition 5.3.2. *For every precisification space $\boldsymbol{S}$ and every formula φ the following holds:*

$$
\begin{aligned}
0 \leq [\![\varphi]\!]_{\boldsymbol{S}} &\leq 1 \\
[\![\neg\varphi]\!]_{\boldsymbol{S}} &= 1 - [\![\varphi]\!]_{\boldsymbol{S}} \\
[\![\varphi \vee \psi]\!]_{\boldsymbol{S}} &= [\![\varphi]\!]_{\boldsymbol{S}} + [\![\psi]\!]_{\boldsymbol{S}} - [\![\varphi \wedge \psi]\!]_{\boldsymbol{S}} \\
\max\left([\![\varphi]\!]_{\boldsymbol{S}}, [\![\psi]\!]_{\boldsymbol{S}}\right) \leq [\![\varphi \vee \psi]\!]_{\boldsymbol{S}} &\leq [\![\varphi]\!]_{\boldsymbol{S}} + [\![\psi]\!]_{\boldsymbol{S}} \\
[\![\varphi]\!]_{\boldsymbol{S}} + [\![\psi]\!]_{\boldsymbol{S}} - 1 \leq [\![\varphi \wedge \psi]\!]_{\boldsymbol{S}} &\leq \min\left([\![\varphi]\!]_{\boldsymbol{S}}, [\![\psi]\!]_{\boldsymbol{S}}\right) \\
\max\left(1 - [\![\varphi]\!]_{\boldsymbol{S}}, [\![\psi]\!]_{\boldsymbol{S}}\right) \leq [\![\varphi \supset \psi]\!]_{\boldsymbol{S}} &\leq 1 - [\![\varphi]\!]_{\boldsymbol{S}} + [\![\psi]\!]_{\boldsymbol{S}} \\
[\![\mathrm{S}\varphi]\!]_{\boldsymbol{S}} &\leq \lfloor [\![\varphi]\!]_{\boldsymbol{S}} \rfloor .
\end{aligned}
$$

If $\boldsymbol{S}$ is a positive precisification space we also have the lower bound

$$\lfloor [\![\varphi]\!]_{\boldsymbol{S}} \rfloor \leq [\![\mathrm{S}\varphi]\!]_{\boldsymbol{S}} .$$

The function $\lfloor \cdot \rfloor$ is the floor function.

Before we give a proof of this statement, note the following:

- The identity for negation corresponds to Łukasiewicz negation.
- The upper bound for conjunction corresponds to the Gödel t-norm, which is the truth function of weak conjunction in all of our fuzzy logics. The lower bound for disjunction corresponds to Gödel t-conorm, which is the truth function of disjunction in all of our fuzzy logics.
- The lower bound for conjunction corresponds to the Łukasiewicz t-norm and the upper bound for disjunction corresponds to Łukasiewicz t-conorm, which is the truth function of strong disjunction in Łukasiewicz logic.
- The upper bound for implication corresponds to the residuum of the Łukasiewicz t-norm.
- The bound for the S-operator corresponds to the truth function of the $\triangle$-operator (see Section 3.6.1).

Proof of Proposition 5.3.2. Let $\boldsymbol{S}$ be a precisification space with a set of precisifications $\boldsymbol{P}$ and a probability measure μ. Most of the inequalities can be obtained by applying Proposition 5.3.1.

- $0 \leq [\![\varphi]\!]_{\boldsymbol{S}} \leq 1$: Directly follows from the definition of a probability measure because $[\![\varphi]\!]_{\boldsymbol{S}} = \mu([\varphi]_{\boldsymbol{S}})$.
- $[\![\neg\varphi]\!]_{\boldsymbol{S}} = 1 - [\![\varphi]\!]_{\boldsymbol{S}}$:
  By Proposition 5.3.1 and Proposition 5.1.2 we get
$$[\![\neg\varphi]\!]_{\boldsymbol{S}} = \mu\left([\neg\varphi]_{\boldsymbol{S}}\right) = \mu\left(\boldsymbol{P}\setminus[\varphi]_{\boldsymbol{S}}\right) = \mu(\boldsymbol{P}) - \mu\left([\varphi]_{\boldsymbol{S}}\right) = 1 - [\![\varphi]\!]_{\boldsymbol{S}}\,.$$
- $[\![\varphi \vee \psi]\!]_{\boldsymbol{S}} = [\![\varphi]\!]_{\boldsymbol{S}} + [\![\psi]\!]_{\boldsymbol{S}} - [\![\varphi \wedge \psi]\!]_{\boldsymbol{S}}$:
  We mainly use Proposition 5.1.2 to prove this part.
$$\begin{aligned}
[\![\varphi \vee \psi]\!]_{\boldsymbol{S}} = \mu\left([\varphi \vee \psi]_{\boldsymbol{S}}\right) &= \mu\left([\varphi]_{\boldsymbol{S}} \cup [\psi]_{\boldsymbol{S}}\right)\\
&= \mu\left([\varphi]_{\boldsymbol{S}} \cup \left([\psi]_{\boldsymbol{S}} \setminus [\varphi]_{\boldsymbol{S}}\right)\right)\\
&= \mu\left([\varphi]_{\boldsymbol{S}}\right) + \mu\left([\psi]_{\boldsymbol{S}} \setminus [\varphi]_{\boldsymbol{S}}\right)\\
&= \mu\left([\varphi]_{\boldsymbol{S}}\right) + \mu\left([\psi]_{\boldsymbol{S}} \setminus \left([\psi]_{\boldsymbol{S}} \cap [\varphi]_{\boldsymbol{S}}\right)\right)\\
&= \mu\left([\varphi]_{\boldsymbol{S}}\right) + \mu\left([\psi]_{\boldsymbol{S}}\right) - \mu\left([\psi]_{\boldsymbol{S}} \cap [\varphi]_{\boldsymbol{S}}\right)\\
&= \mu\left([\varphi]_{\boldsymbol{S}}\right) + \mu\left([\psi]_{\boldsymbol{S}}\right) - \mu\left([\varphi \wedge \psi]_{\boldsymbol{S}}\right)\\
&= [\![\varphi]\!]_{\boldsymbol{S}} + [\![\psi]\!]_{\boldsymbol{S}} - [\![\varphi \wedge \psi]\!]_{\boldsymbol{S}}\,.
\end{aligned}$$
- $\max\left([\![\varphi]\!]_{\boldsymbol{S}}, [\![\psi]\!]_{\boldsymbol{S}}\right) \leq [\![\varphi \vee \psi]\!]_{\boldsymbol{S}} \leq [\![\varphi]\!]_{\boldsymbol{S}} + [\![\psi]\!]_{\boldsymbol{S}}$:
  Due to
$$[\varphi]_{\boldsymbol{S}} \subseteq [\varphi]_{\boldsymbol{S}} \cup [\psi]_{\boldsymbol{S}} = [\varphi \vee \psi]_{\boldsymbol{S}}$$
  and
$$[\psi]_{\boldsymbol{S}} \subseteq [\varphi]_{\boldsymbol{S}} \cup [\psi]_{\boldsymbol{S}} = [\varphi \vee \psi]_{\boldsymbol{S}}$$
  we have $[\![\varphi]\!]_{\boldsymbol{S}} \leq [\![\varphi \vee \psi]\!]_{\boldsymbol{S}}$ as well as $[\![\psi]\!]_{\boldsymbol{S}} \leq [\![\varphi \vee \psi]\!]_{\boldsymbol{S}}$ by Proposition 5.1.2. Therefore $\max([\![\varphi]\!]_{\boldsymbol{S}}, [\![\psi]\!]_{\boldsymbol{S}}) \leq [\![\varphi \vee \psi]\!]_{\boldsymbol{S}}$ holds. The second inequality follows from the previous item:
$$[\![\varphi \vee \psi]\!]_{\boldsymbol{S}} = [\![\varphi]\!]_{\boldsymbol{S}} + [\![\psi]\!]_{\boldsymbol{S}} - [\![\varphi \wedge \psi]\!]_{\boldsymbol{S}} \leq [\![\varphi]\!]_{\boldsymbol{S}} + [\![\psi]\!]_{\boldsymbol{S}}\,.$$
- $[\![\varphi]\!]_{\boldsymbol{S}} + [\![\psi]\!]_{\boldsymbol{S}} - 1 \leq [\![\varphi \wedge \psi]\!]_{\boldsymbol{S}} \leq \min\left([\![\varphi]\!]_{\boldsymbol{S}}, [\![\psi]\!]_{\boldsymbol{S}}\right)$:
  We simply apply relations that we have already proved:
$$\begin{aligned}
[\![\varphi \wedge \psi]\!]_{\boldsymbol{S}} &= [\![\neg(\neg\varphi \vee \neg\psi)]\!]_{\boldsymbol{S}} = 1 - [\![\neg\varphi \vee \neg\psi]\!]_{\boldsymbol{S}} \geq 1 - \left([\![\neg\varphi]\!]_{\boldsymbol{S}} + [\![\neg\psi]\!]_{\boldsymbol{S}}\right)\\
&= 1 - \left(1 - [\![\varphi]\!]_{\boldsymbol{S}} + 1 - [\![\psi]\!]_{\boldsymbol{S}}\right) = [\![\varphi]\!]_{\boldsymbol{S}} + [\![\psi]\!]_{\boldsymbol{S}} - 1\\
[\![\varphi \wedge \psi]\!]_{\boldsymbol{S}} &= 1 - [\![\neg\varphi \vee \neg\psi]\!]_{\boldsymbol{S}} \leq 1 - \max\left([\![\neg\varphi]\!]_{\boldsymbol{S}}, [\![\neg\psi]\!]_{\boldsymbol{S}}\right)\\
&= 1 - \max\left(1 - [\![\varphi]\!]_{\boldsymbol{S}}, 1 - [\![\psi]\!]_{\boldsymbol{S}}\right) = \min\left([\![\varphi]\!]_{\boldsymbol{S}}, [\![\psi]\!]_{\boldsymbol{S}}\right)\,.
\end{aligned}$$

- $\max\left(1 - [\![\varphi]\!]_{\boldsymbol{S}}, [\![\psi]\!]_{\boldsymbol{S}}\right) \leq [\![\varphi \supset \psi]\!]_{\boldsymbol{S}} \leq 1 - [\![\varphi]\!]_{\boldsymbol{S}} + [\![\psi]\!]_{\boldsymbol{S}}$

  Again we apply relations that we already know:

$$[\![\varphi \supset \psi]\!]_{\boldsymbol{S}} = [\![\neg\varphi \vee \psi]\!]_{\boldsymbol{S}} \geq \max\left([\![\neg\varphi]\!]_{\boldsymbol{S}}, [\![\psi]\!]_{\boldsymbol{S}}\right) = \max\left(1 - [\![\varphi]\!]_{\boldsymbol{S}}, [\![\psi]\!]_{\boldsymbol{S}}\right)$$
$$[\![\varphi \supset \psi]\!]_{\boldsymbol{S}} = [\![\neg\varphi \vee \psi]\!]_{\boldsymbol{S}} \leq [\![\neg\varphi]\!]_{\boldsymbol{S}} + [\![\psi]\!]_{\boldsymbol{S}} = 1 - [\![\varphi]\!]_{\boldsymbol{S}} + [\![\psi]\!]_{\boldsymbol{S}}.$$

- $[\![\mathrm{S}\varphi]\!]_{\boldsymbol{S}} \leq \lfloor [\![\varphi]\!]_{\boldsymbol{S}} \rfloor$:

  If $[\![\mathrm{S}\varphi]\!]_{\boldsymbol{S}} = 0$ then the bound holds because in any case $[\![\varphi]\!]_{\boldsymbol{S}} \geq 0$ and therefore also $\lfloor [\![\varphi]\!]_{\boldsymbol{S}} \rfloor \geq 0$. If $[\![\mathrm{S}\varphi]\!]_{\boldsymbol{S}} \neq 0$ then we necessarily have $[\![\mathrm{S}\varphi]\!]_{\boldsymbol{S}} = 1$ which means that $\mu([\mathrm{S}\varphi]_{\boldsymbol{S}}) = 1$. Since $\mu(\emptyset) = 0$ we know that $[\mathrm{S}\varphi]_{\boldsymbol{S}} \neq \emptyset$. Therefore there is some $t \in \boldsymbol{P}$ such that $\|\mathrm{S}\varphi\|_{t,\boldsymbol{S}} = \boldsymbol{P}$ which means that $\|\varphi\|_{s,\boldsymbol{S}} = 1$ for every $s \in \boldsymbol{P}$. Thus we get $[\varphi]_{\boldsymbol{S}} = 1$ and we conclude

$$\lfloor [\![\varphi]\!]_{\boldsymbol{S}} \rfloor = \lfloor \mu([\varphi]_{\boldsymbol{S}}) \rfloor = \lfloor \mu(\boldsymbol{P}) \rfloor = \lfloor 1 \rfloor = 1 = [\![\mathrm{S}\varphi]\!]_{\boldsymbol{S}}$$

- $\lfloor [\![\varphi]\!]_{\boldsymbol{S}} \rfloor \leq [\![\mathrm{S}\varphi]\!]_{\boldsymbol{S}}$:

  Assume now that $\boldsymbol{S}$ is a positive precisification space. If $\lfloor [\![\varphi]\!]_{\boldsymbol{S}} \rfloor = 0$, the bound holds because $[\![\mathrm{S}\varphi]\!]_{\boldsymbol{S}} \geq 0$ in any case. If $\lfloor [\![\varphi]\!]_{\boldsymbol{S}} \rfloor = 1$ we have $[\![\varphi]\!]_{\boldsymbol{S}} = 1$. Due to Proposition 5.1.2 we get

$$1 = \mu(\boldsymbol{P}) = \mu\left([\varphi]_{\boldsymbol{S}} \cup \boldsymbol{P} \setminus [\varphi]_{\boldsymbol{S}}\right) = \mu\left([\varphi]_{\boldsymbol{S}}\right) + \mu\left(\boldsymbol{P} \setminus [\varphi]_{\boldsymbol{S}}\right)$$
$$= [\![\varphi]\!]_{\boldsymbol{S}} + \mu\left(\boldsymbol{P} \setminus [\varphi]_{\boldsymbol{S}}\right) = 1 + \mu\left(\boldsymbol{P} \setminus [\varphi]_{\boldsymbol{S}}\right)$$

  and thus $\mu(\boldsymbol{P} \setminus [\varphi]_{\boldsymbol{S}}) = 0$. Then it must be the case that $\boldsymbol{P} \setminus [\varphi]_{\boldsymbol{S}} = \emptyset$ because if there were some $s \in \boldsymbol{P} \setminus [\varphi]_{\boldsymbol{S}}$ we would, with Proposition 5.1.2, arrive at the contradictory statement $0 = \mu(\boldsymbol{P} \setminus [\varphi]_{\boldsymbol{S}}) \geq \mu(\{s\}) = \mu(s) > 0$ as $\boldsymbol{S}$ is a positive precisification space. This means that $[\varphi]_{\boldsymbol{S}} = \boldsymbol{P}$ and we get $[\![\mathrm{S}\varphi]\!]_{\boldsymbol{S}} = \mu([\varphi]_{\boldsymbol{S}}) = \mu(\boldsymbol{P}) = 1 = \lfloor [\![\varphi]\!]_{\boldsymbol{S}} \rfloor$.

□

We now try to extend the previous result. The fact that all the bounds are available as truth functions in Łukasiewicz logic suggests that a stronger property should hold in SŁ. Due to the normal form of Theorem 5.2.5 we concentrate on formulas where each subformula inside an S-operator is a disjunction of classical literals.

Definition 5.3.3. For a formula φ that is a disjunction of classical literals we denote the formula where all occurrences of the disjunction sign $\vee$ are replaced by the strong disjunction sign $\veebar$ by $\varphi^{\veebar}$.

Lemma 5.3.4. *Let $\boldsymbol{S}$ be a precisification space and φ a formula that is a disjunction of classical literals. Then we have*

$$\|\varphi\|^{Ł}_{\boldsymbol{S}} \leq [\![\varphi]\!]_{\boldsymbol{S}} \leq \|\varphi^{\veebar}\|^{Ł}_{\boldsymbol{S}}$$

Proof. We know that φ is a disjunction of classical literals, i.e., $\varphi = l_1 \vee \ldots \vee l_n$. We prove the claim by induction on the number of literals n. We consider two base cases, $n = 0$ and $n = 1$. For $n = 0$, φ is an empty disjunction that has the truth value 0 in all three cases. For $n = 1$, we have two cases, either $l_1 = p$ or $l_1 = \neg p$ for a propositional variable p, which are both easy to check:

$$\llbracket p \rrbracket_{\boldsymbol{S}} = \|p\|^{Ł}_{\boldsymbol{S}} = \|p^{\veebar}\|^{Ł}_{\boldsymbol{S}}$$
$$\llbracket \neg p \rrbracket_{\boldsymbol{S}} = 1 - \llbracket p \rrbracket_{\boldsymbol{S}} = 1 - \|p\|^{Ł}_{\boldsymbol{S}} = \|\neg p\|^{Ł}_{\boldsymbol{S}} = \|(\neg p)^{\veebar}\|^{Ł}_{\boldsymbol{S}}$$

Now we prove the induction step: Let φ be a disjunction of $n+1$ classical literals, i.e. $\varphi = l_1 \vee \psi$ where ψ is a disjunction of n classical literals. For both bounds we apply the induction hypothesis on l_1 and ψ.

$$\llbracket \varphi \rrbracket_{\boldsymbol{S}} = \llbracket l_1 \vee \psi \rrbracket_{\boldsymbol{S}} \geq \max\left(\llbracket l_1 \rrbracket_{\boldsymbol{S}}, \llbracket \psi \rrbracket_{\boldsymbol{S}}\right) \geq \max\left(\|l_1\|^{Ł}_{\boldsymbol{S}}, \|\psi\|^{Ł}_{\boldsymbol{S}}\right) = \|l_1 \vee \psi\|^{Ł}_{\boldsymbol{S}}$$
$$\llbracket \varphi \rrbracket_{\boldsymbol{S}} = \llbracket l_1 \vee \psi \rrbracket_{\boldsymbol{S}} \leq \llbracket l_1 \rrbracket_{\boldsymbol{S}} + \llbracket \psi \rrbracket_{\boldsymbol{S}} \leq \|l_1^{\veebar}\|^{Ł}_{\boldsymbol{S}} + \|\psi^{\veebar}\|^{Ł}_{\boldsymbol{S}}$$

Since also $\llbracket \varphi \rrbracket_{\boldsymbol{S}} \leq 1$ we get

$$\llbracket \varphi \rrbracket_{\boldsymbol{S}} \leq \min\left(\|l_1^{\veebar}\|^{Ł}_{\boldsymbol{S}} + \|\psi^{\veebar}\|^{Ł}_{\boldsymbol{S}}, 1\right) = \|l_1^{\veebar} \veebar \psi^{\veebar}\|^{Ł}_{\boldsymbol{S}} = \|\varphi^{\veebar}\|^{Ł}_{\boldsymbol{S}}$$

□

With these bounds we can now easily prove the validity of a certain formula.

Proposition 5.3.5. *For all classical literals $l_1, \ldots, l_n$ the formula*

$$\mathrm{S}(l_1 \vee \ldots \vee l_n) \supset (l_1 \veebar \ldots \veebar l_n)$$

is valid in SŁ.

This statement does not hold for SG.

Proof. Define φ as $l_1 \vee \ldots \vee l_n$. Let $\boldsymbol{S}$ be an arbitrary precisification space. By combining the bounds of Proposition 5.3.2 and Lemma 5.3.4 we get

$$\|\mathrm{S}\varphi\|_{\boldsymbol{S}} = \llbracket \mathrm{S}\varphi \rrbracket_{\boldsymbol{S}} \leq \lfloor \llbracket \varphi \rrbracket_{\boldsymbol{S}} \rfloor \leq \llbracket \varphi \rrbracket_{\boldsymbol{S}} \leq \|\varphi^{\veebar}\|^{Ł}_{\boldsymbol{S}} = \|l_1 \veebar \ldots \veebar l_n\|^{Ł}_{\boldsymbol{S}}$$

due to which $\|\mathrm{S}(l_1 \vee \ldots \vee l_n) \supset (l_1 \veebar \ldots \veebar l_n)\|^{Ł}_{\boldsymbol{S}} = 1$.

To prove the second part we construct a simple counterexample. Pick $l_1 = \neg p$ and $l_2 = \neg q$ and consider the precisification space $\boldsymbol{S}$ with probability measure μ and two precisifications s and t such that

- $\|p\|_{s,\boldsymbol{S}} = 1$, $\|q\|_{s,\boldsymbol{S}} = 0$, and $\mu(s) = 0.5$
- $\|p\|_{s,\boldsymbol{S}} = 0$, $\|q\|_{s,\boldsymbol{S}} = 1$, and $\mu(s) = 0.5$

Then clearly $\|\mathrm{S}(\neg p \vee \neg q)\|_{\boldsymbol{S}} = 1$ and $\|p\|_{\boldsymbol{S}} = \|q\|_{\boldsymbol{S}} = 0.5$. The formula $\neg p \veebar \neg q$ is an abbreviation for $\neg(\neg\neg p \,\&\, \neg\neg q)$ and we have

$$\|\neg(\neg\neg p \,\&\, \neg\neg q)\|^{\mathrm{G}}_{\boldsymbol{S}} = \|\neg(\neg\bar{0} \,\&\, \neg\bar{0})\|^{\mathrm{G}}_{\boldsymbol{S}} = \|\neg(\bar{1} \,\&\, \bar{1})\|^{\mathrm{G}}_{\boldsymbol{S}} = \|\neg\bar{1}\|^{\mathrm{G}}_{\boldsymbol{S}} = 0$$

Therefore $\|\mathrm{S}(\neg p \vee \neg q) \supset (\neg p \vee \neg q)\|^{\mathrm{G}}_{\boldsymbol{S}} = 0$. □

5.4 Validity in positive and uniform precisification spaces

In this section, we consider two very natural restrictions that can be imposed on precisification spaces. The first one is the restriction to positive precisification spaces that we have introduced with Definition 5.1.6. Another natural restriction that we consider is giving each precisification equal weight. Under this restriction, the local truth value of a propositional variable can be simply determined by counting the number of precisifications at which it is true.

Definition 5.4.1. A finite precisification space $\boldsymbol{S}$ with probability measure μ is *uniform* iff $\mu(s) = \mu(T)$ for all $s \in \boldsymbol{S}$. In such a case, μ is called a *uniform* probability measure.

Proposition 5.4.2. *Let $\boldsymbol{S}$ be a uniform precisification space with a finite set of precisifications $\boldsymbol{P}$ and probability measure μ. Then the following holds:*

- $\mu(s) = \frac{1}{|\boldsymbol{P}|}$ *for every $s \in \boldsymbol{S}$.*
- $\|p\|_{\boldsymbol{S}} = \frac{|[p]_{\boldsymbol{S}}|}{|\boldsymbol{P}|}$ *for every propositional variable p.*

Proof. By the definition of a probability measure we know that $\sum_{t \in \boldsymbol{P}} \mu(T) = 1$. Since the measures of all precisifications in $\boldsymbol{P}$ are equal we know that $\mu(s) = \mu(T)$ for every $t \in \boldsymbol{P}$ and get $\sum_{t \in \boldsymbol{P}} \mu(s) = 1$. Since $\boldsymbol{P}$ is a finite set we conclude $|\boldsymbol{P}| \cdot \mu(s) = 1$ and because $\boldsymbol{P}$ is nonempty we get $\mu(s) = 1/|\boldsymbol{P}|$.

The second claim then easily follows:

$$\|p\|_{\boldsymbol{S}} = \mu\left([p]_{\boldsymbol{S}}\right) = \sum_{s \in [p]_{\boldsymbol{S}}} \mu(s) = \sum_{s \in [p]_{\boldsymbol{S}}} \frac{1}{|\boldsymbol{P}|} = \left|[p]_{\boldsymbol{S}}\right| \cdot \frac{1}{|\boldsymbol{P}|}.$$

□

Based on these concepts we now define two restricted form of validity.

Definition 5.4.3. Let $*$ be a continuous t-norm and φ a formula. We call φ *p-valid* in S$*$ iff $\|\varphi\|^*_{\boldsymbol{S}} = 1$ for every positive precisification space $\boldsymbol{S}$ and we call φ *u-valid* in S$*$ iff $\|\varphi\|^*_{\boldsymbol{S}} = 1$ for every uniform precisification space $\boldsymbol{S}$.

In the context of p-validity and u-validity we sometimes refer to our unrestricted notion of validity (see Definition 5.1.4) as general validity.

Proposition 5.4.4. *If a formula is generally valid then it is also p-valid and if it is p-valid then it is also u-valid.*

Proof. This follows just from the definitions and the fact that every uniform probability measure is positive. □

In the rest of this section we study the relationship between validity, p-validity and u-validity for different logics.

5.4.1 Equivalence of validity and u-validity in SŁ

The first result that we will see is that in SŁ we can reverse Proposition 5.4.4 Our three variants of validity then coincide, which has to do with the fact that the residuum of the Łukasiewicz t-norm is continuous (see Proposition 3.4.17).

Proposition 5.4.5. *If a formula φ is u-valid in* SŁ*, then φ is also generally valid in* SŁ*.*

For the proof of this proposition we need the following technical lemma which will be proved directly after the proof of Proposition 5.4.5.

Lemma 5.4.6. *Let $\boldsymbol{S}$ be a finite precisification space with probability measure μ such that $\mu(s) \in \mathbb{Q}^{>0}$ for every $s \in \boldsymbol{S}$. Then there is a uniform precisification space $\boldsymbol{S}'$ such that $\|\varphi\|_{\boldsymbol{S}} = \|\varphi\|_{\boldsymbol{S}'}$ for every formula φ.*

Proof of Proposition 5.4.5. The overall structure of the proof is as follows:

- From Lemma 5.4.6 we conclude that u-validity implies truth in all precisification spaces where the probability measure only gives positive, rational measures to precisifications.
- With this in mind, we construct a sequence of precisification spaces with positive rational probability measures in which φ is valid such that the limit of the probability measures approaches an arbitrary real-valued probability measure.
- Finally, we use the fact that the residuum of the Łukasiewicz t-norm is continuous and conclude that if φ is valid "in the limit", it is valid.

Let φ be a formula that is u-valid in SŁ and let $\boldsymbol{S}$ be a precisification space with probability measure μ and a finite number of precisifications $\boldsymbol{P} = \{s_1, \ldots, s_n\}$, which is sufficient due to Corollary 5.1.11. We have to show that $\|\varphi\|^{\text{Ł}}_{\boldsymbol{S}} = 1$.

We define the vector $\vec{\mu} = (\mu_1, \ldots, \mu_n) = (\mu(s_1), \ldots, \mu(s_n))$ which means that $\mu_1, \ldots, \mu_n$ are real numbers that add up to 1. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ [97], there is a sequence of rational numbers $q_i^{(1)}, q_i^{(2)}, \ldots$ such that $\lim_{j\to\infty} q_i^{(j)} = \mu_i$ for every $1 \le i \le n$. If $\mu_i = 0$, then $\lim_{k\to\infty} 1/k = 0 = \mu_i$. Without loss of generality we may assume that $q_i^{(j)} > 0$ for $1 \le i \le n$ and $j \ge 1$. In vector notation, we have $\lim_{j\to\infty} \vec{q}^{(j)} = \vec{\mu}$ where $\vec{q}^{(j)} = (q_1^{(j)}, \ldots, q_n^{(j)})$ for $j \ge 1$.

The problem with $\vec{q}^{(j)}$ is that its components need not necessarily add up to 1. We fix this by defining a sequence $r_i^{(1)}, r_i^{(2)}, \ldots$ for $1 \le i \le n$ by

$$r_i^{(j)} = \frac{q_i^{(j)}}{\sum_{i=1}^n q_i^{(j)}}$$

for $j \ge 1$. Then, for $j \ge 1$, we get that $q_i^{(j)}$ is a rational number such that $0 < r_i^{(j)} \le 1$ and

$$\sum_{i=1}^n r_i^{(j)} = \sum_{i=1}^n \frac{q_i^{(j)}}{\sum_{i'=1}^n q_{i'}^{(j)}} = \frac{1}{\sum_{i'=1}^n q_{i'}^{(j)}} \sum_{i=1}^n q_i^{(j)} = 1\,.$$

We now apply the well-known rules for computing limits of sums and quotients [97] and get

$$\lim_{j\to\infty} r_i^{(j)} = \lim_{j\to\infty} \frac{q_i^{(j)}}{\sum_{i=1}^n q_i^{(j)}} = \frac{\lim_{j\to\infty} q_i^{(j)}}{\sum_{i=1}^n \lim_{j\to\infty} q_i^{(j)}} = \frac{\mu_i}{\sum_{i=1}^n \mu_i} = \frac{\mu_i}{1} = \mu_i\,.$$

In vector notation, we have $\lim_{j\to\infty} \vec{r}^{(j)} = \vec{\mu}$ where $\vec{r}^{(j)} = (r_1^{(j)}, \ldots, r_n^{(j)})$ for $j \geq 1$.

For every vector of real numbers $\vec{x} = (x_1, \ldots, x_n)$ such that $x_1 + \cdots + x_n = 1$ we define the precisification space $\boldsymbol{S}_{\vec{x}}$ as having the same set of precisifications $\boldsymbol{P}$ as $\boldsymbol{S}$ together with the same local truth values and a probability measure $\mu_{\vec{x}}$ that we define as $\mu_{\vec{x}}(s_i) = x_i$ for each $s_i \in \boldsymbol{P}$. Furthermore, we want to define a certain evaluation function $f_\varphi(\vec{x})$ that depends on our initial formula φ. For this purpose, we first define the following constants for every propositional variable p and every formula ψ where $1 \leq i \leq n$:

$$m_{p,i} = \begin{cases} 1 & \text{if } \|p\|_{s_i,\boldsymbol{S}} = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$s_\psi = \begin{cases} 1 & \text{if } \|\psi\|_{s,\boldsymbol{S}} = 1 \text{ for every } s \in \boldsymbol{P} \\ 0 & \text{otherwise}\,. \end{cases}$$

Then we inductively define a function $f_{\varphi^*}(\vec{x})$ for every formula φ^* which also gives us the desired function $f_\varphi(\vec{x})$:

$$\begin{aligned} f_{\bar{0}}\left(\vec{x}\right) &= 0 \\ f_p\left(\vec{x}\right) &= \sum_{i=1}^n m_{p,i} \cdot x_i \text{ for atomic } p \\ f_{\mathrm{S}\psi}\left(\vec{x}\right) &= s_\psi \\ f_{\psi\&\chi}\left(\vec{x}\right) &= f_\psi\left(\vec{x}\right) *_{\text{Ł}} f_\chi\left(\vec{x}\right) \\ f_{\psi\supset\chi}\left(\vec{x}\right) &= f_\psi\left(\vec{x}\right) \Rightarrow_{\text{Ł}} f_\chi\left(\vec{x}\right)\,. \end{aligned}$$

It is easy to see that

$$f_\varphi\left(\vec{x}\right) = \|\varphi\|_{\boldsymbol{S}_{\vec{x}}}\,.$$

Furthermore, f_φ is a continuous function because $*_{\text{Ł}}$, $\Rightarrow_{\text{Ł}}$, addition and multiplication are continuous functions.

By Lemma 5.4.6 we know, for each $j \geq 1$, that for the precisification space $\boldsymbol{S}_{\vec{r}^{(j)}}$ there is a uniform precisification space $\boldsymbol{S}'_{\vec{r}^{(j)}}$ such that $\|\varphi\|_{\boldsymbol{S}_{\vec{r}^{(j)}}} = \|\varphi\|_{\boldsymbol{S}'_{\vec{r}^{(j)}}}$. Since φ is u-valid by assumption we have $\|\varphi\|_{\boldsymbol{S}'_{\vec{r}^{(j)}}} = 1$.

Plugging together these results and using the fact that f is continuous [97], we get

$$\begin{aligned} \|\varphi\|_{\boldsymbol{S}} = \|\varphi\|_{\boldsymbol{S}_{\vec{\mu}}} = f_\varphi\left(\vec{\mu}\right) &= \lim_{j\to\infty} f_\varphi\left(\vec{r}^{(j)}\right) \\ &= \lim_{j\to\infty} \|\varphi\|_{\boldsymbol{S}_{\vec{r}^{(j)}}} = \lim_{j\to\infty} \|\varphi\|_{\boldsymbol{S}'_{\vec{r}^{(j)}}} = \lim_{j\to\infty} 1 = 1 \end{aligned}$$

Since $\boldsymbol{S}$ was an arbitrary finite precisification space we conclude that φ is valid. □

Proof of Lemma 5.4.6. Let $\boldsymbol{S}$ be a precisification space with a finite set of precisifications $\boldsymbol{P} = \{s_1, \dots, s_m\}$ and a probability measure μ such that $\mu(s) \in \mathbb{Q}^{>0}$ for every $s \in \boldsymbol{P}$. Then for every $1 \leq i \leq m$, there are natural numbers k_i and n_i such that $\mu(s_i) = k_i/n_i$ and $0 < k_i \leq n_i$.

Now we define a set of precisifications $\boldsymbol{P}'$ that contains $N = n_1 \cdot n_2 \cdot \ldots \cdot n_m$ precisifications by "duplicating" precisifications of $\boldsymbol{P}$. For this purpose we define the function d for duplicating precisifications by

$$d\left(s_i\right) = \left\{s_i^{(1)}, \dots, s_i^{(N \cdot k_i/n_i)}\right\}$$

and define the set $\boldsymbol{P}'$ as

$$\boldsymbol{P}' = \bigcup_{1 \leq i \leq m} d\left(s_i\right) = \left\{s_1^{(1)}, \dots, s_1^{(N \cdot k_1/n_1)}, \dots, s_m^{(1)}, \dots, s_m^{(N \cdot k_m/n_m)}\right\}$$

Then we define the uniform precisification space $\boldsymbol{S}'$ as having $\boldsymbol{P}'$ as its set of precisifications and the uniform probability measure μ' on $\boldsymbol{P}'$. The local truth values of the precisifications are given by $\|p\|_{s_i^{(j)},\boldsymbol{S}'} = \|p\|_{s_i,\boldsymbol{S}}$ for every propositional variable, $1 \leq j \leq N \cdot k_i/n_i$ and $1 \leq i \leq m$. By our definitions, for every precisification in $s \in \boldsymbol{S}$ there is a precisification $s' \in \boldsymbol{S}'$ that gives the same local truth values to propositional variables, and vice versa. Now let p be a propositional variable and let $s_{i_1}, \dots, s_{i_l}$ denote the precisifications of $\boldsymbol{S}$ at which the local truth value of p is 1, i.e.,

$$\left\{s_{i_1}, \dots s_{i_l}\right\} = [p]_{\boldsymbol{S}}\,.$$

Then we have

$$\begin{aligned}
\|p\|_{\boldsymbol{S}'} &= \mu'\left(\left\{s \in \boldsymbol{S}' \mid \|p\|_{s,\boldsymbol{S}} = 1\right\}\right)\\
&= \mu'\left(d\left(s_{i_1}\right) \cup \ldots \cup d\left(s_{i_l}\right)\right)\\
&= \mu'\left(d\left(s_{i_1}\right)\right) + \cdots + \mu'\left(d\left(s_{i_l}\right)\right)\\
&= \mu'\left(\left\{s_{i_1}^{(1)}, \dots, s_{i_1}^{(N \cdot k_{i_1}/n_{i_1})}\right\}\right) + \cdots + \mu'\left(\left\{s_{i_l}^{(1)}, \dots, s_{i_l}^{(N \cdot k_{i_l}/n_{i_l})}\right\}\right)\\
&= \frac{N \cdot k_{i_1}/n_{i_1}}{N} + \cdots + \frac{N \cdot k_{i_l}/n_{i_l}}{N}\\
&= \frac{k_{i_1}}{n_{i_1}} + \cdots + \frac{k_{i_l}}{n_{i_l}}\\
&= \mu\left(s_{i_1}\right) + \cdots + \mu\left(s_{i_l}\right)\\
&= \mu\left(\left\{s_{i_1}, \dots s_{i_l}\right\}\right)\\
&= \mu\left([p]_{\boldsymbol{S}}\right) = \|p\|_{\boldsymbol{S}}\,.
\end{aligned}$$

Thus by Proposition 5.1.8 and Proposition 5.1.9, we have $\|\varphi\|_{\boldsymbol{S}} = \|\varphi\|_{\boldsymbol{S}'}$ for every formula φ. □

Theorem 5.4.7. *Let φ be a formula. Then the following propositions are equivalent:*

(i) *φ is valid in* SŁ.

(ii) *φ is p-valid in* SŁ.

(iii) *φ is u-valid in* SŁ.

Proof. (i) ⇒ (ii) and (ii) ⇒ (iii) are Proposition 5.4.4 and (iii) ⇒ (i) is Proposition 5.4.5. □

The equivalence between validity and u-validity in SŁ has the further advantage that validity has been reduced to a finitary notion because the possibility of real-valued probability measures has been eliminated.

The natural question to ask now is whether the equivalences of the above theorem also hold for S$*$ when $*$ is not isomorphic to the Łukasiewicz t-norm. For every continuous t-norm that is not isomorphic to the Łukasiewicz t-norm, validity and p-validity in S$*$ do not coincide, as we will show next. We also show that p-validity and u-validity coincide for the Gödel t-norm. The question whether p-validity and u-validity coincide for continuous t-norms different from the Łukasiewicz t-norm and the Gödel t-norm remains open.

5.4.2 Characterization of the equivalence of validity and p-validity

For every continuous t-norm $*$ that is not isomorphic to the Łukasiewicz t-norm we want to find a counterexample formula that is p-valid in S$*$ but not valid in S$*$. Remember that a continuous t-norm is isomorphic to the Łukasiewicz t-norm if and only if its residuum is continuous (see Proposition 3.4.17). This means that we have to find a counterexample for every continuous t-norm with a non-continuous residuum. An important class of continuous t-norms with non-continuous residua are the continuous t-norms that have Gödel negation as their precomplement, as for example the Gödel t-norm and the product t-norm (see Proposition 3.2.15). Our strategy is to distinguish between those continuous t-norms that have Gödel negation as their precomplement and those that have not. For the first case it is relatively easy to find a counterexample. The second case needs a more involved analysis. There we exploit the fact that all such t-norms "start" with an isomorphic copy of the Łukasiewicz t-norm in the generalized ordinal sum representation (see Theorem 3.2.4).

Lemma 5.4.8. *If the precomplement $-_*$ of a continuous t-norm $*$ is equal to Gödel negation $-_G$, then general validity and p-validity do* not *coincide in* S$*$.

Proof. The main idea is that Gödel negation allows us to check whether the truth value of a formula is greater than 0. For positive precisification spaces, we can enforce that a propositional variable p receives a truth value greater than 0. Let $*$ be a continuous t-norm with Gödel negation and define the formula φ as

$$(\neg \mathrm{S}\neg p) \supset (\neg\neg p)\,.$$

We refer to $\neg\mathrm{S}\neg p$ as the antecedent of φ and to $\neg\neg p$ as the succedent of φ.

Gödel negation is the unary truth function given by

$$-_{\mathrm{G}}(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise}. \end{cases}$$

A twofold application of this function gives

$$-_{\mathrm{G}}\left(-_{\mathrm{G}}(x)\right) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{otherwise}. \end{cases}$$

Thus, we can define the projection operator $\triangledown$ with the help of Gödel negation (compare Section 3.6.1). This means that

$$\|\neg\neg p\|^*_{\boldsymbol{S}} = -_{\mathrm{G}}\left(-_{\mathrm{G}}\left(\|p\|_{\boldsymbol{S}}\right)\right) = \lceil \|p\|_{\boldsymbol{S}} \rceil$$

for every precisification space $\boldsymbol{S}$.

We first show that φ is p-valid. Let $\boldsymbol{S}$ be an arbitrary positive precisification space with probability measure μ. The antecedent of φ always has the truth value 1 or 0. If its truth value is 0, then φ trivially has the truth value 1. Assume now that the antecedent of φ has the truth value 1. Then due to $\|\neg\mathrm{S}\neg p\|_{\boldsymbol{S}} = 1$ we know that there is a precisification $s \in \boldsymbol{S}$ such that $\|p\|_{s,\boldsymbol{S}} = 1$. Since $\mu(s) > 0$ and $s \in [p]_{\boldsymbol{S}}$, we get $\|p\|_{\boldsymbol{S}} = \sum_{t\in[p]_{\boldsymbol{S}}} \mu(T) \geq \mu(s) > 0$. Therefore $\|\neg\neg p\|^*_{\boldsymbol{S}} = \lceil \|p\|_{\boldsymbol{S}} \rceil = 1$ which means that the succedent of φ is true. Since both the antecedent and the succedent of φ have the truth value 1, also φ has the truth value 1. Because $\boldsymbol{S}$ was an arbitrary positive precisification space, φ is p-valid.

Finally, we show that φ is not generally valid. Consider the precisification space $\boldsymbol{S}$ consisting of two precisifications s_1 and s_2 with a probability measure μ given by $\mu(s_1) = 1$ and $\mu(s_2) = 0$. We define the interpretation of the propositional variable p in the precisifications as follows: $\|p\|_{s_1,\boldsymbol{S}} = 0$ and $\|p\|_{s_2,\boldsymbol{S}} = 1$. Then $\|p\|_{\boldsymbol{S}} = 0$ and thus we have $\|\neg\neg p\|^*_{\boldsymbol{S}} = \|\neg\neg\bar{0}\|^*_{\boldsymbol{S}} = 0$ for the succedent of φ because t-norm based connectives behave classically for the truth value set $\{0, 1\}$. Clearly, the antecedent of φ has the truth value 1 because $\|\mathrm{S}\neg p\|_{\boldsymbol{S}} = 0$ due to $\|p\|_{s_2,\boldsymbol{S}} = 1$. Therefore $\|\varphi\|^*_{\boldsymbol{S}} = 0$ and thus φ is not generally valid in S$*$ for *any* continuous t-norm $*$. □

Due to Theorem 3.2.4 every continuous t-norm can be represented by a generalized ordinal sum of isomorphic copies of Łukasiewicz or the product t-norm. In the following we need the property that this representation carries over to the residuum of the t-norm. We provide a proof of this fact and then go on with finding our counterexample for the remaining cases.

Lemma 5.4.9. *If f is an order isomorphism between $[a_1, a_2] \subseteq [0, 1]$ and $[b_1, b_2] \subseteq [0, 1]$, then $f(a_1) = b_1$ and $f(a_2) = b_2$.*

Proof. Suppose that $f(a_1) > b_1$. Then by the definition of an order isomorphism we get $a_1 = f^{-1}(f(a_1)) > f^{-1}(b_1) \in [a_1, a_2]$ which contradicts the fact that a_1 is the maximal element of $[a_1, a_2]$. If we suppose that $f(a_2) < b_2$, then we arrive at the contradictory statement $a_2 = f^{-1}(f(a_2)) < f^{-1}(b_2) \in [a_1, a_2]$. □

Lemma 5.4.10. *Let* $[a, b] \subseteq [0, 1]$ *be a subinterval of the unit interval,* $f : [a, b] \to [0, 1]$ *an order isomorphism, and* $*$ *and* $\circ$ *continuous t-norms such that*

$$x * y = \begin{cases} f^{-1}(f(x) \circ f(y)) & \textit{if } x, y \in [a, b] \\ \min(x, y) & \textit{otherwise}. \end{cases}$$

Then for all x, y *with* $a \le y < x \le b$ *we have*

$$\left(x \Rightarrow_* y\right) = f^{-1}\left(f(x) \Rightarrow_\circ f(y)\right) .$$

holds where $\Rightarrow_*$ *is the residuum of* $*$ *and* $\Rightarrow_\circ$ *is the residuum of* $\circ$.

Proof. We define $w = (x \Rightarrow_* y)$, $v = (f(x) \Rightarrow_\circ f(y))$, and $w' = f^{-1}(v)$. By the definition of residua we have

$$w = \max\{z \in [0, 1] \mid x * z \le y\}$$

and

$$v = \max\{z \in [0, 1] \mid f(x) \circ z \le f(y)\} .$$

First of all, we verify that w' is contained in the set $\{z \in [0, 1] \mid x * z \le y\}$.

$$\begin{aligned} x * w' &= x * \left(f^{-1}(v)\right) \\ &= f^{-1}\left(f(x) \circ f\left(f^{-1}(v)\right)\right) \\ &= f^{-1}(f(x) \circ v) \end{aligned}$$

Since $f(x) \circ v \le f(y)$ and f is an order isomorphism we get

$$x * w' = f^{-1}(f(x) \circ v) \le f^{-1}(f(y)) = y .$$

Now we show that $w \in [a, b]$. Suppose first that $w > b$. Then we get the contradictory inequality

$$y \ge x * w = \min(x, w) = x > y .$$

Now suppose that $w < a$. Then $x * w = \min(x, w) = w$. We also have

$$\begin{aligned} x * a &= f^{-1}(f(x) \circ f(a)) \\ &= f^{-1}(f(x) \circ 0) \\ &= f^{-1}(0) \\ &= a \end{aligned}$$

Then, $x * a = a \leq y$ and furthermore $x * a = a > w = x * w$ which contradicts the maximality of w as the residuum $x \Rightarrow_* y$. Thus we conclude that $w \in [a, b]$.

Suppose that $w > w' = f^{-1}(v)$. Since $w \in [a, b]$ and f is an order isomorphism we get

$$f(w) > f\left(f^{-1}(v)\right) = v\,.$$

Because w is the residuum $x \Rightarrow_* y$ we have $x * w \leq y$ and since f is an order isomorphism we get $f(x * w) \leq f(y)$. Thus, we have

$$f(x) \circ f(w) = f(x * w) \leq f(y)\,.$$

But since $f(w) > v$, this statement contradicts the maximality of v as the residuum $f(x) \Rightarrow_\circ f(y)$. Therefore we know that our assumption $w > w'$ was wrong which means that the residuum candidate w' is in fact the residuum $x \Rightarrow_* y$. □

Lemma 5.4.11. *Let $*$ be a continuous t-norm such that the residuum $\Rightarrow_*$ is* not *continuous. If the precomplement $-_*$ is* not *Gödel negation, then validity and p-validity do* not *coincide in* S$*$.

Proof. If $*$ were an isomorphic copy of the product t-norm on the first interval $[0, u]$ (with $u \neq 0$) in the generalized ordinal sum representation (see Theorem 3.2.4), then the precomplement $-_*$ would be Gödel negation. Therefore it must be the case that $*$ is an isomorphic copy of the Łukasiewicz t-norm on $[0, u]$. Furthermore it must be the case that $u < 1$ because otherwise $*$ would be isomorphic to the Łukasiewicz t-norm on the complete unit interval and therefore continuous.

We can now define a formula φ that is p-valid but not generally valid. As in the previous proof, the main idea is that for positive precisification spaces there is a simple way to enforce that p receives a truth value greater than 0 that does not work in general. Define φ as the following formula:

$$(\neg \mathrm{S} \neg p) \supset (\neg\neg q \supset (\neg p \supset q))\,.$$

Then $\neg \mathrm{S} \neg p$ is the antecedent of φ and $\neg\neg q \supset (\neg p \supset q)$ is the succedent of φ.

We first show that φ is p-valid. Let $\boldsymbol{S}$ be an arbitrary positive precisification space. The antecedent of φ either has the truth value 0 or to 1 in $\boldsymbol{S}$. If it has the truth value 0, there is nothing to show because φ is trivially true. Assume now that the antecedent of φ has the truth value 1. Then we have $\|\neg \mathrm{S} \neg p\|_{\boldsymbol{S}} = 1$ and furthermore $\|p\|_{\boldsymbol{S}} > 0$ because $\boldsymbol{S}$ is a positive precisification space. We now have to show that the succedent of φ also has the truth value 1.

Consider first the case that $\|p\|_{\boldsymbol{S}} > u$. We now want to calculate $\|\neg p\|_{\boldsymbol{S}}$. By the definition of the residuum of $*$ we have

$$\|\neg p\|^*_{\boldsymbol{S}} = \|p \supset \bar{0}\|^*_{\boldsymbol{S}} = \left(\|p\|_{\boldsymbol{S}} \Rightarrow_* 0\right) = \max\left\{z \in [0, 1] \mid \|p\|_{\boldsymbol{S}} * z \leq 0\right\}\,.$$

By the generalized ordinal sum representation, $\|p\|_{\boldsymbol{S}}$ lies in an interval $[a, b]$ such that the continuous t-norm $*$ restricted to $[a, b]$ is an isomorphic copy of either Łukasiewicz or the product t-norm. Because the intervals of this representation do not overlap and $\|p\|_S > u$ we know that $a \geq u$. If $z \in [a,b]$, then also $\|p\|_S * z \in [a,b]$ and therefore $\|p\|_S * z \geq u > 0$. If $z \notin [a,b]$ and $z > 0$, then $\|p\|_S * z = \min(\|p\|_S, z) > 0$ because $\|p\|_S > u > 0$. This leaves the case $z = 0$ for which we get $\|p\|_S * z = 0$. This means that $\|\neg p\|^*_S = 0$ and therefore $\|\neg p \supset q\|^*_S = 1$ and $\|\neg\neg q \supset (\neg p \supset p)\|^*_S = 1$.

Consider now the case that $\|p\|_S \leq u$. We have to distinguish two subcases: either $\|q\|_S \geq u$ or $\|q\|_S < u$. Assume that $\|q\|_S \geq u$. Since $0 < \|p\|_S \leq u$ we know by Lemma 5.4.10 that $\|\neg p\|^*_S = \|p \supset \bar{0}\|^*_S \in [0,u]$. This gives $\|\neg p\|^*_S \leq u \leq \|q\|_S$. Therefore $\|\neg p \supset q\|^*_S = 1$ and thus $\|\neg\neg q \supset (\neg p \supset p)\|^*_S = 1$.

Assume that $\|q\|_S < u$. If $\|q\|_S = 0$, then $\|\neg\neg q\|^*_S = \|\neg\neg\bar{0}\|^*_S = \|\neg\bar{1}\|^*_S = 0$ and therefore $\|\neg\neg q \supset (\neg p \supset p)\|^*_S = 1$. Thus we assume in the following that $q > 0$. If $\|\neg p\|^*_S \leq \|q\|_S$, then $\|\neg p \supset q\|^*_S = 1$ and therefore $\|\neg\neg q \supset (\neg p \supset p)\|^*_S = 1$. Hence we assume in the following that $\|\neg p\|^*_S > \|q\|_S$.

Because $0 \leq p \leq u$ and $0 \leq q \leq u$, we now apply Lemma 5.4.10 several times to calculate the truth value of $\neg\neg q \supset (\neg p \supset p)$. Let f denote the order isomorphism between $[0,u]$ and $[0,1]$ of the generalized ordinal sum representation. First of all, since $\|p\|_S > 0$, we have

$$\begin{aligned}\|\neg p\|^*_S = \|p \supset \bar{0}\|^*_S &= f^{-1}\left(\min\left(1 - f\left(\|p\|_S\right) + f(0), 1\right)\right)\\ &= f^{-1}\left(\min\left(1 - f\left(\|p\|_S\right) + 0, 1\right)\right)\\ &= f^{-1}\left(\min\left(1 - f\left(\|p\|_S\right), 1\right)\right)\\ &= f^{-1}\left(1 - f\left(\|p\|_S\right)\right)\end{aligned}$$

and since $\|q\|_S > 0$ we have

$$\|\neg q\|^*_S = f^{-1}\left(1 - f\left(\|q\|_S\right)\right) .$$

Now because $\|\neg p\|^*_S > \|q\|_S$ we get

$$\begin{aligned}\|\neg p \supset q\|^*_S &= f^{-1}\left(\min\left(1 - f\left(\|\neg p\|^*_S\right) + f\left(\|q\|_S\right), 1\right)\right)\\ &= f^{-1}\left(\min\left(1 - f\left(f^{-1}\left(1 - f\left(\|p\|_S\right)\right)\right) + f\left(\|q\|_S\right), 1\right)\right)\\ &= f^{-1}\left(\min\left(1 - \left(1 - f\left(\|p\|_S\right)\right) + f\left(\|q\|_S\right), 1\right)\right)\\ &= f^{-1}\left(\min\left(f\left(\|p\|_S\right) + f\left(\|q\|_S\right), 1\right)\right) .\end{aligned}$$

Since $\|q\|_S > 0$ we get $f(\|q\|_S) > f(0)$ because f is an order isomorphism. Therefore $1 - f(\|q\|_S) > 0$ and thus $\|\neg q\|^*_S = f^{-1}(1 - f(\|q\|_S)) > f^{-1}(0) = 0$. This means that we may apply Lemma 5.4.10 again and we get

$$\begin{aligned}\|\neg\neg q\|^*_S &= f^{-1}\left(1 - f\left(\|\neg q\|^*_S\right)\right)\\ &= f^{-1}\left(1 - f\left(f^{-1}\left(1 - f\left(\|q\|_S\right)\right)\right)\right)\\ &= f^{-1}\left(1 - \left(1 - f\left(\|q\|_S\right)\right)\right)\\ &= f^{-1}\left(f\left(\|q\|_S\right)\right)\\ &= \|q\|_S .\end{aligned}$$

Since $\|p\|_{\boldsymbol{S}} > 0$ and $\|q\|_{\boldsymbol{S}} < u$ we have $f(\|p\|_{\boldsymbol{S}}) > f(0) = 0$ and $f(\|q\|_{\boldsymbol{S}}) < f(u) = 1$ because f is an order isomorphism. Therefore the inequality

$$f\left(\|q\|_{\boldsymbol{S}}\right) < \min\left(f\left(\|p\|_{\boldsymbol{S}}\right) + f\left(\|q\|_{\boldsymbol{S}}\right), 1\right)$$

holds. Since f is an order isomorphism we conclude

$$\|\neg\neg q\|^*_{\boldsymbol{S}} = f^{-1}\left(f\left(\|q\|_{\boldsymbol{S}}\right)\right) < f^{-1}\left(\min\left(f\left(\|p\|_{\boldsymbol{S}}\right) + f\left(\|q\|_{\boldsymbol{S}}\right), 1\right)\right) = \|\neg p \supset q\|^*_{\boldsymbol{S}}.$$

Therefore we get $\|\neg\neg q \supset (\neg p \supset p)\|^*_{\boldsymbol{S}} = 1$.

We have showed that $\|\neg\neg q \supset (\neg p \supset p)\|^*_{\boldsymbol{S}} = 1$ in all possible cases. This means that the succedent of φ is true which means that φ is true in $\boldsymbol{S}$. Since $\boldsymbol{S}$ was an arbitrary positive probability space, we conclude that φ is p-valid.

Finally, we show that φ is not generally valid. Consider the precisification space $\boldsymbol{S}$ consisting of three precisifications s_1, s_2 and s_3 with the probability measure μ given by $\mu(s_1) = 0$ and $\mu(s_2) = u$ and $\mu(s_3) = 1 - u$. The propositional variables are interpreted at the precisifications as follows:

$$\begin{array}{lll} \|p\|_{s_1,\boldsymbol{S}} = 1 & \|p\|_{s_2,\boldsymbol{S}} = 0 & \|p\|_{s_3,\boldsymbol{S}} = 0 \\ \|q\|_{s_1,\boldsymbol{S}} = 0 & \|q\|_{s_2,\boldsymbol{S}} = 1 & \|q\|_{s_2,\boldsymbol{S}} = 0. \end{array}$$

Then we have $\|\neg\mathrm{S}(\neg p)\|^*_{\boldsymbol{S}} = 1$ because $\|p\|_{s_1,\boldsymbol{S}} = 1$. Thus, the antecedent of φ is true. Furthermore, $\|p\|_{\boldsymbol{S}} = 0$ and $\|q\|_{\boldsymbol{S}} = u$. Then $\|\neg p\|^*_{\boldsymbol{S}} = \|\neg\bar{0}\|^*_{\boldsymbol{S}} = 1$ and we get

$$\|\neg p \supset q\|^*_{\boldsymbol{S}} = \|\bar{1} \supset q\|^*_{\boldsymbol{S}} = \|q\|_{\boldsymbol{S}} = u < 1.$$

Since $\|q\|_{\boldsymbol{S}} = u$ we get $\|\neg q\|^*_{\boldsymbol{S}} = 0$ and $\|\neg\neg q\|^*_{\boldsymbol{S}} = 1$. Thus, we get

$$\|\neg\neg q \supset (\neg p \supset q)\|^*_{\boldsymbol{S}} = \|\bar{1} \supset (\neg p \supset q)\|^*_{\boldsymbol{S}} = \|\neg p \supset q\|^*_{\boldsymbol{S}} < 1.$$

Therefore, the succedent of φ does not have the truth value 1 which means that φ does not have the truth value 1. □

Theorem 5.4.7 gave a sufficient condition for the equivalence p-validity and general validity. Our investigation showed that this condition is also necessary.

Theorem 5.4.12. *For every continuous t-norm $*$ the equivalence*

For every formula φ, φ is valid in S$*$ *if and only if φ is p-valid in* S$*$.

holds if and only if $*$ is isomorphic to the Łukasiewicz t-norm.

5.4.3 Equivalence of p-validity and u-validity in SG

In the following we give a prove that p-validity and u-validity coincide in SG. The key idea is that in Gödel logic only the order of evaluations of propositional variables is relevant. In our setting, the evaluations of propositional variables are sums of measures of precisifications. We show that every order on sums of measures that can be expressed with positive precisification spaces can also be expressed with uniform precisification spaces.

Lemma 5.4.13. *Let X be a system set of linear equations and inequalities of the form*

$$\sum_{i=1}^{n} a_{ij} \cdot x_i = 0 \text{ or } \sum_{i=1}^{n} a_{ij} \cdot x_i < 0$$

where each a_{ij} is a rational number, $1 \leq i \leq n$ and $1 \leq j \leq m$. Then X has a positive,[29] *rational solution if and only if X has a positive, real solution.*

Proof. Obviously, every rational solution of X is also a real solution of X. The paper [25] describes an algorithm for finding a positive solution of a system X of linear equations if it has one. From the constructions of the algorithm it can be seen that the found solution is rational if the coefficients of X are rational.

We denote by X' the system that is the result of replacing in X every inequality of the form $\sum_{i=1}^{n} a_{ij} \cdot x_i < 0$ by an equation $\sum_{i=1}^{n} a_i \cdot x_i + t_j = 0$ where t_j is a fresh slack variable. If X has a positive solution $x_1^*, \ldots, x_n^*$, then also X' that has a positive solution. We simply set every slack variable t_j that we introduced to $t_j^* = -\sum_{i=1}^{n} a_{ij} \cdot x_i^*$. We know that t_j^* cannot be negative or 0 because then $x_1^*, \ldots, x_n^*$ would violate the original inequality. Since X' only contains linear equations and all coefficients (including those of the slack variables) are rational, we know by the algorithm mentioned above that X' has a positive, rational solution. This solution of X' then also is a solution of X. □

Lemma 5.4.14. *Let X be a system set of linear equations and inequalities of the form*

$$\sum_{i=1}^{n} a_{ij} \cdot x_i = 0 \text{ or } \sum_{i=1}^{n} a_{ij} \cdot x_i < 0$$

where each a_{ij} is a rational number, $1 \leq i \leq n$ and $1 \leq j \leq m$. If X has a positive, rational solution, then X also has a positive, rational solution such that the constraint $\sum_{i=1}^{n} a_{ij} \cdot x_i = 1$ is fulfilled.

Proof. Let $x_1^*, \ldots, x_n^*$ be a positive, rational solution of X. We define $b = \sum_{i=1}^{n} x_i^*$. Since x_i^* is rational and $x_i^* > 0$ for every $1 \leq i \leq n$, also b is rational and $b > 0$. Now we simply define

$$x_i^\times = \frac{x_i^*}{b}$$

for every $1 \leq i \leq n$. Since x_i^* is a positive rational number for every $1 \leq i \leq n$ and a is a positive rational, also $x_i^\times$ is a positive rational number. We verify that $x_1^\times, \ldots, x_n^\times$ has the desired property:

$$\sum_{i=1}^{n} x_i^\times = \sum_{i=1}^{n} \frac{x_i^*}{b} = \frac{1}{b} \cdot \sum_{i=1}^{n} x_i^* = \frac{1}{b} \cdot b = 1$$

[29] A number is positive iff it is strictly greater than 0.

Now it can easily be checked that $x_1^\times, \dots, x_n^\times$ is indeed a solution of X. We clearly have

$$\sum_{i=1}^{n} a_{ij} \cdot x_i^\times = \sum_{i=1}^{n} a_{ij} \cdot \frac{x_i^*}{b} = \frac{1}{b} \cdot \sum_{i=1}^{n} a_{ij} \cdot x_i^* .$$

If $\sum_{i=1}^n a_{ij}x_i^* = 0$, then $\sum_{i=1}^n a_{ij} \cdot x_i^\times = \frac{1}{b} \cdot \sum_{i=1}^n a_{ij} \cdot x_i^* = \frac{1}{b} \cdot 0 = 0$. If $\sum_{i=1}^n a_{ij}x_i^* < 0$, then $\sum_{i=1}^n a_{ij} \cdot x_i^\times = \frac{1}{b} \cdot \sum_{i=1}^n a_{ij} \cdot x_i^* < \frac{1}{b} \cdot 0 = 0$ because $1/b > 0$. Therefore $x_1^\times, \dots, x_n^\times$ is indeed a solution of X. □

We can now construct from a precisification space with real, positive measures a second precisification space with rational, positive measures such that they are connected by certain conditions. We will subsequently show that these conditions are strong enough to determine the set of true formulas.

Lemma 5.4.15. *Let $\boldsymbol{S}$ be a positive precisification space with a finite set of precisifications and $\mathcal{P}$ a set of propositional variables. Then there is a precisification space $\boldsymbol{S}'$ with probability measure μ' such that $\mu'(s) \in \mathbb{Q}^{>0}$ for every $s \in \boldsymbol{S}'$ and the following conditions hold:*

- $\|\mathrm{S}\varphi\|_{\boldsymbol{S}} = \|\mathrm{S}\varphi\|_{\boldsymbol{S}'}$ *for every formula* φ
- $\|p\|_{\boldsymbol{S}} < \|q\|_{\boldsymbol{S}}$ *if and only if* $\|p\|_{\boldsymbol{S}'} < \|q\|_{\boldsymbol{S}'}$ *for all* $p, q \in \mathcal{P}$
- $\|p\|_{\boldsymbol{S}} = 1$ *if and only if* $\|p\|_{\boldsymbol{S}'} = 1$ *for every* $p \in \mathcal{P}$
- $\|p\|_{\boldsymbol{S}} = 0$ *if and only if* $\|p\|_{\boldsymbol{S}'} = 0$ *for every* $p \in \mathcal{P}$

Proof. Let $\boldsymbol{S}$ be a positive precisification space with a finite set of precisifications $\boldsymbol{P}$ and a probability measure μ. We consider a variable x_s for every $s \in \boldsymbol{P}$. For every propositional variable $p \in \mathcal{P}$ we define the linear combination L_p by

$$L_p = \sum_{s \in [p]_{\boldsymbol{S}}} x_s$$

where the sum of the empty set is 0. We define $x_s^* = \mu(s) > 0$ for every $s \in \boldsymbol{P}$. Note that, for every $p \in \mathcal{P}$ we have $\sum_{s \in [p]_{\boldsymbol{S}}} x_s^* = \|p\|_{\boldsymbol{S}}$. We define the following system of linear equations and inequalities where $p, q \in \mathcal{P}$ that we call X:

$$\begin{aligned} L_p &= L_q \text{ if } \|p\|_{\boldsymbol{S}} = \|q\|_{\boldsymbol{S}} \\ L_p &< L_q \text{ if } \|p\|_{\boldsymbol{S}} < \|q\|_{\boldsymbol{S}} \\ L_q &< L_p \text{ if } \|p\|_{\boldsymbol{S}} > \|q\|_{\boldsymbol{S}} . \end{aligned}$$

It is clear that, by subtracting the right hand sides, the system X is equivalent to a system X' that fulfills the precondition of Lemma 5.4.13. Note that in X' only rational coefficients appear. Since $(x_s^*)_{s \in \boldsymbol{P}}$ is a real, positive solution of X' we know by Lemma 5.4.13 that there exists a rational, positive solution $(x_s^\times)_{s\in\boldsymbol{P}}$ of X'. Due to Lemma 5.4.14 we may assume that $\sum_{s\in\boldsymbol{P}} x_s^\times = 1$.

We define a precisification space $\boldsymbol{S}'$ that is just like $\boldsymbol{S}$ but with a different probability measure. This means that the set of precisifications of $\boldsymbol{S}'$ is $\boldsymbol{P}$ and the local truth value of a propositional variable p, for every $s \in \boldsymbol{P}$ is $\|p\|_{s,\boldsymbol{S}'} = \|p\|_{s,\boldsymbol{S}}$. We define the probability measure μ' of $\boldsymbol{S}'$ by setting $\mu'(s) = x_s^\times$ for every $s \in \boldsymbol{P}$. Then μ' is well-defined because

$$\mu'(\boldsymbol{P}) = \sum_{s\in\boldsymbol{P}} \mu'(s) = \sum_{s\in\boldsymbol{P}} x_s^\times = 1\,.$$

Furthermore $\boldsymbol{S}'$ is a positive precisification space because, for every $s \in \boldsymbol{P}$, $\mu'(s) = x_s^\times > 0$ since $(x_s^\times)_{s\in\boldsymbol{P}}$ is a positive solution. We now show that $\boldsymbol{S}'$ has the desired properties.

Since $\boldsymbol{S}$ and $\boldsymbol{S}'$ have the same sets of precisifications with the same sets of truth values assigned to them we have $[\varphi]_{\boldsymbol{S}} = [\varphi]_{\boldsymbol{S}'}$ for every formula φ. Therefore the following equivalences hold for every formula φ:

$$\begin{aligned} \|\mathrm{S}\varphi\|_{\boldsymbol{S}} &= 1 \text{ if and only if} \\ [\varphi]_{\boldsymbol{S}} &= \boldsymbol{P} \text{ if and only if} \\ [\varphi]_{\boldsymbol{S}'} &= \boldsymbol{P} \text{ if and only if} \\ \|\mathrm{S}\varphi\|_{\boldsymbol{S}'} &= 1 \end{aligned}$$

Since, for every formula φ, $\|\mathrm{S}\varphi\|_{\boldsymbol{S}} \in \{0,1\}$ and $\|\mathrm{S}\varphi\|_{\boldsymbol{S}'} \in \{0,1\}$ we may conclude $\|\mathrm{S}\varphi\|_{\boldsymbol{S}} = \|\mathrm{S}\varphi\|_{\boldsymbol{S}'}$.

We now apply Proposition 5.1.7 and we get the equivalences

$$\begin{aligned} \|p\|_{\boldsymbol{S}} &= 1 \text{ if and only if} \\ \|\mathrm{S}p\|_{\boldsymbol{S}} &= 1 \text{ if and only if} \\ \|\mathrm{S}p\|_{\boldsymbol{S}'} &= 1 \text{ if and only if} \\ \|p\|_{\boldsymbol{S}'} &= 1 \end{aligned}$$

and

$$\begin{aligned} \|p\|_{\boldsymbol{S}} &= 0 \text{ if and only if} \\ \|\mathrm{S}\neg p\|_{\boldsymbol{S}} &= 1 \text{ if and only if} \\ \|\mathrm{S}\neg p\|_{\boldsymbol{S}'} &= 1 \text{ if and only if} \\ \|p\|_{\boldsymbol{S}'} &= 0 \end{aligned}$$

Since $[\varphi]_{\boldsymbol{S}} = [\varphi]_{\boldsymbol{S}'}$ for every formula φ, we in particular have $[p]_{\boldsymbol{S}} = [p]_{\boldsymbol{S}'}$ for every $p \in \mathcal{P}$. Therefore we get, for every $p \in \mathcal{P}$,

$$\|p\|_{\boldsymbol{S}'} = \sum_{s\in[p]_{\boldsymbol{S}'}} \mu'(s) = \sum_{s\in[p]_{\boldsymbol{S}}} \mu'(s) = \sum_{s\in[p]_{\boldsymbol{S}}} x_s^\times\,.$$

Now assume that $\|p\|_{\boldsymbol{S}} < \|q\|_{\boldsymbol{S}}$. Since $\|p\|_{\boldsymbol{S}} = \sum_{s\in[p]_{\boldsymbol{S}}} x^*_s$ and $\|q\|_{\boldsymbol{S}} = \sum_{s\in[q]_{\boldsymbol{S}}} x^*_s$ we then have $\sum_{s\in[p]_{\boldsymbol{S}}} x^*_s < \sum_{s\in[q]_{\boldsymbol{S}}} x^*_s$. Since an inequality that is equivalent to $L_p < L_q$ is contained in X', we get $\sum_{s\in[p]_{\boldsymbol{S}}} x^\times_s < \sum_{s\in[q]_{\boldsymbol{S}}} x^\times_s$. Since $\|p\|_{\boldsymbol{S}'} = \sum_{s\in[p]_{\boldsymbol{S}}} x^\times_s$ and $\|q\|_{\boldsymbol{S}'} = \sum_{s\in[q]_{\boldsymbol{S}}} x^\times_s$, we conclude $\|p\|_{\boldsymbol{S}'} < \|q\|_{\boldsymbol{S}'}$.

Now assume that $\|p\|_{\boldsymbol{S}} \not< \|q\|_{\boldsymbol{S}}$. If $\|p\|_{\boldsymbol{S}} > \|q\|_{\boldsymbol{S}}$, then the same reasoning as before applies and we get $\|p\|_{\boldsymbol{S}'} > \|q\|_{\boldsymbol{S}'}$ and therefore $\|p\|_{\boldsymbol{S}'} \not< \|q\|_{\boldsymbol{S}'}$. If $\|p\|_{\boldsymbol{S}} = \|q\|_{\boldsymbol{S}}$, then also a similar argument as before will give us $\|p\|_{\boldsymbol{S}'} = \|q\|_{\boldsymbol{S}'}$ and therefore $\|p\|_{\boldsymbol{S}'} \not< \|q\|_{\boldsymbol{S}'}$. □

We now show that the conditions of the previous lemma are sufficient for two precisification spaces to have the same sets of true formulas.

Lemma 5.4.16. *Let φ be a formula, $\mathcal{P}$ the set of propositional variables of φ, and $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ precisification spaces such that the following conditions hold:*

- $\|\mathrm{S}\psi\|_{\boldsymbol{S}_1} = \|\mathrm{S}\psi\|_{\boldsymbol{S}_2}$ *for every subformula* $\mathrm{S}\psi$ *of* φ
- $\|p\|_{\boldsymbol{S}_1} < \|q\|_{\boldsymbol{S}_1}$ *if and only if* $\|p\|_{\boldsymbol{S}_2} < \|q\|_{\boldsymbol{S}_2}$ *for all* $p, q \in \mathcal{P}$
- $\|p\|_{\boldsymbol{S}_1} = 0$ *if and only if* $\|p\|_{\boldsymbol{S}_2} = 0$ *for every* $p \in \mathcal{P}$
- $\|p\|_{\boldsymbol{S}_1} = 1$ *if and only if* $\|p\|_{\boldsymbol{S}_2} = 1$ *for every* $p \in \mathcal{P}$

Then $\|\varphi\|^{\mathrm{G}}_{\boldsymbol{S}_1} = 1$ *if and only if* $\|\varphi\|^{\mathrm{G}}_{\boldsymbol{S}_2} = 1$.

Proof. We first show that $\|p\|_{\boldsymbol{S}_1} = \|q\|_{\boldsymbol{S}_1}$ if and only if $\|p\|_{\boldsymbol{S}_2} = \|q\|_{\boldsymbol{S}_2}$ for all $p, q \in \mathcal{P}$.

Assume that $\|p\|_{\boldsymbol{S}_1} = \|q\|_{\boldsymbol{S}_1}$ and suppose that $\|p\|_{\boldsymbol{S}_2} \neq \|q\|_{\boldsymbol{S}_2}$. Then either $\|p\|_{\boldsymbol{S}_2} < \|q\|_{\boldsymbol{S}_2}$ or $\|p\|_{\boldsymbol{S}_2} > \|q\|_{\boldsymbol{S}_2}$. If $\|p\|_{\boldsymbol{S}_2} < \|q\|_{\boldsymbol{S}_2}$, then $\|p\|_{\boldsymbol{S}_1} < \|q\|_{\boldsymbol{S}_1}$ and therefore $\|p\|_{\boldsymbol{S}_1} \neq \|q\|_{\boldsymbol{S}_1}$ which contradicts our assumption. If $\|p\|_{\boldsymbol{S}_2} > \|q\|_{\boldsymbol{S}_2}$, then $\|p\|_{\boldsymbol{S}_1} > \|q\|_{\boldsymbol{S}_1}$ and therefore $\|p\|_{\boldsymbol{S}_1} \neq \|q\|_{\boldsymbol{S}_1}$ which contradicts our assumption. Therefore it must be the case that $\|p\|_{\boldsymbol{S}_2} = \|q\|_{\boldsymbol{S}_2}$. The other direction holds due to the same argument.

We define the following two models $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$, with domains $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$, of the theory of linear orders with endpoints LOE (see Section 4.2) and two evaluations $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$:

$$\boldsymbol{D}_1 = \{0,1\} \cup \left\{\|p\|_{\boldsymbol{S}_1} \mid p \in \mathcal{P}\right\} \qquad \boldsymbol{D}_2 = \{0,1\} \cup \left\{\|p\|_{\boldsymbol{S}_2} \mid p \in \mathcal{P}\right\}$$
$$\|\bar{0}\|_{\boldsymbol{M}_1} = 0 \qquad \|\bar{0}\|_{\boldsymbol{M}_2} = 0$$
$$\|\bar{1}\|_{\boldsymbol{M}_1} = 0 \qquad \|\bar{1}\|_{\boldsymbol{M}_2} = 0$$

We interpret $<$ as the standard strict smaller-than relation on real numbers in both models. Then it is clear that both models satisfy all axioms of the theory LOE.

We denote by φ' the result of replacing every subformula $\mathrm{S}\psi$ of φ that is not in the scope of another S-operator by a new propositional variable p_ψ. Furthermore, we define the $\boldsymbol{D}_1$-evaluation $\boldsymbol{e}_1$ and the $\boldsymbol{D}_2$-evaluation $\boldsymbol{e}_2$ as follows:

$$\boldsymbol{e}_1(v) = \begin{cases} \|\mathrm{S}\psi\|_{\boldsymbol{S}_1} & \text{if } v = p_\psi \\ \|v\|_{\boldsymbol{S}_1} & \text{otherwise} \end{cases} \qquad \boldsymbol{e}_2(v) = \begin{cases} \|\mathrm{S}\psi\|_{\boldsymbol{S}_1} & \text{if } v = p_\psi \\ \|v\|_{\boldsymbol{S}_2} & \text{otherwise} \end{cases}$$

Note that $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are well-defined because $\|\mathrm{S}\psi\|_{\boldsymbol{S}_1} = \|\mathrm{S}\psi\|_{\boldsymbol{S}_2} \in \{0, 1\}$. Furthermore, $\boldsymbol{e}_1(p) = \boldsymbol{e}_2(p)$ for every propositional variable $p \notin \mathcal{P}$ and we clearly have $\|\varphi\|^{\mathrm{G}}_{\boldsymbol{S}_1} = \|\varphi\|^{\mathrm{G}}_{e_1}$ and $\|\varphi\|^{\mathrm{G}}_{\boldsymbol{S}_2} = \|\varphi\|^{\mathrm{G}}_{e_2}$.

We define the following function $f : \boldsymbol{D}_1 \to \boldsymbol{D}_2$:

$$f(d) = \begin{cases} 0 & \text{if } d = 0 \\ 1 & \text{if } d = 1 \\ \|p\|_{\boldsymbol{S}_2} & \text{if there is some } p \in \mathcal{P} \text{ such that } d = \|p\|_{\boldsymbol{S}_1} \end{cases}$$

First, we have to show that f is well-defined. If $\|p\|_{\boldsymbol{S}_1} = 0$, then also $\|p\|_{\boldsymbol{S}_2} = 0$ and therefore $f(\|p\|_{\boldsymbol{S}_1}) = \|p\|_{\boldsymbol{S}_2} = 0 = f(0)$. If $\|p\|_{\boldsymbol{S}_1} = 1$, then also $\|p\|_{\boldsymbol{S}_2} = 1$ and therefore $f(\|p\|_{\boldsymbol{S}_1}) = \|p\|_{\boldsymbol{S}_2} = 1 = f(1)$. If $\|p\|_{\boldsymbol{S}_1} = \|q\|_{\boldsymbol{S}_1}$, then also $\|p\|_{\boldsymbol{S}_2} = \|q\|_{\boldsymbol{S}_2}$ and therefore $f(\|p\|_{\boldsymbol{S}_1}) = \|p\|_{\boldsymbol{S}_2} = \|q\|_{\boldsymbol{S}_2} = f(\|q\|_{\boldsymbol{S}_1})$.

We further show that f is a homomorphism $\boldsymbol{M}_1 \to \boldsymbol{M}_2$. We clearly have

$$f\left(\|\bar{0}\|_{\boldsymbol{M}_1}\right) = f(0) = 0 = \|\bar{0}\|_{\boldsymbol{M}_2}$$
$$f\left(\|\bar{1}\|_{\boldsymbol{M}_1}\right) = f(1) = 1 = \|\bar{1}\|_{\boldsymbol{M}_2}\,.$$

Now let $a, b \in \boldsymbol{D}_1$. We have to show that $a < b$ if and only if $f(a) < f(b)$. We start with the direction from left to right: Assume that $a < b$. Then it is not possible that $a = 1$ or that $b = 0$. We go through all possible cases:

- Assume that $a = 0$ and $b = 1$. Then $f(a) = f(0) = 0 < 1 = f(1) = f(b)$.

- Assume that $a = 0$ and $0 < b < 1$. Then there must be some $p \in \mathcal{P}$ such that $b = \|p\|_{\boldsymbol{S}_1}$. Suppose that $f(b) = 0$. Then $0 = f(b) = f(\|p\|_{\boldsymbol{S}_1}) = \|p\|_{\boldsymbol{S}_2}$. We get $0 = \|p\|_{\boldsymbol{S}_1} = b$ which contradicts $b > 0$. Therefore $f(b) > 0$ and we get $f(a) = f(0) = 0 < f(b)$.

- Assume that $b = 1$ and $0 < a < 1$. Then there must be some $p \in \mathcal{P}$ such that $a = \|p\|_{\boldsymbol{S}_1}$. Suppose that $f(a) = 1$. Then $1 = f(a) = f(\|p\|_{\boldsymbol{S}_1}) = \|p\|_{\boldsymbol{S}_2}$. We get $1 = \|p\|_{\boldsymbol{S}_1} = a$ which contradicts $a < 1$. Therefore $f(a) < 1$ and we get $f(a) < 1 = f(1) = f(b)$.

- Assume that $0 < a < 1$ and $0 < b < 1$. Then it must be the case that $a = \|p\|_{\boldsymbol{S}_1}$ and $b = \|q\|_{\boldsymbol{S}_1}$ for some $p, q \in \mathcal{P}$, then we have $\|p\|_{\boldsymbol{S}_1} < \|q\|_{\boldsymbol{S}_1}$ and therefore

$$f(a) = f\left(\|p\|_{\boldsymbol{S}_1}\right) = \|p\|_{\boldsymbol{S}_2} < \|q\|_{\boldsymbol{S}_2} = f\left(\|q\|_{\boldsymbol{S}_1}\right) = f(b)\,.$$

Now we show the direction from right to left indirectly. Assume that $a \nless b$. Then we know that $a \geq b$. If $a > b$, then we know, due to the proof of the direction from left to right, that $f(a) > f(b)$ and therefore $f(a) \nless f(b)$. If $a = b$, then $f(a) = f(b)$ and thus $f(a) \nless f(b)$.

The last thing that we have to show is that $a = b$ if and only if $f(a) = f(b)$. For the direction from left to right there is nothing to show. We prove the other direction indirectly. If $a \neq b$, then either $a < b$ or $b < a$. If $a < b$, then $f(a) < f(b)$ and if $b < a$, then $f(b) < f(a)$. Thus, in both cases we have $f(a) \neq f(b)$. Note that this also shows that f is injective.

Now we may apply Lemma 4.3.3 and get $f(\|\varphi\|^{\mathrm{G}}_{\boldsymbol{M}_1,\boldsymbol{e}_1}) = \|\varphi\|^{\mathrm{G}}_{\boldsymbol{M}_2,f\circ\boldsymbol{e}_1}$. Note that $f \circ \boldsymbol{e}_1 = \boldsymbol{e}_2$: Let p be a propositional variable. If $p \in \mathcal{P}$, then $f(\boldsymbol{e}_1(p)) = f(\|p\|_{\boldsymbol{S}_1}) = \|p\|_{\boldsymbol{S}_2} = \boldsymbol{e}_2(p)$. If $p \notin \mathcal{P}$, then we know that $\boldsymbol{e}_1(p) = \boldsymbol{e}_2(p) \in \{0, 1\}$. If $\boldsymbol{e}_1(p) = 0$, then $f(\boldsymbol{e}_1(p)) = f(0) = 0 = \boldsymbol{e}_2(p)$ and if $\boldsymbol{e}_1(p) = 1$, then $f(\boldsymbol{e}_1(p)) = f(1) = 1 = \boldsymbol{e}_2(p)$.

Now we get

$$f\left(\|\varphi\|^{\mathrm{G}}_{\boldsymbol{S}_1}\right) = f\left(\|\varphi\|^{\mathrm{G}}_{\boldsymbol{M}_1,\boldsymbol{e}_1}\right) = \|\varphi\|^{\mathrm{G}}_{\boldsymbol{M}_1,f\circ\boldsymbol{e}_1} = \|\varphi\|^{\mathrm{G}}_{\boldsymbol{M}_2,\boldsymbol{e}_2} = \|\varphi\|^{\mathrm{G}}_{\boldsymbol{S}_2}$$

If $\|\varphi\|^{\mathrm{G}}_{\boldsymbol{S}_1} = 1$, we get

$$\|\varphi\|^{\mathrm{G}}_{\boldsymbol{S}_2} = f\left(\|\varphi\|^{\mathrm{G}}_{\boldsymbol{S}_1}\right) = f(1) = 1$$

because f is a homomorphism. We can now show the claim of the lemma. If $\|\varphi\|^{\mathrm{G}}_{\boldsymbol{S}_2} = 1$, then $\|\varphi\|^{\mathrm{G}}_{\boldsymbol{S}_1} = 1$ for suppose $\|\varphi\|^{\mathrm{G}}_{\boldsymbol{S}_1} \neq 1$, then $f(\|\varphi\|^{\mathrm{G}}_{\boldsymbol{S}_1}) \neq 1$ because f is injective and $f(1) = 1$. □

Theorem 5.4.17. *For every formula φ, φ is u-valid in* SG *if and only if φ is p-valid in* SG.

Proof. Assume that φ is u-valid and let $\boldsymbol{S}$ be a positive precisification space with a set of precisifications $\boldsymbol{P}$ and a probability measure μ. By Proposition 5.1.10 we may assume that $\boldsymbol{P}$ is finite. By Lemma 5.4.15 there is a precisification space $\boldsymbol{S}'$ with probability measure μ' such that $\mu'(s) \in \mathbb{Q}^{>0}$ for every $s \in \boldsymbol{P}'$ and the following conditions hold:

- $\|\mathrm{S}\psi\|_{\boldsymbol{S}} = \|\mathrm{S}\psi\|_{\boldsymbol{S}'}$ for every subformula $\mathrm{S}\psi$ of φ
- $\|p\|_{\boldsymbol{S}} < \|q\|_{\boldsymbol{S}}$ if and only if $\|p\|_{\boldsymbol{S}'} < \|q\|_{\boldsymbol{S}'}$ for all $p, q \in \mathcal{P}$
- $\|p\|_{\boldsymbol{S}} = 0$ if and only if $\|p\|_{\boldsymbol{S}'} = 0$ for all $p \in \mathcal{P}$
- $\|p\|_{\boldsymbol{S}} = 1$ if and only if $\|p\|_{\boldsymbol{S}'} = 1$ for all $p \in \mathcal{P}$

By Lemma 5.4.16 we then know that $\|\varphi\|^{\mathrm{G}}_{\boldsymbol{S}} = 1$ if and only if $\|\varphi\|^{\mathrm{G}}_{\boldsymbol{S}'} = 1$. By Lemma 5.4.6 there is a uniform precisification space $\boldsymbol{S}_u$ such that $\|\varphi\|^{\mathrm{G}}_{\boldsymbol{S}'} = \|\varphi\|^{\mathrm{G}}_{\boldsymbol{S}_u}$. Since φ is u-valid we know that $\|\varphi\|^{\mathrm{G}}_{\boldsymbol{S}_u} = 1$. Therefore we get $\|\varphi\|^{\mathrm{G}}_{\boldsymbol{S}'} = 1$ and $\|\varphi\|^{\mathrm{G}}_{\boldsymbol{S}} = 1$. As $\boldsymbol{S}$ was an arbitrary positive precisification space we conclude that φ is p-valid. □

5.5 Embedding SŁ into △Ł

In the following, we show that SŁ can be embedded in Łukasiewicz logic with △-operator. By embedding we mean that for every formula φ of SŁ we can construct a formula φ' of △Ł such that φ is valid in SŁ if and only if φ' is valid in △Ł. For the construction, all propositional variables that occur in φ have to be known beforehand. Therefore the translation cannot be defined purely inductively on the complexity of φ. Furthermore, φ' might be exponentially larger than φ, which means that the embedding cannot be used to obtain complexity results by polynomial-time reductions. For these reasons, our notion of embedding slightly differs from what is usually implicitly understood by an embedding of one logic into another one. Nevertheless the embedding that is considered here shows some interesting connections between SŁ and △Ł.

The starting point for the embedding is the following observation.

Proposition 5.5.1. *For every precisification space $\boldsymbol{S}$ with and every propositional variable p we have*

$$\|p\|_{\boldsymbol{S}} = \sum_{s\in\boldsymbol{S}} \min\left(\mu(s), \|p\|_{s,\boldsymbol{S}}\right)$$
$$\|\neg p\|_{\boldsymbol{S}} = \sum_{s\in\boldsymbol{S}} \min\left(\mu(s), \|\neg p\|_{s,\boldsymbol{S}}\right) .$$

Our main strategy to obtain the embedding is as follows: For every precisification in the precisification space we create a propositional variable that corresponds to the measure of the precisification. Furthermore, we create propositional variables that indicate the truth or falsehood of the classical propositional variables in the precisifications. Since every formula only has finitely many propositional variables and every precisification space is equivalent to a space with a finite number of precisifications, we create only finitely many new propositional variables. By the above proposition, we can ensure that all of the original propositional variables obtain the truth values that correspond to their measure in the precisification space. To make our proof go through we have to restrict ourselves to positive precisification spaces which is no problem because validity and p-validity coincide in SŁ. As a simplification we only consider formulas in normal form. This requirement is not really necessary but simplifies our proves.

Proof of Proposition 5.5.1. Let $\boldsymbol{P}$ denote the set of precisifications of $\boldsymbol{S}$. We show the first part by the following chain of identities:

$$\begin{aligned}
\|p\|_{\boldsymbol{S}} = \mu\left([p]_{\boldsymbol{S}}\right) &= \sum_{s\in[p]_{\boldsymbol{S}}} \mu(s) \\
&= \sum_{s\in[p]_{\boldsymbol{S}}} \mu(s)\cdot\|p\|_{s,\boldsymbol{S}} + \sum_{s\in\boldsymbol{P}\setminus[p]_{\boldsymbol{S}}} \mu(s)\cdot\|p\|_{s,\boldsymbol{S}} \\
&= \sum_{s\in\boldsymbol{P}} \mu(s)\cdot\|p\|_{s,\boldsymbol{S}}
\end{aligned}$$

$$= \sum_{s \in \boldsymbol{P}} \min\left(\mu(s), \|p\|_{s,\boldsymbol{S}}\right) .$$

The second part needs a very similar argument. □

In the following, we will always have the following situation: We have a set of propositional variables $\mathcal{P}$ and a set of precisifications $\boldsymbol{P}$ and for every precisification $s \in \boldsymbol{P}$ we have a propositional variable $\hat{s}$. By *abuse of notation* we will write s instead of $\hat{s}$. Thus $\boldsymbol{P}$ is also a set of propositional variables. We assume that $\mathcal{P}$ and $\boldsymbol{P}$ are disjoint. We also assume that p_s is a new propositional variable for every $p \in \mathcal{P}$ and every $s \in \boldsymbol{P}$.

Lemma 5.5.2. *Let $\boldsymbol{S}$ be a positive precisification space with a set of precisifications $\boldsymbol{P} = \{s_1, \dots s_m\}$ and $\mathcal{P}$ a set of propositional variables. Then there is an evaluation $\boldsymbol{e}$ such that*

$$\begin{aligned} \|p\|_{s,\boldsymbol{S}} &= \boldsymbol{e}(p_s) && \textit{for every } p \in \mathcal{P} \textit{ and every } s \in \boldsymbol{P} \\ \|p\|_{\boldsymbol{S}} &= \boldsymbol{e}(p) && \textit{for every } p \in \mathcal{P} \end{aligned}$$

and for every $p \in \mathcal{P}$

$$\begin{aligned} \|\nabla s\|^{Ł}_{\boldsymbol{e}} &= 1 && \textit{for every } s \in \boldsymbol{P} \\ \|p_s \vee \neg p_s\|^{Ł}_{\boldsymbol{e}} &= 1 && \textit{for every } s \in \boldsymbol{P} \\ \|p \equiv ((p_{s_1} \wedge s_1) \veebar \dots \veebar (p_{s_m} \wedge s_m))\|^{Ł}_{\boldsymbol{e}} &= 1 \\ \|\neg p \equiv ((\neg p_{s_1} \wedge s_1) \veebar \dots \veebar (\neg p_{s_m} \wedge s_m))\|^{Ł}_{\boldsymbol{e}} &= 1 \end{aligned}$$

Proof. We simply define an evaluation $\boldsymbol{e}$ as follows:

$$\begin{aligned} \boldsymbol{e}(p) &= \|p\|_{\boldsymbol{S}} && \text{for every } p \in \mathcal{P} \\ \boldsymbol{e}(p_s) &= \|p\|_{s,\boldsymbol{S}} && \text{for every } p \in \mathcal{P} \text{ and every } s \in \boldsymbol{P} \\ \boldsymbol{e}(s) &= \mu(s) && \text{for every } s \in \boldsymbol{P} \end{aligned}$$

It is clear that $\boldsymbol{e}$ is well-defined because $\|p\|_{\boldsymbol{S}}, \|p\|_{s,\boldsymbol{S}}, \mu(s) \in [0, 1]$.

Since $\boldsymbol{S}$ is a positive precisification space we have $\boldsymbol{e}(s) = \mu(s) > 0$ for every $s \in \boldsymbol{P}$ and therefore $\|\nabla s\|^{Ł}_{\boldsymbol{e}} = 1$. Due to the definition of the local truth value we know that, for every $p \in \mathcal{P}$ and every $s \in \boldsymbol{P}$, $\boldsymbol{e}(p_s) = \|p\|_{s,\boldsymbol{S}} \in \{0, 1\}$ and therefore $\|p_s \vee \neg p_s\|^{Ł}_{\boldsymbol{e}} = 1$.

Furthermore we have $\|p \wedge s\|^{Ł}_{\boldsymbol{S}} = \min(\|p\|^{Ł}_{\boldsymbol{S}}, \|s\|^{Ł}_{\boldsymbol{S}}))$ for every $p \in \mathcal{P}$ and every $s \in \boldsymbol{P}$. Therefore, for every $p \in \mathcal{P}$, we get

$$\begin{aligned} \|(p_{s_1} \wedge s_1) \veebar \dots \veebar (p_{s_m} \wedge s_m)\|^{Ł}_{\boldsymbol{e}} &= \min\left(\sum_{s \in \boldsymbol{P}} \|p_s \wedge s\|^{Ł}_{\boldsymbol{e}}, 1\right) \\ &= \min\left(\sum_{s \in \boldsymbol{P}} \min\left(\|p_s\|^{Ł}_{\boldsymbol{e}}, \|s\|^{Ł}_{\boldsymbol{e}}\right), 1\right) \\ &= \min\left(\sum_{s \in \boldsymbol{P}} \min\left(\|p\|_{s,\boldsymbol{S}}, \mu(s)\right), 1\right) \end{aligned}$$

Due to $\min(\|p\|^{Ł}_{\boldsymbol{S}}, \mu(s)) \leq \mu(s)$ we get

$$\sum_{s\in \boldsymbol{P}} \min\left(\|p\|_{s,\boldsymbol{S}}, \mu(s)\right) \leq \sum_{s\in \boldsymbol{P}} \mu(s) = 1$$

and therefore by Proposition 5.5.1 we get

$$\|(p \wedge s_1) \veebar \ldots \veebar (p \wedge s_m)\|^{Ł}_{\boldsymbol{e}} = \sum_{s\in \boldsymbol{P}} \min\left(\|p\|_{s,\boldsymbol{S}}, \mu(s)\right) = \|p\|_{\boldsymbol{S}}\,.$$

Thus, $\|p \equiv ((p_{s_1} \wedge s_1) \veebar \ldots \veebar (p_{s_m} \wedge s_m))\|^{Ł}_{\boldsymbol{e}} = 1$. A similar argument shows that $\|\neg p \equiv ((\neg p_{s_1} \wedge s_1) \veebar \ldots \veebar (\neg p_{s_m} \wedge s_m))\|^{Ł}_{\boldsymbol{e}} = 1$. □

Lemma 5.5.3. *Let $\boldsymbol{e}$ be an evaluation and $\mathcal{P}$ and $\boldsymbol{P} = \{s_1, \ldots, s_m\}$ sets of propositional variables such that*

$$\begin{aligned}
\|\triangledown s\|^{Ł}_{\boldsymbol{e}} &= 1 && \textit{for every } s \in \boldsymbol{P}\\
\|p_s \vee \neg p_s\|^{Ł}_{\boldsymbol{e}} &= 1 && \textit{for every } s \in \boldsymbol{P}\\
\|p \equiv ((p_{s_1} \wedge s_1) \veebar \ldots \veebar (p_{s_m} \wedge s_m))\|^{Ł}_{\boldsymbol{e}} &= 1 && \textit{for every } p \in \mathcal{P}\\
\|\neg p \equiv ((\neg p_{s_1} \wedge s_1) \veebar \ldots \veebar (\neg p_{s_m} \wedge s_m))\|^{Ł}_{\boldsymbol{e}} &= 1 && \textit{for every } p \in \mathcal{P}
\end{aligned}$$

Then there is a positive precisification space $\boldsymbol{S}$ with $\boldsymbol{P}$ as its set of precisifications such that

$$\begin{aligned}
\|p\|_{s,\boldsymbol{S}} &= \boldsymbol{e}(p_s) && \textit{for every } p \in \mathcal{P} \textit{ and every } s \in \boldsymbol{P}\\
\|p\|_{\boldsymbol{S}} &= \boldsymbol{e}(p) && \textit{for every } p \in \mathcal{P}
\end{aligned}$$

Proof. We will define a precisification space $\boldsymbol{S}$ with $\boldsymbol{P}$ as its set of precisifications and a probability measure μ. In all cases we have to make sure that $\boldsymbol{S}$ is a well-defined.

We define the local truth values by $\|p\|_{s,\boldsymbol{S}} = \boldsymbol{e}(p_s)$ for every $p \in \mathcal{P}$ and every $s \in \boldsymbol{P}$. Since $\|p_s \vee \neg p_s\|^{Ł}_{\boldsymbol{e}} = 1$ we know that $\boldsymbol{e}(p_s) \in \{0, 1\}$: Suppose that $0 < \boldsymbol{e}(p_s) < 1$. Then $\|p_s\|^{Ł}_{\boldsymbol{S}} < 1$ and $\|\neg p_s\|^{Ł}_{\boldsymbol{S}} = 1 - \|p_s\|^{Ł}_{\boldsymbol{S}} < 1$. Thus, $\|p_s \vee \neg p_s\|^{Ł}_{\boldsymbol{e}} = \min(\|p_s\|^{Ł}_{\boldsymbol{S}}, \|\neg p_s\|^{Ł}_{\boldsymbol{S}}) < 1$ which contradicts our assumptions. Therefore, the assignment $\|p\|_{s,\boldsymbol{S}} = \boldsymbol{e}(p_s) \in \{0, 1\}$ is a proper definition of the local truth value.

We consider now first the case that for every $p \in \mathcal{P}$ either $\boldsymbol{e}(p) = 0$ or $\boldsymbol{e}(p) = 1$. We define μ by $\mu(s) = 1/m$ for every $s \in \boldsymbol{P}$. Then clearly $\mu(s) > 0$ and $\sum_{s\in\boldsymbol{P}} \mu(s) = \sum_{s\in\boldsymbol{P}} 1/m = m \cdot (1/m) = 1$. Let $p \in \mathcal{P}$. If $\boldsymbol{e}(p) = 1$, then there cannot be an $s' \in \boldsymbol{P}$ such that $\boldsymbol{e}(p_{s'}) = 0$ for the following reason: If $\boldsymbol{e}(p_{s'}) = 0$, then $\|\neg p_{s'}\|^{Ł}_{\boldsymbol{e}} = 1$ and therefore $\min(\|\neg p_{s'}\|^{Ł}_{\boldsymbol{e}}, \|s'\|^{Ł}_{\boldsymbol{e}}) = \|s'\|^{Ł}_{\boldsymbol{e}} > 0$. Then we arrive at the contradiction

$$\begin{aligned}
0 = \|\neg p\|^{Ł}_{\boldsymbol{e}} = \sum_{s\in\boldsymbol{P}} \min\left(\|\neg p_s\|^{Ł}_{\boldsymbol{e}}, \|s\|^{Ł}_{\boldsymbol{e}}\right) &\geq \min\left(\|\neg p_{s'}\|^{Ł}_{\boldsymbol{e}}, \|s'\|^{Ł}_{\boldsymbol{e}}\right)\\
&= \min\left(1, \|s'\|^{Ł}_{\boldsymbol{e}}\right) = \|s'\|^{Ł}_{\boldsymbol{e}} > 0\,.
\end{aligned}$$

Therefore we may assume that $\|p\|_{s,\boldsymbol{S}} = 1$ for all $s \in \boldsymbol{P}$ and we get $\|p\|_{\boldsymbol{S}} = \mu([p]_{\boldsymbol{S}}) = \mu(\boldsymbol{P}) = 1 = \boldsymbol{e}(p)$. If $\boldsymbol{e}(p) = 0$, then, for a similar reason as before, there cannot be an $s \in \boldsymbol{P}$ such that $\boldsymbol{e}(p_s) = 1$. Therefore, $\|p\|_{s,\boldsymbol{S}} = 0$ for all $s \in \boldsymbol{P}$ and we get $\|p\|_{\boldsymbol{S}} = \mu([p]_{\boldsymbol{S}}) = \mu(\emptyset) = 0 = \boldsymbol{e}(p)$.

The second case that we have to consider is that there is some $p \in \mathcal{P}$ such that $0 < \boldsymbol{e}(p) < 1$. Then we simply define μ by setting $\mu(s) = \boldsymbol{e}(s)$ for every $s \in \boldsymbol{P}$. We first show that μ is well-defined. Since $\|\triangledown s\|^{Ł}_{\boldsymbol{e}} = 1$ we know that $\mu(s) = \boldsymbol{e}(s) > 0$ and thus $\boldsymbol{S}$ is positive. For well-definedness, we still need to show that $\sum_{s \in \boldsymbol{P}} \mu(s) = 1$.

Since $\|p\|^{Ł}_{\boldsymbol{e}} = \boldsymbol{e}(p) < 1$ and $\|\neg p\|^{Ł}_{\boldsymbol{e}} = 1 - \boldsymbol{e}(p) < 1$ our assumptions give us

$$\begin{aligned}\|p\|^{Ł}_{\boldsymbol{e}} &= \min\left(\sum_{s \in \boldsymbol{P}} \min\left(\|p_s\|^{Ł}_{\boldsymbol{e}}, \|s\|^{Ł}_{\boldsymbol{e}}\right), 1\right)\\ &= \sum_{s \in \boldsymbol{P}} \min\left(\|p_s\|^{Ł}_{\boldsymbol{e}}, \|s\|^{Ł}_{\boldsymbol{e}}\right)\\ &= \sum_{s \in \boldsymbol{P}} \min\left(\|p_s\|_{s,\boldsymbol{S}}, \mu(s)\right)\end{aligned}$$

and

$$\begin{aligned}\|\neg p\|^{Ł}_{\boldsymbol{e}} &= \min\left(\sum_{s \in \boldsymbol{P}} \min\left(\|\neg p_s\|^{Ł}_{\boldsymbol{e}}, \|s\|^{Ł}_{\boldsymbol{e}}\right), 1\right)\\ &= \sum_{s \in \boldsymbol{P}} \min\left(\|\neg p_s\|^{Ł}_{\boldsymbol{e}}, \|s\|^{Ł}_{\boldsymbol{e}}\right)\\ &= \sum_{s \in \boldsymbol{P}} \min\left(\|\neg p_s\|_{s,\boldsymbol{S}}, \mu(s)\right)\ .\end{aligned}$$

Since $\|p_s\|_{s,\boldsymbol{S}} \in \{0, 1\}$ and $0 < \mu(s) \leq 1$ we get

$$\min\left(\|p_s\|_{s,\boldsymbol{S}}, \mu(s)\right) + \min\left(\|\neg p_s\|_{s,\boldsymbol{S}}, \mu(s)\right) = \mu(s)$$

and thus we have

$$\begin{aligned}1 &= \|p\|^{Ł}_{\boldsymbol{e}} + 1 - \|p\|^{Ł}_{\boldsymbol{e}}\\ &= \|p\|^{Ł}_{\boldsymbol{e}} + \|\neg p\|^{Ł}_{\boldsymbol{e}}\\ &= \sum_{s \in \boldsymbol{P}} \min\left(\|p_s\|_{s,\boldsymbol{S}}, \mu(s)\right) + \sum_{s \in \boldsymbol{P}} \min\left(\|\neg p_s\|_{s,\boldsymbol{S}}, \mu(s)\right)\\ &= \sum_{s \in \boldsymbol{P}} \left(\min\left(\|p_s\|_{s,\boldsymbol{S}}, \mu(s)\right) + \min\left(\|\neg p_s\|_{s,\boldsymbol{S}}, \mu(s)\right)\right)\\ &= \sum_{s \in \boldsymbol{P}} \mu(s)\end{aligned}$$

which we had to show.

Let $p \in \mathcal{P}$. Since $\min(\|p_s\|_{s,\boldsymbol{S}}, \mu(s)) \leq \mu(s)$ for every $s \in \boldsymbol{P}$ we get

$$\sum_{s \in \boldsymbol{P}} \min\left(\|p_s\|_{s,\boldsymbol{S}}, \mu(s)\right) \leq \sum_{s \in \boldsymbol{P}} \mu(s) = 1$$

and therefore have

$$\begin{aligned} e(p) = \|p\|^{Ł}_{e} &= \min\left(\sum_{s\in P} \min\left(\|p_s\|^{Ł}_{e}, \|s\|^{Ł}_{e}\right), 1\right) \\ &= \min\left(\sum_{s\in P} \min\left(\|p_s\|^{Ł}_{e}, \mu(s)\right), 1\right) \\ &= \sum_{s\in P} \min\left(\|p_s\|_{s,S}, \mu(s)\right) . \end{aligned}$$

By Proposition 5.5.1 the right hand side is equal to $\|p\|_{S}$. Therefore we get $e(p) = \|p\|_{S}$ for every $p \in \mathcal{P}$. □

Note that in this proof it is really necessary to have the restriction to positive precisification spaces to show the well-definedness in the first case.

Definition 5.5.4. Let $\mathcal{P}$ be a set of propositional variables and $P = \{s_1, \ldots, s_m\}$ a finite set of propositional variables. For every formula φ and every $s \in P$, we define $\varphi^{(s)}$ as the result of replacing every occurrence of any propositional variable $p \in \mathcal{P}$ in φ by p_s. For every formula φ that only contains propositional variables of $\mathcal{P}$ we inductively define the formula φ' as follows:

$$\begin{aligned} (\mathrm{S}\psi)' &= \bigwedge_{s\in P} \psi^{(s)} \\ \bar{0}' &= \bar{0} \\ p' &= p \\ (\psi \,\&\, \chi)' &= \psi' \,\&\, \chi' \\ (\psi \supset \chi)' &= \psi' \supset \chi' \end{aligned}$$

Lemma 5.5.5. *Let $*$ be a continuous t-norm, S a precisification space with a set of precisifications P, e an evaluation, and $\mathcal{P}$ a set of propositional variables such that*

$$\begin{aligned} \|p\|_{s,S} &= e(p_s) && \text{for every } p \in \mathcal{P} \text{ and every } s \in P \\ \|p\|_{S} &= e(p) && \text{for every } p \in \mathcal{P} \end{aligned}$$

Then, for every formula φ in normal form, $\|\varphi\|^{*}_{S} = \|\varphi'\|^{*}_{e}$.

Proof. Due to $e(p_s) = \|p\|_{s,S} \in \{0, 1\}$ we obviously have, for every formula ψ that does not contain an S-operator, $\|\psi\|_{s,S} = \|\psi^{(s)}\|^{*}_{e}$ for every $s \in P$ because t-norm based connectives behave classically for the truth value set $\{0, 1\}$. Now consider any subformula of φ that is of the form $\mathrm{S}\psi$. Since φ is in normal form, we know that ψ does not contain an S-operator. It is then easy to show that $\|\mathrm{S}\psi\|_{S} = \|(\mathrm{S}\psi)'\|^{*}_{e}$.

Assume that $\|\mathrm{S}\psi\|_{S} = 1$. Then $\|\psi\|_{s,S} = 1$ for every $s \in P$. By our argument above, then also $\|\psi^{(s)}\|^{*}_{e} = 1$ for every $s \in P$. Thus, we get $\|(\mathrm{S}\psi')\|^{*}_{e} = \|\bigwedge_{s\in P} \psi^{(s)}\|^{*}_{e} = 1 = \|\mathrm{S}\psi\|_{S}$.

Assume that $\|S\psi\|_{\boldsymbol{S}} = 0$. Then $\|\psi\|_{s,\boldsymbol{S}} = 0$ for some $s \in \boldsymbol{P}$. By our argument above, we then have $\|\psi^{(s)}\|^*_{\boldsymbol{e}} = 0$ for some $s \in \boldsymbol{P}$. Thus, we get $\|(S\psi')\|^*_{\boldsymbol{e}} = \|\bigwedge_{s\in\boldsymbol{P}} \psi^{(s)}\|^*_{\boldsymbol{e}} = 0 = \|S\psi\|_{\boldsymbol{S}}$.

A straightforward induction on the structure of φ—similar to the proof of Proposition 5.1.8—then shows that $\|\varphi\|^*_{\boldsymbol{S}} = \|\varphi'\|^*_{\boldsymbol{e}}$. □

Definition 5.5.6. For every formula φ, we construct the formula $\boldsymbol{C}_\varphi$ as follows. We define $\mathcal{P}$ as the set of propositional variables contained in φ, $m = 2^{|\mathcal{P}|}$, and $\boldsymbol{P} = \{s_1, \ldots s_m\}$. The formula $\boldsymbol{C}_\varphi$ is the conjunction of all formulas

$$\nabla s$$

$$p_s \vee \neg p_s$$

$$p \equiv ((p_{s_1} \wedge s_1) \veebar \ldots \veebar (p_{s_m} \wedge s_m))$$

$$\neg p \equiv ((\neg p_{s_1} \wedge s_1) \veebar \ldots \veebar (\neg p_{s_m} \wedge s_m))$$

for all $p \in \mathcal{P}$ and all $s \in \boldsymbol{P}$.

Theorem 5.5.7. *For every formula φ in normal form, φ is p-valid in* SŁ *if and only if $\triangle C_\varphi \supset \varphi'$ is valid in* △Ł.

Proof. Let $\boldsymbol{e}$ be an arbitrary evaluation. We know that $\|\triangle C_\varphi\|^{Ł}_{\boldsymbol{e}} \in \{0,1\}$. In the case $\|\triangle C_\varphi\|^{Ł}_{\boldsymbol{e}} = 0$ we trivially get $\|\triangle C_\varphi \supset \varphi'\|^{Ł}_{\boldsymbol{e}} = 1$. If $\|\triangle C_\varphi\|^{Ł}_{\boldsymbol{e}} = 1$, then $\|C_\varphi\|^{Ł}_{\boldsymbol{e}} = 1$ and by Lemma 5.5.3 and Lemma 5.5.5 there is a positive precisification space $\boldsymbol{S}$ such that $\|\varphi\|^{Ł}_{\boldsymbol{S}} = \|\varphi'\|^{Ł}_{\boldsymbol{e}}$. Since φ is p-valid, we have $\|\varphi\|^{Ł}_{\boldsymbol{S}} = 1$ and therefore get $\|\varphi'\|^{Ł}_{\boldsymbol{e}} = 1$. Hence, $\|\triangle C_\varphi \supset \varphi'\|^{Ł}_{\boldsymbol{e}} = 1$. Since $\boldsymbol{e}$ was an arbitrary evaluation, $\triangle C_\varphi \supset \varphi'$ is valid in △Ł.

Let $\boldsymbol{S}$ be an arbitrary positive precisification space. By Proposition 5.1.10 there is a positive precisification space $\boldsymbol{S}'$ with $m = 2^{|\mathcal{P}|}$ precisifications such that $\|\varphi\|^{Ł}_{\boldsymbol{S}} = \|\varphi\|^{Ł}_{\boldsymbol{S}'}$. Then by Lemma 5.5.2 and Lemma 5.5.5, there is an evaluation $\boldsymbol{e}$ such that $\|\varphi\|_{\boldsymbol{S}'} = \|\varphi'\|_{\boldsymbol{e}}$ and $\|C_\varphi\|^{Ł}_{\boldsymbol{e}} = 1$. Then we also have $\|\triangle C_\varphi\|^{Ł}_{\boldsymbol{e}} = 1$. Since $\triangle C_\varphi \supset \varphi'$ is valid in △Ł, we get $\|\triangle C_\varphi \supset \varphi'\|^{Ł}_{\boldsymbol{e}} = 1$. Therefore we get $\|\varphi'\|^{Ł}_{\boldsymbol{e}} = 1$. We then have $\|\varphi\|^{Ł}_{\boldsymbol{S}} = \|\varphi\|^{Ł}_{\boldsymbol{S}'} = \|\varphi'\|^{Ł}_{\boldsymbol{e}} = 1$. Since $\boldsymbol{S}$ was an arbitrary positive precisification space, we conclude that φ is p-valid in SŁ. □

Since validity and p-validity coincide in SŁ by Theorem 5.4.7 and every formula is valid if and only if its normal form is valid, we have an embedding of SŁ into △Ł.

Corollary 5.5.8. *For every formula φ, φ is valid in* SŁ *if and only if $\triangle C_\varphi \supset \varphi'$ is valid in* △Ł.

Note that our proof was fully constructive: we can give an algorithm to compute φ' from φ. Since checking for validity in △Ł is decidable, we get a simple decision procedure for checking whether a formula φ is valid in SŁ: just check whether φ' is valid in △Ł.

Corollary 5.5.9. *Checking for validity in* SŁ *is decidable.*

5.6 Embedding △G into SŁ

The following investigation is concerned with a certain type of precisification spaces. Remember that in fuzzy logic the truth values of the propositional variables are linearly ordered. We can also introduce the concept of linearity to the framework of precisification spaces.

Definition 5.6.1. A precisification space $\boldsymbol{S}$ is *linear* in a set of propositional variables $\mathcal{P}$ iff for all $p, q \in \mathcal{P}$, $[p]_{\boldsymbol{S}} \subseteq [q]_{\boldsymbol{S}}$ or $[q]_{\boldsymbol{S}} \subseteq [p]_{\boldsymbol{S}}$.

We now show how to construct a precisification space from an evaluation such that the truth values of propositional variables are preserved. This construction will give us a linear precisification space, which is the reason why we include the proof in this section.

Lemma 5.6.2. *Let $\boldsymbol{e}$ be an evaluation of propositional variables and $\mathcal{P} = \{p_1, \dots, p_n,\}$ a finite set of propositional variables. Then there is a positive precisification space $\boldsymbol{S}$ that is linear in $\mathcal{P}$ such that $\|p\|_{\boldsymbol{S}} = \boldsymbol{e}(p)$ for every $p \in \mathcal{P}$. This implies that $\|\varphi\|^*_{\boldsymbol{S}} = \|\varphi\|^*_{\boldsymbol{e}}$ for every* S*-free formula φ containing only propositional variables of $\mathcal{P}$ and every continuous t-norm $*$.*

Proof. Our proof has two parts. First, we prove that a precisification space with the desired properties exists that is not necessarily positive. After that we show how from such a precisification space we obtain a positive precisification space that has the desired properties. If $\|p\|_{\boldsymbol{S}} = \boldsymbol{e}(p)$ for every $p \in \mathcal{P}$, then $\|\varphi\|^*_{\boldsymbol{S}} = \|\varphi\|^*_{\boldsymbol{e}}$ for every S-free formula φ containing only propositional variables of $\mathcal{P}$ because the interpretation of formulas based on the continuous t-norm $*$ is exactly the same in both cases.

Without loss of generality we assume that the truth values of the propositional variables are linearly ordered, i.e., $\boldsymbol{e}(p_1) \leq \dots \leq \boldsymbol{e}(p_n)$. We define a precisification space $\boldsymbol{S}$ that has a set of precisifications $\boldsymbol{P} = \{s_1, \dots, s_{n+1}\}$. The local truth value of a propositional variable p_i at the precisification s_j (with $1 \leq i \leq n$ and $1 \leq j \leq n-1$) is defined as follows:

$$\|p_i\|_{s_j} = \begin{cases} 1 & \text{if } j \leq i \\ 0 & \text{if } j > i\,. \end{cases}$$

Furthermore, we define the probability measure μ of $\boldsymbol{S}$ as follows:

$$\mu\left(s_j\right) = \begin{cases} \boldsymbol{e}\left(p_1\right) & \text{if } j = 1 \\ \boldsymbol{e}\left(p_j\right) - \boldsymbol{e}\left(p_{j-1}\right) & \text{if } 1 < j < n+1 \\ 1 - \boldsymbol{e}\left(p_n\right) & \text{if } j = n+1\,. \end{cases}$$

It can easily be verified that μ is well-defined:

$$\begin{aligned}
\sum_{s\in \boldsymbol{S}} \mu(s) &= \sum_{j=1}^{n+1} \mu\left(s_j\right) \\
&= \mu\left(s_1\right) + \sum_{j=2}^{n} \mu\left(s_j\right) + \mu\left(s_{n+1}\right) \\
&= \boldsymbol{e}\left(p_1\right) + \sum_{j=2}^{n} \left(\boldsymbol{e}\left(p_j\right) - \boldsymbol{e}\left(p_{j-1}\right)\right) + 1 - \boldsymbol{e}\left(p_n\right) \\
&= \boldsymbol{e}\left(p_1\right) + \sum_{j=2}^{n} \boldsymbol{e}\left(p_j\right) - \sum_{j=2}^{n} \boldsymbol{e}\left(p_{j-1}\right) + 1 - \boldsymbol{e}\left(p_n\right) \\
&= \boldsymbol{e}\left(p_1\right) + \sum_{j=2}^{n} \boldsymbol{e}\left(p_j\right) - \sum_{j=1}^{n-1} \boldsymbol{e}\left(p_j\right) + 1 - \boldsymbol{e}\left(p_n\right) \\
&= \sum_{j=1}^{n} \boldsymbol{e}\left(p_j\right) - \sum_{j=1}^{n} \boldsymbol{e}\left(p_j\right) + 1 = 1\,.
\end{aligned}$$

Furthermore, the precisification space $\boldsymbol{S}$ has the desired properties. For every propositional variable $p_i \in \mathcal{P}$ we have

$$\begin{aligned}
\|p_i\|_{\boldsymbol{S}} &= \mu\left([p_i]_{\boldsymbol{S}}\right) \\
&= \mu\left(\left\{s_j \in \boldsymbol{S} \mid j \leq i\right\}\right) \\
&= \sum_{j=1}^{i} \mu\left(s_j\right) \\
&= \mu\left(s_1\right) + \sum_{j=2}^{i} \mu\left(s_j\right) \\
&= \boldsymbol{e}\left(p_1\right) + \sum_{j=2}^{i} \left(\boldsymbol{e}\left(p_j\right) - \boldsymbol{e}\left(p_{j-1}\right)\right) \\
&= \boldsymbol{e}\left(p_1\right) + \sum_{j=2}^{i} \boldsymbol{e}\left(p_j\right) - \sum_{j=2}^{i} \boldsymbol{e}\left(p_{j-1}\right) \\
&= \sum_{j=1}^{i} \boldsymbol{e}\left(p_j\right) - \sum_{j=1}^{i-1} \boldsymbol{e}\left(p_j\right) \\
&= \boldsymbol{e}\left(p_i\right)\,.
\end{aligned}$$

We also verify that $\boldsymbol{S}$ is linear in $\mathcal{P}$. Let $p_{i_1}, p_{i_2} \in \mathcal{P}$. Then $[p_{i_1}]_{\boldsymbol{S}} = \{s_j \in \boldsymbol{S} \mid j \leq i_1\}$ and $[p_{i_2}]_{\boldsymbol{S}} = \{s_j \in \boldsymbol{S} \mid j \leq i_2\}$. If $i_1 \leq i_2$, then $[p_{i_1}]_{\boldsymbol{S}} \subseteq [p_{i_2}]_{\boldsymbol{S}}$ and if $i_1 > i_2$, then $[p_{i_2}]_{\boldsymbol{S}} \subseteq [p_{i_1}]_{\boldsymbol{S}}$.

We now show that we can also find a positive precisification space with the desired properties. Let $\boldsymbol{S}$ be a precisification space with a set of precisifications $\boldsymbol{P}$ and a probability measure μ that is linear in $\mathcal{P}$ such that $\|p\|_{\boldsymbol{S}} = \boldsymbol{e}(p)$ for every $p \in \mathcal{P}$. Then there is a positive precisification space $\boldsymbol{S}'$ that also has these properties. We define $\boldsymbol{S}'$ by giving its set of precisifications $\boldsymbol{P}'$, its probability measure μ' and its local truth values:

$$\begin{aligned} \boldsymbol{P}' &= \{s \in \boldsymbol{P} \mid \mu(s) > 0\} \\ \mu'(s) &= \mu(s) \text{ for every } s \in \boldsymbol{P}' \\ \|p\|_{s,\boldsymbol{S}'} &= \|p\|_{s,\boldsymbol{S}} \text{ for every } s \in \boldsymbol{P}' \text{ and atomic } p\,. \end{aligned}$$

Then we have

$$1 = \sum_{s \in \boldsymbol{P}} \mu(s) = \sum_{s \in \boldsymbol{P}'} \mu(s) + \sum_{s \in \boldsymbol{P} \setminus \boldsymbol{P}'} \mu(s) = \sum_{s \in \boldsymbol{P}'} \mu(s) + \sum_{s \in \boldsymbol{P} \setminus \boldsymbol{P}'} 0 = \sum_{s \in \boldsymbol{P}'} \mu(s)$$

and therefore $\boldsymbol{S}$ is well-defined. Clearly, $\boldsymbol{S}$ is positive by the definitions of μ' and $\boldsymbol{P}'$.

Now we show that the desired properties still hold. Due to Proposition 5.1.2, for every $p \in \mathcal{P}$, we have

$$\begin{aligned} \boldsymbol{e}(p) = \|p\|_{\boldsymbol{S}} = \mu\left([p]_{\boldsymbol{S}}\right) &= \mu\left(\left([p]_{\boldsymbol{S}} \cap \boldsymbol{P}'\right) \cup \left([p]_{\boldsymbol{S}} \cap \left(\boldsymbol{P} \setminus \boldsymbol{P}'\right)\right)\right) \\ &= \mu\left([p]_{\boldsymbol{S}} \cap \boldsymbol{P}'\right) + \mu\left([p]_{\boldsymbol{S}} \cap \left(\boldsymbol{P} \setminus \boldsymbol{P}'\right)\right) \\ &= \mu\left([p]_{\boldsymbol{S}'}\right) + 0 \\ &= \mu\left([p]_{\boldsymbol{S}'}\right) = \|p\|_{\boldsymbol{S}'}\,. \end{aligned}$$

Now let $p, q \in \boldsymbol{P}'$ and assume that $[p]_{\boldsymbol{S}} \subseteq [q]_{\boldsymbol{S}}$. Since $[p]_{\boldsymbol{S}'} = [p]_{\boldsymbol{S}} \cap \boldsymbol{P}'$ and $[q]_{\boldsymbol{S}'} = [q]_{\boldsymbol{S}} \cap \boldsymbol{P}'$, we have $[p]_{\boldsymbol{S}'} \subseteq [q]_{\boldsymbol{S}'}$. Since $[p]_{\boldsymbol{S}} \subseteq [q]_{\boldsymbol{S}}$ or $[q]_{\boldsymbol{S}} \subseteq [p]_{\boldsymbol{S}}$ for all $p, q \in \mathcal{P}$, we then also have $[p]_{\boldsymbol{S}'} \subseteq [q]_{\boldsymbol{S}}$ or $[q]_{\boldsymbol{S}'} \subseteq [p]_{\boldsymbol{S}}$ for all $p, q \in \mathcal{P}$. Thus, $\boldsymbol{S}'$ is linear in $\mathcal{P}$. □

It is an easy, but crucial, observation that the concept of linearity can be expressed by a formula.

Lemma 5.6.3. *A precisification space $\boldsymbol{S}$ is linear in a set of propositional variables $\mathcal{P} = \{p_1, \dots p_n\}$ if and only if there is a permutation π of $\{1, \dots, n\}$ such that the formula $\mathrm{S}(p_{\pi(1)} \supset p_{\pi(2)}) \wedge \ldots \wedge \mathrm{S}(p_{\pi(n-1)} \supset p_{\pi(n)})$ is true in $\boldsymbol{S}$.*

Proof. It is clear that for every precisification space $\boldsymbol{S}$ and all formulas φ and ψ we have $\|\mathrm{S}(\varphi \supset \psi)\|_{\boldsymbol{S}} = 1$ if and only if $[\varphi]_{\boldsymbol{S}} \subseteq [\psi]_{\boldsymbol{S}}$.

If $\boldsymbol{S}$ is a precisification space that is linear in $\mathcal{P}$, then $\subseteq$ is a linear order on the set $\{[p]_{\boldsymbol{S}} \mid p \in \mathcal{P}\}$. Therefore every subset of $\{[p]_{\boldsymbol{S}} \mid p \in \mathcal{P}\}$ contains a minimum with respect to $\subseteq$. This means that we can label the elements of $\mathcal{P}$ by a permutation π of $\{1, \dots, n\}$ such that $[p_{\pi(1)}]_{\boldsymbol{S}} \subseteq \ldots \subseteq [p_{\pi(n)}]_{\boldsymbol{S}}$. Thus, that the formula $\mathrm{S}(p_{\pi(1)} \supset p_{\pi(2)}) \wedge \ldots \wedge \mathrm{S}(p_{\pi(n-1)} \supset p_{\pi(n)})$ is true in $\boldsymbol{S}$.

Now assume that there is a permutation π of $\{1, \dots, n\}$ such that $\|\mathrm{S}(p_{\pi(1)} \supset p_{\pi(2)}) \wedge \ldots \wedge \mathrm{S}(p_{\pi(n-1)} \supset p_{\pi(n)})\|_{\boldsymbol{S}} = 1$. Then we know that $[p_{\pi(1)}]_{\boldsymbol{S}} \subseteq [p_{\pi(2)}]_{\boldsymbol{S}}$, …, $[p_{\pi(n-1)}]_{\boldsymbol{S}} \subseteq$

$[p_{\pi(n)}]_{\boldsymbol{S}}$. For two propositional variables $p_{\pi(i)}, p_{\pi(j)} \in \mathcal{P}$ we then get $[p_{\pi(i)}]_{\boldsymbol{S}} \subseteq [p_{\pi(j)}]_{\boldsymbol{S}}$ if $i \leq j$ and $[p_{\pi(j)}]_{\boldsymbol{S}} \subseteq [p_{\pi(i)}]_{\boldsymbol{S}}$ if $j \leq i$. Therefore $\boldsymbol{S}$ is linear in $\mathcal{P}$. □

Since $\mathcal{P}$ is a finite set, there are only finitely many permutations. We therefore can characterize the property of being linear in $\mathcal{P}$ by transforming the existential statement of the previous lemma into a disjunctive formula.

Corollary 5.6.4. *Let $\mathcal{P} = \{p_1, \ldots p_n\}$ be a set of propositional variables and define the formula $\psi_{\mathcal{P}}$ by*

$$\psi_{\mathcal{P}} = \bigvee_{\pi \in \Pi_{\{1,\ldots,n\}}} \mathrm{S}\left(p_{\pi(1)} \supset p_{\pi(2)}\right) \wedge \ldots \wedge \mathrm{S}\left(p_{\pi(n-1)} \supset p_{\pi(n)}\right)$$

where $\Pi_{\{1,\ldots,n\}}$ denotes the set of all permutations of $\{1, \ldots, n\}$. Then a precisification space $\boldsymbol{S}$ is linear in $\mathcal{P}$ if and only if $\psi_{\mathcal{P}}$ is true in $\boldsymbol{S}$.

In the following we show tight connections between Gödel logic with △-operator and the logic S$*$, for any continuous t-norm $*$, restricted to linear precisification spaces. We consider all "levels" of our semantics: extensions, measures, and truth values. This will give us an embedding of △G into S$*$ in the sense that we discussed at the beginning of Section 5.5. We first define two syntactical translations of formulas of △G to formulas of S$*$. As usual we treat formulas of Gödel logic that contain disjunction or negation symbols as abbreviations for formulas that do not contain them.

Definition 5.6.5. Let φ be a formula of Gödel logic with △-operator. The *local* S-*translation* φ^{l} of φ inductively defined as follows:

$$\begin{aligned}
\bar{0}^{\mathrm{l}} &= \bar{0} \\
p^{\mathrm{l}} &= p \text{ for atomic } p \\
(\triangle\psi)^{\mathrm{l}} &= \mathrm{S}\psi^{\mathrm{l}} \\
(\psi \wedge \chi)^{\mathrm{l}} &= \psi^{\mathrm{l}} \wedge \chi^{\mathrm{l}} \\
(\psi \supset \chi)^{\mathrm{l}} &= \mathrm{S}\left(\psi^{\mathrm{l}} \supset \chi^{\mathrm{l}}\right) \vee \chi^{\mathrm{l}}\,.
\end{aligned}$$

The *global* S-*translation* φ^{g} of φ is defined as follows: We denote by φ' the result of iteratedly replacing every maximal[30] subformula of φ of the form $\psi \supset \chi$ that is not in the scope of a △-operator by $(\triangle(\psi \supset \chi)) \vee \chi$. The global S-translation then is $\varphi^{\mathrm{g}} = (\varphi')^{\mathrm{l}}$.

Note that we assume that the formulas of Gödel logic only have one conjunction sign $\wedge$ and that in the translated formulas $\wedge$ is read as the *weak* conjunction sign.

Our next observation is that for certain formulas φ, the "value" of $[\varphi]_{\boldsymbol{S}}$ in a linear precisification space $\boldsymbol{S}$ can be computed according to the truth functions of Gödel logic.

[30]Maximality of a subformula $\psi \supset \chi$ of φ means that there is no subformula $\psi^* \supset \chi^*$ of φ such that $\psi \supset \chi$ is a subformula of ψ^* or $\psi \supset \chi$ is a subformula of χ^*.

Lemma 5.6.6. *Let $\boldsymbol{S}$ be a precisification space that is linear in a set of propositional variables $\mathcal{P}$ and let φ be a formula of Gödel logic with $\triangle$ consisting only of propositional variables contained in $\mathcal{P}$. Then the following holds:*

(i) One of the following conditions holds:

- $[\varphi^1]_{\boldsymbol{S}} = \boldsymbol{P}$
- $[\varphi^1]_{\boldsymbol{S}} = \emptyset$
- *There is some $p \in \mathcal{P}$ such that $[\varphi^1]_{\boldsymbol{S}} = [p^1]_{\boldsymbol{S}}$.*

(ii) $[\varphi^1]_{\boldsymbol{S}}$ can be computed as follows:

- *If $\varphi = \psi \mathbin{\&} \chi$, then* $[\varphi^1]_{\boldsymbol{S}} = \begin{cases} [\psi^1]_{\boldsymbol{S}} & \text{if } [\psi^1]_{\boldsymbol{S}} \subseteq [\chi^1]_{\boldsymbol{S}} \\ [\chi^1]_{\boldsymbol{S}} & \text{otherwise.} \end{cases}$
- *If $\varphi = \triangle\psi$, then* $[\varphi^1]_{\boldsymbol{S}} = \begin{cases} \boldsymbol{P} & \text{if } [\psi^1]_{\boldsymbol{S}} = \boldsymbol{P} \\ \emptyset & \text{otherwise.} \end{cases}$
- *If $\varphi = \psi \supset \chi$, then* $[\varphi^1]_{\boldsymbol{S}} = \begin{cases} \boldsymbol{P} & \text{if } [\psi^1]_{\boldsymbol{S}} \subseteq [\chi^1]_{\boldsymbol{S}} \\ [\chi^1]_{\boldsymbol{S}} & \text{otherwise.} \end{cases}$

Proof. We prove both claims in conjunction by induction on the structure of φ.

- If $\varphi = \bar{1}$, $\varphi = \bar{0}$ or $\varphi = p$ for some propositional variable p, then the first claim trivially holds due to $[\bar{1}]_{\boldsymbol{S}} = \boldsymbol{P}$ and $[\bar{0}]_{\boldsymbol{S}} = \emptyset$ and there is nothing to show for the second claim.

- $\varphi = \psi \mathbin{\&} \chi$: Note that in any case we have

  $$[\varphi^1]_{\boldsymbol{S}} = [(\psi \mathbin{\&} \chi)^1]_{\boldsymbol{S}} = [\psi^1 \wedge \chi^1]_{\boldsymbol{S}} = [\psi^1]_{\boldsymbol{S}} \cap [\chi^1]_{\boldsymbol{S}}\,.$$

  The crucial observation now is that for two sets $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ such that $\boldsymbol{S}_1 \subseteq \boldsymbol{S}_2$ we get $\boldsymbol{S}_1 \cap \boldsymbol{S}_2 = \boldsymbol{S}_1$.

  If $[\psi^1]_{\boldsymbol{S}} \subseteq [\chi^1]_{\boldsymbol{S}}$, then we get $[\varphi^1]_{\boldsymbol{S}} = [\psi^1]_{\boldsymbol{S}} \cap [\chi^1]_{\boldsymbol{S}} = [\psi^1]_{\boldsymbol{S}}$. Now assume that $[\psi^1]_{\boldsymbol{S}} \nsubseteq [\chi^1]_{\boldsymbol{S}}$. Then, by the induction hypothesis, $[\psi^1]_{\boldsymbol{S}} = \boldsymbol{P}$ or $[\psi^1]_{\boldsymbol{S}} = [p]$ for some $p \in \mathcal{P}$ and $[\chi^1]_{\boldsymbol{S}} = \emptyset$ or $[\chi^1]_{\boldsymbol{S}} = [q]$ for some $q \in \mathcal{P}$.[31] If $[\psi^1]_{\boldsymbol{S}} = \boldsymbol{P}$, then we get

  $$[\varphi^1]_{\boldsymbol{S}} = [\psi^1]_{\boldsymbol{S}} \cap [\chi^1]_{\boldsymbol{S}} = \boldsymbol{P} \cap [\chi^1]_{\boldsymbol{S}} = [\chi^1]_{\boldsymbol{S}}$$

  because $[\chi^1]_{\boldsymbol{S}} \subseteq \boldsymbol{P}$. If $[\chi^1]_{\boldsymbol{S}} = \emptyset$, then we get

  $$[\varphi^1]_{\boldsymbol{S}} = [\psi^1]_{\boldsymbol{S}} \cap [\chi^1]_{\boldsymbol{S}} = [\psi^1]_{\boldsymbol{S}} \cap \emptyset = \emptyset = [\chi^1]_{\boldsymbol{S}}\,.$$

[31] The cases $[\psi^1]_{\boldsymbol{S}} = \emptyset$ and $[\chi^1]_{\boldsymbol{S}} = \boldsymbol{P}$ cannot occur because $\emptyset \subseteq [\psi^1]_{\boldsymbol{S}} \subseteq \boldsymbol{P}$.

The remaining case now is that $[\psi^1]_{\boldsymbol{S}} = [p]_{\boldsymbol{S}}$ and $[\chi^1]_{\boldsymbol{S}} = [q]_{\boldsymbol{S}}$ for some propositional variables $p, q \in \mathcal{P}$. Since $\boldsymbol{S}$ is linear, we know that $[p]_{\boldsymbol{S}} \subseteq [q]_{\boldsymbol{S}}$ or $[q]_{\boldsymbol{S}} \subseteq [p]_{\boldsymbol{S}}$. It cannot be the case that $[p]_{\boldsymbol{S}} \subseteq [q]_{\boldsymbol{S}}$ due to $[p]_{\boldsymbol{S}} = [\psi^1]_{\boldsymbol{S}} \nsubseteq [\chi^1]_{\boldsymbol{S}} = [q]_{\boldsymbol{S}}$. Therefore we know that $[q]_{\boldsymbol{S}} \subseteq [p]_{\boldsymbol{S}}$ and get

$$\left[\varphi^1\right]_{\boldsymbol{S}} = \left[\psi^1\right]_{\boldsymbol{S}} \cap \left[\chi^1\right]_{\boldsymbol{S}} = [p]_{\boldsymbol{S}} \cap [q]_{\boldsymbol{S}} = [q]_{\boldsymbol{S}} = \left[\chi^1\right]_{\boldsymbol{S}} .$$

Having considered all cases we conclude the following:

$$\left[\varphi^1\right]_{\boldsymbol{S}} = \begin{cases} \left[\psi^1\right]_{\boldsymbol{S}} & \text{if } \left[\psi^1\right]_{\boldsymbol{S}} \subseteq \left[\chi^1\right]_{\boldsymbol{S}} \\ \left[\chi^1\right]_{\boldsymbol{S}} & \text{otherwise} . \end{cases}$$

This means that our second claim holds in the case that φ is a disjunction.

By the induction hypothesis the result of $[\psi^1]_{\boldsymbol{S}}$ as well as $[\chi^1]_{\boldsymbol{S}}$ is either $\boldsymbol{P}$, $\varnothing$ or $[p]_{\boldsymbol{S}}$ for some propositional variable $p \in \mathcal{P}$. Since the result of $[\varphi^1]_{\boldsymbol{S}}$ is either $[\psi^1]_{\boldsymbol{S}}$ or $[\chi^1]_{\boldsymbol{S}}$, the same property holds for $[\varphi^1]_{\boldsymbol{S}}$. Thus, also our first claim holds in the case that φ is a disjunction.

- $\varphi = \triangle\psi$: We prove this case by showing that for every formula χ we have

$$[\mathrm{S}\chi]_{\boldsymbol{S}} = \begin{cases} \boldsymbol{P} & \text{if } [\chi]_{\boldsymbol{S}} = \boldsymbol{P} \\ \varnothing & \text{otherwise} . \end{cases}$$

Note that the induction hypothesis will not be needed to prove this statement.

If $[\chi]_{\boldsymbol{S}} = \boldsymbol{P}$, then by the definition of the S-operator the formula $\mathrm{S}\chi$ is true at every precisification $s \in \boldsymbol{P}$ and we get $[\mathrm{S}\chi]_{\boldsymbol{S}} = \boldsymbol{P}$. If $[\chi]_{\boldsymbol{S}} \neq \boldsymbol{P}$, then there is a precisification $s \in \boldsymbol{P}$ at which χ is not true and therefore $[\mathrm{S}\chi]_{\boldsymbol{S}} = \varnothing$. Thus, we get

$$[\mathrm{S}\chi]_{\boldsymbol{S}} = \begin{cases} \boldsymbol{P} & \text{if } [\chi]_{\boldsymbol{S}} = \boldsymbol{P} \\ \varnothing & \text{otherwise} . \end{cases}$$

Since $\varphi^1 = (\triangle\psi)^1 = \mathrm{S}\psi^1$ we can set $\chi = \psi^1$ to prove that both claims hold for this case.

- $\varphi = \psi \supset \chi$: We use the observation of the previous case and obtain

$$\left[\mathrm{S}\left(\psi^1 \supset \chi^1\right)\right]_{\boldsymbol{S}} = \begin{cases} \boldsymbol{P} & \text{if } \left[\psi^1 \supset \chi^1\right]_{\boldsymbol{S}} = \boldsymbol{P} \\ \varnothing & \text{otherwise} . \end{cases}$$

It can be easily seen that $[\psi^1 \supset \chi^1]_{\boldsymbol{S}} = \boldsymbol{P}$ if and only if $[\psi^1]_{\boldsymbol{S}} \subseteq [\chi^1]_{\boldsymbol{S}}$ and therefore we get

$$\left[\mathrm{S}\left(\psi^1 \supset \chi^1\right)\right]_{\boldsymbol{S}} = \begin{cases} \boldsymbol{P} & \text{if } \left[\psi^1\right]_{\boldsymbol{S}} \subseteq \left[\chi^1\right]_{\boldsymbol{S}} \\ \varnothing & \text{otherwise} . \end{cases}$$

Now we have

$$\begin{aligned}
[\varphi^1]_{\boldsymbol{S}} = [\mathrm{S}\,(\psi^1 \supset \chi^1) \vee \chi^1]_{\boldsymbol{S}} &= [\mathrm{S}\,(\psi^1 \supset \chi^1)]_{\boldsymbol{S}} \cup [\chi^1]_{\boldsymbol{S}} \\
&= \begin{cases} \boldsymbol{P} \cup [\chi^1]_{\boldsymbol{S}} & \text{if } [\psi^1]_{\boldsymbol{S}} \subseteq [\chi^1]_{\boldsymbol{S}} \\ \emptyset \cup [\chi^1]_{\boldsymbol{S}} & \text{otherwise} \end{cases} \\
&= \begin{cases} \boldsymbol{P} & \text{if } [\psi^1]_{\boldsymbol{S}} \subseteq [\chi^1]_{\boldsymbol{S}} \\ [\chi^1]_{\boldsymbol{S}} & \text{otherwise} \end{cases}
\end{aligned}$$

which proves the second claim. By the induction hypothesis we know that the result of $[\chi^1]_{\boldsymbol{S}}$ is $\boldsymbol{P}$, $\emptyset$ or $[p]_{\boldsymbol{S}}$ for some $p \in \mathcal{P}$. Therefore also the first claim has been shown.

□

Note that it was necessary to prove both statements at once because of the conjunctive case.

The next result states that linearity in a set of propositional variables induces linearity in a certain set of formulas.

Lemma 5.6.7. *Let $\boldsymbol{S}$ be a positive precisification space that is linear in a set of propositional variables $\mathcal{P}$ and φ and ψ formulas in the language of Gödel logic with $\triangle$-operator consisting only of propositional variables contained in $\mathcal{P}$. Then $[\varphi^1]_{\boldsymbol{S}} \subseteq [\psi^1]_{\boldsymbol{S}}$ or $[\psi^1]_{\boldsymbol{S}} \subseteq [\varphi^1]_{\boldsymbol{S}}$.*

Proof. Let $\boldsymbol{P}$ denote the set of precisifications of $\boldsymbol{S}$. If $[\varphi^1]_{\boldsymbol{S}} = \emptyset$, $[\varphi^1]_{\boldsymbol{S}} = \boldsymbol{P}$, $[\psi^1]_{\boldsymbol{S}} = \emptyset$, or $[\psi^1]_{\boldsymbol{S}} = \boldsymbol{P}$, then the claim trivially holds. Otherwise, we know by Lemma 5.6.6 that $[\varphi^1]_{\boldsymbol{S}} = [p]_{\boldsymbol{S}}$ and $[\psi^1]_{\boldsymbol{S}} = [q]_{\boldsymbol{S}}$ for some propositional variables $p, q \in \mathcal{P}$. Since $\boldsymbol{S}$ is linear in $\mathcal{P}$, we have $[p]_{\boldsymbol{S}} \subseteq [q]_{\boldsymbol{S}}$ or $[q]_{\boldsymbol{S}} \subseteq [p]_{\boldsymbol{S}}$ and therefore $[\varphi^1]_{\boldsymbol{S}} \subseteq [\psi^1]_{\boldsymbol{S}}$ or $[\psi^1]_{\boldsymbol{S}} \subseteq [\varphi^1]_{\boldsymbol{S}}$. □

The next step now consists in extending the relation to Gödel logic with $\triangle$ to the level of measures of formulas. Note that from now on linearity is not the only requirement, we also demand that the precisification spaces be positive.[32]

Lemma 5.6.8. *Let $\boldsymbol{S}$ be a positive precisification space that is linear in a set of propositional variables $\mathcal{P}$ and φ and ψ formulas in the language of Gödel logic with $\triangle$-operator consisting only of propositional variables contained in $\mathcal{P}$. Then $[\![\varphi^1]\!]_{\boldsymbol{S}} \leq [\![\psi^1]\!]_{\boldsymbol{S}}$ if and only if $[\varphi^1]_{\boldsymbol{S}} \subseteq [\psi^1]_{\boldsymbol{S}}$.*

---

[32]It should be possible to weaken the requirement that the precisification space $\boldsymbol{S}$ is positive to the following two conditions that have to hold for every propositional variable $p \in \mathcal{P}$ (see Proposition 5.1.7):

- If $\|p\|_{\boldsymbol{S}} = 1$, then $\|\mathrm{S}p\|_{\boldsymbol{S}} = 1$.
- If $\|p\|_{\boldsymbol{S}} = 0$, then $\|\mathrm{S}\neg p\|_{\boldsymbol{S}} = 1$.

The proof of the case $\varphi = \psi \wedge \chi$ for Lemma 5.6.9 will then be more complicated but should still go through.

Proof. Let μ denote the probability measure of $\boldsymbol{S}$. Assume that $[\varphi^1]_{\boldsymbol{S}} \subseteq [\psi^1]_{\boldsymbol{S}}$. Then $[\![\varphi^1]\!]_{\boldsymbol{S}} = \mu([\varphi^1]_{\boldsymbol{S}}) \leq \mu([\psi^1]_{\boldsymbol{S}}) = [\![\psi]\!]_{\boldsymbol{S}}$ by Proposition 5.1.2.

Now assume that $[\varphi^1]_{\boldsymbol{S}} \nsubseteq [\psi^1]_{\boldsymbol{S}}$. Then $[\psi^1]_{\boldsymbol{S}} \subseteq [\varphi^1]_{\boldsymbol{S}}$ by Lemma 5.6.8. Since $[\psi^1]_{\boldsymbol{S}} \neq [\varphi^1]_{\boldsymbol{S}}$, there is an $s \in [\varphi^1]_{\boldsymbol{S}}$ such that $s \notin [\psi^1]_{\boldsymbol{S}}$. Since $\boldsymbol{S}$ is positive we know that $\mu(s) > 0$ and therefore $[\![\psi]\!]_{\boldsymbol{S}} < [\![\varphi]\!]_{\boldsymbol{S}}$. This means that $[\![\varphi]\!]_{\boldsymbol{S}} \nleq [\![\psi]\!]_{\boldsymbol{S}}$. □

Lemma 5.6.9. *Let $\boldsymbol{S}$ be a positive precisification space that is linear in a set of propositional variables $\mathcal{P}$ and φ a formula in the language of Gödel logic with △-operator consisting only of propositional variables contained in $\mathcal{P}$. Then $[\![\varphi^1]\!]_{\boldsymbol{S}} = \|\varphi\|^{\mathrm{G}}_{\boldsymbol{S}}$.*

Proof. We prove the claim by induction on the structure of φ.

- $\varphi = \bar{0}$: $[\![\varphi^1]\!]_{\boldsymbol{S}} = [\![\bar{0}]\!]_{\boldsymbol{S}} = \mu([\bar{0}]_{\boldsymbol{S}}) = \mu(\emptyset) = 0 = \|\bar{0}\|^{\mathrm{G}}_{\boldsymbol{S}} = \|\varphi\|^{\mathrm{G}}_{\boldsymbol{S}}$.
- $\varphi = p$ for a propositional variable p: Then by the definition of the truth value of the propositional variable p we have $\|\varphi\|^{\mathrm{G}}_{\boldsymbol{S}} = \|p\|^{\mathrm{G}}_{\boldsymbol{S}} = [\![p]\!]_{\boldsymbol{S}} = [\![\varphi^1]\!]_{\boldsymbol{S}}$.
- $\varphi = \triangle\psi$: By Lemma 5.6.6 we get

$$[\varphi^1]_{\boldsymbol{S}} = \begin{cases} \boldsymbol{P} & \text{if } [\psi^1]_{\boldsymbol{S}} = \boldsymbol{P} \\ \emptyset & \text{otherwise} \end{cases}$$

  and the induction hypothesis gives us $\|\psi\|^{\mathrm{G}}_{\boldsymbol{S}} = [\![\psi^1]\!]_{\boldsymbol{S}}$. Furthermore, since $\boldsymbol{S}$ is a positive precisification space, we know that $[\![\psi^1]\!]_{\boldsymbol{S}} = 1$ if and only if $[\psi^1]_{\boldsymbol{S}} = \boldsymbol{P}$. Therefore we get

$$\begin{aligned} [\![\varphi^1]\!]_{\boldsymbol{S}} = \mu\left([\varphi^1]_{\boldsymbol{S}}\right) &= \begin{cases} \mu(\boldsymbol{P}) & \text{if } [\psi^1]_{\boldsymbol{S}} = \boldsymbol{P} \\ \mu(\emptyset) & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 & \text{if } [\psi^1]_{\boldsymbol{S}} = \boldsymbol{P} \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 & \text{if } [\![\psi^1]\!]_{\boldsymbol{S}} = 1 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 & \text{if } \|\psi\|^{\mathrm{G}}_{\boldsymbol{S}} = 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

  which is exactly the truth function of the △-operator. Thus, we have $[\![\varphi^1]\!]_{\boldsymbol{S}} = \|\triangle\psi\|^{\mathrm{G}}_{\boldsymbol{S}} = \|\varphi\|^{\mathrm{G}}_{\boldsymbol{S}}$.
- $\varphi = \psi \wedge \chi$: By Lemma 5.6.6 we get

$$[\varphi^1]_{\boldsymbol{S}} = \begin{cases} [\psi^1]_{\boldsymbol{S}} & \text{if } [\psi^1]_{\boldsymbol{S}} \subseteq [\chi^1]_{\boldsymbol{S}} \\ [\chi^1]_{\boldsymbol{S}} & \text{otherwise}. \end{cases}$$

Furthermore, the induction hypothesis gives us $\|\psi\|^{\mathrm{G}}_{\boldsymbol{S}} = [\![\psi^1]\!]_{\boldsymbol{S}}$ and $\|\chi\|^{\mathrm{G}}_{\boldsymbol{S}} = [\![\chi^1]\!]_{\boldsymbol{S}}$. By Lemma 5.6.8, $[\![\psi^1]\!]_{\boldsymbol{S}} \leq [\![\chi^1]\!]_{\boldsymbol{S}}$ if and only if $[\psi^1]_{\boldsymbol{S}} \subseteq [\chi^1]_{\boldsymbol{S}}$.

Therefore we get

$$
\begin{aligned}
[\![\varphi^1]\!]_{\boldsymbol{S}} = \mu\left([\varphi^1]_{\boldsymbol{S}}\right) &= \begin{cases} \mu\left([\psi^1]_{\boldsymbol{S}}\right) & \text{if } [\psi^1]_{\boldsymbol{S}} \subseteq [\chi^1]_{\boldsymbol{S}} \\ \mu\left([\chi^1]_{\boldsymbol{S}}\right) & \text{otherwise} \end{cases} \\
&= \begin{cases} [\![\psi^1]\!]_{\boldsymbol{S}} & \text{if } [\psi^1]_{\boldsymbol{S}} \subseteq [\chi^1]_{\boldsymbol{S}} \\ [\![\chi^1]\!]_{\boldsymbol{S}} & \text{otherwise} \end{cases} \\
&= \begin{cases} [\![\psi^1]\!]_{\boldsymbol{S}} & \text{if } [\![\psi^1]\!]_{\boldsymbol{S}} \leq [\![\chi^1]\!]_{\boldsymbol{S}} \\ [\![\chi^1]\!]_{\boldsymbol{S}} & \text{otherwise} \end{cases} \\
&= \begin{cases} \|\psi\|^{\mathrm{G}}_{\boldsymbol{S}} & \text{if } \|\psi\|^{\mathrm{G}}_{\boldsymbol{S}} \leq \|\chi\|^{\mathrm{G}}_{\boldsymbol{S}} \\ \|\chi\|^{\mathrm{G}}_{\boldsymbol{S}} & \text{otherwise} \end{cases}
\end{aligned}
$$

which is exactly the truth function of conjunction in Gödel logic. Thus, we have $[\![\varphi^1]\!]_{\boldsymbol{S}} = \|\psi \wedge \chi\|^{\mathrm{G}}_{\boldsymbol{S}} = \|\varphi\|^{\mathrm{G}}_{\boldsymbol{S}}$.

- The case $\varphi = \psi \supset \chi$ can be proved with exactly the same arguments.

□

Finally, we get an even stronger result at the level of truth values.

Lemma 5.6.10. *Let $*$ be a continuous t-norm, $\boldsymbol{S}$ a positive precisification space that is linear in a set of propositional variables $\mathcal{P}$ and φ a formula in the language of Gödel logic with $\triangle$-operator consisting only of propositional variables contained in $\mathcal{P}$. Then $\|\varphi\|^{\mathrm{G}}_{\boldsymbol{S}} = \|\varphi^{\mathrm{g}}\|^{*}_{\boldsymbol{S}}$, where φ^{g} is the global* S*-translation of φ (see Definition 5.6.5).*

Proof. The proof is by induction on the structure of φ.

- $\varphi = \bar{0}$ or $\varphi = p$ for a propositional variable p: Then it is clear that $\|\varphi\|^{\mathrm{G}}_{\boldsymbol{S}} = \|\varphi\|^{*}_{\boldsymbol{S}} = \|\varphi^{\mathrm{g}}\|^{*}_{\boldsymbol{S}}$.

- $\varphi = \psi \wedge \chi$: By the induction hypothesis we know that $\|\psi\|^{\mathrm{G}}_{\boldsymbol{S}} = \|\psi^{\mathrm{g}}\|^{*}_{\boldsymbol{S}}$ and $\|\chi\|^{\mathrm{G}}_{\boldsymbol{S}} = \|\chi^{\mathrm{g}}\|^{*}_{\boldsymbol{S}}$. Since the for any continuous t-norm the truth function of weak conjunction is exactly the minimum-conjunction in Gödel logic we get

$$
\begin{aligned}
\|\varphi^{\mathrm{g}}\|^{*}_{\boldsymbol{S}} = \|\psi^{\mathrm{g}} \wedge \chi^{\mathrm{g}}\|^{*}_{\boldsymbol{S}} &= \min\left(\|\psi^{\mathrm{g}}\|^{*}_{\boldsymbol{S}}, \|\chi^{\mathrm{g}}\|^{*}_{\boldsymbol{S}}\right) \\
&= \min\left(\|\psi\|^{\mathrm{G}}_{\boldsymbol{S}}, \|\chi\|^{\mathrm{G}}_{\boldsymbol{S}}\right) = \|\psi \wedge \chi\|^{\mathrm{G}}_{\boldsymbol{S}} = \|\varphi\|^{\mathrm{G}}_{\boldsymbol{S}}
\end{aligned}
$$

- $\varphi = \triangle\psi$: Then we have $\varphi^{\mathrm{g}} = (\triangle\psi)^{\mathrm{g}} = (\triangle\psi)^{1} = \mathrm{S}\psi^{1}$. Since $\boldsymbol{S}$ is a positive precisification space we know that $[\psi^1]_{\boldsymbol{S}} = \boldsymbol{P}$ if and only if $[\![\psi^1]\!]_{\boldsymbol{S}} = 1$. By

Lemma 5.6.9 we also know that $[\![\psi^1]\!]_{\boldsymbol{S}} = \|\psi\|^{\mathrm{G}}_{\boldsymbol{S}}$. Thus we get

$$\begin{aligned}\|\varphi\|^{\mathrm{G}}_{\boldsymbol{S}} = \|\triangle\psi\|^{\mathrm{G}}_{\boldsymbol{S}} &= \begin{cases}1 & \text{if } \|\psi\|^{\mathrm{G}}_{\boldsymbol{S}} = 1\\ 0 & \text{otherwise}\end{cases}\\ &= \begin{cases}1 & \text{if } [\![\psi^1]\!]_{\boldsymbol{S}} = 1\\ 0 & \text{otherwise}\end{cases}\\ &= \begin{cases}1 & \text{if } [\psi^1]_{\boldsymbol{S}} = \boldsymbol{P}\\ 0 & \text{otherwise}\end{cases}\\ &= \|\mathrm{S}\psi^1\|^*_{\boldsymbol{S}} = \|\varphi^{\mathrm{g}}\|^*_{\boldsymbol{S}}\ .\end{aligned}$$

Note that we do not need the induction hypothesis in this case.

- $\varphi = \psi \supset \chi$: By the induction hypothesis we know that $\|\chi\|^{\mathrm{G}}_{\boldsymbol{S}} = \|\chi^{\mathrm{g}}\|^*_{\boldsymbol{S}}$. It is obvious that $\|\varphi\|^{\mathrm{G}}_{\boldsymbol{S}} = \|\psi \supset \chi\|^{\mathrm{G}}_{\boldsymbol{S}} = \|\triangle(\psi \supset \chi) \vee \chi\|^{\mathrm{G}}_{\boldsymbol{S}}$. We use the previous case, which needed no induction hypothesis, to obtain $\|\triangle(\psi \supset \chi)\|^{\mathrm{G}}_{\boldsymbol{S}} = \|(\triangle(\psi \supset \chi))^{\mathrm{g}}\|^*_{\boldsymbol{S}}$. Since for all continuous t-norms the truth function of disjunction is the same, we get

$$\begin{aligned}\|\varphi\|^{\mathrm{G}}_{\boldsymbol{S}} = \|\triangle(\psi \supset \chi) \vee \chi\|^{\mathrm{G}}_{\boldsymbol{S}} &= \max\left(\|\triangle(\psi \supset \chi)\|^{\mathrm{G}}_{\boldsymbol{S}}, \|\chi\|^{\mathrm{G}}_{\boldsymbol{S}}\right)\\ &= \max\left(\|(\triangle(\psi \supset \chi))^{\mathrm{g}}\|^*_{\boldsymbol{S}}, \|\chi^{\mathrm{g}}\|^*_{\boldsymbol{S}}\right)\\ &= \|(\triangle(\psi \supset \chi))^{\mathrm{g}} \vee \chi^{\mathrm{g}}\|^*_{\boldsymbol{S}}\\ &= \|\varphi^{\mathrm{g}}\|^*_{\boldsymbol{S}}\end{aligned}$$

□

It is now possible to embed Gödel logic with △ into the logic given by all positive, linear precisification spaces. Since linearity can be expressed by a formula, we can embed Gödel logic with △ into the logic of all positive precisification spaces, which corresponds to the concept of p-validity.

Theorem 5.6.11. *Let φ be a formula in the language of Gödel logic with △-operator and $*$ a continuous t-norm. Let $\mathcal{P}$ denote the finite set of propositional variables contained in φ and let $\psi_{\mathcal{P}}$ denote the characterizing formula of precisification spaces that are linear in $\mathcal{P}$ as defined in Corollary 5.6.4. Then the following holds:*

(i) φ is valid in Gödel logic with △ if and only if $\psi_{\mathcal{P}} \supset \varphi^{\mathrm{g}}$ is p-valid in S$*$.

(ii) φ is valid in Gödel logic with △ if and only if $\psi_{\mathcal{P}} \supset \varphi^{\mathrm{g}}$ is valid in SŁ.

Proof. The second statement follows from the first statement due to Theorem 5.4.7. To prove the first statement, assume that φ is valid in Gödel logic with △. Let $\boldsymbol{S}$ be an arbitrary positive precisification space. We want to show that $\|\psi_{\mathcal{P}} \supset \varphi^{\mathrm{S}}\|^*_{\boldsymbol{S}} = 1$.

If $\|\psi_{\mathcal{P}}\|^*_{\boldsymbol{S}} = 0$, then trivially $\|\psi_{\mathcal{P}} \supset \varphi^{\mathrm{S}}\|^*_{\boldsymbol{S}} = 1$. Assume now that $\|\psi_{\mathcal{P}}\|^*_{\boldsymbol{S}} \neq 0$. Then it must be the case that $\|\psi_{\mathcal{P}}\|^*_{\boldsymbol{S}} = 1$. By Corollary 5.6.4 we then know that $\boldsymbol{S}$ is linear in $\mathcal{P}$. Therefore we may apply Lemma 5.6.10 and get $\|\varphi\|^{G}_{\boldsymbol{S}} = \|\varphi^{S}\|^{*}_{\boldsymbol{S}}$. The formula φ does not contain any S-operator which means that its truth value only depends on the truth values of the propositional variables. Therefore we can define an evaluation $\boldsymbol{e}$ by $\boldsymbol{e}(p) = \|p\|^{G}_{\boldsymbol{S}}$ for every propositional variable p and get $\|\varphi\|^{G}_{\boldsymbol{S}} = \|\varphi\|^{G}_{e}$. Since φ is valid in Gödel logic with $\triangle$ we get $\|\varphi\|^{G}_{\boldsymbol{e}} = 1$. Putting all things together we get $\|\varphi^{S}\|^{*}_{\boldsymbol{S}} = 1$ and then also $\|\psi_{\mathcal{P}} \supset \varphi^{S}\|^{*}_{\boldsymbol{S}} = 1$.

Now assume that $\psi_{\mathcal{P}} \supset \varphi^{S}$ is p-valid in S∗ and let $\boldsymbol{e}$ be an arbitrary evaluation. We want to show that $\|\varphi\|^{G}_{\boldsymbol{e}} = 1$. By Lemma 5.6.2 there is a positive precisification space $\boldsymbol{S_e}$ that is linear in $\mathcal{P}$ such that $\|\varphi\|^{G}_{\boldsymbol{e}} = \|\varphi\|^{G}_{\boldsymbol{S_e}}$. Since $\psi_{\mathcal{P}} \supset \varphi^{S}$ is p-valid we get $\|\psi_{\mathcal{P}} \supset \varphi^{S}\|^{*}_{\boldsymbol{S_e}} = 1$. Because $\boldsymbol{S_e}$ is linear in $\mathcal{P}$ we know that $\|\psi_{\mathcal{P}}\|^{*}_{\boldsymbol{S_e}} = 1$ and therefore also $\|\varphi^{S}\|^{*}_{\boldsymbol{S_e}} = 1$ must hold. By Lemma 5.6.10 we then get $\|\varphi\|_{\boldsymbol{e}} = \|\varphi\|^{G}_{\boldsymbol{S_e}} = \|\varphi^{S}\|^{*}_{\boldsymbol{S_e}} = 1$. Since $\boldsymbol{e}$ was an arbitrary evaluation we conclude that φ^{S} is valid in Gödel logic with $\triangle$. □

Conclusions

Supervaluationism and fuzzy logic are two concepts that can both be applied to dealing with vagueness; however the corresponding research areas greatly differ in their focus. Research in supervaluationism primarily compares supervaluationism—or one of its variants—to other proposed theories of vagueness by arguing about its advantages and disadvantages. Traditional logical questions, like model theory and proof theory, are secondary. Fuzzy logics are nowadays highly developed from a logical point of view due to many technical contributions in the last two decades. However, fuzzy logic as a theory of vagueness in the philosophical sense is only a narrow field. Therefore, topics like higher-order vagueness, which is for example very central in supervaluationism, are barely discussed in the "mainstream" literature on fuzzy logics.

Many philosophers reject fuzzy logic as a suitable way of modeling vagueness, mostly due to concerns about the nature of truth values. In some sense, this thesis has shown two ways of coping with this criticism. The first approach is to relax the "strength" of the truth values: if only the order of the truth values counts, we arrive at Gödel logic. The second approach introduced truth values to the supervaluational model in a natural way.

We outlined a very general approach of specifying propositional logics of comparison and showed that all these specified logics are already captured by Gödel logic. We defined our logics of comparison using the framework of projective logics as introduced by Baaz and Fermüller [3], for which they presented analytic Gentzen-style proof systems. A natural question is how this framework can be extended to first-order logics. This also leads to the question how quantifiers can be generalized.

By adding a means of measuring truth to precisification spaces we arrived at a hybrid logic for which we were concerned with several problems revolving around the notion of validity. We pointed out connections between some of the hybrid logics to certain fuzzy logics with globalization operator by studying embeddings between them. It seems plausible that more embeddings of that kind exist. In another effort, we related different forms of validity for the hybrid logic arising from restrictions imposed on precisification spaces. We showed that p-validity and u-validity coincide for both the Łukasiewicz and the Gödel variant of our hybrid logic. The arguments for obtaining this result are very different for those two logics. For the Łukasiewicz variant even more holds: u-validity and general validity coincide due to the continuity of all truth functions. The question whether p-validity and u-validity also coincide for continuous t-norms different from Łukasiewicz and the Gödel t-norm remains open, in particular for the product t-norm. It must be noted that some of our arguments involving positive precisification spaces only need the property of a certain "faithfulness" of fuzzy truth with respect to supervaluational truth (see Proposition 5.1.7). It would therefore make sense to study the class of precisification spaces that have this property in its own right.

The next important step would be to properly develop the proof theory of the hybrid logic. A tableaux system based on a dialogue game characterization for the logic **SŁ** was given by Fermüller and Kosik [38]. The fundamental open question is to find Hilbert- and Gentzen-style proof systems for these logics.[33] We saw that the variant of the hybrid logic based on the Łukasiewicz t-norm has many properties that make it interesting. Therefore it might be a good idea to focus on this special case first.

Supervaluationism and fuzzy logic are just two starting points for a "logic of vagueness". Further tasks arise from the logical analysis of other theories of vagueness, in particular the rich frameworks of Shapiro [93] and Smith [94]. Building upon research in linguistics it should also be possible to define appropriate logical modalities for vague modifiers like "very" and "clearly" [6, 39, 71] that follow other ideas than the fuzzy hedges presented in Section 3.6.2.

[33] A Hilbert-style prove system for **SŁ** was conjectured by Fermüller and Kosik [38].

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