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Results in Mathematics

On Positive-Characteristic Semi-parametric Local Uniform Reductions of Varieties over Finitely Generated $\mathbb{Q}$-Algebras

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Abstract. We present a non-standard proof of the fact that the existence of a local (i.e. restricted to a point) characteristic-zero, semi-parametric lifting for a variety defined by the zero locus of polynomial equations over the integers is equivalent to the existence of a collection of local semi-parametric (positive-characteristic) reductions of such variety for almost all primes (i.e. outside a finite set), and such that there exists a global complexity bounding all the corresponding structures involved. Results of this kind are a fundamental tool for transferring theorems in commutative algebra from a characteristic-zero setting to a positive-characteristic one.

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1. Introduction

In this article we present a characterization of the fact that a finite system of polynomial equations over the integers has a local solution (i.e. a punctual one) over a characteristic zero k-algebra, where k is a field, such that the first n-components of it represent a system of quasi-parameters (i.e. they generate a maximal ideal which induces a natural 'residual' isomorphism with k). This equivalence is given in terms of the existence of positive-characteristic solutions with analogous properties and whose complexity can be uniformly bounded. This result can be seen as a kind of local-global criterion for the existence of punctual solutions of elementary 'Diophantine' varieties, which satisfies a sort

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of ‘semi-parametric’ condition. A similar result was obtained by Hochster [3, Pag. 22] with the restrictions that the algebras involved should be integral domains and the first m-components should generate a maximal ideal. In addition, Hochster’s original proof uses quite intricate algebraic and homological (standard) methods. As we will see in the last section, the former constraints can be essentially avoided due to the powerful non-standard methods that we use in our proof, which are essentially based in the remarkable introduction of ultraproducts in commutative algebra due to the work of Schoutens [8, §1, 5], [9, Ch. 4].

2. Preliminary Facts

We start by recalling the following definitions (see [8]). Throughout this discussion, we will fix a monomial order in the polynomial ring $k[x_1,\ldots,x_n]$, where k denotes a field.

Definition 1. Let R be a finitely generated k-algebra.

1. Let I be an ideal of $k[x_1,\ldots,x_n]$. We will say that I has **complexity at most** d, if $n \le d$, and it is possible to choose generators for I, $f_1,\ldots,f_s$, with $\deg f_i \le d$, for $i=1,\ldots,s$.
2. We say R has **complexity at most** d if there is a presentation of R as a quotient $k[x_1,\ldots,x_n]/I$, with I an ideal of complexity at most d.
3. If $J \subseteq R$ is an ideal, we will say that J **has complexity at most** d, if R has complexity less than or equal to d, and there exists a lifting of J in $k[x_1,\ldots,x_n]$, let us say J', with complexity at most d.

Remark 1. In (1), the number of generators of I may always be bounded in terms of d . In fact, without loss of generality we can assume that all the f_i are monic, and also that the leading terms of f_i and f_j are different from each other, when $i \neq j$ (if they have same leading term, we can change f_j by $f_j - f_i$ and get a new set of generators for I satisfying this last property). So, $s \le D$, where D is the number of monomials of degree d, $D = |\{x_1^{r_1}\cdots x_n^{r_n} \mid \Sigma_{i=1}^n r_i \le d\}|$. It is then easy to see that $D = \binom{n+d}{n} \le \binom{2d}{d}$, since $d \ge n$.

Let $A = k[x_1,\ldots,x_n]$ be the polynomial ring with a fixed monomial order. For any polynomial $f \in A$ we will denote by a_f the tuple of all the coefficients of f. When the complexity of I is at most d, and $I=(f_1,\ldots,f_s)$, by adding zeroes if necessary, we may always assume that $s=D$, where D is the number defined above. Then, the ideal I can be encoded by a tuple of the form

$$a_I = \left(n, \underbrace{\text{--------}}_{\substack{\text{D coefficients}\\ \text{of } f_1}}, \ldots, \underbrace{\text{--------}}_{\substack{\text{D coefficients}\\ \text{of } f_s}} \right) \in \mathbb{N} \times k^{D^2},$$

where the monomials are listed according to the fixed monomial order. Conversely, given one of those tuples, a, we can always reconstruct the ideal it comes from. This ideal we shall denote by $\mathcal{I}(a)$. Similarly, if R is a k-algebra with complexity at most d, then R can be written as $k[x_1, \ldots, x_n]/\mathcal{I}(a)$. We will express this fact as $R = \mathcal{R}(a)$.

For the sake of clarity for the reader and for introducing some important terminology used later, we will state explicitly some seminal results described in [8]. Let us recall that if $\Phi(x_1, \cdots, x_n)$ is a first-order formula, where $x_1, \cdots, x_n$ are the free variables, then the support of Φ with respect to a fixed interpretation A, denoted by $|\Phi|_A$, consists of all the $n-$tuples $(a_1, \cdots, a_n) \in A^n$ such that $\Phi(a_1, \cdots, a_n)$ is true in A.

Theorem 1 *([8], Proposition 5.1.). For each $d, h > 0$, there exists a formula $\mathrm{Height}_d = h$ such that for any field k, any finitely generated k-algebra R of complexity at most d, and any ideal $I \subseteq R$ of complexity at most d, the height of I is equal to h if and only if $(a, b) \in |\mathrm{Height}_d = h|_k$, where a, b are codes for R and I , respectively.*

Example 1. Let k be an algebraic closed field of characteristic 0, let $I \subseteq k[x_1, \ldots, x_n]$ be an ideal of height h and complexity at most d. Let a_I be a code for I (so that, $k \models \exists(\xi)(\mathrm{Height}_d(\xi) = h)$, namely $\mathrm{Height}_d(a_I) = h$, holds in k). So, by Lefschetz's Principle $\mathbb{F}_p^{alg} \models \exists(\xi)(\mathrm{Height}_d(\xi) = h)$, for all $p > m$, where m is some (fixed) integer. Let $a'_{I'}$ be a tuple in $\mathbb{F}_p^{alg}$ for which, after substitution, the sentence $\exists(\xi)(\mathrm{Height}_d(\xi) = h)$ holds true in $\mathbb{F}_p^{alg}$ for $p > m$. Then, by decoding $a'_{I'}$, we may find an ideal $I' \subseteq \mathbb{F}_p^{alg}[x_1, \ldots, x_n]$ of height h and with complexity at most d.

Corollary 1 *([9], Theorem 4.4.1.).Given $d > 0$, there exists a formula $IdMem_d$ such that for any field k, any ideal $I \subseteq k[x_1, \ldots, x_n]$ and any k-algebra R, both of complexity at most d over k, it holds that $f \in IR$ if an only if $k \models IdMem_d(a_f, a_I)$. Here a_f and a_I denote codes for f and I respectively.*

Theorem 2 *([9] Theorem 4.4.6). For any pair of integers $d, n > 0$, there exists a bound $b = b(d, n)$ such that for any field k, and any ideal $I \subseteq k[x_1, \ldots, x_n]$ of complexity at most d, its radical $J = \mathrm{Rad}(I)$ has complexity at most b. Moreover, $J^b \subseteq I$, and I has at most b distinct minimal primes all of which are generated by polynomials of degree at most b.*

Example 2. Given $d, n > 0$, there exists a formula Rad_d such that for any field k and any pair of ideals $P, I \subseteq k[x_1, \ldots, x_n]$ of complexity at most d, with P a prime ideal containing I, it holds: the radical of I is P (i.e., $\mathrm{Rad}(I) = P$) if and only if $k \models \mathrm{Rad}_d(a_I, a_P)$. Here a_I, a_P are codes for I and P respectively. In fact, by the last theorem, we know that there exists a bound $b = b(d, n)$, depending only on d and n such that $P^b \subseteq I$. But this is equivalent to saying that $\mathrm{Rad}(I) = P$, since $\mathrm{Rad}(I)$ is the intersection of all prime ideals containing I. It is then sufficient for the formula Rad_d to express that the product of any set of b elements between the bounded generators of P lies in I. Now, this can be done by means of a first order formula, by using Corollary 1.

Remark 2. Using Corollary 1, it is easy to get for each d, formulas Inc_d and Equal_d such that if R is a finitely generated k-algebra with complexity at most d, and if J and I are ideals of R with complexity less than d, then $(a_I, a_J) \in |\mathrm{Inc}_d|_k$ (resp. $(a_I, a_J) \in |\mathrm{Equal}_d|_k$) if and only if $I \subseteq J$ (resp. $I = J$).

Theorem 3 *([9] Theorem 4.4.4). For any pair of integers $d, n > 0$, there exists a bound $b = b(d, n)$ such that for any field k and any ideal $P \subseteq k[x_1, \ldots, x_n]$ of complexity at most d, P is a prime ideal if and only if for any two polynomials f, g of complexity at most b which do not belong to P then neither does their product.*

Remark 3. Given $d, n > 0$ there exists a formula Prime_d such that for any field k and any ideal $P \subseteq k[x_1, \ldots, x_n]$ of complexity at most d, P is a prime ideal if and only if $k \models \mathrm{Prime}_d(a_P)$. Where a_P is a code for P.

The existence of this formula follows from the last theorem, and from Corollary 1.

Example 3. Let k be an algebraic closed field with $\mathrm{char}(k) = 0$, let P be a prime ideal in $k[x_1, \ldots, x_n]$ of complexity at most d, and let a_P be a code for P.

So, $k \models \exists(\xi)\mathrm{Prime}_d(\xi)$, since $\mathrm{Prime}_d(a_P)$ holds in k. By Lefschetz's Principle $\mathbb{F}_p^{alg} \models \exists(\xi)\ \mathrm{Prime}_d(\xi)$ for all $p > m$, for some m.

Furthermore, if $a'_{P'}$ is a tuple in $\mathbb{F}_p^{alg}$ for which the sentence $\exists(\xi)\mathrm{Prime}_d(\xi)$ is true in $\mathbb{F}_p^{alg}$, for a fix prime number $p > m$, then, by decoding $a'_{P'}$, we may find a prime ideal $P' \subseteq \mathbb{F}_p^{alg}[x_1, \ldots, x_n]$ with complexity at most d.

Example 4. Given $d, n > 0$ there exists a formula $\mathrm{MaxIdeal}_{d,n}$ such that for any algebraic closed field k and any ideal $m \subseteq k[x_1, \ldots, x_n]$ of complexity at most d we have:
m is a maximal ideal if and only if $k \models \mathrm{MaxIdeal}_{d,n}(a_m)$, where a_m is a code for m. In fact, by the Nullstellensatz m is maximal if and only if there exist $b_1, \ldots, b_n \in k$ such that $m = (x_1 - b_1, \ldots, x_n - b_n)$. Let us call $J_{\underline{b}} = (x_1 - b_1, \ldots, x_n - b_n)$. Then, the required formula is:

$$\mathrm{MaxIdeal}(\xi) : (\exists b_1, \ldots, b_n)(\mathrm{Equal}_d(\xi, a_{J_{\underline{b}}})),$$

where ξ and a_J must be replaced by the codes a_m of m, and $a_{J_{\underline{b}}}$ of $J_{\underline{b}}$, respectively.

Lemma 1. *Let $\{F_\alpha(\underline{X}, \underline{Y})\}_{\alpha=1,\ldots,l}$ be a polynomial system of equations with coefficients in $\mathbb{Z}$. Then, the following two conditions are equivalent:*

(a) There exists a $k-$algebra $S = k[T_1, \ldots, T_\nu]/I$ (resp. integral domain) over an algebraic closed field k of characteristic 0, where $I \subseteq k[T_1, \ldots, T_\nu]$ is an ideal (resp. prime ideal) of complexity at most d; and $m \subseteq k[T_1, \ldots, T_\nu]$ a

prime ideal (resp. maximal) with complexity at most d and height n in S. And, there exists a tuple $(\underline{x}, \underline{y}) = (x_1, \ldots, x_n, y_1, \ldots, y_r)$ of elements in S such that:

1. $\mathrm{Rad}(\underline{x}) = m$ *in* S.
2. $F_\alpha(\underline{x}, \underline{y}) = 0$, *for all* $\alpha = 1, \ldots, l$.

(b) There exists $c, d \in \mathbb{N}$, such that for any prime number $p \geq c$, we can always construct a $\mathbb{F}_p^{alg}$-algebra (resp. $\mathbb{F}_p^{alg}$-domain) $S' = \mathbb{F}_p^{alg}[T_1, \ldots, T_\nu]/I'$, with $I' \subseteq \mathbb{F}_p^{alg}[T_1, \ldots, T_\nu]$ an ideal (resp. prime ideal) of complexity at most d, such that:
There exists $m' \subseteq \mathbb{F}_p^{alg}[T_1, \ldots, T_\nu]$, a prime ideal (resp. maximal) with complexity at most d and height n in S', and $(\underline{x'}, \underline{y'}) = (x'_1, \ldots, x'_n, y'_1, \ldots, y'_r)$, a tuple of elements in S' such that:

1. $\mathrm{Rad}(\underline{x'}) = m'$ *in* S'.
2. $F_\alpha(\underline{x'}, \underline{y'}) = 0$, *for all* $\alpha = 1, \ldots, l$.

Proof. ($\Rightarrow$) First, let us consider the cases where I is a prime ideal and m is a prime ideal, i.e, $I \subseteq m$, and $\mathrm{ht}(m) = \omega = n + \mathrm{ht}(I)$ in $k[T_1, \ldots, T_\nu]$, where ω is an integer $\leq \nu$ (resp. $\omega = \nu$, if (and only if) m is maximal). The hypothesis above may be expressed by means of a first order formula Φ_d such that when we evaluate it on the codes of $S, m, I, \{F_\alpha(\underline{X}, \underline{Y})\}$, respectively; $k \models \Phi_d$ if and only if m is a maximal ideal of height n in S, I is a prime ideal contained in m, and (1), (2) are satisfied. This formula may be explicitly given as:

$$\begin{aligned}\Phi_d : &(\exists a_m, a_I, a_{(\underline{x},\underline{y})})(\mathrm{Prime}_d(a_I) \wedge \mathrm{Prime}_d(a_m)\\ &\mathrm{Inc}_d(a_I, a_m) \wedge \mathrm{Height}_d(a_m) = \omega \wedge \mathrm{Height}_d(a_I) = \omega - n \wedge \mathrm{Rad}_d(a_{(\underline{x})}, a_m) \wedge\\ &\mathrm{IdMem}_d(F_\alpha(\underline{x}, \underline{y}), a_I)),\end{aligned}$$

where the degrees of parameters $(\underline{x}, \underline{y})$ in Φ_d is bounded by the degrees of fixed liftings in $k[T_1, \cdots, T_\nu]$ of the actual existing parameters $(\underline{x}, \underline{y})$ in S.

Now, by Lefschetz's principle, $k \models \Phi_d$, if and only if $\mathbb{F}_p^{alg} \models \Phi_d$, for any prime p large enough. As we discussed in Examples 2, 3, 1 and Remark 2; there are $\mathbb{F}_p^{alg}-$tuples a'_m, a'_I and $a'_{(\underline{x},\underline{y})}$ which codify a prime ideal m' (resp. $\mathrm{Height}_d(a_m) = \nu$ codifies additionally the maximality of m), a prime ideal I', and a system of elements $\underline{x}, \underline{y}$, satisfying all the required conditions in $\mathbb{F}_p^{alg}[T_1, \ldots T_\nu]/I'$.

So, $S' = \mathbb{F}_p^{alg}[T_1, \ldots T_\nu]/I'$ is the required $\mathbb{F}_p^{alg}$-algebra. Finally, it is clear from above that S' might be constructed of characteristic equal to p, for any prime p big enough.

For the cases where I is not necessarily a prime ideal we just eliminate the formula $Prime_d(-)$ on $\Phi(d)$ from the former proof, accordingly.

($\Leftarrow$) If there exists a global complexity d satisfying (b) for all prime numbers $p \geq c$, then we can construct in each of the former cases a suitable formula Φ_d, such that $\mathbb{F}_p^{alg} \models \phi_d$ for all $p \geq r$. Thus, by Lefschetz's principle for any algebraic closed field k of characteristic zero, we get $k \models \Phi_d$. So, as we have seen before in the examples we can construct the corresponding k-structures described in condition (a), which finishes the proof. $\square$

Remark 4. Note that in the former lemma the condition (a) can be re-phrased in a more general way requiring that for a fixed complexity d and any algebraic closed field k characteristic zero, those k-structures mention there exists. In this case, the proof would be essentially the same due to the usage of the Lefschetz's principle.

3. Main Result

The main theorem of this paper is the following:

Theorem 4. *Let $\{F_\alpha(\underline{X},\underline{Y})\}$, with $\alpha = 1,\ldots,l$, be a polynomial system of equations with coefficients in $\mathbb{Z}$. Then, the following two conditions are equivalent:*

- *(a) There exists a field of characteristic zero k, a finitely generated k-algebra (resp. domain) S; a prime ideal $m \subseteq S$ (resp. maximal) with height n and $(\underline{x},\underline{y}) = (x_1,\ldots,x_n,y_1,\ldots,y_r)$ a tuple of elements in S such that:*
  - *1.* $\mathrm{Rad}(\underline{x}) = m$.
  - *2.* $F_\alpha(\underline{x},\underline{y}) = 0$, *for every α.*
  - *3. (If m is a maximal ideal) $k \subseteq S \xrightarrow{\pi} S/m$ is an isomorphism.*
- *(b) There exists a global complexity d such that for all prime numbers p not belonging to a finite set, there exists a field L of prime characteristic, a finitely generated L-algebra (resp. domain) S', a prime ideal m' of S' (resp. maximal) with height n, and elements $(\underline{x}',\underline{y}') = (x'_1,\ldots,x'_n, y'_1,\ldots,y'_r)$ in S', all with complexity at most d such that:*
  - *1.* $\mathrm{Rad}(\underline{x}') = m'$.
  - *2.* $F_\alpha(\underline{x}',\underline{y}') = 0$, *for all α.*
  - *3. (If m' is a maximal ideal) $L \overset{i}{\hookrightarrow} S' \xrightarrow{\pi} S'/m'$, with $\pi\circ i$ an isomorphism.*

Proof. **First step**: Reduction of the condition (a) to the case where k is algebraically closed, when m is a maximal ideal. The remaining cases can be proved in a basically the same way.

Let R be a finitely generated k-algebra, where k is any field of characteristic zero. Let $m \subseteq R$ a maximal ideal of height n, and let $x_1,\ldots,x_n$ be elements in R such that $\mathrm{Rad}(x_1,\ldots,x_n) = m$, and such that conditions (1)-(3) hold in R. Let $\overline{k}$ be an algebraic closure of k, and define $R' = \overline{k}\otimes_k R$. We notice that $R \longrightarrow R'$ is a faithfully flat extension, and therefore R injects into R'. Moreover, $mR'\cap R = m$ ([7] Theorem 7.5, (2), page 49). Consequently, there is a prime ideal $q \subseteq R'$ such that $q\cap R = m$. We notice that q has to be maximal: Since R is a finitely generated k-algebra domain, all its maximal ideals have the same height, equal to the Krull dimension of R. Hence, by Noether Normalization Theorem, there are algebraically independent elements

$a_1, \ldots, a_n$ in R such that $A = k[a_1, \ldots, a_n] \subseteq R$ is a module-finite extension. But this implies that $\overline{k}[a_1, \cdots, a_n] \subseteq \overline{k} \otimes_k R$ is also module-finite extension, and consequently $\dim(\overline{k} \otimes_k R) = \dim R$. From this, we see that q has to be a maximal ideal in R', since R'/q is an integral domain of dimension zero, i.e., a field.

Now, in R'_q, the ideal mR'_q is qR'_q -primary. Thus, there is a power $n > 0$ such that $q^n R'_q \subseteq mR'_q$. On the other hand, there is a power $l > 0$, such that $m^l R \subseteq (x_1, \ldots, x_n)R$. Thus, $q^{n+l} R'_q \subseteq (x_1, \ldots, x_n)R'_q$. After inverting a finite number of elements in R' we may assume that by localizing at a single element $u \in R' - q$, the inclusion $q^{n+l} R'_u \subseteq (x_1, \ldots, x_n)R'_u$ still holds. We let R'' be the localized ring R'_u.

This ring is a finitely generated $\overline{k}$-algebra extension of R of the same dimension. Moreover, the ideal $m'' = qR''$ is maximal, and

$$q^{n+l} R'' \subseteq (x_1, \ldots, x_n)R''.$$

Therefore, $\mathrm{Rad}(x_1, \ldots, x_n)R'' = m''R''$. Let $Q \subseteq R''$ be a minimal prime ideal of R'' included in m'', and such that $\dim(R''/Q) = \dim R''$. Thus, if we let S be the ring R''/Q, and let $\eta = m''S$, then, S is a f.g. $\overline{k}$-algebra domain, with $ht(\eta) = n$, and $\mathrm{Rad}(x_1, \ldots, x_n)S = \eta S$. Thus, condition 1 holds in S.

Besides, since there is a ring homomorphism $R \longrightarrow S$, it is then clear that condition 2 also holds in S. Finally, the Nullstellensatz implies that condition 3 is true in S. So, we may replace R by S.

Second step:

($\Rightarrow$) Let us take a presentation for S, say $S = k[T_1, \ldots, T_\nu]/I$. Since S is an algebra (resp. integer domain), we have that $I \subseteq k[T_1, \ldots, T_\nu]$ is a ideal (resp. prime ideal). By the hypothesis, there exists a prime ideal $m \subseteq k[T_1, \ldots, T_\nu]$ with height n in S (resp. maximal, i.e., $\mathrm{ht}(m) = \nu = n + \mathrm{ht}(I)$ in $k[T_1, \ldots, T_\nu]$), and there exists a tuple of elements in S, $(\underline{x}, \underline{y}) = (x_1(\underline{t}), \ldots, x_n(\underline{t}), y_{1,} \ldots, y_r(\underline{t}))$ such that $\mathrm{Rad}(\underline{x}) = m$ and $F_\alpha(\underline{x}, \underline{y}) = 0$, for all $\alpha = 1, \ldots, l$. Let us note that since we can suppose by the first step that k is algebraically closed, the condition requiring that $k \subseteq S \xrightarrow{\pi} S/m$ is an isomorphism, turns out to be (trivially) satisfied.

Let $d > 0$ be an integer that bounds all the complexities of the objects mentioned above. So, the proof follows from Lemma 1.

($\Leftarrow$) Assume that there exists a uniform complexity d, and $c \in \mathbb{N}$ such that for any characteristic $p \geq c$ the L-structures of condition (b) exist. Then, by Lemma 1 there exists an (algebraically closed) field of characteristic zero k, a finitely generated k-algebra (resp. domain) S; a prime ideal $m \subseteq S$ (resp. maximal) with height n, and $(\underline{x}, \underline{y}) = (x_1, \ldots, x_n, y_{1,} \ldots, y_r)$ a tuple of elements in S such that $\mathrm{Rad}(\underline{x}) = m$, and $F_\alpha(\underline{x}, \underline{y}) = 0$, for every α. Finally, if the m' are maximal ideals, then by the Nullstellensatz, $k \subseteq S \xrightarrow{\pi} S/m$ is an isomorphism. $\square$

Remark 5. As pointed out in the introduction, the importance of the former Theorem lies in the fact that it generalizes Hochster's theorem, as it first appeared in [3, Pag. 22]. Now, Hochster's remarkable result provides a natural bridge for proving statements about equicharacteristic rings in characteristic zero, by first reducing to the case of prime characteristic. In addition, Hochster's theorem combined with M. Artin's approximation theorem [1] turns out to be especially powerful. The full existence of Big Cohen Macaulay modules, for instance, depends directly on Hochster's reduction method. For a survey of different applications of this technique the reader may consult [5, Ch. 4, 5, 7-9].

On the other hand, Hochster's theorem becomes especially relevant for the construction of a theory of Tight Closure in characteristic zero [4, Ch. 3], [5, Pag. 94] (for further readings see also [6] and [2]).

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