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DISSERTATION

Rate Optimality of Adaptive Algorithms

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Kurzfassung. Diese Arbeit schafft einen axiomatischen Rahmen für den Beweis von optimalen Konvergenzraten adaptiver Algorithmen. Das Hauptanwendungsfeld hierfür sind die Finite-Element-Methode sowie auch die Randelement-Methode. Drei Axiome für den Fehlerschätzer und drei weitere für die zugehörige Netzverfeinerung garantieren optimale Konvergenzraten. Der axiomatische Zugang erlaubt es, spezielle Fragen nach der Notwendigkeit von (diskreten) unteren Fehlerschranken, dem Einsatz von approximativen Lösern, der Einbindung von inhomogenen Randdaten oder auch der Verwendung von äquivalenten Fehlerschätzern zu beantworten. Die Weiterentwicklungen und Verbesserungen im Vergleich zum aktuellen Stand der Forschung (ausgenommen der eigenen Arbeit [**24**], welche in dieser Dissertation teilweise erweitert wird) werden im Folgenden zusammengefasst:

- Es wird ein einheitlicher und komplett abstrakter theoretischer Rahmen geschaffen, der die aktuelle Literatur zum Thema optimaler Konvergenzraten umfasst. Die abstrakte Form erlaubt es, lineare sowie nichtlineare Probleme zu behandeln, und sie ist unabhängig von der zugrundeliegenden (konformen, nicht-konformen, gemischten) Methode. Verwendet und analysiert wird einzig der Fehlerschätzer, welcher als Funktion der Triangulierung betrachtet wird. Dieser Zugang ermöglicht es, Axiome zu formulieren, die unabhängig von allen Annahmen an das konkrete Modell sind.
- Die Beweise für Konvergenz und Konvergenz mit optimaler Rate kommen ohne Effizienz des Fehlerschätzers aus. Effizienz wird in dieser Arbeit nur verwendet, um die Approximationsklasse mittels Best-Approximationsfehler und Datenfehler zu charakterisieren. Als Konsequenz davon und im Unterschied zur gegenwärtigen Literatur hängt die obere Schranke für optimale Markierungsparameter nicht mehr von der Effizienzkonstante ab.
- Die Arbeit führt eine allgemeine Quasi-Galerkinorthogonalität ein, die nicht nur hinreichend, sondern auch notwendig für die R-lineare Konvergenz des Fehlerschätzers ist. Betrachtet man die optimale Konvergenzrate des Fehlerschätzers bezüglich der Komplexität des Verfahrens (das heißt: die Komplexität der Berechnung des aktuellen Schritts und die Komplexität aller vorausgegangenen Schritte), so stellt sich die R-lineare Konvergenz selbst als notwendig heraus. Die optimale Komplexität wird dann als Konsequenz der optimalen Konvergenzraten des Fehlerschätzers bewiesen.
- Anstatt der *Overlay*-Eigenschaft (eine übliche Annahme in aktueller Literatur) verwendet diese Arbeit eine tieferliegende Eigenschaft der Netzverfeinerung. Dies erlaubt es, auch für populäre Verfeinerungsmethoden wie die Rot-Grün-Blau-Verfeinerung, optimale Konvergenzraten zu beweisen.
- Schlussendlich behandelt diese Arbeit equivalente Fehlerschätzer, approximative Löser sowie inhomogene und gemischte Randdaten. Zusätzlich wird eine neue Methode zur adaptiven Geometrie-Approximation für eine spezielle Randelement-Methode eingeführt und deren Konvergenz bewiesen.

ABSTRACT. This work aims first at the development of an axiomatic framework for the proof of optimal convergence rates for adaptive algorithms, with the main field of application being the finite element method and the boundary element method. Second, the axiomatic view allows refinements of particular questions like the avoidance of (discrete) lower bounds, inexact solvers, inhomogeneous boundary data, or the use of equivalent error estimators. Three axioms which are related to the estimator guarantee optimal convergence rates in terms of the error estimator for any refinement strategy which satisfies additional three triangulation related axioms. Compared to the state of the art in the literature (except for the recent own work **[24]** which is partially generalized), the improvements of this work can be summarized as follows:

• First, a general and completely abstract framework is presented which covers the existing literature on rate optimality of adaptive algorithms. The abstract analysis covers linear as well as nonlinear problems and is independent of the underlying (conforming, non-conforming, or mixed) finite element or boundary element method. Solely, the error estimator, considered as a function of the underlying triangulation, is used and analyzed. This allows to formulate axioms which are not restricted to any concrete model assumption.

• Second, efficiency of the error estimator is neither needed to prove convergence nor quasi-optimal convergence behavior of the error estimator. In this work, efficiency exclusively characterizes the approximation classes involved in terms of the best-approximation error and data resolution. Therefore, the upper bound on the optimal marking parameters does not depend on the efficiency constant.

• Third, some general quasi-Galerkin orthogonality is not only sufficient, but also necessary for the R-linear convergence of the error estimator, which turns out to be necessary itself when it comes to optimal complexity estimates. The latter means the optimality of the adaptive algorithm when considering the overall cost of the algorithm (which includes the computation of all previous steps) and is proved as a by-product of rate optimality.

• Fourth, we circumvent the use of the overlay estimate of the refinement strategy, which is a standard assumption in the recent literature, to include popular refinement schemes like red-green-blue refinement into the analysis.

• Finally, the general analysis allows for equivalent error estimators and inexact solvers as well as different non-homogeneous and mixed boundary conditions and is even employed to prove convergence of some novel adaptive geometry approximation for a certain boundary element method.

Contents

CHAPTER 1

Outline & Introduction

1.1. Adaptivity

In this work, adaptivity is understood as the property of some numerical algorithm to adapt its behavior to the given instance of a problem. In contrast to that, a uniform algorithm is assumed to show more or less the same behavior for any given problem in a certain class for which the algorithm is designed. This means, that the algorithm uses a priori knowledge of the problem only. One example for that difference is the numerical integration, i.e, the approximation of $\int_0^1 f(x)\,dx$ for some given function $f\colon [0,1] \to \mathbb{R}$. A uniform algorithm evaluates the function f at a priori determined grid points and computes an approximation. An adaptive quadrature, on the other hand, tries to add grid points, where f appears to be rough, and to remove grid points, where f appears to be smooth. This is done with the overall goal of reducing the computational cost to reach a certain accuracy (see Figure 1 for an example). The key difference of both approaches is that the uniform algorithm uses all evaluations of f for the computation of the approximation. The adaptive algorithm, invests some of the evaluations in the determination of better evaluation points. This strategy makes only sense, if the additional investment of computational time pays at some point in terms of an improved accuracy. Therefore, an adaptive algorithm is only useful, if the problem at hand benefits from a non-uniform approach. In terms of the quadrature example above, this is the case if one wants to design a black-box algorithm, which integrates a large class of functions equally well in terms of accuracy, since for any particular function, one could design an optimal grid of evaluation points a priori.

But also for very specific problems, an adaptive approach can make sense. An illustrative example for this situation (which however is way beyond the current state of theory), is the following: Assume one wants to predict how a car will deform under a front impact. It is obvious that the front bumpers and the hood will suffer from major deformation and thus require high computational accuracy. However, in low speed crashes, the strong cylinder block could survive without any deformation and thus it suffices to compute how the cylinder block translates and rotates within the car. This is, of course, much cheaper in terms of computational time, than computing the local deformations of the block. For high speed crashes, when even the cylinder block deforms, this might not be sufficiently accurate any more. Therefore, a detailed computation is necessary. The particular threshold speed, which separates those two cases, may not be known a priori. Hence, it might not be possible to design a uniform algorithm, which uses only a priori knowledge of the problem, but still computes the solution efficiently.

An often heard argument in favor of uniform algorithms is that computing power and memory have become so cheap that one just increases the size of the computing facility, if a given algorithm does not produce the desired accuracy. This argument is misleading for two reasons: First, even the upgraded computers can benefit from an adaptive approach which focuses the computational power on where it is needed most. Second, it might be not even possible to reach a given accuracy just by upscaling the facilities. To illustrate that, assume

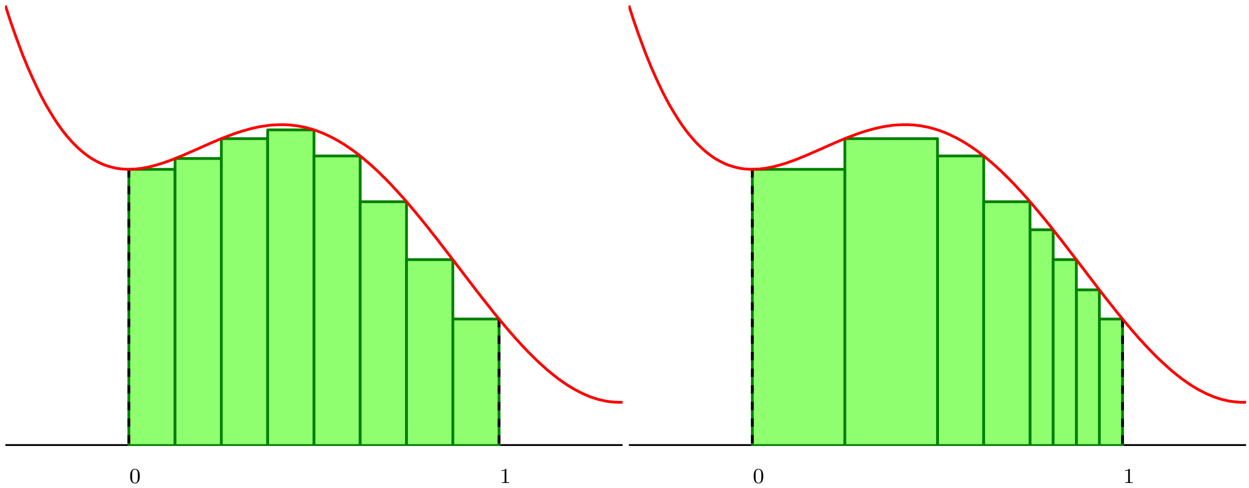

Figure 1. Numerical integration of some given function with uniform grid (left) and adaptively generated (grid).

that the approximation error (e.g., the quadrature error or the geometric differences of the simulated crash compared to an actual crash test) behaves as a function of the degrees of freedom of the discretized system, i.e.,

$$\mathrm{err}(N) \simeq N^{-s}$$

for some $s > 0$ and $N \in \mathbb{N}$ denoting the degrees of freedom (e.g., the number of evaluation points). This is a very realistic assumption for many problem classes. Note that the convergence rate s does not only depend on the problem itself, but also on the method of approaching this problem. A quadrature algorithm which wastes computational time on smooth parts of the integrand, will achieve a lower rate $s' < s$. Furthermore, assume that the computational time needed to compute the approximate solution is related to the degrees of freedom in the sense of

$$\mathrm{time}(N) \simeq N^t \,\mathrm{seconds}$$

for some $t > 0$ (for the direct solution of a densely populated linear system of N equations we have, e.g., $t = 3$). If the exact solution is known, one can design custom made grids to approximate the exact solution with some optimal rate $s_{\mathrm{opt}} > 0$, i.e.,

$$\mathrm{err}(N) \simeq N^{-s_{\mathrm{opt}}}.$$

Hence, to reach a desired accuracy of, e.g. 10^{-5}, it suffices to use $N \approx 10^{5/s_{\mathrm{opt}}}$ degrees of freedoms, when they are optimally distributed. In terms of computational time, we obtain

$$\mathrm{time} \simeq 10^{5t/s_{\mathrm{opt}}} \,\mathrm{seconds}.$$

Under realistic assumptions of the involved parameters, i.e., $t = 1$ (linear time) and $s_{\mathrm{opt}} = 1$ (e.g., lowest order finite element method), this results in

$$10^5 \,\mathrm{seconds} \approx 1 \,\mathrm{day}.$$

However, it is entirely possible, that due to non-uniformities in the solution a uniform approach will reveal a reduced rate of convergence of $s = 1/2$ (due to degrees of freedom wasted for mostly uniform parts of the solution, whereas non-uniform parts lack the necessary resolution). Then, we end up with

$$10^{10}\,\text{seconds} \approx 316\,\text{years}.$$

Even increasing the computational power by an order of magnitude does not bring the uniform approach anywhere near feasibility. This is the reason why the understanding of adaptivity plays a crucial role.

The concept of adaptivity aims to provide a method which automatically, without user intervention, reaches optimal convergence rates, i.e., $s = s_{\rm opt}$. Moreover, it aims to rigorously prove that this optimal convergence is achieved for a given problem. The existing literature on adaptivity focuses on very specific model problems (see the historical overview in Section 2.8 for references), i.e., certain types of (elliptic) partial differential equations. In contrast to that, this work provides a framework, sort of a construction guide, for adaptive algorithms which realize optimal convergence rates. To that end, certain requirements on the algorithm (later called axioms) are derived, which are sufficient and even necessary to prove the optimal convergence behavior. This allows to apply the abstract theory to a large number of model problems and particularly determines what are the key properties of an optimally convergent adaptive algorithm. This might help in the design of new algorithms for complex problems and situations.

1.2. An exemplary adaptive algorithm

This introductory section demonstrates an adaptive refinement algorithm for a very simple approximation problem. To that end, consider some function $u \in L^2(0,1)$ and a partition $\mathcal{T}$ of $[0,1]$ into compact intervals $T \in \mathcal{T}$ such that $[0,1] = \bigcup_{T\in\mathcal{T}} T$. Let $U(\mathcal{T}) \in \mathcal{P}^0(\mathcal{T})$ denote the L^2-orthogonal projection of u onto the space of $\mathcal{T}$-piecewise constant functions

$$\mathcal{P}^0(\mathcal{T}) := \big\{ V \in L^2(0,1) \,:\, V|_T \in \mathbb{R}, \text{ for all } T \in \mathcal{T} \big\}$$

defined by

$$b(U(\mathcal{T})\,,\,V) := \int_0^1 U(\mathcal{T}) V\,dx = \int_0^1 uV\,dx \quad \text{for all } V \in \mathcal{P}^0(\mathcal{T}). \tag{1.2.1}$$

Suppose that one is interested in the weighted error measure

$$\mathrm{err}(\mathcal{T}) := \Big(\sum_{T\in\mathcal{T}} |T|^2 \|u - U(\mathcal{T})\|_{L^2(T)}^2\Big)^{1/2} = \|h(\mathcal{T})(u - U(\mathcal{T}))\|_{L^2(0,1)},$$

where $h(\mathcal{T})|_T := |T|$ for all $T \in \mathcal{T}$ and $|T|$ denotes the length of the interval T. This could be of interest, if one wants to approximate the volume force of some second-order elliptic PDE (which usually has to be approximated in the $H^{-1}(0,1)$-norm). Standard results show that for $u \in L^2(0,1) \subset H^{-1}(0,1)$ it holds $\|u - U(\mathcal{T})\|_{H^{-1}(0,1)} \lesssim \mathrm{err}(\mathcal{T})$.

Provided that $u \in H^1(0,1)$, the Poincaré inequality proves that

$$\mathrm{err}(\mathcal{T}) \le C_{\rm apriori} \|h(\mathcal{T})^2 u'\|_{L^2(0,1)} \le C_{\rm apriori} \|u'\|_{L^2(0,1)} \max_{T\in\mathcal{T}} |T|^2. \tag{1.2.2}$$

Thus, the naive strategy is to uniformly reduce $|T|$ in some sequence of partitions $(\mathcal{T}_\ell^{\rm unif})_{\ell\in\mathbb{N}_0}$ such that $\max_{T\in\mathcal{T}_\ell^{\rm unif}} |T| \le 2^{-\ell}$. If $u \in H^1(0,1)$, this results in a convergence rate of

$$\|u - U(\mathcal{T}_\ell^{\rm unif})\|_{L^2(0,1)} \lesssim 2^{-2\ell} \quad \text{for all } \ell \in \mathbb{N}_0,$$

which one could call *exponential* convergence. The reason why we do not consider this as exponential convergence, is because the number of steps ℓ has nothing to do with the degrees of freedom of the linear system (1.2.1). However, the computational effort involved to get $U(\mathcal{T}_\ell^{\rm unif})$ is directly related to the degrees of freedom, since the linear system (even if it is diagonal in this case) has $|\mathcal{T}|$ many rows and columns (here $|\mathcal{T}|$ denotes the counting measure, i.e., the number of elements). In terms of degrees of freedom, the convergence rate decreases to

$$\|u - U(\mathcal{T}_\ell^{\rm unif})\|_{L^2(0,1)} \lesssim |\mathcal{T}_\ell^{\rm unif}|^{-2} \quad \text{for all } \ell \in \mathbb{N}_0.$$

This shows algebraic convergence rate $s = 2$ if $u \in H^1(0,1)$. If u has less regularity, e.g., $u(x) := x^\alpha$ for some $-1/2 < \alpha < 1/2$, the convergence rate is even slower, see Figure 2 for an example. However, one can construct graded partitions $\mathcal{T}_\ell^{\rm grad}$, such that the function $u(x) := x^\alpha$ can be approximated with rate $s = 2$. To that end, a uniform partition $\mathcal{T}_\ell^{\rm unif}$ is mapped via an appropriate function $x \mapsto x^\beta$ for $\beta := 3/(2+\alpha)$, i.e., $\mathcal{T}_\ell^{\rm grad} = (\mathcal{T}_\ell^{\rm unif})^\beta$; see Figure 2–3 for an example. Standard estimates prove

$$\|u - U(\mathcal{T}_\ell^{\rm grad})\|_{L^2(0,1)} \leq C_{\rm grad}|\mathcal{T}_\ell^{\rm grad}|^{-2} \quad \text{for all } \ell \in \mathbb{N}_0 \tag{1.2.3}$$

for some uniform $C_{\rm grad} > 0$, even though the exact solution is not in $H^1(0,1)$ for $\alpha < 1/2$. The ultimate goal of adaptivity is to automatically generate such partitions for a general class of exact solutions u. To that end, the following algorithm is widely used in the literature:

Algorithm 1.2.1. Input: *Initial partition $\mathcal{T}_0$ and bulk parameter $0 < \theta \leq 1$.*
Loop: *For $\ell = 0, 1, 2, \dots$ do* (i) – (iii).

(i) *Compute the refinement indicators $\eta_T(\mathcal{T}_\ell) := |T|\|u - U(\mathcal{T}_\ell)\|_{L^2(T)}$ for all $T \in \mathcal{T}_\ell$.*

(ii) *Determine some set $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ of minimal cardinality such that*

$$\frac{1}{2}\sum_{T\in\mathcal{T}_\ell} \eta_T(\mathcal{T}_\ell)^2 \leq \sum_{T\in\mathcal{M}_\ell} \eta_T(\mathcal{T}_\ell)^2. \tag{1.2.4}$$

(iii) *Define the next triangulation $\mathcal{T}_{\ell+1}$ by bisection of all marked elements.*

Output: *Sequence of approximations $U(\mathcal{T}_\ell)$ for all $\ell \in \mathbb{N}_0$.*

Figure 2 shows the performance of this algorithm in terms of error reduction and Figure 3 plots the generated partitions $\mathcal{T}_\ell$.

We aim to prove the observed convergence behavior of Algorithm 1.2.1 in Figure 2, i.e., the fact that $\mathrm{err}(\mathcal{T}_\ell) \lesssim |\mathcal{T}_\ell|^{-2}$ for all $\ell \in \mathbb{N}_0$. To that end, we first prove a contraction property of the error as illustrated in Figure 2, i.e.,

$$\|h(\mathcal{T}_{\ell+1})(u - U(\mathcal{T}_{\ell+1}))\|_{L^2(0,1)} \leq \kappa\|h(\mathcal{T}_\ell)(u - U(\mathcal{T}_\ell))\|_{L^2(0,1)} \quad \text{for all } \ell \in \mathbb{N}_0 \tag{1.2.5}$$

for some $0 < \kappa < 1$. This follows with the fact that bisection halves the element lengths and that $U(\mathcal{T}_\ell)|_T$ depends only on $u|_T$ by

$$\begin{aligned}
&\|h(\mathcal{T}_{\ell+1})(u - U(\mathcal{T}_{\ell+1}))\|_{L^2(0,1)}^2 \\
&\qquad = \|h(\mathcal{T}_{\ell+1})(u - U(\mathcal{T}_{\ell+1}))\|_{L^2(\cup(\mathcal{T}_{\ell+1}\setminus\mathcal{T}_\ell))}^2 + \|h(\mathcal{T}_{\ell+1})(u - U(\mathcal{T}_{\ell+1}))\|_{L^2(\cup(\mathcal{T}_{\ell+1}\cap\mathcal{T}_\ell))}^2 \\
&\qquad \leq 1/4\|h(\mathcal{T}_\ell)(u - U(\mathcal{T}_\ell))\|_{L^2(\cup(\mathcal{T}_{\ell+1}\setminus\mathcal{T}_\ell))}^2 + \|h(\mathcal{T}_\ell)(u - U(\mathcal{T}_\ell))\|_{L^2(\cup(\mathcal{T}_{\ell+1}\cap\mathcal{T}_\ell))}^2 \\
&\qquad \leq (1/4 - 1)\|h(\mathcal{T}_\ell)(u - U(\mathcal{T}_\ell))\|_{L^2(\cup(\mathcal{T}_{\ell+1}\setminus\mathcal{T}_\ell))}^2 + \|h(\mathcal{T}_\ell)(u - U(\mathcal{T}_\ell))\|_{L^2(0,1)}^2.
\end{aligned}$$

With the marking criterion (1.2.4), the fact that $\mathcal{M}_\ell = \mathcal{T}_\ell \setminus \mathcal{T}_{\ell+1}$, and $\bigcup(\mathcal{T}_\ell \setminus \mathcal{T}_{\ell+1}) = \bigcup(\mathcal{T}_{\ell+1} \setminus \mathcal{T}_\ell)$, this implies

$$\|h(\mathcal{T}_{\ell+1})(u - U(\mathcal{T}_{\ell+1}))\|_{L^2(0,1)}^2 \leq (1 - (1 - 1/4)/2)\|h(\mathcal{T}_\ell)(u - U(\mathcal{T}_\ell))\|_{L^2(0,1)}^2,$$

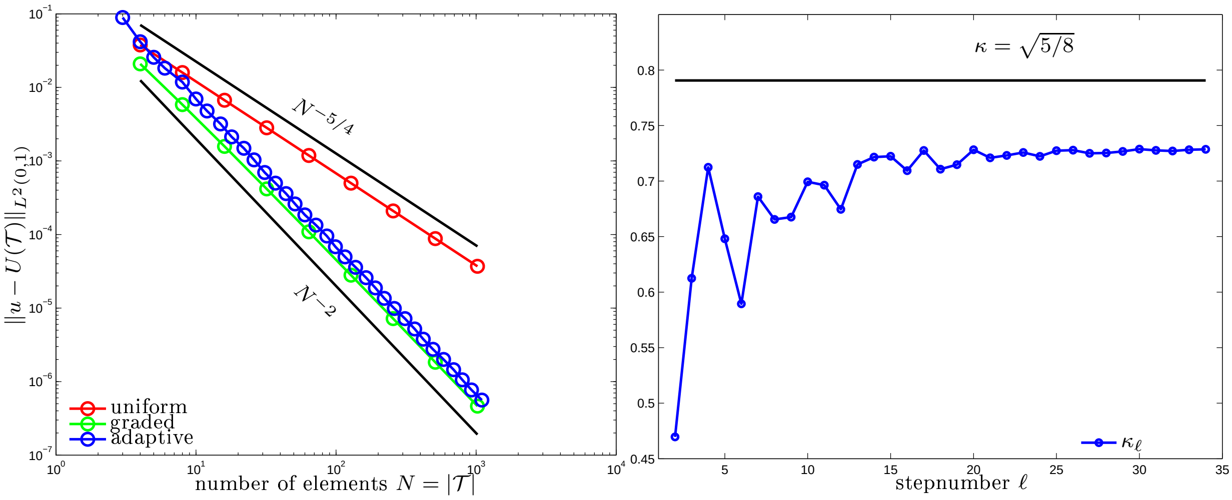

FIGURE 2. The left-hand side figure compares uniform refinement, graded partitions, and adaptively generated partitions to approximate $u(x) := x^{-1/4}$. The right-hand side figure shows the experimental contraction constant $\kappa_\ell := \mathrm{err}(\mathcal{T}_\ell)/\mathrm{err}(\mathcal{T}_{\ell-1})$ as well as the theoretical bound $\kappa = \sqrt{5/8}$ for adaptive refinement.

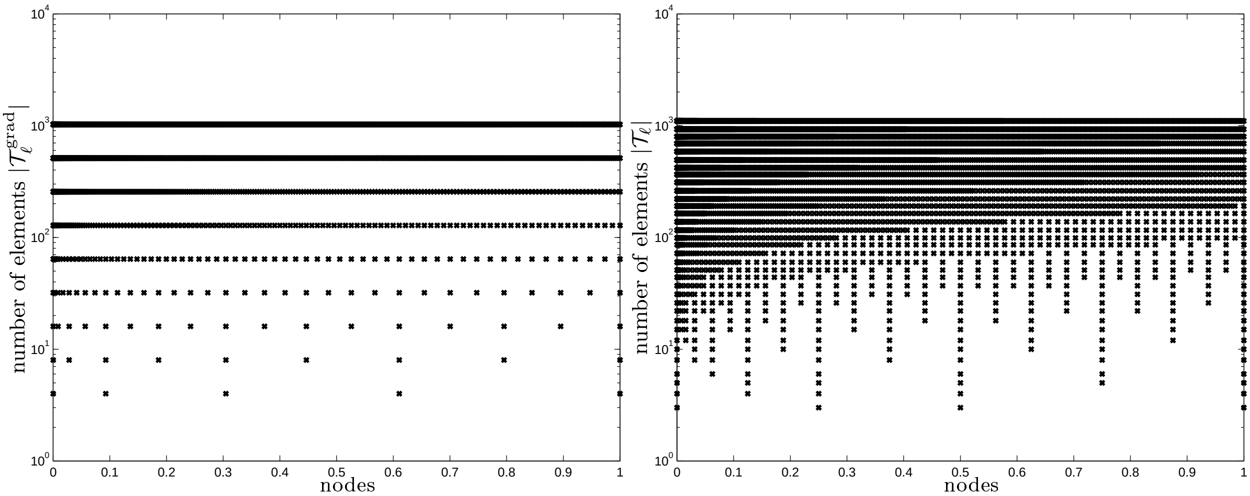

FIGURE 3. Graded partitions with $\beta = 4/3$ (left) and adaptively generated partitions (right). Each row shows the nodes of a partition $\mathcal{T}_\ell^{\mathrm{grad}}$ resp. $\mathcal{T}_\ell$ in the interval $[0, 1]$. The height of the row indicates the number of elements in the particular partition.

which is (1.2.5) with $\kappa = \sqrt{5/8}$ (see also Figure 2 for the comparison with the experimental results). Hence, the error converges linearly to zero. This linear convergence is the backbone of the optimality analysis. The next step is to compare the adaptively generated partitions with some optimal partitions. As discussed above (and demonstrated in Figure 2), there exist graded partitions $\mathcal{T}_\ell^{\mathrm{grad}}$, which realize the optimal convergence rate $s = 2$ in (1.2.3). Hence, the necessary thing to do is to look at the difference of $\mathcal{T}_\ell$ and $\mathcal{T}_\ell^{\mathrm{grad}}$. To that end,

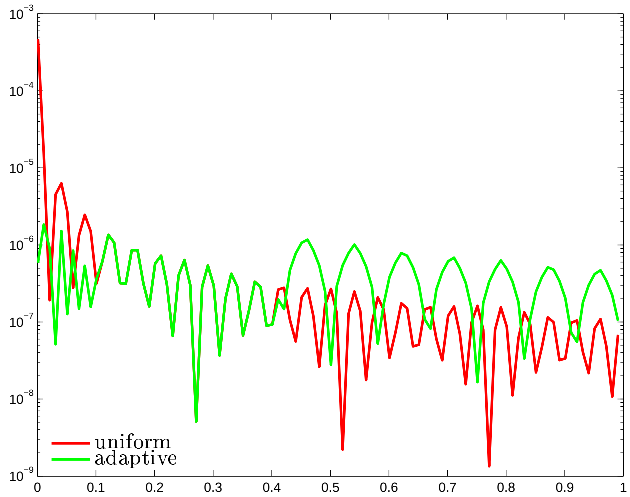

FIGURE 4. The error distribution $h(\mathcal{T})(u - U(\mathcal{T}))$ for the uniform approach with $|\mathcal{T}_{10}^{\rm unif}| = 1024$ and the adaptive approach with $|\mathcal{T}_{34}| = 928$ shows that the error is more or less equidistributed on the adaptive partitions, whereas for the uniform partitions the error is concentrated around the singularity at $x = 0$. In this example, there holds $\mathrm{err}(\mathcal{T}_{34}) \approx 10^{-7}$ and $\mathrm{err}(\mathcal{T}_{10}^{\rm unif}) \approx 10^{-5}$.

choose the minimal $k \in \mathbb{N}$ such that

$$|\mathcal{T}_k^{\rm grad}|^{-2} \le C_{\rm grad}^{-1}\mathrm{err}(\mathcal{T}_\ell)/4. \tag{1.2.6}$$

For simplicity assume that $k > 1$ in this case. Minimality of k then implies $|\mathcal{T}_{k-1}^{\rm grad}|^{-2} > C_{\rm grad}^{-1}\mathrm{err}(\mathcal{T}_\ell)/4$, i.e., $|\mathcal{T}_{k-1}^{\rm grad}| < 2C_{\rm grad}^{1/2}\mathrm{err}(\mathcal{T}_\ell)^{-1/2}$. Since we have by construction $|\mathcal{T}_k^{\rm grad}| = |\mathcal{T}_k^{\rm unif}| = 2|\mathcal{T}_{k-1}^{\rm unif}| = 2|\mathcal{T}_{k-1}^{\rm grad}|$, the minimality of k shows

$$|\mathcal{T}_k^{\rm grad}| = 2|\mathcal{T}_{k-1}^{\rm grad}| \le 4C_{\rm grad}^{1/2}\mathrm{err}(\mathcal{T}_\ell)^{-1/2}. \tag{1.2.7}$$

The overlay of $\mathcal{T}_k^{\rm grad}$ and $\mathcal{T}_\ell$ gives some measure of the distance of those two partitions, i.e., $\mathcal{T}_k^{\rm grad} \oplus \mathcal{T}_\ell := \big\{T \cap T' \,:\, T \in \mathcal{T}_k^{\rm grad},\, T' \in \mathcal{T}_\ell,\, |T \cap T'| > 0\big\}$ is the coarsest common refinement of $\mathcal{T}_\ell$ and $\mathcal{T}_k^{\rm grad}$. Assume $T_0 \in (\mathcal{T}_k^{\rm grad} \oplus \mathcal{T}_\ell) \setminus \mathcal{T}_\ell$. By definition, there exist $T \in \mathcal{T}_k^{\rm grad}$ and $T' \in \mathcal{T}_\ell$ such that $T_0 = T \cap T'$ and $|T \cap T'| > 0$. Moreover, since T is not in $\mathcal{T}_\ell$, there holds $T \not\subseteq T'$. This shows that there holds

$$(\mathcal{T}_k^{\rm grad} \oplus \mathcal{T}_\ell) \setminus \mathcal{T}_\ell = \big\{T \cap T' \,:\, T \in \mathcal{T}_k^{\rm grad},\, T' \in \mathcal{T}_\ell,\, |T \cap T'| > 0,\, T' \not\subseteq T\big\}.$$

Since $T \in \mathcal{T}_k^{\rm grad}$ is an interval, there exist at most two $T' \in \mathcal{T}_\ell$ with $|T \cap T'| > 0$ and $T' \not\subseteq T$ (the elements T' must contain at least one endpoint of T). This, however, implies

$$\begin{aligned}|(\mathcal{T}_k^{\rm grad} \oplus \mathcal{T}_\ell) \setminus \mathcal{T}_\ell| &= |\big\{T \cap T' \,:\, T \in \mathcal{T}_k^{\rm grad},\, T' \in \mathcal{T}_\ell,\, |T \cap T'| > 0,\, T' \not\subseteq T\big\}| \\ &\le 2|\mathcal{T}_k^{\rm grad}|.\end{aligned} \tag{1.2.8}$$

On the other hand, each $T \in \mathcal{T}_\ell \setminus (\mathcal{T}_k^{\rm grad} \oplus \mathcal{T}_\ell)$ has at least two sons $T' \subsetneq T$ with $T' \in (\mathcal{T}_k^{\rm grad} \oplus \mathcal{T}_\ell) \setminus \mathcal{T}_\ell$. This implies

$$|\mathcal{T}_\ell \setminus (\mathcal{T}_k^{\rm grad} \oplus \mathcal{T}_\ell)| \le |(\mathcal{T}_k^{\rm grad} \oplus \mathcal{T}_\ell) \setminus \mathcal{T}_\ell|. \tag{1.2.9}$$

Together with (1.2.7) this shows

$$|\mathcal{T}_\ell \setminus (\mathcal{T}_k^{\mathrm{grad}} \oplus \mathcal{T}_\ell)| \overset{(1.2.8)}{\leq} 2|\mathcal{T}_k^{\mathrm{grad}}| \overset{(1.2.7)}{\leq} 8C_{\mathrm{grad}}^{1/2}\mathrm{err}(\mathcal{T}_\ell)^{-1/2}. \tag{1.2.10}$$

It remains to relate $|(\mathcal{T}_k^{\mathrm{grad}} \oplus \mathcal{T}_\ell) \setminus \mathcal{T}_\ell|$ to $|\mathcal{M}_\ell|$. To that end, note that the element-wise best approximation property $U(\mathcal{T}_\ell)$ shows

$$\mathrm{err}(\mathcal{T}_k^{\mathrm{grad}} \oplus \mathcal{T}_\ell) \leq \mathrm{err}(\mathcal{T}_k^{\mathrm{grad}}) \overset{(1.2.3)}{\leq} C_{\mathrm{grad}}|\mathcal{T}_k^{\mathrm{grad}}|^{-2} \overset{(1.2.6)}{\leq} \mathrm{err}(\mathcal{T}_\ell)/4.$$

With $\mathrm{err}(\mathcal{T}_k^{\mathrm{grad}} \oplus \mathcal{T}_\ell) = \|h(\mathcal{T}_k^{\mathrm{grad}} \oplus \mathcal{T}_\ell)(u - U(\mathcal{T}_k^{\mathrm{grad}} \oplus \mathcal{T}_\ell))\|_{L^2(0,1)}$, this implies

$$\begin{aligned}
\mathrm{err}(\mathcal{T}_\ell)^2 &= \|h(\mathcal{T}_\ell)(u - U(\mathcal{T}_\ell))\|^2_{L^2(\cup((\mathcal{T}_k^{\mathrm{grad}} \oplus \mathcal{T}_\ell)\setminus\mathcal{T}_\ell))} \\
&\qquad + \|h(\mathcal{T}_\ell)(u - U(\mathcal{T}_\ell))\|^2_{L^2(\cup((\mathcal{T}_k^{\mathrm{grad}} \oplus \mathcal{T}_\ell)\cap\mathcal{T}_\ell))} \\
&\leq \|h(\mathcal{T}_\ell)(u - U(\mathcal{T}_\ell))\|^2_{L^2(\cup((\mathcal{T}_k^{\mathrm{grad}} \oplus \mathcal{T}_\ell)\setminus\mathcal{T}_\ell))} \\
&\qquad + \|h(\mathcal{T}_k^{\mathrm{grad}} \oplus \mathcal{T}_\ell)(u - U(\mathcal{T}_k^{\mathrm{grad}} \oplus \mathcal{T}_\ell))\|^2_{L^2(0,1)} \\
&\leq \sum_{T \in \mathcal{T}_\ell \setminus (\mathcal{T}_k^{\mathrm{grad}} \oplus \mathcal{T}_\ell)} \eta_T(\mathcal{T}_\ell)^2 + \mathrm{err}(\mathcal{T}_\ell)^2/16.
\end{aligned} \tag{1.2.11}$$

Hence, we derive

$$\frac{1}{2}\sum_{T\in\mathcal{T}_\ell} \eta_T(\mathcal{T}_\ell)^2 \leq \frac{15}{16}\mathrm{err}(\mathcal{T}_\ell)^2 \leq \sum_{T \in \mathcal{T}_\ell \setminus (\mathcal{T}_k^{\mathrm{grad}} \oplus \mathcal{T}_\ell)} \eta_T(\mathcal{T}_\ell)^2. \tag{1.2.12}$$

Since $\mathcal{M}_\ell$ is a set of minimal cardinality with (1.2.4), we obtain

$$|\mathcal{M}_\ell| \leq |\mathcal{T}_\ell \setminus (\mathcal{T}_k^{\mathrm{grad}} \oplus \mathcal{T}_\ell)| \overset{(1.2.10)}{\leq} 8C_{\mathrm{grad}}^{1/2}\mathrm{err}(\mathcal{T}_\ell)^{-1/2} \quad \text{for all } \ell \in \mathbb{N}_0.$$

By definition of the refinement in Step (iii) of Algorithm 1.2.1, there holds

$$|\mathcal{T}_\ell| - |\mathcal{T}_0| = \sum_{k=0}^{\ell-1}(|\mathcal{T}_{k+1}| - |\mathcal{T}_k|) = \sum_{k=0}^{\ell-1}|\mathcal{M}_k| \leq 8C_{\mathrm{grad}}^{-1/2}\sum_{k=0}^{\ell-1}\mathrm{err}(\mathcal{T}_k)^{-1/2}.$$

By induction, the linear convergence (1.2.5) proves

$$\mathrm{err}(\mathcal{T}_\ell) \leq \kappa^{\ell-k}\mathrm{err}(\mathcal{T}_k).$$

Hence, by convergence of the geometric series, we obtain

$$\sum_{k=0}^{\ell-1}\mathrm{err}(\mathcal{T}_k)^{-1/2} \leq \mathrm{err}(\mathcal{T}_\ell)^{-1/2}\sum_{k=0}^{\ell-1}\kappa^{(\ell-k)/2} \leq (1-\sqrt{\kappa})^{-1}\mathrm{err}(\mathcal{T}_\ell)^{-1/2}.$$

Altogether, this yields

$$|\mathcal{T}_\ell| - |\mathcal{T}_0| \leq 8C_{\mathrm{grad}}^{1/2}(1-\sqrt{\kappa})^{-1}\mathrm{err}(\mathcal{T}_\ell)^{-1/2},$$

and we end up with convergence rate $s = 2$, i.e.,

$$\mathrm{err}(\mathcal{T}_\ell) \leq (1-\sqrt{\kappa})^{-2}8^2C_{\mathrm{grad}}(|\mathcal{T}_\ell| - |\mathcal{T}_0|)^{-2} \quad \text{for all } \ell \in \mathbb{N}.$$

1.3. Discussion of the example

The sketch of the optimality proof above reveals certain interesting things. First, we extensively used the fact that the error estimator $\sum_{T\in\mathcal{T}}\eta_T(\mathcal{T})^2$ and the error $\|h(\mathcal{T})(u-U(\mathcal{T}))\|_{L^2(0,1)}^2$ coincide for this example, since we approximate a known function. If one thinks of u as the solution of some PDE, it is more likely that one computes the approximations to u without knowing u itself (i.e., by solving a finite element system). Then, the error estimator differs from the error, but can be related to it by reliability

$$\mathrm{err}(\mathcal{T}) \leq C_{\mathrm{rel}}\Big(\sum_{T\in\mathcal{T}}\eta_T(\mathcal{T})^2\Big)^{1/2} \tag{1.3.1}$$

and/or efficiency

$$C_{\mathrm{eff}}^{-1}\Big(\sum_{T\in\mathcal{T}}\eta_T(\mathcal{T})^2\Big)^{1/2} \leq \mathrm{err}(\mathcal{T}) + \mathrm{data}(\mathcal{T}) \tag{1.3.2}$$

for some uniform constants $C_{\mathrm{rel}}, C_{\mathrm{eff}} > 0$ and some perturbation term $\mathrm{data}(\mathcal{T})$, which often depends on the given data.

The linear convergence (1.2.5) is an important tool for the analysis. To prove it, we used that fact that $U(\mathcal{T}_\ell)$ satisfies the orthogonality

$$\|u-U(\mathcal{T}_\ell)\|_{L^2(0,1)}^2 = \|u-U(\mathcal{T}_{\ell+1})\|_{L^2(0,1)}^2 + \|U(\mathcal{T}_{\ell+1})-U(\mathcal{T}_\ell)\|_{L^2(0,1)}^2.$$

This identity holds only for the case of a bilinear form $b(\cdot\,,\,\cdot)$ which is a scalar product on the given Hilbert space and hence restricts the applicability of the analysis.

The overlay estimate (1.2.8) bounds the difference between the optimal partition $\mathcal{T}_k^{\mathrm{grad}}$ and the adaptively generated partition $\mathcal{T}_\ell$. In the 1D case, the overlay estimate seems almost trivial, however for 2D and 3D refinement strategies, it is not straightforward to prove, and it is even wrong for some strategies (see Section 3.2.9 below for a counterexample for red-green-blue refinement in 2D).

Finally, the identity

$$|\mathcal{T}_\ell| - |\mathcal{T}_0| = \sum_{k=0}^{\ell-1}|\mathcal{M}_k|$$

is trivial in our case, but poses a real issue in the case of certain practical refinement strategies. The main problem here is, that usual refinement strategies have to refine more elements than only the marked ones, to keep the partition regular in a certain sense (e.g., avoidance of hanging nodes; see Section 3.2 for details). Then, the question is how to bound the number of refined elements by the number of marked elements.

Chapter 2 states exactly, what is necessary to prove optimal convergence rates for some given problem in a very abstract and general framework and will thus focus on the error estimator instead of the error.

1.4. Outline

This section states the main results of the following chapters and sections.

Chapter 2:

The chapter introduces an abstract framework for adaptive algorithms and formulates a particular algorithm (Algorithm 2.2.1). Within this framework, the adaptive approximation problem formulated in Section 2.2.3, is stated. This problem assumes a certain quantity $\eta(\cdot)$ (the error estimator) which is a function of an underlying discretization (the triangulation).

The goal is to drive the error estimator to zero as fast as possible, i.e., $\lim_{\ell\to\infty}\eta(\mathcal{T}_\ell) = 0$ with optimal rate for a sequence of triangulations $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}_0}$. We state six axioms (E1)–(E3) & (T1)–(T3) which determine the behavior of the adaptive algorithm and suffice to show that optimal convergence rates are obtained, i.e.,

$$\eta(\mathcal{T}_\ell) \lesssim |\mathcal{T}_\ell|^{-s} \quad \text{for all } \ell \in \mathbb{N}_0,$$

where $|\mathcal{T}_\ell|$ denotes the number of elements in the triangulation $\mathcal{T}_\ell$ and $s > 0$ denotes the best possible convergence rate which is achievable for a particular problem. The latter is the main result of this chapter and stated formally in Theorem 2.3.3. The axioms can roughly be categorized into estimator related axioms (E1)–(E3) and triangulation related axioms (T1)–(T3). The first category (E1)–(E3) can be paraphrased as follows:

(E1) Stability and reduction: The estimator is a Lipschitz continuous function of the triangulation, and it is contractive up to a perturbation when the triangulation is locally refined.
(E2) General quasi-orthogonality: The perturbation from (E1) is ℓ_2-summable and also bounded by the estimator on the coarsest triangulation.
(E3) Discrete reliability: The error estimator is a local upper bound of the perturbation from (E1).

The triangulation related axioms (T1)–(T3) can be heuristically formulated as follows:

(T1) Son estimate: The refinement strategy increases the number of elements at most linearly.
(T2) Closure estimate: The number of elements is bounded by the number of marked elements.
(T3) Uniform approximability: The problem allows for a certain convergence rate.

Chapter 3:
This chapter applies the abstract theory from Chapter 2 to certain model problems. We consider the conforming finite element method (FEM) for the Poisson problem with bisection based refinement and red-green-blue refinement. The optimality result for general second-order elliptic PDEs marks the main achievement of this chapter (Section 3.6.1). This includes also an adaptive algorithm for problems which satisfy a Gårding inequality only, where the difficulty is, that the discrete system is not necessarily solvable in each step (Section 3.6.2). Therefore, we propose an algorithm which guarantees unique solvability after a finite number of steps. Moreover, we consider non-linear problems with quite general coefficients. Altogether, we prove optimality results for the following problem classes:

- FEM for the Poisson problem (Consequence 3.5.2–3.5.5),
- FEM for general second-order elliptic PDEs with
  - ellipticity estimate (Consequence 3.6.2),
  - Gårding inequality (Consequence 3.6.15),
  - non-linear coefficients (Consequence 3.7.5),
- boundary element method (BEM) for
  - weakly-singular integral equation (Consequence 3.5.9),
  - hyper-singular integral equation (Consequence 3.5.11–3.5.12).

Chapter 4:
This chapter extends the abstract theory of Chapter 2 to equivalent error estimators, where Theorem 4.3.1 states the main result. We consider error estimators which satisfy the axioms only in average, but not in every single step of the adaptive algorithm. This abstract setting

covers inexact solve, i.e., the case of iterative solvers, where instead of the error estimator only an approximation

$$\widetilde{\eta}(\mathcal{T}) \approx \eta(\mathcal{T})$$

is computed but the axioms are only satisfied for the exact error estimator. Moreover, we cover estimators which are equivalent to some weighted error estimator, i.e.,

$$\widetilde{\eta}(\mathcal{T}) \simeq \|h(\mathcal{T})\mathrm{res}(\mathcal{T})\|,$$

where $h(\mathcal{T})$ is a triangulation related weight function and $\mathrm{res}(\cdot)$ is some quantity which measures the error in the appropriate norm, e.g., the residual in case of a weighted-residual error estimator. To that end, we exploit certain properties which are automatically satisfied by weighted error estimators and develop a super contractive weight function (Proposition 4.5.4) which enables us to control the equivalence constants.

Chapter 5:

This chapter applies the extended theory of Chapter 4 to certain model problems. The main result of this section is the incorporation of inhomogeneous boundary data into the FEM optimality analysis. This is possible by use of the super contractive weight function from Chapter 4 in combination with the Scott-Zhang projection. Altogether, we consider the following problems:

- FEM for non-residual error estimators in the frame of the Poisson problem (Consequence 5.2.3–5.2.11),
- FEM for the p-Laplacian (Consequence 5.3.3),
- FEM for non-trivial boundary conditions (Consequence 5.4.3).

Chapter 6:

This chapter steps out of the line of the other chapters, as we introduce a new adaptive algorithm (Algorithm 6.2.2) for the solution of integral equations on piecewise smooth geometries. The idea is to approximate the exact geometry with piecewise affine line segments and to solve a standard BEM problem on the approximate geometry. A posteriori analysis for this kind of problem is available for FEM, but is missing entirely for BEM, where very different techniques are necessary. We introduce an error estimator

$$\eta(\mathcal{T})^2 = \rho(\mathcal{T})^2 + \mathrm{geo}(\mathcal{T})^2,$$

where $\rho(\mathcal{T})$ is a standard residual error estimator for the weakly singular integral equation on piecewise affine geometries and $\mathrm{geo}(\mathcal{T})$ is a geometric error estimator which measures the approximation quality of the approximate geometry. We prove that the error estimator provides an upper error bound and use this to prove convergence of the corresponding adaptive algorithm (Consequence 6.4.2). The convergence proof is done within the frame of Chapter 2. Although we are convinced that optimal convergence rates are possible with the given algorithm, the proof requires additional ideas which are beyond the scope of this work.

Chapter 7:

The final chapter is focused on the general quasi-orthogonality (E2). The reason for this is that for many problem classes (e.g., for non-symmetric or non conforming approaches) the general quasi-orthogonality is the most difficult axiom to verify. We show that the general quasi-orthogonality holds for the non-symmetric and non-linear example problems in Chapter 3.

CHAPTER 2

Abstract Theory

2.1. Introduction, state of the art & outline

The purpose of this chapter is to find an abstract framework within, e.g., the results of the introductory chapter can be reproduced. The reproduction of existing results is, of course, not the main reason for developing the abstract framework. The abstract point of view sheds new light on this terrain and enables us to prove new results for a very general class of problems (as is demonstrated in the applications of Chapter 3, 5, 6). To that end, we abandon the framework of exact solutions and their discrete approximations and focus completely on the error estimator. The function $\eta(\cdot)$ can be seen as a function on the underlying triangulations with some specific properties. Then, the goal of the adaptive algorithm is to manipulate the triangulation in such a way, that the error estimator converges to zero as fast as possible. An immediate consequence of this viewpoint is that it removes the need for the lower error bound (1.3.2). An earlier version of this abstract framework can be found in [**24**]. However, this work takes one step further into the abstraction of the concrete problems. This, for example, enables us to prove optimal convergence rates of the adaptive algorithm for refinement strategies which do not satisfy the overlay property (1.2.8) (e.g., red-green-blue refinement). Moreover, the conditions (axioms) which we derive in this chapter turn out to be sufficient for optimal convergence rates, and, under realistic assumptions, even necessary. Therefore, we obtain explicit criteria which determine if a given problem or problem class will reveal optimal convergence behavior. For the state of the art in the literature, we refer the reader to the historic overview of Section 2.8. The remainder of this chapter is organized as follows: Section 2.2 describes the abstract framework which is necessary to formulate the axioms. This includes a formal definition of the error estimator, the triangulations, the approximation problem of driving the estimator to zero, and the adaptive algorithm to solve the approximation problem. Section 2.3 states the main theorem (Theorem 2.3.3) of this chapter as well as the axioms which are then used to prove optimal convergence rates. Section 2.4–2.5 give alternative approximation problems (optimal convergence of the error and optimal complexity in terms of computational work) and state the respective results. Section 2.6 proves that the axioms are not only sufficient, but even necessary for proving optimal convergence rates. Section 2.7 demonstrates certain problem classes, for which one or more of the axioms are a priori satisfied. Finally, Section 2.8 concludes with a historic overview and motivates the particular choice of axioms in Section 2.3.1.

2.2. Abstract setting

This section is devoted to the definition of the problem and the precise statement of the adaptive algorithm.

2.2.1. Triangulations. Let $\mathcal{T}_\infty$ be a countable set. Each finite subset $\mathcal{T} \subseteq \mathcal{T}_\infty$ with $|\mathcal{T}| < \infty$ elements is called a triangulation. Let $\mathbb{T}$ be a set of triangulations (which is countable since the set of all triangulations is countable) with the corresponding refinement strategy $\mathbb{T}(\cdot,\cdot) : \big\{(\mathcal{T},\mathcal{M}) : \mathcal{T} \in \mathbb{T}, \mathcal{M} \subseteq \mathcal{T}\big\} \to \mathbb{T}$. This is a function which satisfies $\mathbb{T}(\mathcal{T},\mathcal{M}) \cap \mathcal{M} = \emptyset$ for all $\mathcal{M} \subseteq \mathcal{T}$ and all $\mathcal{T} \in \mathbb{T}$. Here, $\mathcal{M}$ is called the set of marked elements. Given $\mathcal{T} \in \mathbb{T}$, define $\mathbb{T}(\mathcal{T}) \subseteq \mathbb{T}$ such that $\widehat{\mathcal{T}} \in \mathbb{T}(\mathcal{T})$ if and only if there exists a sequence of triangulations $\mathcal{T}_0 = \mathcal{T}, \mathcal{T}_1, \ldots, \mathcal{T}_\ell = \widehat{\mathcal{T}}$ as well as a sequence of marked elements $\mathcal{M}_0, \ldots, \mathcal{M}_{\ell-1}$ with $\mathcal{M}_j \subseteq \mathcal{T}_j$ for all $j = 0, \ldots, \ell-1$ such that $\mathcal{T}_{j+1} = \mathbb{T}(\mathcal{T}_j, \mathcal{M}_j)$ for all $j = 0, \ldots, \ell-1$. We call $\mathbb{T}(\mathcal{T})$ the set of refinements of $\mathcal{T}$. We assume that there exists an initial triangulation $\mathcal{T}_0 \in \mathbb{T}$ such that $\mathbb{T}(\mathcal{T}_0) = \mathbb{T}$. Additionally, we assume that $T \in \widehat{\mathcal{T}} \cap \mathcal{T}$ if and only if $T \in \mathcal{T}_j$ for all $j = 0, \ldots, \ell$.

The subset of all refinements which have at most $N \in \mathbb{N}$ elements more than a triangulation $\mathcal{T} \in \mathbb{T}$ reads

$$\mathbb{T}(\mathcal{T}, N) := \big\{\widehat{\mathcal{T}} \in \mathbb{T}(\mathcal{T}) : |\widehat{\mathcal{T}} \setminus \mathcal{T}| \le N\big\},$$

where $|\cdot| = \mathrm{card}(\cdot)$ is the counting measure. Since each triangulation $\mathcal{T} \in \mathbb{T}$ allows for at most $2^{|\mathcal{T}|}$ sets of marked elements, there holds $|\mathbb{T}(\mathcal{T}, N)| < \infty$. Moreover, we write $\mathbb{T}(N) := \mathbb{T}(\mathcal{T}_0, N)$.

2.2.2. Error estimator. The error estimator is a function $\eta(\cdot) : \mathbb{T} \to \bigcup_{\mathcal{T}\in\mathbb{T}} [0,\infty)^{\mathcal{T}}$ (where A^B denotes the set of functions mapping B to A) with $\eta(\mathcal{T}) : \mathcal{T} \to [0,\infty)$ for all $\mathcal{T} \in \mathbb{T}$. By $\eta_T(\mathcal{T})$ for some $T \in \mathcal{T}$, we denote the evaluation of the function $\eta_{(\cdot)}(\mathcal{T}) := \eta(\mathcal{T})$. For brevity of notation, we also write $\eta(\mathcal{T}) := \big(\sum_{T\in\mathcal{T}} \eta_T(\mathcal{T})^2\big)^{1/2} \ge 0$, which is the global error estimator.

2.2.3. Adaptive approximation problem. The goal of the adaptive approximation problem is to find a sequence of triangulations $\mathcal{T}_\ell$, $\ell \in \mathbb{N}_0$ such that

$$\sup_{\ell\in\mathbb{N}_0} \eta(\mathcal{T}_\ell)(|\mathcal{T}_\ell| + 1)^s < \infty$$

for $s > 0$ as large as possible. This implies that the error estimator converges to zero with rate s, i.e., there exists a constant $C > 0$ such that

$$\eta(\mathcal{T}_\ell) \le C|\mathcal{T}_\ell|^{-s} \quad \text{for all } \ell \in \mathbb{N}_0.$$

2.2.4. Adaptive algorithm. The algorithm to solve the adaptive approximation problem from Section 2.2.3 reads

ALGORITHM 2.2.1. INPUT: *Initial triangulation $\mathcal{T}_0$ and bulk parameter $0 < \theta \le 1$.*
Loop: *For $\ell = 0, 1, 2, \ldots$ do* (i) – (iii).

(i) *Compute refinement indicators $\eta_T(\mathcal{T}_\ell)$ for all $T \in \mathcal{T}_\ell$.*
(ii) *Determine set $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ of (up to the multiplicative constant $C_{\min}$) minimal cardinality such that*

$$\theta\, \eta(\mathcal{T}_\ell)^2 \le \sum_{T\in\mathcal{M}_\ell} \eta_T(\mathcal{T}_\ell)^2. \tag{2.2.1}$$

(iii) *Define the next triangulation $\mathcal{T}_{\ell+1} := \mathbb{T}(\mathcal{T}_\ell, \mathcal{M}_\ell)$.*

OUTPUT: *Error estimators $\eta(\mathcal{T}_\ell)$ for all $\ell \in \mathbb{N}_0$.*

REMARK 2.2.2. *Suppose that $\mathcal{S}_\ell \subseteq \mathcal{T}_\ell$ is some (not necessarily unique) set of minimal cardinality which satisfies the Dörfler marking criterion* (2.2.1). *In step (iii) the phrase* up to the multiplicative constant minimal cardinality *means that $|\mathcal{M}_\ell| \le C_{\min}|\mathcal{S}_\ell|$ with some ℓ-independent constant $C_{\min} \ge 1$.*

Remark 2.2.3. *A greedy algorithm for* (2.2.1)*, sorts the elements* $\mathcal{T}_\ell = \{T_1, \dots, T_N\}$ *such that* $\eta_{T_1}(\mathcal{T}_\ell) \geq \eta_{T_2}(\mathcal{T}_\ell) \geq \dots \geq \eta_{T_N}(\mathcal{T}_\ell)$ *and takes the minimal* $1 \leq J \leq N$ *such that* $\theta\eta(\mathcal{T}_\ell)^2 \leq \sum_{j=1}^{J} \eta_{T_j}(\mathcal{T}_\ell)^2$. *This results in logarithmic-linear growth of the complexity. The relaxation to almost minimal cardinality of* $\mathcal{M}_\ell$ *allows to employ a sorting algorithm based on binning so that* $\mathcal{M}_\ell$ *in* (2.2.1) *can be determined in linear complexity* [**78**, Section 5] *with* $C_{\min} = 2$.

Remark 2.2.4. *Small adaptivity parameters* $0 < \theta \ll 1$ *lead to only few marked elements and so to possibly very local refinements. The other extreme,* $\theta = 1$*, basically leads to uniform refinement, where (almost) all elements are refined.*

2.2.5. Approximability. Given $\mathcal{T} \in \mathbb{T}$ and $s > 0$, define

$$\|\eta, \mathbb{T}(\mathcal{T})\|_s := \sup_{N \in \mathbb{N}_0} \min_{\widehat{\mathcal{T}} \in \mathbb{T}(\mathcal{T}, N)} ((N+1)^s \eta(\widehat{\mathcal{T}})). \tag{2.2.2}$$

The fact $\|\eta, \mathbb{T}(\mathcal{T})\|_s < \infty$ implies that there exists a sequence of triangulations $(\mathcal{T}_\ell^{\mathrm{opt}})_{\ell \in \mathbb{N}}$ in $\mathbb{T}(\mathcal{T})$ which satisfies convergence

$$\lim_{\ell \to \infty} \eta(\mathcal{T}_\ell^{\mathrm{opt}}) = 0$$

and the convergence rate

$$\eta(\mathcal{T}_\ell^{\mathrm{opt}}) \lesssim (|\mathcal{T}_\ell^{\mathrm{opt}} \setminus \mathcal{T}|)^{-s} \quad \text{for all } \ell \in \mathbb{N}.$$

Remark 2.2.5. *The quantity* $\|\eta, \mathbb{T}(\mathcal{T})\|_s$ *measures how fast the error estimator can be driven to zero when starting from the triangulation* $\mathcal{T}$. *The main interest, of course, lies in the approximability when starting from the initial triangulation* $\|\eta, \mathbb{T}\|_s$.

2.3. The axioms

This section introduces the set of axioms and states the main result (Theorem 2.3.3) derived from these axioms. In the following, $\mathcal{T}_\ell$ denotes a triangulation generated in the ℓ-th step of Algorithm 2.2.1.

2.3.1. Set of axioms. The following axioms (E1)–(E3), (T1)–(T3) act on the function $\eta(\cdot) : \mathbb{T} \to \bigcup_{\mathcal{T} \in \mathbb{T}} \left([0,\infty)^{\mathcal{T}}\right)$ with $\eta(\mathcal{T}) : \mathcal{T} \to [0,\infty)$ for all $\mathcal{T} \in \mathbb{T}$, some perturbation function $\varrho(\cdot,\cdot) : \mathbb{T} \times \mathbb{T} \to [0,\infty)$, $\mathbb{T}(\cdot) : \mathbb{T} \to 2^{\mathbb{T}}$, and involve the set $\mathbb{T}$ as well as the constants $s > 0$, $C_{\mathrm{drel}}, C_{\mathrm{ref}}, C_{\mathrm{qo}}, C_{\mathrm{son}}, C_{\mathrm{closure}} \geq 1$, $0 < \kappa_{\mathrm{dlr}} \leq \infty$, and $0 \leq \rho_{\mathrm{red}}, \varepsilon_{\mathrm{qo}}, \varepsilon_{\mathrm{drel}} < 1$.

(E1) **Stability and reduction**: For all refinements $\widehat{\mathcal{T}} \in \mathbb{T}(\mathcal{T})$ of a triangulation $\mathcal{T} \in \mathbb{T}$, there exist sets $\mathcal{S}(\mathcal{T}, \widehat{\mathcal{T}}) \subseteq \mathcal{T}$ and $\widehat{\mathcal{S}}(\mathcal{T}, \widehat{\mathcal{T}}) \subseteq \widehat{\mathcal{T}}$ with $\mathcal{T} \setminus \widehat{\mathcal{T}} \subseteq \mathcal{S}(\mathcal{T}, \widehat{\mathcal{T}})$ such that (E1a)–(E1b) hold

(a) $$\Big| \Big(\sum_{T \in \widehat{\mathcal{T}} \setminus \widehat{\mathcal{S}}(\mathcal{T}, \widehat{\mathcal{T}})} \eta_T(\widehat{\mathcal{T}})^2 \Big)^{1/2} - \Big(\sum_{T \in \mathcal{T} \setminus \mathcal{S}(\mathcal{T}, \widehat{\mathcal{T}})} \eta_T(\mathcal{T})^2 \Big)^{1/2} \Big| \leq \varrho(\mathcal{T}, \widehat{\mathcal{T}}),$$

(b) $$\sum_{T \in \widehat{\mathcal{S}}(\mathcal{T}, \widehat{\mathcal{T}})} \eta_T(\widehat{\mathcal{T}})^2 \leq \rho_{\mathrm{red}} \sum_{T \in \mathcal{S}(\mathcal{T}, \widehat{\mathcal{T}})} \eta_T(\mathcal{T})^2 + \varrho(\mathcal{T}, \widehat{\mathcal{T}})^2.$$

(E2) **General quasi-orthogonality**: There holds

$$0 \leq \varepsilon_{\mathrm{qo}} < \sup_{\delta > 0} \frac{1 - (1+\delta)(1 - (1-\rho_{\mathrm{red}})\theta)}{2 + \delta^{-1}}$$

and the sequence of triangulations $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}_0}$ from Algorithm 2.2.1 satisfies for all $\ell, N \in \mathbb{N}_0$

$$\sum_{k=\ell}^{\ell+N} \big(\varrho(\mathcal{T}_k,\mathcal{T}_{k+1})^2 - \varepsilon_{\rm qo}\eta(\mathcal{T}_k)^2\big) \le C_{\rm qo}\,\eta(\mathcal{T}_\ell)^2.$$

(E3) **Discrete reliability**: For all refinements $\widehat{\mathcal{T}} \in \mathbb{T}(\mathcal{T})$ of a triangulation $\mathcal{T} \in \mathbb{T}$ with $\eta(\widehat{\mathcal{T}}) \le \kappa_{\rm dlr}\eta(\mathcal{T})$, there exists a subset $\mathcal{R}(\mathcal{T},\widehat{\mathcal{T}}) \subseteq \mathcal{T}$ with $\mathcal{S}(\mathcal{T},\widehat{\mathcal{T}}) \subseteq \mathcal{R}(\mathcal{T},\widehat{\mathcal{T}})$ and $|\mathcal{R}(\mathcal{T},\widehat{\mathcal{T}})| \le C_{\rm ref}|\mathcal{T}\backslash\widehat{\mathcal{T}}|$ such that

$$\varrho(\mathcal{T},\widehat{\mathcal{T}})^2 \le \varepsilon_{\rm drel}\eta(\mathcal{T})^2 + C_{\rm drel}^2 \sum_{T\in\mathcal{R}(\mathcal{T},\widehat{\mathcal{T}})} \eta_T(\mathcal{T})^2.$$

(T1) **Son estimate**: The sequence of triangulations $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}_0}$ from Algorithm 2.2.1 satisfies $|\mathcal{T}_{\ell+1}| \le C_{\rm son}|\mathcal{T}_\ell|$ for all $\ell \in \mathbb{N}_0$.

(T2) **Closure estimate**: The sequence of triangulations $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}_0}$ from Algorithm 2.2.1 satisfies $|\mathcal{T}_\ell \setminus \mathcal{T}_0| \le C_{\rm closure}\sum_{j=0}^{\ell-1}|\mathcal{M}_j|$ for all $\ell \in \mathbb{N}_0$.

(T3) **Uniform approximability**: The sequence of triangulations $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}_0}$ from Algorithm 2.2.1 satisfies $C_{\rm approx}(s) := \sup_{\ell\in\mathbb{N}_0}\|\eta,\mathbb{T}(\mathcal{T}_\ell)\|_s < \infty$ for all $\ell \in \mathbb{N}_0$.

Definition 2.3.1. *We say that a certain subset of the axioms defined above* $\mathcal{A} \subseteq \{\text{(E1)},\dots,\text{(E3)},\text{(T1)},\dots,\text{(T3)}\}$ *is satisfied, if the error estimator* $\eta(\cdot)$ *and the refinement strategy* $\mathbb{T}(\cdot)$ *(which are clear from the context if not mentioned otherwise) allow for the necessary functions and constants from Section 2.3.1, which are involved in the axioms of* $\mathcal{A}$*, to exist.*

Remark 2.3.2. *Proposition 2.6.2 below shows that general quasi-orthogonality* (E2) *together with* (E1) *implies* (E2) *even with* $\varepsilon_{\rm qo} = 0$ *and* $0 < C_{\rm qo} < \infty$.

2.3.2. Optimal convergence rates for the error estimator.

The main results of this Section state convergence and optimality of the adaptive algorithm in the sense that the error estimator converges with optimal convergence rate. This is a generalization of existing results as discussed in Section 2.4. On the other hand, Theorem 2.3.3 (iii) shows that the adaptive algorithm characterizes the approximability of a problem in the sense of Section 2.2.5.

Theorem 2.3.3. (i) *Suppose* (E1) *is satisfied and assume* $\lim_{\ell\to\infty}\varrho(\mathcal{T}_\ell,\mathcal{T}_{\ell+1}) = 0$. *Then, for all* $0 < \theta \le 1$*, the estimator is convergent in the sense*

$$\lim_{\ell\to\infty}\eta(\mathcal{T}_\ell) = 0. \qquad (2.3.1)$$

(ii) *Suppose* (E1)–(E2) *are satisfied. Then, for all* $0 < \theta \le 1$*, the estimator is R-linear convergent in the sense that there exists* $0 < \rho_{\rm conv} < 1$ *and* $C_{\rm conv} > 0$ *such that*

$$\eta(\mathcal{T}_{\ell+j})^2 \le C_{\rm conv}\rho_{\rm conv}^j\,\eta(\mathcal{T}_\ell)^2 \quad \textit{for all } j,\ell \in \mathbb{N}_0. \qquad (2.3.2)$$

(iii) *Suppose* (E1)–(E3) *and* (T1)–(T3) *are satisfied for some* $s > 0$. *Then* $0 < \theta < \theta_\star := (1-\varepsilon_{\rm drel})/(1+C_{\rm drel}^2)$ *implies quasi-optimal convergence of the estimator in the sense of*

$$c_{\rm opt}C_{\rm approx}(s) \le \sup_{\ell\in\mathbb{N}_0}\frac{\eta(\mathcal{T}_\ell)}{(|\mathcal{T}_\ell\setminus\mathcal{T}_0|+1)^{-s}} \le C_{\rm opt}C_{\rm approx}(s), \qquad (2.3.3)$$

where the lower bound requires only (T1) *to hold.*

The constants C_{conv}, $\rho_{\mathrm{conv}} > 0$ depend only on ρ_{red}, C_{qo}, $\varepsilon_{\mathrm{qo}}$, and on θ. The constant $C_{\mathrm{opt}} > 0$ depends additionally on $C_{\min}, C_{\mathrm{ref}}, C_{\mathrm{closure}}, C_{\mathrm{drel}}$, $\varepsilon_{\mathrm{drel}}$, and on s, while $c_{\mathrm{opt}} > 0$ depends only on C_{son} and $|\mathcal{T}_0|$.

Remark 2.3.4. *The upper bound in (2.3.3) states that given $C_{\mathrm{approx}}(s) < \infty$, the estimator sequence $\eta(\mathcal{T}_\ell)$ of Algorithm 2.2.1 will decay with order s, i.e., if a decay with order s is possible if the optimal triangulations are chosen, this decay will in fact be realized by the adaptive algorithm. The lower bound in (2.3.3) states that the asymptotic convergence rate of the estimator sequence characterizes the theoretically optimal convergence rate.*

2.3.3. Estimator reduction and convergence of $\eta(\mathcal{T}_\ell)$. We start with the observation that stability (E1a) and reduction (E1b) lead to a perturbed contraction of the error estimator in each step of the adaptive loop.

Lemma 2.3.5. *Let $0 < \theta \le 1$ and let $\widehat{\mathcal{T}} \in \mathbb{T}(\mathcal{T})$ denote a refinement of $\mathcal{T} \in \mathbb{T}$ such that*

$$\theta \eta(\mathcal{T})^2 \le \sum_{T \in \mathcal{S}(\mathcal{T},\widehat{\mathcal{T}})} \eta_T(\mathcal{T})^2. \tag{2.3.4}$$

Then, the following relaxation of (E1a)

$$\Big(\sum_{T \in \widehat{\mathcal{T}} \setminus \widehat{\mathcal{S}}(\mathcal{T},\widehat{\mathcal{T}})} \eta_T(\widehat{\mathcal{T}})^2 \Big)^{1/2} \le \Big(\sum_{T \in \mathcal{T} \setminus \mathcal{S}(\mathcal{T},\widehat{\mathcal{T}})} \eta_T(\mathcal{T})^2 \Big)^{1/2} + \varrho(\mathcal{T},\widehat{\mathcal{T}}) \tag{2.3.5}$$

and reduction (E1b) *imply the estimator reduction*

$$\eta(\widehat{\mathcal{T}})^2 \le \rho_{\mathrm{est}}\, \eta(\mathcal{T})^2 + C_{\mathrm{est}}\, \varrho(\widehat{\mathcal{T}},\mathcal{T})^2 \tag{2.3.6}$$

with the constants $0 < \rho_{\mathrm{est}} < 1$ and $C_{\mathrm{est}} > 0$ which relate via

$$\rho_{\mathrm{est}} = (1+\delta)(1-(1-\rho_{\mathrm{red}})\theta) \quad \text{and} \quad C_{\mathrm{est}} = 2 + \delta^{-1} \tag{2.3.7}$$

for all sufficiently small $\delta > 0$ such that $\rho_{\mathrm{est}} < 1$. This particularly implies

$$\eta(\mathcal{T}_{\ell+1})^2 \le \rho_{\mathrm{est}}\, \eta(\mathcal{T}_\ell)^2 + C_{\mathrm{est}}\, \varrho(\mathcal{T}_\ell,\mathcal{T}_{\ell+1})^2 \tag{2.3.8}$$

for all $\ell \in \mathbb{N}_0$.

Proof. The Young inequality together with stability (2.3.5) and reduction (E1b) shows for each $\delta > 0$ and $C_{\mathrm{est}} = 2 + \delta^{-1}$ that

$$\begin{aligned} \eta(\widehat{\mathcal{T}})^2 &= \sum_{T \in \widehat{\mathcal{S}}(\mathcal{T},\widehat{\mathcal{T}})} \eta_T(\widehat{\mathcal{T}})^2 + \sum_{T \in \widehat{\mathcal{T}} \setminus \widehat{\mathcal{S}}(\mathcal{T},\widehat{\mathcal{T}})} \eta_T(\widehat{\mathcal{T}})^2 \\ &\le \rho_{\mathrm{red}} \sum_{T \in \mathcal{S}(\mathcal{T},\widehat{\mathcal{T}})} \eta_T(\mathcal{T})^2 + (1+\delta) \sum_{T \in \mathcal{T} \setminus \mathcal{S}(\mathcal{T},\widehat{\mathcal{T}})} \eta_T(\mathcal{T})^2 + C_{\mathrm{est}} \varrho(\mathcal{T},\widehat{\mathcal{T}})^2. \end{aligned}$$

Therefore, the Dörfler marking (2.3.4) leads to

$$\begin{aligned} \eta(\widehat{\mathcal{T}})^2 &\le (1+\delta)\Big(\eta(\mathcal{T})^2 - (1-\rho_{\mathrm{red}}) \sum_{T \in \mathcal{S}(\mathcal{T},\widehat{\mathcal{T}})} \eta_T(\mathcal{T})^2\Big) + C_{\mathrm{est}} \varrho(\mathcal{T},\widehat{\mathcal{T}})^2 \\ &\le (1+\delta)\big(1-(1-\rho_{\mathrm{red}})\theta\big)\eta(\mathcal{T})^2 + C_{\mathrm{est}} \varrho(\mathcal{T},\widehat{\mathcal{T}})^2. \end{aligned}$$

The choice of a sufficiently small $\delta > 0$ allows for $\rho_{\mathrm{est}} = (1+\delta)\big(1-(1-\rho_{\mathrm{red}})\theta\big) < 1$. This shows (2.3.6). By definition of the refinement strategy $\mathbb{T}(\cdot,\cdot)$ in Section 2.2.1, there holds $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell \setminus \mathcal{T}_{\ell+1} \subseteq \mathcal{S}(\mathcal{T}_\ell,\mathcal{T}_{\ell+1})$. Hence, Dörfler marking (2.2.1) for $\mathcal{M}_\ell$ implies Dörfler marking (2.3.4) for $\mathcal{S}(\mathcal{T}_\ell,\mathcal{T}_{\ell+1})$. This concludes the proof. ∎

The *estimator reduction concept* used in the the following proof is studied in [**5**] and applies to a general class of problems and error estimators.

Lemma 2.3.6. *Suppose that the estimator satisfies estimator reduction* (2.3.8) *and suppose that*

$$\lim_{\ell\to\infty} \varrho(\mathcal{T}_\ell, \mathcal{T}_{\ell+1}) = 0.$$

Then, there holds estimator convergence in the sense $\lim_{\ell\to\infty} \eta(\mathcal{T}_\ell) = 0$.

Proof. Mathematical induction on ℓ proves with (2.3.8) for all $\ell \in \mathbb{N}_0$

$$\begin{aligned}
\eta(\mathcal{T}_{\ell+1})^2 &\le \rho_{\mathrm{est}}^{\ell+1}\eta(\mathcal{T}_0)^2 + C_{\mathrm{est}}\sum_{j=0}^{\ell}\rho_{\mathrm{est}}^{\ell-j}\varrho(\mathcal{T}_j,\mathcal{T}_{j+1})^2 \\
&\le \eta(\mathcal{T}_0)^2 + C_{\mathrm{est}}\sup_{j\in\mathbb{N}_0}\varrho(\mathcal{T}_j,\mathcal{T}_{j+1})^2\sum_{j=0}^{\ell}\rho_{\mathrm{est}}^{\ell-j} \\
&\le \eta(\mathcal{T}_0)^2 + C_{\mathrm{est}}\sup_{j\in\mathbb{N}_0}\varrho(\mathcal{T}_j,\mathcal{T}_{j+1})^2(1-\rho_{\mathrm{est}})^{-1}.
\end{aligned} \tag{2.3.9}$$

The assumption $\varrho(\mathcal{T}_\ell, \mathcal{T}_{\ell+1}) \to 0$ implies $\sup_{\ell\in\mathbb{N}} \eta(\mathcal{T}_\ell) < \infty$. Moreover, (2.3.8) yields

$$\begin{aligned}
\limsup_{\ell\to\infty}\eta(\mathcal{T}_{\ell+1})^2 &\le \limsup_{\ell\to\infty}\big(\rho_{\mathrm{est}}\,\eta(\mathcal{T}_\ell)^2 + C_{\mathrm{est}}\,\varrho(\mathcal{T}_\ell,\mathcal{T}_{\ell+1})^2\big) \\
&= \rho_{\mathrm{est}}\limsup_{\ell\to\infty}\eta(\mathcal{T}_{\ell+1})^2.
\end{aligned}$$

This shows $\limsup_{\ell\to\infty}\eta(\mathcal{T}_\ell)^2 = 0$, and hence elementary calculus proves convergence $\eta(\mathcal{T}_\ell) \to 0$. ☐

Proof of Theorem 2.3.3 (i). Lemma 2.3.6 is applicable and concludes the proof. ☐

2.3.4. Uniform R-linear convergence of $\eta(\mathcal{T}_\ell)$ on any level.

The general quasi-orthogonality (E2) allows to improve (2.3.1) to R-linear convergence on any level. To that end, we prove the following auxiliary lemma.

Remark 2.3.7. *The term uniform R-linear convergence on any level needs some explanation. A sequence* $(a_k)_{k\in\mathbb{N}_0}$ *is said to converge (Q-)linearly to zero, if*

$$\limsup_{k\to\infty}\frac{|a_{k+1}|}{|a_k|} = q < 1.$$

A sequence $(b_k)_{k\in\mathbb{N}_0}$ *is said to converge R-linearly to zero if there exists a Q-linearly convergent sequence* $(a_k)_{k\in\mathbb{N}_0}$ *with*

$$|b_k| \le |a_k| \quad \textit{for all } k\in\mathbb{N}_0. \tag{2.3.10}$$

The R stands for root, *since the definition above is equivalent to*

$$\limsup_{k\to\infty}|b_k|^{1/k} = q < 1. \tag{2.3.11}$$

To see that, note that (2.3.10) *implies* (2.3.11) *since* $|a_k| \le q^{k-k_0}|a_{k_0}|$ *for all* $k \ge k_0$ *and some sufficiently large* $k_0 \in \mathbb{N}$. *On the other hand,* (2.3.11) *implies* (2.3.10) *with* $a_k := (\sup_{j\ge k}|b_j|^{1/j})^k$.

Uniform R-linear convergence on any level of a sequence $(b_k)_{k\in\mathbb{N}_0}$ *(in the following denoted by R-linear convergence) means that there exists a constant* $C>0$ *and some* $0<q<1$ *such that*

$$|b_{\ell+k}| \le Cq^k |b_\ell| \quad \textit{for all } \ell, k \in \mathbb{N}_0.$$

This particularly implies (2.3.11) *for all sequences* $(b_{k+\ell})_{k\in\mathbb{N}_0}$, $\ell\in\mathbb{N}_0$.

Lemma 2.3.8. *Given a real sequence* $(a_\ell)_{\ell\in\mathbb{N}_0}$ *with* $a_\ell \ge 0$ *for all* $\ell\in\mathbb{N}_0$ *such that* $a_\ell = 0$ *implies* $a_k = 0$ *for all* $k \ge \ell$*. Then, the statements* (i)–(iii) *are pairwise equivalent.*

(i) *Uniform summability: There exists a constant* $C_1 > 0$ *such that*

$$\sum_{k=\ell+1}^{\infty} a_k^2 \le C_1 a_\ell^2 \quad \textit{for all } \ell\in\mathbb{N}_0. \tag{2.3.12}$$

(ii) *Inverse summability: For all* $s>0$*, there exists a constant* $C_2>0$ *such that*

$$\sum_{k=0}^{\ell-1} a_k^{-1/s} \le C_2 a_\ell^{-1/s} \quad \textit{for all } \ell\in\mathbb{N} \textit{ with } a_\ell > 0. \tag{2.3.13}$$

(iii) *Uniform R-linear convergence on any level: There exist constants* $0<\rho_1<1$ *and* $C_3>0$ *such that*

$$a_{\ell+k}^2 \le C_3 \rho_1^k \, a_\ell^2 \quad \textit{for all } k,\ell\in\mathbb{N}_0. \tag{2.3.14}$$

The relation between the respective constants is given by

$$\begin{aligned} C_2 &\le \frac{C_3^{1/(2s)}}{1-\rho_1^{1/(2s)}}, & \rho_1 &\le \frac{C_1}{1+C_1}, & C_3 &\le 1+C_1,\\ C_1 &\le \frac{C_3\rho_1}{1+\rho_1}, & \rho_1 &\le \Big(\frac{C_2}{1+C_2}\Big)^{2s}, & C_3 &\le (1+C_2)^{2s}. \end{aligned} \tag{2.3.15}$$

Proof. For sake of simplicity, we show the equivalence of (i)–(iii) by proving the equivalences (iii) $\iff$ (i) and (iii) $\iff$ (ii).

For the proof of the implication (iii) $\Rightarrow$ (i), suppose (iii) and use the convergence of the geometric series to see

$$\sum_{k=\ell+1}^{\infty} a_k^2 \le C_3 a_\ell^2 \sum_{k=\ell+1}^{\infty} \rho_1^{k-\ell} = C_3\rho_1(1-\rho_1)^{-1} a_\ell^2.$$

This proves (i) with $C_1 = C_3\rho_1(1-\rho_1)^{-1}$.

Similarly, the implication (iii) $\Rightarrow$ (ii) follows via

$$\begin{aligned} \sum_{k=0}^{\ell-1} a_k^{-1/s} &\le C_3^{1/(2s)} a_\ell^{-1/s} \sum_{k=0}^{\ell-1} \rho_1^{(\ell-k)/(2s)} \\ &\le C_3^{1/(2s)} (1-\rho_1^{1/(2s)})^{-1} a_\ell^{-1/s}. \end{aligned}$$

This shows (ii) with $C_2 = C_3^{1/(2s)}(1-\rho_1^{1/(2s)})^{-1}$.

For the proof of the implication (i) $\Rightarrow$ (iii), suppose (i) and conclude

$$(1+C_1^{-1}) \sum_{j=\ell+1}^{\infty} a_j^2 \le \sum_{j=\ell+1}^{\infty} a_j^2 + a_\ell^2 = \sum_{j=\ell}^{\infty} a_j^2.$$

By mathematical induction, this implies

$$\sum_{j=\ell+k}^{\infty} a_j^2 \le (1+C_1^{-1})^{-1} \sum_{j=\ell+k-1}^{\infty} a_j^2 \le (1+C_1^{-1})^{-k} \sum_{j=\ell}^{\infty} a_j^2$$

and hence

$$\begin{aligned} a_{\ell+k}^2 \le \sum_{j=\ell+k}^{\infty} a_j^2 &\le (1+C_1^{-1})^{-k} \sum_{j=\ell}^{\infty} a_j^2 \\ &\le (1+C_1)(1+C_1^{-1})^{-k} a_\ell^2. \end{aligned}$$

This proves (iii) with $\rho_1 = (1+C_1^{-1})^{-1}$ and $C_3 = (1+C_1)$.

The implication (ii) $\Rightarrow$ (iii) follows analogously. To that end, assume $a_{\ell+k} > 0$. Then, there holds

$$(1+C_2^{-1}) \sum_{j=0}^{\ell-1} a_j^{-1/s} \le \sum_{j=0}^{\ell} a_j^{-1/s}.$$

Mathematical induction shows then shows

$$\sum_{j=0}^{\ell} a_j^{-1/s} \le (1+C_2^{-1})^{-1} \sum_{j=0}^{\ell+1} a_j^{-1/s} \le (1+C_2^{-1})^{-k} \sum_{j=0}^{\ell+k} a_j^{-1/s}$$

and hence

$$\begin{aligned} a_\ell^{-1/s} \le \sum_{j=0}^{\ell} a_j^{-1/s} &\le (1+C_2^{-1})^{-k} \sum_{j=0}^{\ell+k} a_j^{-1/s} \\ &\le (1+C_2)(1+C_2^{-1})^{-k} a_{\ell+k}^{-1/s}. \end{aligned}$$

With the assumption that $a_{\ell+k} = 0$ implies $a_{\ell+k+n} = 0$ for all $n \in \mathbb{N}_0$, this proves $a_{\ell+k}^2 \le (1+C_2)^{2s}(1+C_2^{-1})^{-2sk} a_\ell^2$ for all $\ell, k \in \mathbb{N}_0$. This is (iii) with $\rho_1 = (1+C_2^{-1})^{-2s}$ and $C_3 = (1+C_2)^{2s}$. $\square$

Proposition 2.3.9. *Suppose estimator reduction* (2.3.8)*. Then, general quasi-orthogonality* (E2) *implies* (2.3.12)–(2.3.14) *with* $a_\ell = \eta(\mathcal{T}_\ell)$ *for all* $\ell \in \mathbb{N}_0$*. The constant* $C_1 > 0$ *depends only on* ρ_{est}*,* C_{est}*, and* $\varepsilon_{\mathrm{qo}}$*, whereas the constants* C_2*,* $C_3 > 0$*, and* $0 < \rho_1 < 1$ *are given by* (2.3.15)*.*

Proof. In the following, the general quasi-orthogonality (E2) implies each the statements (2.3.12)–(2.3.14) since (E2) implies (2.3.12). To that end, the estimator reduction (2.3.8) from Lemma 2.3.5 yields for any $\nu > 0$ that

$$\begin{aligned} \sum_{k=\ell+1}^{\ell+N+1} \eta(\mathcal{T}_k)^2 &\le \sum_{k=\ell+1}^{\ell+N+1} \big(\rho_{\mathrm{est}} \eta(\mathcal{T}_{k-1})^2 + C_{\mathrm{est}} \varrho(\mathcal{T}_{k-1}, \mathcal{T}_k)^2\big) \\ &= \sum_{k=\ell+1}^{\ell+N+1} \Big((\rho_{\mathrm{est}} + \nu)\eta(\mathcal{T}_{k-1})^2 + C_{\mathrm{est}}\big(\varrho(\mathcal{T}_{k-1}, \mathcal{T}_k)^2 - \nu C_{\mathrm{est}}^{-1} \eta(\mathcal{T}_{k-1})^2\big)\Big). \end{aligned} \tag{2.3.16}$$

With the constants ρ_{est} and C_{est} from (2.3.7), the constraint on $\varepsilon_{\mathrm{qo}}$ in (E2) reads

$$0 \le \varepsilon_{\mathrm{qo}} < \frac{1-\rho_{\mathrm{est}}}{C_{\mathrm{est}}} \le \sup_{\delta>0} \frac{1-(1+\delta)(1-(1-\rho_{\mathrm{red}})\theta}{2+\delta^{-1}}$$

for some choice of $\delta > 0$. Note that this choice is valid since $\rho_{\rm est} < 1$. In particular, it exists $\nu < 1 - \rho_{\rm est}$ such that $\varepsilon_{\rm qo} \le \nu C_{\rm est}^{-1}$. This allows to apply general quasi-orthogonality (E2) to the last term of (2.3.16), i.e.,

$$\sum_{k=\ell+1}^{\ell+N+1} \varrho(\mathcal{T}_{k-1}, \mathcal{T}_k)^2 - \nu C_{\rm est}^{-1} \eta(\mathcal{T}_{k-1})^2 \le C_{\rm qo} \eta(\mathcal{T}_\ell)^2. \tag{2.3.17}$$

The combination of (2.3.16)–(2.3.17) and passing to the limit $N \to \infty$ proves

$$\sum_{k=\ell+1}^{\infty} \eta(\mathcal{T}_k)^2 \le \sum_{k=\ell+1}^{\infty} (\rho_{\rm est} + \nu)\eta(\mathcal{T}_{k-1})^2 + C_{\rm est} C_{\rm qo} \eta(\mathcal{T}_\ell)^2.$$

Some rearrangement leads to

$$(1 - (\rho_{\rm est} + \nu)) \sum_{k=\ell+1}^{\infty} \eta(\mathcal{T}_k)^2 \le (\rho_{\rm est} + \nu + C_{\rm est} C_{\rm qo}) \eta(\mathcal{T}_\ell)^2.$$

This shows that $a_\ell := \eta(\mathcal{T}_\ell)$ satisfies that $a_\ell = 0$ implies $a_k = 0$ for all $k \ge \ell$. Hence, we have (2.3.12) with $C_1 = (\rho_{\rm est} + \nu + C_{\rm est} C_{\rm qo})/(1 - (\rho_{\rm est} + \nu))$ and conclude the proof of (E2) $\Rightarrow$ (2.3.12). Lemma 2.3.8 yields the equivalence (2.3.12)–(2.3.14). □

Proof of Theorem 2.3.3, (ii). Stability and reduction (E1) guarantee estimator reduction (2.3.8) for $\eta(\mathcal{T}_\ell)$ by Lemma 2.3.5. Together with quasi-orthogonality (E2), Proposition 2.3.9 shows (2.3.14) for $a_\ell = \eta(\mathcal{T}_\ell)$. This proves Theorem 2.3.3 (ii) with $C_{\rm conv} = C_3$ and $\rho_{\rm conv} = \rho_1$. □

2.3.5. Optimality of Dörfler marking. Theorem 2.3.3 (i)–(ii) state that Dörfler marking (2.2.1) essentially guarantees $\lim_{\ell\to\infty} \eta(\mathcal{T}_\ell) = 0$ or even R-linear convergence to zero. The next statement asserts the converse.

Proposition 2.3.10. *Let $\widehat{\mathcal{T}} \in \mathbb{T}(\mathcal{T})$ denote a refinement of $\mathcal{T} \in \mathbb{T}$. Stability* (E1a) *and discrete reliability* (E3) *imply that for all $0 < \theta_0 < \theta_\star := (1 - \varepsilon_{\rm drel})/(1 + C_{\rm drel}^2)$, there exists some $0 < \kappa_0 < \min\{\kappa_{\rm dlr}, 1\}$ such that*

$$\eta(\widehat{\mathcal{T}})^2 \le \kappa_0 \eta(\mathcal{T})^2 \quad \Longrightarrow \quad \theta\, \eta(\mathcal{T})^2 \le \sum_{T \in \mathcal{R}(\mathcal{T}, \widehat{\mathcal{T}})} \eta_T(\mathcal{T})^2 \tag{2.3.18}$$

holds for all $0 < \theta \le \theta_0$, where $\mathcal{S}(\mathcal{T}, \widehat{\mathcal{T}}) \subseteq \mathcal{R}(\mathcal{T}, \widehat{\mathcal{T}}) \subseteq \mathcal{T}$ with $|\mathcal{T} \setminus \widehat{\mathcal{T}}| \le |\mathcal{R}(\mathcal{T}, \widehat{\mathcal{T}})| \le C_{\rm ref} |\mathcal{T} \setminus \widehat{\mathcal{T}}|$ from (E3). *The constant κ_0 depends only on $C_{\rm drel}$, $\varepsilon_{\rm drel}$, and θ_0.*

Remark 2.3.11. *Note that the proof requires* (E3) *to hold only for the particular $\mathcal{T}$ and $\widehat{\mathcal{T}}$ in* (2.3.18).

Proof. The Young inequality and stability (E1a) show, for any $\delta > 0$, that

$$\begin{aligned} \eta(\mathcal{T})^2 &= \sum_{T \in \mathcal{S}(\mathcal{T}, \widehat{\mathcal{T}})} \eta_T(\mathcal{T})^2 + \sum_{T \in \mathcal{T} \setminus \mathcal{S}(\mathcal{T}, \widehat{\mathcal{T}})} \eta_T(\mathcal{T})^2 \\ &\le \sum_{T \in \mathcal{S}(\mathcal{T}, \widehat{\mathcal{T}})} \eta_T(\mathcal{T})^2 + (1 + \delta^{-1}) \sum_{T \in \widehat{\mathcal{T}} \setminus \widehat{\mathcal{S}}(\mathcal{T}, \widehat{\mathcal{T}})} \eta_T(\widehat{\mathcal{T}})^2 + (1 + \delta) \varrho(\mathcal{T}, \widehat{\mathcal{T}})^2. \end{aligned}$$

Recall $\mathcal{S}(\mathcal{T},\widehat{\mathcal{T}}) \subseteq \mathcal{R}(\mathcal{T},\widehat{\mathcal{T}})$ by (E3). The application of the discrete reliability (E3) and the assumption $\eta(\widehat{\mathcal{T}})^2 \leq \kappa_0 \eta(\mathcal{T})^2$ yield

$$\begin{aligned}\eta(\mathcal{T})^2 \leq (1+\delta^{-1})\kappa_0\eta(\mathcal{T})^2 + (1+\delta)\varepsilon_{\mathrm{drel}}\eta(\mathcal{T})^2 \\ + \big(1+(1+\delta)C_{\mathrm{drel}}^2\big) \sum_{T\in\mathcal{R}(\mathcal{T},\widehat{\mathcal{T}})} \eta_T(\mathcal{T})^2.\end{aligned}$$

Some rearrangement of those terms reads

$$\frac{1-(1+\delta^{-1})\kappa_0-(1+\delta)\varepsilon_{\mathrm{drel}}}{1+(1+\delta)C_{\mathrm{drel}}^2}\,\eta(\mathcal{T})^2 \leq \sum_{T\in\mathcal{R}(\mathcal{T},\widehat{\mathcal{T}})} \eta_T(\mathcal{T})^2.$$

Recall $\varepsilon_{\mathrm{drel}} < 1$ by (E3), choose $\delta > 0$ sufficiently small, and determine $0 < \kappa_0 < 1$ such that

$$\theta_0 \leq \frac{1-(1+\delta^{-1})\kappa_0-(1+\delta)\varepsilon_{\mathrm{drel}}}{1+(1+\delta)C_{\mathrm{drel}}^2} < \frac{1-\varepsilon_{\mathrm{drel}}}{1+C_{\mathrm{drel}}^2} = \theta_\star. \tag{2.3.19}$$

□

The next result is a variant of Proposition 2.3.10 which is not actually needed in the forthcoming analysis. However, we include it for completeness.

Corollary 2.3.12. *Let $\widehat{\mathcal{T}} \in \mathbb{T}(\mathcal{T})$ denote a refinement of $\mathcal{T} \in \mathbb{T}$. For all $0 < \kappa_0 < 1$ with $\kappa_0 \leq \kappa_{\mathrm{dlr}}$, there exists a constant $0 < \theta_0 < 1$ and some $0 < \varepsilon_0 < 1$ such that stability* (E1a)*, discrete reliability* (E3) *with $\varepsilon_{\mathrm{drel}} \leq \varepsilon_0$, and $0 < \theta \leq \theta_0$ imply* (2.3.18)*. The constants θ_0, ε_0 depend only on C_{drel} and κ_0.*

Proof. For arbitrary $0 < \kappa_0 < 1$ with $\kappa_0 \leq \kappa_{\mathrm{dlr}}$ choose $\delta, \varepsilon_0 > 0$ sufficiently small such that (2.3.19) becomes

$$\theta_0 := \frac{1-(1+\delta^{-1})\kappa_0-(1+\delta)\varepsilon_{\mathrm{drel}}}{1+(1+\delta)C_{\mathrm{drel}}^2} \geq \frac{1-(1+\delta^{-1})\kappa_0-(1+\delta)\varepsilon_0}{1+(1+\delta)C_{\mathrm{drel}}^2} > 0.$$

As in the proof of Proposition 2.3.10, this concludes (2.3.18). □

2.3.6. Quasi-optimality of adaptive algorithm. This section proves optimal convergence rates for the estimator and thereby renders the theoretical heart of the proof of Theorem 2.3.3 (iii).

Lemma 2.3.13. *Let $\mathcal{T} \in \mathbb{T}$ such that $\eta(\mathcal{T}) > 0$. Then, for $s > 0$ with $\|\eta,\mathbb{T}(\mathcal{T})\|_s < \infty$, there exists a refinement $\widehat{\mathcal{T}} \in \mathbb{T}(\mathcal{T})$ with*

$$\eta(\widehat{\mathcal{T}})^2 \leq \kappa_0 \eta(\mathcal{T})^2, \tag{2.3.20a}$$

$$|\widehat{\mathcal{T}} \setminus \mathcal{T}| < \|\eta,\mathbb{T}(\mathcal{T})\|_s^{1/s}\,\kappa_0^{-1/s}\eta(\mathcal{T})^{-1/s}. \tag{2.3.20b}$$

Assume that the implication (2.3.18) *is valid for one particular choice of $0 < \kappa_0, \theta_0 < 1$ and the triangulations $\mathcal{T}$ and $\widehat{\mathcal{T}}$. Then, the set $\mathcal{R}(\mathcal{T},\widehat{\mathcal{T}}) \supseteq \mathcal{T}\setminus\widehat{\mathcal{T}}$ from Proposition 2.3.10 satisfies*

$$|\mathcal{R}(\mathcal{T},\widehat{\mathcal{T}})| < C_{\mathrm{ref}}\kappa_0^{1/(-2s)}\eta(\mathcal{T})^{-1/s}\|\eta,\mathbb{T}(\mathcal{T})\|_s^{1/s} \tag{2.3.21a}$$

and satisfies the Dörfler marking for all $0 < \theta \leq \theta_0$, i.e.,

$$\theta\eta(\mathcal{T})^2 \leq \sum_{T\in\mathcal{R}(\mathcal{T},\widehat{\mathcal{T}})} \eta_T(\mathcal{T})^2. \tag{2.3.21b}$$

PROOF. Choose a minimal $N \in \mathbb{N}_0$, such that $\|\eta, \mathbb{T}(\mathcal{T})\|_s (N+1)^{-s} \leq \kappa_0^{1/2} \eta(\mathcal{T})$ (note that $N > 0$ by the fact that $\eta(\mathcal{T}) \leq \|\eta, \mathbb{T}(\mathcal{T})\|_s$ and $\kappa_0 < 1$). By assumption (and the fact that $\mathbb{T}(\mathcal{T}, N)$ is finite), there holds

$$\min_{\widehat{\mathcal{T}} \in \mathbb{T}(\mathcal{T}, N)} ((N+1)^s \eta(\widehat{\mathcal{T}})) \leq \|\eta, \mathbb{T}(\mathcal{T})\|_s$$

and hence, there exists a triangulation $\widehat{\mathcal{T}} \in \mathbb{T}(\mathcal{T}, N)$ with $(N+1)^s \eta(\widehat{\mathcal{T}}) \leq \|\eta, \mathbb{T}(\mathcal{T})\|_s$. This implies

$$\eta(\widehat{\mathcal{T}}) \leq (N+1)^{-s} \|\eta, \mathbb{T}(\mathcal{T})\|_s \leq \kappa_0^{1/2} \eta(\mathcal{T}).$$

The minimality of N implies $N^{-s} > \kappa_0^{1/2} \eta(\mathcal{T}) \|\eta, \mathbb{T}(\mathcal{T})\|_s^{-1}$ and hence

$$N < \kappa_0^{1/(-2s)} \eta(\mathcal{T})^{-1/s} \|\eta, \mathbb{T}(\mathcal{T})\|_s^{1/s}. \tag{2.3.22}$$

Since $\widehat{\mathcal{T}} \in \mathbb{T}(\mathcal{T}, N)$, this concludes (2.3.20). The implication (2.3.18) thus guarantees that the set $\mathcal{R}(\mathcal{T}, \widehat{\mathcal{T}}) \subseteq \mathcal{T}$ with $|\mathcal{R}(\mathcal{T}, \widehat{\mathcal{T}})| \simeq |(\mathcal{T} \setminus \widehat{\mathcal{T}})|$ satisfies the Dörfler marking (2.3.21b). Estimate (2.3.21a) follows from (2.3.22), i.e.,

$$C_{\mathrm{ref}}^{-1} |\mathcal{R}(\mathcal{T}, \widehat{\mathcal{T}})| \leq |(\mathcal{T} \backslash \widehat{\mathcal{T}})| \leq N < \kappa_0^{1/(-2s)} \eta(\mathcal{T})^{-1/s} \|\eta, \mathbb{T}(\mathcal{T})\|_s^{1/s}.$$

This concludes the proof. □

The following two propositions state the optimality of the adaptive algorithm.

PROPOSITION 2.3.14. *The son estimate* (T1) *implies*

$$c_{\mathrm{opt}} C_{\mathrm{approx}}(s) \leq \sup_{\ell \in \mathbb{N}_0} \frac{\eta(\mathcal{T}_\ell)}{(|\mathcal{T}_\ell \setminus \mathcal{T}_0| + 1)^{-s}}, \tag{2.3.23}$$

where the constant $c_{\mathrm{opt}} > 0$ depends only on C_{son} and $|\mathcal{T}_0|$.

PROPOSITION 2.3.15. *Suppose that* (2.3.20)–(2.3.21a) *of Lemma 2.3.13 are valid for one particular $0 < \kappa_0 < 1$ and $s > 0$, as well as for all $\mathcal{T} = \mathcal{T}_\ell$, $\ell \in \mathbb{N}_0$ with $\eta(\mathcal{T}_\ell) > 0$. Assume that there holds* (T2)–(T3) *and that* (2.3.13) *from Lemma 2.3.8 holds for $\alpha_\ell := \eta(\mathcal{T}_\ell)$. Then, $|\mathcal{M}_\ell| \leq C_{\min} |\mathcal{R}(\mathcal{T}_\ell, \widehat{\mathcal{T}}_\ell)|$ for all $\ell \in \mathbb{N}_0$ (with $\mathcal{R}(\mathcal{T}_\ell, \widehat{\mathcal{T}}_\ell)$ from Lemma 2.3.13) implies*

$$\sup_{\ell \in \mathbb{N}_0} \frac{\eta(\mathcal{T}_\ell)}{(|\mathcal{T}_\ell \setminus \mathcal{T}_0| + 1)^{-s}} \leq C_{\mathrm{opt}} C_{\mathrm{approx}}(s). \tag{2.3.24}$$

There holds $C_{\mathrm{opt}} = 2^s C_2^s C_{\mathrm{closure}}^s C_{\min}^s C_{\mathrm{ref}}^s \kappa_0^{-1/2}$ and $c_{\mathrm{opt}} > 0$ depends only on C_{son} and $|\mathcal{T}_0|$.

PROOF OF PROPOSITION 2.3.14. Choose $N \in \mathbb{N}_0$, $\ell \in \mathbb{N}_0$, and the largest possible $k \in \mathbb{N}_0$ with $|\mathcal{T}_{\ell+k} \setminus \mathcal{T}_\ell| \leq N$. Due to the maximality of k and (T1), there holds $N + 1 < |\mathcal{T}_{\ell+k+1} \setminus \mathcal{T}_\ell| + 1 \leq |\mathcal{T}_{\ell+k+1}| + 1 \lesssim C_{\mathrm{son}}(|\mathcal{T}_{\ell+k}| + 1) \lesssim C_{\mathrm{son}}(|\mathcal{T}_{\ell+k} \setminus \mathcal{T}_0| + 1)$, where the hidden constant depends only on $|\mathcal{T}_0|$. This leads to

$$\inf_{\widehat{\mathcal{T}} \in \mathbb{T}(\mathcal{T}_\ell, N)} (N+1)^s \eta(\widehat{\mathcal{T}}) \lesssim (|\mathcal{T}_{\ell+k} \setminus \mathcal{T}_0| + 1)^s \eta(\mathcal{T}_{\ell+k})$$

and concludes the proof. □

PROOF OF PROPOSITION 2.3.15. If $\eta(\mathcal{T}_{\ell_0}) = 0$. Then, (2.3.13) implies $\eta(\mathcal{T}_\ell) = 0$ for all $\ell \geq \ell_0$. Hence, we may consider $0 \leq \ell \leq \ell_0$ only. By assumption (2.3.21a), there holds

$$|\mathcal{M}_\ell| \leq C_{\min} |\mathcal{R}(\mathcal{T}_\ell, \widehat{\mathcal{T}}_\ell)| \leq C_{\min} C_{\mathrm{ref}} \kappa_0^{1/(-2s)} \eta(\mathcal{T}_\ell)^{-1/s} \|\eta, \mathbb{T}(\mathcal{T}_\ell)\|_s^{1/s}.$$

The uniform approximability (T3) shows

$$|\mathcal{M}_\ell| \le C_{\min} C_{\text{ref}} C_{\text{approx}}(s)^{1/s} \kappa_0^{1/(-2s)} \eta(\mathcal{T}_\ell)^{-1/s} \quad \text{for all } \ell \in \mathbb{N}_0. \tag{2.3.25}$$

The inverse summability (2.3.13) together with (2.3.25) and the closure estimate (T2) show for all $\ell \in \mathbb{N}_0$

$$\begin{aligned} |\mathcal{T}_\ell \setminus \mathcal{T}_0| + 1 \le 2(|\mathcal{T}_\ell \setminus \mathcal{T}_0|) &\le 2C_{\text{closure}} \sum_{j=0}^{\ell-1} |\mathcal{M}_j| \\ &\le 2C_{\text{closure}} C_{\min} C_{\text{ref}} C_{\text{approx}}(s)^{1/s} \kappa_0^{1/(-2s)} \sum_{j=0}^{\ell-1} \eta(\mathcal{T}_j)^{-1/s} \\ &\le 2C_2 C_{\text{closure}} C_{\min} C_{\text{ref}} C_{\text{approx}}(s)^{1/s} \kappa_0^{1/(-2s)} \eta(\mathcal{T}_\ell)^{-1/s}. \end{aligned} \tag{2.3.26}$$

Consequently,

$$\eta(\mathcal{T}_\ell)(|\mathcal{T}_\ell \setminus \mathcal{T}_0| + 1)^s \le 2^s C_2^s C_{\text{closure}}^s C_{\min}^s C_{\text{ref}}^s \kappa_0^{-1/2} C_{\text{approx}}(s) \quad \text{for all } \ell \in \mathbb{N}.$$

This leads to the upper bound in (2.3.24). □

Proof of Theorem 2.3.3 (iii). Choose $\theta_0 := \theta < \theta_\star$. Stability (E1a) and discrete reliability (E3) guarantee that (2.3.18) holds for θ_0, some $0 < \kappa_0 < 1$, and in particular for all $\mathcal{T} = \mathcal{T}_\ell$, $\ell \in \mathbb{N}_0$. This implies that (2.3.20)–(2.3.21) of Lemma 2.3.13 are valid particularly for all $\mathcal{T} = \mathcal{T}_\ell$, $\ell \in \mathbb{N}_0$. Step (iii) of Algorithm 2.2.1 selects some set $\mathcal{M}_\ell$ with *(almost) minimal* cardinality which satisfies the Dörfler marking (2.2.1) for θ. The Dörfler marking (2.3.21b) for $\theta = \theta_0$ implies $|\mathcal{M}_\ell| \le C_{\min} |\mathcal{R}(\mathcal{T}_\ell, \widehat{\mathcal{T}}_\ell)|$. Reduction and stability (E1) proves the estimator reduction (2.3.8) from Lemma 2.3.5. This and quasi-orthogonality (E2) allow to employ Proposition 2.3.9 which ensures that (2.3.12)–(2.3.14) hold for $\alpha_\ell := \eta(\mathcal{T}_\ell)$. Finally, Proposition 2.3.14–2.3.15 conclude the proof. □

Remark 2.3.16. *Note that the proof of Theorem 2.3.3 (iii) requires* (2.3.18) *only for* $\mathcal{T} = \mathcal{T}_\ell$, $\ell \in \mathbb{N}_0$. *Hence, Remark 2.3.11 shows that it is sufficient to claim* (E3) *for all* $\mathcal{T} = \mathcal{T}_\ell$, $\ell \in \mathbb{N}_0$ *to obtain Theorem 2.3.3 (iii). This relaxation is exploited in Section 3.6.2, below.*

2.4. Equivalent approximation problems

Assume that there exist constants $C_{\text{rel}}, C_{\text{eff}} > 0$ as well as functions $\text{err}(\cdot) : \mathbb{T} \to [0, \infty)$ and $\text{data}(\cdot) : \mathbb{T} \to [0, \infty)$ such that there holds reliability

$$\text{err}(\mathcal{T}) \le C_{\text{rel}} \eta(\mathcal{T}) \quad \text{for all } \mathcal{T} \in \mathbb{T}. \tag{2.4.1}$$

as well as efficiency

$$C_{\text{eff}}^{-1} \eta(\mathcal{T}) \le \text{err}(\mathcal{T}) + \text{data}(\mathcal{T}) \quad \text{for all } \mathcal{T} \in \mathbb{T}. \tag{2.4.2}$$

Suppose that the functions $\text{err}(\cdot)$ and $\text{data}(\cdot)$ are quasi-monotone (see also (2.7.6) below) in the sense that there exists a constant $C_{\text{mon}} > 0$ such that all $\widehat{\mathcal{T}} \in \mathbb{T}(\mathcal{T})$ and all $\mathcal{T} \in \mathbb{T}$ satisfy

$$\text{err}(\widehat{\mathcal{T}}) \le C_{\text{mon}} \text{err}(\mathcal{T}) \quad \text{and} \quad \text{data}(\widehat{\mathcal{T}}) \le C_{\text{mon}} \text{data}(\mathcal{T}). \tag{2.4.3}$$

We define the corresponding approximability norms analogously to (2.2.5) as

$$\|\mathrm{err}, \mathbb{T}(\mathcal{T})\|_s := \sup_{N\in\mathbb{N}_0} \min_{\widehat{\mathcal{T}}\in\mathbb{T}(\mathcal{T},N)} ((N+1)^s \mathrm{err}(\widehat{\mathcal{T}})),$$

$$\|\mathrm{data}, \mathbb{T}(\mathcal{T})\|_s := \sup_{N\in\mathbb{N}_0} \min_{\widehat{\mathcal{T}}\in\mathbb{T}(\mathcal{T},N)} ((N+1)^s \mathrm{data}(\widehat{\mathcal{T}})).$$

Analogously to (T3), we say that $\mathrm{err}(\cdot)$ and $\mathrm{data}(\cdot)$ satisfy uniform approximability if

$$C_{\mathrm{approx}}^{\mathrm{err}}(s) := \sup_{\mathcal{T}\in\mathbb{T}} \|\mathrm{err}, \mathbb{T}(\mathcal{T})\|_s < \infty, \tag{2.4.4a}$$

$$C_{\mathrm{approx}}^{\mathrm{data}}(s) := \sup_{\mathcal{T}\in\mathbb{T}} \|\mathrm{data}, \mathbb{T}(\mathcal{T})\|_s < \infty. \tag{2.4.4b}$$

for some $s > 0$.

Proposition 2.4.1. *Assume that there holds reliability* (2.4.1)*, efficiency* (2.4.2)*, and quasi-monotonicity* (2.4.3)*. Then, the uniform approximability statements in* (2.4.4) *and* (T3) *are equivalent in the sense that*

(i) $2^{-s} C_{\mathrm{eff}}^{-1} C_{\mathrm{approx}}(s) \le C_{\mathrm{approx}}^{\mathrm{err}}(s) + C_{\mathrm{mon}} C_{\mathrm{approx}}^{\mathrm{data}}(s)$,

(ii) $C_{\mathrm{approx}}^{\mathrm{err}}(s) \le C_{\mathrm{rel}} C_{\mathrm{approx}}(s)$.

Remark 2.4.2. *The literature, e.g.,* **[78, 35]***, usually assumes* $\|\mathrm{err}, \mathbb{T}\|_s + \|\mathrm{data}, \mathbb{T}\|_s < \infty$ *and uses the equivalence* (2.4.1)*–*(2.4.2) *as well as the overlay estimate* (2.5.1) *below to obtain rate optimality of the error estimator and the so called total error* $\mathrm{err}(\mathcal{T}) + \mathrm{data}(\mathcal{T})$*. Our approach, however, is much more fundamental as we only use properties of the error estimator itself to deduce the rate optimality of Theorem 2.3.3 (iii). The statements on error convergence are derived in this section by bootstrapping the results on the estimator. This point of view allows to include a much broader class of applications as is shown in the examples of Chapter 3, 5, 6, below.*

Proof. The upper bound (2.4.1) shows

$$\|\mathrm{err}, \mathbb{T}(\mathcal{T})\|_s \le C_{\mathrm{rel}} \|\eta, \mathbb{T}(\mathcal{T})\|_s \quad \text{for all } s > 0.$$

This proves (ii).

To see (i), suppose (2.4.4) for some $s > 0$. For all even $N \in \mathbb{N}_0$, this guarantees the existence of a triangulation $\mathcal{T}_{N/2} \in \mathbb{T}(\mathcal{T}, N/2)$ with

$$\mathrm{err}(\mathcal{T}_{N/2})(N/2+1)^s \le C_{\mathrm{approx}}^{\mathrm{err}}(s)$$

and also the existence of a triangulation $\mathcal{T}_N \in \mathbb{T}(\mathcal{T}_{N/2}, N/2)$ with

$$\mathrm{data}(\mathcal{T}_N)(N/2+1)^s \le C_{\mathrm{approx}}^{\mathrm{data}}(s). \tag{2.4.5}$$

With quasi-monotonicity (2.4.3), there holds

$$\mathrm{err}(\mathcal{T}_N) \le C_{\mathrm{mon}} \mathrm{err}(\mathcal{T}_{N/2}) \le C_{\mathrm{mon}} (N/2+1)^{-s} C_{\mathrm{approx}}^{\mathrm{err}}(s).$$

This and the lower bound (2.4.2) yield

$$\begin{aligned} C_{\mathrm{eff}}^{-1} \eta(\mathcal{T}_N) &\le \mathrm{err}(\mathcal{T}_N) + \mathrm{data}(\mathcal{T}_N) \\ &\le (C_{\mathrm{approx}}^{\mathrm{data}}(s) + C_{\mathrm{mon}} C_{\mathrm{approx}}^{\mathrm{err}}(s))(N/2+1)^{-s} \\ &\le 2^s (C_{\mathrm{approx}}^{\mathrm{data}}(s) + C_{\mathrm{mon}} C_{\mathrm{approx}}^{\mathrm{err}}(s))(N+1)^{-s}. \end{aligned}$$

By definition, there holds $|\mathcal{T}_N \setminus \mathcal{T}| \le |\mathcal{T}_N \setminus \mathcal{T}_{N/2}| + |\mathcal{T}_{N/2} \setminus \mathcal{T}| \le N$. This shows $\mathcal{T}_N \in \mathbb{T}(\mathcal{T}, N)$ and hence proves $\|\eta, \mathbb{T}(\mathcal{T})\|_s \le 2^s C_{\mathrm{eff}} (C_{\mathrm{approx}}^{\mathrm{data}}(s) + C_{\mathrm{mon}} C_{\mathrm{approx}}^{\mathrm{err}}(s))$. This concludes the proof. □

In the frame of this section, we prove following analog of Theorem 2.3.3 which provides convergence results for the error instead of the estimator.

Theorem 2.4.3. (i) *Suppose* (E1) *is satisfied and assume* $\lim_{\ell\to\infty} \varrho(\mathcal{T}_\ell, \mathcal{T}_{\ell+1}) = 0$ *(with* $\varrho(\cdot,\cdot)$ *from Section 2.3.1). Then, for all* $0 < \theta \le 1$, *the error is convergent in the sense*

$$\lim_{\ell\to\infty} \mathrm{err}(\mathcal{T}_\ell) = 0. \tag{2.4.6}$$

(ii) *Suppose* (E1)–(E2) *are satisfied. Then, for all* $0 < \theta \le 1$, *the error is R-linear convergent in the sense that there exists* $0 < \rho_{\mathrm{conv}} < 1$ *and* $C_{\mathrm{conv}} > 0$ *such that*

$$\mathrm{err}(\mathcal{T}_{\ell+j})^2 \le C_{\mathrm{eff}}^2 C_{\mathrm{conv}} \rho_{\mathrm{conv}}^j \left(\mathrm{err}(\mathcal{T}_\ell) + \mathrm{data}(\mathcal{T}_\ell)\right)^2 \quad \text{for all } j, \ell \in \mathbb{N}_0. \tag{2.4.7}$$

(iii) *Suppose* (E1)–(E3) *and* (T1)–(T3) *are satisfied for some* $s > 0$. *Then* $0 < \theta < \theta_\star := (1 - \varepsilon_{\mathrm{drel}})/(1 + C_{\mathrm{drel}}^2)$ *implies quasi-optimal convergence of the error in the sense of*

$$\begin{aligned} c_{\mathrm{opt}} C_{\mathrm{approx}}^{\mathrm{err}}(s) &\le \sup_{\ell\in\mathbb{N}_0} \frac{\mathrm{err}(\mathcal{T}_\ell)}{(|\mathcal{T}_\ell \setminus \mathcal{T}_0| + 1)^{-s}} \\ &\le 2^s C_{\mathrm{opt}} C_{\mathrm{rel}} C_{\mathrm{eff}} (C_{\mathrm{approx}}^{\mathrm{data}}(s) + C_{\mathrm{mon}} C_{\mathrm{approx}}^{\mathrm{err}}(s)), \end{aligned} \tag{2.4.8}$$

where the lower bound requires only (T1) *to hold.*

The constants C_{conv}, ρ_{conv}, c_{opt}, C_{opt} *are defined in Theorem 2.3.3.*

Proof. The statements (i)–(ii) follow immediately from Theorem 2.3.3 (i)–(ii) and the equivalences (2.4.1)–(2.4.2). To see the upper bound in (iii), combine the upper bound in Theorem 2.3.3 (iii) with Proposition 2.4.1 and the upper bound (2.4.1). For the lower bound in (iii), choose $N \in \mathbb{N}_0$, $\ell \in \mathbb{N}_0$, and the largest possible $k \in \mathbb{N}_0$ with $|\mathcal{T}_{\ell+k} \setminus \mathcal{T}_\ell| \le N$. Due to maximality of ℓ and (T1), there holds $N + 1 < |\mathcal{T}_{\ell+k+1} \setminus \mathcal{T}_\ell| + 1 \le |\mathcal{T}_{\ell+k+1}| + 1 \lesssim C_{\mathrm{son}}(|\mathcal{T}_{\ell+k}| + 1) \lesssim C_{\mathrm{son}}(|\mathcal{T}_{\ell+k} \setminus \mathcal{T}_0| + 1)$, where the hidden constant depends only on $|\mathcal{T}_0|$. This leads to

$$\inf_{\mathcal{T}\in\mathbb{T}(\mathcal{T}_\ell,N)} (N+1)^s \mathrm{err}(\mathcal{T}) \lesssim (|\mathcal{T}_{\ell+k} \setminus \mathcal{T}_0| + 1)^s \mathrm{err}(\mathcal{T}_{\ell+k})$$

and concludes the proof. □

Before we conclude the section, we provide a criterion, under which reliability (2.4.1) follows from discrete reliability (E3).

Proposition 2.4.4. *Suppose a constant* $C > 0$ *such that the following holds. Given* $\mathcal{T} \in \mathbb{T}$, *there exists a sequence of triangulations* $\widehat{\mathcal{T}}_\ell \in \mathbb{T}(\mathcal{T})$ *with* $\lim_{\ell\to\infty} \eta(\widehat{\mathcal{T}}_\ell) = 0$ *such that*

$$C^{-1}\mathrm{err}(\mathcal{T}) \le \lim_{\ell\to\infty} \varrho(\mathcal{T}, \widehat{\mathcal{T}}_\ell)$$

with $\varrho(\cdot,\cdot)$ *from Section 2.3.1. Then, discrete reliability* (E3) *(where the restriction* $\varepsilon_{\mathrm{drel}} < 1$ *is not necessary) and quasi-monotonicity* (2.7.6) *imply reliability* (2.4.1) *with* $C_{\mathrm{rel}}^2 = C^2(C_{\mathrm{drel}}^2 + \varepsilon_{\mathrm{drel}})$.

Proof. Assume $\eta(\mathcal{T}) = 0$. Then, (2.7.6) implies $\eta(\widehat{\mathcal{T}}_\ell) = 0$ for all $\ell \in \mathbb{N}$ and hence $\eta(\widehat{\mathcal{T}}) \le \kappa_{\mathrm{dlr}}\eta(\mathcal{T})$ for all $\ell \in \mathbb{N}$. Assume $\eta(\mathcal{T}) > 0$. Then, $\lim_{\ell\to\infty} \eta(\widehat{\mathcal{T}}_\ell) = 0$ shows $\eta(\widehat{\mathcal{T}}) \le \kappa_{\mathrm{dlr}}\eta(\mathcal{T})$ for all $\ell \ge \ell_0$ for some sufficiently large $\ell_0 \in \mathbb{N}$. In either case, (E3) is applicable and shows

$$C^{-2}\mathrm{err}(\mathcal{T})^2 \le \lim_{\ell\to\infty} \varrho(\mathcal{T}, \widehat{\mathcal{T}}_\ell)^2 \le (\varepsilon_{\mathrm{drel}} + C_{\mathrm{drel}}^2)\eta(\mathcal{T})^2.$$

This concludes the proof. □

2.5. Optimal complexity

This section understands complexity as a measure of computational effort necessary to compute one step of Algorithm 2.2.1. We assume that the effort is related to

$$|\mathcal{T}_\ell|^\gamma$$

for some $\gamma > 0$ and call this quantity single-step complexity. This is a reasonable assumption, since usually the solution of some linear or nonlinear systems is involved where the complexity is related to the degrees of freedom. To compute the ℓ-th step of Algorithm 2.2.1, it is necessary to compute all the previous steps, too. Therefore, we define the cumulative complexity of the ℓ-th step of Algorithm 2.2.1 by

$$\sum_{j=0}^{\ell} |\mathcal{T}_j|^\gamma.$$

The following theorem shows that for the adaptive algorithm, both measures coincide. To that end, we define the overlay estimate which states that there exists a constant $C_4 > 0$ such that any two triangulations $\mathcal{T}, \widehat{\mathcal{T}} \in \mathbb{T}$ have a coarsest common refinement $\mathcal{T} \oplus \widehat{\mathcal{T}} \in \mathbb{T}(\mathcal{T}) \cap \mathbb{T}(\widehat{\mathcal{T}})$ with

$$|(\mathcal{T} \oplus \widehat{\mathcal{T}}) \setminus \mathcal{T}| \le C_4 |\widehat{\mathcal{T}} \setminus \mathcal{T}_0|. \tag{2.5.1}$$

Theorem 2.5.1. *Suppose a sequence $(\mathcal{T}_\ell^{\mathrm{opt}})_{\ell\in\mathbb{N}_0} \subset \mathbb{T}$ with $\mathcal{T}_{\ell+1}^{\mathrm{opt}} \in \mathbb{T}(\mathcal{T}_\ell^{\mathrm{opt}})$ and $|\mathcal{T}_{\ell+1}^{\mathrm{opt}}| \le C_{\mathrm{son}}|\mathcal{T}_\ell|$ for all $\ell \in \mathbb{N}_0$ such that $\mathcal{T}_0^{\mathrm{opt}} = \mathcal{T}_0$ and that there holds the single-step complexity rate*

$$\sup_{\ell\in\mathbb{N}_0} \frac{\eta(\mathcal{T}_\ell^{\mathrm{opt}})}{(|\mathcal{T}_\ell^{\mathrm{opt}}|^\gamma)^{-s}} < \infty \tag{2.5.2}$$

for some $s > 0$ and some $\gamma > 0$. Suppose (E1)–(E3) *and* (T1)–(T2) *as well as the overlay estimate* (2.5.1). *Then, given $0 < \theta < \theta_\star := (1-\varepsilon_{\mathrm{drel}})/(1+C_{\mathrm{drel}}^2)$, the output of Algorithm 2.2.1 satisfies the same cumulative complexity rate*

$$\sup_{\ell\in\mathbb{N}_0} \frac{\eta(\mathcal{T}_\ell)}{(\sum_{j=0}^{\ell} |\mathcal{T}_j|^\gamma)^{-s}} < \infty. \tag{2.5.3}$$

Remark 2.5.2. *The above result shows that Algorithm 2.2.1 realizes any possible single-step complexity rate even with respect to the cumulative complexity $\sum_{j=0}^{\ell} |\mathcal{T}_j|^\gamma$. This means that the overall investment of computational time is asymptotically optimal and the iterative steps of Algorithm 2.2.1 do not spoil the performance. Particularly, it shows that under the assumptions of Theorem 2.5.1, the adaptive approach converges faster or at least with the same complexity rate as the uniform refinement strategy which realizes $\mathcal{T}_{\ell+1}^{\mathrm{unif}} := \mathbb{T}(\mathcal{T}_\ell^{\mathrm{unif}}, \mathcal{T}_\ell^{\mathrm{unif}})$. To see this, note that the uniform refinement does not require to compute each previous step of the algorithm. Hence, its complexity to compute the ℓ-th step is best measured by the single-step complexity $|\mathcal{T}_\ell^{\mathrm{unif}}|^\gamma$. If uniform refinement satisfies the single-step complexity rate $s > 0$, i.e.,*

$$\sup_{\ell\in\mathbb{N}_0} \frac{\eta(\mathcal{T}_\ell^{\mathrm{unif}})}{(|\mathcal{T}_j^{\mathrm{unif}}|^\gamma)^{-s}} < \infty,$$

Theorem 2.5.1 (with $\mathcal{T}_\ell^{\mathrm{unif}} = \mathcal{T}_\ell^{\mathrm{opt}}$) shows that Algorithm 2.2.1 converges with at least the same rate of cumulative complexity. Particularly, the same effort in terms of computational time leads to asymptotically better approximation accuracy.

PROOF. The assumption (2.5.2) implies $\|\eta, \mathbb{T}\|_{s\gamma} < \infty$. To see this, we follow the proof of Proposition 2.3.14. Choose $N \in \mathbb{N}_0$ and the largest possible $\ell \in \mathbb{N}_0$ with $|\mathcal{T}_\ell^{\rm opt} \setminus \mathcal{T}_0| \leq N$. Due to the maximality of ℓ and by $|\mathcal{T}_{\ell+1}^{\rm opt}| \leq C_{\rm son}|\mathcal{T}_\ell^{\rm opt}|$, there holds $N+1 < |\mathcal{T}_{\ell+1}^{\rm opt} \setminus \mathcal{T}_0| + 1 \lesssim C_{\rm son}(|\mathcal{T}_\ell^{\rm opt} \setminus \mathcal{T}_0| + 1)$, where the hidden constant depends only on $|\mathcal{T}_0|$. This leads to

$$\min_{\widehat{\mathcal{T}} \in \mathbb{T}(N)} (N+1)^{s\gamma} \eta(\widehat{\mathcal{T}}) \lesssim (|\mathcal{T}_\ell^{\rm opt} \setminus \mathcal{T}_0| + 1)^{s\gamma} \eta(\mathcal{T}_\ell^{\rm opt})$$

and concludes

$$\|\eta, \mathbb{T}\|_{s\gamma} = \sup_{N \in \mathbb{N}_0} \min_{\widehat{\mathcal{T}} \in \mathbb{T}(N)} (N+1)^{s\gamma} \eta(\widehat{\mathcal{T}}) < \infty.$$

Lemma 2.7.5 below shows quasi-monotonicity (2.7.6) of $\eta(\cdot)$. With the above, Lemma 2.7.4 implies $C_{\rm approx}(s\gamma) < \infty$. This shows that (T3) holds. Therefore, Theorem 2.3.3 (i)–(iii) apply and prove

$$\eta(\mathcal{T}_j) \leq C_{\rm opt} C_{\rm approx}(s\gamma)(|\mathcal{T}_j \setminus \mathcal{T}_0| + 1)^{-s\gamma} \lesssim C_{\rm opt} C_{\rm approx}(s\gamma)|\mathcal{T}_j|^{-s\gamma}, \tag{2.5.4}$$

where the hidden constant depends only on $|\mathcal{T}_0|$ and $s\gamma$. Moreover, there holds R-linear convergence (2.3.2). We assume $\eta(\mathcal{T}_\ell) > 0$ for all $\ell \in \mathbb{N}_0$, since otherwise R-linear convergence (2.3.2) implies $\eta(\mathcal{T}_\ell) = 0$ for all $\ell \geq \ell_0$ for some $\ell_0 \in \mathbb{N}$ and hence (2.5.3) follows immediately. With (2.5.4), this implies

$$|\mathcal{T}_j|^\gamma \lesssim \eta(\mathcal{T}_j)^{-1/s} \quad \text{for all } j \in \mathbb{N}_0.$$

Together with R-linear convergence (2.3.2) and the equivalent inverse summability from Lemma 2.3.8 (ii), this shows

$$\sum_{j=0}^{\ell} |\mathcal{T}_j|^\gamma \lesssim \sum_{j=0}^{\ell} \eta(\mathcal{T}_j)^{-1/s} \lesssim \eta(\mathcal{T}_\ell)^{-1/s}.$$

We obtain immediately (2.5.3) and conclude the proof. $\square$

2.6. Necessity of the axioms

The convergence results in Theorem 2.3.3 show that the axioms (E1)–(E3), (T1)–(T3) are sufficient for rate optimality. By definition of the axioms (E1)–(E3), it is clear that if there exists a function $\varrho(\cdot,\cdot)$ such that (E1)–(E3) hold, we can choose the point wise minimal $\varrho_{\min}(\cdot,\cdot) \leq \varrho(\cdot,\cdot)$ to satisfy (E1), without violating (E2)–(E3). Given a triangulation $\mathcal{T} \in \mathbb{T}$, a refinement $\widehat{\mathcal{T}} \in \mathbb{T}(\mathcal{T})$, $\rho_{\rm red}$, and sets $\mathcal{T} \setminus \widehat{\mathcal{T}} \subseteq \mathcal{S}(\mathcal{T}, \widehat{\mathcal{T}}) \subseteq \mathcal{T}$, $\widehat{\mathcal{S}}(\mathcal{T}, \widehat{\mathcal{T}}) \subseteq \widehat{\mathcal{T}}$, this reads

$$\begin{aligned} \varrho_{\min}(\mathcal{T}, \widehat{\mathcal{T}}) := \max\Big\{ \Big| \Big(\sum_{T \in \widehat{\mathcal{T}} \setminus \widehat{\mathcal{S}}(\mathcal{T}, \widehat{\mathcal{T}})} \eta_T(\widehat{\mathcal{T}})^2 \Big)^{1/2} - \Big(\sum_{T \in \mathcal{T} \setminus \mathcal{S}(\mathcal{T}, \widehat{\mathcal{T}})} \eta_T(\mathcal{T})^2 \Big)^{1/2} \Big|, \\ \Big| \sum_{T \in \widehat{\mathcal{S}}(\mathcal{T}, \widehat{\mathcal{T}})} \eta_T(\widehat{\mathcal{T}})^2 - \rho_{\rm red} \sum_{T \in \mathcal{S}(\mathcal{T}, \widehat{\mathcal{T}})} \eta_T(\mathcal{T})^2 \Big|^{1/2} \Big\}. \end{aligned}$$

This section examines the necessity of the axioms with $\varrho(\cdot,\cdot) = \varrho_{\min}(\cdot,\cdot)$.

2.6.1. Convergence implies (E1)**.** The stability and reduction (E1) leads to the convergence result of Theorem 2.3.3 (i) and provides the basis for all the other convergence results. The following result shows that (E1) is even necessary.

Proposition 2.6.1. *Assume convergence* (2.3.1). *Then,* (E1) *holds for arbitrary* $0 \leq \rho_{\mathrm{red}} < 1$ *and arbitrary sets* $\widehat{\mathcal{S}}(\cdot,\cdot)$, $\mathcal{S}(\cdot,\cdot)$ *with* $\lim_{\ell\to\infty} \varrho_{\min}(\mathcal{T}_{\ell+1},\mathcal{T}_\ell) = 0$.

Proof. Stability and reduction (E1) is satisfied by definition of $\varrho_{\min}(\cdot,\cdot)$. By convergence (2.3.1), we obtain $\lim_{\ell\to\infty} \varrho(\mathcal{T}_\ell,\mathcal{T}_{\ell+1}) \lesssim \lim_{\ell\to\infty}(\eta(\mathcal{T}_\ell) + \eta(\mathcal{T}_{\ell+1})) = 0$. This concludes the proof. $\square$

2.6.2. R**-linear convergence implies** (E2)**.** Theorem 2.3.3 (ii) proves that (E1)–(E2) yield linear convergence (2.3.2). The following proposition shows that linear convergence (2.3.14) implies the general quasi-orthogonality (E2). In view of Proposition 2.6.1–2.6.2, linear convergence (2.3.14) is equivalent (E1)–(E2).

Proposition 2.6.2. *The R-linear convergence* (2.3.2) *implies general quasi-orthogonality* (E2) *with* $\varepsilon_{\mathrm{qo}} = 0$ *and* $C_{\mathrm{qo}} > 0$.

Proof. Since $\varrho_{\min}(\mathcal{T},\widehat{\mathcal{T}}) \lesssim \eta(\mathcal{T}) + \eta(\widehat{\mathcal{T}})$, R-linear convergence (2.3.2) together with Lemma 2.3.8 (where $\alpha_k = \eta(\mathcal{T}_k)$) show

$$\sum_{k=\ell}^{\ell+N} \varrho(\mathcal{T}_k,\mathcal{T}_{k+1})^2 \lesssim \sum_{k=\ell}^{\ell+N+1} \eta(\mathcal{T}_k)^2 \lesssim \eta(\mathcal{T}_\ell)^2$$

for all $\ell, N \in \mathbb{N}_0$. This is (E2) with $\varepsilon_{\mathrm{qo}} = 0$. $\square$

2.6.3. R**-linear convergence implies** (E3)**.** The discrete reliability (E3) proves the optimality of the Dörfler marking in Proposition 2.3.10. The following result shows that, under some minor assumptions, also the converse is true.

Proposition 2.6.3. *Assume R-linear convergence* (2.3.2) *and* $\mathcal{S}(\mathcal{T},\widehat{\mathcal{T}}) \leq C_{\mathrm{ref}}|\mathcal{T}\setminus\widehat{\mathcal{T}}|$. *Then, discrete reliability* (E3) *holds on the sequence of triangulations* $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}_0}$ *generated by Algorithm 2.2.1 with* $\varepsilon_{\mathrm{drel}} = 0$, $C_{\mathrm{drel}} = C_{\mathrm{conv}}\rho_{\mathrm{conv}}/\theta$, *and* $\mathcal{R}(\mathcal{T}_\ell,\mathcal{T}_{\ell+1}) = \mathcal{S}(\mathcal{T}_\ell,\mathcal{T}_{\ell+1})$, *i.e.,*

$$\varrho_{\min}(\mathcal{T}_\ell,\mathcal{T}_{\ell+1})^2 \leq C_{\mathrm{conv}}\rho_{\mathrm{conv}}\theta^{-1} \sum_{T\in\mathcal{S}(\mathcal{T}_\ell,\mathcal{T}_{\ell+1})} \eta_T(\mathcal{T}_\ell)^2$$

for all $\ell \in \mathbb{N}_0$.

Proof. The definition of $\varrho_{\min}(\cdot,\cdot)$ implies that either (E1a) holds with equality, i.e.,

$$\Big(\sum_{T\in\mathcal{T}_{\ell+1}\setminus\widehat{\mathcal{S}}(\mathcal{T}_\ell,\mathcal{T}_{\ell+1})} \eta_T(\mathcal{T}_{\ell+1})^2\Big)^{1/2} = \Big(\sum_{T\in\mathcal{T}_\ell\setminus\mathcal{S}(\mathcal{T}_\ell,\mathcal{T}_{\ell+1})} \eta_T(\mathcal{T}_\ell)^2\Big)^{1/2} + \varrho(\mathcal{T}_\ell,\mathcal{T}_{\ell+1}) \tag{2.6.1}$$

or (E1b) holds with equality, i.e.,

$$\sum_{T\in\widehat{\mathcal{S}}(\mathcal{T}_\ell,\mathcal{T}_{\ell+1})} \eta_T(\mathcal{T}_{\ell+1})^2 = \rho_{\mathrm{red}} \sum_{T\in\mathcal{S}(\mathcal{T}_\ell,\mathcal{T}_{\ell+1})} \eta_T(\mathcal{T}_\ell)^2 + \varrho(\mathcal{T}_\ell,\mathcal{T}_{\ell+1})^2. \tag{2.6.2}$$

In case of (2.6.1), we obtain

$$C_{\mathrm{conv}}^{1/2}\rho_{\mathrm{conv}}^{1/2}\eta(\mathcal{T}_\ell) \geq \eta(\mathcal{T}_{\ell+1}) \geq \Big(\sum_{T\in\mathcal{T}_{\ell+1}\setminus\widehat{\mathcal{S}}(\mathcal{T}_\ell,\mathcal{T}_{\ell+1})} \eta_T(\mathcal{T}_{\ell+1})^2\Big)^{1/2} \geq \varrho(\mathcal{T}_\ell,\mathcal{T}_{\ell+1}).$$

Analogously, (2.6.2) implies

$$C_{\rm conv}^{1/2}\rho_{\rm conv}^{1/2}\eta(\mathcal{T}_\ell) \geq \eta(\mathcal{T}_{\ell+1}) \geq \Big(\sum_{T\in\widehat{\mathcal{S}}(\mathcal{T}_\ell,\mathcal{T}_{\ell+1})} \eta_T(\mathcal{T}_{\ell+1})^2\Big)^{1/2} \geq \varrho(\mathcal{T}_\ell,\mathcal{T}_{\ell+1}).$$

Since the triangulations $\mathcal{T}_\ell$ satisfy the Dörfler marking (2.2.1), the above implies

$$\varrho(\mathcal{T}_\ell,\mathcal{T}_{\ell+1})^2 \leq C_{\rm conv}\rho_{\rm conv}\eta(\mathcal{T}_\ell)^2 \leq C_{\rm conv}\rho_{\rm conv}\theta^{-1}\sum_{T\in\mathcal{M}_\ell}\eta_T(\mathcal{T})^2. \tag{2.6.3}$$

Since, by definition of the refinement strategy, there holds $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell\setminus\mathcal{T}_{\ell+1} \subseteq \mathcal{S}(\mathcal{T}_\ell,\mathcal{T}_{\ell+1})$, we obtain (E3) with $\varepsilon_{\rm drel}=0$, $C_{\rm drel}=C_{\rm conv}\rho_{\rm conv}/\theta$, and $\mathcal{R}(\mathcal{T}_\ell,\mathcal{T}_{\ell+1})=\mathcal{S}(\mathcal{T}_\ell,\mathcal{T}_{\ell+1})$. □

The following result shows that Proposition 2.3.10 is sharp in the sense that (E3) is even equivalent to (2.3.18).

Proposition 2.6.4. *Assume stability and reduction* (E1) *with* $\varrho(\cdot,\cdot):=\varrho_{\min}(\cdot,\cdot)$. *Assume that for* $\kappa_0=\kappa_{\rm dlr}$ *exists some* θ_0 *such that the implication* (2.3.18) *holds. Then, discrete reliability* (E3) *is satisfied with* $\varepsilon_{\rm drel}=0$ *and* $\mathcal{R}(\mathcal{T},\widehat{\mathcal{T}})$ *from Proposition 2.3.10 and* $C_{\rm drel}=\theta_0^{-1/2}$.

Proof. Let $\widehat{\mathcal{T}}\in\mathbb{T}(\mathcal{T})$ such that $\eta(\widehat{\mathcal{T}})\leq\kappa_{\rm dlr}\eta(\mathcal{T})$. By assumption, there exists $0<\theta_0<1$, which depends on $\kappa_{\rm dlr}$, such that the implication (2.3.18) holds and shows that $\mathcal{R}(\mathcal{T},\widehat{\mathcal{T}})$ satisfies the Dörfler marking (2.2.1). As in (2.6.3), we obtain

$$\varrho(\mathcal{T},\widehat{\mathcal{T}})^2 \leq \eta(\widehat{\mathcal{T}})^2 < \eta(\mathcal{T})^2 \leq \theta_0^{-1}\sum_{T\in\mathcal{R}(\mathcal{T},\widehat{\mathcal{T}})}\eta_T(\mathcal{T})^2.$$

This concludes the proof. □

2.6.4. Optimal complexity implies R-linear convergence. The optimal complexity result of Theorem 2.5.1 implies R-linear convergence (2.3.2) in the following sense. Assume that the error estimator converges with a certain rate

$$|\mathcal{T}_\ell|^{-s} \lesssim \eta(\mathcal{T}_\ell) \lesssim |\mathcal{T}_\ell|^{-s} \quad \text{for all } \ell\in\mathbb{N}_0 \tag{2.6.4}$$

and assume that the implication of Theorem 2.5.1, i.e., (2.5.2) implies (2.5.3), is true. Under (T1), we may use $\mathcal{T}_\ell^{\rm opt}:=\mathcal{T}_\ell$ and obtain

$$\sup_{\ell\in\mathbb{N}_0}\frac{\eta(\mathcal{T}_\ell)}{(\sum_{j=0}^{\ell}|\mathcal{T}_j|^{\gamma})^{-s/\gamma}}<\infty.$$

With this, (2.5.3) shows

$$\eta(\mathcal{T}_\ell)^{-\gamma/s} \gtrsim \sum_{j=0}^{\ell}|\mathcal{T}_j|^{\gamma} \gtrsim \sum_{j=0}^{\ell}\eta(\mathcal{T}_j)^{-\gamma/s}$$

for all $\ell\in\mathbb{N}_0$. Lemma 2.3.8 with $\alpha_\ell=\eta(\mathcal{T}_\ell)$ concludes R-linear convergence (2.3.2).

Remark 2.6.5. *Although it is possible to construct examples which satisfy rate optimality* (2.3.3) *but fail to satisfy* (2.6.4), *there are many practical examples with* (2.6.4). *In this sense, R-linear convergence might not be necessary for any particular instance of the approximation problem, but is definitely necessary for the general case.*

2.6.5. The refinement axioms (T1)&(T3)**.** The assumption (T1) is not necessary from a theoretical point of view. However, since $|\mathcal{T}|$ is usually related to the degrees of freedom, a reasonable refinement strategy will aim to produce a refinement with $|\mathcal{T}_{\ell+1}| \simeq |\mathcal{T}_\ell|$. The uniform approximability (T3) is necessary since it follows immediately from (2.3.3).

2.7. Particular realizations of the axioms

In many cases, some of the axioms (E1)–(E3), (T1)–(T3) hold due to some more specific properties of the estimator $\eta(\cdot)$ or the refinement strategy $\mathbb{T}(\cdot,\cdot)$.

2.7.1. A priori convergence. Suppose a Banach space $\mathcal{X}$ with norm $\|\cdot\|_{\mathcal{X}}^2$ as well as a *solver* function $U(\cdot):\mathbb{T}\to\mathcal{X}$. Assume that

$$\varrho(\mathcal{T},\widehat{\mathcal{T}})^2 := \alpha\|U(\mathcal{T})-U(\widehat{\mathcal{T}})\|_{\mathcal{X}}^2$$

for some $\alpha>0$.

Lemma 2.7.1. *Suppose that there exist subspaces $\mathcal{X}(\mathcal{T}_\ell)\subseteq\mathcal{X}$ for all $\ell\in\mathbb{N}_0$ (where $\mathcal{T}_\ell$ denotes the output of Algorithm 2.2.1) and a function $U_\infty\in\mathcal{X}_\infty := \overline{\bigcup_{\ell\in\mathbb{N}_0}\mathcal{X}(\mathcal{T}_\ell)}$ such that the Céa lemma holds, i.e.,*

$$\|U_\infty-U(\mathcal{T}_\ell)\|_{\mathcal{X}} \le C_{\text{Céa}}\min_{V\in\mathcal{X}(\mathcal{T}_\ell)}\|U_\infty-V\|_{\mathcal{X}}\quad\textit{for all } \ell\in\mathbb{N}_0, \tag{2.7.1}$$

where $C_{\text{Céa}}>0$ is some constant which does not depend on $\ell\in\mathbb{N}_0$. Then, there holds a priori convergence

$$\lim_{\ell\to\infty}\|U_\infty-U(\mathcal{T}_\ell)\|_{\mathcal{X}}=0=\lim_{\ell\to\infty}\varrho(\mathcal{T}_\ell,\mathcal{T}_{\ell+1}). \tag{2.7.2}$$

Proof. By definition of $\mathcal{X}_\infty$, the right-hand side of (2.7.1) converges towards zero as $\ell\to\infty$. The convergence $\lim_{\ell\to\infty}\varrho(\mathcal{T}_\ell,\mathcal{T}_{\ell+1})=0$ follows immediately with the triangle inequality. This concludes the proof. □

2.7.2. $\varrho(\cdot,\cdot)$ is a Hilbert norm. If the perturbation has the structure of a Hilbert norm, the general quasi-orthogonality follows immediately.

Lemma 2.7.2. *Suppose a Hilbert space $\mathcal{X}$ with $\|\cdot\|_{\mathcal{X}}^2 := \langle\cdot\,,\cdot\rangle_{\mathcal{X}}$ and $U(\cdot):\mathbb{T}\to\mathcal{X}$. Let $\varrho(\cdot,\cdot)$ be given as in Section 2.7.1 and suppose that the solver $U(\cdot)$ satisfies Galerkin orthogonality*

$$\langle U(\mathcal{T}_{\ell+k})-U(\mathcal{T}_{\ell+1})\,,\,U(\mathcal{T}_{\ell+1})-U(\mathcal{T}_\ell)\rangle_{\mathcal{X}}=0\quad\textit{for all } k,\ell\in\mathbb{N}_0. \tag{2.7.3}$$

Then, discrete reliability (E3) *with $\kappa_{\rm dlr}=\infty$ (where the restriction $\varepsilon_{\rm drel}<1$ is not necessary) implies the general quasi-orthogonality* (E2) *with $\varepsilon_{\rm qo}=0$ and $C_{\rm qo}=\varepsilon_{\rm drel}+C_{\rm drel}$. Moreover, there holds a priori convergence*

$$\lim_{\ell\to\infty}\varrho(\mathcal{T}_\ell,\mathcal{T}_{\ell+1})=0. \tag{2.7.4}$$

Proof. The Galerkin orthogonality (2.7.3) implies for $k,N\in\mathbb{N}_0$

$$\begin{aligned}\|U(\mathcal{T}_k)-U(\mathcal{T}_{k+1})\|_{\mathcal{X}}^2 &= \|U(\mathcal{T}_{\ell+N})-U(\mathcal{T}_k)\|_{\mathcal{X}}^2-\|U(\mathcal{T}_{\ell+N})-U(\mathcal{T}_{k+1})\|_{\mathcal{X}}^2\\ &\quad-2\langle U(\mathcal{T}_{\ell+N})-U(\mathcal{T}_{k+1})\,,\,U(\mathcal{T}_k)-U(\mathcal{T}_{k+1})\rangle_{\mathcal{X}}\\ &= \|U(\mathcal{T}_{\ell+N})-U(\mathcal{T}_k)\|_{\mathcal{X}}^2-\|U(\mathcal{T}_{\ell+N})-U(\mathcal{T}_{k+1})\|_{\mathcal{X}}^2.\end{aligned}$$

Hence, there holds for $\ell \in \mathbb{N}_0$

$$
\begin{aligned}
\sum_{k=\ell}^{\ell+N} \varrho(\mathcal{T}_k, \mathcal{T}_{k+1})^2 &\le \alpha \lim_{N\to\infty} \sum_{k=\ell}^{\ell+N} \|U(\mathcal{T}_k) - U(\mathcal{T}_{k+1})\|_{\mathcal{X}}^2 \\
&= \alpha \lim_{N\to\infty} \sum_{k=\ell}^{\ell+N} \big(\|U(\mathcal{T}_{\ell+N}) - U(\mathcal{T}_k)\|_{\mathcal{X}}^2 - \|U(\mathcal{T}_{\ell+N}) - U(\mathcal{T}_{k+1})\|_{\mathcal{X}}^2\big) \\
&= \alpha \lim_{N\to\infty} \big(\|U(\mathcal{T}_{\ell+N}) - U(\mathcal{T}_\ell)\|_{\mathcal{X}}^2 - \|U(\mathcal{T}_{\ell+N}) - U(\mathcal{T}_{\ell+N+1})\|_{\mathcal{X}}^2\big) \\
&\le \alpha \lim_{N\to\infty} \|U(\mathcal{T}_{\ell+N}) - U(\mathcal{T}_\ell)\|_{\mathcal{X}}^2 \\
&= \lim_{N\to\infty} \varrho(\mathcal{T}_\ell, \mathcal{T}_{\ell+N})^2 \le (\varepsilon_{\rm drel} + C_{\rm drel})\eta(\mathcal{T}_\ell)^2.
\end{aligned}
$$

The above for $\ell = 0$ concludes also (2.7.4) and hence the proof. □

2.7.3. Quasi-orthogonality implies general quasi-orthogonality. In the literature, one often finds the following quasi-orthogonality: Let $0 \le \varepsilon < 1$, and $C_{\rm rel} > 0$ such that all $\ell \in \mathbb{N}_0$ satisfy

$$
C_{\rm rel}^{-1} \varrho(\mathcal{T}_\ell, \mathcal{T}_{\ell+1})^2 \le (1-\varepsilon)^{-1}\alpha_\ell^2 - \alpha_{\ell+1}^2, \tag{2.7.5a}
$$

for some $\alpha_\ell \in \mathbb{R}$ with

$$
\alpha_\ell^2 \le C_{\rm rel}\eta(\mathcal{T}_\ell)^2. \tag{2.7.5b}
$$

Lemma 2.7.3. *The quasi-orthogonality* (2.7.5) *with* $0 \le \varepsilon < 1$ *and* $C_{\rm rel} > 0$ *implies the general quasi-orthogonality* (E2) *with* $\varepsilon_{\rm qo} = C_{\rm rel}\varepsilon/(1-\varepsilon)$ *and* $C_{\rm qo} = C_{\rm rel}$.

Proof. There holds with $\varepsilon_{\rm qo} = C_{\rm rel}\varepsilon/(1-\varepsilon)$ and (2.7.5)

$$
\begin{aligned}
\sum_{k=\ell}^{N} \big(\varrho(\mathcal{T}_k, \mathcal{T}_{k+1})^2 - \varepsilon_{\rm qo}\eta(\mathcal{T}_k)^2\big) &\le \sum_{k=\ell}^{N} \Big(\frac{\alpha_k^2}{1-\varepsilon} - \alpha_{k+1}^2 - \frac{C_{\rm rel}\varepsilon\eta(\mathcal{T}_k)^2}{1-\varepsilon}\Big) \\
&\le \sum_{k=\ell}^{N} \Big(\frac{\alpha_k^2}{1-\varepsilon} - \alpha_{k+1}^2 - \frac{\varepsilon\alpha_k^2}{1-\varepsilon}\Big) \\
&\le \sum_{k=\ell}^{N} (\alpha_k^2 - \alpha_{k+1}^2) \le \alpha_\ell^2 \le C_{\rm rel}\eta(\mathcal{T}_\ell)^2.
\end{aligned}
$$

□

2.7.4. Quasi-monotonicity and the overlay estimate. We say that a function $\lambda(\cdot) : \mathbb{T} \to [0,\infty)$ is quasi-monotone, if there exists a constant $C_{\rm mon} > 0$ such that all triangulations $\mathcal{T} \in \mathbb{T}$ satisfy

$$
\lambda(\widehat{\mathcal{T}}) \le C_{\rm mon}\lambda(\mathcal{T}) \quad \text{for all } \widehat{\mathcal{T}} \in \mathbb{T}(\mathcal{T}). \tag{2.7.6}
$$

Lemma 2.7.4. *Assume that the refinement strategy* $\mathbb{T}(\cdot,\cdot)$ *satisfies the overlay estimate* (2.5.1) *and that the function* $\lambda(\cdot) : \mathbb{T} \to [0,\infty)$ *is quasi-monotone* (2.7.6). *Then,* $\|\lambda, \mathbb{T}\|_s < \infty$ *for some* $s > 0$ *implies*

$$
\sup_{\mathcal{T}\in\mathbb{T}} \|\lambda, \mathbb{T}(\mathcal{T})\|_s \le C_{\rm mon}(C_4+1)^s\|\lambda, \mathbb{T}\|_s.
$$

Particularly, for $\lambda(\cdot) = \eta(\cdot)$, $\|\eta, \mathbb{T}\|_s < \infty$ *implies* (T3).

Proof. Let $N \in \mathbb{N}_0$ and define $M := \text{floor}(N/C_4)$. The fact $\|\lambda, \mathbb{T}\|_s < \infty$ allows to choose some triangulation $\mathcal{T}^N \in \mathbb{T}(M)$ with

$$\lambda(\mathcal{T}^N)(M+1)^s \leq \|\lambda, \mathbb{T}\|_s.$$

Given any $\mathcal{T} \in \mathbb{T}$, the overlay estimate (2.5.1) states $|(\mathcal{T}^N \oplus \mathcal{T}) \setminus \mathcal{T}| \leq N$ and hence $\mathcal{T}^N \oplus \mathcal{T} \in \mathbb{T}(\mathcal{T}, N)$. The quasi-monotonicity (2.7.6) and $N+1 \leq (M+1)(C_4+1)$ shows

$$\lambda(\mathcal{T}^N \oplus \mathcal{T})(N+1)^s \leq C_{\text{mon}}(C_4+1)^s \lambda(\mathcal{T}^N)(M+1)^s \leq C_{\text{mon}}(C_4+1)^s \|\lambda, \mathbb{T}\|_s.$$

This implies

$$\inf_{\widehat{\mathcal{T}} \in \mathbb{T}(\mathcal{T}, N)} (N+1)^s \lambda(\widehat{\mathcal{T}}) \leq C_{\text{mon}}(C_4+1)^s \|\lambda, \mathbb{T}\|_s$$

and concludes the proof. □

The quasi-monotonicity (2.7.6) follows from the stability and reduction (E1) together with discrete reliability (E3) or quasi-orthogonality (2.7.5).

Lemma 2.7.5. *Assume* (E1) *(where the restriction $\rho_{\text{red}} < 1$ is not necessary) as well as* (E3) *with $\kappa_{\text{dlr}} = \infty$. Then, there holds* (2.7.6) *with $\lambda(\cdot) = \eta(\cdot)$ and $C_{\text{mon}} = \big(\max\{\rho_{\text{red}}, 2\} + 3(\varepsilon_{\text{drel}} + C_{\text{drel}}^2)\big)^{1/2}$.*

Proof. The stability (E1a) and the reduction estimate (E1b) imply

$$\eta(\widehat{\mathcal{T}})^2 \leq \rho_{\text{red}} \sum_{T \in \mathcal{S}(\mathcal{T}, \widehat{\mathcal{T}})} \eta_T(\mathcal{T})^2 + 2 \sum_{T \in \mathcal{T} \setminus \mathcal{S}(\mathcal{T}, \widehat{\mathcal{T}})} \eta_T(\mathcal{T})^2 + 3\varrho(\mathcal{T}, \widehat{\mathcal{T}})^2.$$

The discrete reliability (E3), leads to

$$\begin{aligned}\eta(\widehat{\mathcal{T}})^2 &\leq (\max\{\rho_{\text{red}}, 2\} + 3\varepsilon_{\text{drel}})\eta(\mathcal{T})^2 + 3C_{\text{drel}}^2 \sum_{T \in \mathcal{R}(\mathcal{T}, \widehat{\mathcal{T}})} \eta_T(\mathcal{T})^2 \\ &\leq \big(\max\{\rho_{\text{red}}, 2\} + 3(\varepsilon_{\text{drel}} + C_{\text{drel}}^2)\big)\eta(\mathcal{T})^2.\end{aligned}$$

This is (2.7.6) with $C_{\text{mon}} := \big(\max\{\rho_{\text{red}}, 2\} + 3(\varepsilon_{\text{drel}} + C_{\text{drel}}^2)\big)^{1/2}$. □

Lemma 2.7.6. *Assume* (E1) *(where the restriction $\rho_{\text{red}} < 1$ is not necessary) as well as the quasi-orthogonality* (2.7.5) *for $\mathcal{T}_\ell = \mathcal{T}$ and $\mathcal{T}_{\ell+1} = \widehat{\mathcal{T}}$. Then, there holds* (2.7.6) *with $\lambda(\cdot) = \eta(\cdot)$ and $C_{\text{mon}} = \big((\max\{\rho_{\text{red}}, 2\} + 3C_{\text{rel}}^2(1-\varepsilon)^{-1})\big)^{1/2}$.*

Proof. The stability (E1a) and the reduction estimate (E1b) imply

$$\begin{aligned}\eta(\widehat{\mathcal{T}})^2 &\leq \rho_{\text{red}} \sum_{T \in \mathcal{S}(\mathcal{T}, \widehat{\mathcal{T}})} \eta_T(\mathcal{T})^2 + 2 \sum_{T \in \mathcal{T} \setminus \mathcal{S}(\mathcal{T}, \widehat{\mathcal{T}})} \eta_T(\mathcal{T})^2 + 3\varrho(\mathcal{T}, \widehat{\mathcal{T}})^2 \\ &\overset{(2.7.5)}{\leq} \max\{\rho_{\text{red}}, 2\}\eta(\mathcal{T})^2 + 3C_{\text{rel}}((1-\varepsilon)^{-1}\alpha_\ell^2 - \alpha_{\ell+1}^2) \\ &\leq (\max\{\rho_{\text{red}}, 2\} + 3C_{\text{rel}}^2(1-\varepsilon)^{-1})\eta(\mathcal{T})^2.\end{aligned}$$

This concludes the proof. □

2.7.5. Other versions of the overlay estimate (2.5.1) **and of** (T2)**.** The following estimate provides a lower bound for the number of newly generated elements, i.e.,

$$|\mathcal{T}\setminus\widehat{\mathcal{T}}| \le |\widehat{\mathcal{T}}| - |\mathcal{T}| \quad \text{for all } \widehat{\mathcal{T}} \in \mathbb{T}(\mathcal{T}) \text{ and all } \mathcal{T} \in \mathbb{T}. \tag{2.7.7}$$

This is particularly satisfied if each refined element $T \in \mathcal{T}\setminus\widehat{\mathcal{T}}$ generates at least two sons $T_1, T_2 \in \widehat{\mathcal{T}}\setminus\mathcal{T}$.

Lemma 2.7.7. *Let the refinement strategy satisfy* (2.7.7)*, then there holds for all refinements* $\widehat{\mathcal{T}} \in \mathbb{T}(\mathcal{T})$

$$|\widehat{\mathcal{T}}| - |\mathcal{T}| \le |\widehat{\mathcal{T}}\setminus\mathcal{T}| \le 2(|\widehat{\mathcal{T}}| - |\mathcal{T}|). \tag{2.7.8}$$

Proof. The first inequality follows from

$$|\widehat{\mathcal{T}}\setminus\mathcal{T}| = |\widehat{\mathcal{T}}| - |\widehat{\mathcal{T}}\cap\mathcal{T}| \ge |\widehat{\mathcal{T}}| - |\mathcal{T}|.$$

The second inequality follows similarly by

$$|\widehat{\mathcal{T}}\setminus\mathcal{T}| = |\widehat{\mathcal{T}}| - |\widehat{\mathcal{T}}\cap\mathcal{T}| = |\widehat{\mathcal{T}}| - (|\mathcal{T}| - |\mathcal{T}\setminus\widehat{\mathcal{T}}|) \le 2(|\widehat{\mathcal{T}}| - |\mathcal{T}|),$$

where we used (2.7.7). □

Lemma 2.7.8. *Under* (2.7.7)*, the closure estimate* (T2) *is equivalent to*

$$|\mathcal{T}_\ell| - |\mathcal{T}_0| \le \widetilde{C_{\rm closure}} \sum_{j=0}^{\ell-1} |\mathcal{M}_j| \quad \textit{for all } \ell \in \mathbb{N}_0, \tag{2.7.9}$$

where the closure estimate (T2) *implies* (2.7.9) *with* $\widetilde{C_{\rm closure}} = C_{\rm closure}$ *and* (2.7.9) *implies* (T2) *with* $C_{\rm closure} = 2\widetilde{C_{\rm closure}}$*. Moreover, the overlay estimate* (2.5.1) *is equivalent to*

$$|(\mathcal{T}\oplus\widehat{\mathcal{T}})| \le \widetilde{C_4}(|\widehat{\mathcal{T}}| - |\mathcal{T}_0|) + |\mathcal{T}| \quad \textit{for all } \widehat{\mathcal{T}} \in \mathbb{T}(\mathcal{T}), \tag{2.7.10}$$

where (2.5.1) *implies* (2.7.10) *with* $\widetilde{C_4} = 2C_4$ *and* (2.7.10) *implies* (2.5.1) *with* $C_4 = 2\widetilde{C_4}$*.*

Proof. Both statements follow directly with (2.7.8). □

2.8. Historical remarks

This section is based on and extends [**24**, Section 3.2]. This work provides some unifying framework on the theory of adaptive algorithms and the related convergence and quasi-optimality analysis. Some historic remarks are in order on the development of the arguments over the years. In one way or another, the axioms arose in various works throughout the literature. We aim to motivate the specific choice of axioms (which turn out to be even necessary in Section 2.6) in terms of historic development of the field.

2.8.1. Reliability (2.4.1)**.** Reliability basically states that the unknown error tends to zero if the computable and hence known error bound is driven to zero by smart adaptive algorithms. As the main result of this chapter (Theorem 2.3.3) focuses solely on the error estimator, the reliability is not explicitly used in the analysis. However, Section 2.4 introduces reliability to prove optimal convergence of the error. Since the invention of adaptive FEM in the 1970s, the question of reliability was thus a pressing matter and first results for FEM date back to the early works of Babuska & Rheinboldt [**7**] in 1D and Babuska & Miller [**6**] in 2D. Therein, the error is estimated by means of the residual. In the context of BEM, reliable residual-based error estimators date back to the works of Carstensen &

Stephan [**34, 33, 20**]. Since the actual adaptive algorithm only knows the estimator, reliability estimates have been a crucial ingredient for convergence proofs of adaptive schemes of any kind.

2.8.2. Efficiency (2.4.2)**.** Compared to reliability (2.4.1), efficiency (2.4.2) provides the converse estimate and states that the error is not overestimated by the estimator, up to some oscillation terms data(·) determined from the given data. An error estimator which satisfies both, reliability and efficiency, is mathematically guaranteed to asymptotically behave like the error, i.e., it decays with the same rate as the actual computational error. Consequently, efficiency is a desirable property as soon as it comes to convergence rates. For FEM with residual error estimators, efficiency has first been proved by Verfürth [**82**]. He used appropriate inverse estimates and localization by means of bubble functions. In the frame of BEM, however, efficiency (2.4.2) of the residual error estimators is widely open and only known for particular problems [**3, 19**], although observed empirically, see also Section 3.5.3.

In terms of convergence proofs, efficiency is often a useful tool as is mentioned in the following section. However, the main result of this chapter (Theorem 2.3.3) does not require the efficiency estimate (2.4.2) and thus allows applications to a much wider problem class.

2.8.3. Discrete local efficiency and first convergence analysis of [40, 65]. Reliability (2.4.1) and efficiency (2.4.2) are nowadays standard topics in textbooks on a posteriori FEM error estimation [**1, 82**], in contrast to the convergence of adaptive algorithms. Babuska & Vogelius [**8**] already observed for conforming discretizations, that the sequence of discrete approximations $U(\mathcal{T}_\ell)$ always converges (see Section 2.7.1 for an abstract form of this a priori convergence). The work of Dörfler [**40**] introduced the marking strategy (2.2.1) for the Poisson model problem

$$-\Delta u = f \text{ in } \Omega \quad \text{and} \quad u = 0 \text{ on } \Gamma = \partial\Omega \tag{2.8.1}$$

and conforming first-order FEM to show convergence up to any given tolerance. Morin, Nochetto & Siebert [**65**] refined this and the arguments of Verfürth [**82**] and Dörfler [**40**] and proved the discrete variant

$$C_{\rm eff}^{-2}\eta(\mathcal{T}_\ell)^2 \le \|\nabla(U(\mathcal{T}_{\ell+1}) - U(\mathcal{T}_\ell))\|_{L^2(\Omega)}^2 + \text{data}(\mathcal{T}_\ell)^2 \tag{2.8.2}$$

of the efficiency (2.4.2). See also [**50**] for the explicit statement and proof. The proof relies on discrete bubble functions and thus required an *interior node property* of the refinement strategy, which is ensured, e.g., by bisection for $d = 2$ from Section 3.2.8 and five bisections for each refined element. With this, [**65**] proved *error reduction* up to data oscillation terms in the sense of

$$\|\nabla(u - U(\mathcal{T}_{\ell+1}))\|_{L^2(\Omega)}^2 \le \kappa\, \|\nabla(u - U(\mathcal{T}_\ell))\|_{L^2(\Omega)}^2 + C\,\text{data}(\mathcal{T}_\ell) \tag{2.8.3}$$

with some ℓ-independent constants $0 < \kappa < 1$ and $C > 0$. This and additional enrichment of the marked elements $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ to ensure $\text{data}(\mathcal{T}_\ell) \to 0$ as $\ell \to \infty$ leads to convergence

$$\|\nabla(u - U(\mathcal{T}_\ell))\|_{L^2(\Omega)} \xrightarrow{\ell\to\infty} 0. \tag{2.8.4}$$

The reason why this work neglects the discrete local efficiency (2.8.2) is that it can only be proven for a very narrow class of model problems, and thus does not allow for some general framework. Moreover, the over refinement due to the *interior node property* is practically observed to be unnecessary.

2.8.4. Quasi-orthogonality (E2). The approach of [**65**] has been generalized to non-symmetric operators in [**64**], to nonconforming and mixed methods in [**26, 25**], as well as to the nonlinear obstacle problem in BRAESS, CARSTENSEN & HOPPE [**17, 18**]. One additional difficulty is the lack of the Galerkin orthogonality which is circumvented with the general quasi-orthogonality axiom (2.7.5) in Section 2.7.3. Stronger variants of quasi-orthogonality have been used in [**26, 25, 64**] and imply (2.7.5) in Section 2.7.3. In its current form, however, the general quasi-orthogonality (E2) goes back to [**46**] for non-symmetric operators without artificial assumptions on the initial triangulation as in [**36, 64**], see also Section 3.6.1. Proposition 2.6.2 shows that the present form (E2) of the quasi-orthogonality cannot be weakened if one aims to follow the analysis of [**35, 78**] to prove quasi-optimal convergence rates. Moreover, Section 2.6.4 shows that the optimal complexity result of Theorem 2.5.1 necessarily implies R-linear convergence and thus general quasi-orthogonality (E2) by Proposition 2.6.2.

2.8.5. Optimal convergence rates and discrete reliability (E3). The seminal work of BINEV, DAHMEN & DEVORE [**14**] was the first one to prove algebraic convergence rates for adaptive FEM of the Poisson model problem (2.8.1) and lowest-order FEM. They extended the adaptive algorithm of [**65**] by additional coarsening steps to avoid over-refinement. STEVENSON [**78**] removed this artificial coarsening step and introduced the basic form of the axiom (E3) on discrete reliability, i.e., with $\varepsilon_{\rm drel} = 0$ and $\kappa_{\rm dlr} = \infty$. He implicitly introduced the concept of *separate Dörfler marking*: If the data oscillations $\mathrm{data}(\mathcal{T}_\ell)$ are small compared to the error estimator $\eta(\mathcal{T}_\ell)$, he used the common Dörfler marking (2.2.1) to single out the elements for refinement. Otherwise, he suggested the Dörfler marking (2.2.1) for the local contributions of the data oscillation term $\mathrm{data}(\mathcal{T}_\ell)$. The core proof of [**78**] then uses the observation from [**64**] that the so-called *total error* is contracted in each step of the adaptive loop in the sense of

$$\Delta_{\ell+1} \le \kappa\,\Delta_\ell \quad \text{for} \quad \Delta_\ell := \|\nabla(u - U(\mathcal{T}_\ell))\|_{L^2(\Omega)}^2 + \gamma\,\mathrm{data}(\mathcal{T}_\ell)^2 \tag{2.8.5}$$

with some ℓ-independent constants $0 < \kappa < 1$ and $\gamma > 0$.

Moreover, the analysis of [**78**] shows that the Dörfler marking (2.2.1) is not only sufficient to guarantee contraction (2.8.5), but somehow even necessary, see Section 2.3.5 for the refined analysis which avoids the use of efficiency (2.4.2).

2.8.6. Stability and reduction (E1). Anticipating the convergence of [**39**] for the p-Laplacian, the AFEM analysis of [**78**] was simplified by CASCON, KREUZER, NOCHETTO & SIEBERT [**35**] with the introduction of the *estimator reduction* in the sense of

$$\eta(\mathcal{T}_{\ell+1})^2 \le \kappa\,\eta(\mathcal{T}_\ell)^2 + C\,\|\nabla(U(\mathcal{T}_{\ell+1}) - U(\mathcal{T}_\ell))\|_{L^2(\Omega)}^2 \tag{2.8.6}$$

with constants $0 < \kappa < 1$ and $C > 0$. This is an immediate consequence of stability and reduction (E1b) in Section 2.3.3 and also ensures contraction of the so-called quasi-error

$$\Delta_{\ell+1} \le \kappa\,\Delta_\ell \quad \text{for} \quad \Delta_\ell := \|\nabla(u - U(\mathcal{T}_\ell))\|_{L^2(\Omega)}^2 + \gamma\,\eta(\mathcal{T}_\ell)^2 \tag{2.8.7}$$

with some ℓ-independent constants $0 < \kappa < 1$ and $\gamma > 0$. The analysis of [**35**] removed the *discrete local lower bound* from the set of necessary axioms (and hence the *interior node property* [**65**]). Implicitly, the axiom (E1) is part of the proof of (2.8.6) in [**35**]. While (E1a) essentially follows from the triangle inequality and appropriate inverse estimates in practice, the reduction (E1b) builds on the observation that the element sizes of the sons of a refined element uniformly decreases. For instance, bisection-based refinement strategies yield $|T'| \le |T|/2$, if $T' \in \mathcal{T}_{\ell+1} \backslash \mathcal{T}_\ell$ is a son of $T \in \mathcal{T}_\ell \backslash \mathcal{T}_{\ell+1}$.

2.8.7. Extensions of the analysis of [35]. The work [**60**] considers lowest-order AFEM for the Poisson problem (2.8.1) for error estimators which are locally equivalent to the residual error estimator. The works [**36, 46**] analyze optimality of AFEM for linear, but non-symmetric elliptic operators. While [**36**] required that the corresponding bilinear form induces a norm, such an assumption is dropped in [**46**], so that the latter work concluded the AFEM analysis for linear second-order elliptic PDEs. Convergence with optimal rates for adaptive boundary element methods has independently been proved in [**47, 80**]. The main additional difficulty was the development of appropriate *local* inverse estimates for the *nonlocal* operators involved. The BEM analysis, however, still hinges on symmetric and elliptic integral operators and excludes boundary integral formulations of mixed boundary value problems as well as the FEM-BEM coupling. AFEM with nonconforming and mixed FEMs is considered for various problems in [**71, 32, 29, 31, 12, 61**]. AFEM with non-homogeneous Dirichlet and mixed Dirichlet-Neumann boundary conditions are analyzed in [**48**] for 2D and in [**4**] for 3D. The latter work adapts the separate Dörfler marking from [**78**] to decide whether the refinement relies on the error estimator for the discretization error or the approximation error of the given continuous Dirichlet data, see Section 5.4. The results of those works are reproduced and partially even improved in the frame of the abstract axioms of Section 2.3.1. Finally, the proofs of [**4, 46**] simplified the core analysis of [**78, 35**] in the sense that the optimality analysis avoids the use of the *total error* and solely works with the error estimator. The work [**24**] on which this work is based, derives a first set of axioms to unify the theory of the mentioned works. In this work, we take one step further and also drop the notion of exact solution and approximate solution, to solely focus on the error estimator. Moreover, we relax some standard assumptions on the refinement strategy to include a more general class of triangulations into the optimality analysis.

CHAPTER 3

Applications I

3.1. Introduction, state of the art & outline

This chapter applies the abstract machinery of the previous chapter to concrete model problems. This means that for each problem, the axioms of Section 2.3.1 are checked and the abstract results are interpreted. We reproduce well-known optimality results (e.g., for the Poisson problem of Section 3.5.1 which was firstly proved in [**78**] and then generalized by [**35**]), improve recent results for general elliptic second-order operators from Section 3.6 (which was firstly proved in [**46**] but is generalized in this work for operators which satisfy a Gårding inequality), and even derive completely new results as for example the optimality result for reg-green-blue refinement from Section 3.5.2. Some of the examples are already found in similar manner in [**24**]. The remainder of this chapter is organized as follows: Section 3.2 introduces usual properties of concrete refinement strategies and gives some examples. Section 3.3 proves the uniform approximability (T3) for a certain class of problems. Section 3.4 introduces the notion of weighted error estimators, for which some of the axioms follow from simpler assumptions. Section 3.5 validates the axioms for examples from finite element and boundary element methods. Section 3.6 extends the problem class to general second-order elliptic equations and Section 3.7 introduces nonlinear model problems for which optimal convergence rates can be proven.

3.2. Real world triangulations and refinement strategies

The following Sections 3.2.1–3.2.7 describe properties which refinement strategies from Section 2.2.1 can additionally satisfy. Below, we provide several examples of possible refinement strategies $\mathbb{T}(\cdot,\cdot)$.

3.2.1. General assumptions. We consider a piecewise smooth d-dimensional Lipschitz manifold $\Omega \subseteq \mathbb{R}^D$ for some $d \le D$ with surface measure $|\cdot|$ such that there exists a constant $C_\omega > 0$ with

$$|B_\delta(x)| \le C_\Omega \delta^d \quad \text{for all } x \in \Omega \quad \text{and} \quad B_\delta(x) := \big\{ z \in \Omega \,:\, |x - z| \le \delta \big\}. \tag{3.2.1}$$

We assume that all triangulations $\mathcal{T} \in \mathbb{T}$ consist of compact elements $T \in \mathcal{T} \subseteq \mathcal{T}_\infty$ (where $\mathcal{T}_\infty$ is the set of all possible elements defined in Section 2.2.1) with $\bigcup_{T\in\mathcal{T}} T = \overline{\Omega}$ and $|T \cap T'| = 0$ for all $T, T' \in \mathcal{T}$ with $T \neq T'$.

3.2.2. K-mesh property. The K-mesh property relates the size of neighboring elements in the sense

$$K(\mathcal{T}) := \max \big\{ |T|/|T'| \,:\, T, T' \in \mathcal{T},\, T \cap T' \neq \emptyset \big\}. \tag{3.2.2}$$

We say that a refinement strategy preserves the K-mesh property, if there exists a constant $C_{\rm K} > 0$ such that

$$K(\mathcal{T}) \le C_{\rm K} K(\mathcal{T}_0) \quad \text{for all } \mathcal{T} \in \mathbb{T}. \tag{3.2.3}$$

3.2.3. Shape regularity. In the following applications, the shape regularity of triangulations plays an important role. Define for $d \geq 2$

$$\gamma(\mathcal{T}) := \max\big\{\mathrm{diam}(T)/|T|^{1/d} \,:\, T \in \mathcal{T}\big\}. \tag{3.2.4}$$

We say that a refinement strategy preserves shape regularity, if there exists a constant $C_{\mathrm{shp}} > 0$ such that

$$\gamma(\mathcal{T}) \leq C_{\mathrm{shp}}\gamma(\mathcal{T}_0) \quad \text{for all } \mathcal{T} \in \mathbb{T}. \tag{3.2.5}$$

Lemma 3.2.1. *Let $\mathcal{T}$ be shape regular and satisfy the K-mesh property. Then, all $z \in \overline{\Omega}$ and all $T \in \mathcal{T}$ satisfy*

$$\begin{aligned} |\big\{T' \in \mathcal{T} \,:\, z \in T'\big\}| &\leq K(\mathcal{T})\gamma(\mathcal{T})^d C_\Omega, \\ |\big\{T' \in \mathcal{T} \,:\, T \cap T' \neq \emptyset\big\}| &\leq K(\mathcal{T})^2\gamma(\mathcal{T})^d C_\Omega. \end{aligned}$$

Proof. Let $\delta := \mathrm{diam}(T_0)$, $z \in T_0$ denote the maximal diameter of all $T \in \mathcal{T}$ with $z \in T$. Then, $\bigcup\big\{T \in \mathcal{T} \,:\, z \in T\big\} \subseteq B_\delta(z) := \big\{x \in \mathbb{R}^d \,:\, |z - x| \leq \delta\big\}$. Shape regularity and the K-mesh property imply $|T| \geq K(\mathcal{T})^{-1}|T_0| \geq K(\mathcal{T})^{-1}\gamma(\mathcal{T})^{-d}\delta^d$. Altogether, this shows

$$|\big\{T \in \mathcal{T} \,:\, z \in T\big\}| \leq |B_\delta(z)|\delta^{-d}K(\mathcal{T})\gamma(\mathcal{T})^d \leq K(\mathcal{T})\gamma(\mathcal{T})^d C_\Omega.$$

Analogously, we obtain for $T' \cap T \neq \emptyset$ and $T_0 \cap T \neq \emptyset$, that $|T'| \geq K(\mathcal{T})^{-1}|T| \geq K(\mathcal{T})^{-2}|T_0| \geq K(\mathcal{T})^{-2}\gamma(\mathcal{T})^{-d}\delta^d$. This and the above conclude the proof. □

3.2.4. Existence of a reference element. Most of the practically used shape regular triangulations allow for a reference element $T_{\mathrm{ref}} \subseteq \mathbb{R}^d$ such that there exist bijective functions $F_T : T_{\mathrm{ref}} \to T$ for all $T \in \mathcal{T}_\infty$. The functions are smooth and uniformly bounded, i.e., all $p \in \mathbb{N}$ satisfy

$$\sup_{T \in \mathcal{T}_\infty} \big(|T|^{-p/d}\|D^p F_T\|_{L^\infty(T_{\mathrm{ref}})} + |T|^{p/d}\|D^p F_T^{-1}\|_{L^\infty(T)}\big) < \infty, \tag{3.2.6}$$

where $D^p(\cdot)$ denotes the p-th order derivative which is defined on $\mathbb{R}^d$ and on Ω (as a surface derivative) such that there holds $(DF_T^{-1}) \circ F_T = (DF_T)^{-1}$ with pointwise regular matrices in $\mathbb{R}^{d\times d}$. This particularly implies bi-Lipschitz continuity

$$C_5^{-1}|x - y| \leq |T|^{-1/d}|F_T(x) - F_T(y)| \leq C_5|x - y| \quad \text{for all } x, y \in T_{\mathrm{ref}} \tag{3.2.7}$$

for some constant $C_5 > 0$. Moreover, we suppose that all $T, T' \in \mathcal{T}$ with $z \in T \cap T' \neq \emptyset$ satisfy

$$F_T \circ F_{T'}^{-1}(z) = z. \tag{3.2.8}$$

This allows to define the usual spaces of piecewise polynomials

$$\mathcal{P}^p(\mathcal{T}) := \big\{V \in L^2(\Omega) \,:\, V \circ F_T \text{ is polynomial of degree } \leq p \text{ for all } T \in \mathcal{T}\big\} \tag{3.2.9}$$

and

$$\mathcal{S}^p(\mathcal{T}) := \mathcal{P}^p(\mathcal{T}) \cap C(\overline{\Omega}). \tag{3.2.10}$$

3.2.5. Father-son relation. Often, a refinement strategy allows for a unique father son relation, i.e., for all $\widehat{\mathcal{T}} \in \mathbb{T}(\mathcal{T})$ and all $T \in \mathcal{T} \setminus \widehat{\mathcal{T}}$, there exist son elements $T_0', \ldots, T_n' \in \widehat{\mathcal{T}} \setminus \mathcal{T}$ for some $2 \leq n \leq n_{\rm son} \in \mathbb{N}$ such that

$$T = \bigcup_{i=1}^{n} T_i'. \tag{3.2.11}$$

We call T the father of $T_0', \ldots, T_n'$. Note that (3.2.11) particularly implies (T1). Each of the sons satisfies

$$q_{\rm con}'|T| \leq |T'| \leq q_{\rm con}|T|, \tag{3.2.12}$$

for some constants $0 < q_{\rm con}' \leq q_{\rm con} < 1$.

3.2.6. Closure estimate. The axiom (T2) states that the output of Algorithm 2.2.1 satisfies the closure estimate. However, a generally defined refinement strategy often satisfies the closure estimate for any refinement $\widehat{\mathcal{T}} \in \mathbb{T}(\mathcal{T})$ and $\mathcal{T} \in \mathbb{T}$, i.e.,

$$|\widehat{\mathcal{T}} \setminus \mathcal{T}| \leq C_{\rm closure} \sum_{j=0}^{\ell-1} |\mathcal{M}_j|, \tag{3.2.13}$$

where $\mathcal{T} = \widehat{\mathcal{T}}_0, \ldots, \widehat{\mathcal{T}}_\ell = \widehat{\mathcal{T}}$ for some $\widehat{\mathcal{T}}_j \in \mathbb{T}$ and $\widehat{\mathcal{M}}_j \subseteq \mathcal{T}_j$ with $\widehat{\mathcal{T}}_{j+1} = \mathbb{T}(\widehat{\mathcal{T}}_j, \widehat{\mathcal{M}}_j)$ for all $j = 0, \ldots, \ell-1$. By Lemma 2.7.8, this is also equivalent to (2.7.9) if Section 3.2.5 applies.

3.2.7. Simplicial triangulations. Under the assumptions of Section 3.2.1–3.2.5, we assume that $T_{\rm ref}$ is a simplex of dimension d with set of nodes $\mathcal{K}(T_{\rm ref})$. By $\mathcal{K}(T) := F_T(\mathcal{K}(\mathcal{T}_{\rm ref}))$, we denote the nodes of the elements $T \in \mathcal{T}_\infty$ and $\mathcal{K}(\mathcal{T}) := \bigcup_{T\in\mathcal{T}} \mathcal{K}(T)$ denotes the nodes of the triangulation. We prohibit hanging nodes, i.e., all $T, T' \in \mathcal{T}$ satisfy $\mathcal{K}(T) \cap T' \subseteq \mathcal{K}(T')$. The element mappings $F_T \colon T_{\rm ref} \to T$ are affine functions.

The following result is well-known in the literature

Lemma 3.2.2. *Let $T \in \mathcal{T}$ and $z \in \mathcal{K}(\mathcal{T})$ such that $z \notin T$. Then, there holds*

$$\operatorname{diam}(T) \leq C_6 \min_{z' \in T} |z - z'|, \tag{3.2.14}$$

where the constant $C_6 > 0$ depends only on $\gamma(\mathcal{T})$, d, and $K(\mathcal{T})$.

3.2.8. Example 1: Bisection. For $d \geq 1$, the elements in $\mathcal{T}_\infty$ are compact simplices $T \subseteq \mathbb{R}^d$, i.e., affine line segments for $d = 1$, triangles for $d = 2$, and tetrahedra for $d = 3$. All triangulations $\mathcal{T} \in \mathbb{T}$ are regular in the sense that all vertices $z \in \mathcal{K}(\mathcal{T})$ are vertices of all elements $T \in \mathcal{T}$ with $z \in T$ (no hanging nodes).

For $d = 1$, bisection splits the elements $T \in \mathcal{M} \subseteq \mathcal{T}$ marked for refinement at a generic point $x_T \in T$ (e.g., the barycenter) to generate two new elements T_1 and T_2 which both share the endpoint x_T. Additional bisections have to be imposed to ensure that the bisection preserves the K-mesh property (3.2.3). We refer to **[3]** for some extended 1D bisection algorithm.

For $d \geq 2$, the bisection is described in **[78]** (called newest vertex bisection for $d = 2$) and **[79]** (for $d \geq 3$). Each element $T \in \mathcal{T}$ has a distinguished edge (the reference edge). If the element is refined, first the reference edge is split. See Figure 1 for an illustration of the refinement rules for $d = 2$.

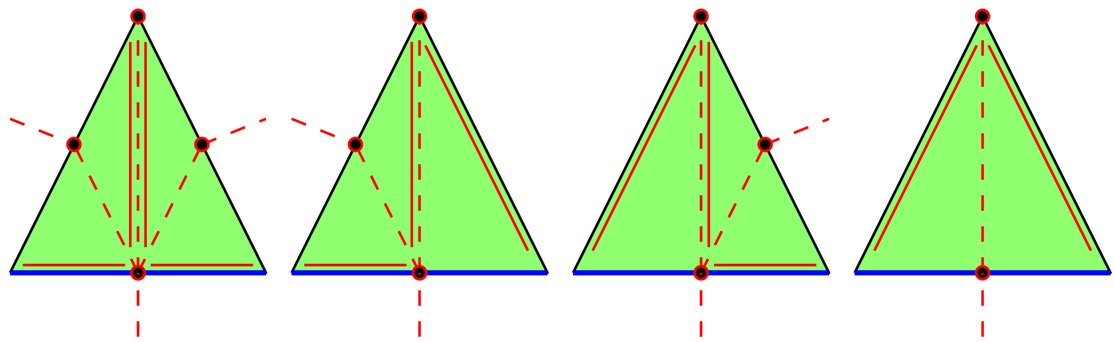

FIGURE 1. Refinement rules for 2D bisection (newest vertex bisection). The reference edge is indicated in blue. The leftmost triangle depicts the *bisec3* refinement of a marked element. The other three refinement rules (*bisec2* and *bisec1*) are recursively applied to avoid hanging nodes. The dashed line outside of the triangles indicates where the neighboring triangle is refined.

LEMMA 3.2.3. *The bisection strategies for $d \geq 1$ are refinement strategies in the sense of Section 3.2.1–3.2.7 and satisfy* (T1)–(T2) *as well as the overlay estimate* (2.5.1) *and the son estimate* (2.7.7)*. For $d \geq 3$, an appropriate labeling of the edges of the initial triangulation $\mathcal{T}_0$ is necessary to guarantee* (T2) *(see* [**14, 79**] *for details).*

PROOF. The $d = 1$ case is proved in [**3**]. The estimate (2.7.7) holds since each of the refinement strategies generates at least two son elements for each refined element. The proof of (2.7.10) with $\widetilde{C_4} = 1$ is found in [**78**, Proof of Lemma 5.2] for $d = 2$ and [**35**, Lemma 3.7] for $d \geq 2$. By Lemma 2.7.8, this is equivalent to (2.5.1) with $C_4 = 2$. However, since [**35**, Lemma 3.7] shows that the coarsest common refinement $\mathcal{T} \oplus \widehat{\mathcal{T}} \in \mathbb{T}(\mathcal{T}) \cap \mathbb{T}(\widehat{\mathcal{T}})$ is given by

$$\mathcal{T} \oplus \widehat{\mathcal{T}} := \{T \in \mathcal{T} \,:\, \exists \widehat{T} \in \widehat{\mathcal{T}}, T \subseteq \widehat{T}\} \cup \{\widehat{T} \in \widehat{\mathcal{T}} \,:\, \exists T \in \mathcal{T}, \widehat{T} \subseteq T\}, \tag{3.2.15}$$

counting the elements reveals

$$|(\mathcal{T} \oplus \widehat{\mathcal{T}}) \setminus \mathcal{T}| = |\{\widehat{T} \in \widehat{\mathcal{T}} \,:\, \exists T \in \mathcal{T}, \widehat{T} \subsetneq T\}| \leq |\widehat{\mathcal{T}} \setminus \mathcal{T}| \leq |\widehat{\mathcal{T}} \setminus \mathcal{T}_0|.$$

This, however, is (2.5.1) with $C_4 = 1$.

For the proof of (2.7.9) and hence (T2) and (3.2.13) (by Lemma 2.7.8), we refer to [**14**] for $d = 2$ and [**79**] for $d \geq 2$. The works [**14, 79**] assume an appropriate labeling of the edges of the initial triangulation $\mathcal{T}_0$ to prove (T2). This poses a combinatorial problem on the initial triangulation $\mathcal{T}_0$ but does not concern any of the following triangulations $\mathcal{T}_\ell$, $\ell \geq 1$. For $d = 2$, it can be proven that each conforming triangular triangulation $\mathcal{T}$ allows for such a labeling, while no efficient algorithm is known to compute this in linear complexity. For $d \geq 3$, such a result is missing. However, it is known that an appropriate uniform refinement of an arbitrary conforming simplicial triangulation $\mathcal{T}$ for $d \geq 2$ allows for such a labeling [**79**]. Moreover, for $d = 2$, it has recently been proved in [**59**] that (T2) even holds without any further assumption on the initial triangulation $\mathcal{T}_0$. The axiom (T1) is proved by use of [**52**, Corollary 3.5], which shows the level difference between some $T \in \mathbb{T}(\mathcal{T}, \mathcal{M})$ for some $\mathcal{M} \subseteq \mathcal{T}$ and its father element $T' \in \mathcal{T}$ with $T \subseteq T'$ is uniformly bounded. Since the level measures the number of bisections used to generate the element from $\mathcal{T}_0$, this implies that each father element $T' \in \mathcal{T}$ has uniformly bounded number of sons in $\mathbb{T}(\mathcal{T}, \mathcal{M})$. This concludes the proof. □

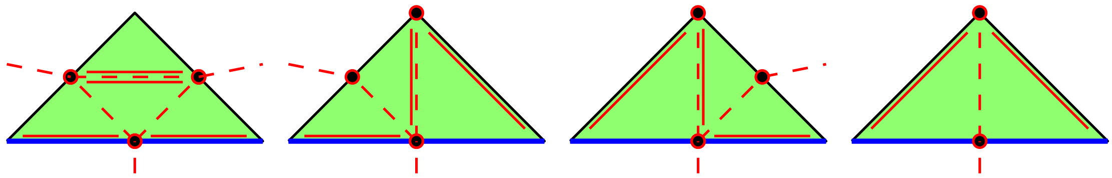

FIGURE 2. Refinement rules for 2D red-green-blue refinement. The leftmost triangle is red-refined, i.e., all of its edges are bisected, the right most triangle is blue-refined, i.e., only its reference edge is refined, and the other triangles are green-refined. The reference edges of the son triangles are indicated with a solid red line. Red refinement is used for marked elements, green and blue refinement are used to avoid hanging nodes. There are two methods to determine the reference edge. The simplest one is to take the longest edge of the triangle. The second one (also known as modified red-green-blue refinement) is to choose a labeling of the initial triangulation $\mathcal{T}_0$ as for bisection from Section 3.2.8. The reference edge of each son triangle is then chosen such that it is congruent with its father triangle. Under certain conditions on the interior angles of the triangles, [**70**, Satz 4.17] (in German) shows that both methods coincide as is the case in the example above.

3.2.9. Example 2: Red-green-blue refinement. For $d \geq 2$, the elements are compact simplices $T \subseteq \mathbb{R}^d$.

The red-green-blue refinement (discussed e.g., in [**82**]) refines a given triangulation for $d = 2$ according to Figure 2. For $d = 3$, the situation is more complicated as a tetrahedron is split into four similar tetrahedra at the parents vertices plus an octahedron in the center which has to be split furthermore. This is laid out in detail in [**9**]. In contrast to bisection from Section 3.2.8, red-green-blue refinement fails to satisfy (2.5.1) as seen from a counterexample in [**70**, Satz 4.15] (in German). For illustration purposes, we provide a slightly simplified example in Figure 3

LEMMA **3.2.4.** *The red-green-blue refinement strategies for $d = 2, 3$ are refinement strategies in the sense of Section 3.2.1–3.2.7 and satisfy* (T1)–(T2) *as well as the son estimate* (2.7.7) *at least for $d = 2$ (if reference edges are inherited as for 2D bisection and the initial triangulation satisfies an appropriate labeling of the edges; see* [**14, 79**] *for details).*

PROOF. For the proof of (2.7.9) and hence (T2) and (3.2.13) (by Lemma 2.7.8), we refer to [**53**, Appendix A] or [**70**, Satz 4.14] for $d = 2$ under the assumption of an appropriate labeling of the edges of the initial triangulation $\mathcal{T}_0$ as is Section 3.2.8. The axiom (T1) is obvious for $d = 2$, since all possibilities are depicted in Figure 2. The estimate (2.7.7) follows since each refinement produces at least two sons. This concludes the proof. ☐

3.2.10. Example 3: Quad refinement with one hanging node. If one admits hanging nodes, also quad-refinement is an option. The elements $T \in \mathcal{T}_\infty$ are quadrilaterals for $d = 2$ an hexahedra for $d = 3$. The refinement of an element is realized by dividing the element into 2^d congruent sons. This strategy is described in [**16**].

LEMMA **3.2.5.** *The quad refinement strategies for $d = 2, 3$ are refinement strategies in the sense of Section 3.2.1–3.2.6 and satisfy* (T1)–(T2) *as well as the overlay estimate* (2.5.1) *and the son estimate* (2.7.7).

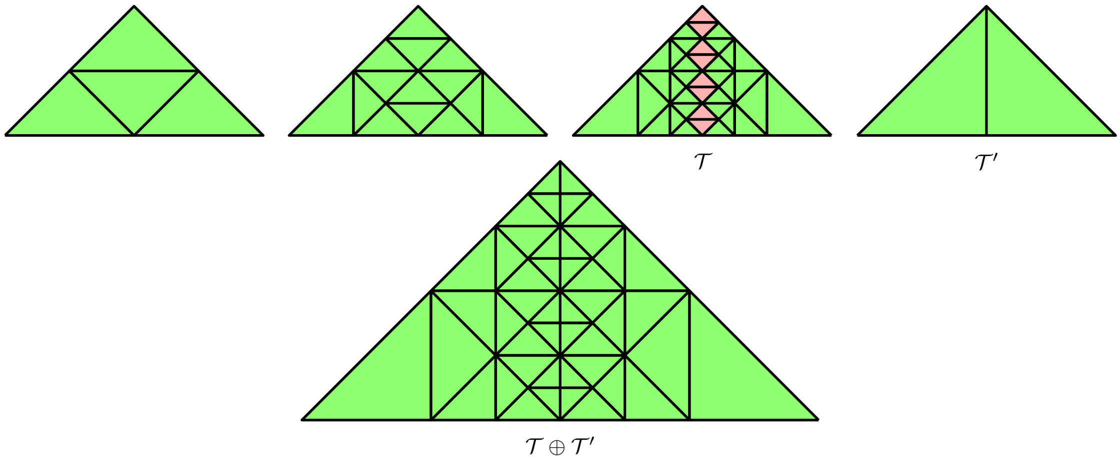

FIGURE 3. Counter-example to (2.5.1) for red-green-blue refinement. With $j = 4$, there holds $|\mathcal{T} \oplus \mathcal{T}' \setminus \mathcal{T}| = 4j$ and $|\mathcal{T}'| = 2$. Obviously, the construction can be extended to any $j = 2^n$, $n \in \mathbb{N}$ by red-refinement of the marked triangles in $\mathcal{T}$ and thus contradicts (2.5.1) for any constant.

PROOF. The closure estimate (2.7.9) and hence (T2) and (3.2.13) (by Lemma 2.7.8) is proved [**16**, Section 6.3]. The overlay estimate (2.5.1) follows from the fact that it is a binary refinement strategy, i.e., there holds (3.2.15). The estimate (2.7.7) follows from the fact that each refinement produces four sons. Finally, (T1) follows by consideration of all possible element intersections. □

3.2.11. Example 4: Facet based refinement strategies. The refinement strategies from Section 3.2.8 and Section 3.2.9 can be formulated in a facet based way. In this case, $\mathcal{T}_\infty$ is the set of facets which can be generated and $\mathcal{T} \subseteq \mathbb{T}$ is a triangulation represented by the element facets. For refinement, we mark facets $\mathcal{M} \subseteq \mathcal{T}$ and generate the refinement $\mathbb{T}(\mathcal{T}, \mathcal{M})$ according to the rules depicted in Figure 1–2 for $d = 2$. For $d \geq 3$, we refer to [**79**] for bisection and [**9**] for red-green-blue refinement. The results of Lemma 3.2.3 and Lemma 3.2.4 hold also for facet based refinement.

3.3. Uniform approximability

Apart from Lemma 2.7.4, the uniform approximability axiom (T3) is relatively unaccessible without looking at concrete problems. To that end we aim to provide a characterization of (T3) for a certain class of problems in terms of Proposition 2.4.1, where $\mathrm{err}(\cdot) := \min_{V \in \mathcal{S}^p(\cdot)} \|u - V\|_{H^1(\Omega)}$ measures the best approximation error of some given function $u \in H^1(\Omega)$. The key problem is that the results on the characterization of approximability, e.g., [**55, 56**], usually show $\|\mathrm{err}, \mathbb{T}(\mathcal{T}_0)\|_s < \infty$ under certain assumptions on the function u. However, the proofs in [**55, 56**] do not give explicit dependence of the constants with respect to $\mathcal{T}_0$ and work only for bisection from Section 3.2.8. In the following, we generalize the result from [**55**] to general refinement strategies and with explicit constants. It might also be possible to generalize [**56**] with similar techniques as shown in this section, however, this is beyond the scope of this work.

THEOREM 3.3.1. *Assume $\mathbb{T}$ and a corresponding refinement strategy $\mathbb{T}(\cdot,\cdot)$ in the sense of Section 3.2.1–3.2.7. Let $\Omega \subseteq \mathbb{R}^d$ for $d = 2, 3$ denote a polyhedral domain (not necessarily Lipschitz) and let $\mathcal{T}_0$ be an initial triangulation of Ω. Given $p \in \mathbb{N}$, suppose $u, u_0 \in H^1(\Omega)$*

such that $u_0|_T \in H^{p+1}(T)$ *for all* $T \in \mathcal{T}_0$ *and*

$$u = u_0 + \sum_{i=1}^{N} u_i \quad \textit{with} \quad u_i(r_i, \theta_i) := c_i \log(r_i)^{\mu_i} r_i^{\gamma_i} g_i(\theta_i) \chi_i \quad \textit{for all } i = 1, \dots, N. \tag{3.3.1}$$

Here, $N \in \mathbb{N}_0$*,* $c_i \in \mathbb{R}$*,* $\mu_i \geq 0$*,* $0 < \gamma_i < 1$*, and*

(i) $\chi_i \in C^\infty(\overline{\Omega})$ *is an arbitrary function,*
(ii) $(r_i, \theta_i) \in [0, \infty) \times [0, 2\pi) \times [0, \pi]^{d-2}$ *denote the polar (spherical) coordinates with respect to some origin* $x_i \in \overline{\Omega}$ *with* $x_i \in \mathcal{K}(\mathcal{T}_0)$*,*
(iii) $g_i \in W^{1,\infty}(\Omega)$ *are constant with respect to* r_i*, i.e.,* $g_i(r_i, \theta_i) := g_i(\theta_i)$*, and satisfy* $g_i|_T \in W^{p+1,\infty}(T)$ *for all* $T \in \mathcal{T}_0$*.*

Then, given $p \in \mathbb{N}$*, there exists* $C_7 > 0$ *such that for all* $\mathcal{T} \in \mathbb{T}$ *and all* $\varepsilon > 0$*, there exists* $\widehat{\mathcal{T}} \in \mathbb{T}(\mathcal{T})$ *with*

$$\operatorname{err}(\widehat{\mathcal{T}}) := \min_{V \in \mathcal{S}^p(\widehat{\mathcal{T}})} \|u - V\|_{H^1(\Omega)} \leq \varepsilon \quad \textit{and} \quad |\widehat{\mathcal{T}} \setminus \mathcal{T}| \leq C_7 \varepsilon^{-d/p}.$$

The constant C_7 *depends only on* u*,* p*, and* $\mathcal{T}_0$*, but not on* $\mathcal{T} \in \mathbb{T}$*.*

We postpone the proof of Theorem 3.3.1 to the end of the section and collect several intermediate results.

3.3.0.1. *Scott-Zhang projection.* The Scott-Zhang projection was introduced in [**76**]. We give a slightly modified definition.

Definition 3.3.2 (Scott-Zhang projection)**.** *Assume a triangulation* $\mathcal{T}$ *in the sense of Section 3.2.1–3.2.7 and let* $p \in \mathbb{N}$*. For each* $z \in \mathcal{K}(\mathcal{T})$ *choose* $T_z \in \mathcal{T}$ *with* $z \in T_z$*. Consider the nodal basis* $\{\phi_z \in \mathcal{S}^1(\mathcal{T}) \,:\, z \in \mathcal{K}(\mathcal{T})\}$ *with* $\phi_z(z') = 0$ *for all* $z' \neq z$ *and* $\phi_z(z) = 1$*. Let* $p \geq 1$ *and consider the extended basis* $\{b_1, \dots, b_n\} \in \mathcal{P}^p(T_{\rm ref})$ *for some* $n \in \mathbb{N}$ *with* $\|b_i\|_{L^\infty(T_{\rm ref})} \leq 1$ *such that*

$$\operatorname{span}\big\{\{\phi_z \,:\, z \in \mathcal{K}(\mathcal{T})\} \cup \{b_{T,i} := b_i \circ F_T^{-1} \,:\, i = 1, \dots, n,\, T \in \mathcal{T}\}\big\} = \mathcal{S}^p(\mathcal{T}).$$

For each $T \in \mathcal{T}$ *let* $\{\phi^\star_{T,z}, b^\star_{T,1}, \dots, b^\star_{T,n}\} \subseteq \mathcal{P}^p(T)$ *denote the dual basis functions with respect to* $\{\phi_z|_T, b_{T,1}, \dots, b_{T,n}\}$*. Define for* $v \in L^2(\Omega)$

$$J(\mathcal{T})v := \sum_{z \in \mathcal{K}(\mathcal{T})} \phi_z \int_{T_z} \phi^\star_{T_z,z} v \, dx + \sum_{T \in \mathcal{T}} \sum_{i=1}^{n} b_{T,i} \int_T b^\star_{T,i} v \, dx.$$

Moreover, define the patch $\omega(T, \mathcal{T}) := \{T' \in \mathcal{T} \,:\, T \cap T' \neq \emptyset\}$*.*

The stability estimates (3.3.2a)–(3.3.2b) are known since the seminal work [**76**]. However, the optimality estimate (3.3.2c) was first derived in [**4**] for triangulations which are generated by bisection from Section 3.2.8. Later, this result was generalized in [**81**] to shape regular triangulations. Below, we provide a simplified proof with the techniques of the original proof in [**4**].

Lemma 3.3.3 (Scott-Zhang projection)**.** *Assume a triangulation* $\mathcal{T}$ *in the sense of Section 3.2.1–3.2.7 and let* $p \in \mathbb{N}$*. The Scott-Zhang projection from Definition 3.3.2 satisfies for all* $T \in \mathcal{T}$ *and all* $v \in H^1(\Omega)$

$$\|J(\mathcal{T})v\|_{L^2(T)} \leq C_{\rm sz} \|v\|_{L^2(\cup\omega(T,\mathcal{T}))}, \tag{3.3.2a}$$

$$\|\nabla J(\mathcal{T})v\|_{L^2(T)} \leq C_{\rm sz} \|\nabla v\|_{L^2(\cup\omega(T,\mathcal{T}))}, \tag{3.3.2b}$$

$$\|\nabla(1 - J(\mathcal{T}))v\|_{L^2(T)} \leq C_{\rm sz} \min_{V \in \mathcal{P}^{p-1}_\nabla(\mathcal{T})} \|\nabla v - V\|_{L^2(\cup\omega(T,\mathcal{T}))}, \tag{3.3.2c}$$

where

$$\mathcal{P}^{p-1}_{\nabla}(\mathcal{T}) := \big\{ V \in L^2(\Omega)^d \,:\, V|_T = W\, DF_T^{-1},\, W \in \mathcal{P}^{p-1}(T)^d,\, T \in \mathcal{T} \big\}. \tag{3.3.2d}$$

The constant $C_{\rm sz} > 0$ depends only on the constants in Section 3.2.1–3.2.7, $\mathbb{T}$, and $p \in \mathbb{N}$.

Before we prove Lemma 3.3.3, we state the following auxiliary lemma from [**41**].

Lemma 3.3.4 (Generalized Poincaré-Friedrichs inequality)**.** *Assume a triangulation $\mathcal{T}$ in the sense of Section 3.2.1–3.2.7. Let $v \in H^1(\Omega)$, $T, T' \in \mathcal{T}$ with $T \cap T' \neq \emptyset$. Then, there holds with $v_T := |T|^{-1} \int_T v\, dx$*

$$\|v - v_T\|_{L^2(T)} + |T|^{1/2} |v_T - v_{T'}| \leq C_8 |T|^{1/d} \|\nabla v\|_{L^2(\cup\omega(T,\mathcal{T}))},$$

where $C_8 > 0$ depends only the constants in Section 3.2.1–3.2.7. □

Proof of (3.3.2a)–(3.3.2b). By definition of the dual basis, $J(\mathcal{T})$ is a projection. To see (3.3.2b), consider $T \in \mathcal{T}$ and $b^\star \in \{\phi^\star_{T,z}, b^\star_{T,1}, \ldots, b^\star_{T,n}\}$. A scaling argument proves

$$\|b^\star\|_{L^\infty(T)} \lesssim |T|^{-1},$$

where the hidden constant depends only on $\gamma(\mathcal{T})$, p, and the reference element $T_{\rm ref}$ from Section 3.2.4. With this, there holds

$$\Big| \int_T b^\star v\, dx \Big| \leq \|b^\star\|_{L^\infty(T)} \|v\|_{L^1(T)} \lesssim |T|^{-1/2} \|v\|_{L^2(T)}.$$

An inverse estimate shows for any basis function $b \in \big\{\phi_z \,:\, z \in \mathcal{K}(\mathcal{T})\big\} \cup \big\{b_{T,i} \,:\, i = 1, \ldots, n,\, T \in \mathcal{T}\big\}$ with $|\mathrm{supp}(b) \cap T| > 0$

$$\|\nabla b\|_{L^2(T)} \lesssim |T|^{1/2 - 1/d},$$

where the hidden constant depends only on the constants in Section 3.2.1–3.2.7 and p. Altogether, this implies

$$\begin{aligned} \|\nabla J(\mathcal{T}) v\|_{L^2(T)} &\leq \sum_{z \in \mathcal{K}(T)} \|\nabla \phi_z\|_{L^2(T_z)} \Big| \int_{T_z} \phi^\star_{T_z,z} v\, dx \Big| + \sum_{i=1}^n \|\nabla b_{T,i}\|_{L^2(T)} \Big| \int_T b^\star_{T,i} v\, dx \Big| \\ &\lesssim |T|^{-1/d} \|v\|_{L^2(\cup\omega(T,\mathcal{T}))}, \end{aligned}$$

where the hidden constant depends only on the constants in Section 3.2.1–3.2.7, T, and p. Define $v_T := |T|^{-1} \int_T v\, dx$. Then, there holds with the last estimate and the projection property $J(\mathcal{T}) v_T = v_T$

$$\|\nabla J(\mathcal{T}) v\|_{L^2(T)} = \|\nabla J(\mathcal{T})(v - v_T)\|_{L^2(T)} \lesssim |T|^{-1/d} \|v - v_T\|_{L^2(\cup\omega(T,\mathcal{T}))}.$$

Lemma 3.3.4 implies

$$\begin{aligned} \|v - v_T\|^2_{L^2(\cup\omega(T,\mathcal{T}))} &\leq 2 \sum_{T' \in \omega(T,\mathcal{T})} \|v - v_{T'}\|^2_{L^2(T')} + |T'| |v_T - v_{T'}|^2 \\ &\leq 2 C_8^2 K(\mathcal{T})^{2/d} |T|^{2/d} \|\nabla v\|^2_{L^2(\cup\omega(T,\mathcal{T}))}. \end{aligned}$$

Altogether, this proves (3.3.2b). The same argument shows also (3.3.2a). □

Lemma 3.3.5. *Assume a set of triangulations $\mathbb{T}$ in the sense of Section 3.2.1–3.2.4 Let $v \in H^1(\Omega)$ with $\nabla v \in \mathcal{P}^{p-1}_{\nabla}(\mathcal{T})$. Then, $v \in \mathcal{S}^p(\mathcal{T})$ and $\nabla \mathcal{S}^p(\mathcal{T}) \subseteq \mathcal{P}^{p-1}_{\nabla}(\mathcal{T})$.*

Proof. Let $v \in \mathcal{S}^p(\mathcal{T})$, then $v \circ F_T \in \mathcal{P}^p(T_{\rm ref})$ and hence $\nabla v = \nabla(v \circ F_T) \circ F_T^{-1}\, DF_T^{-1}$. Since $\nabla(v\circ F_T) \in \mathcal{P}^{p-1}(T_{\rm ref})^d$, this shows $\nabla\mathcal{S}^p(\mathcal{T}) \subseteq \mathcal{P}^{p-1}_\nabla(\mathcal{T})$.

By definition of $\mathcal{P}^{p-1}_\nabla(\mathcal{T})$, there holds for $T \in \mathcal{T}$, $\nabla v|_T = W|_T DF_T^{-1}$ for some $W \in \mathcal{P}^{p-1}(\mathcal{T})^d$. This shows

$$\nabla(v \circ F_T)(DF_T)^{-1} = (\nabla v)\circ F_T = W \circ F_T\,(DF_T^{-1})\circ F_T.$$

By assumption in Section 3.2.4, there holds $(DF_T)^{-1} = (DF_T^{-1})\circ F_T$ with point wise regular matrices. Hence, we end up with $\nabla(v\circ F_T) = W\circ F_T \in \mathcal{P}^{p-1}(T_{\rm ref})^d$, which implies $v \circ F_T \in \mathcal{P}^p(T_{\rm ref})$. Since $v \in H^1(\Omega)$, this concludes the proof. $\square$

Lemma 3.3.6. *Assume a set of triangulations $\mathbb{T}$ in the sense of Section 3.2.1–3.2.4 such that there exists a set $\mathcal{E}_{\rm ref} := \{\emptyset, E_1,\ldots,E_{M_{\rm ref}}\}$ of boundary parts $E_i \subseteq \partial T_{\rm ref}$ such that for all $T,T' \in \mathcal{T}$ holds $F_T^{-1}(T\cap T') \in \mathcal{E}_{\rm ref}$. Then, for all $T\in\mathcal{T}$ and all $\mathcal{T}\in\mathbb{T}$, there exists a bi-Lipschitz continuous map $G_T\colon \bigcup\omega_{\rm ref}(T) \to \bigcup\omega(T,\mathcal{T})$ with*

$$C_9^{-1}|x-y| \le |T|^{-1/d}|G_T(x)-G_T(y)| \le C_9|x-y| \quad \text{for all } x,y\in\omega_{\rm ref}(T),$$

where $\omega_{\rm ref}(T)\in\omega(\mathbb{T})$ for a finite subset $\omega(\mathbb{T}) \subseteq \big\{\omega(T,\mathcal{T}) \,:\, \mathcal{T}\in\mathbb{T}, T\in\mathcal{T}\big\}$. For $T'\in\omega_{\rm ref}(T)$, there holds $G_T|_{T'} = F_{T''}\circ F_{T'}^{-1}$ for $T'' := G_T(T') \in \omega(T,\mathcal{T})$. This particularly implies that G_T maps polynomials onto polynomials, i.e., $V\circ G_T \in \mathcal{P}^p(\omega_{\rm ref}(T))$ for all $V\in\mathcal{P}^p(\omega(T,\mathcal{T}))$ and $V\circ G_T^{-1}\in\mathcal{P}^p(\omega(T,\mathcal{T}))$ for all $V \in \mathcal{P}^p(\omega_{\rm ref}(T))$. The constant $C_9>0$ depends only on $\mathbb{T}$, $\mathcal{E}_{\rm ref}$, and the constants in Section 3.2.1–3.2.4.

Remark 3.3.7. *This result is only applied in the case of triangulations in the sense of Section 3.2.7 for which the proof would simplify vastly. However, we include the general result as we think it might be of independent interest, as it holds for a huge class of possible triangulations including non-regular ones.*

Proof. The first step is to sort the patches into certain equivalence classes. With Lemma 3.2.1, there holds $|\omega(T,\mathcal{T})| \le n_1 := K(\mathcal{T})^2\gamma(\mathcal{T})^d C_\Omega$ for all $T\in\mathcal{T}$. Define $\mathcal{G} := \mathcal{E}_{\rm ref}^2\times\{1,\ldots,n_1\}^2$. Each patch $\omega(T,\mathcal{T})$ has a signature

$$\mathcal{G}_T := \big\{(E_1,E_2,T_1,T_2)\in\mathcal{E}_{\rm ref}^2\times\mathcal{T}^2 \,:\, T_1,T_2\in\omega(T,\mathcal{T}),\ F_{T_i}^{-1}(T_1\cap T_2) = E_i,\ i=1,2\big\}.$$

For $\mathcal{G}'\subseteq\mathcal{G}$ and $T\in\mathcal{T}$, we write $\mathcal{G}'\sim\mathcal{G}_T$ if and only if there exist an injective map $M_T\colon\omega(T,\mathcal{T})\to\{1,\ldots,n_1\}$ with

$$(E_1,E_2,T_1,T_2)\in\mathcal{G}_T \iff (E_1,E_2,M_T(T_1),M_T(T_2))\in\mathcal{G}'. \tag{3.3.3}$$

Define $\mathcal{G}_{\rm ref} := \big\{\mathcal{G}'\subseteq\mathcal{G} \,:\, \exists\mathcal{T}\in\mathbb{T}, \exists T\in\mathcal{T}, \mathcal{G}_T\sim\mathcal{G}'\big\}$. The set $\mathcal{G}_{\rm ref}\subseteq 2^{\mathcal{G}}$ is finite by definition. For each $\mathcal{G}'\in\mathcal{G}_{\rm ref}$, choose one $T'\in\mathcal{T}'\in\mathbb{T}$ with $\mathcal{G}_{T'}\sim\mathcal{G}'$ and maximal element measure $|T|$ to define the finite set

$$\omega(\mathbb{T}) := \big\{\omega(T',\mathcal{T}') \,:\, \mathcal{G}'\in\mathcal{G}_{\rm ref}\big\}.$$

Define the function G_T as follows: Given $T\in\mathcal{T}$ for some $\mathcal{T}\in\mathbb{T}$, choose some $\mathcal{G}'\in\mathcal{G}_{\rm ref}$ with $\mathcal{G}_T\sim\mathcal{G}'$ as well as $\omega(T',\mathcal{T}')\in\omega(\mathbb{T})$ such that $\mathcal{G}_{T'}\sim\mathcal{G}'$. For all $T_1\in\omega(T,\mathcal{T})$ determine $T_2 := M_{T'}^{-1}\circ M_T(T_1)$ and let

$$G_T|_{T_2} := F_{T_1}\circ F_{T_2}^{-1}.$$

This defines a function $G_T\colon\bigcup\omega(T',\mathcal{T}')\to\bigcup\omega(T,\mathcal{T})$. To show that G_T is continuous, consider $T_3\in\omega(T,\mathcal{T})$ with $T_4 := M_{T'}^{-1}\circ M_T(T_3)$. Since $(E,E',T_2,T_4)\in\mathcal{G}_{T'}$ for some $E,E'\in\mathcal{E}_{\rm ref}$, and since $\mathcal{G}_{T'}\sim\mathcal{G}'\sim\mathcal{G}_T$, there holds $(E,E',T_1,T_3)\in\mathcal{G}_T$. This implies

$$F_{T_2}^{-1}(T_2\cap T_4) = F_{T_1}^{-1}(T_1\cap T_3) \quad\text{and}\quad T_2\cap T_4\neq\emptyset. \tag{3.3.4}$$

Let $z \in T_2 \cap T_4$. Then, (3.3.4) implies $F_{T_1} \circ F_{T_2}^{-1}(z) \in T_1 \cap T_3$. By the continuity assumption (3.2.8), this shows

$$F_{T_3} \circ F_{T_2}^{-1}(z) = (F_{T_3} \circ F_{T_1}^{-1}) \circ F_{T_1} \circ F_{T_2}^{-1}(z) \overset{(3.2.8)}{=} F_{T_1} \circ F_{T_2}^{-1}(z) \in T_1 \cap T_3. \tag{3.3.5}$$

Another application of the continuity (3.2.8) (with $z \in T_2 \cap T_4$) then concludes

$$\begin{aligned} G_T|_{T_4}(z) &= F_{T_3} \circ F_{T_4}^{-1}(z) = F_{T_3} \circ F_{T_4}^{-1} \circ F_{T_4} \circ F_{T_2}^{-1}(z) = F_{T_3} \circ F_{T_2}^{-1}(z) \\ &\overset{(3.3.5)}{=} F_{T_1} \circ F_{T_2}^{-1}(z) = G_T|_{T_2}(z). \end{aligned}$$

This proves continuity of G_T. The element wise bi-Lipschitz continuity of the F_T (3.2.7) together with the K-mesh property (3.2.3) and the fact that Ω is Lipschitz conclude the bi-Lipschitz continuity of G_T. The fact that G_T is defined element wise with the element mappings F_T implies $V \circ G_T \in \mathcal{P}^p(\omega_{\mathrm{ref}}(T))$ for all $V \in \mathcal{P}^p(\omega(T,\mathcal{T}))$ and $V \circ G_T^{-1} \in \mathcal{P}^p(\omega(T,\mathcal{T}))$ for all $V \in \mathcal{P}^p(\omega_{\mathrm{ref}}(T))$. This concludes the proof. □

Proof of (3.3.2c). Lemma 3.3.6 with $\mathcal{E}_{\mathrm{ref}} := \{\emptyset\} \cup \{\text{facets, edges, nodes of } T_{\mathrm{ref}}\}$ is applicable due to the assumptions in Section 3.2.7 and allows to prove the statement on the finitely many reference patches $\omega_{\mathrm{ref}} \in \omega(\mathbb{T})$ and to obtain the general result (3.3.2c) by transformation. Assume that (3.3.2c) holds for $G_T^{-1}(T)$ and $\omega_{\mathrm{ref}} \in \omega(\mathbb{T})$. Then, $w = v \circ G_T \in H^1(\bigcup \omega_{\mathrm{ref}})$ implies

$$\begin{aligned} |T|^{1/d-1/2} \|\nabla(1 - J(\mathcal{T}))v\|_{L^2(T)} &\lesssim \|\nabla(1 - J(\omega_{\mathrm{ref}}))w\|_{L^2(G_T^{-1}(T))} \\ &\le C_{\mathrm{sz}} \min_{W \in \mathcal{P}_\nabla^{p-1}(\omega_{\mathrm{ref}})} \|\nabla w - W\|_{L^2(\cup\omega_{\mathrm{ref}})} \\ &\lesssim |T|^{-1/2} C_{\mathrm{sz}} \min_{W \in \mathcal{P}_\nabla^{p-1}(\omega_{\mathrm{ref}})} \|(\nabla w) \circ G_T^{-1} - W \circ G_T^{-1}\|_{L^2(\cup\omega(T,\mathcal{T}))}, \end{aligned} \tag{3.3.6}$$

where $J(\omega_{\mathrm{ref}})w|_{G_T^{-1}(T)} := (J(\mathcal{T})v) \circ G_T$. By definition of G_T and since the F_T are affine, $J(\omega_{\mathrm{ref}})$ is a Scott-Zhang projection on ω_{ref} in the sense of Definition 3.3.2. By definition of w, there holds

$$\begin{aligned} &\min_{W \in \mathcal{P}_\nabla^{p-1}(\omega_{\mathrm{ref}})} \|(\nabla w) \circ G_T^{-1} - W \circ G_T^{-1}\|_{L^2(\cup\omega(T,\mathcal{T}))} \\ &\qquad \le \|DG_T\|_{L^\infty(\cup\omega_{\mathrm{ref}})} \min_{W \in \mathcal{P}_\nabla^{p-1}(\omega_{\mathrm{ref}})} \|\nabla v - W \circ G_T^{-1}(DG_T)^{-1} \circ G_T^{-1}\|_{L^2(\cup\omega(T,\mathcal{T}))}. \end{aligned} \tag{3.3.7}$$

By definition of $\mathcal{P}_\nabla^{p-1}(\cdot)$ and G_T, there holds $W|_{T'} = V DF_{T'}^{-1}$ for some $V \in \mathcal{P}^{p-1}(\omega_{\mathrm{ref}})^d$. By Lemma 3.3.6, there holds $DG_T|_{T'} = DF_{T''} \circ F_{T'}^{-1} DF_{T'}^{-1}$ for $T'' = G_T(T') \in \omega(T,\mathcal{T})$ and hence

$$\begin{aligned} (DG_T)^{-1} \circ G_T^{-1}|_{T''} &= (DF_{T'}^{-1})^{-1} \circ G_T^{-1}|_{T''} (DF_{T''})^{-1} \circ F_{T'}^{-1} \circ G_T^{-1}|_{T''} \\ &= (DF_{T'}^{-1})^{-1} \circ G_T^{-1}|_{T''} (DF_{T''})^{-1} \circ F_{T''}^{-1} \\ &= (DF_{T'}^{-1})^{-1} \circ G_T^{-1}|_{T''} (DF_{T''}^{-1})|_{T''}. \end{aligned}$$

This shows that

$$(W \circ G_T^{-1}(DG_T)^{-1} \circ G_T^{-1})|_{T''} = (V \circ G_T^{-1} DF_{T''}^{-1})|_{T''},$$

and hence $W \circ G_T^{-1}(DG_T)^{-1} \circ G_T^{-1} \in \mathcal{P}_\nabla^{p-1}(\omega(T,\mathcal{T}))$. Since this relation between the spaces $\mathcal{P}_\nabla^{p-1}(\omega(T,\mathcal{T}))$ and $\mathcal{P}_\nabla^{p-1}(\omega_{\rm ref})$ is bijective, we proved together with (3.3.7)

$$\begin{aligned}\min_{W\in\mathcal{P}_\nabla^{p-1}(\omega_{\rm ref})} &\|(\nabla w)\circ G_T^{-1} - W\circ G_T^{-1}\|_{L^2(\cup\omega(T,\mathcal{T}))}\\ &\lesssim |T|^{1/d} \min_{W\in\mathcal{P}_\nabla^{p-1}(\omega(T,\mathcal{T}))} \|\nabla v - W\|_{L^2(\cup\omega(T,\mathcal{T}))}.\end{aligned}$$

With (3.3.6), this shows (3.3.2c). It remains to show

$$\|\nabla(1-J(\omega_{\rm ref}))w\|_{L^2(G_T^{-1}(T))} \le C_{\rm sz} \min_{W\in\mathcal{P}_\nabla^{p-1}(\omega_{\rm ref})} \|\nabla w - W\|_{L^2(\cup\omega_{\rm ref})}, \tag{3.3.8}$$

for some constant $C_{\rm sz}>0$. We proceed by contradiction. Assume (3.3.8) is false for any constant $C_{\rm sz}>0$. Then, there exists a sequence $w_n \in H^1(\bigcup\omega_{\rm ref})$ with

$$\|\nabla(1-J(\omega_{\rm ref}))w_n\|_{L^2(G_T^{-1}(T))} > n \min_{W\in\mathcal{P}_\nabla^{p-1}(\omega_{\rm ref})} \|\nabla w_n - W\|_{L^2(\cup\omega_{\rm ref})}$$

for all $n\in\mathbb{N}$. Without loss of generality, we may assume $\|w_n\|_{H^1(\cup\omega_{\rm ref})}=1$ for all $n\in\mathbb{N}$. Let $\mathbb{Q}\colon H^1(\bigcup\omega_{\rm ref}) \to \mathcal{S}^p(\omega_{\rm ref})$ denote the H^1-orthogonal projection. The sequence $v_n := (1-\mathbb{Q})w_n$ satisfies

$$\|\nabla(1-J(\omega_{\rm ref}))v_n\|_{L^2(G_T^{-1}(T))} > n \min_{W\in\mathcal{P}_\nabla^{p-1}(\omega_{\rm ref})} \|\nabla v_n - W\|_{L^2(\cup\omega_{\rm ref})}$$

for all $\ell\in\mathbb{N}$ since $J(\omega_{\rm ref})$ is a projection and hence $(1-J(\omega_{\rm ref})\mathbb{Q}w_n = 0$ on $G_T^{-1}(T)$ as well as $\nabla\mathbb{Q}w_n \in \mathcal{P}_\nabla^{p-1}(\omega_{\rm ref})$. The above together with the stability of $J(\omega_{\rm ref})$ imply

$$\min_{W\in\mathcal{P}_\nabla^{p-1}(\omega_{\rm ref})} \|\nabla v_n - W\|_{L^2(\cup\omega_{\rm ref})} \le C_{\rm sz}\|v_n\|_{H^1(\cup\omega_{\rm ref})}/n \le C_{\rm sz}/n, \tag{3.3.9}$$

and hence there exists a sequence $W_n \in \mathcal{P}_\nabla^{p-1}(\omega_{\rm ref})$ with

$$\lim_{n\to\infty} \|\nabla v_n - W_n\|_{L^2(\cup\omega_{\rm ref})} = 0 \tag{3.3.10}$$

and $\|W_n\|_{L^2(\cup\omega_{\rm ref})} \le C_{\rm sz}/n + 1$ for all $n\in\mathbb{N}$. Since $\mathcal{P}_\nabla^{p-1}(\omega_{\rm ref})$ is a finite dimensional space, we may extract a convergent subsequence $W_{n_k} \in \mathcal{P}_\nabla^{p-1}(\omega_{\rm ref})$ with $\lim_{k\to\infty} W_{n_k} = W_0 \in \mathcal{P}_\nabla^{p-1}(\omega_{\rm ref})$. By (3.3.10), there holds $\lim_{k\to\infty}\|\nabla v_{n_k} - W_0\|_{L^2(\cup\omega_{\rm ref})} = 0$. The boundedness $\|v_n\|_{H^1(\cup\omega_{\rm ref})} \le 1$ allows to extract another subsequence (also denoted with n_k) such that $v_{n_k} \rightharpoonup v_0 \in H^1(\cup\omega_{\rm ref})$ weakly and (by Rellich compactness) $\|v_{n_k} - v_0\|_{L^2(\cup\omega_{\rm ref})} \to 0$. This implies $\nabla v_{n_k} \to \nabla v_0 \in L^2(\cup\omega_{\rm ref})$ weakly, and by uniqueness of limits also $\nabla v_0 = W_0$. With (3.3.10), we obtain

$$\lim_{k\to\infty} \|v_{n_k} - v_0\|_{H^1(\cup\omega_{\rm ref})} = 0.$$

This implies $\|v_0\|_{H^1(\cup\omega_{\rm ref})} = 1$, and by definition of the $v_n = (1-\mathbb{Q})w_n$, also $\mathbb{Q}v_0 = 0$. On the other hand, $\nabla v_0 = W_0 \in \mathcal{P}_\nabla^{p-1}(\omega_{\rm ref})$. Therefore, Lemma 3.3.5 shows $v_0 \in \mathcal{S}^p(\omega_{\rm ref})$ and hence $(1-\mathbb{Q})v_0 = 0$. Altogether, we have $v_0 = 0$, which contradicts $\|v_0\|_{H^1(\cup\omega_{\rm ref})} = 1$. This concludes the proof. □

3.3.0.2. *Proof of Theorem 3.3.1.* For all triangulations $\mathcal{T} \in \mathbb{T}$ and all $T \in \mathcal{T}$, define $r_T := \min_{i=1,\dots,N} \min_{(r_i,\theta_i)\in T} r_i$, where r_i denotes the radius as defined in Theorem 3.3.1.

Lemma 3.3.8. *Let $\mathcal{S} \subseteq \mathcal{T}_\infty$ denote a set of elements which is shape regular in the sense $\gamma(\mathcal{S}) < \infty$ (where $\gamma(\cdot)$ is defined in Section 3.2.3), satisfies $|T|^{1/d} \le Cr_T$ for all $T \in \mathcal{T}$ with $r_T > 0$ and some $C > 0$, and $|T \cap T'| = 0$ for all $T, T' \in \mathcal{S}$. Given $\alpha > -d$, there holds for all $\mathcal{T} \in \mathbb{T}$*

$$\int_{\cup\{T\in\mathcal{S}\,:\,r_T>0\}} r_T^\alpha \le C_{10}.$$

The constant $C_{10} > 0$ depends only on C, $\gamma(\mathcal{S})$, α, N, d, and Ω.

Proof. With $B_i(a,b) := \{z \in \Omega \,:\, a \le |x_i - z| \le b\}$, there holds

$$\int_{\cup\{T\in\mathcal{S}\,:\,r_T>0\}} r_T^\alpha \le \sum_{i=1}^N \sum_{n=-\log_2(\mathrm{diam}(\Omega))-1}^\infty \int_{B_i(2^{-n},2^{-n+1})} r_T^\alpha$$
$$\lesssim \sum_{i=1}^N \sum_{n=-\log_2(\mathrm{diam}(\Omega))-1}^\infty 2^{-n\alpha} \sum_{\substack{T\in\mathcal{S}\\ T\cap B_i(2^{-n},2^{-n+1})\neq\emptyset}} |T|,$$

where the hidden constant is 1 in case of $\alpha \le 0$ and depends only on $\mathrm{diam}(\Omega)$ and α for $\alpha > 0$. For all $T \in \mathcal{S}$ with $T \cap B_i(2^{-n}, 2^{-n+1}) \neq \emptyset$ holds $|T|^{1/d} \le Cr_T \le C2^{-n+1}$. The shape regularity (3.2.5) shows $\mathrm{diam}(T) \le \gamma(\mathcal{S})C2^{-n+1}$ and hence $T \subseteq B_i(0, (1+\gamma(\mathcal{S})C)2^{-n+1})$. This and (3.2.1) imply

$$\sum_{\substack{T\in\mathcal{S}\\ T\cap B_i(2^{-n},2^{-n+1})\neq\emptyset}} |T| \le |B_i(0,(1+\gamma(\mathcal{S})C)2^{-n+1})| \simeq 2^{-dn}.$$

Altogether, this shows

$$\int_{\cup\{T\in\mathcal{S}\,:\,r_T>0\}} r_T^\alpha \lesssim \sum_{i=1}^N \sum_{n=-\log_2(\mathrm{diam}(\Omega))-1}^\infty 2^{-n(\alpha+d)} \lesssim N\big(\frac{1}{1-2^{\alpha+d}} + 2^{(|\log_2(\mathrm{diam}(\Omega))|+1)(\alpha+d)}\big).$$

This concludes the proof. ∎

Lemma 3.3.9. *Assume $\mathbb{T}$ and a corresponding refinement strategy $\mathbb{T}(\cdot,\cdot)$ in the sense of Section 3.2.1–3.2.4. Let u be given as in (3.3.1) and define $\gamma := \min_{i=1,\dots,N} \gamma_i/2 > 0$. Given $i = 1,\dots,N$, all triangulations $\mathcal{T} \in \mathbb{T}$ and all $T \in \mathcal{T}$ with $x_i \notin T$ satisfy*

$$\min_{V\in\mathcal{P}^{p-1}_\nabla(T)} \|\nabla u_i - V\|_{L^2(T)} \le C_{11}|T|^{p/d} r_T^{\gamma-p-1}\|1\|_{L^2(T)}.$$

The constant $C_{11} > 0$ depends only on $\mathrm{diam}(\Omega)$, p, the constants in the definition of u_i, the constants in Section 3.2.1–3.2.4, $\mathbb{T}$, as well as on $\|g_i\|_{W^{\infty,p+1}(T)}$ and $\|\chi_i\|_{W^{\infty,p+1}(T)}$.

Proof. The first step is to bound the derivative $D^{p+1}u_i$ on $T \in \mathcal{T}$ with $r_T > 0$. To that end, let $e_1, \dots, e_d \in \mathbb{R}^d$ denote the unit vectors. Moreover, given a point $z_0 = (r_i, \theta_i) \in T$, let $e_{r_i}, e_{\theta_i,1}, \dots, e_{\theta_i,d-1} \in \mathbb{R}^d$ denote the unit vectors associated with (r_i, θ_i) in the sense that $z_0 = x_i + r_i e_{r_i}$ and that the $e_{\theta_i,j}$ are orthogonal onto e_{r_i} and onto each other. Define

$u_{s,i}(r_i) := \log(r_i)^{\mu_i} r_i^{\gamma_i}$ and the operator norm $\|\cdot\|_{L(\otimes_{n=1}^p \mathbb{R}^d,\mathbb{R})}$ in the space of linear operators from $\otimes_{n=1}^p \mathbb{R}^d$ to $\mathbb{R}$. Then, there holds

$$\begin{aligned}
&\|D^{p+1} u_i(z_0)\|_{L(\otimes_{n=1}^{p+1}\mathbb{R}^d,\mathbb{R})} \\
&\quad\lesssim \max_{i=1,\dots,N} |c_i| \Big(\sum_{k=0}^{p+1} \|D^k (g_i \chi_i)(z_0)\|_{L(\otimes_{n=1}^{k}\mathbb{R}^d,\mathbb{R})} \Big) \Big(\sum_{k=0}^{p+1} \|D^k u_{s,i}(z_0)\|_{L(\otimes_{n=1}^{k}\mathbb{R}^d,\mathbb{R})} \Big),
\end{aligned}$$

where the hidden constant depends only on p. The derivatives $D^k(g_i\chi_i)$ are uniformly bounded on T by $\max_{i=0,\dots,N}(\|\chi_i\|_{W^{p+1,\infty}(\Omega)} + \|g_i\|_{W^{p+1,\infty}(\Omega)})$. Let $D_{v_1,\dots,v_m}$ denote the derivative matrix (tensor) with respect to the vectors $v_1, \dots, v_m \in \mathbb{R}^d$ and $m \le d$. Since $D_{e_{\theta_i,1},\dots,e_{\theta_i,d-1}} u_{s,i}(z_0) = 0$ by definition, change of basis shows for any matrix (tensor) norm $\|\cdot\|_F$ that

$$\begin{aligned}
\|D^k u_{s,i}(z_0)\|_{L(\otimes_{n=1}^{p}\mathbb{R}^d,\mathbb{R})} &\simeq \|D^k_{e_1,\dots,e_d} u_{s,i}(z_0)\|_{\mathrm{F}} \\
&\simeq \|D^k_{e_{r_i},\dots,e_{\theta_i,d-1}} u_{s,i}(z_0)\|_{\mathrm{F}} \simeq |\partial^k_{e_{r_i}} u_{s,i}(z_0)|,
\end{aligned}$$

where the hidden constants depend only on d and p. A straightforward computation shows

$$\partial^k_{e_{r_i}} u_{s,i} = r_i^{\gamma_i - k} \sum_{j=0}^{k} \alpha_{i,j,k} \log(r_i)^{\mu_i - j}$$

for some constants $\alpha_{i,j,k} \in \mathbb{R}$ which depend only on γ_i, μ_i, p, k. This shows

$$\begin{aligned}
\sum_{k=0}^{p+1} \|D^k u_{s,i}(z_0)\|_{L(\otimes_{n=1}^{k}\mathbb{R}^d,\mathbb{R})} &\lesssim \sum_{k=0}^{p+1} r_i^{\gamma_i - k} \sum_{j=0}^{k} |\alpha_{i,j,k} \log(r_i)^{\mu_i - j}| \\
&\lesssim \sum_{k=0}^{p+1} r_i^{\gamma - k} \sum_{j=0}^{k} |\alpha_{i,j,k} r_i^{\gamma_i - \gamma} \log(r_i)^{\mu_i - j}|.
\end{aligned}$$

For each $j = 0, \dots, p+1$, there holds

$$r_i^{\gamma_i - \gamma} |\log(r_i)|^{\mu_i - j} \le \max_{0 \le r \le \mathrm{diam}(\Omega)} r^{\gamma_i - \gamma} |\log(r)|^{\mu_i - j} < \infty,$$

since $\gamma_i - \gamma > 0$. Moreover, there holds for all $k = 0, \dots, p+1$

$$r_i^{\gamma - k} \lesssim r_i^{\gamma - p - 1},$$

where the hidden constant depends only on $\mathrm{diam}(\Omega)$. The above estimates imply

$$\sum_{k=0}^{p+1} \|D^k u_{s,i}(z_0)\|_{L(\otimes_{n=1}^{k}\mathbb{R}^d,\mathbb{R})} \lesssim r_i^{\gamma - p - 1}. \tag{3.3.11}$$

Altogether, for $i = 1, \dots, N$, we end up with

$$\|D^{p+1} u_i(z_0)\|_{L(\otimes_{n=1}^{p+1}\mathbb{R}^d,\mathbb{R})} \lesssim r_i^{\gamma - p - 1},$$

where the hidden constant depends only on $\mathrm{diam}(\Omega)$, p, the constants in the definition of u_i as well as on $\|g_i\|_{W^{\infty,p+1}(T)}$ and $\|\chi_i\|_{W^{\infty,p+1}(T)}$. A scaling argument and the Bramble-Hilbert lemma show

$$\begin{aligned}\min_{V\in\mathcal{P}^{p-1}_\nabla(T)}\|\nabla u_i - V\|_{L^2(T)} &\simeq |T|^{1/2}\min_{W\in\mathcal{P}^{p-1}(T_{\rm ref})^d}\|(\nabla(u_i\circ F_T)-W)DF_T^{-1}\circ F_T\|_{L^2(T_{\rm ref})}\\ &\lesssim |T|^{1/2-1/d}\min_{W\in\mathcal{P}^{p-1}(T_{\rm ref})^d}\|\nabla(u_i\circ F_T)-W\|_{L^2(T_{\rm ref})}\\ &\lesssim |T|^{1/2-1/d}\|D^{p+1}(u_i\circ F_T)\|_{L^2(T_{\rm ref})}\\ &\lesssim |T|^{p/d}\|D^{p+1}u_i\|_{L^2(T)}\lesssim |T|^{p/d}r_T^{\gamma-p-1}\|1\|_{L^2(T)}.\end{aligned}\tag{3.3.12}$$

This concludes the proof. □

Lemma 3.3.10. *Assume $\mathbb{T}$ and a corresponding refinement strategy $\mathbb{T}(\cdot,\cdot)$ in the sense of Section 3.2.1–3.2.4. Let u be defined as in (3.3.1). Given $i=1,\dots,N$, all triangulations $\mathcal{T}\in\mathbb{T}$ and all $T\in\mathcal{T}$ with $x_i\in T$ satisfy*

$$\min_{V\in\mathcal{P}^{p-1}_\nabla(T)}\|\nabla u_i - V\|_{L^2(T)}\leq C_{12}|T|^{(2\gamma+d-2)/(2d)}.$$

The constant $C_{12}>0$ depends only on $\operatorname{diam}(\Omega)$, the constants in the definition of u_i, the constants in Section 3.2.1–3.2.4, $\mathbb{T}$, as well as on $\|g_i\|_{W^{1,\infty}(T)}$ and $\|\chi_i\|_{W^{1,\infty}(T)}$.

Proof. With $\gamma:=\min_{i=1,\dots,N}\gamma_i/2$, there holds point wise in T

$$|\nabla u_i|\lesssim r_i^{\gamma-1},$$

where the hidden constant depends only on $\operatorname{diam}(\Omega)$, the constants in the definition of u_i as well as on $\|g_i\|_{W^{1,\infty}(T)}$ and $\|\chi_i\|_{W^{1,\infty}(T)}$. This implies

$$\|\nabla u_i\|^2_{L^2(T)}\lesssim\int_0^{\operatorname{diam}(T)} r_i^{d-1}r_i^{2\gamma-2}\,dr_i\lesssim\operatorname{diam}(T)^{2\gamma+d-2}\simeq|T|^{(2\gamma+d-2)/d}$$

and concludes the proof. □

Proposition 3.3.11. *Assume $\mathbb{T}$ and a corresponding refinement strategy $\mathbb{T}(\cdot,\cdot)$ in the sense of Section 3.2.1–3.2.7. Let u be given as in (3.3.1). Then, there exists $C_{13}>0$ such that all $\mathcal{T}\in\mathbb{T}$ and all $0<\varepsilon<1$, $p\in\mathbb{N}$ with*

$$C^{-1}|T|^{1/d}\leq\begin{cases}\varepsilon^{1/p}r_T^{1-\gamma/(2p)} & \text{for all } T\in\mathcal{T}\text{ with } r_T>0,\\ \min\{\varepsilon^{2/(2\gamma+d-2)},\varepsilon^{1/p}\} & \text{for all } T\in\mathcal{T}\text{ with } r_T=0,\end{cases}\tag{3.3.13}$$

for some constant $C>0$ satisfy

$$\operatorname{err}(\mathcal{T})\leq C_{13}\varepsilon.$$

The constant $C_{13}>0$ depends only on p, N, C, C_{11}, C_{12}, the constants in Section 3.2.1–3.2.7, $\mathbb{T}$, and on Ω.

Proof. The approximation result (3.3.2c) implies

$$\begin{aligned}\operatorname{err}(\mathcal{T})^2&\lesssim\min_{V\in\mathcal{P}^{p-1}_\nabla(\mathcal{T})}\|\nabla u - V\|^2_{L^2(\Omega)}\\ &\lesssim\sum_{T\in\mathcal{T}}\Big(\min_{V\in\mathcal{P}^{p-1}_\nabla(T)}\|\nabla u_0 - V\|^2_{L^2(T)}+\sum_{i=1}^N\min_{V\in\mathcal{P}^{p-1}_\nabla(T)}\|\nabla u_i - V\|^2_{L^2(T)}\Big).\end{aligned}$$

With Lemma 3.3.9–3.3.10 and (3.3.13), this shows

$$\operatorname{err}(\mathcal{T})^2 \lesssim \sum_{T\in\mathcal{T}} \min_{V\in\mathcal{P}^{p-1}_\nabla(T)} \|\nabla u_0 - V\|^2_{L^2(T)} + \varepsilon \sum_{\substack{T\in\mathcal{T}\\ r_T>0}} r_T^{\gamma/2-1}\|1\|^2_{L^2(T)} + \varepsilon \sum_{\substack{T\in\mathcal{T}\\ r_T=0}} 1.$$

Assumption (3.2.14) implies $|T|^{1/d} \lesssim r_T$ for all $T \in \mathcal{T}$ with $r_T > 0$. Hence, Lemma 3.3.8 shows

$$\sum_{\substack{T\in\mathcal{T}\\ r_T>0}} r_T^{\gamma/2-1}\|1\|^2_{L^2(T)} = \int_{\cup\{T\in\mathcal{T}\,:\,r_T>0\}} r_T^{\gamma/2-1}\,dx \lesssim C_{10}.$$

As in (3.3.12), one obtains

$$\min_{V\in\mathcal{S}^{p-1}(T)} \|\nabla u_0 - V\|^2_{L^2(T)} \lesssim |T|^{p/d}\|D^{p+1}u_0\|_{L^2(T)}.$$

Altogether, we obtain

$$\operatorname{err}(\mathcal{T})^2 \lesssim \varepsilon(C_{10} + |\{T\in\mathcal{T}\,:\, r_T = 0\}| + \|D^{p+1}u_0\|_{L^2(\Omega)}).$$

Lemma 3.2.1 bounds $|\{T\in\mathcal{T}\,:\, r_T = 0\}|$ and hence concludes the proof. □

Proposition 3.3.12. *Assume $\mathbb{T}$ and a corresponding refinement strategy $\mathbb{T}(\cdot,\cdot)$ in the sense of Section 3.2.1–3.2.7. Suppose u as defined in (3.3.1). Given $\varepsilon > 0$, $p \in \mathbb{N}$ and $\mathcal{T} \in \mathbb{T}$, there exists a triangulation $\widehat{\mathcal{T}} \in \mathbb{T}(\mathcal{T}, C_{14}\varepsilon^{-d/p})$ which satisfies (3.3.13). The constant $C_{14} \geq 1$ depends only on q_{con}, u, p, d, the constants in Section 3.2.1–3.2.7, $\mathbb{T}$, and Ω.*

Proof. Define $h_{\min} := \min\{\varepsilon^{2/\gamma}, \varepsilon^{1/p}\}$. In the following, we construct an almost minimal refinement of $\mathcal{T}$ such that all elements satisfy

$$|T|^{1/d} \leq \max\{h_{\min}, \varepsilon^{1/p} r_T^{1-\gamma/(2p)}\}. \tag{3.3.14}$$

Generate the triangulation $\widehat{\mathcal{T}} \in \mathbb{T}(\mathcal{T})$ with the following algorithm:

Algorithm 3.3.13. *Set $\widehat{\mathcal{T}}_0 = \mathcal{T}$ and $\ell = 0$*

(i) *Define $\widehat{\mathcal{M}}_\ell := \{T \in \widehat{\mathcal{T}}_\ell \,:\, T$ does not satisfy (3.3.14)$\}$.*

(ii) *If $\widehat{\mathcal{M}}_\ell = \emptyset$, set $\widehat{\mathcal{T}} = \widehat{\mathcal{T}}_\ell$ and stop, else goto (iii).*

(iii) *Define $\widehat{\mathcal{T}}_{\ell+1} := \mathbb{T}(\widehat{\mathcal{T}}_\ell, \widehat{\mathcal{M}}_\ell)$, $\ell = \ell + 1$, and goto (i).*

The algorithm stops after a finite number of steps, since $|T|^{1/d}$ is reduced by $q_{\mathrm{con}}^{1/d}$ with each refinement and eventually is smaller than $h_{\min}$. Hence (3.3.14) is satisfied for all elements $T \in \widehat{\mathcal{T}} = \widehat{\mathcal{T}}_\ell$ after a finite number of steps. If for some element $T \in \widehat{\mathcal{T}}$ holds $\varepsilon^{1/p} r_T^{1-\gamma/(2p)} \geq h_{\min}$, then (3.3.13) follows directly from (3.3.14). If there holds $\varepsilon^{1/p} r_T^{1-\gamma/(2p)} < h_{\min}$, then we obtain

$$r_T \leq r_{\max} := h_{\min}^{2p/(2p-\gamma)} \varepsilon^{-2/(2p-\gamma)} \tag{3.3.15}$$

and since $h_{\min} \leq \varepsilon^{2/\gamma}$, it follows

$$r_T^{\gamma/(2p)} \leq h_{\min}^{\gamma/(2p-\gamma)} \varepsilon^{-\gamma/(p(2p-\gamma))} \leq \varepsilon^{2/(2p-\gamma)-\gamma/(p(2p-\gamma))} = \varepsilon^{1/p} \quad \text{for all } T \in \mathcal{T}.$$

With (3.2.14), this implies

$$C_{\mathrm{shp}}^{-1}\gamma(\mathcal{T}_0)^{-1}C_6^{-1}|T|^{1/d} \leq r_T \leq \varepsilon^{1/p} r_T^{1-\gamma/(2p)} \qquad \text{for } r_T > 0,$$
$$|T|^{1/d} \leq h_{\min} \qquad \text{for } r_T = 0.$$

Since $2/\gamma \geq 2/(2\gamma+d-2)$ and $0<\varepsilon<1$, there holds $h_{\min} \leq \min\{\varepsilon^{2/(2\gamma+d-2)}, \varepsilon^{1/p}\}$. Thus, the above implies (3.3.13) with the constant $C := \max\{1, C_{\mathrm{shp}}\gamma(\mathcal{T}_0)C_6\}$.

It remains to count the elements of $\widehat{\mathcal{T}} \setminus \mathcal{T}$. To that end, recall $\widehat{\mathcal{T}} = \widehat{\mathcal{T}}_\ell$ and define the function $M\colon \mathcal{T}_\infty \to [0,\infty]$ by $M(T) := \max\{h_{\min}, \varepsilon^{1/p} r_T^{1-\gamma/(2p)}\}$ as well as

$$\mathcal{S}_j := \big\{T \in \bigcup_{k=0}^{\ell} \widehat{\mathcal{M}}_k \,:\, q_{\mathrm{con}}^{-j} < |T|/M(T)^d \leq q_{\mathrm{con}}^{-j-1}\big\}.$$

Note that $T \in \widehat{\mathcal{M}}_j$ implies $|T| > M(T)^d$ and hence $\bigcup_{j=0}^\infty \mathcal{S}_j = \bigcup_{j=0}^\ell \widehat{\mathcal{M}}_j$. Assume $T, T' \in \mathcal{S}_j$ with $|T \cap T'| > 0$. Without loss of generality assume $T' \subseteq T$, then (3.2.12) and $r_{T'} \geq r_T$ imply the contradiction $q_{\mathrm{con}}^{-j} < |T'|/M(T')^d \leq q_{\mathrm{con}}|T|/M(T)^d \leq q_{\mathrm{con}}^{-j}$. Hence $|T \cap T'| = 0$ for all $T, T' \in \mathcal{S}_j$. Given j, split $\mathcal{S}_j = \mathcal{M}_r \cup \mathcal{M}_h$ with

$$\mathcal{M}_r := \big\{T \in \mathcal{S}_j \,:\, \varepsilon^{1/p} r_T^{1-\gamma/(2p)} \geq h_{\min}\big\} \quad \text{and} \quad \mathcal{M}_h := \mathcal{S}_j \setminus \mathcal{M}_r.$$

Define the function $A : \Omega \to [0,\infty)$ by $A|_T := |T|$ for all $T \in \mathcal{M}_r$. Then, there holds with Lemma 3.3.8 and $|T|^{1/d} \lesssim r_T$ for all $T \in \mathcal{M}_r$ from (3.2.14) that

$$\begin{aligned} |\mathcal{M}_r| &\leq \int_{\cup \mathcal{M}_r} A^{-1} \leq \int_{\cup \mathcal{M}_r} M(T)^{-d} q_{\mathrm{con}}^j \\ &= q_{\mathrm{con}}^j \varepsilon^{-d/p} \int_{\cup\left\{T \in \widehat{\mathcal{S}}_j \,:\, r_T > 0\right\}} r_T^{d(\gamma/(2p)-1)} \leq C_{10} q_{\mathrm{con}}^j \varepsilon^{-d/p}. \end{aligned} \tag{3.3.16}$$

On the other hand, $T \in \mathcal{M}_h$ implies $|T| \geq q_{\mathrm{con}}^{-j} h_{\min}^d$. Together with (3.2.14), (3.3.15), and $B_i(b) := \big\{x \in \Omega \,:\, |x - x_i| \leq b\big\}$, this shows $T \subseteq B_i((1+C_6)r_{\max})$ for some $i \in \{1,\dots,N\}$ and hence

$$|\mathcal{M}_h| \leq q_{\mathrm{con}}^j \sum_{i=1}^N \frac{|B_i((1+C_6)r_{\max})|}{h_{\min}^d} \lesssim q_{\mathrm{con}}^j \sum_{i=1}^N \frac{r_{\max}^d}{h_{\min}^d} \lesssim q_{\mathrm{con}}^j h_{\min}^{2pd/(2p-\gamma)-d} \varepsilon^{-2d/(2p-\gamma)}.$$

Since $h_{\min} \leq \varepsilon^{1/p}$, we end up with

$$|\mathcal{M}_h| \lesssim q_{\mathrm{con}}^j h_{\min}^{\frac{d\gamma}{2p-\gamma}} \varepsilon^{\frac{2d}{2p-\gamma}} \leq q_{\mathrm{con}}^j \varepsilon^{\frac{d\gamma}{p(2p-\gamma)} + \frac{-2d}{2p-\gamma}} = q_{\mathrm{con}}^j \varepsilon^{\frac{-d(2p-\gamma)}{p(2p-\gamma)}} = q_{\mathrm{con}}^j \varepsilon^{-d/p}. \tag{3.3.17}$$

The combination of (3.3.16) and (3.3.17) shows

$$|\mathcal{S}_j| \lesssim q_{\mathrm{con}}^j \varepsilon^{-d/p} \quad \text{for all } j = 0, \dots, \ell. \tag{3.3.18}$$

The closure estimate (3.2.13) implies

$$|\widehat{\mathcal{T}} \setminus \mathcal{T}| \leq C_{\mathrm{closure}} \sum_{j=0}^{\ell-1} |\mathcal{M}_j| = \sum_{j=0}^{\infty} |\mathcal{S}_j| \lesssim \varepsilon^{-d/p} \sum_{j=0}^{\infty} q_{\mathrm{con}}^j.$$

The convergence of the geometric series concludes the proof. ☐

Proof of Theorem 3.3.1. Given $\varepsilon > 0$ and $p \in \mathbb{N}$, Proposition 3.3.12 provides a triangulation $\widehat{\mathcal{T}} \in \mathbb{T}(\mathcal{T}, \mathrm{floor}(C_{14}(\varepsilon/C_{13})^{-d/p}))$ such that (3.3.13) is satisfied for ε/C_{13}. Therefore, Proposition 3.3.11 concludes the proof. ☐

3.3.0.3. *Proof of uniform approximability.* Recall the uniform approximability constants for the error $C_{\rm approx}^{\rm err}(s)$ as well as for the data $C_{\rm approx}^{\rm data}(s)$ defined in Section 2.4.

Theorem 3.3.14. *With* $\mathrm{err}(\cdot) := \min_{V\in\mathcal{S}^p(\cdot)}\|u-V\|_{H^1(\Omega)}$ *and under the assumptions of Theorem 3.3.1, there holds* $C_{\rm approx}^{\rm err}(p/d) < \infty$.

Proof. Let $\mathcal{T} \in \mathbb{T}$. Given $N \in \mathbb{N}$, define $\varepsilon = N^{-p/d}C_7^{p/d}$. Theorem 3.3.1 provides a triangulation $\widehat{\mathcal{T}} \in \mathbb{T}(\mathcal{T}, N)$ with $\mathrm{err}(\widehat{\mathcal{T}}) \le \varepsilon$. Hence, there holds

$$(N+1)^{p/d}\mathrm{err}(\widehat{\mathcal{T}}) \le (N+1)^{p/d}\varepsilon \le (N+1)^{p/d}N^{-p/d}C_7^{p/d} \le 2C_7^{p/d}.$$

This concludes the proof. □

The following result is the analog of Theorem 3.3.14 for the approximability of the data.

Theorem 3.3.15. *Given* $f \in L^2(\Omega)$ *and* $\alpha \ge 0$, *define*

$$\mathrm{data}(\mathcal{T})^2 := \min_{V\in\mathcal{P}^p(\mathcal{T})}\sum_{T\in\mathcal{T}}|T|^{2\alpha/d}\|f-V\|_{L^2(\Omega)}^2.$$

Assume $f|_T \in H^p(T)$ *for all* $T \in \mathcal{T}_0$. *Let the refinement strategy* $\mathbb{T}(\cdot,\cdot)$ *satisfy the assumptions from Section 3.2.1–3.2.6. Then,* $C_{\rm approx}^{\rm data}((p+\alpha)/d) < \infty$.

Proof. Given $\varepsilon > 0$ and $\mathcal{T} \in \mathbb{T}$, generate the triangulation $\widehat{\mathcal{T}}$ in $\mathbb{T}(\mathcal{T})$ with the following algorithm:

Algorithm 3.3.16. *Set* $\widehat{\mathcal{T}}_0 = \mathcal{T}$ *and* $\ell = 0$

(i) *Define* $\widehat{\mathcal{M}}_\ell := \{T \in \widehat{\mathcal{T}}_\ell \,:\, |T|^{(\alpha+p)/d} > \varepsilon\}$.
(ii) *If* $\widehat{\mathcal{M}}_\ell = \emptyset$, *set* $\widehat{\mathcal{T}} = \widehat{\mathcal{T}}_\ell$ *and stop, else goto (iii).*
(iii) *Define* $\widehat{\mathcal{T}}_{\ell+1} := \mathbb{T}(\widehat{\mathcal{T}}_\ell, \widehat{\mathcal{M}}_\ell)$, $\ell = \ell + 1$, *and goto (i).*

By definition of Algorithm 3.3.16 and (3.2.12), the algorithm stops after finitely many steps, i.e., $\widehat{\mathcal{T}} = \widehat{\mathcal{T}}_\ell$. Define the sets

$$\mathcal{S}_j := \{T \in \bigcup_{k=0}^{\ell}\widehat{\mathcal{M}}_k \,:\, q_{\rm con}^{-j} < |T|/\varepsilon^{d/(\alpha+p)} \le q_{\rm con}^{-j-1}\}.$$

Assume $T, T' \in \mathcal{S}_j$ with $|T \cap T'| > 0$. Without loss of generality, there holds $T' \subseteq T$. The assumption (3.2.12) implies the contradiction

$$\varepsilon^{d/(\alpha+p)}q_{\rm con}^{-j} < |T'| \le q_{\rm con}|T| \le \varepsilon^{d/(\alpha+p)}q_{\rm con}^{-j}.$$

Hence $|T \cap T'| = 0$ for all $T, T' \in \mathcal{S}_j$. This implies immediately

$$|\mathcal{S}_j| \le |\Omega| q_{\rm con}^j \varepsilon^{-d/(\alpha+p)}.$$

With the closure estimate (3.2.13), this shows

$$\begin{aligned}|\widehat{\mathcal{T}} \setminus \mathcal{T}| &\le C_{\rm closure}\sum_{j=0}^{\ell}|\widehat{\mathcal{M}}_j| = C_{\rm closure}\sum_{j=0}^{\infty}|\mathcal{S}_j| \lesssim C_{\rm closure}\varepsilon^{-d/(\alpha+p)}\sum_{j=0}^{\infty}q_{\rm con}^j\\ &\le C_{\rm closure}(1-q_{\rm con})^{-1}\varepsilon^{-d/(\alpha+p)}.\end{aligned}$$

A scaling argument, (3.2.6), and the Bramble-Hilbert lemma show

$$\begin{aligned}\mathrm{data}(\widehat{\mathcal{T}})^2 &= \sum_{T\in\mathcal{T}} |T|^{2\alpha/d} \min_{V\in\mathcal{P}^p(\widehat{\mathcal{T}})} \|f-V\|_{L^2(T)}^2 \\ &\lesssim \sum_{T\in\widehat{\mathcal{T}}} |T|^{2(\alpha+p)/d}\|D^p f\|_{L^2(T)}^2 \leq \varepsilon^2 \sum_{T\in\mathcal{T}_0} \|D^p f\|_{L^2(T)}^2.\end{aligned} \tag{3.3.19}$$

Finally, given $N\in\mathbb{N}$, define $\varepsilon>0$ by $C_{\mathrm{closure}}(1-q_{\mathrm{con}})^{-1}\varepsilon^{-d/(\alpha+p)} = N$. Then, the above construction provides $\widehat{\mathcal{T}}\in\mathbb{T}(\mathcal{T},N)$ such that $\mathrm{data}(\widehat{\mathcal{T}})\lesssim\varepsilon = C^{-p/d}N^{-(\alpha+p)/d}$. This shows $C_{\mathrm{approx}}^{\mathrm{data}}((p+\alpha)/d)<\infty$. ☐

As a straightforward but important consequence, we obtain the following result.

COROLLARY 3.3.17. *Suppose that $\eta(\cdot)$ satisfies reliability* (2.4.1) *and efficiency* (2.4.2) *with* $\mathrm{err}(\mathcal{T}) := \min_{V\in\mathcal{S}^p(\mathcal{T})}\|u-V\|_{H^1(\Omega)}$ *and* $\mathrm{data}(\mathcal{T})^2 := \min_{V\in\mathcal{P}^p(\mathcal{T})}\sum_{T\in\mathcal{T}}|T|^{2\alpha/d}\|f-V\|_{L^2(\Omega)}^2$ *for some $\alpha\geq 0$. Then, under the assumptions of Theorem 3.3.1 and with $f|_T\in H^{\mathrm{ceil}(p-\alpha)}(T)\cap L^2(T)$ for all $T\in\mathcal{T}_0$, there holds $C_{\mathrm{approx}}(p/d)<\infty$ and hence* (T3).

PROOF. Theorem 3.3.14 and Theorem 3.3.15 show $C_{\mathrm{approx}}^{\mathrm{err}}(p/d)+C_{\mathrm{approx}}^{\mathrm{data}}(p/d)<\infty$. The quasi-monotonicity (2.4.3) holds by definition of $\mathrm{err}(\cdot)$ and $\mathrm{data}(\cdot)$ with $C_{\mathrm{mon}}=1$. Proposition 2.4.1 (i) implies $C_{\mathrm{approx}}(p/d)<\infty$ and hence (T3). This concludes the proof. ☐

3.4. Weighted error estimators

Under the general assumption in Section 3.2.1, this section assumes that the error estimator $\eta(\cdot)$ depends not only on the triangulation, but also on a certain weight function $h\in L^\infty(\Omega)$. We call the error estimator $\eta(\cdot,h)$ a weighted error estimator with weight h. In the applications below, we define for each $\mathcal{T}\in\mathbb{T}$ a certain natural weight function $h(\mathcal{T}):\Omega\to(0,\infty)$ for which we write $\eta(\mathcal{T}) := \eta(\mathcal{T},h(\mathcal{T}))$. This natural weight function must be continuous on $\Omega\setminus\bigcup_{T\in\mathcal{T}}\partial T$. Suppose that $\eta(\cdot,\cdot)$ satisfies the following homogeneity condition: There exist constants $0<r_+\leq r_-<\infty$ such that all $T\in\mathcal{T}\in\mathbb{T}$, and all $\alpha:\Omega\to[0,1]$ with $\alpha\in L^\infty(\Omega)$ satisfy

$$\min_{x\in T}|\alpha(x)|^{r_-}\,\eta_T(\mathcal{T},h) \leq \eta_T(\mathcal{T},\alpha h) \leq \max_{x\in T}|\alpha(x)|^{r_+}\,\eta_T(\mathcal{T},h). \tag{3.4.1}$$

Suppose stability in the following sense: All refinements $\widehat{\mathcal{T}}\in\mathbb{T}(\mathcal{T})$ of a triangulation $\mathcal{T}\in\mathbb{T}$ and all subsets $\mathcal{S}\subseteq\mathcal{T}$ with $\widehat{\mathcal{S}} := \big\{T\in\widehat{\mathcal{T}}\,:\,T\subseteq\bigcup\mathcal{S}\big\}$ satisfy

$$\Big|\Big(\sum_{T\in\widehat{\mathcal{S}}}\eta_T(\widehat{\mathcal{T}},h)^2\Big)^{1/2} - \Big(\sum_{T\in\mathcal{S}}\eta_T(\mathcal{T},h)^2\Big)^{1/2}\Big| \leq \widetilde{\varrho}(\mathcal{T},\widehat{\mathcal{T}}), \tag{3.4.2}$$

where $h:\Omega\to(0,\infty)$ is a weight function with $h|_T\leq h(\widehat{\mathcal{T}})|_T$ for all $T\in\mathcal{S}$ and $\widetilde{\varrho}(\cdot,\cdot):\mathbb{T}\times\mathbb{T}\to[0,\infty)$.

PROPOSITION 3.4.1. *Let the error estimator $\eta(\cdot)$ be a weighted error estimator which satisfies homogeneity* (3.4.1) *and stability* (3.4.2) *and define $\mathcal{S}(\mathcal{T},\widehat{\mathcal{T}}) := \big\{T\in\mathcal{T}\,:\,h(\widehat{\mathcal{T}})|_T\leq q_{\mathrm{con}}h(\mathcal{T})|_T\big\}$ for some $0<q_{\mathrm{con}}<1$. With $\widehat{\mathcal{S}}(\mathcal{T},\widehat{\mathcal{T}}) := \big\{T\in\widehat{\mathcal{T}}\,:\,T\subseteq\bigcup\mathcal{S}(\mathcal{T},\widehat{\mathcal{T}})\big\}$, $\rho_{\mathrm{red}}=(1+\delta)q_{\mathrm{con}}^{2r_+}$, and $\varrho(\mathcal{T},\widehat{\mathcal{T}}) := (1+\delta^{-1})^{1/2}\widetilde{\varrho}(\mathcal{T},\widehat{\mathcal{T}})$ for all $\delta>0$, this implies* (E1b). *If additionally $h(\widehat{\mathcal{T}})|_T = h(\mathcal{T})|_T$ for all $T\in\mathcal{T}\setminus\mathcal{S}(\mathcal{T},\widehat{\mathcal{T}})$. This implies even* (E1a).

PROOF. Let h be a weight function. The homogeneity (3.4.1) implies for some $T \in \mathcal{T}$ and

$$\widetilde{h} := \begin{cases} h|_T & \text{on } T, \\ 0 & \text{on } \Omega \setminus T \end{cases}$$

that

$$\begin{aligned} \eta_T(\mathcal{T}, h) &= \min_{x \in T} |\widetilde{h}(x)/h(x)|^{r_-}\, \eta_T(\mathcal{T}, h) \le \eta_T(\mathcal{T}, \widetilde{h}) \\ &\le \max_{x \in T} |\widetilde{h}(x)/h(x)|^{r_+}\, \eta_T(\mathcal{T}, h) = \eta_T(\mathcal{T}, h). \end{aligned}$$

Hence $\eta_T(\mathcal{T}, h)$ depends only on $h|_T$. With this, stability (E1a) follows from (3.4.2) with $\mathcal{S} := \mathcal{T} \setminus \mathcal{S}(\mathcal{T}, \widehat{\mathcal{T}})$ and $h := h(\mathcal{T})$, since $\eta_T(\widehat{\mathcal{T}}, h(\mathcal{T})) = \eta_T(\mathcal{T}, h(\widehat{\mathcal{T}}))$ for all $T \in \mathcal{S}$.

Reduction (E1b) follows with (3.4.1) and (3.4.2). For $\delta > 0$, there holds

$$\begin{aligned} \sum_{T \in \widehat{\mathcal{S}}(\mathcal{T}, \widehat{\mathcal{T}})} \eta_T(\widehat{\mathcal{T}})^2 &\le (1+\delta) \sum_{T \in \mathcal{S}(\mathcal{T}, \widehat{\mathcal{T}})} \eta_T(\mathcal{T}, h(\widehat{\mathcal{T}}))^2 + (1+\delta^{-1})\widetilde{\varrho}(\mathcal{T}, \widehat{\mathcal{T}})^2 \\ &\le (1+\delta) \sum_{T \in \mathcal{S}(\mathcal{T}, \widehat{\mathcal{T}})} \max_{x \in T} \frac{h(\widehat{\mathcal{T}})^{2r_+}(x)}{h(\mathcal{T})^{2r_+}(x)} \eta_T(\mathcal{T})^2 + (1+\delta^{-1})\widetilde{\varrho}(\mathcal{T}, \widehat{\mathcal{T}})^2 \\ &\le (1+\delta) q_{\rm con}^{2r_+} \sum_{T \in \mathcal{S}(\mathcal{T}, \widehat{\mathcal{T}})} \eta_T(\mathcal{T})^2 + (1+\delta^{-1})\widetilde{\varrho}(\mathcal{T}, \widehat{\mathcal{T}})^2. \end{aligned}$$

This concludes the proof. □

3.5. Example 1: Laplace problem with residual error estimator

This section applies the abstract analysis of the preceding sections to different discretizations of the Laplace problem. The examples are taken from conforming finite element methods (FEM) as well as the boundary element methods (BEM) for weakly-singular and hypersingular integral equations. More examples, e.g., non-conforming or mixed methods (with the error estimator from [**21**]), are found and discussed in [**24**]. A general review on error estimators for finite element methods is found in [**23**].

3.5.1. Conforming FEM. This section is based on [**24**, Section 5]. In the context of conforming FEM for symmetric operators, the convergence and quasi-optimality of the adaptive algorithm has finally been analyzed in the seminal works [**35, 78**]. In this section, we show that their results can be reproduced and even extended in the abstract framework developed.

Let $\Omega \subset \mathbb{R}^d$, $d \ge 2$, be a bounded Lipschitz domain with polyhedral boundary $\Gamma := \partial\Omega$. With given volume force $f \in L^2(\Omega)$, we consider the Poisson model problem

$$-\Delta u = f \quad \text{in } \Omega \quad \text{and} \quad u = 0 \quad \text{on } \Gamma. \tag{3.5.1}$$

For the weak formulation, let $\mathcal{X} := H^1_0(\Omega)$ denote the usual Sobolev space, with the equivalent H^1-norm $\|v\|_{H^1_0(\Omega)} := \|\nabla v\|_{L^2(\Omega)}$ associated with the scalar product

$$b(u, v) := \int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \quad \text{for all } v \in H^1_0(\Omega). \tag{3.5.2}$$

Then, the weak form of (3.5.1) admits a unique solution $u \in H_0^1(\Omega)$. Based on a triangulation $\mathcal{T}$ of Ω generated by bisection (Section 3.2.8), we use the conforming finite element spaces $\mathcal{S}_0^p(\mathcal{T}) := \mathcal{P}^p(\mathcal{T}) \cap H_0^1(\Omega)$ of fixed polynomial order $p \geq 1$. The discrete form

$$b(U(\mathcal{T}), V) = \int_\Omega fV\,dx \quad \text{for all } V \in \mathcal{S}_0^p(\mathcal{T}) \tag{3.5.3}$$

also admits a unique FE solution $U(\mathcal{T}) \in \mathcal{S}_0^p(\mathcal{T})$. Following **[35]**, we use the local weight function

$$h(\mathcal{T}) \in \mathcal{P}^0(\mathcal{T}) \quad \text{with} \quad h(\mathcal{T})|_T := |T|^{1/d}, \tag{3.5.4}$$

where $|T|$ denotes the volume of an element $T \in \mathcal{T}$. The standard residual error estimator consists of the local contributions for all $T \in \mathcal{T}$

$$\eta_T(\mathcal{T})^2 := h(\mathcal{T})|_T^2\, \|f + \Delta U(\mathcal{T})\|_{L^2(T)}^2 + h(\mathcal{T})|_T\, \|[\partial_n U(\mathcal{T})]\|_{L^2(\partial T \cap \Omega)}^2, \tag{3.5.5}$$

see, e.g., **[1, 82]** as well as **[35, 78]**.

Here, $[\partial_n(\cdot)]$ denotes the jump of the normal derivative over interior facets of $\mathcal{T}$. Hence, $\eta(\cdot)$ is a weighted error estimator in the sense of Section 3.4 (the proofs of (3.4.1) and (3.4.2) follow below).

Since the admissible triangulations $\mathcal{T} \in \mathbb{T}$ are uniformly shape regular (3.2.5), we note that $h(\mathcal{T})|_T \simeq \operatorname{diam}(T)$ with the Euclidean diameter $\operatorname{diam}(T)$. In particular, $\eta(\cdot)$ coincides, up to a multiplicative constant, with the usual definition found in textbooks, cf., e.g., **[1, 82]**. We refer to Section 5.2.2 for the proof that the choice of the weight function does not affect convergence and quasi-optimality of the adaptive algorithm.

Proposition 3.5.1. *The conforming discretization of the Poisson problem* (3.5.1) *with residual error estimator* (3.5.5) *and bisection as refinement strategy* $\mathbb{T}(\cdot,\cdot)$ *satisfies*

(i) *stability and reduction* (E1) *with* $\rho_{\rm red} = 2^{-1/d}$, $\mathcal{S}(\mathcal{T}, \widehat{\mathcal{T}}) := \mathcal{T} \setminus \widehat{\mathcal{T}}$ *as well as* $\widehat{\mathcal{S}}(\mathcal{T}, \widehat{\mathcal{T}}) := \widehat{\mathcal{T}} \setminus \mathcal{T}$, *and* $\varrho(\mathcal{T}, \widehat{\mathcal{T}}) := C_{\rm pert}\|U(\mathcal{T}) - U(\widehat{\mathcal{T}})\|_{H_0^1(\Omega)}$,
(ii) *general quasi-orthogonality* (E2) *with* $\varepsilon_{\rm qo} = 0$,
(iii) *discrete reliability* (E3) *with* $\mathcal{R}(\mathcal{T}, \widehat{\mathcal{T}}) = \mathcal{T} \setminus \widehat{\mathcal{T}}$, $\kappa_{\rm dlr} = \infty$, *and* $\varepsilon_{\rm drel} = 0$,
(iv) *the refinement axioms* (T1)–(T3) *with* $C_{\rm approx}(s) \leq C_{\rm mon}(C_4 + 1)^s \|\eta, \mathbb{T}\|_s$ *for all* $s > 0$ *and the overlay estimate* (2.5.1).

Moreover, the estimator satisfies reliability and efficiency (2.4.1)–(2.4.2) *with* $\operatorname{err}(\mathcal{T}) := \|u - U(\mathcal{T})\|_{H_0^1(\Omega)}$ *and*

$$\operatorname{data}(\mathcal{T}) := \min_{F \in \mathcal{P}^{p-1}(\mathcal{T})} \|h(\mathcal{T})\,(f - F)\|_{L^2(\Omega)}, \tag{3.5.6}$$

where $C_{\rm approx}^{\rm data}(p/d) < \infty$ *(defined in Section 2.4) is guaranteed if* $f|_T \in H^{p-1}(T)$ *for all* $T \in \mathcal{T}_0$. *The constants* $C_{\rm drel}, C_{\rm qo}, C_{\rm pert}, C_{\rm eff}, C_{\rm rel}$ *depend only on the polynomial degree* $p \in \mathbb{N}$, $\mathcal{T}_0$, *and on* Ω.

Proof. Stability (E1a) as well as reduction (E1b) are part of the proof of **[35**, Corollary 3.4**]**. The discrete reliability (E3) is found in **[35**, Lemma 3.6**]** with $\varepsilon_{\rm drel} = 0$ and $\kappa_{\rm dlr} = \infty$. Since $\varrho(\mathcal{T}, \widehat{\mathcal{T}})$ is a Hilbert norm and the Galerkin orthogonality (2.7.3) is satisfied by definition, Lemma 2.7.2 implies (E2) with $\varepsilon_{\rm qo} = 0$ and $C_{\rm qo} = C_{\rm drel}$. Lemma 3.2.3 shows (T1)–(T2) & (2.5.1), (2.7.7). Lemma 2.7.5 shows quasi-monotonicity (2.7.6). Hence, Lemma 2.7.4 proves (iv).

The bounds (2.4.1)–(2.4.2) are well exposed in text books on a posteriori error estimation, see, e.g., [**1, 82**]. Theorem 3.3.15 implies $C_{\mathrm{approx}}^{\mathrm{data}}(p/d) < \infty$ and hence concludes the proof. $\square$

Consequence 3.5.2. *Let $s > 0$ with $\|\eta, \mathbb{T}\|_s < \infty$. Then, the adaptive algorithm leads to convergence with optimal rate for the estimator $\eta(\cdot)$ in the sense of Theorem 2.3.3 and with optimal complexity in the sense of Theorem 2.5.1. Moreover, the error converges in the sense of Theorem 2.4.3 for $s = p/d$ if $f|_T \in H^{p-1}(T)$ for all $T \in \mathcal{T}_0$.* $\square$

Numerical examples for the 2D Laplacian with mixed Dirichlet-Neumann boundary conditions are found in [**51**] together with a detailed discussion of the implementation. Examples for 3D are found in [**35**].

3.5.2. Conforming FEM without bisection. A major drawback of the current results on adaptive finite element methods, is the restriction to bisection (Section 3.2.8) as a refinement strategy. This comes from the fact that other popular refinement strategies (i.e., red-green-blue refinement from Section 3.2.9) do not satisfy the overlay estimate (2.5.1), which is a key ingredient in state of the art literature. However, the present abstract framework circumvents the use of (2.5.1) by using (T3) instead. The results from Section 3.3 allow to proof optimal convergence for refinement strategies in the sense of Section 3.2.1–3.2.7.

We consider the Poisson problem (3.5.1) on a polygonal domain $\Omega \subseteq \mathbb{R}^2$. The following result from [**55**, Section 2] proves that Theorem 3.3.14 is applicable.

Proposition 3.5.3. *Given $p \in \mathbb{N}$, let $f \in H^{p-1+\varepsilon}(\Omega)$ for some $\varepsilon > 0$ if $p > 1$ and $f \in L^2(\Omega)$ for $p = 1$. Then, the solution $u \in H^1(\Omega)$ of* (3.5.1) *allows for the decomposition* (3.3.1). $\square$

We suppose that $\mathbb{T}(\cdot,\cdot)$ is a refinement strategy which satisfies the assumptions from Section 3.2.1–3.2.7. Additionally to the bisection strategy which was treated in Section 3.5.1, this particularly includes the red-green-blue refinement from Section 3.2.9.

Proposition 3.5.4. *Let $\mathbb{T}(\cdot,\cdot)$ denote an arbitrary refinement strategy in the sense of Section 3.2.1–3.2.7. The conforming discretization of the Poisson problem* (3.5.1) *with residual error estimator* (3.5.5) *satisfies*

(i) *stability and reduction* (E1) *with $\rho_{\mathrm{red}} = 2^{-1/d}$, $\mathcal{S}(\mathcal{T}, \widehat{\mathcal{T}}) := \mathcal{T} \setminus \widehat{\mathcal{T}}$ as well as $\widehat{\mathcal{S}}(\mathcal{T}, \widehat{\mathcal{T}}) := \widehat{\mathcal{T}} \setminus \mathcal{T}$, and $\varrho(\mathcal{T}, \widehat{\mathcal{T}}) := C_{\mathrm{pert}}\|U(\mathcal{T}) - U(\widehat{\mathcal{T}})\|_{H_0^1(\Omega)}$,*
(ii) *general quasi-orthogonality* (E2) *with $\varepsilon_{\mathrm{qo}} = 0$,*
(iii) *discrete reliability* (E3) *with $\mathcal{R}(\mathcal{T}, \widehat{\mathcal{T}}) = \mathcal{T} \setminus \widehat{\mathcal{T}}$, $\kappa_{\mathrm{dlr}} = \infty$, and $\varepsilon_{\mathrm{drel}} = 0$,*
(iv) *the refinement axioms* (T1)–(T3) *with $C_{\mathrm{approx}}(p/d) < \infty$ for all $p \in \mathbb{N}$ with*

$$f \in \begin{cases} H^{p-1+\varepsilon}(\Omega) \text{ for some } \varepsilon > 0 & p > 1, \\ L^2(\Omega) & p = 1. \end{cases} \tag{3.5.7}$$

Moreover, the estimator satisfies (2.4.1)–(2.4.2) *with $\mathrm{err}(\mathcal{T}) := \|u - U(\mathcal{T})\|_{H_0^1(\Omega)}$ and*

$$\mathrm{data}(\mathcal{T}) := \min_{F \in \mathcal{P}^{p-1}(\mathcal{T})} \|h(\mathcal{T})\,(f - F)\|_{L^2(\Omega)}, \tag{3.5.8}$$

where $C_{\mathrm{approx}}^{\mathrm{data}}(p/d) < \infty$ (defined in Section 2.4) is guaranteed if $p \in \mathbb{N}$ satisfies (3.5.7). *The constants C_{drel}, C_{qo}, C_{pert}, C_{eff}, C_{rel} depend only on the polynomial degree $p \in \mathbb{N}$, $\mathcal{T}_0$, and on Ω.*

Proof. The statements (i)–(iii) follow as in Proposition 3.5.1. The assumptions in Section 3.2.5–3.2.6 imply the axioms (T1)–(T2). Moreover, Proposition 3.5.3 shows that

Theorem 3.3.1 is applicable if (3.5.7) is satisfied. With the Céa lemma (5.4.5) below, we obtain even

$$\mathrm{err}(\mathcal{T}) \simeq \min_{V \in \mathcal{S}^p(\mathcal{T})} \|u - V\|_{H^1(\Omega)}$$

(note the lack of boundary conditions on the right-hand side) and hence Theorem 3.3.14 shows $C_{\mathrm{approx}}^{\mathrm{err}}(p/d) < \infty$. Under the same assumptions, Theorem 3.3.15 is applicable and shows that $C_{\mathrm{approx}}^{\mathrm{data}}(p/d) < \infty$. Moreover, Corollary 3.3.17 implies $C_{\mathrm{approx}}(p/d) < \infty$. This concludes the proof. □

Consequence 3.5.5. *Let $p \in \mathbb{N}$ satisfy (3.5.7). Then, the adaptive algorithm leads to convergence with optimal rate for the estimator $\eta(\cdot)$ in the sense of Theorem 2.3.3 for $s \le p/d$. Moreover, the error converges in the sense of Theorem 2.4.3 for $s = p/d$.* □

3.5.3. Conforming BEM for weakly-singular integral equation. In this section (which is based on [**24**, Section 5]), we consider the weighted-residual error estimator in the context of BEM for integral operators of order -1. Unlike FEM, the efficiency of this error estimator is still an open question in general and mathematically guaranteed only for particular situations [**3**] while typically observed throughout, see, e.g. [**20, 28, 33, 34**]. Nevertheless, the abstract framework of Chapter 2 provides the means to analyze convergence and quasi-optimality of the adaptive algorithm. Non-residual error estimators are proposed in [**30, 50**], which are numerically straightforward to implement but lack the necessary properties to prove optimality.

In a specific setting, optimal convergence of adaptive algorithms has independently first been proved by [**47, 80**] for lowest-order BEM. While the analysis of [**80**] covers general operators, but is restricted to smooth boundaries Γ, the analysis of [**47**] focuses on the Laplace equation only, but allows for polyhedral boundaries. In [**44**], these results are generalized to BEM with ansatz functions of arbitrary, but fixed polynomial order.

Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain with polyhedral boundary $\partial\Omega$ and $d = 2, 3$. Let $\Gamma \subseteq \partial\Omega$ be a relatively open subset which has a Lipschitz boundary itself. For given $f \in H^{1/2}(\Gamma) := \big\{\phi|_\Gamma \,:\, \phi \in H^1(\Omega)\big\}$, we consider the weakly-singular first-kind integral equation

$$\mathcal{V}u(x) = f(x) \quad \text{for } x \in \Gamma. \tag{3.5.9}$$

The sought solution satisfies $u \in \widetilde{H}^{-1/2}(\Gamma)$. The negative-order Sobolev space $\widetilde{H}^{-1/2}(\Gamma)$ is the dual space of $H^{1/2}(\Gamma)$ with respect to the extended $L^2(\Gamma)$-scalar product $\langle \cdot\,,\,\cdot\rangle_{L^2(\Gamma)}$. We refer to the monographs [**58, 62, 75**] for details and proofs of this as well as of the following facts on the functional analytic setting: With the fundamental solution of the Laplacian

$$G(z) := \begin{cases} -\frac{1}{2\pi}\log|z| & \text{for } d = 2, \\ +\frac{1}{4\pi}\frac{1}{|z|} & \text{for } d = 3, \end{cases} \tag{3.5.10}$$

the *simple-layer potential* reads

$$\mathcal{V}u(x) := \int_\Gamma G(x-y)u(y)\,d\Gamma(y) \quad \text{for } x \in \Gamma. \tag{3.5.11}$$

We note that $\mathcal{V}\colon H^{-1/2+s}(\Gamma) \to H^{1/2+s}(\Gamma)$ is a linear, continuous, and symmetric operator for all $-1/2 \le s \le 1/2$. For 2D, we assume $\mathrm{diam}(\Omega) < 1$ which can always be achieved by scaling. Then, $\mathcal{V}$ is also elliptic (see also Proposition 6.2.23, below) , i.e.,

$$b(u,v) := \langle \mathcal{V}u\,,\,v\rangle_{L^2(\Gamma)} \tag{3.5.12}$$

defines an equivalent scalar product on $\mathcal{X} := \widetilde{H}^{-1/2}(\Gamma)$. We equip $\widetilde{H}^{-1/2}(\Gamma)$ with the induced Hilbert space norm $\|v\|^2_{\widetilde{H}^{-1/2}(\Gamma)} := \langle \mathcal{V}v\,,\,v\rangle_{L^2(\Gamma)}$. According to the Hahn-Banach theorem, (3.5.9) is equivalent to the variational formulation

$$b(u,v) = \langle f\,,\,v\rangle_{L^2(\Gamma)} \quad \text{for all } v \in \widetilde{H}^{-1/2}(\Gamma). \tag{3.5.13}$$

It relies on the scalar product $b(\cdot,\cdot)$ and hence admits a unique solution $u \in \widetilde{H}^{-1/2}(\Gamma)$ of (3.5.13).

Let $\mathcal{T}$ be a regular triangulation of Γ, generated by bisection from Section 3.2.8 from some initial triangulation $\mathcal{T}_0$. We employ conforming boundary elements $\mathcal{P}^p(\mathcal{T}) \subset H^{-1/2}(\Gamma)$ of order $p \geq 0$. The discrete formulation

$$b(U(\mathcal{T}),V) = \langle f\,,\,V\rangle_{L^2(\Gamma)} \quad \text{for all } V \in \mathcal{P}^p(\mathcal{T})$$

admits a unique BE solution $U(\mathcal{T}) \in \mathcal{P}^p(\mathcal{T})$.

Under additional regularity of the data $f \in H^1(\Gamma)$, we consider the weighted-residual error estimator of **[20, 28, 33, 34]** with local contributions

$$\eta_T(\mathcal{T})^2 := h(\mathcal{T})|_T\,\|\nabla_\Gamma(f - \mathcal{V}U(\mathcal{T}))\|^2_{L^2(T)} \quad \text{for all } T \in \mathcal{T}. \tag{3.5.14}$$

Here, $\nabla_\Gamma(\cdot)$ denotes the surface gradient and $h(\mathcal{T}) \in \mathcal{P}^0(\mathcal{T})$ denotes the weight function defined by $h(\mathcal{T})|_T = |T|^{1/(d-1)}$ for all $T \in \mathcal{T}$ as Γ is a $(d-1)$-dimensional manifold. We note that the analysis of **[20, 28, 33, 34]** relies on a Poincaré-type estimate $\|R(\mathcal{T})\|_{H^{1/2}(\Gamma)} \lesssim \|h(\mathcal{T})^{1/2}\nabla_\Gamma R(\mathcal{T})\|_{L^2(\Gamma)}$ for the Galerkin residual $R(\mathcal{T}) = f - \mathcal{V}U(\mathcal{T})$ and requires shape-regularity of the triangulation $\mathcal{T}$ for $d = 3$, in particular, the fact that $h(\mathcal{T})|_T \simeq \operatorname{diam}(T)$.

Proposition 3.5.6. *The conforming discretization of the Poisson problem* (3.5.9) *with residual error estimator* (3.5.14) *satisfies*

- (i) *stability and reduction* (E1) *with* $\varrho(\mathcal{T},\widehat{\mathcal{T}}) := C_{\rm pert}\|U(\mathcal{T}) - U(\widehat{\mathcal{T}})\|_{\widetilde{H}^{-1/2}(\Gamma)}$, $\rho_{\rm red} = 2^{-1/(d-1)}$, *and* $\mathcal{S}(\mathcal{T},\widehat{\mathcal{T}}) := \mathcal{T}\setminus\widehat{\mathcal{T}}$ *as well as* $\widehat{\mathcal{S}}(\mathcal{T},\widehat{\mathcal{T}}) := \widehat{\mathcal{T}}\setminus\mathcal{T}$,
- (ii) *general quasi-orthogonality* (E2) *with* $\varepsilon_{\rm qo} = 0$,
- (iii) *discrete reliability* (E3) *with*

$$\mathcal{R}(\mathcal{T},\widehat{\mathcal{T}}) := \big\{T \in \mathcal{T} \,:\, \exists T' \in \mathcal{T}\backslash\widehat{\mathcal{T}} \quad T \cap T' \neq \emptyset\big\}, \tag{3.5.15}$$

  $\kappa_{\rm dlr} = \infty$, *and* $\varepsilon_{\rm drel} = 0$,
- (iv) *the refinement axioms* (T1)–(T3) *with* $C_{\rm approx}(s) \leq C_{\rm mon}(C_4 + 1)^s\|\eta,\mathbb{T}\|_s$ *for all* $s > 0$ *and the overlay estimate* (2.5.1).

Moreover, the estimator satisfies reliability (2.4.1) *with* $\operatorname{err}(\mathcal{T}) := \|u - U(\mathcal{T})\|_{\widetilde{H}^{-1/2}(\Gamma)}$. *The constants* $C_{\rm drel}, C_{\rm qo}, C_{\rm pert}, C_{\rm rel}$ *depend only on the polynomial degree* $p \in \mathbb{N}$, $\mathcal{T}_0$, *and on* Γ.

Proof. Reliability (2.4.1) is well-known in the literature (e.g. **[28, 33, 34]**). Stability (E1a) as well as reduction (E1b) are part of the proof of [**47**, Proposition 4.2] and also found in [**44**]. The proof essentially follows [**35**], but additionally relies on the novel inverse-type estimate

$$\|h(\mathcal{T})^{1/2}\nabla_\Gamma\mathcal{V}V\|_{L^2(\Gamma)} \lesssim \|V\|_{\widetilde{H}^{-1/2}(\Gamma)} \quad \text{for all } V \in \mathcal{P}^p(\mathcal{T}).$$

While the work [**47**] is concerned with the lowest-order case $p = 0$ only, we refer to [**2**, Corollary 2] for general $p \geq 0$ so that [**47**, Proposition 4.2] transfers to $p \geq 0$. Discrete reliability (E3) is proved in [**47**, Proposition 5.3] for $p = 0$, but the proof holds accordingly for arbitrary $p \geq 0$. Lemma 2.7.2 implies general quasi-orthogonality (E2) with

$\varepsilon_{\mathrm{qo}} = 0$. Lemma 3.2.3 shows (T1)–(T2) & (2.5.1), (2.7.7). Lemma 2.7.5 shows quasi-monotonicity (2.7.6). Hence, Lemma 2.7.4 proves (iv). □

Consequence 3.5.7. *Let $s > 0$ with $\|\eta, \mathbb{T}\|_s < \infty$. Then, the adaptive algorithm leads to convergence with optimal rate for the estimator $\eta(\cdot)$ in the sense of Theorem 2.3.3 and optimal complexity in the sense of Theorem 2.5.1.* □

Numerical examples that underline the above result can be found in [**33**].

The lower bound (2.4.2) for the weighted-residual error estimator (3.5.14) remains an open question. The only result available [**3**] is for $d = 2$, and it exploits the equivalence of (3.5.9) to some Dirichlet-Laplace problem: Assume $\Gamma = \partial\Omega$ and let

$$\mathcal{K}g(x) := \int_\Gamma \partial_{n(y)} G(x-y)\, g(y)\, dy \tag{3.5.16}$$

denote the double-layer potential $\mathcal{K}\colon H^{1/2+s}(\Gamma) \to H^{1/2+s}(\Gamma)$, for all $-1/2 \le s \le 1/2$. Then, the weakly-singular integral equation (3.5.17) for given Dirichlet data $g \in H^{1/2}(\Gamma)$ and $f := (\mathcal{K} + 1/2)g$ is an equivalent formulation of the Dirichlet-Laplace problem

$$-\Delta\phi = 0 \quad \text{in } \Omega \quad \text{and} \quad \phi = g \quad \text{on } \Gamma = \partial\Omega. \tag{3.5.17}$$

The density $u \in \widetilde{H}^{-1/2}(\Gamma)$, which is sought in (3.5.9), is the normal derivative $u = \partial_n\phi$ to the potential $\phi \in H^1(\Omega)$ of (3.5.17).

For this special situation and lowest-order elements $p = 0$, the lower bound (2.4.2) is proved in [**3**, Theorem 4].

Proposition 3.5.8. *We consider lowest-order BEM $p = 0$ for $d = 2$ and $\Gamma = \partial\Omega$. Let $\sigma > 2$ and $g \in H^\sigma(\partial\Omega) := \big\{\phi|_{\partial\Omega} : \phi \in H^{\sigma+1/2}(\Omega)\big\}$. For $f := (\mathcal{K} + 1/2)g$, the weighted-residual error estimator* (3.5.14) *satisfies* (2.4.1)–(2.4.2) *for some (in general non-computable)* data$(\cdot)$ *with $C_{\mathrm{approx}}^{\mathrm{data}}(3/2) < \infty$ (defined in Section 2.4).*

Proof. The statement (2.4.2) is found in [**3**, Theorem 4], where data$(\mathcal{T})$ is based on the regular part of the exact solution u. The definition [**3**, Definition 15] shows data$(\mathcal{T}) \lesssim \|h(\mathcal{T}))^{3/2+\varepsilon}\|_{L^\infty(\Gamma)}$ for $\mathcal{T} \in \mathbb{T}$ and some σ-dependent $\varepsilon > 0$. The same argumentation as in the proof of Theorem 3.3.15 shows $C_{\mathrm{approx}}^{\mathrm{data}}(3/2) < \infty$ and concludes the proof. □

For some smooth exact solution u, the generically optimal order of convergence is $\mathcal{O}(h^{3/2})$ for lowest-order elements $p = 0$, where h denotes the maximal element size. For quasi-uniform triangulations with N elements and 2D BEM, this corresponds to $\mathcal{O}(N^{-3/2})$ and hence $s = 3/2$. With the foregoing proposition and according to Theorem 2.4.3, the adaptive algorithm attains any possible convergence order $0 < s \le 3/2$ and the generically optimal rate is thus achieved.

Consequence 3.5.9. *Let $0 < s \le 3/2$ with $\|\eta, \mathbb{T}\|_s < \infty$. Under the assumptions of Proposition 3.5.8, the adaptive algorithm leads to the generically optimal rate for the error in the sense of Theorem 2.4.3.* □

Numerical examples that underline the above result can be found in [**3, 20, 28, 33, 34, 47**].

3.5.4. Conforming BEM for hyper-singular integral equation. In this section (which is based on [**24**, Section 5]), we consider adaptive BEM for hyper-singular integral equations, where the hyper-singular operator is of order +1. In this frame, convergence and quasi-optimality of the adaptive algorithm has first been proved in [**80**], while the necessary technical tools have independently been developed in [**2**]. While the analysis of [**80**] only covers the lowest-order case $p = 1$ and smooth boundaries, the recent work **[45]** generalizes this to BEM with ansatz functions of arbitrary, but fixed polynomial order $p \geq 1$ and polyhedral boundaries.

Throughout, we use the notation from Section 3.5.3. Additionally, we assume that $\Gamma \subseteq \partial\Omega$ is connected. We consider the hyper-singular integral equation

$$\mathcal{W}u(x) = f(x) \quad \text{for } x \in \Gamma, \tag{3.5.18}$$

where the hyper-singular integral operator formally reads

$$\mathcal{W}v(x) := \partial_{n(x)} \int_\Gamma \partial_{n(y)} G(x-y) v(y) \, d\Gamma(y). \tag{3.5.19}$$

By definition, there holds $\mathcal{W}g(x) = \partial_n \mathcal{K}g(x)$ if the double-layer potential $\mathcal{K}g(x)$ is considered as a function on Ω by evaluating (3.5.16) for $x \in \Omega$. Again, we refer to the monographs **[58, 62, 75]** for details and proofs of the following facts on the functional analytic setting: The hyper-singular integral operator $\mathcal{W}$ is symmetric as well as positive semi-definite and has a one-dimensional kernel which consists of the constant functions, i.e., $\mathcal{W}1 = 0$. To deal with this kernel and to obtain an elliptic formulation, we distinguish the cases $\Gamma \subsetneqq \partial\Omega$ and $\Gamma = \partial\Omega$.

3.5.4.1. *Screen problem* $\Gamma \subsetneqq \partial\Omega$. On the screen, the hyper-singular integral operator $\mathcal{W} : \widetilde{H}^{1/2+s}(\Gamma) \to H^{-1/2+s}(\Gamma)$ is a continuous mapping for all $-1/2 \leq s \leq 1/2$. Here, $\widetilde{H}^{1/2+s}(\Gamma) := \{v|_\Gamma \,:\, v \in H^{1/2+s}(\partial\Omega) \text{ with } \operatorname{supp}(v) \subseteq \overline{\Gamma}\}$ denotes the space of functions which can be extended by zero to the entire boundary, and $H^{-1/2+s}(\Gamma)$ denotes the dual space of $\widetilde{H}^{1/2-s}(\Gamma)$. For given $f \in H^{-1/2}(\Gamma)$, we seek the solution $u \in \widetilde{H}^{1/2}(\Gamma)$ of (3.5.18).

We note that $1 \notin \widetilde{H}^{1/2}(\Gamma)$ and $\mathcal{W} : \widetilde{H}^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is a symmetric and elliptic operator. In particular,

$$b(u,v) := \langle \mathcal{W}u \,,\, v\rangle_{L^2(\Gamma)} \tag{3.5.20}$$

defines an equivalent scalar product on $\mathcal{X} := \widetilde{H}^{1/2}(\Gamma)$. We equip $\widetilde{H}^{1/2}(\Gamma)$ with the induced Hilbert space norm $\|v\|^2_{\widetilde{H}^{1/2}(\Gamma)} := b(v,v)$. The hyper-singular integral equation is thus equivalently stated as

$$b(u,v) = \langle f \,,\, v\rangle_{L^2(\Omega)} \quad \text{for all } v \in \widetilde{H}^{1/2}(\Gamma) \tag{3.5.21}$$

and admits a unique solution.

Given a regular triangulation $\mathcal{T}$ generated by bisection from Section 3.2.8 and a polynomial degree $p \geq 1$, we employ conforming boundary elements $\mathcal{S}^p_0(\mathcal{T}) := \mathcal{P}^p(\mathcal{T}) \cap \widetilde{H}^{1/2}(\Gamma)$. The discrete formulation

$$b(U(\mathcal{T}), V) = \langle f \,,\, V\rangle_{L^2(\Gamma)} \quad \text{for all } V \in \mathcal{S}^p_0(\mathcal{T})$$

admits a unique BE solution $U(\mathcal{T}) \in \mathcal{S}^p_0(\mathcal{T})$.

Under additional regularity of the data $f \in L^2(\Gamma)$, we may define the weighted-residual error estimator from **[20, 27, 33, 34]** with local contributions

$$\eta_T(\mathcal{T})^2 := h(\mathcal{T})|_T \|f - \mathcal{W}U(\mathcal{T})\|^2_{L^2(T)} \quad \text{for all } T \in \mathcal{T}. \tag{3.5.22}$$

As for the weakly-singular integral equation from Section 3.5.3, the lower bound (2.4.2) is only observed empirically **[20, 27, 33, 34]**, but a rigorous mathematical proof remains as an open question.

Proposition 3.5.10. *The conforming BEM discretization of the hyper-singular integral equation* (3.5.18) *on the screen with weighted-residual error estimator* (3.5.22) *satisfies*

(i) *stability and reduction* (E1) *with* $\varrho(\mathcal{T},\widehat{\mathcal{T}}) := C_{\rm pert}\|U(\mathcal{T}) - U(\widehat{\mathcal{T}})\|_{\widetilde{H}^{1/2}(\Gamma)}$, $\rho_{\rm red} = 2^{-1/(d-1)}$, *and* $\mathcal{S}(\mathcal{T},\widehat{\mathcal{T}}) := \mathcal{T}\setminus\widehat{\mathcal{T}}$ *as well as* $\widehat{\mathcal{S}}(\mathcal{T},\widehat{\mathcal{T}}) := \widehat{\mathcal{T}}\setminus\mathcal{T}$,
(ii) *general quasi-orthogonality* (E2) *with* $\varepsilon_{\rm qo} = 0$,
(iii) *discrete reliability* (E3) *with* $\mathcal{R}(\mathcal{T},\widehat{\mathcal{T}}) := \mathcal{T}\backslash\widehat{\mathcal{T}}$, $\kappa_{\rm dlr} = \infty$, *and* $\varepsilon_{\rm drel} = 0$,
(iv) *the refinement axioms* (T1)–(T3) *with* $C_{\rm approx}(s) \le C_{\rm mon}(C_4+1)^s\|\eta,\mathbb{T}\|_s$ *for all* $s > 0$ *and the overlay estimate* (2.5.1).

Moreover, the estimator satisfies reliability (2.4.1) *with* $\mathrm{err}(\mathcal{T}) := \|u - U(\mathcal{T})\|_{\widetilde{H}^{(1/2)(\Gamma)}}$. *The constants* $C_{\rm drel}, C_{\rm qo}, C_{\rm pert}, C_{\rm rel}$ *depend only on the polynomial degree* $p\in\mathbb{N}$, $\mathcal{T}_0$, *and on* Γ.

Proof. The reliability (2.4.1) is well-known in the literature (e.g. [**20, 27, 33, 34**]). The discrete reliability (E3) follows with the techniques from [**35**] which are combined with the localization techniques for the $H^{1/2}(\Gamma)$-norm from [**27**]. We refer to [**45**] for details. For the lowest-order case $p = 1$, an alternate proof is found in [**80**, Section 4], where $\mathcal{R}(\mathcal{T},\widehat{\mathcal{T}})$ are the refined elements $\mathcal{T}\backslash\widehat{\mathcal{T}}$ plus one additional layer of elements, see (3.5.15). Stability (E1a) and reduction (E1b) are proved in [**45**] and use the inverse estimate from [**2**, Corollary 2]. The remaining statements follow as in Proposition 3.5.6. □

Consequence 3.5.11. *Let* $s > 0$ *with* $\|\eta,\mathbb{T}\|_s < \infty$. *Then, the adaptive algorithm leads to convergence with optimal rate for the estimator* $\eta(\cdot)$ *in the sense of Theorem 2.3.3 and optimal complexity in the sense of Theorem 2.5.1.* □

Numerical examples that underline the above result can be found in [**33**].

3.5.4.2. *Laplace-Neumann problem* $\Gamma = \partial\Omega$. On the closed boundary $\Gamma = \partial\Omega$, the hyper-singular integral operator (3.5.19) is continuous for all $-1/2 \le s \le 1/2$

$$\mathcal{W}:\ H^{1/2+s}(\Gamma) \to H^{-1/2+s}(\Gamma).$$

Due to $1 \in H^{1/2}(\Gamma)$, we have to stabilize $\mathcal{W}$, e.g., with the rank-one operator $\mathcal{S}v := \langle v\,,\,1\rangle_{L^2(\Omega)}\,1$. Alternatively, it is possible to consider $\mathcal{W}$ on the factor space $H^{1/2}(\Gamma)/\mathbb{R} \simeq H^{1/2}_\star(\Gamma) := \big\{v\in H^{1/2}(\Gamma)\ :\ \int_\Gamma v\,ds = 0\big\}$. The (stabilized) hyper-singular integral equation reads

$$(\mathcal{W}+\mathcal{S})u(x) = f(x)\quad \text{for } x\in\Gamma. \tag{3.5.23}$$

The sought solution satisfies $u\in\mathcal{X} := H^{1/2}(\Gamma)$. The stabilization $\mathcal{S}$ allows to define an equivalent scalar product on $H^{1/2}(\Gamma)$ by

$$b(u,v) := \langle\mathcal{W}u\,,\,v\rangle_{L^2(\Gamma)} + \langle u\,,\,1\rangle_{L^2(\Gamma)}\langle v\,,\,1\rangle_{L^2(\Gamma)}.$$

We equip $H^{1/2}(\Gamma)$ with the induced Hilbert space norm $\|v\|^2_{H^{1/2}(\Gamma)} = b(v,v)$. Then, the equation (3.5.23) is equivalent to

$$b(u,v) = \langle f\,,\,v\rangle_{L^2(\Gamma)}\quad\text{for all } v\in H^{1/2}(\Gamma). \tag{3.5.24}$$

In case of $\langle f\,,\,1\rangle_{L^2(\Gamma)} = 0$, we see that $\langle u\,,\,1\rangle_{L^2(\Gamma)} = 0$ by choice of the test function $v = 1$. Then, the above formulation (3.5.23) resp. (3.5.24) is equivalent to (3.5.18).

For given $g\in H^{-1/2}(\Gamma)$ and the special right-hand side $f = (1/2 - \mathcal{K}')g$, it holds $\langle f,1\rangle_{L^2(\Gamma)} = 0$. Moreover, (3.5.18) resp. (3.5.23) is an equivalent formulation of the Laplace-Neumann problem

$$-\Delta\phi = 0\quad\text{in }\Omega\quad\text{and}\quad\partial_n\phi = g\quad\text{on }\Gamma=\partial\Omega. \tag{3.5.25}$$

Clearly, the solution $\phi \in H^1(\Omega)$ is only unique up to an additive constant. If we fix this constant by $\langle \phi \, , \, 1 \rangle_{L^2(\Gamma)} = 0$, the density $u \in H^{1/2}(\Gamma)$ which is sought in (3.5.18) for $f = (1/2 - \mathcal{K}')g$, is the trace $u = \phi|_\Gamma$ of the potential ϕ.

For fixed $p \geq 1$ and a regular triangulation $\mathcal{T}$ generated by bisection from Section 3.2.8 of Γ, we employ conforming boundary elements $\mathcal{S}^p(\mathcal{T}) := \mathcal{P}^p(\mathcal{T}) \cap H^{1/2}(\Gamma)$. The discrete formulation

$$b(U(\mathcal{T}), V) = \langle f \, , \, V \rangle_{L^2(\Gamma)} \quad \text{for all } V \in \mathcal{S}^p(\mathcal{T}) \tag{3.5.26}$$

admits a unique solution $U(\mathcal{T}) \in \mathcal{S}^p(\mathcal{T})$. In case of $\langle f \, , \, 1 \rangle_{L^2(\Gamma)} = 0$, it follows as for the continuous case that $\langle U(\mathcal{T}) \, , \, 1 \rangle_\Gamma = 0$ and therefore $\mathcal{S}U(\mathcal{T}) = 0$. Hence, the definition of the error estimator as well as the proof of the axioms (E1)–(E3), (T1)–(T3) is verbatim to the screen problem in Section 3.5.4.1 and therefore omitted.

Consequence 3.5.12. *Let $s > 0$ with $\|\eta, \mathbb{T}\|_s < \infty$. Then, the adaptive algorithm leads to convergence with optimal rate for the estimator $\eta(\cdot)$ in the sense of Theorem 2.3.3 and optimal complexity in the sense of Theorem 2.5.1.* □

Numerical examples that underline the above result can be found in [**27**].

Although one may expect a lower bound (2.4.2) similar to that from [**3**] for Symm's integral equation from Section 3.5.3, see Consequence 3.5.9, the details have not been worked out yet. In particular, quasi-optimality of the adaptive algorithm in the sense of Theorem 2.4.3 remains as an open question.

3.6. Example 2: General second-order elliptic equations

This section collects further fields of applications for the abstract theory developed in Chapter 2 beyond the Laplace model problem from Section 3.5. The results of Section 3.6.1 appear first in [**46**]. A first version of this section can be found in the recent own work [**24**, Section 6].

3.6.1. Conforming FEM for non-symmetric, elliptic linear problems. On the bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, we consider the following linear second-order PDE

$$\mathcal{L}u := -\mathrm{div}\boldsymbol{A}\nabla u + \boldsymbol{b} \cdot \nabla u + cu = f \quad \text{in } \Omega \quad \text{and} \quad u = 0 \quad \text{on } \Gamma. \tag{3.6.1}$$

For all $x \in \Omega$, $\boldsymbol{A}(x) \in \mathbb{R}^{d \times d}$ is a symmetric matrix with $\boldsymbol{A} \in W^{1,\infty}(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})$. Moreover, $\boldsymbol{b}(x) \in \mathbb{R}^d$ is a vector with $\boldsymbol{b} \in L^\infty(\Omega; \mathbb{R}^d)$ and $c(x) \in \mathbb{R}$ is a scalar with $c \in L^\infty(\Omega)$. Note that $\mathcal{L}$ is non-symmetric as

$$\mathcal{L} \neq \mathcal{L}^T = -\mathrm{div}\boldsymbol{A}\nabla u - \boldsymbol{b} \cdot \nabla u + (c - \mathrm{div}\boldsymbol{b})u.$$

We assume that the induced bilinear form

$$b(u \, , \, v) := \langle \mathcal{L}u \, , \, v \rangle = \int_\Omega \boldsymbol{A}\nabla u \cdot \nabla v + \boldsymbol{b} \cdot \nabla u v + cuv \, dx \quad \text{for } u, v \in \mathcal{X} := H^1_0(\Omega) \tag{3.6.2}$$

is continuous and $H^1_0(\Omega)$-elliptic and denote by $\|v\|^2 := b(v \, , \, v)$ the induced *quasi-norm* on $H^1_0(\Omega)$, which satisfies $\|\nabla(\cdot)\|_{L^2(\Omega)} \leq C_{\mathrm{norm}}\| \cdot \|$ for some $C_{\mathrm{norm}} > 0$. According to the Lax-Milgram lemma and for given $f \in L^2(\Omega)$, the weak formulation

$$b(u \, , \, v) = \int_\Omega fv \, dx \quad \text{for all } v \in H^1_0(\Omega) \tag{3.6.3}$$

admits a unique solution $u \in H^1_0(\Omega)$.

Historically, the convergence and quasi-optimality analysis for the adaptive algorithm has first been developed for elliptic and symmetric operators, e.g., [**40, 65, 14, 78, 35**] to name some milestones, and the analysis strongly used the fact that $\|v\|$ is a Hilbert norm and hence Lemma 2.7.2 applies. The work **[64]** introduced an appropriate quasi-orthogonality (2.7.5) in the H^1-norm to prove linear convergence of the so-called *total error* which is the weighted sum of error plus oscillations. Later, **[36]** used this approach to prove quasi-optimal convergence rates. However, **[64, 36]** are restricted to div $\boldsymbol{b} = 0$ and sufficiently fine initial triangulations $\mathcal{T}_0$ to prove this quasi-orthogonality. The recent work **[46]** removes these artificial assumption by proving the general quasi-orthogonality (E2) with respect to the induced energy *quasi-norm* $\|\cdot\|$. Moreover, the latter analysis also provides a framework for convergence and quasi-optimality if $b(\cdot\,,\,\cdot)$ is not uniformly elliptic, but only satisfies some Gårding inequality. For details, the reader is referred to Section 3.6.2

The discretization of (3.6.3) is done as in Section 3.5.1, from where we adopt the notation: For a given regular triangulation $\mathcal{T}$ generated by bisection from Section 3.2.8 and a polynomial degree $p \geq 1$, we consider $\mathcal{S}_0^p(\mathcal{T}) := \mathcal{P}^p(\mathcal{T}) \cap H_0^1(\Omega)$ with $\mathcal{P}^p(\mathcal{T})$. The discrete formulation also fits into the frame of the Lax-Milgram lemma and

$$b(U(\mathcal{T}), V) = \int_\Omega fV\,dx \quad \text{for all } V \in \mathcal{S}_0^p(\mathcal{T}) \tag{3.6.4}$$

hence admits a unique FE solution $U(\mathcal{T}) \in \mathcal{S}_0^p(\mathcal{T})$. Moreover, one has the Céa lemma

$$\|u - U(\mathcal{T})\| \leq C_{\text{Céa}} \min_{V \in \mathcal{S}_0^p(\mathcal{T})} \|u - V\| \quad \text{for all } \mathcal{T} \in \mathbb{T}, \tag{3.6.5}$$

where $C_{\text{Céa}} > 0$ depends only on $b(\cdot\,,\,\cdot)$.

The residual error-estimator $\eta(\cdot)$ differs slightly from the one in Section 3.5.1, namely

$$\eta_T(\mathcal{T})^2 := h(\mathcal{T})|_T^2 \|\mathcal{L}|_T U(\mathcal{T}) - f\|_{L^2(T)}^2 + h(\mathcal{T})|_T \|[\boldsymbol{A}\nabla U(\mathcal{T}) \cdot n]\|_{L^2(\partial T \cap \Omega)}^2 \tag{3.6.6}$$

for all $T \in \mathcal{T}$ and $\mathcal{L}|_T V := -\text{div}|_T \boldsymbol{A}(\nabla V) + \boldsymbol{b} \cdot \nabla V + cV$, see e.g. **[1, 82]**.

Proposition 3.6.1. *The conforming discretization of problem* (3.6.1) *with residual error estimator* (3.6.6) *satisfies*

(i) *stability and reduction* (E1) *with* $\rho_{\text{red}} = 2^{-1/d}$, $\varrho(\mathcal{T}, \widehat{\mathcal{T}}) := C_{\text{pert}}\|U(\mathcal{T}) - U(\widehat{\mathcal{T}})\|$, *and* $\mathcal{S}(\mathcal{T}, \widehat{\mathcal{T}}) := \mathcal{T} \setminus \widehat{\mathcal{T}}$ *as well as* $\widehat{\mathcal{S}}(\mathcal{T}, \widehat{\mathcal{T}}) := \widehat{\mathcal{T}} \setminus \mathcal{T}$,
(ii) *general quasi-orthogonality* (E2),
(iii) *discrete reliability* (E3) *with* $\mathcal{R}(\mathcal{T}, \widehat{\mathcal{T}}) = \mathcal{T} \setminus \widehat{\mathcal{T}}$, $\kappa_{\text{dlr}} = \infty$, *and* $\varepsilon_{\text{drel}} = 0$,
(iv) *the refinement axioms* (T1)–(T3) *with* $C_{\text{approx}}(s) \leq C_{\text{mon}}(C_4 + 1)^s \|\eta, \mathbb{T}\|_s$ *for all* $s > 0$ *and the overlay estimate* (2.5.1).

Moreover, the estimator satisfies reliability and efficiency (2.4.1)–(2.4.2) *with* $\text{err}(\mathcal{T}) := \|u - U(\mathcal{T})\|$ *and*

$$\begin{aligned} \text{data}(\mathcal{T})^2 := &\min_{F \in \mathcal{P}^q(\mathcal{T})} \sum_{T \in \mathcal{T}} h_T^2 \|\mathcal{L}|_T U(\mathcal{T}) - f - F\|_{L^2(T)}^2 \\ &+ \min_{F \in \mathcal{P}^{q'}(\mathcal{T})} \sum_{T \in \mathcal{T}} h_T \|[\boldsymbol{A}\nabla U(\mathcal{T}) \cdot n] - F\|_{L^2(\partial T \cap \Omega)}^2, \end{aligned} \tag{3.6.7}$$

where $q, q' \in \mathbb{N}_0$ *are arbitrary. If the differential operator* $\mathcal{L}$ *has piecewise polynomial coefficients, sufficiently large* $q, q' \in \mathbb{N}_0$ *even provides* (2.4.2) *with*

$$\text{data}(\mathcal{T}) = \min_{F \in \mathcal{P}^{p-1}(\mathcal{T})} \|h(\mathcal{T})\,(f - F)\|_{L^2(\Omega)}. \tag{3.6.8}$$

In this case, there holds $C_{\text{approx}}^{\text{data}}(p/d) < \infty$ *(defined in Section 2.4) if* $f|_T \in H^{p-1}(T)$ *for all* $T \in \mathcal{T}_0$. *The constants* $C_{\text{drel}}, C_{\text{qo}}, C_{\text{pert}}, C_{\text{eff}}, C_{\text{rel}}$ *depend only on the polynomial degrees* $p, q, q' \in \mathbb{N}$, $\mathcal{T}_0$, Ω, *and on* $\mathcal{L}$.

PROOF. The statements (i),(iii)–(iv) follow as for the Poisson model problem from Section 3.5.1. Standard arguments from, e.g., [**1, 82**] provide (2.4.1)–(2.4.2). The bound on $C_{\rm approx}^{\rm data}(p/d)$ follows as in Proposition 3.5.1. The general quasi-orthogonality (E2) is proved in Theorem 7.2.5. The solution of (3.6.4) with $\mathcal{X}_\infty := \overline{\bigcup_{\ell\in\mathbb{N}_0}\mathcal{S}_0^p(\mathcal{T}_\ell)}$ instead of $\mathcal{S}_0^p(\mathcal{T}_\ell)$ satisfies the assumptions of Lemma 2.7.1. Hence, (2.7.2) and Theorem 2.3.3 (i) prove $\lim_{\ell\to\infty}\eta(\mathcal{T}_\ell)=0$. Together with reliability (2.4.1), this implies $\lim_{\ell\to\infty}\|u-U(\mathcal{T}_\ell)\|=0$. Thus, all requirements of Theorem 7.2.5 are satisfied. This concludes the proof. □

CONSEQUENCE 3.6.2. *Let $s>0$ with $\|\eta,\mathbb{T}\|_s<\infty$. Then, the adaptive algorithm leads to convergence with optimal rate for the estimator $\eta(\cdot)$ in the sense of Theorem 2.3.3 and optimal complexity in the sense of Theorem 2.5.1. Moreover, the error converges in the sense of Theorem 2.4.3 at least for $s=1/d$. This is the optimal rate for lowest-order elements $p=1$. For piecewise polynomial coefficients of $\mathcal{L}$ and $f|_T\in H^{p-1}(T)$ for all $T\in\mathcal{T}_0$, one obtains even $s=p/d$.* □

Numerical examples for the symmetric case that underline the above result can be found in [**64**].

3.6.2. Conforming FEM for non-symmetric problems which satisfy a Gårding inequality. We consider the setting of Section 3.6.1 with the difference that the bilinear form $b(\cdot\,,\,\cdot)$ from (3.6.2) satisfies only the Gårding inequality

$$b(u,u)+C_{\rm grd}\|u\|_{L^2(\Omega)}^2\ge q_{\rm grd}\|\nabla u\|_{L^2(\Omega)}^2\quad\text{for all } u\in H^1(\Omega) \tag{3.6.9}$$

with constants $C_{\rm grd},q_{\rm grd}>0$. Suppose that $\mathbb{T}(\cdot,\cdot)$ denotes bisection from Section 3.2.8. We have to assume that $b(\cdot,\cdot)$ is definite on the continuous level, i.e., for all $v\in H_0^1(\Omega)$, it holds

$$b(v,w)=0\quad\text{for all } w\in H_0^1(\Omega)\qquad\Longrightarrow\qquad v=0. \tag{3.6.10}$$

This together with Fredholm's alternative guarantees the unique solvability of (3.6.3) and implies a continuous inf-sup condition, i.e.,

$$\inf_{v\in H_0^1(\Omega)\setminus\{0\}}\ \sup_{w\in H_0^1(\Omega)\setminus\{0\}}\frac{b(v,w)}{\|\nabla v\|_{L^2(\Omega)}\|\nabla w\|_{L^2(\Omega)}}\ge\delta>0. \tag{3.6.11}$$

To account for the fact that not each triangulation $\mathcal{T}\in\mathbb{T}$ allows for a solution of (3.6.4) and hence for an error estimator, we set $\eta(\mathcal{T}):=1$ if (3.6.4) is not uniquely solvable. With this, $\|\eta,\mathbb{T}\|_s$ is well-defined.

We propose a modified adaptive algorithm to solve this particular problem.

ALGORITHM 3.6.3. INPUT: *Initial triangulation $\mathcal{T}_0$, bulk parameter $0<\theta\le 1$, expected convergence rate $s>0$ with $\|\eta,\mathbb{T}\|_s<\infty$.*
Loop: *For $\ell=0,1,2,\dots$ do* (i) – (iii).

(i) *Try to solve* (3.6.4) *on $\mathcal{T}=\mathcal{T}_\ell$:*
  (i1) *If* (3.6.4) *is not uniquely solvable, set $\mathcal{T}_{\ell+1}=\mathbb{T}(\mathcal{T}_\ell,\mathcal{T}_\ell)$ and goto* (i).

(ii) *Compute $\eta_T(\mathcal{T}_\ell)$ for all $T\in\mathcal{T}_\ell$.*

(iii) *Determine set $\mathcal{M}_\ell\subseteq\mathcal{T}_\ell$ of (almost) minimal cardinality such that*

$$\theta\,\eta(\mathcal{T}_\ell)^2\le\sum_{T\in\mathcal{M}_\ell}\eta_T(\mathcal{T}_\ell)^2. \tag{3.6.12}$$

(iv) *Define the next triangulation as follows:*
  (i2) *If $\sum_{k=0}^{\ell-1}|\mathcal{M}_k|>(1+\log(\ell+1))\eta(\mathcal{T}_\ell)^{-1/s}$, set $\mathcal{T}_{\ell+1}:=\mathbb{T}(\mathcal{T}_\ell,\mathcal{T}_\ell)$.*
  (i3) *If not* (i2), *set $\mathcal{T}_{\ell+1}:=\mathbb{T}(\mathcal{T}_\ell,\mathcal{M}_\ell)$.*

OUTPUT: *Error estimators $\eta(\mathcal{T}_\ell)$ for all $\ell \in \mathbb{N}_0$.*

REMARK 3.6.4. *The algorithm requires the expected optimal rate of convergence $s > 0$ as an input parameter. This may be regarded as a drawback of the analysis. On the other hand, we do not assume any discrete inf-sup condition and Lemma 3.6.11 below shows that Algorithm 3.6.3 leads to convergence even for arbitrary $s > 0$.*

REMARK 3.6.5. *Case* (i1) *requires the algorithm to decide whether the linear system* (3.6.4) *is uniquely solvable. Due to finite dimension, this is equivalent to solvability. However, an iterative solver usually produces an approximation regardless of the solvability of the system. In this case, on may skip case* (i1) *and only check for case* (i2)–(i3). *The analysis and all the results from this section remain valid.*

LEMMA 3.6.6. *There exists a constant $C_{\rm rel} > 0$ such that all $\mathcal{T} \in \mathbb{T}$ for which* (3.6.4) *is uniquely solvable satisfy*

$$\|\nabla(u - U(\mathcal{T}))\|_{L^2(\Omega)} \leq C_{\rm rel}\eta(\mathcal{T}), \tag{3.6.13}$$

where $\eta(\cdot)$ is defined in (3.6.6).

PROOF. The reliability of $\eta(\cdot)$ is well-known and depends only on the continuous inf-sup condition (3.6.11), see also Proposition 3.5.1 for references. □

REMARK 3.6.7. *Due to Lemma 3.6.6, we may assume that $\eta(\mathcal{T}_\ell) > 0$ for all $\ell \in \mathbb{N}_0$, since otherwise $u = U(\mathcal{T}_\ell)$ and the adaptive algorithm converges with any rate by definition.*

PROPOSITION 3.6.8. *The conforming discretization of problem* (3.6.1) *with residual error estimator* (3.6.6) *satisfies under the assumptions of this section*

(i) *stability and reduction* (E1) *with $\varrho(\mathcal{T}, \widehat{\mathcal{T}}) := C_{\rm pert}\|\nabla(U(\mathcal{T}) - U(\widehat{\mathcal{T}}))\|_{L^2(\Omega)}$, $\rho_{\rm red} = 2^{-1/d}$, and $\mathcal{S}(\mathcal{T}, \widehat{\mathcal{T}}) := \mathcal{T} \setminus \widehat{\mathcal{T}}$ as well as $\widehat{\mathcal{S}}(\mathcal{T}, \widehat{\mathcal{T}}) := \widehat{\mathcal{T}} \setminus \mathcal{T}$ if* (3.6.4) *is uniquely solvable on $\mathcal{T}$ and $\widehat{\mathcal{T}}$,*

(ii) *the refinement axiom* (T1) *and the closure estimate* (3.2.13).

The constant $C_{\rm pert} > 0$ depends only on the polynomial degree $p \in \mathbb{N}$, Ω, and on $\mathcal{L}$.

PROOF. The proof of (i) in Proposition 3.6.1 (with $\|\cdot\| = \|\nabla(\cdot)\|_{L^2(\Omega)}$) is independent of the bilinear form and thus remains valid. Moreover, (T1) and (3.2.13) are proved in Lemma 3.2.3. □

LEMMA 3.6.9. *Let $\mathbb{T}' \subseteq \mathbb{T}$ denote a set of triangulations with the following property: Any sequence $(\mathcal{T}'_\ell)_{\ell\in\mathbb{N}_0} \subseteq \mathbb{T}'$ with $\mathcal{T}'_\ell \neq \mathcal{T}'_k$ for all $\ell \neq k$ satisfies $\lim_{\ell\to\infty} \|h(\mathcal{T}'_\ell)\|_{L^\infty(\Omega)} = 0$. Then, there exists $\varepsilon_0 > 0$ such that all but finitely many $\mathcal{T} \in \mathbb{T}'$ satisfy*

$$\inf_{V\in\mathcal{S}^p_0(\mathcal{T})\setminus\{0\}} \sup_{W\in\mathcal{S}^p_0(\mathcal{T})\setminus\{0\}} \frac{b(V,W)}{\|\nabla V\|_{L^2(\Omega)}\|\nabla W\|_{L^2(\Omega)}} \geq \varepsilon_0 \tag{3.6.14}$$

as well as the Céa Lemma

$$\|\nabla(u - U(\mathcal{T}))\|_{L^2(\Omega)} \leq C_{\rm Céa} \min_{V\in\mathcal{S}^p_0(\mathcal{T})} \|\nabla(u - V)\|_{L^2(\Omega)} \tag{3.6.15}$$

for some constant $C_{\rm Céa} > 0$.

PROOF. Assume that the statement (3.6.14) is wrong. Then, there exists a sequence of triangulations $\mathcal{T}'_\ell$ and corresponding $V_\ell \in \mathcal{S}^p_0(\mathcal{T}'_\ell)$ with $\|\nabla V_\ell\|_{L^2(\Omega)} = 1$ for all $\ell \in \mathbb{N}_0$ such that

$$\lim_{\ell\to\infty} \sup_{W\in\mathcal{S}_0^p(\mathcal{T}_\ell)\setminus\{0\}} \frac{|b(V_\ell, W)|}{\|\nabla W\|_{L^2(\Omega)}} = 0. \tag{3.6.16}$$

The boundedness implies the existence of a weak convergent subsequence $V_{\ell_k} \rightharpoonup V \in H_0^1(\Omega)$ where we assume without loss of generality that $\mathcal{T}_{\ell_k} \neq \mathcal{T}_{\ell_j}$ for all $k \neq j$.

By assumption, there holds $\lim_{\ell\to\infty} \|h(\mathcal{T}'_\ell)\|_{L^\infty(\Omega)} = 0$ and hence $\overline{\bigcup_{\ell\in\mathbb{N}_0} \mathcal{S}_0^p(\mathcal{T}'_\ell)} = H_0^1(\Omega)$. Let $w \in H_0^1(\Omega)$ and $\varepsilon > 0$. Then, the above guarantees some $W \in \mathcal{S}_0^p(\mathcal{T}'_\ell)$ such that

$$|b(V,w)| \le |b(V,W)| + |b(V,w-W)| \le |b(V,W)| + \varepsilon = \lim_{\ell\to\infty} |b(V_\ell, W)| + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, and with (3.6.16), this shows $b(V,w) = 0$ for all $w \in H_0^1(\Omega)$. Definiteness (3.6.10) then implies $V = 0$. On the other hand, the Gårding inequality shows

$$|b(V_{\ell_k}, V_{\ell_k})| + C_{\rm grd}\|V_{\ell_k}\|^2_{L^2(\Omega)} \ge q_{\rm grd} \quad \text{for all } k \in \mathbb{N}_0.$$

The Rellich compactness theorem implies $V_{\ell_k} \to 0$ in $L^2(\Omega)$. Hence, the above together with (3.6.16) shows the contradiction

$$0 = \lim_{k\to\infty} \big(|b(V_{\ell_k}, V_{\ell_k})| + C_{\rm grd}\|V_{\ell_k}\|^2_{L^2(\Omega)}\big) \ge q_{\rm grd}.$$

This concludes the proof of (3.6.14). The Céa lemma 3.6.15 follows by standard arguments. □

Lemma 3.6.10. *There exists $\ell_0 \in \mathbb{N}$ such that case* (i1) *in Algorithm 3.6.3 is not executed for any step $\ell \ge \ell_0$.*

Proof. Assume that case (i1) is executed in infinitely many steps $\ell \in \mathbb{N}_0$. Since case (i1) triggers a uniform refinement, this implies that $\lim_{\ell\to\infty} \|h(\mathcal{T}_\ell)\|_{L^\infty(\Omega)} = 0$. Lemma 3.6.9 with $\mathbb{T}' = \{\mathcal{T}_\ell : \ell \in \mathbb{N}_0\}$ shows that for all but finitely many $\mathcal{T}_\ell$ there holds (3.6.14). This implies that (3.6.4) is uniquely solvable for all $\mathcal{T} = \mathcal{T}_\ell$ and $\ell \ge k$ for some $k \in \mathbb{N}_0$ and contradicts the assumption that case (i1) is executed in infinitely many steps $\ell \in \mathbb{N}_0$. □

Lemma 3.6.11. *Algorithm 3.6.3 guarantees convergence of estimator and error, i.e.* $\lim_{\ell\to\infty} \eta(\mathcal{T}_\ell) = 0 = \lim_{\ell\to\infty} \|\nabla(u - U(\mathcal{T}_\ell))\|_{L^2(\Omega)}$.

Proof. First, we prove convergence

$$\|\nabla(u - U(\mathcal{T}_\ell))\|_{L^2(\Omega)} \to 0 \quad \text{as } \ell \to \infty. \tag{3.6.17}$$

To that end, we distinguish two cases. First, assume that case (i2) is executed for infinitely many steps $\ell \ge \ell_0$. Then, since case (i2) triggers uniform refinement, it holds $\lim_{\ell\to\infty} \|h(\mathcal{T}_\ell)\|_{L^\infty(\Omega)} = 0$. Lemma 3.6.9 with $\mathbb{T}' = \{\mathcal{T}_\ell : \ell \in \mathbb{N}_0\}$ provides some $k \in \mathbb{N}_0$ such that the Céa lemma (3.6.15) holds for all $\mathcal{T}_\ell$ with $\ell \ge k$. The fact $u \in H_0^1(\Omega) = \overline{\bigcup_{\ell=0}^\infty \mathcal{S}_0^p(\mathcal{T}_\ell)}$ implies $\min_{V\in\mathcal{S}_0^p(\mathcal{T}_\ell)} \|\nabla(u - V)\|_{L^2(\Omega)} \to 0$ as $\ell \to \infty$ and particularly (3.6.17).

Second, assume that case (i2) is not executed after some $k \ge \ell_0$. Then, by definition, there holds

$$\sum_{k=0}^{\ell-1} |\mathcal{M}_k| \le (1 + \log(\ell+1))\eta(\mathcal{T}_\ell)^{-1/s} \quad \text{for all } \ell \ge k. \tag{3.6.18}$$

Since $|\mathcal{M}_k| \ge 1$, this implies

$$\eta(\mathcal{T}_\ell) \le \Big(\frac{(1 + \log(\ell+1))}{\ell}\Big)^s \to 0 \quad \text{as } \ell \to \infty.$$

With (3.6.13), this shows (3.6.17). It remains to show $\lim_{\ell\to\infty}\eta(\mathcal{T}_\ell) = 0$ in the case that case (i2) is executed infinitely many times. To that end, recall that Proposition 3.6.8 shows (E1). Convergence (3.6.17) and Lemma 3.6.10 show $\lim_{\ell\to\infty}\varrho(\mathcal{T}_\ell, \mathcal{T}_{\ell+1}) = 0$ and since Dörfler marking (3.6.12) is satisfied for each step, Lemma 2.3.6 implies $\lim_{\ell\to\infty}\eta(\mathcal{T}_\ell) = 0$. This concludes the proof. □

Lemma 3.6.12. *Assume that there holds* $\lim_{\ell\to\infty}\|h(\mathcal{T}_\ell)\|_{L^\infty(\Omega)} = 0$. *Then,* $\|\eta, \mathbb{T}\|_s < \infty$ *for some* $s > 0$ *implies* (T3).

Proof. We mimic the proof of Lemma 2.7.4. Let $N \in \mathbb{N}_0$ and define the integer $M := \text{floor}(N/(2C_4^2))$. The fact $\|\eta, \mathbb{T}\|_s < \infty$ allows to choose some triangulation $\mathcal{T}_0^N \in \mathbb{T}(M)$ with

$$\eta(\mathcal{T}_0^N)(M+1)^s \le \|\eta, \mathbb{T}\|_s.$$

If $\lim_{N\to\infty}\|h(\mathcal{T}_0^N)\|_{L^\infty(\Omega)} = 0$, set $\mathcal{T}^N := \mathcal{T}_0^N$. Otherwise, consider a sequence of uniformly refined triangulations $\mathcal{T}_\ell^{\text{unif}}$ with $\mathcal{T}_0^{\text{unif}} = \mathcal{T}_0$ and $\mathcal{T}_{\ell+1}^{\text{unif}} := \mathbb{T}(\mathcal{T}_\ell^{\text{unif}}, \mathcal{T}_\ell^{\text{unif}})$. Given $N \in \mathbb{N}_0$, define $\mathcal{T}^N := \mathcal{T}_0^N \oplus \mathcal{T}_\ell^{\text{unif}}$, where ℓ is maximal with $|\mathcal{T}_\ell^{\text{unif}} \setminus \mathcal{T}_0| \le N/(2C_4)$. The overlay estimate (2.5.1) shows

$$|\mathcal{T}^N \setminus \mathcal{T}_0| \le |\mathcal{T}^N \setminus \mathcal{T}_\ell^{\text{unif}}| + |\mathcal{T}_\ell^{\text{unif}} \setminus \mathcal{T}_0| \le C_4|\mathcal{T}_0^N \setminus \mathcal{T}_0| + N/(2C_4) \le N/C_4.$$

Moreover, there holds $\lim_{N\to\infty}\|h(\mathcal{T}^N)\|_{L^\infty(\Omega)} = 0$. Given any $\mathcal{T}_\ell$, $\ell \in \mathbb{N}_0$, the overlay estimate (2.5.1) states $|(\mathcal{T}^N \oplus \mathcal{T}_\ell) \setminus \mathcal{T}_\ell| \le N$ and hence $\mathcal{T}^N \oplus \mathcal{T}_\ell \in \mathbb{T}(\mathcal{T}_\ell, N)$. Lemma 3.6.9 with $\mathbb{T}' := \{\mathcal{T}_\ell : \ell \in \mathbb{N}_0\} \cup \{\mathcal{T}^N \oplus \mathcal{T}_\ell : \ell, N \in \mathbb{N}_0\}$ shows that (3.6.4) is uniquely solvable and the Céa lemma (3.6.15) holds for all but finitely many $\mathcal{T} \in \mathbb{T}'$. This, together with (3.6.13) and (E1) from Proposition 3.6.8, implies

$$\eta(\mathcal{T}^N \oplus \mathcal{T}_\ell) \lesssim \eta(\mathcal{T}^N) + \varrho(\mathcal{T}^N, \mathcal{T}^N \oplus \mathcal{T}_\ell) \lesssim \eta(\mathcal{T}^N) + \|\nabla(u - U(\mathcal{T}^N))\|_{L^2(\Omega)} \lesssim \eta(\mathcal{T}^N)$$

for all $N, \ell \ge k$ and some $k \in \mathbb{N}_0$. Consequently, there holds

$$\eta(\mathcal{T}^N \oplus \mathcal{T})(N+1)^s \lesssim \eta(\mathcal{T}^N)(M+1)^s \le \|\lambda, \mathbb{T}\|_s$$

and we obtain

$$\inf_{\widehat{\mathcal{T}} \in \mathbb{T}(\mathcal{T}_\ell, N)} (N+1)^s \eta(\widehat{\mathcal{T}}) \lesssim \|\lambda, \mathbb{T}\|_s.$$

This concludes the proof. □

Lemma 3.6.13. *There exists* $\ell_1 \in \mathbb{N}$ *such that case* (i2) *in Algorithm 3.6.3 is not executed for any step* $\ell \ge \ell_1$.

Proof. Assume that case (i2) is executed infinitely many times. Then, there holds $\overline{\bigcup_{\ell=0}^\infty \mathcal{S}_0^p(\mathcal{T}_\ell)} = H_0^1(\Omega)$ or equivalently $\lim_{\ell\to\infty}\|h(\mathcal{T}_\ell)\|_{L^\infty(\Omega)} = 0$. With this, Theorem 7.3.4 proves (E2) for all $\ell \ge \ell_0$.

Proposition 3.6.8 together with Lemma 3.6.10 and Lemma 3.6.12 prove (E1) and (T1)–(T3) for the parameter s chosen in Algorithm 3.6.3. Lemma 2.3.13 then shows that for all $\mathcal{T} = \mathcal{T}_\ell$, there exists $\widehat{\mathcal{T}}_\ell \in \mathbb{T}(\mathcal{T}_\ell)$ with (2.3.20). Moreover, Lemma 3.6.9 with $\mathbb{T}' := \{\mathcal{T}_\ell : \ell \in \mathbb{N}_0\} \cup \{\widehat{\mathcal{T}}_\ell : \ell \in \mathbb{N}_0\}$ implies the discrete inf-sup condition (3.6.14) for all $\mathcal{T}_\ell$ and $\widehat{\mathcal{T}}_\ell$ with $\ell \ge k$ for some $k \in \mathbb{N}_0$.

Therefore, the proof of discrete reliability (E3) of Proposition 3.6.1 remains valid for all $\mathcal{T} = \mathcal{T}_\ell$ and $\widehat{\mathcal{T}} = \widehat{\mathcal{T}}_\ell$, $\ell \geq k$ since (3.6.14) implies

$$\|\nabla(U(\mathcal{T}_\ell) - U(\widehat{\mathcal{T}}_\ell))\|_{L^2(\Omega)} \lesssim \sup_{W \in \mathcal{S}_0^p(\widehat{\mathcal{T}}_\ell)} \frac{b(U(\mathcal{T}_\ell) - U(\widehat{\mathcal{T}}_\ell)\,,\,W)}{\|\nabla W\|_{L^2(\Omega)}}.$$

The remaining proof of (E3) follows as in the references given in the proof of Proposition 3.6.1. With this, Proposition 2.3.10 (and Remark 2.3.11) shows the implication (4.2.2) for $\mathcal{T} = \mathcal{T}_\ell$ and $\widehat{\mathcal{T}} = \widehat{\mathcal{T}}_\ell$ for all $\ell \geq k$ and therefore (2.3.21) holds, too.

Since $\mathcal{M}_\ell$ satisfies Dörfler marking (3.6.12) for all $\ell \geq \ell_0$ with (almost) minimal cardinality, there holds $|\mathcal{M}_\ell| \lesssim |\mathcal{R}(\mathcal{T}_\ell, \widehat{\mathcal{T}}_\ell)|$ with the set $\mathcal{R}(\mathcal{T}_\ell, \widehat{\mathcal{T}}_\ell)$ from (2.3.21).

Theorem 2.3.3 (ii) implies R-linear convergence (2.3.2) for all $\ell \geq k$ and Lemma 2.3.8 shows

$$\sum_{k=\ell_0}^{\ell-1} \eta(\mathcal{T}_k)^{-1/s} \leq C_2 \eta(\mathcal{T}_\ell)^{-1/s}.$$

With this and (2.3.21), we obtain

$$\sum_{k=\ell_0}^{\ell-1} |\mathcal{M}_k| \lesssim \sum_{k=\ell_0}^{\ell-1} |\mathcal{R}(\mathcal{T}_k, \widehat{\mathcal{T}}_k)| \lesssim C_{\rm approx}(s) \sum_{k=\ell_0}^{\ell-1} \eta(\mathcal{T}_k)^{-1/s} \lesssim C_{\rm approx}(s) \eta(\mathcal{T}_\ell)^{-1/s}.$$

Since $C_{\rm approx}(s) < \infty$ by (T3), the above implies for all $\ell \geq \ell_0$ for which case (i2) is executed

$$(1 + \log(\ell+1)) \lesssim C_{\rm approx}(s)^{1/s}.$$

Hence, the number of steps $\ell \geq \ell_0$ for which case (i2) is executed, must be finite. This, however, contradicts the assumption and thus concludes the proof. $\square$

Theorem 3.6.14. *Given $\theta < \theta_\star = (1 - \varepsilon_{\rm drel})/(1 + C_{\rm drel}^2)$, Algorithm 3.6.3 converges with almost optimal rate $s - \varepsilon$ for all $\varepsilon > 0$ (where s is chosen in Algorithm 3.6.3 such that $\|\eta, \mathbb{T}\|_s < \infty$) in the sense*

$$c_{\rm opt} C_{\rm approx}(s - \varepsilon) \leq \sup_{\ell \in \mathbb{N}_0} \frac{\eta(\mathcal{T}_\ell)}{(|\mathcal{T}_\ell \setminus \mathcal{T}_0| + 1)^{-s+\varepsilon}} \leq C_{\rm opt}, \tag{3.6.19}$$

where $C_{\rm opt} > 0$ depends only on ε, ℓ_0, ℓ_1, $|\mathcal{T}_0|$, $C_{\rm closure}$ and $c_{\rm opt}$ is defined in Theorem 2.3.3.

Proof. Lemma 3.6.10 and Lemma 3.6.13 show that after step $k := \max\{\ell_0, \ell_1\}$ only case (i3) is executed. This particularly implies

$$\sum_{k=0}^{\ell-1} |\mathcal{M}_k| \leq (1 + \log(\ell+1)) \eta(\mathcal{T}_\ell)^{-1/s} \quad \text{for all } \ell > k.$$

The closure estimate (T2) and the fact that case (i1)–(i2) is executed only finitely many times show

$$|\mathcal{T}_\ell \setminus \mathcal{T}_0| + 1 \lesssim \sum_{k=0}^{\ell-1} |\mathcal{M}_k| + 1 \lesssim (1 + \log(\ell+1)) \eta(\mathcal{T}_\ell)^{-1/s} \quad \text{for all } \ell \in \mathbb{N}_0.$$

This implies

$$\eta(\mathcal{T}_\ell) \lesssim (1 + \log(\ell+1))^s (|\mathcal{T}_\ell \setminus \mathcal{T}_0| + 1)^{-s}.$$

Since $|\mathcal{T}_\ell \setminus \mathcal{T}_0| + 1 \geq \ell + 1$, and $\sup_{\ell \in \mathbb{N}_0} \log(\ell+1)^s (\ell+1)^{-\varepsilon} < \infty$, this implies the upper bound in (3.6.19). The lower bound follows as in the proof of Theorem 2.3.3 (iii). $\square$

Consequence 3.6.15. *Algorithm 3.6.3 converges with optimal rates in the sense of Theorem* (3.6.14).

3.7. Example 3: Conforming FEM for certain strongly-monotone operators

The result of this section is first found in [**46**]. A first version of this section can be found in the recent own work [**24**, Section 10]. We consider the following *non-linear* operator

$$\mathcal{L}u(x) := -\mathrm{div}\boldsymbol{A}(x, \nabla u(x)) + g(x, u(x), \nabla u(x)),$$

for functions $\boldsymbol{A} : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ and $g : \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$. We assume that $\boldsymbol{A}(\cdot, \nabla u)$, $g(\cdot, u, \nabla u) \in L^2(\Omega)$ for all $u \in H^1_0(\Omega)$. On the polyhedral domain $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$ and given $f \in L^2(\Omega)$, the weak formulation of

$$\begin{aligned} \mathcal{L}u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{3.7.1}$$

reads: Find $u \in H^1_0(\Omega)$ such that

$$\langle \mathcal{L}u \, , \, v \rangle = \int_\Omega \boldsymbol{A}(x, \nabla u(x)) \cdot \nabla v(x) + g(x, u(x), \nabla u(x)) v(x) \, dx = \int_\Omega f v \, dx \tag{3.7.2}$$

for all $v \in H^1_0(\Omega)$. Define two auxiliary operators $\mathcal{A}, \mathcal{C} : H^1_0(\Omega) \to H^{-1}(\Omega)$ as

$$\mathcal{A}v := -\mathrm{div}\boldsymbol{A}(\cdot, \nabla v) \quad \text{and} \quad \mathcal{C}v := g(\cdot, v, \nabla v) \quad \text{for all } v \in H^1_0(\Omega).$$

Let $\mathbb{T}(\cdot, \cdot)$ denote the bisection strategy from Section 3.2.8. Given $\mathcal{T} \in \mathbb{T}$ and $p \in \mathbb{N}$, the discrete form of (3.7.2) reads: Find $U(\mathcal{T}) \in \mathcal{S}^p_0(\mathcal{T})$ such that

$$\langle \mathcal{L}U(\mathcal{T}) \, , \, V \rangle = \int_\Omega f V \, dx \quad \text{for all } V \in \mathcal{S}^p_0(\mathcal{T}). \tag{3.7.3}$$

We formally define the residual error estimator for a triangulation $\mathcal{T} \in \mathbb{T}$ and all $T \in \mathcal{T}$ by

$$\eta_T(\mathcal{T})^2 := |T|^{2/d} \|\mathcal{L}|_T U_\ell - f\|^2_{L^2(T)} + |T|^{1/d} \|[\boldsymbol{A}(\cdot, \nabla U_\ell) \cdot n]\|^2_{L^2(\partial T \cap \Omega)}. \tag{3.7.4}$$

The solvability and uniqueness of (3.7.2) as well as the regularity assumptions needed such that (3.7.4) is well-defined are part of the subsequent sections.

3.7.0.1. *Regularity assumptions.* We consider the frame of strongly monotone operators and require the following regularity assumptions on $\mathcal{L}$:

$$\|\mathcal{A}\nabla w - \mathcal{A}\nabla v\|_{H^{-1}(\Omega)} \leq C_{15} \|\nabla(w - v)\|_{L^2(\Omega)}, \tag{3.7.5a}$$

$$\|\mathcal{C}w - \mathcal{C}v\|_{L^2(\Omega)} \leq C_{15} \|\nabla(w - v)\|_{L^2(\Omega)} \tag{3.7.5b}$$

for all $w, v \in H^1_0(\Omega)$ and some constant $C_{15} > 0$ as well as

$$\langle \mathcal{L}w - \mathcal{L}v \, , \, w - v \rangle \geq C_{16} \|\nabla(w - v)\|^2_{L^2(\Omega)} \tag{3.7.6}$$

for all $w, v \in H^1_0(\Omega)$ and some constant $C_{16} > 0$. These assumptions, in particular, allow to apply the main theorem on strongly monotone operators [**86**, Theorem 26.A] and to obtain the unique solvability of (3.7.2) as well as of (3.7.3). Additionally, (3.7.5)–(3.7.6) guarantee that the norms of the residual and the error are equivalent, i.e.,

$$\begin{aligned} \|\mathcal{L}u - \mathcal{L}U(\mathcal{T})\|_{H^{-1}(\Omega)} &\simeq \|\nabla(u - U(\mathcal{T}))\|_{L^2(\Omega)} \quad \text{for all } \mathcal{T} \in \mathbb{T}, \\ \|\mathcal{L}U(\widehat{\mathcal{T}}) - \mathcal{L}U(\mathcal{T})\|_{H^{-1}(\Omega)} &\simeq \|\nabla(U(\widehat{\mathcal{T}}) - U(\mathcal{T}))\|_{L^2(\Omega)} \quad \text{for all } \widehat{\mathcal{T}} \in \mathbb{T}(\mathcal{T}). \end{aligned} \tag{3.7.7}$$

We also obtain the Céa lemma

$$\|\nabla(u - U(\mathcal{T}))\|_{L^2(\Omega)} \leq 2C_{15}C_{16}^{-1} \min_{V \in \mathcal{S}^p_0(\mathcal{T})} \|\nabla(u - V)\|_{L^2(\Omega)}. \tag{3.7.8}$$

Moreover, we require that (3.7.4) is well-defined and satisfies (E1) with $\varrho(\mathcal{T},\widehat{\mathcal{T}}) \simeq \|\nabla(U(\mathcal{T})-U(\widehat{\mathcal{T}}))\|_{L^2(\Omega)}$. For possible non-linearities $\boldsymbol{A}$ which allow for (2.3.6), we refer to Lemma 3.7.2 below.

We assume that $\mathcal{L}:\ H_0^1(\Omega)\to H^{-1}(\Omega)$ as well as $\mathcal{A}:\ H_0^1(\Omega)\to H^{-1}(\Omega)$ are twice Fréchet differentiable, i.e., there exist

$$\begin{aligned} D\mathcal{L}, D\mathcal{A} &: H_0^1(\Omega)\to L(H_0^1(\Omega),H^{-1}(\Omega)),\\ D^2\mathcal{L}, D^2\mathcal{A} &: H_0^1(\Omega)\to L\big(H_0^1(\Omega),L(H_0^1(\Omega),H^{-1}(\Omega))\big). \end{aligned} \tag{3.7.9}$$

The second derivative should be bounded locally around the solution u of (3.7.2), i.e., there exists $\varepsilon_{\ell oc}>0$ with

$$\begin{aligned} C_{17} := \sup_{\|\nabla(u-v)\|_{L^2(\Omega)}<\varepsilon_{\ell oc}} \Big(&\|D^2\mathcal{L}(v)\|_{L\left(H_0^1(\Omega),L(H_0^1(\Omega),H^{-1}(\Omega))\right)} \\ &+ \|D^2\mathcal{A}(v)\|_{L\left(H_0^1(\Omega),L(H_0^1(\Omega),H^{-1}(\Omega))\right)}\Big) < \infty. \end{aligned} \tag{3.7.10}$$

Finally, we assume that $D\mathcal{A}(v):\ H_0^1(\Omega)\to H^{-1}(\Omega)$ is symmetric for all $v\in H_0^1(\Omega)$, i.e., for all $w_1,w_2\in H_0^1(\Omega)$ holds

$$\langle D\mathcal{A}(v)(w_1)\,,\,w_2\rangle = \langle D\mathcal{A}(v)(w_2)\,,\,w_1\rangle. \tag{3.7.11}$$

Remark 3.7.1. *Note that if $\boldsymbol{A}:\ \Omega\times\mathbb{R}^d\to\mathbb{R}^d$ and $g:\ \Omega\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}$ are twice differentiable, and if the Jacobian $J_y\boldsymbol{A}(x,y)\in\mathbb{R}^{d\times d}$ additionally is a symmetric matrix, then $\mathcal{L}$ and $\mathcal{A}$ satisfy* (3.7.9) *as well as* (3.7.10). *Moreover, $D\mathcal{A}(v)$ is symmetric for all $v\in H_0^1(\Omega)$, since there holds for $w\in H_0^1(\Omega)$*

$$D\mathcal{A}(v)(w) = \mathrm{div}_x\Big(\big(J_y\boldsymbol{A}(x,\nabla v(x))\big)\big(\nabla_x w(x)\big)\Big).$$

We stress that the symmetry assumption (3.7.11) posed on $D\mathcal{A}$ covers in particular the operator class from **[54]**, where

$$\boldsymbol{A}(x,y) = \alpha(x,|y|^2)y$$

for some function $\alpha:\ \Omega\times\mathbb{R}\to\mathbb{R}$ with continuous derivative $t\mapsto\partial_t\alpha(x,t)$. In contrast to **[54]** where $\alpha(x,\cdot)\in C^1(\mathbb{R})$ is sufficient, the analysis here covers a wider class of operators, however, for this special case needs $\alpha(x,\cdot)\in C^2(\mathbb{R})$ to guarantee (3.7.10).

Lemma 3.7.2. *Sufficient regularity assumptions in addition to* (3.7.5b) *and* (3.7.6) *to guarantee that the error estimator* (3.7.4) *is well-defined and satisfies* (E1) *are, for instance, either of the following conditions (i) and (ii):*

(i) $\boldsymbol{A}(\cdot,\cdot):\ \Omega\times\mathbb{R}^d\to\mathbb{R}^d$ *is Lipschitz continuous and there exists a constant $C_{18}>0$ such that for all $\mathcal{T}\in\mathbb{T}$ and all $V,W\in\mathcal{S}_0^p(\mathcal{T})$ there holds* $\mathrm{div}\boldsymbol{A}(\cdot,V(\cdot))\in L^2(\Omega)$ *as well as*

$$\|\mathrm{div}|_T\big(\boldsymbol{A}(\cdot,V(\cdot))-\boldsymbol{A}(\cdot,W(\cdot))\big)\|_{L^2(T)} \le C_{18}\|V-W\|_{H^2(T)} \quad \textit{for all } T\in\mathcal{T}. \tag{3.7.12}$$

(ii) *There holds $p=1$ (lowest-order case) as well as*

$$\boldsymbol{A}(x,y)=\boldsymbol{A}(y) \quad \textit{for all } x\in\Omega,\ y\in\mathbb{R}^d,$$

and additionally $\boldsymbol{A}(\cdot):\ \mathbb{R}^d\to\mathbb{R}^d$ is Lipschitz continuous.

PROOF. The jump terms in (3.7.4) are well-defined in both cases (i) and (ii) since $\boldsymbol{A}(\cdot, \nabla U(\cdot))$ is a piecewise Lipschitz continuous function. Moreover, this immediately shows that $\operatorname{div}\boldsymbol{A}(\cdot, \nabla U(\mathcal{T})(\cdot)) \in L^\infty(T) \subset L^2(T)$ for all $T \in \mathcal{T}$. Therefore, (3.7.4) is well-defined.

Given $T_+, T_- \in \mathcal{T}$ as well as $W, V \in \mathcal{S}_0^p(\mathcal{T}_\ell)$, the Lipschitz continuity also proves the following point wise estimate for all $x \in T_+ \cap T_-$

$$
\begin{aligned}
&|[(\boldsymbol{A}(x, \nabla W(x)) - \boldsymbol{A}(x, \nabla V(x))) \cdot n]| \\
&\qquad \leq \Big|\big(\boldsymbol{A}(x, (\nabla W)|_{T_+}(x)) - \boldsymbol{A}(x, (\nabla V)|_{T_+}(x))\big) \cdot n|_{T_+}\Big| \\
&\qquad\qquad + \Big|\big(\boldsymbol{A}(x, (\nabla W)|_{T_-}(x)) - \boldsymbol{A}(x, (\nabla V)|_{T_-}(x))\big) \cdot n|_{T_-}\Big| \\
&\qquad \lesssim \Big|(\nabla W)|_{T_+}(x) - (\nabla V(x))|_{T_+}\Big| + \Big|(\nabla W)|_{T_-}(x) - (\nabla V)|_{T_-}(x)\Big|.
\end{aligned}
$$

Combining the estimate above with the trace inequality for polynomials, we obtain

$$
|T_+|^{1/d} \|[(\boldsymbol{A}(\cdot, \nabla W) - \boldsymbol{A}(\cdot, \nabla V)) \cdot n]\|_{L^2(T_+ \cap T_-)}^2 \lesssim \|\nabla(W - V)\|_{L^2(T_+ \cup T_-)}^2. \tag{3.7.13}
$$

This hidden constant depends only on the polynomial degree $p \in \mathbb{N}$ as well as the Lipschitz continuity of $\boldsymbol{A}(\cdot, \cdot)$ and the shape regularity $\gamma(\mathcal{T})$. It remains to prove a similar estimate for the volume residual in (3.7.4), i.e.,

$$
\sum_{T \in \mathcal{T}} |T|^{2/d} \|\mathcal{L}|_T W - \mathcal{L}|_T V\|_{L^2(T)}^2 \lesssim \|\nabla(W - V)\|_{L^2(\Omega)}^2 \quad \text{for all } T \in \mathcal{T}. \tag{3.7.14}
$$

In case of (i), this follows immediately from the combination of (3.7.12) and (3.7.5b) together with a standard inverse estimate. In case of (ii), we observe that ∇U_ℓ is piecewise constant. Therefore, $\boldsymbol{A}(\nabla V)$ is piecewise constant and hence $\mathcal{A}(\nabla V) = \operatorname{div}\boldsymbol{A}(\nabla V(\cdot)) = 0$. Thus, $\mathcal{L}|_T = (\mathcal{C}V)|_T$, and it suffices to apply (3.7.5b) to prove (3.7.14). The estimates (3.7.13)–(3.7.14) imply stability and reduction (E1) with $\varrho(\mathcal{T}, \widehat{\mathcal{T}}) \simeq \|\nabla(U(\mathcal{T}) - U(\widehat{\mathcal{T}}))\|_{L^2(\Omega)}$ and $\mathcal{S}(\mathcal{T}, \widehat{\mathcal{T}}) = \mathcal{T} \setminus \widehat{\mathcal{T}}$ as well as $\widehat{\mathcal{S}}(\mathcal{T}, \widehat{\mathcal{T}}) = \widehat{\mathcal{T}} \setminus \mathcal{T}$. To see this, note that $\eta(\cdot)$ is a weighted error estimator in the sense of Section 3.4 and satisfies homogeneity (3.4.1) with $r_- = 1$ and $r_+ = 1/2$. Moreover, stability (3.4.2) holds for some $\mathcal{S} \subseteq \mathcal{T}$ and $h \leq h(\mathcal{T})$ by

$$
\begin{aligned}
&\Big|\Big(\sum_{T \in \widehat{\mathcal{S}}} \eta_T(\widehat{\mathcal{T}}, h)^2\Big)^{1/2} - \Big(\sum_{T \in \mathcal{S}} \eta_T(\mathcal{T}, h)^2\Big)^{1/2}\Big| \\
&\qquad \leq \Big(\sum_{T \in \mathcal{S}} h(\mathcal{T})|_T^2 \|\mathcal{L}|_T U(\mathcal{T}) - \mathcal{L}|_T U(\widehat{\mathcal{T}})\|_{L^2(T)}^2\Big)^{1/2} \\
&\qquad\qquad + \Big(\sum_{T \in \mathcal{S}} h(\mathcal{T})|_T \|[(\boldsymbol{A}(\cdot, \nabla U(\mathcal{T})) - \boldsymbol{A}(\cdot, \nabla U(\widehat{\mathcal{T}}))) \cdot n]\|_{L^2(\partial T \cap \Omega)}^2\Big)^{1/2} \\
&\qquad \lesssim \|\nabla(U(\mathcal{T}) - U(\widehat{\mathcal{T}}))\|_{L^2(\Omega)}^2.
\end{aligned}
$$

Therefore, Proposition 3.4.1 applies and proves (E1). □

3.7.0.2. *Proof of the axioms.*

LEMMA 3.7.3. *The residual error estimator $\eta(\cdot)$ satisfies discrete reliability* (E3) *and reliability* (2.4.1) *with $\operatorname{err}(\mathcal{T}) := \|\nabla(u - U(\mathcal{T}))\|_{L^2(\Omega)}$. Moreover, there holds convergence*

$$
\|\nabla(u - U(\mathcal{T}_\ell))\|_{L^2(\Omega)} \to 0 \quad \textit{as } \ell \to \infty. \tag{3.7.15}
$$

PROOF. The residual error estimator $\eta(\cdot)$ is well-defined under the assumptions in Section 3.7.0.1. With the equivalence (3.7.7), the standard arguments from [**35**] apply to prove discrete reliability (E3). Also the reliability (2.4.1) follows with standard arguments from the literature. The estimator reduction (2.3.6) holds by assumption in Section 3.7.0.1. The assumptions for a priori convergence of Section 2.7.1 are satisfied. The main theorem on strongly monotone operators [**86**, Theorem 26.A] proves that there exists a solution U_∞ of (3.7.3) when $\mathcal{S}_0^p(\mathcal{T})$ is exchanged with $\mathcal{X}_\infty := \overline{\bigcup_{\ell\in\mathbb{N}_0} \mathcal{S}_0^p(\mathcal{T}_\ell)}$. Since the $U(\mathcal{T}_\ell)$ are also Galerkin approximations to $U_\infty \in \mathcal{X}_\infty$, the Céa lemma (3.7.8) implies (2.7.1). Hence the requirements of Lemma 2.7.1 are satisfied and we obtain $\lim_{\ell\to\infty} \varrho(\mathcal{T}_\ell, \mathcal{T}_{\ell+1}) = 0$. Lemma 2.3.6 together with reliability (2.4.1) proves the convergence. □

PROPOSITION 3.7.4. *The conforming discretization of* (3.7.1) *with residual error estimator* (3.7.4) *satisfies*

(i) *stability and reduction* (E1) *with* $\varrho(\mathcal{T},\widehat{\mathcal{T}}) \simeq \|\nabla(U(\mathcal{T}) - U(\widehat{\mathcal{T}}))\|_{L^2(\Omega)}$ *as well as* $\mathcal{S}(\mathcal{T},\widehat{\mathcal{T}}) = \mathcal{T}\setminus\widehat{\mathcal{T}}$ *and* $\widehat{\mathcal{S}}(\mathcal{T},\widehat{\mathcal{T}}) = \widehat{\mathcal{T}}\setminus\mathcal{T}$,
(ii) *general quasi-orthogonality* (E2),
(iii) *discrete reliability* (E3) *with* $\mathcal{R}(\mathcal{T},\widehat{\mathcal{T}}) = \mathcal{T}\setminus\widehat{\mathcal{T}}$, $\kappa_{\rm dlr} = \infty$, *and* $\varepsilon_{\rm drel} = 0$,
(iv) *the refinement axioms* (T1)–(T3) *with* $C_{\rm approx}(s) \leq C_{\rm mon}(C_4+1)^s\|\eta,\mathbb{T}\|_s$ *for all* $s > 0$ *and the overlay estimate* (2.5.1).

The constants $C_{\rm drel}, C_{\rm qo}$ *depend only on the polynomial degree* $p \in \mathbb{N}$, $\mathcal{T}_0$, Ω, *and on* $\mathcal{L}$.

PROOF. Stability and reduction (i) follows by assumption. Discrete reliability (iii) is proved in Lemma 3.7.3. The refinement axioms (iv) follow as for the Poisson model problem from Section 3.5.1. The proof of the general quasi-orthogonality (E2) follows with Theorem 7.4.5. This concludes the proof. □

CONSEQUENCE 3.7.5. *Let* $s > 0$ *with* $\|\eta,\mathbb{T}\|_s < \infty$. *Then, the adaptive algorithm leads to convergence with optimal rate for the estimator* $\eta(\cdot)$ *in the sense of Theorem 2.3.3 and optimal complexity in the sense of Theorem 2.5.1.* □

CHAPTER 4

Abstract Theory: Equivalent Error Estimators

4.1. Introduction, state of the art & outline

This sector extends the abstract approach of Chapter 2 and includes equivalent error estimators. The idea behind is that the axioms do not have to be satisfied by the error estimator itself, but only by an *equivalent* error estimator. Of course, this observation could be included directly into the axioms in Chapter 2. However, we think that this separate presentation of the arguments is clearer and is easier to understand. The overall idea is the following: If a certain estimator is used for computations, this is often because it is easy to implement or it possesses some nice numerical features. This, however, is often in stark contrast with the analytic features in terms of Chapter 2 of the error estimator. For example, an error estimator might satisfy the contraction in (E1) on average, but fails to satisfy it in each single step (see, e.g., Section 5.2 for some examples). Moreover, any computation is prone to numerical errors (e.g., round-off errors). This means that any implementation of the adaptive algorithm will, in fact, compute an approximate error estimator (this is of even more significance if iterative solvers are used; see Section 4.4 for details). Hence, the computed error estimator will satisfy the axioms only up to some error and only the exact (theoretical) error estimator fits into the abstract framework of Chapter 2.

The framework of this chapter allows to prove the axioms for some equivalent, well-behaving, error estimator, and gives results for the error estimator in use. This idea firstly appeared in [**60**], where several error estimators equivalent to the residual error estimator for the Poisson problem of Section 3.5.1 are analyzed (see also the examples in Section 5.2). A similar version of this chapter can be found in the recent own work [**24**]. However, this work simplifies the arguments and generalizes the results.

The remainder of the chapter is organized as follows: Section 4.2 states the assumptions on the equivalent error estimator and Section 4.3 given the main result on optimal convergence rates. Section 4.4 treats the particular case of approximate computations and Section 4.5 proves the assumptions of Section 4.2 for the special case of weighted error estimators. Finally, Section 4.5.4 proves the existence of a super contractive weight function, which might be of independent interest.

4.2. Abstract setting

4.2.1. Equivalent error estimator. Recall the sets $\mathcal{T}_\infty$ and $\mathbb{T}$ from Section 2.2.1. We assume that $\widetilde{\mathbb{T}}$ is a set of triangulations which is based on a set $\widetilde{\mathcal{T}}_\infty$ (where we allow $\widetilde{\mathcal{T}}_\infty = \mathcal{T}_\infty$ as well as $\widetilde{\mathbb{T}} = \mathbb{T}$) and a refinement strategy $\widetilde{\mathbb{T}}(\cdot,\cdot)$ (also $\widetilde{\mathbb{T}}(\cdot,\cdot) = \mathbb{T}(\cdot,\cdot)$ is allowed). We assume that there is a one-to-one correspondence between $\mathcal{T} \in \mathbb{T}$ and $\widetilde{\mathcal{T}} \in \widetilde{\mathbb{T}}$ and that there exists a constant $C_{\rm eq} \geq 1$ such that $C_{\rm eq}^{-1}|\mathcal{T}| \leq |\widetilde{\mathcal{T}}| \leq C_{\rm eq}|\mathcal{T}|$.

Additionally to the error estimator from Section 2.2.2, we define an equivalent error estimator as a function $\widetilde{\eta}(\cdot) : \widetilde{\mathbb{T}} \to \bigcup_{\widetilde{\mathcal{T}}\in\widetilde{\mathbb{T}}} \big([0,\infty)^{\widetilde{\mathcal{T}}}\big)$ (where A^B denotes the set of functions mapping B to A) with $\widetilde{\eta}(\widetilde{\mathcal{T}}) : \widetilde{\mathcal{T}} \to [0, \infty)$ for all $\widetilde{\mathcal{T}} \in \widetilde{\mathbb{T}}$. As for the error estimator, we also write $\widetilde{\eta}(\widetilde{\mathcal{T}}) := \big(\sum_{T \in \widetilde{\mathcal{T}}} \widetilde{\eta}_T(\mathcal{T})^2\big)^{1/2}$, which is the global equivalent error estimator.

Suppose that the error estimators are equivalent in the sense that there exists $C_{\rm eq} \geq 1$ such that

$$C_{\rm eq}^{-1}\widetilde{\eta}(\widetilde{\mathcal{T}})^2 \leq \eta(\mathcal{T})^2 \leq C_{\rm eq}\widetilde{\eta}(\widetilde{\mathcal{T}})^2 \quad \text{for all } \mathcal{T} \in \mathbb{T}, \tag{4.2.1}$$

and such that for all $\widetilde{\mathcal{M}} \subseteq \widetilde{\mathcal{T}} \in \widetilde{\mathbb{T}}$ and all $0 < \widetilde{\theta} \leq 1$, there exists $\mathcal{M} \subseteq \mathcal{T}$ (where $\mathcal{T}$ is uniquely determined by $\widetilde{\mathcal{T}}$) with $C_{\rm eq}^{-1}|\widetilde{\mathcal{M}}| \leq |\mathcal{M}| \leq C_{\rm eq}|\widetilde{\mathcal{M}}|$ and

$$\widetilde{\theta}\,\widetilde{\eta}(\widetilde{\mathcal{T}})^2 \leq \sum_{T \in \widetilde{\mathcal{M}}} \widetilde{\eta}_T(\widetilde{\mathcal{T}})^2 \quad \Longrightarrow \quad C_{\rm eq}^{-1}\widetilde{\theta}\,\eta(\mathcal{T})^2 \leq \sum_{T \in \mathcal{M}} \eta_T(\mathcal{T})^2. \tag{4.2.2a}$$

Conversely, for all $\mathcal{M} \subseteq \mathcal{T} \in \mathbb{T}$ and all $0 < \theta \leq 1$, there exists $\overline{\mathcal{M}} \subseteq \widetilde{\mathcal{T}}$ (where $\widetilde{\mathcal{T}}$ is uniquely determined by $\mathcal{T}$) with $C_{\rm eq}^{-1}|\overline{\mathcal{M}}| \leq |\mathcal{M}| \leq C_{\rm eq}|\overline{\mathcal{M}}|$ and

$$\theta\,\eta(\widetilde{\mathcal{T}})^2 \leq \sum_{T \in \mathcal{M}} \eta_T(\mathcal{T})^2 \quad \Longrightarrow \quad C_{\rm eq}^{-1}\theta\,\widetilde{\eta}(\widetilde{\mathcal{T}})^2 \leq \sum_{T \in \overline{\mathcal{M}}} \widetilde{\eta}_T(\widetilde{\mathcal{T}})^2. \tag{4.2.2b}$$

4.2.2. Equivalent adaptive approximation problem. The goal of the equivalent adaptive approximation problem is to find a sequence of triangulations $\widetilde{\mathcal{T}}_\ell$, $\ell \in \mathbb{N}_0$ such that

$$\sup_{\ell \in \mathbb{N}_0} \widetilde{\eta}(\widetilde{\mathcal{T}}_\ell)(|\widetilde{\mathcal{T}}_\ell| + 1)^s < \infty$$

for $s > 0$ as large as possible.

4.2.3. Adaptive algorithm. The algorithm to solve the equivalent adaptive approximation problem from Section 4.2.2 reads

Algorithm 4.2.1. Input: *Initial triangulation* $\widetilde{\mathcal{T}}_0$ *and bulk parameter* $0 < \widetilde{\theta} \leq 1$.
Loop: *For* $\ell = 0, 1, 2, \dots$ *do* (i) – (iii).

(i) *Compute refinement indicators* $\widetilde{\eta}_T(\widetilde{\mathcal{T}}_\ell)$ *for all* $T \in \widetilde{\mathcal{T}}_\ell$.
(ii) *Determine set* $\widetilde{\mathcal{M}}_\ell \subseteq \widetilde{\mathcal{T}}_\ell$ *of (up to the multiplicative constant* $C_{\rm min}$*) minimal cardinality such that*

$$\widetilde{\theta}\,\widetilde{\eta}(\widetilde{\mathcal{T}}_\ell)^2 \leq \sum_{T \in \widetilde{\mathcal{M}}_\ell} \widetilde{\eta}_T(\widetilde{\mathcal{T}}_\ell)^2. \tag{4.2.3}$$

(iii) *Define the next triangulation as* $\widetilde{\mathcal{T}}_{\ell+1} := \widetilde{\mathbb{T}}(\widetilde{\mathcal{T}}_\ell, \widetilde{\mathcal{M}}_\ell)$.

Output: *Error estimators* $\widetilde{\eta}(\widetilde{\mathcal{T}}_\ell)$ *for all* $\ell \in \mathbb{N}_0$.

4.3. Optimal convergence

In the following, the notion that a certain subset $\mathcal{A} \subseteq \{(\mathrm{E1}), \dots, (\mathrm{E3}), (\mathrm{T1}), \dots, (\mathrm{T3})\}$ is satisfied means that the axioms in $\mathcal{A}$ are satisfied for the error estimator $\eta(\cdot)$, the corresponding refinement strategy $\mathbb{T}(\cdot, \cdot)$, and the respective constants from Section 2.3.1. The triangulations $(\mathcal{T}_\ell)_{\ell \in \mathbb{N}_0}$ in (E2), (T1)–(T3) are determined by $(\widetilde{\mathcal{T}}_\ell)_{\ell \in \mathbb{N}_0}$ via the function $\widetilde{(\cdot)}$.

Theorem 4.3.1. *Suppose that the error estimator* $\eta(\cdot)$ *satisfies the estimator reduction* (2.3.8)*. Then, (i)–(iii) holds*

(i) *Assume* $\lim_{\ell\to\infty} \varrho(\mathcal{T}_\ell, \mathcal{T}_{\ell+1}) = 0$ *(with* $\varrho(\cdot,\cdot)$ *from Section 2.3.1). Then, for all* $0 < \widetilde{\theta} \le 1$*, the equivalent estimator is convergent in the sense*

$$\lim_{\ell\to\infty} \widetilde{\eta}(\widetilde{\mathcal{T}}_\ell) = 0. \tag{4.3.1}$$

(ii) *Suppose* (E2) *is satisfied by* $\eta(\cdot)$*. Then, for all* $0 < \widetilde{\theta} \le 1$*, the equivalent estimator is R-linear convergent in the sense that there exists* $0 < \widetilde{\rho}_{\rm conv} < 1$ *and* $\widetilde{C}_{\rm conv} > 0$ *such that*

$$\widetilde{\eta}(\widetilde{\mathcal{T}}_{\ell+j})^2 \le \widetilde{C}_{\rm conv}\widetilde{\rho}_{\rm conv}^{\,j}\, \widetilde{\eta}(\widetilde{\mathcal{T}}_\ell)^2 \quad \textit{for all } j,\ell \in \mathbb{N}_0. \tag{4.3.2}$$

(iii) *Suppose that R-linear convergence* (4.3.2) *holds and that* (E1a), (E3) *and* (T1)–(T3) *are satisfied by* $\eta(\cdot)$ *and some* $s > 0$*. Then* $0 < \widetilde{\theta} < C_{\rm eq}^{-1}\theta_\star = C_{\rm eq}^{-1}(1-\varepsilon_{\rm drel})/(1+C_{\rm drel}^2)$ *implies quasi-optimal convergence of the estimator in the sense of*

$$\widetilde{c}_{\rm opt} C_{\rm approx}(s) \le \sup_{\ell\in\mathbb{N}_0} \frac{\widetilde{\eta}(\widetilde{\mathcal{T}}_\ell)}{(|\widetilde{\mathcal{T}}_\ell \setminus \widetilde{\mathcal{T}}_0| + 1)^{-s}} \le \widetilde{C}_{\rm opt} C_{\rm approx}(s), \tag{4.3.3}$$

where the lower bound requires only (T1) *to hold.*

The constants $\widetilde{C}_{\rm conv}$*,* $\widetilde{\rho}_{\rm conv} > 0$ *depend only on* $\rho_{\rm red}$*,* $C_{\rm qo}$*,* $\varepsilon_{\rm qo}$*,* $C_{\rm eq}$*, and on* $\widetilde{\theta}$*. The constant* $\widetilde{C}_{\rm opt} > 0$ *depends additionally on* $\widetilde{C}_{\rm conv}$*,* $\widetilde{\rho}_{\rm conv}$*,* $C_{\min}, C_{\rm ref}, C_{\rm closure}, C_{\rm drel}$*,* $\varepsilon_{\rm drel}$*, and on* s*, while* $\widetilde{c}_{\rm opt} > 0$ *depends only on* $C_{\rm son}$ *and* $|\mathcal{T}_0|$*.*

Proof of Theorem 4.3.1 (i). Lemma 2.3.6 for $\eta(\cdot)$ shows $\lim_{\ell\to\infty}\eta(\mathcal{T}_\ell) = 0$. The global equivalence (4.2.1) concludes the proof. □

Proof of Theorem 4.3.1 (ii). Proposition 2.3.9 together with the global equivalence estimate (4.2.1) implies

$$\widetilde{\eta}(\widetilde{\mathcal{T}}_{\ell+j})^2 \le C_{\rm eq}\eta(\mathcal{T}_{\ell+j})^2 \le C_{\rm eq}C_3\rho_1^j\eta(\mathcal{T}_\ell)^2 \le C_{\rm eq}^2 C_3 \rho_1^j \widetilde{\eta}(\widetilde{\mathcal{T}}_\ell)^2$$

for all $\ell, j \in \mathbb{N}_0$. This concludes the proof. □

Lemma 4.3.2. *Recall* $\widetilde{\mathcal{M}}_\ell \subseteq \widetilde{\mathcal{T}}_\ell$ *from Algorithm 4.2.1. Let* $\mathcal{M}_\ell^0 \subseteq \mathcal{T}_\ell$ *(where* $\mathcal{T}_\ell$ *is uniquely determined by* $\widetilde{\mathcal{T}}_\ell$*, cf. Section 4.2.1) be a set with minimal cardinality which satisfies*

$$C_{\rm eq}\widetilde{\theta}\eta(\mathcal{T}_\ell)^2 \le \sum_{T\in\mathcal{M}_\ell^0} \eta_T(\mathcal{T}_\ell)^2. \tag{4.3.4}$$

Then, the set $\mathcal{M}_\ell$ *from* (4.2.2a) *satisfies* $|\mathcal{M}_\ell| \le C_{\min}C_{\rm eq}|\mathcal{M}_\ell^0|$ *as well as*

$$C_{\rm eq}^{-1}\widetilde{\theta}\eta(\mathcal{T}_\ell)^2 \le \sum_{T\in\mathcal{M}_\ell} \eta_T(\mathcal{T}_\ell)^2. \tag{4.3.5}$$

Proof. With (4.3.4), the implication (4.2.2b) states the existence of $\overline{\mathcal{M}}_\ell^0 \subseteq \widetilde{\mathcal{T}}$ with $|\overline{\mathcal{M}}_\ell^0| \le C_{\rm eq}|\mathcal{M}_\ell^0|$ and

$$\widetilde{\theta}\,\widetilde{\eta}(\widetilde{\mathcal{T}}_\ell)^2 \le \sum_{T\in\overline{\mathcal{M}}_\ell^0} \widetilde{\eta}_T(\widetilde{\mathcal{T}}_\ell)^2.$$

Since $\widetilde{\mathcal{M}}_\ell$ is a set of almost minimal cardinality which satisfies (4.2.3), there holds $C_{\rm eq}^{-1}|\mathcal{M}_\ell| \le |\widetilde{\mathcal{M}}_\ell| \le C_{\min}|\overline{\mathcal{M}}_\ell^0| \le C_{\min}C_{\rm eq}|\mathcal{M}_\ell^0|$. The implication (4.2.2a) shows (4.3.5). □

Proof of Theorem 4.3.1 (iii). Stability (E1a) and discrete reliability (E3) guarantee that (2.3.18) holds for all $0 < \theta_0 < \theta_\star$ and some $0 < \kappa_0 < 1$. The assumption $\widetilde{\theta} < C_{\rm eq}^{-1}\theta$ allows to choose $\theta_0 = C_{\rm eq}\widetilde{\theta}$. This implies that (2.3.20)–(2.3.21) of Lemma 2.3.13 are valid for $\theta = C_{\rm eq}\widetilde{\theta}$. Since $\mathcal{R}(\mathcal{T}_\ell, \widehat{\mathcal{T}}_\ell)$ from 2.3.21 satisfies (2.3.21b) for all $0 < \theta \le \theta_0 = C_{\rm eq}\widetilde{\theta}$, (4.3.4) shows that $|\mathcal{M}_\ell^0| \le |\mathcal{R}(\mathcal{T}_\ell, \widehat{\mathcal{T}}_\ell)|$. Hence, Lemma 4.3.2 implies $|\mathcal{M}_\ell| \le C_{\rm min}C_{\rm eq}|\mathcal{R}(\mathcal{T}_\ell, \widehat{\mathcal{T}}_\ell)|$. By assumption (4.3.2), Lemma 2.3.8 implies that (2.3.12)–(2.3.14) hold for $\alpha_\ell := \eta(\mathcal{T}_\ell)$. The application of Proposition 2.3.14–2.3.15 shows (2.3.3) for all $\widetilde{\theta} < C_{\rm eq}^{-1}\theta_\star$. Additionally, there holds

$$|\mathcal{T}_\ell \setminus \mathcal{T}_0| + 1 \le |\mathcal{T}_\ell| + 1 \le C_{\rm eq}(|\widetilde{\mathcal{T}}_\ell| + 1) \lesssim |\widetilde{\mathcal{T}}_\ell \setminus \widetilde{\mathcal{T}}_0| + 1 \lesssim |\mathcal{T}_\ell \setminus \mathcal{T}_0| + 1,$$

where the hidden constants depend only on $C_{\rm eq}$ and $|\mathcal{T}_0|$. Together with (4.2.1), this concludes the proof. □

4.4. Inexact Solve

This section covers a particular case of the abstract theory from Section 4.2. To that end, let $\widetilde{\mathbb{T}} = \mathbb{T}$ and $\widetilde{\mathcal{T}} = \mathcal{T}$. We assume that there exists an *approximate* error estimator $\widetilde{\eta}(\cdot)$, which results from an inexact computation of the *exact* error estimator $\eta(\cdot)$ and satisfies for all $\mathcal{T} \in \mathbb{T}$ and all $\mathcal{S} \subseteq \mathcal{T}$

$$\Big|\Big(\sum_{T\in\mathcal{S}} \eta_T(\mathcal{T})^2\Big)^{1/2} - \Big(\sum_{T\in\mathcal{S}} \widetilde{\eta}_T(\mathcal{T})^2\Big)^{1/2}\Big| \le \vartheta\widetilde{\eta}(\mathcal{T}) \tag{4.4.1}$$

for some constant $0 < \vartheta < 1$. Naturally, it is convenient to check the axioms (E1)–(E3) for the exact error estimator rather than incorporating the numerical error bounds into the analysis.

4.4.1. Local and global equivalence.

Lemma 4.4.1. *Under* (4.4.1)*, there exists $C_{\rm eq} > 0$ which depends only on $\vartheta < 1$, such that the approximate error estimator $\widetilde{\eta}(\cdot)$ satisfies* (4.2.1) *as well as* (4.2.2) *with $\mathcal{M} = \widetilde{\mathcal{M}} = \overline{\mathcal{M}}$.*

Proof. The global equivalence (4.2.1) follows directly from (4.4.1) with $\mathcal{S} = \mathcal{T}$, i.e.,

$$(1-\vartheta)\widetilde{\eta}(\mathcal{T}) \le \eta(\mathcal{T}) \le (1+\vartheta)\widetilde{\eta}(\mathcal{T}).$$

For (4.2.2a), set $\mathcal{S} = \mathcal{M}$ to obtain for all $\delta > 0$ with $(1+\delta)\vartheta < 1$

$$\sum_{T\in\mathcal{M}} \widetilde{\eta}_T(\mathcal{T})^2 \le (1+\delta^{-1}) \sum_{T\in\mathcal{M}} \eta_T(\mathcal{T})^2 + (1+\delta)\vartheta^2 \sum_{T\in\widetilde{\mathcal{M}}} \widetilde{\eta}_T(\mathcal{T})^2.$$

Moreover, there holds

$$\widetilde{\theta}\eta(\mathcal{T})^2 \le \widetilde{\theta}(1+\vartheta)^2\widetilde{\eta}(\mathcal{T})^2 \le (1+\vartheta)^2 \sum_{T\in\widetilde{\mathcal{M}}} \widetilde{\eta}_T(\mathcal{T})^2.$$

Together, this implies

$$\widetilde{\theta}\eta(\mathcal{T})^2 \le (1+\vartheta)^2(1-(1+\delta)\vartheta^2)^{-1}(1+\delta^{-1}) \sum_{T\in\mathcal{M}} \eta_T(\mathcal{T})^2.$$

Analogously, one derives (4.2.2b), i.e.,

$$\theta\widetilde{\eta}(\mathcal{T})^2 \le (1-\vartheta)^2(1+2\vartheta^2)^{-1}2\sum_{T\in\widetilde{\mathcal{M}}}\widetilde{\eta}_T(\mathcal{T})^2.$$

With $C_{\rm eq} := \max\{(1+\vartheta),(1-\vartheta)^{-1},(1+\vartheta)^2(1-(1+\delta)\vartheta^2)^{-1}(1+\delta^{-1}),2(1-\vartheta)^2(1+2\vartheta^2)^{-1}\}$, we conclude the proof. ☐

4.4.2. Optimal convergence.

Proposition 4.4.2. *Let stability and reduction* (E1) *be satisfied. Then, $\eta(\cdot)$ satisfies estimator reduction* (2.3.8).

Proof. Lemma 4.4.1 shows that Dörfler marking (2.2.1) holds with $\theta = C_{\rm eq}\widetilde{\theta}$. Hence, Lemma 2.3.5 concludes the proof. ☐

In the following, the notion that a certain subset $\mathcal{A} \subseteq \{(E1),\dots,(E3),(T1),\dots,(T3)\}$ is satisfied means that the axioms in $\mathcal{A}$ are satisfied for the error estimator $\eta(\cdot)$, the corresponding refinement strategy $\mathbb{T}(\cdot,\cdot)$, and the respective constants from Section 2.3.1. The triangulations $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}_0}$ in (E2), (T1)–(T3) are determined by $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}_0} = (\widetilde{\mathcal{T}}_\ell)_{\ell\in\mathbb{N}_0}$ from Algorithm 4.2.1.

Theorem 4.4.3. *Suppose that the error estimator $\eta(\cdot)$ satisfies* (E1).

(i) *Assume $\lim_{\ell\to\infty}\varrho(\mathcal{T}_\ell,\mathcal{T}_{\ell+1}) = 0$ (with $\varrho(\cdot,\cdot)$ from Section 2.3.1). Then, for all $0 < \widetilde{\theta} \le 1$, the equivalent estimator is convergent in the sense*

$$\lim_{\ell\to\infty}\widetilde{\eta}(\widetilde{\mathcal{T}}_\ell) = 0.$$

(ii) *Suppose* (E2) *is satisfied by $\eta(\cdot)$. Then, for all $0 < \widetilde{\theta} \le 1$, the equivalent estimator is R-linear convergent in the sense that there exists $0 < \widetilde{\rho}_{\rm conv} < 1$ and $\widetilde{C}_{\rm conv} > 0$ such that*

$$\widetilde{\eta}(\widetilde{\mathcal{T}}_{\ell+j})^2 \le \widetilde{C}_{\rm conv}\widetilde{\rho}_{\rm conv}^{\,j}\,\widetilde{\eta}(\widetilde{\mathcal{T}}_\ell)^2 \quad \text{for all } j,\ell\in\mathbb{N}_0.$$

(iii) *Suppose that* (E1a), (E2)–(E3) *and* (T1)–(T3) *are satisfied by $\eta(\cdot)$ for some $s > 0$. Then $0 < \widetilde{\theta} < C_{\rm eq}^{-1}\theta_\star = C_{\rm eq}^{-1}(1-\varepsilon_{\rm drel})/(1+C_{\rm drel}^2)$ implies quasi-optimal convergence of the estimator in the sense of*

$$\widetilde{c}_{\rm opt}C_{\rm approx}(s) \le \sup_{\ell\in\mathbb{N}_0}\frac{\widetilde{\eta}(\widetilde{\mathcal{T}}_\ell)}{(|\widetilde{\mathcal{T}}_\ell\setminus\widetilde{\mathcal{T}}_0|+1)^{-s}} \le \widetilde{C}_{\rm opt}C_{\rm approx}(s),$$

where the lower bound requires only (T1) *to hold.*

The constants $\widetilde{C}_{\rm conv}$, $\widetilde{\rho}_{\rm conv} > 0$ depend only on $\rho_{\rm red}$, $C_{\rm qo}$, $\varepsilon_{\rm qo}$, and on θ,ϑ. The constant $\widetilde{C}_{\rm opt} > 0$ depends additionally on $C_{\rm min}, C_{\rm ref}, C_{\rm closure}, C_{\rm drel}$, $\varepsilon_{\rm drel}$, and on s, while $\widetilde{c}_{\rm opt} > 0$ depends only on $C_{\rm son}$ and $|\mathcal{T}_0|$.

Proof. Lemma 4.4.1 proves that the assumptions in Section 4.2.1 are satisfied and Proposition 4.4.2 shows that the estimator reduction holds. Hence, the requirements of Theorem 4.3.1 are fulfilled. This concludes the proof. ☐

4.5. Weighted error estimators

This section covers the particular case of weighted error estimators of the abstract theory from Section 4.2. Examples which fit in the abstract framework are presented in Section 5.2. To that end, we assume the conventions and notation from Section 3.4, particularly, the existence of a certain natural weight function $h(\mathcal{T}) : \Omega \to (0, \infty)$ for all $\mathcal{T} \in \mathbb{T}$ such that $\|h(\mathcal{T})\|_{L^\infty(\Omega)} < \infty$ and $h(\mathcal{T})$ is continuous on $\Omega \setminus \bigcup_{T\in\mathcal{T}} \partial T$ as well as the assumptions on the triangulations in Section 3.2.1. In the following $\max_{x\in T} g := \operatorname{ess\,sup}_{x\in T} g(x)$ and $\min_{x\in T} g := \operatorname{ess\,inf}_{x\in T} g(x)$ denote the essential supremum resp. essential infimum of the function g on the element $T \in \mathcal{T}$. In addition to Section 3.4, this section assumes the following: There exist constants $0 < q_{\rm con} < 1$ and $C_{\rm sum} \geq 1$ such that

(i) The weight function $h(\cdot)$ satisfies for all $T \in \mathcal{T} \in \mathbb{T}$, all $\widehat{\mathcal{T}} \in \mathbb{T}(\mathcal{T})$

$$
\begin{gathered}
h(\widehat{\mathcal{T}})|_T \neq h(\mathcal{T})|_T \text{ or } T \notin \widehat{\mathcal{T}} \\
\Longrightarrow \\
\max_{x\in T} h(\widehat{\mathcal{T}}) = \|h(\widehat{\mathcal{T}})\|_{L^\infty(T)} \leq q_{\rm con} \min_{x\in T} h(\mathcal{T}),
\end{gathered}
\tag{4.5.1}
$$

where $\neq$ is understood in the sense *not equal on a set with positive measure.* Note that this assumptions implies particularly $h(\widehat{\mathcal{T}}) \leq h(\mathcal{T})$ almost everywhere in Ω.

(ii) All $T \in \mathcal{T} \in \mathbb{T}$ and each sequence $T_i \in \widehat{\mathcal{T}}_i \in \mathbb{T}(\mathcal{T})$, $i = 1, \ldots N$ for some $N \in \mathbb{N}$ with $|T_i \cap T_j| = 0$ and $|T \cap T_i| > 0$ for $1 \leq i \neq j \leq N$ satisfy

$$
\sum_{i=1}^{N} \max_{x\in T_i} h(\widehat{\mathcal{T}}_i)^d \leq C_{\rm sum} \min_{x\in T} h(\mathcal{T})^d. \tag{4.5.2}
$$

Remark 4.5.1. *Assumption* (4.5.2) *implies that the abstract area of an element $h(\mathcal{T})|_T^d$ derived from the weight function, is additive up to constants.*

4.5.1. Definition of patches.

Given a constant $C_{\rm patch} > 0$ and a weight function $h(\mathcal{T})$ for all $\mathcal{T} \in \mathbb{T}$, a patch $\omega(\cdot, \cdot)$ satisfies the following properties:

(i) All $\mathcal{T} \in \mathbb{T}$ and all $\mathcal{S}, \mathcal{S}' \subseteq \mathcal{T}$ satisfy $\mathcal{S} \subseteq \omega(\mathcal{S}, \mathcal{T}) \subseteq \mathcal{T}$ and $\omega(\mathcal{S}, \mathcal{T}) \cup \omega(\mathcal{S}', \mathcal{T}) \subseteq \omega(\mathcal{S} \cup \mathcal{S}', \mathcal{T})$.

(ii) All $\mathcal{T} \in \mathbb{T}$ and all $\mathcal{S} \subseteq \mathcal{T}$ satisfy

$$
|\mathcal{S}| \leq |\omega(\mathcal{S}, \mathcal{T})| \leq C_{\rm patch} |\mathcal{S}|. \tag{4.5.3}
$$

(iii) All $\mathcal{S} \subseteq \mathcal{T} \in \mathbb{T}$ and all $\widehat{\mathcal{T}} \in \mathbb{T}(\mathcal{T})$ with $\mathcal{S} \subseteq \widehat{\mathcal{T}}$ satisfy

$$
\bigcup \omega(\mathcal{S}, \widehat{\mathcal{T}}) \subseteq \bigcup \omega^2(\mathcal{S}, \mathcal{T}), \tag{4.5.4}
$$

where $\omega^2(\mathcal{S}, \mathcal{T}) := \omega(\omega(\mathcal{S}, \mathcal{T}), \mathcal{T})$.

(iv) There holds for all $T \in \mathcal{T} \in \mathbb{T}$ and all $T' \in \omega(\{T\}, \mathcal{T})$

$$
C_{\rm patch}^{-1} \min_{x\in T'} h(\mathcal{T}) \leq h(\mathcal{T})|_T \leq C_{\rm patch} \max_{x\in T'} h(\mathcal{T}). \tag{4.5.5}
$$

For brevity of notation, we also write $\omega_k(T, \mathcal{T}) := \omega_k(\{T\}, \mathcal{T})$ for elements $T \in \mathcal{T}$.

4.5.2. Error estimators. Additionally to $\eta(\cdot)$ let $\widetilde{\eta}(\cdot)$ denote the equivalent error estimator from Section 4.2.1. Suppose that for all $\widetilde{\mathcal{M}}_\ell$ from Algorithm 4.2.1, the set $\mathcal{M}_\ell$ from (4.2.2a) satisfies

$$\mathcal{M}_\ell \subseteq \omega(\mathcal{T}_\ell \setminus \mathcal{T}_{\ell+1}, \mathcal{T}_\ell). \tag{4.5.6}$$

Finally, suppose that $\eta(\cdot)$ is a weighted error estimator as defined in Section 3.4.

Remark 4.5.2. *Examples of error estimators which fit in the abstract framework of this section can be found in Section 5.2.*

4.5.3. Optimal convergence. In the following, the notion that a certain subset of the axioms $\mathcal{A} \subseteq \{(\mathrm{E1}), \dots, (\mathrm{E3}), (\mathrm{T1}), \dots, (\mathrm{T3})\}$ is satisfied means that the axioms in $\mathcal{A}$ are satisfied for the error estimator $\eta(\cdot)$, the quantities from (4.5.7) below, the corresponding refinement strategy $\mathbb{T}(\cdot,\cdot)$, and the respective constants from Section 2.3.1. The triangulations $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}_0}$ in (E2), (T1)–(T3) are determined by $(\widetilde{\mathcal{T}}_\ell)_{\ell\in\mathbb{N}_0}$ via the function $\widetilde{(\cdot)}$.

The following theorem allows to drop the assumption of estimator reduction in Theorem 4.3.1 due to the additional assumptions in this section.

Theorem 4.5.3. *Under the assumptions of Section 4.5 (particularly* (4.5.1)–(4.5.5)*) and with homogeneity* (3.4.1) *and stability* (3.4.2)*, $\eta(\cdot)$ satisfies* (E1) *with*

$$\begin{aligned}
\mathcal{S}(\mathcal{T},\widehat{\mathcal{T}}) &:= \big\{T\in\mathcal{T} \,:\, h(\widehat{\mathcal{T}})|_T \le q_{\rm con} h(\mathcal{T})|_T\big\},\\
\widehat{\mathcal{S}}(\mathcal{T},\widehat{\mathcal{T}}) &:= \big\{T\in\widehat{\mathcal{T}} \,:\, T\subseteq \bigcup \mathcal{S}(\mathcal{T},\widehat{\mathcal{T}})\big\},\\
\rho_{\rm red} &= (1+\delta)q_{\rm con}^{2r_+},\\
\varrho(\mathcal{T},\widehat{\mathcal{T}}) &:= (1+\delta^{-1})^{1/2}\widetilde{\varrho}(\mathcal{T},\widehat{\mathcal{T}})
\end{aligned} \tag{4.5.7}$$

for all $\delta>0$ such that $\rho_{\rm red}<1$. Moreover, there holds the following:

(i) *Assume $\lim_{\ell\to\infty}\varrho(\mathcal{T}_{\ell+1},\mathcal{T}_\ell)=0$. Then, for all $0<\widetilde{\theta}\le 1$, the equivalent estimator is convergent in the sense*

$$\lim_{\ell\to\infty}\widetilde{\eta}(\widetilde{\mathcal{T}}_\ell)=0.$$

(ii) *Suppose* (E2) *is satisfied by $\eta(\cdot)$. Then, for all $0<\widetilde{\theta}\le 1$, the equivalent estimator is R-linear convergent in the sense that there exists $0<\widetilde{\rho}_{\rm conv}<1$ and $\widetilde{C}_{\rm conv}>0$ such that*

$$\widetilde{\eta}(\widetilde{\mathcal{T}}_{\ell+j})^2 \le \widetilde{C}_{\rm conv}\widetilde{\rho}_{\rm conv}^{\,j}\,\widetilde{\eta}(\widetilde{\mathcal{T}}_\ell)^2 \quad \textit{for all } j,\ell\in\mathbb{N}_0.$$

(iii) *Suppose that* (E2)–(E3) *and* (T1)–(T3) *are satisfied by $\eta(\cdot)$ for some $s>0$. Then $0<\widetilde{\theta}<C_{\rm eq}^{-1}\theta_\star = C_{\rm eq}^{-1}(1-\varepsilon_{\rm drel})/(1+C_{\rm drel}^2)$ implies quasi-optimal convergence of the estimator in the sense of*

$$\widetilde{c}_{\rm opt}C_{\rm approx}(s) \le \sup_{\ell\in\mathbb{N}_0}\frac{\widetilde{\eta}(\widetilde{\mathcal{T}}_\ell)}{(|\widetilde{\mathcal{T}}_\ell\setminus\widetilde{\mathcal{T}}_0|+1)^{-s}} \le \widetilde{C}_{\rm opt}C_{\rm approx}(s),$$

where the lower bound requires only (T1) *to hold.*

The constants $\widetilde{C}_{\rm conv}$, $\widetilde{\rho}_{\rm conv}>0$ depend only on $q_{\rm con}$, r_+, r_-, $C_{\rm qo}$, $\varepsilon_{\rm qo}$, $q_{\rm con}$, $C_{\rm patch}$, $C_{\rm sum}$, and on $\widetilde{\theta}$. The constant $\widetilde{C}_{\rm opt}>0$ depends additionally on $C_{\rm min}, C_{\rm ref}, C_{\rm closure}, C_{\rm drel}$, $\varepsilon_{\rm drel}$, and on s, while $\widetilde{c}_{\rm opt}>0$ depends only on $C_{\rm son}$ and $|\mathcal{T}_0|$.

PROOF. The assumption (4.5.1) implies that $h(\widehat{\mathcal{T}}) = h(\mathcal{T})$ on $\Omega \setminus \bigcup \mathcal{S}(\mathcal{T}, \widehat{\mathcal{T}})$. Therefore, Proposition 3.4.1 proves (E1) with (4.5.7). Since $\eta(\cdot)$ is a weighted error estimator, consider $\eta(\cdot, h_\omega(\cdot))$, where $h_\omega(\cdot)$ denotes the super contractive weight function $h_\omega(\cdot)$ from Proposition 4.5.4 below. The homogeneity (3.4.1) of $\eta(\cdot)$ and the equivalence (4.5.9) show for all $T \in \mathcal{T}$.

$$\min_{x\in T} |h_\omega(\mathcal{T})/h(\mathcal{T})|^{r_-}\, \eta_T(\mathcal{T}) \le \eta_T(\mathcal{T}, h_\omega(\mathcal{T})) \le \max_{x\in T} |h_\omega(\mathcal{T})/h(\mathcal{T})|^{r_+}\, \eta_T(\mathcal{T})$$

and hence

$$C_{19}^{-r_-}\, \eta_T(\mathcal{T}) \le \eta_T(\mathcal{T}, h_\omega(\mathcal{T})) \le \eta_T(\mathcal{T}). \tag{4.5.8}$$

Proposition 3.4.1 shows reduction (E1b) for the estimator $\eta(\cdot, h_\omega(\cdot))$ with $\mathcal{S}_\omega(\mathcal{T}, \widehat{\mathcal{T}}) := \big\{T \in \mathcal{T} : h_\omega(\widehat{\mathcal{T}})|_T \le q_{\mathrm{sc}} h_\omega(\mathcal{T})|_T\big\}$, $\widehat{\mathcal{S}}_\omega(\mathcal{T}, \widehat{\mathcal{T}}) := \big\{T \in \widehat{\mathcal{T}} : T \subseteq \bigcup \mathcal{S}(\mathcal{T}, \widehat{\mathcal{T}})\big\}$, and $\varrho(\cdot,\cdot)$ from (4.5.7). Moreover, monotonicity (4.5.11), homogeneity (3.4.1), and stability of the weighted error estimator (3.4.2) show

$$\begin{aligned}\Big(\sum_{T\in\widehat{\mathcal{T}}\setminus\widehat{\mathcal{S}}_\omega(\mathcal{T},\widehat{\mathcal{T}})} \eta_T(\widehat{\mathcal{T}}, h_\omega(\widehat{\mathcal{T}}))^2\Big)^{1/2} &\le \Big(\sum_{T\in\widehat{\mathcal{T}}\setminus\widehat{\mathcal{S}}_\omega(\mathcal{T},\widehat{\mathcal{T}})} \eta_T(\widehat{\mathcal{T}}, h_\omega(\mathcal{T}))^2\Big)^{1/2}\\ &\le \Big(\sum_{T\in\mathcal{T}\setminus\mathcal{S}_\omega(\mathcal{T},\widehat{\mathcal{T}})} \eta_T(\mathcal{T}, h_\omega(\mathcal{T}))^2\Big)^{1/2} + \widetilde{\varrho}(\mathcal{T}, \widehat{\mathcal{T}}).\end{aligned}$$

Since $\widetilde{\varrho}(\cdot,\cdot) \le \varrho(\cdot,\cdot)$, this shows stability (2.3.5). By (4.5.1) and Proposition 4.5.4 (ii), one obtains $\omega(T, \mathcal{T}) \subseteq \mathcal{S}_\omega(\mathcal{T}, \widehat{\mathcal{T}})$ for all $T \in \mathcal{T}\setminus\widehat{\mathcal{T}}$. By assumption (i) in Section 4.5.1, this shows $\omega(\mathcal{T} \setminus \widehat{\mathcal{T}}, \mathcal{T}) \subseteq \mathcal{S}_\omega(\mathcal{T}, \widehat{\mathcal{T}})$ and the assumption (4.5.6) implies $\mathcal{M}_\ell \subseteq \mathcal{S}_\omega(\mathcal{T}_\ell, \mathcal{T}_{\ell+1})$. Since $\widetilde{\mathcal{M}}_\ell$ satisfies Dörfler marking (4.2.3), (4.2.2a) shows for all $\ell \in \mathbb{N}_0$

$$C_{\mathrm{eq}}^{-1}\widetilde{\theta}\, \eta(\mathcal{T}_\ell)^2 \le \sum_{T\in\mathcal{S}_\omega(\mathcal{T}_\ell,\mathcal{T}_{\ell+1})} \eta_T(\mathcal{T}_\ell)^2.$$

This and (4.5.8) imply immediately for all $\ell \in \mathbb{N}_0$

$$C_{\mathrm{eq}}^{-1} C_{19}^{-r_-}\widetilde{\theta}\, \eta(\mathcal{T}_\ell, h_\omega(\mathcal{T}_\ell))^2 \le \sum_{T\in\mathcal{S}_\omega(\mathcal{T}_\ell,\mathcal{T}_{\ell+1})} \eta_T(\mathcal{T}_\ell, h_\omega(\mathcal{T}_\ell))^2.$$

Therefore, Lemma 2.3.5 with $\widehat{\mathcal{T}} = \mathcal{T}_{\ell+1}$ and $\mathcal{T} = \mathcal{T}_\ell$ shows that estimator reduction (2.3.6) and hence (2.3.8) holds for all $\ell \in \mathbb{N}_0$ and $\eta(\mathcal{T}_\ell, h_\omega(\mathcal{T}_\ell))$. Since $\widetilde{\varrho}(\cdot,\cdot) \simeq \varrho(\cdot,\cdot)$, Lemma 2.3.6 shows $\lim_{\ell\to\infty} \eta(\mathcal{T}_\ell, h_\omega(\mathcal{T}_\ell)) = 0$ under the assumptions of (i). Equivalence (4.5.8) shows $\lim_{\ell\to\infty} \eta(\mathcal{T}_\ell) = 0$ and (4.2.1) implies (i).

Since (2.3.8) holds for all $\ell \in \mathbb{N}_0$ and $\eta(\mathcal{T}_\ell, h_\omega(\mathcal{T}_\ell))$, Proposition 2.3.9 shows that the general quasi-orthogonality (E2) implies R-linear convergence (2.3.14) with $\alpha_\ell = \eta(\mathcal{T}_\ell, h_\omega(\mathcal{T}_\ell))$. Again (4.5.8) and (4.2.1) imply (ii).

The R-linear convergence from (ii), (4.5.7) and the assumptions from (iii) imply the assumptions of Theorem 4.3.1 (iii). This proves (iii) and concludes the proof. □

4.5.4. Super contractive weight function. The next proposition defines an equivalent weight function $h_\omega(\cdot)$, which contracts even if $h(\cdot)$ contracts only nearby (namely within the patch). To that end, recall the definition of $\max_{x\in T}$ and $\min_{x\in T}$ from Section 4.5.

PROPOSITION 4.5.4. *Suppose a weight function $h(\cdot)$ with $h(\mathcal{T}) \in L^\infty(\Omega)$ for all $\mathcal{T} \in \mathbb{T}$. Moreover, we assume that* (4.5.1) *and* (4.5.2) *are satisfied and that $h(\mathcal{T})$ is continuous on $\Omega \setminus \bigcup_{T\in\mathcal{T}} \partial T$. Let $\omega(\cdot,\cdot)$ denote a patch function which satisfies* (4.5.3)–(4.5.5)*. Then, there*

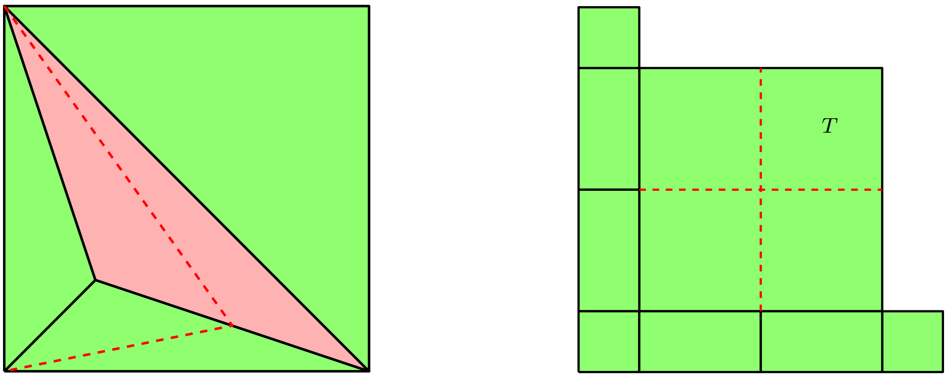

FIGURE 1. Consider the standard patch from Remark 4.5.5. Then, the patch area of the red triangle in the left figure coincides with the patch area of each of its two sons after two bisections. The area of the large green square in the right figure is 1. The average of areas in its patch is smaller than 0.22. After two bisections, the average of areas of the patch of T is 0.25.

exists a super contractive weight function $h_\omega(\cdot)$ such that $h_\omega(\mathcal{T})$ is $\mathcal{T}$-piecewise constant for all $\mathcal{T} \in \mathbb{T}$, which satisfies (i)–(iii).

(i) ***Equivalence**: For all $\mathcal{T} \in \mathbb{T}$ and all $T \in \mathcal{T}$, it holds:*

$$C_{\mathrm{eq}}^{-1} \min_{x \in T} h(\mathcal{T}) \leq h_\omega(\mathcal{T})|_T \leq \min_{x \in T} h(\mathcal{T}). \tag{4.5.9}$$

(ii) ***Contraction on the patch**: All refinements $\widehat{\mathcal{T}} \in \mathbb{T}(\mathcal{T})$ and all $T \in \mathcal{T}$ satisfy*

$$h_\omega(\widehat{\mathcal{T}})|_T \leq q_{\mathrm{sc}} h_\omega(\mathcal{T})|_T \quad \text{if } h(\mathcal{T})|_{\cup\omega(T,\mathcal{T})} \neq h(\widehat{\mathcal{T}})|_{\cup\omega(T,\mathcal{T})}. \tag{4.5.10}$$

(iii) ***Monotonicity**: All refinements $\widehat{\mathcal{T}} \in \mathbb{T}$ of a triangulation $\mathcal{T} \in \mathbb{T}$ satisfy*

$$h_\omega(\widehat{\mathcal{T}}) \leq h_\omega(\mathcal{T}) \quad \text{almost everywhere in } \Omega. \tag{4.5.11}$$

The constants $C_{19} \geq 1$ and $0 < q_{\mathrm{sc}} < 1$ depend only on C_{patch}, C_{sum}, d, and q_{con}.

REMARK **4.5.5.** *A typical example would be $h(\mathcal{T})|_T := |T|^{1/d}$ and the standard patch function $\omega(\mathcal{S}, \mathcal{T}) := \{T \in \mathcal{T} : \exists T' \in \mathcal{S},\, T \cap T' \neq \emptyset\}$ for some $\mathbb{T}$ generated by bisection from Section 3.2.8. Then, Proposition 4.5.4 provides a super contractive weight function $h_\omega(\mathcal{T})$ which satisfies $h_\omega(\widehat{\mathcal{T}})|_T \leq q_{\mathrm{sc}} h_\omega(\mathcal{T})|_T$ for all $T \in \omega(\mathcal{T} \setminus \widehat{\mathcal{T}}, \mathcal{T})$.*

Even for very specific refinement strategies, i.e., bisection from Section 3.2.8, the straightforward constructions of $h_\omega(\cdot)$ by averaging over the patch or by considering the area of the patch fail to satisfy (i)–(iii). See Figure 1 for some counterexamples.

The proof of Proposition 4.5.4 requires the next three lemmas, which consider an arbitrary sequence of consecutive triangulations

$$(\mathcal{T}_\ell)_{\ell \in \mathbb{N}} \subset \mathbb{T} \quad \text{with} \quad \mathcal{T}_{\ell+1} \in \mathbb{T}(\mathcal{T}_\ell) \quad \text{for all } \ell \in \mathbb{N}_0. \tag{4.5.12}$$

Note that throughout this section $(\mathcal{T}_\ell)$ is not necessarily the sequence generated by Algorithm 2.2.1.

LEMMA **4.5.6.** *Under the assumptions of Proposition 4.5.4 and given* (4.5.12) *and $\ell, N \in \mathbb{N}_0$, suppose a strictly monotone sequence $0 \leq m_0 < m_1 < \ldots < m_N \in \mathbb{N}_0$ with $h(\mathcal{T}_{\ell+m_N})|_T = h(\mathcal{T}_\ell)|_T$ for some $T \in \bigcap_{j=\ell}^{m_N} \mathcal{T}_j$. Suppose there exist elements $T_i \in \omega(T, \mathcal{T}_{\ell+m_i})$,*

$i = 0, \ldots, N$ *such that all* $i = 0, \ldots, N-1$ *satisfy*

$$\min_{x \in T_{i+1}} h(\mathcal{T}_{\ell+m_{i+1}}) \le \max_{x \in T_i} h(\mathcal{T}_{\ell+m_{i+1}}) \le q_{\rm con} \min_{x \in T_i} h(\mathcal{T}_{\ell+m_i}). \tag{4.5.13}$$

Then, $N \le 2\log(C_{\rm patch})/|\log(q_{\rm con})|$.

Proof. The assumptions imply $\max_{x\in T_N} h(\mathcal{T}_{\ell+m_N}) \le q_{\rm con}^N \min_{x\in T_0} h(\mathcal{T}_\ell)$. The assumption (4.5.5) shows

$$\begin{aligned} h(\mathcal{T}_\ell)|_T = h(\mathcal{T}_{\ell+m_N})|_T &\le C_{\rm patch} \max_{x\in T_N} h(\mathcal{T}_{\ell+m_N}) \\ &\le C_{\rm patch} q_{\rm con}^N \min_{x\in T_0} h(\mathcal{T}_\ell) \le C_{\rm patch}^2 q_{\rm con}^N h(\mathcal{T}_\ell)|_T. \end{aligned} \tag{4.5.14}$$

This implies that N is bounded above by the restriction $1 \le C_{\rm patch}^2 q_{\rm con}^N$. □

Lemma 4.5.7. *Under the assumptions of Proposition 4.5.4 and given* (4.5.12) *and* $\ell, N \in \mathbb{N}_0$, *suppose a strictly monotone sequence* $0 \le m_0 < m_1 < \ldots < m_N \in \mathbb{N}_0$ *with* $h(\mathcal{T}_{\ell+m_N})|_T = h(\mathcal{T}_\ell)|_T$ *for some* $T \in \bigcap_{j=\ell}^{m_N} \mathcal{T}_j$. *Suppose that for all* $i = 0, \ldots, N-1$ *exists* $T_i \in \omega(T, \mathcal{T}_{\ell+m_i})$ *with*

$$\max_{x\in T_i} h(\mathcal{T}_{\ell+m_{i+1}}) \le q_{\rm con} \min_{x\in T_i} h(\mathcal{T}_{\ell+m_i}). \tag{4.5.15}$$

Then, $N \le 2\log(C_{\rm patch})/|\log(q_{\rm con})| C_{\rm sum} C_{\rm patch}^{2d+2}$.

Proof. For all $T' \in \omega^2(T, \mathcal{T}_\ell)$ define

$$\alpha_{T'} := \big\{T_i \text{ from (4.5.15)} \,:\, |T_i \cap T'| > 0\big\}.$$

Since $\bigcup \omega(T, \mathcal{T}_{\ell+m_i}) \subseteq \bigcup \omega^2(T, \mathcal{T}_\ell)$ for all $i = 0, \ldots, N$ by definition of the patch, and $|\omega^2(T, \mathcal{T}_\ell)| \le C_{\rm patch}^2$, there exists at least one $T_0' \in \omega^2(T, \mathcal{T}_\ell)$ with $n := |\alpha_{T_0'}| \ge N/C_{\rm patch}^2$. Let now $\alpha_{T_0'} = \{T_{i_1}, \ldots, T_{i_n}\}$ such that $i_1 \le i_2 \le \ldots \le i_n$. We define a directed graph $\mathcal{G}$ with set of vertices $\alpha_{T_0'}$. Two vertices $T_{i_j}, T_{i_k} \in \alpha_{T_0'}$ are connected by an edge $E_{jk} \in \mathcal{G}$ if and only if there holds

$$\min_{x\in T_{i_k}} h(\mathcal{T}_{\ell+m_{i_k}}) \le \max_{x\in T_{i_j}} h(\mathcal{T}_{\ell+m_{i_k}}) \le q_{\rm con} \min_{x\in T_{i_j}} h(\mathcal{T}_{\ell+m_{i_j}}). \tag{4.5.16}$$

With (4.5.1), the fact $E_{jk} \in \mathcal{G}$ implies immediately $k > j$ and hence prohibits $E_{kj} \in \mathcal{G}$. Therefore, any path $\mathcal{E} := \{E_{j_0 j_1}, E_{j_1, j_2}, \ldots, E_{j_{m-1} j_m}\} \subseteq \mathcal{G}$ satisfies $j_1 < j_2 < \ldots < j_m$ and thus can't be closed. Moreover, the corresponding vertices $T_{i_{j_k}}$, $k = 0, \ldots, m$ satisfy the requirements of Lemma 4.5.6. This shows

$$|\mathcal{E}| = m \le m_{\max} := 2\log(C_{\rm patch})/|\log(q_{\rm con})|. \tag{4.5.17}$$

Consider the set of leafs $\mathcal{L}_0 := \big\{T_{i_j} \in \alpha_{T_0'} \,:\, \forall E_{j_1 j_2} \in \mathcal{G},\, j_1 \neq j\big\}$ of $\mathcal{G}$. Moreover, for $k \in \mathbb{N}$ define the set of leafs $\mathcal{L}_k$ of the subgraph $\mathcal{G}_k$ on the reduced vertices set $\alpha_{T_0'} \setminus \bigcup_{j=0}^{k-1} \mathcal{L}_j$. Since no closed path $\mathcal{E}$ can exist, any path $\mathcal{E}$ which is maximal with respect to $\subseteq$, must end with a leaf.

First, we prove

$$\bigcup_{j=0}^{m_{\max}} \mathcal{L}_j = \alpha_{T_0'}. \tag{4.5.18}$$

To that end, we show by induction that any path $\mathcal{E} \subseteq \mathcal{G}_k$ satisfies

$$|\mathcal{E}| \le m_{\max} - k. \tag{4.5.19}$$

For $k = 0$ and $\mathcal{G}_0 := \mathcal{G}$ this is (4.5.17). Assume the induction hypothesis (4.5.19) holds for $k > 0$. Since a path $\mathcal{E} \subseteq \mathcal{G}_{k+1}$, which is maximal with respect to $\subseteq$, must end with a leaf, it can not be maximal in $\mathcal{G}_k$ (otherwise the leaf is in $\mathcal{L}_k$ and hence not in $\alpha_{T_0'} \setminus \bigcup_{j=0}^{k} \mathcal{L}_j$ which is the vertex set of $\mathcal{G}_{k+1}$). This implies the existence of a path $\mathcal{E}' \subseteq \mathcal{G}_k$ with $|\mathcal{E}| < |\mathcal{E}'| \leq m_{\max} - k$ and hence proves the hypothesis (4.5.19) for $k+1$. Induction concludes (4.5.19) for all $0 \leq k \leq m_{\max}$. Since no path of positive length can exist in $\mathcal{G}_{m_{\max}}$, there holds $\mathcal{L}_{m_{\max}} = \mathcal{G}_{m_{\max}}$. This implies $\mathcal{L}_{m_{\max}+1} = \emptyset$ and hence (4.5.18).

By definition, the $\mathcal{L}_j$ are disjoint. Therefore (4.5.18) implies that there exists $0 \leq j_0 \leq m_{\max}$ such that

$$|\mathcal{L}_{j_0}| \geq |\alpha_{T_0'}|/m_{\max}. \tag{4.5.20}$$

Assume there holds $|T_{i_j} \cap T_{i_k}| > 0$ for $T_{i_j}, T_{i_k} \in \mathcal{L}_{j_0}$ with $T_{i_j} \neq T_{i_k}$. Then, by definition in Section 3.2.1, there holds $i_j \neq i_k$. Without loss of generality, assume $i_j < i_k$. Since $|T_{i_j} \cap T_{i_k}| > 0$, there holds $T_{i_j} \notin \mathcal{T}_{\ell+m_{i_k}}$, and hence by (4.5.1), there holds $\max_{x\in T_{i_j}} h(\mathcal{T}_{\ell+m_{i_k}}) \leq q_{\mathrm{con}} \min_{T_{i_j}} h(\mathcal{T}_{\ell+m_{i_j}})$. This and $|T_{i_j} \cap T_{i_k}| > 0$ imply (4.5.16) and hence $E_{jk} \in \mathcal{G}_{j_0}$. This, however, contradicts the definition of $\mathcal{L}_{j_0}$ as a set of leafs. Therefore, all elements of $\mathcal{L}_{j_0}$ have pairwise intersections with measure zero. Hence, (4.5.5) and (4.5.2) imply

$$\begin{aligned} C_{\mathrm{patch}}^{-d} \sum_{T_{i_j}\in\mathcal{L}_{j_0}} \min_{x\in T} h(\mathcal{T}_{\ell+m_{i_j}})^d &\leq \sum_{T_{i_j}\in\mathcal{L}_{j_0}} \max_{x\in T_{i_j}} h(\mathcal{T}_{\ell+m_{i_j}})^d \\ &\leq C_{\mathrm{sum}} \min_{x\in T_0'} h(\mathcal{T}_\ell)^d \leq C_{\mathrm{sum}} C_{\mathrm{patch}}^d \min_{x\in T} h(\mathcal{T}_\ell)^d. \end{aligned}$$

This and the assumption $h(\mathcal{T}_{\ell+m_N})|_T = h(\mathcal{T}_\ell)|_T = h(\mathcal{T}_{\ell+m_i})|_T$ for all $i = 0, \ldots, N$ imply

$$|\mathcal{L}_{j_0}| \leq C_{\mathrm{sum}} C_{\mathrm{patch}}^{2d}.$$

Together with (4.5.20), this implies

$$N/C_{\mathrm{patch}}^2 \leq |\alpha_{T_0'}| \leq m_{\max} C_{\mathrm{sum}} C_{\mathrm{patch}}^{2d}$$

and concludes the proof. □

Lemma 4.5.8. *Under the assumptions of Proposition 4.5.4 and given* (4.5.12), *there exists a weight function* $\widetilde{h}_\omega(\mathcal{T}_\ell)$ *which satisfies for all* $\ell \in \mathbb{N}_0$ (i)–(iii)

(i) *All* $T \in \mathcal{T}_\ell$ *satisfy:*

$$q_{\mathrm{con}}^{N_{\max}/(N_{\max}+1)} \min_{x\in T} h(\mathcal{T}_\ell) \leq \widetilde{h}_\omega(\mathcal{T}_\ell)|_T \leq h(\mathcal{T}_\ell)|_T \quad \textit{pointwise almost everywhere.}$$

(ii) *All* $T \in \mathcal{T}_\ell$ *and all* $k \geq \ell$ *satisfy*

$$\max_{x\in T} \widetilde{h}_\omega(\mathcal{T}_k) \leq q_{\mathrm{con}}^{1/(N_{\max}+1)} \min_{x\in T} \widetilde{h}_\omega(\mathcal{T}_\ell)|_T \quad \textit{if } h(\mathcal{T}_\ell)|_{\cup\omega(T,\mathcal{T}_\ell)} \neq h(\mathcal{T}_k)|_{\cup\omega(T,\mathcal{T}_\ell)}.$$

(iii) *All* $k \geq \ell$ *satisfy*

$$\widetilde{h}_\omega(\mathcal{T}_k) \leq \widetilde{h}_\omega(\mathcal{T}_\ell) \quad \textit{almost everywhere in } \Omega.$$

There holds $N_{\max} := 2\log(C_{\mathrm{patch}})/|\log(q_{\mathrm{con}})| C_{\mathrm{sum}} C_{\mathrm{patch}}^{2d+2}$.

PROOF. For $\ell = 0$, set $\widetilde{h}_\omega(\mathcal{T}_0) = h(\mathcal{T}_0)$. For $\ell \geq 0$ and for all $T \in \mathcal{T}_\ell$ set

$$\widetilde{h}_\omega(\mathcal{T}_{\ell+1})|_T := \begin{cases} h(\mathcal{T}_{\ell+1})|_T & \text{case 1: } h(\mathcal{T}_{\ell+1})|_T \neq h(\mathcal{T}_\ell)|_T, \\ q_{\mathrm{con}}^{1/(N_{\max}+1)} \min_{x\in T} \widetilde{h}_\omega(\mathcal{T}_\ell) & \text{case 2: } \substack{h(\mathcal{T}_\ell)|_{\cup\omega(T,\mathcal{T}_\ell)} \neq h(\mathcal{T}_{\ell+1})|_{\cup\omega(T,\mathcal{T}_\ell)} \\ h(\mathcal{T}_\ell)|_T = h(\mathcal{T}_{\ell+1})|_T}, \\ \widetilde{h}_\omega(\mathcal{T}_\ell)|_T & \text{case 3: else.} \end{cases}$$

The upper bound in (i) follows immediately by induction on $\ell \in \mathbb{N}$: It holds for $\ell = 0$. Assume the upper bound holds for $\ell \in \mathbb{N}$. Then, the definition of $\widetilde{h}_\omega(\mathcal{T}_{\ell+1})$ implies for $T \in \mathcal{T}_{\ell+1}$ and all $T' \in \mathcal{T}_\ell$ with $|T' \cap T| > 0$

$$\widetilde{h}_\omega(\mathcal{T}_{\ell+1})|_{T\cap T'} \leq \begin{cases} h(\mathcal{T}_{\ell+1})|_{T\cap T'} & \text{case 1,} \\ \widetilde{h}_\omega(\mathcal{T}_\ell)|_{T\cap T'} & \text{case 2 and 3.} \end{cases}$$

The induction hypothesis for case 2–3 and the monotonicity from (4.5.1) for case 1 prove $\widetilde{h}_\omega(\mathcal{T}_{\ell+1})|_{T\cap T'} \leq h(\mathcal{T}_\ell)|_{T\cap T'}$. This concludes the induction. The lower bound (i) follows by contradiction. Consider an element $T \in \mathcal{T}_j$, $j \in \mathbb{N}$, with

$$\min_{x\in T} \widetilde{h}_\omega(\mathcal{T}_j) < q_{\mathrm{con}}^{N_{\max}/(N_{\max}+1)} \min_{x\in T} h(\mathcal{T}_j). \tag{4.5.21}$$

Let $\ell \leq j$ be the minimal index with $T \in \mathcal{T}_\ell$. If $\ell = 0$, there holds $\widetilde{h}_\omega(\mathcal{T}_0)|_T = h(\mathcal{T}_0)|_T$ by definition. For $\ell > 0$, the assumption (4.5.1) implies $h(\mathcal{T}_\ell)|_{T'} \neq h(\mathcal{T}_{\ell-1})|_{T'}$ for all $T' \in \mathcal{T}_{\ell-1}$ with $|T' \cap T| > 0$ and hence by definition $\widetilde{h}_\omega(\mathcal{T}_\ell)|_{T'} = h(\mathcal{T}_\ell)|_{T'}$ (case 1). Altogether, we have an index $0 \leq \ell \leq j$ with $\widetilde{h}_\omega(\mathcal{T}_\ell)|_T = h(\mathcal{T}_\ell)|_T$. We redefine $\ell \leq j$ to denote the largest index smaller or equal to j with $\widetilde{h}_\omega(\mathcal{T}_\ell)|_T = h(\mathcal{T}_\ell)|_T$. Therefore, *case 1* cannot occur for any index $\ell < i < j$. This implies also $T \in \bigcap_{i=\ell}^{j-1} \mathcal{T}_i$. To obtain (4.5.21), there must exist at least $N_{\max} + 1$ indices $\ell + m_i < j$ with *case 2*. This particularly implies $h(\mathcal{T}_{\ell+m_{N_{\max}+1}})|_T = h(\mathcal{T}_\ell)|_T$ and $T \in \bigcap_{j=\ell}^{\ell+m_{N_{\max}}} \mathcal{T}_j$. We aim to verify the remaining assumptions of Lemma 4.5.7. To that end, note that *case 2* for $T \in \mathcal{T}_{\ell+m_i}$ and (4.5.1) imply the existence of $T_i \in \omega(T, \mathcal{T}_{\ell+m_i})$ with $\max_{x\in T_i} h(\mathcal{T}_{\ell+m_i+1}) \leq q_{\mathrm{con}} \min_{x\in T_i} h(\mathcal{T}_{\ell+m_i})$. The monotonicity of $h(\mathcal{T}_\ell)$ from (4.5.1) and $\ell + m_i + 1 \leq \ell + m_{i+1}$ imply even (4.5.15). Hence, the requirements of Lemma 4.5.7 are satisfied and the contradiction $N_{\max} + 1 \leq 2\log(C_{\mathrm{patch}})/|\log(q_{\mathrm{con}})| C_{\mathrm{sum}} C_{\mathrm{patch}}^{2d+2} = N_{\max}$ follows. This proves the lower bound in (i).

To prove the contraction estimate (ii), distinguish two cases. If $T \in \mathcal{T}_\ell$ satisfies *case 1* in the definition of $\widetilde{h}_\omega(\cdot)$, then, with the lower bound in (i) and (4.5.1), it holds

$$\begin{aligned} \max_{x\in T} \widetilde{h}_\omega(\mathcal{T}_{\ell+1}) &= \max_{x\in T} h(\mathcal{T}_{\ell+1}) \leq q_{\mathrm{con}} \min_{x\in T} h(\mathcal{T}_\ell) \\ &\leq q_{\mathrm{con}}\, q_{\mathrm{con}}^{-N_{\max}/(N_{\max}+1)} \min_{x\in T} \widetilde{h}_\omega(\mathcal{T}_\ell) = q_{\mathrm{con}}^{1/(N_{\max}+1)} \min_{x\in T} \widetilde{h}_\omega(\mathcal{T}_\ell). \end{aligned} \tag{4.5.22}$$

If $T \in \mathcal{T}_\ell$ satisfies *case 2* in the definition of $\widetilde{h}_\omega(\cdot)$, then, it holds

$$\max_{x\in T} \widetilde{h}_\omega(\mathcal{T}_{\ell+1}) = q_{\mathrm{con}}^{1/(N_{\max}+1)} \min_{x\in T} \widetilde{h}_\omega(\mathcal{T}_\ell). \tag{4.5.23}$$

Each case leads to some contraction with constant $q_{\mathrm{sc}} = q_{\mathrm{con}}^{1/(N_{\max}+1)} \in (0, 1)$.

This also implies monotonicity (iii) for *case 1* and *case 2*. Let $T \in \mathcal{T}_\ell$ which satisfies *case 3*. The definition shows

$$\widetilde{h}_\omega(\mathcal{T}_{\ell+1})|_T = \widetilde{h}_\omega(\mathcal{T}_\ell)|_T$$

and hence (iii). This concludes the proof. $\square$

Proof of Proposition 4.5.4. The weight function $\widetilde{h}_\omega(\cdot)$ depends on the sequence $(\mathcal{T}_\ell)$ from (4.5.12). Hence, we write

$$\widetilde{h}_\omega(\mathcal{T}_\ell) = \widetilde{h}_\omega(\mathcal{T}_0, \ldots, \mathcal{T}_\ell).$$

Given $\mathcal{T} \in \mathbb{T}$, define the set of all sequences which lead to that particular triangulation, i.e.,

$$\mathbb{T}(\mathcal{T}_0, \mathcal{T}) := \big\{(\mathcal{T}_0, \ldots, \mathcal{T}_\ell = \mathcal{T}) \,:\, \ell \in \mathbb{N},\, \mathcal{T}_{j+1} \in \mathbb{T}(\mathcal{T}_j) \setminus \{\mathcal{T}_j\} \text{ for all } j = 0, \ldots, \ell - 1\big\}.$$

The definition of the refinement strategy $\mathbb{T}(\cdot,\cdot)$ in Section 2.2.1 implies that $\mathbb{T}(\mathcal{T}_0, \mathcal{T})$ is finite. Define $h_\omega(\mathcal{T}_0)|_T := \min_{x\in T} h(\mathcal{T}_0)$ for all $T \in \mathcal{T}_0$ and for $\mathcal{T} \in \mathbb{T} \setminus \{\mathcal{T}_0\}$ by

$$h_\omega(\mathcal{T})|_T := \min_{(\mathcal{T}_0,\ldots,\mathcal{T}_\ell)\in\mathbb{T}(\mathcal{T}_0,\mathcal{T})} \min_{x\in T} \widetilde{h}_\omega(\mathcal{T}_0, \ldots, \mathcal{T}_\ell) \quad \text{for all } T \in \mathcal{T}.$$

We denote by $(\mathcal{T}_0^T, \ldots, \mathcal{T}_\ell^T) \in \mathbb{T}(\mathcal{T}_0, \mathcal{T})$ a sequence which satisfies

$$\min_{x\in T} \widetilde{h}_\omega(\mathcal{T}_0^T, \ldots, \mathcal{T}_\ell^T) = h_\omega(\mathcal{T})|_T.$$

To see the equivalence (4.5.9), Lemma 4.5.8 (i) shows

$$\min_{x\in T} h(\mathcal{T}) \lesssim \min_{x\in T} \widetilde{h}_\omega(\mathcal{T}_0^T, \ldots, \mathcal{T}_\ell^T) \le \min_{x\in T} h(\mathcal{T}),$$

where the hidden constants do not depend on the particular sequence $\mathcal{T}_0^T, \ldots, \mathcal{T}_\ell^T$. This implies (4.5.9).

The contraction property (4.5.10) follows with Lemma 4.5.8 (ii). To see that, let $T \in \mathcal{T}$ with $h(\mathcal{T})|_{\cup\omega(T,\mathcal{T})} \neq h(\widehat{\mathcal{T}})|_{\cup\omega(T,\mathcal{T})}$. There holds $(\mathcal{T}_0^T, \ldots, \mathcal{T}_\ell^T, \widehat{\mathcal{T}}) \in \mathbb{T}(\mathcal{T}_0, \widehat{\mathcal{T}})$ and hence for all $T' \in \widehat{\mathcal{T}}$ with $|T' \cap T| > 0$

$$\begin{aligned} h_\omega(\widehat{\mathcal{T}})|_{T'} &\le \min_{x\in T'} \widetilde{h}_\omega(\mathcal{T}_0^T, \ldots, \mathcal{T}_\ell^T, \widehat{\mathcal{T}}) \\ &\le \max_{x\in T} \widetilde{h}_\omega(\mathcal{T}_0^T, \ldots, \mathcal{T}_\ell^T, \widehat{\mathcal{T}}) \\ &\le q_{\rm con}^{1/(N_{\max}+1)} \min_{x\in T} \widetilde{h}_\omega(\mathcal{T}_0^T, \ldots, \mathcal{T}_\ell^T) = q_{\rm con}^{1/(N_{\max}+1)} h_\omega(\mathcal{T})|_T. \end{aligned} \tag{4.5.24}$$

Since the involved constants do not depend on the particular sequence $\mathcal{T}_0^T, \ldots, \mathcal{T}_\ell^T$, this shows (4.5.10) with $q_{\rm sc} = q_{\rm con}^{1/(N_{\max}+1)}$.

Finally, we show (4.5.11). Therefore, let $T \in \mathcal{T}$ and $\widehat{\mathcal{T}} \in \mathbb{T}(\mathcal{T})$. If $T \neq \widehat{\mathcal{T}}$, the contraction (4.5.24) applies and shows monotonicity (4.5.11) on T. If $T \in \widehat{\mathcal{T}}$, Lemma 4.5.8 (iii) implies

$$\begin{aligned} h_\omega(\widehat{\mathcal{T}})|_T &\le \min_{x\in T} \widetilde{h}_\omega(\mathcal{T}_0^T, \ldots, \mathcal{T}_\ell^T, \widehat{\mathcal{T}}) \\ &\le \min_{x\in T} \widetilde{h}_\omega(\mathcal{T}_0^T, \ldots, \mathcal{T}_\ell^T) = h_\omega(\mathcal{T})|_T. \end{aligned}$$

This concludes the proof. □

CHAPTER 5

Applications II

5.1. Introduction, state of the art & outline

This chapter applies the abstract results from the previous chapter to certain model problems. The examples below are found in a similar manner in [**24**]. Note that the super contractive weight function from Section 4.5.4 allows to prove optimal convergence rates, even if the equivalence of the error estimators is only patch wise. This is a major improvement over [**60**], where all the patches are refined, too. Moreover the super contractive patch function is used in Section 5.4 to prove the contraction of data oscillations. This improves the work [**4**], where a modified marking strategy is employed to overcome this problem. The remainder of the chapter is organized as follows: Section 5.2 shows rate optimality for certain estimators which are equivalent to the residual estimator from Section 3.5.1. Section 5.3 reproduces the results of [**13**] for the p-Laplacian and Section 5.4 demonstrates the incorporation of inhomogeneous boundary data into the optimality analysis.

5.2. Example 1: Locally equivalent error estimators for the Poisson problem

This section applies the analysis Chapter 4 to a specific model problem, where the adaptive algorithm is steered by some locally equivalent and possibly non-residual error estimator.

5.2.1. Poisson model problem. In the spirit of [**60**], consider the Poisson model problem (3.5.1) in $\Omega \subseteq \mathbb{R}^d$,

$$-\Delta u = f \quad \text{in } \Omega \quad \text{and} \quad u = 0 \quad \text{on } \Gamma,$$

and recall the weak formulation (3.5.2), and the FE discretization (3.5.3) by means of piecewise polynomials $\mathcal{S}_0^p(\mathcal{T}) = \mathcal{P}^p(\mathcal{T}) \cap H_0^1(\Omega)$ of degree $p \geq 1$. The residual error estimator $\eta(\cdot)$ with local contributions

$$\eta_T(\mathcal{T})^2 = \eta_T(\mathcal{T}, h(\mathcal{T}))^2 := h(\mathcal{T})|_T^2 \, \|f + \Delta_{\mathcal{T}} V\|_{L^2(T)}^2 + h(\mathcal{T})|_T \, \|[\partial_n V]\|_{L^2(\partial T \cap \Omega)}^2 \qquad (5.2.1)$$

with $h(\mathcal{T})|_T := |T|^{1/d}$ for all $T \in \mathcal{T}$ and $\Delta_{\mathcal{T}}$ the $\mathcal{T}$-element wise Laplacian serves as a theoretical tool. Under the assumptions of Section 3.5.1 or Section 3.5.2 (particularly that $\mathbb{T}(\cdot,\cdot)$ is a refinement strategy in the sense of Section (3.2.1)–(3.2.7)), the following result holds.

Proposition 5.2.1. *In addition to the properties stated in Proposition 3.5.1, the residual error estimator* (5.2.1) *satisfies homogeneity* (3.4.1) *with* $r_+ = 1/2$ *and* $r_- = 1$ *and stability* (3.4.2) *with* $\widetilde{\varrho}(\cdot,\cdot) = \varrho(\cdot,\cdot)$.

Proof. Stability (3.4.2) is well-known and follows by use of the triangle inequality as well as standard inverse estimates analogously to the proof of [**35**, Corollary 3.4]. The homogeneity (3.4.1) is obvious. □

The following sections concern different error estimators $\widetilde{\eta}(\cdot)$ which are equivalent to $\eta(\cdot)$ and fit into the framework of Section 4.5. Section 5.2.2 studies the influence of equivalent choices of the weight function $h(\mathcal{T})$ for the residual error estimator (This is well-known by experts but does not appear in the literature except for the recent own work [**24**]. Moreover, it fits perfectly into the abstract framework of Chapter 4). Section 5.2.3 concerns a facet-based formulation of $\eta(\cdot)$, while Section 5.2.4 analyzes recovery-based error estimators. Further examples for the lowest-order case $p = 1$, which also fit in the frame of the analysis from Section 4.5, are found in [**60**].

5.2.2. Estimator based on equivalent weight function. This section is based on the recent own work [**24**, Section 9]. Instead of $|T|^{1/d}$ for weighting the local contributions of $\eta(\cdot)$, one can also use the local diameter $\mathrm{diam}(T)$. This leads to

$$\widetilde{\eta}_T(\mathcal{T})^2 := \mathrm{diam}(T)^2 \, \|f + \Delta V\|_{L^2(T)}^2 + \mathrm{diam}(T) \, \|[\partial_n V]\|_{L^2(\partial T \cap \Omega)}^2.$$

This variant of $\eta(\cdot)$ is usually found in textbooks as e.g. [**1, 82**]. Under the assumptions of Section 3.5.1 or Section 3.5.2 the shape regularity (3.2.5) leads to $h(\mathcal{T})|_T \leq \mathrm{diam}(T) \leq C_{\mathrm{shp}}\gamma(\mathcal{T}_0)\, h(\mathcal{T})|_T$ for all $T \in \mathcal{T} \in \mathbb{T}$. In particular, $\eta(\cdot)$ and $\widetilde{\eta}(\cdot)$ are element wise equivalent.

Proposition 5.2.2. *The estimators $\eta(\cdot)$ and $\widetilde{\eta}(\cdot)$ are globally equivalent in the sense that* (4.2.1) *with $\widetilde{\mathbb{T}} = \mathbb{T}$, $\widetilde{\mathbb{T}}(\cdot,\cdot) = \mathbb{T}(\cdot,\cdot)$ and $C_{\mathrm{eq}} = C_{\mathrm{shp}}^2\gamma(\mathcal{T}_0)^2$. Moreover,* (4.2.2) *holds with $\mathcal{M} = \widetilde{\mathcal{M}} = \overline{\mathcal{M}}$. The weight-function $h(\mathcal{T})$ satisfies* (4.5.1) *and* (4.5.2)*. Moreover,* (4.5.6) *is satisfied with the trivial patch function $\omega(\mathcal{S},\mathcal{T}) = \mathcal{S}$ for all $\mathcal{S} \subseteq \mathcal{T}$ and all $\mathcal{T} \in \mathbb{T}$. Together with Proposition 5.2.1, all the assumptions of Theorem 4.5.3 are satisfied.*

Proof. Define the weight function $h : \Omega \to (0,\infty)$ by $h|_T := \mathrm{diam}(T)$ for all $\mathcal{T} \in \mathbb{T}$. Then, there holds $\widetilde{\eta}_T(\mathcal{T}) = \eta_T(\mathcal{T}, h)$ for all $T \in \mathcal{T}$. The homogeneity (3.4.1) of $\eta(\cdot)$ shows

$$\min_{x \in T} |(h(\mathcal{T})/h)(x)|^{r_-} \, \widetilde{\eta}_T(\mathcal{T}) \leq \eta_T(\mathcal{T}, h(\mathcal{T})) \leq \max_{x \in T} |(h(\mathcal{T})/h)(x)|^{r_+} \, \widetilde{\eta}_T(\mathcal{T})$$

and hence

$$C_{\mathrm{shp}}^{-1}\gamma(\mathcal{T}_0)^{-1}\widetilde{\eta}_T(\mathcal{T}) \leq \eta_T(\mathcal{T}) \leq \widetilde{\eta}_T(\mathcal{T}) \quad \text{for all } T \in \mathcal{T}.$$

From this element wise equivalence, the statements (4.2.1) and (4.2.2) follow immediately. The estimate (3.2.12) implies (4.5.1) and (4.5.6) follows from $\widetilde{\mathcal{M}} = \mathcal{M}$. Finally, the estimate (4.5.2) follows with $C_{\mathrm{sum}} = 1$. □

Consequence 5.2.3. *Let $s > 0$ with $\|\eta, \mathbb{T}\|_s < \infty$. Then, the adaptive algorithm leads to convergence with optimal rate for the estimator $\widetilde{\eta}(\cdot)$ in the sense of Theorem 4.5.3. If the assumptions of Section 3.5.2 are satisfied, then the adaptive algorithm leads to convergence with optimal rate for the estimator $\widetilde{\eta}(\cdot)$ in the sense of Theorem 4.5.3 for all $s \leq p/d$.* □

5.2.3. Facet-based formulation of residual error estimator. This section is based on [**24**, Section 9]. For a given triangulation $\mathcal{T} \in \mathbb{T}$ generated by bisection from Section 3.2.8, let $\widetilde{\mathcal{T}} := \mathcal{E}(\mathcal{T})$ denote the corresponding set of facets which lie inside Ω, i.e., for each $E \in \widetilde{\mathcal{T}}$ there are two unique elements $T, T' \in \mathcal{T}$ with $T \neq T'$ and $E = T \cap T'$. Let

$$\omega(E, \mathcal{T}) := \{T, T'\} \quad \text{and} \quad \bigcup \omega(E, \mathcal{T}) = T \cup T' \tag{5.2.2}$$

denote the patch of $E \in \widetilde{\mathcal{T}}$. Let $\mathbb{T}(\cdot,\cdot)$ denote bisection (Section 3.2.8) and let $\widetilde{\mathbb{T}}(\cdot,\cdot)$ denote the corresponding facet based version from Section 3.2.11. Assume that each element $T \in \mathcal{T}$ has at most one facet on the boundary $\Gamma = \partial\Omega$ which is a minor additional assumption on the initial triangulation $\mathcal{T}_0$ to exclude pathological cases. In particular, each element $T \in \mathcal{T}$

has at least one node $z \in \mathcal{K}(\mathcal{T})$ inside Ω. For each facet $E \in \widetilde{\mathcal{T}}$, let $F_E \in \mathcal{P}^{p-1}(\bigcup \omega(E,\mathcal{T}))$ be the unique polynomial of degree $p-1$ such that

$$\|\Delta_{\mathcal{T}} V - f - F_E\|_{L^2(\cup\omega(E,\mathcal{T}))} = \min_{F\in\mathcal{P}^{p-1}(\cup\omega(E,\mathcal{T}))} \|\Delta_{\mathcal{T}} V - f - F\|_{L^2(\cup\omega(E,\mathcal{T}))}. \tag{5.2.3}$$

With the introduced notation, consider the following facet-based variant of the residual error estimator (5.2.1)

$$\widetilde{\eta}(\widetilde{\mathcal{T}})^2 = \sum_{E\in\widetilde{\mathcal{T}}} \eta_E(\widetilde{\mathcal{T}})^2, \tag{5.2.4a}$$

$$\widetilde{\eta}_E(\widetilde{\mathcal{T}})^2 = \mathrm{diam}(E)^2 \, \|\Delta_{\mathcal{T}} V - f - F_E\|^2_{L^2(\cup\omega(E,\mathcal{T}))} + \mathrm{diam}(E) \, \|[\partial_n V]\|^2_{L^2(E)}. \tag{5.2.4b}$$

Convergence and quasi-optimality for this estimator is directly proved for $d = 2$ and $p = 1$ in [**48**] via the technical and non-obvious observation that the edge oscillations are contractive [**69, 68**]. The novel approach of this paper generalizes the mentioned works to arbitrary dimension $d \geq 2$ and polynomial degree $p \geq 1$.

Proposition 5.2.4. *The estimators $\eta(\cdot)$ and $\widetilde{\eta}(\cdot)$ are globally equivalent in the sense of* (4.2.1)*. Moreover,* (4.2.2) *holds with*

$$\mathcal{M} := \bigcup_{E\in\widetilde{\mathcal{M}}} \omega(E,\mathcal{T}) \quad \textit{and} \quad \overline{\mathcal{M}} := \big\{E \in \widetilde{\mathcal{T}} \,:\, \exists T \in \mathcal{M},\, E\cap T \neq \emptyset\big\}.$$

The weight-function $h(\mathcal{T})$ satisfies (4.5.1) *as well as* (4.5.2) *and* (4.5.6) *is satisfied with the patch function*

$$\omega(\mathcal{S},\mathcal{T}) := \big\{T \in \mathcal{T} \,:\, \exists T' \in \mathcal{S},\, T\cap T' \neq \emptyset\big\}$$

for all $\mathcal{S} \subseteq \mathcal{T}$ and all $\mathcal{T} \in \mathbb{T}$. Together with Proposition 5.2.1, all the assumptions of Theorem 4.5.3 are satisfied.

The proof of Proposition 5.2.4 requires some technical lemmas and some further notation: For an interior node $z \in \mathcal{K}(\mathcal{T})\cap\Omega$ of $\mathcal{T}$, define the star $\Sigma(z,\mathcal{T}) := \big\{E \in \widetilde{\mathcal{T}} \,:\, z \in E\big\}$ as well as the patch $\omega(z,\mathcal{T}) := \big\{T \in \mathcal{T} \,:\, z \in T\big\}$. Let $F_z \in \mathcal{P}^{p-1}(\bigcup\omega(z,\mathcal{T}))$ denote the unique polynomial of degree $p-1$ such that

$$\|\Delta_{\mathcal{T}} V - f - F_z\|_{L^2(\cup\omega(z,\mathcal{T}))} = \min_{F\in\mathcal{P}^{p-1}(\cup\omega(z,\mathcal{T}))} \|\Delta_{\mathcal{T}} V - f - F\|_{L^2(\cup\omega(z,\mathcal{T}))}. \tag{5.2.5}$$

To abbreviate notation, write $r(\mathcal{T}) := \Delta_{\mathcal{T}} U(\mathcal{T}) - f$ for the residual.

Lemma 5.2.5. *Any interior node $z \in \mathcal{K}(\mathcal{T})\cap\Omega$ and $T \in \mathcal{T}$ with $z \in T$ satisfies*

$$C_{20}^{-1}\, \|r(\mathcal{T})\|^2_{L^2(T)} \leq h(\mathcal{T})|_T^{-1} \, \|[\partial_n U(\mathcal{T})]\|^2_{L^2(\cup\Sigma(z,\mathcal{T}))} + \|r(\mathcal{T}) - F_z\|^2_{L^2(\cup\omega(z,\mathcal{T}))}. \tag{5.2.6}$$

The constant $C_{20} > 0$ depends only on $\gamma(\mathcal{T})$ and hence on $\mathbb{T}$.

Proof. Consider the nodal basis function $\phi_z \in \mathcal{S}^1(\mathcal{T})$ characterized by $\phi_z(z) = 1$ and $\phi_z(z') = 0$ for all $z' \in \mathcal{K}(\mathcal{T})$ with $z \neq z'$. In particular, $\mathrm{supp}(\phi_z) = \bigcup\omega(z,\mathcal{T})$. Let $\Pi^{p-1} : L^2(\bigcup\omega(z,\mathcal{T})) \to \mathcal{P}^{p-1}(\bigcup\omega(z,\mathcal{T}))$ be the L^2-orthogonal projection and note that

$F_z = \Pi^{p-1} r(\mathcal{T})$. A scaling argument and $\|\phi_z\|_{L^\infty(\Omega)} = 1$ prove

$$\begin{aligned}\|F_z\|^2_{L^2(\cup\omega(z,\mathcal{T}))} &\lesssim \|\phi_z^{1/2} F_z\|^2_{L^2(\cup\omega(z,\mathcal{T}))} \\ &= \int_{\cup\omega(z,\mathcal{T})} r(\mathcal{T})\phi_z F_z\,dx - \int_{\cup\omega(z,\mathcal{T})} \big((1-\Pi^{p-1})r(\mathcal{T})\big)\phi_z F_z\,dx \\ &\leq \int_{\cup\omega(z,\mathcal{T})} r(\mathcal{T})\phi_z F_z\,dx + \|(1-\Pi^{p-1})r(\mathcal{T})\|_{L^2(\cup\omega(z,\mathcal{T}))}\|F_z\|_{L^2(\cup\omega(z,\mathcal{T}))}.\end{aligned}$$

Consider the first term on the right-hand side and use that $V := \phi_z F_z \in \mathcal{S}^p_0(\mathcal{T})$ is a suitable test function. With the Galerkin formulation (3.5.3) and element wise integration by parts, it follows that

$$\begin{aligned}\int_{\cup\omega(z,\mathcal{T})} r(\mathcal{T})\phi_z F_z\,dx &= \int_{\cup\omega(z,\mathcal{T})} r(\mathcal{T}) V\,dx \\ &= \int_{\cup\omega(z,\mathcal{T})} \Delta_{\mathcal{T}} U(\mathcal{T})\, V\,dx + \int_{\cup\omega(z,\mathcal{T})} \nabla U(\mathcal{T})\cdot\nabla V\,dx \\ &= \int_{\cup\Sigma(z,\mathcal{T})} [\partial_n U(\mathcal{T})]\,\phi_z F_z\,dx \\ &\leq \|[\partial_n U(\mathcal{T})]\|_{L^2(\cup\Sigma(z,\mathcal{T}))}\|F_z\|_{L^2(\cup\Sigma(z,\mathcal{T}))}.\end{aligned}$$

Since $F_z \in \mathcal{P}^{p-1}(\bigcup\omega(z,\mathcal{T}))$, an inverse-type inequality with $h_z := \operatorname{diam}(\bigcup\omega(z,\mathcal{T}))$ shows

$$\|F_z\|_{L^2(\cup\Sigma(z,\mathcal{T}))} \lesssim h_z^{-1/2}\,\|F_z\|_{L^2(\cup\omega(z,\mathcal{T}))}.$$

The hidden constant depends only on $\gamma(\mathcal{T})$. The combination of the previous arguments implies

$$\|F_z\|^2_{L^2(\cup\omega(z,\mathcal{T}))} \lesssim \big(h_z^{-1/2}\,\|[\partial_n U(\mathcal{T})]\|_{L^2(\cup\Sigma(z,\mathcal{T}))} + \|r(\mathcal{T}) - F_z\|_{L^2(\cup\omega(z,\mathcal{T}))}\big)\|F_z\|_{L^2(\cup\omega(z,\mathcal{T}))}.$$

The triangle inequality together with $h_z \simeq h(\mathcal{T})|_T$ proves

$$\begin{aligned}&h(\mathcal{T})|_T^2\|\Delta_{\mathcal{T}}U(\mathcal{T}) + f\|^2_{L^2(T)} \\ &\quad\lesssim h(\mathcal{T})|_T^2\|F_z\|^2_{L^2(\cup\omega(z,\mathcal{T}))} + h(\mathcal{T})|_T^2\|r(\mathcal{T}) - F_z\|_{L^2(\cup\omega(z,\mathcal{T}))} \\ &\quad\lesssim h(\mathcal{T})|_T\|[\partial_n U(\mathcal{T})]\|^2_{L^2(\cup\Sigma(z,\mathcal{T}))} + h(\mathcal{T})|_T^2\|r(\mathcal{T}) - F_z\|^2_{L^2(\cup\omega(z,\mathcal{T}))}.\end{aligned}$$

This concludes the proof. □

The following lemma shows that edge oscillations (5.2.3) and node oscillations (5.2.5) are equivalent on patches.

Lemma 5.2.6. *Any interior node $z \in \mathcal{K}(\mathcal{T}) \cap \Omega$ and $T \in \mathcal{T}$ with $z \in T$ satisfies*

$$\begin{aligned}C_{21}^{-1}\,\|r(\mathcal{T}) - F_z\|^2_{L^2(\cup\omega(z,\mathcal{T}))} &\leq \sum_{E\in\Sigma(z,\mathcal{T})} \|r(\mathcal{T}) - F_E\|^2_{L^2(\cup\omega(E,\mathcal{T}))} \\ &\leq C_{22}\,\|r(\mathcal{T}) - F_z\|^2_{L^2(\cup\omega(z,\mathcal{T}))}.\end{aligned} \tag{5.2.7}$$

The constants $C_{21}, C_{22} > 0$ depend only on $\mathbb{T}$, the polynomial degree $p \geq 1$, and the use of bisection.

Proof. The upper bound in (5.2.7) follows from

$$\|r(\mathcal{T}) - F_E\|_{L^2(\cup\omega(E,\mathcal{T}))} \leq \|r(\mathcal{T}) - F_z\|_{L^2(\cup\omega(E,\mathcal{T}))} \leq \|r(\mathcal{T}) - F_z\|_{L^2(\cup\omega(z,\mathcal{T}))}$$

for all $E \in \Sigma(z,\mathcal{T})$ and the fact that the cardinality $|\Sigma(z,\mathcal{T})|$ is uniformly bounded by $\gamma(\mathcal{T}) \leq C_{\mathrm{shp}}\gamma(\mathcal{T}_0)$.

The lower bound in (5.2.7) is first proved for a piecewise polynomial $r(\mathcal{T}) \in \mathcal{P}^{p-1}(\mathcal{T})$. We employ equivalence of seminorms on finite dimensional spaces and scaling arguments. Note that both terms in (5.2.7) define seminorms on the finite dimensional space $\mathcal{P}^{p-1}(\omega(z,\mathcal{T}))$ with the kernel $\mathcal{P}^{p-1}(\bigcup\omega(z,\mathcal{T}))$ and hence are equivalent with constants $C_{21}, C_{22} > 0$. A scaling argument proves that these constants depend only on the shape of $\bigcup\omega(E,\mathcal{T})$ or $\bigcup\Sigma(z,\mathcal{T})$. Since bisection from Section 3.2.8 only leads to finitely many shapes of triangles and hence patches and facet stars, this proves that C_{21} and C_{22} depend only on $\mathbb{T}$, p, and the use of bisection.

It remains to prove the lower bound in (5.2.7) for general $f \in L^2(\Omega)$. Let $\Pi^{p-1} : L^2(\Omega) \to \mathcal{P}^{p-1}(\mathcal{T})$ denote the L^2-projection so that $F(\mathcal{T}) = \Pi^{p-1} r(\mathcal{T})$ is the unique solution to

$$\|r(\mathcal{T}) - F(\mathcal{T})\|_{L^2(T)} = \min_{F \in \mathcal{P}^{p-1}(T)} \|r(\mathcal{T}) - F\|_{L^2(T)} \quad \text{for all } T \in \mathcal{T}.$$

Note that $\mathcal{P}^{p-1}(\bigcup\omega(E,\mathcal{T})) \subset \mathcal{P}^{p-1}(\omega(E,\mathcal{T}))$ and hence

$$\langle (1-\Pi^{p-1}) r(\mathcal{T})\,,\, F(\mathcal{T}) - F_z\rangle_{L^2(T)} = 0 = \langle (1-\Pi^{p-1}) r(\mathcal{T})\,,\, F(\mathcal{T}) - F_E\rangle_{L^2(T)}.$$

According to the $\mathcal{T}$-element wise Pythagoras theorem and the foregoing discussion for $\mathcal{T}$-piecewise polynomial $r(\mathcal{T})$, it follows

$$\begin{aligned}
\|r(\mathcal{T}) - F_z\|^2_{L^2(\cup\omega(z,\mathcal{T}))} &= \|r(\mathcal{T}) - F(\mathcal{T})\|^2_{L^2(\cup\omega(z,\mathcal{T}))} + \|F(\mathcal{T}) - F_z\|^2_{L^2(\cup\omega(z,\mathcal{T}))} \\
&\lesssim \sum_{E \in \Sigma(z,\mathcal{T})} \big(\|r(\mathcal{T}) - F(\mathcal{T})\|^2_{L^2(\cup\omega(E,\mathcal{T}))} + \|F(\mathcal{T}) - F_E\|^2_{L^2(\cup\omega(E,\mathcal{T}))}\big) \\
&= \sum_{E \in \Sigma(\mathcal{T};z)} \|r(\mathcal{T}) - F_E\|^2_{L^2(\cup\omega(E,\mathcal{T}))}.
\end{aligned}$$

This concludes the proof. $\square$

Proof of Proposition 5.2.4. Shape regularity (3.2.5) yields $h_E = \mathrm{diam}(E) \simeq h(\mathcal{T})|_T$ for all $E \in \widetilde{\mathcal{T}}$ and $T \in \mathcal{T}$ with $E \subseteq T$. Hence

$$\begin{aligned}
\widetilde{\eta}_E(\widetilde{\mathcal{T}})^2 &= h_E^2\,\|r(\mathcal{T}) - F_E\|^2_{L^2(\cup\omega(E,\mathcal{T}))} + h_E\,\|[\partial_n U(\mathcal{T})]\|^2_{L^2(E)} \\
&\le \sum_{T \in \omega(E,\mathcal{T})} \big(h_E^2 \|r(\mathcal{T})\|^2_{L^2(T)} + h_E\,\|[\partial_n U(\mathcal{T})]\|^2_{L^2(\partial T \cap \Omega)}\big) \\
&\simeq \sum_{T \in \omega(E,\mathcal{T}))} \eta_T(\mathcal{T})^2.
\end{aligned}$$

Lemma 5.2.5 and 5.2.6 imply

$$\begin{aligned}
\eta_T(\mathcal{T})^2 &= h(\mathcal{T})|_T^2\,\|r(\mathcal{T})\|^2_{L^2(T)} + h(\mathcal{T})|_T\,\|[\partial_n U(\mathcal{T})]\|^2_{L^2(\partial T \cap \Omega)} \\
&\lesssim \sum_{z \in \mathcal{K}(T) \cap \Omega} \big(h(\mathcal{T})|_T^2\,\|r(\mathcal{T}) - F_z\|^2_{L^2(\cup\omega(T,z))} + h(\mathcal{T})|_T\,\|[\partial_n U(\mathcal{T})]\|^2_{L^2(\cup\Sigma(z,\mathcal{T}))}\big) \\
&\simeq \sum_{z \in \mathcal{K}(T) \cap \Omega} \sum_{E \in \Sigma(z,\mathcal{T})} \big(h_E^2\,\|r(\mathcal{T}) - F_E\|^2_{L^2(\cup\omega(E,\mathcal{T}))} + E_T\,\|[\partial_n U(\mathcal{T})]\|^2_{L^2(E)}\big) \\
&\le \sum_{z \in \mathcal{K}(T) \cap \Omega} \sum_{E \in \Sigma(z,\mathcal{T})} \widetilde{\eta}_E(\widetilde{\mathcal{T}})^2.
\end{aligned}$$

The last two estimates imply immediately (4.2.1). The first implication (4.2.2a) follows by

$$\widetilde{\theta}\eta(\mathcal{T})^2 \lesssim \widetilde{\theta}\widetilde{\eta}(\widetilde{\mathcal{T}})^2 \le \sum_{E\in\widetilde{\mathcal{M}}} \widetilde{\eta}_E(\widetilde{\mathcal{T}})^2 \lesssim \sum_{E\in\widetilde{\mathcal{M}}} \sum_{T\in\omega(E,\mathcal{T})} \eta_T(\mathcal{T}) = \sum_{T\in\mathcal{M}} \eta_T(\mathcal{T})^2.$$

To see the second implication (4.2.2b), consider

$$\theta\widetilde{\eta}(\widetilde{\mathcal{T}})^2 \lesssim \theta\eta(\mathcal{T})^2 \le \sum_{T\in\mathcal{M}} \eta_T(\mathcal{T})^2 \lesssim \sum_{T\in\mathcal{M}} \sum_{z\in\mathcal{K}(T)\cap\Omega} \sum_{\substack{E\in\Sigma(z,\mathcal{T})\\ z\in\mathcal{K}(\mathcal{T})\cap T}} \widetilde{\eta}_E(\widetilde{\mathcal{T}})^2 \le \sum_{E\in\overline{\mathcal{M}}} \widetilde{\eta}_E(\widetilde{\mathcal{T}})^2.$$

The remaining statements follow as in Section 5.2.2. □

Consequence 5.2.7. *Let $s > 0$ with $\|\eta, \mathbb{T}\|_s < \infty$. Then, the adaptive algorithm leads to convergence with optimal rate for the facet based estimator $\widetilde{\eta}(\cdot)$ in the sense of Theorem 4.5.3.*

Numerical examples that underline the above result can be found in for 2D and lowest-order elements in [**49**]. Moreover, numerical examples for the obstacle problem with the facet-based estimator are found in [**69, 68**].

5.2.4. Recovery-based error estimator. This section is based on [**24**, Section 9]. We consider recovery-based error estimators for FEM which are occasionally also called ZZ-estimators after Zienkiewicz and Zhu [**87**]. These estimators are popular in computational science and engineering because of their implementational ease and striking performance in many applications. Reliability has independently been shown by [**72, 22**] for lowest-order elements $p = 1$ and later generalized to higher-order elements $p \ge 1$ in [**10**]. For the lowest-order case, convergence and quasi-optimality of the related adaptive algorithm has been analyzed in [**60**]. In the following, the result of [**60**] is reproduced and even generalized to higher-order elements $p \ge 1$. Moreover, the abstract analysis of Section 4.5 removes the artificial refinements in [**60**].

Let $G(\mathcal{T}) : L^2(\Omega) \to \mathcal{S}^p_0(\mathcal{T})$ denote the local averaging operator which is defined as follows:

- For lowest-order polynomials $p = 1$, define $G(\mathcal{T})(v) \in \mathcal{S}^1_0(\mathcal{T})$ by
$$G(\mathcal{T})(v)(z) := \frac{1}{|\omega(z,\mathcal{T})|} \int_{\cup\omega(z,\mathcal{T})} v\,dx \quad \text{for all inner nodes } z \in \mathcal{K}(\mathcal{T}) \cap \Omega.$$
- For the general case $p \ge 1$, define $G(\mathcal{T}) = J(\mathcal{T}) : H^1_0(\Omega) \to \mathcal{S}^p_0(\mathcal{T})$ as the Scott-Zhang projection from [**76**], see also Definition 3.3.2.

Based on $G(\mathcal{T})$, the local estimator contributions of the recovery-based error estimator $\widetilde{\eta}(\cdot)$ read

$$\widetilde{\eta}_\tau(\mathcal{T})^2 := \begin{cases} \|(1 - G(\mathcal{T}))\nabla U(\mathcal{T})\|^2_{L^2(T)} & \text{for } \tau = T \in \mathcal{T}, \\ \mathrm{diam}(E)^2\, \|\Delta_{\mathcal{T}} U(\mathcal{T}) - f - F_E\|^2_{L^2(\omega(E,\mathcal{T}))} & \text{for } \tau = E \in \mathcal{E}(\mathcal{T}), \end{cases} \tag{5.2.8}$$

where F_E is defined in (5.2.3). Given a set of triangulations $\mathbb{T}$ with the bisection refinement strategy $\mathbb{T}(\cdot,\cdot)$ from Section 3.2.8, the recovery-based error estimator acts on the set $\widetilde{\mathbb{T}} := \{\widetilde{\mathcal{T}} : \mathcal{T} \in \mathbb{T}\}$ and $\widetilde{\mathcal{T}} := \mathcal{T} \cup \mathcal{E}(\mathcal{T})$. The refinement strategy $\widetilde{\mathbb{T}}(\cdot,\cdot)$ employs facet based variant from Section 3.2.11, where each marked element $T \in \mathcal{T}$ marks the corresponding facets $E \subseteq \partial T$. Moreover, given $\mathcal{T} \in \mathbb{T}$ and $\mathcal{S} \subseteq \mathcal{T}$ define the 2-patch

$$\omega^2(\mathcal{S},\mathcal{T}) := \{T \in \mathcal{T} : \exists T_0, T_1 \in \mathcal{T},\, T_0 \in \mathcal{S},\, T_0 \cap T_1 \neq \emptyset,\, T_1 \cap T \neq \emptyset\}. \tag{5.2.9}$$

PROPOSITION 5.2.8. *For general polynomial degree $p \geq 1$, the error estimators $\eta(\cdot)$ from (5.2.1) and $\widetilde{\eta}(\cdot)$ from (5.2.8) satisfy for all $E \in \mathcal{E}(\mathcal{T})$ with $E = T_0 \cap T_1$ for some $T_0, T_1 \in \mathcal{T}$*

$$\widetilde{\eta}_E(\widetilde{\mathcal{T}})^2 + \widetilde{\eta}_{T_0}(\widetilde{\mathcal{T}})^2 \leq C_{23} \sum_{T \in \omega^2(T_0, \mathcal{T})} \eta_T(\mathcal{T})^2, \tag{5.2.10a}$$

as well as

$$\eta_{T_0}(\mathcal{T})^2 \leq C_{23} \sum_{\substack{\tau \in \widetilde{\mathcal{T}} \\ \tau \cap T_0 \neq \emptyset}} \widetilde{\eta}_\tau(\widetilde{\mathcal{T}})^2. \tag{5.2.10b}$$

The constant $C_{23} > 0$ depends only on $\gamma(\mathcal{T})$, the use of bisection, and p.

The proof requires the following lemma which states that the normal jumps are locally equivalent to averaging. The result is well-known for the lowest-order case, and its proof is included for the convenience of the reader.

LEMMA 5.2.9. *For some interior node $z \in \mathcal{K}(\mathcal{T}) \cap \Omega$, it holds*

$$\begin{aligned} C_{24}^{-1}\, h_T\, \|[\partial_n U(\mathcal{T})]\|^2_{L^2(\cup\Sigma(z,\mathcal{T}))} &\leq \|(1 - G(\mathcal{T}))\nabla U(\mathcal{T})\|^2_{L^2(\cup\omega(z,\mathcal{T}))} \\ &\leq C_{25} \sum_{z' \in \Sigma(z,\mathcal{T}) \cap \mathcal{K}(\mathcal{T}) \cap \Omega} h_{z'} \|[\partial_n U(\mathcal{T})]\|^2_{L^2(\cup\Sigma(z',\mathcal{T}))}. \end{aligned} \tag{5.2.11}$$

The constants $C_{24}, C_{25} > 0$ depend only on $\mathcal{T}_0$, the polynomial degree $p \geq 1$, and the use of bisection.

PROOF. We use equivalence of seminorms on finite dimensional spaces and scaling arguments. To prove (5.2.11), it thus suffices to show that the chain of inequalities holds true if one term is zero.

First, assume $(1 - G(\mathcal{T}))\nabla U(\mathcal{T}) = 0$ on $\bigcup \omega(z, \mathcal{T})$. This implies $\nabla U(\mathcal{T}) \in \mathcal{S}^p(\omega(z, \mathcal{T}))$ and hence $[\partial_n U(\mathcal{T})] = 0$ on $\bigcup \Sigma(z, \mathcal{T})$.

Second, assume $[\partial_n U(\mathcal{T})] = 0$ on $\bigcup \Sigma(z', \mathcal{T})$ for all inner nodes z' of $\Sigma(z, \mathcal{T})$. This shows that the normal jumps of $\nabla U(\mathcal{T})$ are zero over $\bigcup \Sigma(z', \mathcal{T})$. Since $U(\mathcal{T}) \in H^1(\Omega)$, the tangential jumps of $\nabla U(\mathcal{T})$ also vanish over $\Sigma(z', \mathcal{T})$. Altogether, this implies $\nabla U(\mathcal{T}) \in \mathcal{S}^{p-1}(\omega(z', \mathcal{T}))$ for all z'. If the Scott-Zhang projection defines the averaging, $G(\mathcal{T})\nabla U(\mathcal{T})(z')$ depends only on $\nabla U(\mathcal{T})|_{\omega(z',\mathcal{T})}$, this implies $G(\mathcal{T})\nabla U(\mathcal{T}) = \nabla U(\mathcal{T})$. In the particular case $p = 1$ and patch averaging, $\nabla U(\mathcal{T})$ is constant on $\omega(z', \mathcal{T})$. In any case, we thus derive $(1 - G(\mathcal{T}))\nabla U(\mathcal{T}) = 0$ on $\bigcup \omega(z, \mathcal{T})$.

The constants in (5.2.11) depend on the shapes of patches $\bigcup \omega(z', \mathcal{T})$ involved. Since bisection from Section 3.2.8 leads to only finitely many patch shapes, we deduce that the these constants depend only on the polynomial degree $p \in \mathbb{N}$ and on $\mathcal{T}_0$. □

PROOF OF PROPOSITION 5.2.8. In order to prove the local equivalence (5.2.10), let $z \in \mathcal{K}(\mathcal{T}) \cap \Omega$ be an interior node of $T \in \mathcal{T}$. The upper estimate in (5.2.11) yields

$$\widetilde{\eta}_T(\widetilde{\mathcal{T}})^2 \lesssim \sum_{T' \in \omega^2(T,\mathcal{T})} \eta_{T'}(\mathcal{T})^2.$$

For $E = T_0 \cap T_1 \in \widetilde{\mathcal{T}}$, it holds

$$\widetilde{\eta}_E(\widetilde{\mathcal{T}})^2 = \operatorname{diam}(E)^2 \|r(\mathcal{T})\|^2_{L^2(T_0)} + \operatorname{diam}(E)^2 \|r(\mathcal{T})\|^2_{L^2(T_1)} \lesssim \sum_{T' \in \omega(T_0,\mathcal{T})} \eta_{T'}(\mathcal{T})^2.$$

The combination of the last two estimates proves (5.2.10a). The proof of (5.2.10b) employs Lemma 5.2.5 and 5.2.6 as well as the lower bound in (5.2.11). For an interior node $z \in \mathcal{K}(\mathcal{T}) \cap \Omega$ of $T \in \mathcal{T}$, it follows

$$\begin{aligned}\eta_T(\mathcal{T})^2 &\lesssim h(\mathcal{T})|_T \,\|[\partial_n U(\mathcal{T})]\|^2_{L^2(\cup\Sigma(z,\mathcal{T}))} + h(\mathcal{T})|_T^2 \sum_{E\in\Sigma(z,\mathcal{T})} \|r(\mathcal{T}) - F_E\|^2_{L^2(\cup\omega(E,\mathcal{T})}\\ &\lesssim \sum_{\substack{\tau\in\widetilde{\mathcal{T}}\\ \tau\cap T\neq\emptyset}} \widetilde{\eta}_\tau(\widetilde{\mathcal{T}})^2.\end{aligned}$$

This concludes the proof. □

Proposition 5.2.10. *With the patch functions from* (5.2.2) *and* (5.2.9)*, the estimators* $\eta(\cdot)$ *and* $\widetilde{\eta}(\cdot)$ *are globally equivalent in the sense of* (4.2.1)*. Moreover,* (4.2.2) *holds with*

$$\mathcal{M} := \bigcup_{E\in\widetilde{\mathcal{M}}\cap\mathcal{E}(\mathcal{T})} \omega^2(\omega(E,\mathcal{T}),\mathcal{T}) \cup \bigcup_{T\in\widetilde{\mathcal{M}}\cap\mathcal{T}} \omega^2(T,\mathcal{T})$$

and

$$\overline{\mathcal{M}} := \big\{\tau \in \widetilde{\mathcal{T}} \,:\, \exists T \in \mathcal{M},\, \tau\cap T \neq \emptyset\big\}.$$

The weight-function $h(\mathcal{T})$ *satisfies* (4.5.1) *and* (4.5.2)*. Moreover,* (4.5.6) *is satisfied with the patch function* $\omega^2(\cdot,\cdot)$*. Together with Proposition 5.2.1, all the assumptions of Theorem 4.5.3 are satisfied.*

Proof. The global equivalence follows from Proposition 5.2.8. The implication (4.2.2a) follows by (5.2.10a) and

$$\begin{aligned}\widetilde{\theta}\eta(\mathcal{T})^2 \lesssim \widetilde{\theta}\widetilde{\eta}(\widetilde{\mathcal{T}})^2 &\le \sum_{E\in\widetilde{\mathcal{M}}\cap\mathcal{E}(\mathcal{T})} \widetilde{\eta}_E(\widetilde{\mathcal{T}})^2 + \sum_{T\in\widetilde{\mathcal{M}}\cap\mathcal{T}} \widetilde{\eta}_T(\widetilde{\mathcal{T}})^2\\ &\lesssim \sum_{E\in\widetilde{\mathcal{M}}} \sum_{T\in\omega^2(\omega(E,\mathcal{T}),\mathcal{T})} \eta_T(\mathcal{T})^2 = \sum_{T\in\mathcal{M}} \eta_T(\mathcal{T})^2.\end{aligned}$$

To see the second implication (4.2.2b), consider (5.2.10b) and

$$\theta\widetilde{\eta}(\widetilde{\mathcal{T}})^2 \lesssim \theta\eta(\mathcal{T})^2 \le \sum_{T\in\mathcal{M}} \eta_T(\mathcal{T})^2 \lesssim \sum_{T\in\mathcal{M}} \sum_{\substack{\tau\in\widetilde{\mathcal{T}}\\ \tau\cap T\neq\emptyset}} \widetilde{\eta}_\tau(\widetilde{\mathcal{T}})^2 = \sum_{\tau\in\overline{\mathcal{M}}} \widetilde{\eta}_\tau(\widetilde{\mathcal{T}})^2.$$

The remaining statements follow as in Section 5.2.2. □

Consequence 5.2.11. *Let* $s > 0$ *with* $\|\eta,\mathbb{T}\|_s < \infty$*. Then, the adaptive algorithm leads to convergence with optimal rate for the facet based estimator* $\widetilde{\eta}(\cdot)$ *in the sense of Theorem 4.5.3.*

5.3. Example 2: Conforming FEM for the p-Laplacian

This section is based on [**24**, Section 10]. The p-Laplacian allows for a review of the results of [**13**] in terms of the abstract framework of Chapter 4. Since no lower error bound is required, the present analysis provides some slight improvement over [**13**]. The following allows generalizations to N-functions as in [**13**], which we, however, omit in favor of a straightforward presentation.

Consider the energy minimization problem

$$\mathcal{J}(u) = \min_{v\in W_0^{1,p}(\Omega)} \mathcal{J}(v) \quad \text{with} \quad \mathcal{J}(v) := \frac{1}{p}\int_\Omega |\nabla v|^p\,dx - \int_\Omega fv\,dx \tag{5.3.1}$$

for $p > 1$ and $W_0^{1,p}(\Omega)$ equipped with the norm $\|v\|_{W^{1,p}(\Omega)} := \big(\|v\|_{L^p(\Omega)}^2 + \|\nabla v\|_{L^p(\Omega)}^2\big)^{1/2}$. The direct method of the calculus of variations yields existence and strict convexity of $\mathcal{J}(\cdot)$ even uniqueness of the solution $u \in W_0^{1,p}(\Omega)$. With the nonlinearity

$$\boldsymbol{A} : \mathbb{R}^d \to \mathbb{R}^d,\ \boldsymbol{A}(Q) = |Q|^{p-2}Q,$$

the Euler-Lagrange equations associated to (5.3.1) read

$$\langle \mathcal{L}u\,,\,v\rangle = \int_\Omega \boldsymbol{A}(\nabla u)\cdot\nabla v = \int_\Omega fv\,dx \quad \text{for } u,v \in \mathcal{X} := W_0^{1,p}(\Omega). \tag{5.3.2}$$

The discretization of (5.3.2) and the notation follows Section 3.5.1. For a given regular triangulation $\mathcal{T} \in \mathbb{T}$ (where $\mathbb{T}$ is generated by bisection from Section 3.2.8), we consider the lowest-order Courant finite element space $\mathcal{S}_0^1(\mathcal{T}) := \mathcal{P}^1(\mathcal{T}) \cap H_0^1(\Omega)$. Arguing as in the continuous case, we obtain that the minimization problem

$$\mathcal{J}(U(\mathcal{T})) = \min_{V\in\mathcal{S}_0^1(\mathcal{T})} \mathcal{J}(V) \tag{5.3.3}$$

admits a unique discrete solution $U(\mathcal{T}) \in \mathcal{S}_0^1(\mathcal{T})$, which satisfies

$$\langle \mathcal{L}U(\mathcal{T})\,,\,V\rangle = \int_\Omega fV\,dx \quad \text{for all } V \in \mathcal{S}_0^1(\mathcal{T}). \tag{5.3.4}$$

Define $\boldsymbol{F}(Q) := |Q|^{p/2-1}Q$ for all $Q \in \mathbb{R}^d$. There holds the Céa Lemma [**13**, Lemma 3.1] for all $\mathcal{T} \in \mathbb{T}$

$$\|F(|\nabla u|) - F(|\nabla U(\mathcal{T})|)\|_{L^2(\Omega)} \le C_{\text{Céa}} \min_{V\in\mathcal{S}_0^p(\mathcal{T})} \|F(|\nabla u|) - F(|\nabla V|)\|_{L^2(\Omega)}. \tag{5.3.5}$$

In terms of Chapter 4, we define $\widetilde{\mathbb{T}} = \mathbb{T}$ and $\widetilde{\mathcal{T}} = \mathcal{T}$. With $1/p + 1/q = 1$, the residual error estimator $\widetilde{\eta}(\cdot)$ reads

$$\begin{aligned}\widetilde{\eta}_T(\mathcal{T})^2 := |T|^{2/d}\int_T \big(|\nabla U(\mathcal{T})|^{p-1} + |T|^{1/d}|f|\big)^{q-2}|f|^2\,dx \\ + |T|^{1/d}\|[\boldsymbol{F}(\nabla U(\mathcal{T}))\cdot n]\|_{L^2(\partial T\cap\Omega)}^2\end{aligned} \tag{5.3.6}$$

for all $T \in \mathcal{T}$ and all $\mathcal{T} \in \mathbb{T}$ (see [**13**, Section 3.2]).

Since the first term of $\widetilde{\eta}(\cdot)$ depends nonlinearly on $U(\mathcal{T})$, [**13**, Section 3.2] introduces an equivalent error estimator $\eta(\cdot)$ with local contributions

$$\begin{aligned}\eta_T(\mathcal{T})^2 := |T|^{2/d}\int_T \big(|\nabla u|^{p-1} + |T|^{1/d}|f|\big)^{q-2}|f|^2\,dx \\ + |T|^{1/d}\|[\boldsymbol{F}(\nabla U(\mathcal{T}))\cdot n]\|_{L^2(\partial T\cap\Omega)}^2\end{aligned} \tag{5.3.7}$$

for all $T \in \mathcal{T}$ and all $\mathcal{T} \in \mathbb{T}$. Note that $\eta(\cdot)$ can only serve as a theoretical tool as it employs the unknown solution u.

Proposition 5.3.1. *The residual error estimator* (5.3.7) *is a weighted error estimator in the sense of Section 3.4, i.e.,*

$$\eta_T(\mathcal{T},h)^2 := \int_T h|_T^2\big(|\nabla u|^{p-1} + h|_T|f|\big)^{q-2}|f|^2\,dx + h|_T\|[\boldsymbol{F}(\nabla U(\mathcal{T}))\cdot n]\|_{L^2(\partial T\cap\Omega)}^2$$

and satisfies

(i) *homogeneity* (3.4.1) *with* $r_+ = 1/2$ *and* $r_- = 1$ *and stability* (3.4.2) *with*

$$\widetilde{\varrho}(\mathcal{T},\widehat{\mathcal{T}}) := C_{\rm pert}\|F(|\nabla U(\mathcal{T})|) - F(|\nabla U(\widehat{\mathcal{T}})|)\|_{L^2(\Omega)},$$

(ii) *general quasi-orthogonality* (E2) *with* $\varrho(\cdot,\cdot)$ *given by Proposition 3.4.1,*

(iii) *discrete reliability* (E3) *for all* $\varepsilon_{\rm drel} > 0$ *with* $C_{\rm drel} := C_{\rm drel}(\varepsilon_{\rm drel})$ *and* $\mathcal{R}(\mathcal{T},\widehat{\mathcal{T}}) = \mathcal{T}\backslash\widehat{\mathcal{T}}$ *as well as* $\kappa_{\rm dlr} = \infty$,

(iv) *the refinement axioms* (T1)–(T3) *with* $C_{\rm approx}(s) \le C_{\rm mon}(C_4+1)^s\|\eta,\mathbb{T}\|_s$ *for all* $s > 0$.

Moreover, the estimator is reliable (2.4.1) *with* $\mathrm{err}(\mathcal{T}) := \|F(|\nabla u|) - F(|\nabla U(\mathcal{T})|)\|_{L^2(\Omega)}$. *The constants* $C_{\rm drel}, C_{\rm qo}, C_{\rm pert}, C_{\rm rel}$ *depend only on the parameter* $p > 1$, $\mathcal{T}_0$, *and on* Ω.

Proof. To see the homogeneity (3.4.1), consider the function $g(t) := t^2b^2(a+tb)^{q-2}$ for some $a,b \ge 0$. The function g is convex and hence there holds for $0 \le \alpha \le 1$ that

$$g(\alpha t) \le \alpha g(t) + (1-\alpha)g(0) = \alpha g(t).$$

This shows $g(\alpha t) \le \alpha^{2r_+}g(t)$ for $r_+ = 1/2$. Moreover, we have

$$\frac{\alpha^{2r_-}g(t)}{g(\alpha t)} = \frac{\alpha^{2r_-}t^2b^2(a+tb)^{q-2}}{\alpha^2t^2b^2(a+\alpha tb)^{q-2}} = \alpha^{2r_- -2}\frac{(a+tb)^{q-2}}{(a+\alpha tb)^{q-2}} \le \begin{cases}\alpha^{2r_- -2} & q \le 2,\\ \alpha^{2r_- -q} & q > 2.\end{cases}$$

For $q \le 2$, choose $r_- = 1$ and for $q > 2$, choose $r_- = q/2$ to ensure $\alpha^{2r_-}g(t) \le g(\alpha t)$. Since the first term of $\eta_T(\mathcal{T},h)$ reads $\int_T g(h|_T)\,dx$ with $a = |\nabla u|^{p-1}$ and $b = |f|$ pointwise, the above considerations imply

$$\begin{aligned}\min_{x\in T}|\alpha(x)|^{2r_-}&\int_T h|_T^2\big(|\nabla u|^{p-1} + h|_T|f|\big)^{q-2}|f|^2\,dx\\ &\le \int_T (\alpha h)|_T^2\big(|\nabla u|^{p-1} + (\alpha h)|_T|f|\big)^{q-2}|f|^2\,dx\\ &\le \max_{x\in T}|\alpha(x)|^{2r_+}\int_T h|_T^2\big(|\nabla u|^{p-1} + h|_T|f|\big)^{q-2}|f|^2\,dx.\end{aligned}$$

Since the second term in the definition of $\eta(\cdot)$ behaves analogously, this implies homogeneity (3.4.1). Since the first term of $\eta(\cdot,h)$ does not depend on $\mathcal{T}$, standard inverse estimates as for the linear case (Proposition 5.2.1) prove stability (3.4.2) (see also [**13**, Proposition 4.4]). Reliability (2.4.1) is proved in [**13**, Lemma 3.5]. The discrete reliability (E3) with $\mathcal{R}(\mathcal{T},\widehat{\mathcal{T}}) = \mathcal{T}\backslash\widehat{\mathcal{T}}$ for $\widetilde{\eta}(\cdot)$ follows from [**13**, Lemma 3.7]. Together with the equivalence from [**13**, Proposition 4.2], there holds for all $\delta > 0$

$$\varrho(\mathcal{T},\widehat{\mathcal{T}}) \lesssim \sum_{T\in\mathcal{R}(\mathcal{T},\widehat{\mathcal{T}})}\widetilde{\eta}_T(\mathcal{T})^2 \lesssim C_\delta \sum_{T\in\mathcal{R}(\mathcal{T},\widehat{\mathcal{T}})}\eta_T(\mathcal{T})^2 + \delta\,\mathrm{err}(\mathcal{T})^2.$$

The constant $C_\delta > 0$ is defined in [**13**, Proposition 4.2]. Together with reliability (E3), this proves discrete reliability (E3) for all $\varepsilon_{\rm drel} > 0$, where $C_{\rm drel} > 0$ depends on $\varepsilon_{\rm drel}$. The statement (iv) follows as in Proposition 3.5.1. To see general quasi-orthogonality (E2), consider [**13**, Lemma 3.2], which implies for all refinements $\widehat{\mathcal{T}} \in \mathbb{T}(\mathcal{T})$

$$\begin{aligned}\mathcal{J}(U(\widehat{\mathcal{T}})) - \mathcal{J}(u) &\simeq \|F(|\nabla u|) - F(|\nabla U(\mathcal{T})|)\|_{L^2(\Omega)}^2,\\ \mathcal{J}(U(\mathcal{T})) - \mathcal{J}(U(\widehat{\mathcal{T}})) &\simeq \|F(|\nabla U(\widehat{\mathcal{T}})|) - F(|\nabla U(\mathcal{T})|)\|_{L^2(\Omega)}^2\end{aligned}$$

with hidden constants, which depend only on $p > 1$. This immediately implies for all $\ell \le N \in \mathbb{N}$ that

$$\begin{aligned}\sum_{k=\ell}^{N} \varrho(\mathcal{T}, \widehat{\mathcal{T}})^2 &\lesssim \sum_{k=\ell}^{N} \mathcal{J}(U(\mathcal{T}_\ell)) - \mathcal{J}(U(\mathcal{T}_{\ell+1}))\\ &= \mathcal{J}(U(\mathcal{T}_\ell)) - \mathcal{J}(U(\mathcal{T}_{N+1}))\\ &\le \mathcal{J}(U(\mathcal{T}_\ell)) - \mathcal{J}(u) \simeq \|F(|\nabla u|) - F(|\nabla U(\mathcal{T}_\ell)|)\|_{L^2(\Omega)}^2.\end{aligned}$$

Together with reliability (2.4.1), this implies (E2) with $\varepsilon_{\rm qo} = 0$. □

Proposition 5.3.2. *The estimators $\eta(\cdot)$ and $\widetilde{\eta}(\cdot)$ are globally equivalent in the sense of* (4.2.1). *Moreover,* (4.2.2) *holds with*

$$\mathcal{M} = \widetilde{\mathcal{M}} = \overline{\mathcal{M}}.$$

The weight-function $h(\mathcal{T})$ satisfies (4.5.1) *as well as* (4.5.2) *and* (4.5.6) *is satisfied with the trivial patch function $\omega(\mathcal{S}, \mathcal{T}) := \mathcal{S}$. Together with Proposition 5.3.1, all the assumptions of Theorem 4.5.3 are satisfied.*

Proof. The global equivalence (4.2.1) is proved in [**13**, Corollary 4.3]. The equivalence from [**13**, Proposition 4.2] implies for all $\delta > 0$ and all $T \in \mathcal{T}$

$$\begin{aligned}\eta_T(\mathcal{T})^2 &\le C_\delta \widetilde{\eta}_T(\mathcal{T})^2 + \delta\|F(|\nabla u|) - F(|\nabla U(\mathcal{T})|)\|_{L^2(T)}^2,\\ \widetilde{\eta}_T(\mathcal{T})^2 &\le C_\delta \eta_T(\mathcal{T})^2 + \delta\|F(|\nabla u|) - F(|\nabla U(\mathcal{T})|)\|_{L^2(T)}^2,\end{aligned}$$

where $C_\delta > 0$ depends only on $p > 1$ and on δ. With this, the implication (4.2.2a) follows from reliability (2.4.1) and global equivalence (4.2.1) by

$$\begin{aligned}\widetilde{\theta}\eta(\mathcal{T})^2 &\le \widetilde{\theta} C_{\delta_1} \widetilde{\eta}(\mathcal{T})^2 + \widetilde{\theta}\delta_1 \|F(|\nabla u|) - F(|\nabla U(\mathcal{T})|)\|_{L^2(\Omega)}^2\\ &\le \widetilde{\theta}(C_{\delta_1} + \delta_1 C_{\rm rel}^2 C_{\rm eq})\widetilde{\eta}(\mathcal{T})^2\\ &\le (C_{\delta_1} + \delta_1 C_{\rm rel}^2 C_{\rm eq}) \sum_{T \in \widetilde{\mathcal{M}}} \widetilde{\eta}_T(\mathcal{T})^2\\ &\le (C_{\delta_1} + \delta_1 C_{\rm rel}^2 C_{\rm eq}) \Big(C_{\delta_2} \sum_{T \in \widetilde{\mathcal{M}}} \eta_T(\mathcal{T})^2 + \delta_2 \|F(|\nabla u|) - F(|\nabla U(\mathcal{T})|)\|_{L^2(\Omega)}^2\Big)\\ &\le (C_{\delta_1} + \delta_1 C_{\rm rel}^2 C_{\rm eq}) \Big(C_{\delta_2} \sum_{T \in \widetilde{\mathcal{M}}} \eta_T(\mathcal{T})^2 + \delta_2 C_{\rm rel}^2 \eta(\mathcal{T})^2\Big).\end{aligned}$$

For arbitrary $\delta_1 > 0$, choose δ_2 sufficiently small such that $(C_{\delta_1} + \delta_1 C_{\rm rel}^2 C_{\rm eq})\delta_2 < \widetilde{\theta}$ to conclude (4.2.2a). The analogous argument shows also (4.2.2b). The remaining statements follow as in Section 5.2.2. □

Consequence 5.3.3. *Let $s > 0$ with $\|\eta, \mathbb{T}\|_s < \infty$. Then, the adaptive algorithm leads to convergence with optimal rate for $\widetilde{\eta}(\cdot)$ in the sense of Theorem 4.5.3.* □

Numerical examples for 2D that underline the above result can be found in [**13**].

5.4. Example 3: Non-homogeneous and mixed boundary conditions

The literature on adaptive finite elements focuses on homogeneous Dirichlet conditions with the exception of [**11, 66, 48, 4**]. This section extends the previous results to non-homogeneous boundary conditions of mixed Dirichlet-Neumann-Robin type, where inhomogeneous Dirichlet conditions enforce some additional discretization error. The present section is based on [**24**, Section 11] and improves [**4**] since we show that standard Dörfler marking (2.2.1) leads to convergence with optimal rates if the Scott-Zhang projection [**76**] is used for the discretization of the Dirichlet data [**4, 74**]. The heart of the analysis is the application of the super-contractive weight function $h_\omega(\mathcal{T})$ from Proposition 4.5.4.

5.4.1. Model problem. The Laplace model problem in $\mathbb{R}^d$ for $d \geq 2$ with mixed Dirichlet-Neumann-Robin boundary conditions splits the boundary Γ of the Lipschitz domain $\Omega \subset \mathbb{R}^d$ into three (relatively) open and pairwise disjoint boundary parts $\partial\Omega = \overline{\Gamma_D \cup \Gamma_N \cup \Gamma_R}$. Given data $f \in L^2(\Omega)$, $g_D \in H^1(\Gamma_D)$, $\phi_N \in L^2(\Gamma_N)$, $\phi_R \in L^2(\Gamma_R)$, and $\alpha \in L^\infty(\Gamma_R)$ with $\alpha \geq \alpha_0 > 0$ almost everywhere on Γ_R, the problem seeks $u \in H^1(\Omega)$ with

$$-\Delta u = f \qquad \text{in } \Omega, \tag{5.4.1a}$$
$$u = g_D \qquad \text{on } \Gamma_D, \tag{5.4.1b}$$
$$\partial_n u = \phi_N \qquad \text{on } \Gamma_N, \tag{5.4.1c}$$
$$\phi_R - \alpha u = \partial_n u \qquad \text{on } \Gamma_R. \tag{5.4.1d}$$

The presentation focuses on the case that $|\Gamma_D|, |\Gamma_R| > 0$, with possibly $\Gamma_N = \emptyset$. However, the cases $\Gamma_D = \emptyset$ and $|\Gamma_R| > 0$, $|\Gamma_D| > 0$ and $\Gamma_R = \emptyset$, as well as the pure Neumann problem $\Gamma_N = \partial\Omega$ are also covered by the abstract analysis.

5.4.2. Weak formulation. The weak formulation of (5.4.1) seeks $u \in \mathcal{X} := H^1(\Omega)$ such that

$$u = g_D \text{ on } \Gamma_D \text{ in the sense of traces} \tag{5.4.2a}$$

and all $v \in H^1_D(\Omega) := \big\{ v \in H^1(\Omega) \,:\, v = 0 \text{ on } \Gamma_D \big\}$ satisfy

$$b(u, v) := \int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\Gamma_R} \alpha u v \, ds = RHS(v) \tag{5.4.2b}$$

with

$$RHS(v) := \int_\Omega f v \, dx + \int_{\Gamma_N} \phi_N v \, ds + \int_{\Gamma_R} \phi_R v \, ds. \tag{5.4.2c}$$

Since $|\Gamma_R| > 0$ and $\alpha \geq \alpha_0 > 0$, the norm $\| \cdot \| := b(\cdot, \cdot)^{1/2}$ is equivalent to the $H^1(\Omega)$-norm.

Let $u_D \in H^1(\Omega)$ with $u_D|_\Gamma = g_D$ be an arbitrary lifting of the given Dirichlet data and set $u_0 := u - u_D \in H^1_D(\Omega)$. Then, (5.4.2) is equivalent to seek $u_0 \in H^1_D(\Omega)$ with

$$b(u_0, v) = RHS(v) - b(u_D, v) \quad \text{for all } v \in H^1_D(\Omega). \tag{5.4.3}$$

According to the Lax-Milgram lemma, the auxiliary problem (5.4.3) admits a unique solution $u_0 \in H^1(\Omega)$ and thus $u := u_0 + u_D$ is the unique solution of (5.4.2).

5.4.3. FEM discretization and approximation of Dirichlet data. Assume the initial triangulation $\mathcal{T}_0$, and hence all triangulations $\mathcal{T} \in \mathbb{T}$ of Ω, to resolve the boundary conditions in the sense that for all facets $E \subset \partial\Omega$ on the boundary, there holds $E \subseteq \overline{\gamma}$ for some $\gamma \in \{\Gamma_D, \Gamma_N, \Gamma_R\}$ and let $\mathbb{T}(\cdot,\cdot)$ denote bisection from Section 3.2.8. Let $\mathcal{S}^p_D(\mathcal{T}) := \mathcal{P}^p(\mathcal{T}) \cap H^1_D(\Omega)$ with fixed polynomial order $p \geq 1$. To discretize the given Dirichlet data g_D, for any given triangulation $\mathcal{T} \in \mathbb{T}$, choose an approximation

$$G_D(\mathcal{T}) \in \mathcal{S}^p(\mathcal{T}|_{\Gamma_D}) := \big\{V|_{\Gamma_D} \,:\, V \in \mathcal{S}^p(\mathcal{T})\big\}$$

of the Dirichlet data g_D. Here and throughout this section, let $\mathcal{T}|_{\Gamma_D} := \big\{T|_{\Gamma_D} \,:\, T \in \mathcal{T}\big\}$ denote the restriction of the volume triangulation to the Dirichlet boundary Γ_D, and $\mathcal{S}^p(\mathcal{T}|_{\Gamma_D})$ is the discrete trace space. A convenient way to choose this approximation independently of the spatial dimension is the Scott-Zhang projection $J(\mathcal{T}) : H^1(\Omega) \to \mathcal{S}^p(\mathcal{T})$ from [**76**]. The formal definition also allows for an operator $J(\mathcal{T}|_{\Gamma_D}) : L^2(\Gamma_D) \to \mathcal{S}^p(\mathcal{T}|_{\Gamma_D})$ on the boundary (see also Definition 3.3.2 for details). The reader is referred to [**4**] for details and further discussions.

The discrete counterpart of (5.4.2) seeks $U(\mathcal{T}) \in \mathcal{S}^p(\mathcal{T})$ such that

$$U(\mathcal{T})|_{\Gamma_D} = G_D(\mathcal{T}), \tag{5.4.4a}$$

$$b(U(\mathcal{T}), V) = f(V) \quad \text{for all } V \in \mathcal{S}^p_D(\mathcal{T}). \tag{5.4.4b}$$

As in the continuous case, (5.4.4) admits a unique solution and there holds a general Céa lemma

$$\|u - U(\mathcal{T})\|_{H^1(\Omega)} \leq C_{26} \min_{V \in \mathcal{S}^p(\mathcal{T})} \|u - V\|_{H^1(\Omega)}, \tag{5.4.5}$$

where $C_{26} > 0$ depends only on the boundary parts, p, the shape regularity (3.2.5), and on α. The Céa lemma (5.4.5) is proved in [**4**, Proposition 2] for the case $\Gamma_R = \emptyset$. The proof, however, transfers to the present case with the obvious modifications.

5.4.4. Quasi-optimal convergence. The derivation of the residual-based error estimator $\eta(\mathcal{T})$ follows similarly to the homogeneous case and differs only by adding an oscillation term to control the approximation of the Dirichlet data [**4, 11, 48, 74**]. With the weight function $h(\mathcal{T})|_T := |T|^{1/d}$ for all $T \in \mathcal{T}$, the local contributions read

$$\begin{aligned}\eta_T(\mathcal{T}) := h(\mathcal{T})|_T^2 \|f + \Delta_{\mathcal{T}} U(\mathcal{T})\|^2_{L^2(T)} + h(\mathcal{T})|_T \|[\partial_n U(\mathcal{T})]\|^2_{L^2(\partial T \cap \Omega)} \\ + \|h(\mathcal{T})^{1/2}(\phi_R - \alpha U(\mathcal{T}) - \partial_n U(\mathcal{T}))\|^2_{L^2(\partial T \cap \Gamma_R)} \\ + \|h(\mathcal{T})^{1/2}(\phi_N - \partial_n U(\mathcal{T}))\|^2_{L^2(\partial T \cap \Gamma_N)} + \mathrm{dir}_T(\mathcal{T})^2,\end{aligned}$$

where

$$\mathrm{dir}_T(\mathcal{T})^2 := h(\mathcal{T})|_T \|(1 - \Pi_{p-1}(\mathcal{T}|_{\Gamma_D}))\nabla_\Gamma g_D\|^2_{L^2(\partial T \cap \Gamma_D)}$$

and $\Pi_{p-1}(\mathcal{T}|_{\Gamma_D}) \,:\, L^2(\Gamma_D) \to \mathcal{P}^{p-1}(\mathcal{T}|_{\Gamma_D}) := \big\{V|_{\Gamma_D} \,:\, V \in \mathcal{P}^{p-1}(\mathcal{T})\big\}$ is the (piecewise) L^2-orthogonal projection, and $\nabla_\Gamma(\cdot)$ denotes the surface gradient.

For each facet $E \subset \partial\Omega$, there exists a unique element $T \in \mathcal{T}$ such that $E \subset \partial T$. In particular, $h(\mathcal{T})$ also induces a weight function on $\gamma \in \{\Gamma_D, \Gamma_N, \Gamma_R\}$.

The following proposition shows that inhomogeneous (and mixed) boundary data fit in the framework of our abstract analysis. Emphasis is on the novel quasi-orthogonality (E2) which improves the analysis of [**4**] on separate Dörfler marking. The super-contractive weight function $h_\omega(\mathcal{T})$ from Proposition 4.5.4 establishes optimal convergence of Algorithm 2.2.1 with the standard Dörfler marking (2.2.1).

Given $\mathcal{T} \in \mathbb{T}$ and $\mathcal{S} \subseteq \mathcal{T}$, define the 5-patch by

$$\omega^5(\mathcal{S}, \mathcal{T}) := \big\{T \in \mathcal{T} : \exists T_0, \dots, T_4 \in \mathcal{T},\, T_0 \in \mathcal{S},\, T_4 \cap T \neq \emptyset, \\ T_i \cap T_{i+1} \neq \emptyset,\, i = 0, \dots, 3\big\}. \tag{5.4.6}$$

Proposition 5.4.1. *The conforming discretization of the Poisson problem* (5.4.2) *with residual error estimator* $\eta(\cdot)$ *satisfies*

(i) *stability and reduction* (E1) *with* $\varrho(\mathcal{T}, \widehat{\mathcal{T}}) := C_{\mathrm{pert}} \|U(\mathcal{T}) - U(\widehat{\mathcal{T}})\|_{H^1_0(\Omega)}$ *as well as* $\mathcal{S}(\mathcal{T}, \widehat{\mathcal{T}}) := \mathcal{T} \setminus \widehat{\mathcal{T}}$ *and* $\widehat{\mathcal{S}}(\mathcal{T}, \widehat{\mathcal{T}}) := \widehat{\mathcal{T}} \setminus \mathcal{T}$,
(ii) *general quasi-orthogonality* (E2),
(iii) *discrete reliability* (E3) *with* $\mathcal{R}(\mathcal{T}, \widehat{\mathcal{T}}) = \omega^5(\mathcal{T}\backslash\widehat{\mathcal{T}}, \mathcal{T})$, $\kappa_{\mathrm{dlr}} = \infty$, *and* $\varepsilon_{\mathrm{drel}} = 0$,
(iv) *the refinement axioms* (T1)–(T3) *with* $C_{\mathrm{approx}}(s) \leq C_{\mathrm{mon}}(C_4 + 1)^s \|\eta, \mathbb{T}\|_s$ *for all* $s > 0$.

Moreover, the estimator satisfies (2.4.1)–(2.4.2) *with* $\mathrm{err}(\mathcal{T}) := \|u - U(\mathcal{T})\|_{H^1(\Omega)}$ *and*

$$\begin{aligned} \mathrm{data}(\mathcal{T})^2 := \mathrm{dir}(\mathcal{T})^2 &+ \min_{F \in \mathcal{P}^{p-1}(\mathcal{T})} \|h(\mathcal{T})(f - F)\|^2_{L^2(\Omega)} \\ &+ \min_{\Phi \in \mathcal{P}^{p-1}(\mathcal{T}|_{\Gamma_N})} \|h(\mathcal{T})^{1/2}(\phi_N - \Phi)\|^2_{L^2(\Gamma_N)} \\ &+ \min_{\Phi \in \mathcal{P}^{p-1}(\mathcal{T}|_{\Gamma_R})} \|h(\mathcal{T})^{1/2}(\phi_R - \Phi)\|^2_{L^2(\Gamma_R)}. \end{aligned} \tag{5.4.7}$$

The constants $C_{\mathrm{drel}}, C_{\mathrm{qo}}, C_{\mathrm{pert}}, C_{\mathrm{rel}}, C_{\mathrm{eff}}$ *depend only on the parameter* $p > 1$, $\mathcal{T}_0$, *and on* Ω.

Proof. Efficiency (2.4.2) can be found in [**11, 74**] or [**4**, Proposition 3]. The proof of (5.4.7) follows similarly to that of Proposition 3.5.1 and exploits that $\Delta_{\mathcal{T}} U(\mathcal{T})|_T$ is a polynomial of degree $\leq p - 2$.

The proofs of stability and reduction (E1) are verbatim to the case with $\Gamma_R = \emptyset$ from [**4**, Proposition 11]. The proof of discrete reliability (E3) is more involved, however, the difficulties arise only due to the approximation of the Dirichlet data and the non-local $H^{1/2}(\Gamma_D)$-norm. The proof in [**4**, Proposition 21] for $\Gamma_R = \emptyset$ generalizes to the present case. The statement (iv) follows as for the homogeneous case in Section 3.5.1.

It remains to verify the quasi-orthogonality (2.7.5) which implies (E2) by virtue of Lemma 2.7.3. The 5-patch $\omega^5(\cdot, \cdot)$ is a patch function in the sense of Section 4.5.1. Moreover, the weight function $h(\mathcal{T})$ satisfies the assumptions of Section 4.5. Hence, Proposition 4.5.4 provides a super contractive weight function $h_{\omega^5}(\cdot)$. It is proved in [**4**, Lemma 20] for $\Gamma_R = \emptyset$ that there holds for all $\varepsilon_{\mathrm{qo}} > 0$ and all $\widehat{\mathcal{T}} \in \mathbb{T}(\mathcal{T})$, $\mathcal{T} \in \mathbb{T}$, that

$$\begin{aligned} \|U(\widehat{\mathcal{T}}) - U(\mathcal{T})\|^2 \leq \|u - U(\mathcal{T})\|^2 - (1 - \varepsilon_{\mathrm{qo}})\|u - U(\widehat{\mathcal{T}})\|^2 \\ + C_{\mathrm{pyth}} {\varepsilon_{\mathrm{qo}}}^{-1} \|(J(\widehat{\mathcal{T}}|_{\Gamma_D}) - J(\mathcal{T}|_{\Gamma_D})) g_D\|^2_{H^{1/2}(\Gamma_D)}, \end{aligned} \tag{5.4.8}$$

where $C_{\mathrm{pyth}} > 0$ depends only on $\mathbb{T}$ and Γ_D. Although [**4**] considers $\Gamma_R = \emptyset$ and hence $\|\cdot\| = \|\nabla(\cdot)\|_{L^2(\Omega)}$, the proof transfers to the present case.

The focus in the derivation of quasi-orthogonality (2.7.5) is on the last term on the right-hand side. First, let $\omega^5_D(\mathcal{T}\backslash\widehat{\mathcal{T}}, \mathcal{T}) \subseteq \mathcal{T}|_{\Gamma_D}$ denote the set of all facets E of $\mathcal{T}$ with $E \subseteq \overline{\Gamma}_D \cap \bigcup \omega^5(\mathcal{T}\backslash\widehat{\mathcal{T}}, \mathcal{T})$. It is part of the proof of [**4**, Proposition 21] that there exists a uniform constant $C_{27} > 0$ such that any triangulation $\mathcal{T} \in \mathbb{T}$ and all refinements $\widehat{\mathcal{T}} \in \mathbb{T}(\mathcal{T})$ of $\mathcal{T} \in \mathbb{T}$ satisfy

$$\|(J(\widehat{\mathcal{T}}|_{\Gamma_D}) - J(\mathcal{T}|_{\Gamma_D}))v\|_{H^{1/2}(\Gamma_D)} \leq C_{27} \|h(\mathcal{T})^{1/2}(1 - \Pi^{p-1}(\mathcal{T}|_{\Gamma_D}))\nabla_\Gamma v\|_{L^2(\cup \omega^5_D(\mathcal{T}\backslash\widehat{\mathcal{T}}, \mathcal{T}))}$$

for all $v \in H^1(\Gamma_D)$. We note that this result hinges on the use of bisection (Section 3.2.8) in the sense that the constant $C_{27} > 0$ depends on the shape of all possible patches. By means of Lemma 3.3.3, the proof of [**4**, Proposition 21] can be extended to triangulations in the sense of Section 3.2.1–3.2.7.

This estimate is applied for $v = g_D$. The definition of $h_{\omega^5}(\mathcal{T})$ in Proposition 4.5.4 implies

$$\begin{aligned} h_{\omega^5}(\widehat{\mathcal{T}}) &\le h_{\omega^5}(\mathcal{T}) \quad \text{pointwise on all } T \in \mathcal{T}, \\ h_{\omega^5}(\widehat{\mathcal{T}}) &\le q_{\mathrm{sc}} h_{\omega^5}(\mathcal{T}) \quad \text{pointwise on all } T \in \mathcal{T} \text{ with } h(\mathcal{T})|_{\cup\omega^5(T,\widehat{\mathcal{T}})} \neq h(\widehat{\mathcal{T}})|_{\cup\omega^5(T,\widehat{\mathcal{T}})}. \end{aligned}$$

Recall that $h(\mathcal{T})|_{\cup\omega^5(T,\widehat{\mathcal{T}})} \neq h(\widehat{\mathcal{T}})|_{\cup\omega^5(T,\widehat{\mathcal{T}})}$ is in the present case equivalent to $\omega^5(T,\mathcal{T}) \cap \mathcal{T} \setminus \widehat{\mathcal{T}} \neq \emptyset$ or $T \in \omega^5(\mathcal{T} \setminus \widehat{\mathcal{T}}, \mathcal{T})$. Hence, we obtain

$$h_{\omega^5}(\widehat{\mathcal{T}}) \le q_{\mathrm{sc}} h_{\omega^5}(\mathcal{T}) \quad \text{pointwise on all } T \in \omega^5(\mathcal{T} \setminus \widehat{\mathcal{T}}, \mathcal{T}).$$

This implies

$$(1 - q_{\mathrm{sc}})\, h_{\omega^5}(\mathcal{T})|_{\cup\omega^5(\mathcal{T}\setminus\widehat{\mathcal{T}},\mathcal{T})} \le h_{\omega^5}(\mathcal{T}) - h_{\omega^5}(\widehat{\mathcal{T}}) \quad \text{pointwise in } \Omega.$$

The contraction above allows to write

$$\begin{aligned} &(1 - q_{\mathrm{sc}})\|h_{\omega^5}(\mathcal{T})^{1/2}(1 - \Pi^{p-1}(\mathcal{T}|_{\Gamma_D}))\nabla_\Gamma g_D\|^2_{L^2(\cup\omega^5_D(\mathcal{T}\setminus\widehat{\mathcal{T}},\mathcal{T}))} \\ &\le \|h_{\omega^5}(\mathcal{T})^{1/2}(1 - \Pi^{p-1}(\mathcal{T}|_{\Gamma_D}))\nabla_\Gamma g_D\|^2_{L^2(\Gamma_D)} - \|h_{\omega^5}(\mathcal{T})^{1/2}(1 - \Pi^{p-1}(\mathcal{T}|_{\Gamma_D}))\nabla_\Gamma g_D\|^2_{L^2(\Gamma_D)}. \end{aligned}$$

This and the element wise best-approximation property of $\Pi^{p-1}(\widehat{\mathcal{T}}|_{\Gamma_D})$ prove that

$$\|h_{\omega^5}(\mathcal{T})^{1/2}(1 - \Pi^{p-1}(\widehat{\mathcal{T}}|_{\Gamma_D}))\nabla_\Gamma g_D\|^2_{L^2(\Gamma_D)} \le \|h_{\omega^5}(\mathcal{T})^{1/2}(1 - \Pi^{p-1}(\mathcal{T}|_{\Gamma_D}))\nabla_\Gamma g_D\|^2_{L^2(\Gamma_D)}.$$

With $h(\mathcal{T}) \le C_{19} h_{\omega^5}(\mathcal{T})$ from Proposition 4.5.4, we obtain

$$\begin{aligned} (1 - q_{\mathrm{sc}})C_{19}^{-1}&\|h(\mathcal{T})^{1/2}(1 - \Pi^{p-1}(\mathcal{T}|_{\Gamma_D}))\nabla_\Gamma g_D\|^2_{L^2(\cup\omega^5_D(\mathcal{T}\setminus\widehat{\mathcal{T}},\mathcal{T}))} \\ &\le \|h_{\omega^5}(\mathcal{T})^{1/2}(1 - \Pi^{p-1}(\mathcal{T}|_{\Gamma_D}))\nabla_\Gamma g_D\|^2_{L^2(\Gamma_D)} \\ &\qquad - \|h_{\omega^5}(\mathcal{T})^{1/2}(1 - \Pi^{p-1}(\mathcal{T}|_{\Gamma_D}))\nabla_\Gamma g_D\|^2_{L^2(\Gamma_D)}. \end{aligned}$$

The combination of the previous arguments leads to

$$\|(J(\widehat{\mathcal{T}}|_{\Gamma_D}) - J(\mathcal{T}|_{\Gamma_D}))g_D\|^2_{H^{1/2}(\Gamma_D)} \le \alpha(\mathcal{T})^2 - \alpha(\widehat{\mathcal{T}})^2,$$

where

$$\alpha(\mathcal{T}) := C_{27}^{1/2} C_{19}^{1/2}(1 - q_{\mathrm{sc}})^{-1/2}\|h_{\omega^5}(\mathcal{T})^{1/2}(1 - \Pi^{p-1}(\mathcal{T}|_{\Gamma_D}))\nabla_\Gamma g_D\|_{L^2(\Gamma_D)}.$$

By equivalence (4.5.9), one obtains (2.7.5b) and hence Lemma 2.7.3 proves general quasi-orthogonality (E2). This concludes the proof. □

Remark 5.4.2. *We briefly comment on the case $\Gamma_R = \emptyset$ with*

$$\|v\|^2 := \|\nabla v\|^2_{L^2(\Omega)} + \|v\|^2_{H^{1/2}(\Gamma_D)} \neq b(v,v).$$

The Rellich compactness theorem guarantees that $\|\cdot\|$ is an equivalent norm in $H^1(\Omega)$. The combination with [**4**, Lemma 20] *(i.e.* (5.4.8) *with $\|\cdot\| = \|\nabla(\cdot)\|_{L^2(\Omega)}$) proves for sufficiently small $\varepsilon_{\mathrm{qo}} \ll 1$ that*

$$\begin{aligned} \|U(\widehat{\mathcal{T}}) - U(\mathcal{T})\|^2 \le \|\nabla(u - U(\mathcal{T}))\|^2_{L^2(\Omega)} - (1 - \varepsilon_{\mathrm{qo}})\|\nabla(u - U(\widehat{\mathcal{T}}))\|^2_{L^2(\Omega)} \\ + \widetilde{C}_{\mathrm{pyth}}{\varepsilon_{\mathrm{qo}}}^{-1}\|(J(\widehat{\mathcal{T}}|_{\Gamma_D}) - J(\mathcal{T}|_{\Gamma_D}))g_D\|^2_{H^{1/2}(\Gamma_D)}. \end{aligned} \tag{5.4.9}$$

With (5.4.9) *instead of* (5.4.8)*, the arguments in the proof of Proposition 5.4.1 remain valid.*

The adaptive FEM for the mixed boundary value boundary (5.4.1) satisfies all assumptions of the abstract framework.

Consequence 5.4.3. *The adaptive algorithm leads to convergence with optimal rate for the estimator $\eta(\mathcal{T})$ in the sense of Theorem 2.3.3. For optimal rates of the discretization error in the sense of Theorem 2.4.3, additional regularity of the data has to be imposed for higher-order elements $p \geq 1$, cf. Consequence 3.5.2.* □

CHAPTER 6

Applications III: Adaptive BEM with Geometry Approximation

6.1. Introduction, state of the art & outline

This chapter treats the weakly-singular integral equation from Section 3.5.3 for general boundaries. Most of the literature concerns piecewise polynomial boundary geometries [**20, 28, 33, 34, 27, 47**]. One way to circumvent this, is to employ the isogeometric approach, where the boundary is given in terms of B-splines or NURBS which stem from computer aided design systems. This, however, involves the drawback, that one has to compute the integral operators on nonstandard geometries, which is at the moment not supported by available BEM libraries, and moreover is expensive. The approach proceeded here, is to approximate the boundary by piecewise affine line segments and to perform the computation on the approximate polygonal boundary. This allows to employ standard BEM implementations and moreover enables to compute the operator matrices analytically in 2D. To estimate the approximation error, we develop an error estimator, which reliably estimates the discretization error of the approximation spaces as well as the geometric approximation error introduced by the approximate boundary. While there are some results on geometry approximation for the finite element method [**15, 63, 38, 42**], this is the first a posteriori analysis of geometry approximation for the boundary element method (several a priori results for BEM are available in, e.g. [**75, 67**]). Under some assumptions, we are able to prove plain convergence in the sense of (2.3.1) of the error estimator and the approximate solutions. The remainder of the chapter is organized as follows: Section 6.2 states the assumptions on the geometry and introduces the geometric error estimator. The main result of this chapter is stated in Section 6.4 and the convergence proof is given in Section 6.3.

6.2. Setting

Consider the weakly-singular integral equation on the boundary $\Gamma := \partial\Omega$ of a connected Lipschitz domain $\Omega \subseteq \mathbb{R}^2$ with $\mathrm{diam}(\Omega) < 1$

$$\mathcal{V}u = f,$$

where the weakly-singular integral operator $\mathcal{V} :\ H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is given by (3.5.11).

6.2.0.1. *Exact and approximate geometry.* Let the exact boundary $\Gamma := \partial\Omega$ allow for a piecewise smooth parametrization $\gamma :\ [0,1] \to \Gamma$ such that both γ and γ^{-1} are Lipschitz continuous with constant $C_\gamma > 0$ and $|\gamma'(s)| = |\Gamma|$ for all $s \in [0,1]$ (to avoid problems with the endpoints of $[0,1]$, we identify $\{0\}$ and $\{1\}$ and consider the metric $d(s,t) := \min\{|s-t|, |1-s| + |0-t|, |0-s| + |1-t|\}$ on $[0,1]$). Let t_Γ denote the unit tangent on Γ and let n_Γ denote the unit normal. By ∂_Γ, we denote the arc-length derivative on Γ (see Definition 6.2.5 below). We assume that Γ has bounded curvature in the sense that $\|\partial_\Gamma t_\Gamma\|_{L^\infty(\Gamma)} \leq \kappa_\Gamma$ (where ∂_Γ is understood piecewise on smooth parts of Γ) for some $\kappa_\Gamma > 0$. Any approximate boundary $\Gamma_\star$ must be a nodal interpolation of Γ with nodes $\mathcal{K}_\star \subseteq \Gamma \cap \Gamma_\star$. The finitely many non-smooth points $\mathcal{P}_\Gamma$ of Γ have to satisfy $\mathcal{P}_\Gamma \subset \mathcal{K}_\star$ and the enclosed domain $\Omega_\star$ (i.e., $\partial\Omega_\star = \Gamma_\star$) must satisfy $\mathrm{diam}(\Omega_\star) \leq 1 - \varepsilon_{\mathrm{scale}}$ for some uniform

$\varepsilon_{\text{scale}} > 0$ (Note that this can always be achieved by scaling of the exact boundary Γ). The approximation $\Gamma_\star$ is associated with the partition $\mathcal{T}_\star$ which consists of the compact line segments of $\Gamma_\star$. We call the pair $(\mathcal{T}_\star, \Gamma_\star)$ an approximate geometry. Each element $T \in \mathcal{T}_\star$ satisfies

$$T \cap \Gamma \subseteq \mathcal{K}_\star \quad \text{or} \quad T \cap \Gamma = T,$$

i.e., the exact boundary touches elements only at the nodes or coincides exactly with the element. Each $T \in \mathcal{T}_\star$ defines a unique compact curve segment $T^\Gamma \subseteq \Gamma$ with the same endpoints as T. The collection of all this curve segments defines a partition $\mathcal{T}_\star^\Gamma$ of Γ. To avoid degenerate cases, we consider only partitions with $|T^\Gamma| < |\Gamma|/2$ for all $T \in \mathcal{T}_\star$.

Consider the map $\gamma_\star : \Gamma \to \Gamma_\star$ (see Figure 2 for an illustration) implicitly defined by

$$\begin{aligned} \gamma_\star(T) &\subseteq T^\Gamma \quad \text{for all } T \in \mathcal{T}_\star, \\ (x - \gamma_\star(x)) \cdot t_\Gamma(x) &= 0 \quad \text{for all } x \in \Gamma \setminus \mathcal{P}_\Gamma, \\ \gamma_\star(x) &= x \quad \text{for all } x \in \mathcal{K}_\star. \end{aligned} \tag{6.2.1}$$

Note that the subscript $\star$ denotes the relation to the approximate geometry $(\mathcal{T}_\star, \Gamma_\star)$.

Remark 6.2.1. *In Lemma 6.2.17 below, we introduce an extension $\gamma_\star\colon \mathbb{R}^2 \to \mathbb{R}^2$. Hence, after Lemma 6.2.17, $\gamma_\star$ is also used to denote its extension, where the meaning will be clear from the context.*

The approximate geometry $(\mathcal{T}_\star, \Gamma_\star)$ must be sufficiently close to Γ such that (Γ2)–(Γ4) hold for uniform constants $C_{\text{Lip}}, C_\mu > 0$

(Γ1) The orthogonal projection $\gamma_\star : \Gamma \to \Gamma_\star$ from (6.2.1) is well-defined and uniquely determined, piecewise smooth, and is a continuous one-to-one map.

(Γ2) All $x, y \in \Gamma$ satisfy

$$C_{\text{Lip}}^{-1}|x - y| \le |\gamma_\star(x) - \gamma_\star(y)| \le C_{\text{Lip}}|x - y|.$$

(Γ3) All $T \in \mathcal{T}_\star$ with endpoints $x_T, y_T \in \Gamma$ satisfy that each $x \in T$ defines a unique $y \in T^\Gamma$ with

$$(x - y) \cdot (x_T - y_T) = 0.$$

This defines a map $\mu_\star : \Gamma \to \Gamma_\star$ by $\mu_\star(y) := x$ (see Figure 2 for an illustration).

(Γ4) There holds

$$C_\mu^{-1}\|\text{id}_\Gamma - \gamma_\star\|_{L^\infty(\Gamma)}^2 \le \|\text{id}_\Gamma - \mu_\star\|_{L^\infty(\Gamma)}^2 \le \|\text{id}_\Gamma - \gamma_\star\|_{L^\infty(\Gamma)}^2.$$

Note that the upper bound holds for any geometry $\Gamma_\star$, since $\mu_\star$ is the orthogonal projection onto $\Gamma_\star$.

Lemma 6.2.9 below gives some sufficient conditions which imply (Γ1)–(Γ4).

6.2.0.2. *Approximate solution.* With the $\mathcal{T}_\star$-piecewise constant functions $\mathcal{P}^0(\mathcal{T}_\star)$, the Galerkin approximation $U(\mathcal{T}_\star) \in \mathcal{P}^0(\mathcal{T}_\star)$ is the solution of

$$\int_{\Gamma_\star} \mathcal{V}_\star U(\mathcal{T}_\star)\, V\, dx = \int_{\Gamma_\star} f_\star V\, dx \quad \text{for all } V \in \mathcal{P}^0(\mathcal{T}_\star), \tag{6.2.2}$$

where

$$\mathcal{V}_\star w(x) := -\frac{1}{2\pi}\int_{\Gamma_\star} \log|x - y| w(y)\, dy$$

denotes the weakly-singular integral operator on $\Gamma_\star$ and $f_\star := f \circ \gamma_\star^{-1}$.

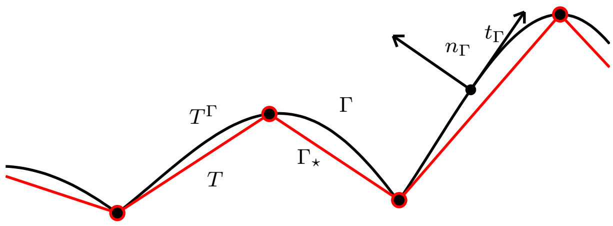

FIGURE 1. Boundary Γ with tangent vector t_Γ and normal vector n_Γ as well as approximate geometry $(\mathcal{T}_\star, \Gamma_\star)$ with element $T \in \mathcal{T}_\star$ and corresponding $T^\Gamma \subseteq \Gamma$.

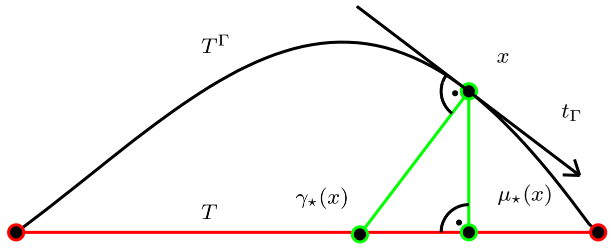

FIGURE 2. The mappings $\gamma_\star$ and $\mu_\star$.

We propose to approximate the exact solution $u \approx U(\mathcal{T}_\star)^\Gamma$ by

$$U(\mathcal{T}_\star)^\Gamma := U(\mathcal{T}_\star) \circ \gamma_\star |\partial_\Gamma \gamma_\star|.$$

6.2.1. Error estimator. The partition $\mathcal{T}_\star$ induces a local weight function $h_\star|_T := |T| := \text{length}(T)$ for all $T \in \mathcal{T}_\star$. The error quantity of interest is

$$\|u - U(\mathcal{T}_\star)^\Gamma\|_{H^{-1/2}(\Gamma)}.$$

With the identity mapping $\text{id}_\Gamma : \Gamma \to \Gamma$ and the geometric error

$$\text{geo}(\mathcal{T}_\star) := \max\{\|\text{id}_\Gamma - \gamma_\star\|_{L^\infty(\Gamma)}^{1/2}, \|t_\Gamma - \partial_\Gamma \gamma_\star\|_{L^\infty(\Gamma)}\}, \tag{6.2.3}$$

the error estimator reads

$$\begin{aligned}\eta(\mathcal{T}_\star) := \Big(&\|h_\star^{1/2} \partial_{\Gamma_\star}(\mathcal{V}U(\mathcal{T}_\star) - f_\star)\|_{L^2(\Gamma_\star)}^2 \\ &+ \text{geo}(\mathcal{T}_\star)^3 (1 + |\log(\text{geo}(\mathcal{T}_\star))|^2) \|U(\mathcal{T}_\star)\|_{L^2(\Gamma_\star)}^2\Big)^{1/2}.\end{aligned} \tag{6.2.4}$$

For brevity of notation, we write $\rho(\mathcal{T}_\star) := \|h_\star^{1/2} \partial_{\Gamma_\star}(\mathcal{V}U(\mathcal{T}_\star) - f_\star)\|_{L^2(\Gamma_\star)}$ and define the element wise contributions for all $T \in \mathcal{T}_\star$

$$\begin{aligned}\rho_T(\mathcal{T}_\star) &:= h_\star|_T^{1/2} \|\partial_{\Gamma_\star}(\mathcal{V}U(\mathcal{T}_\star) - f_\star)\|_{L^2(T)}, \\ \text{geo}_T(\mathcal{T}_\star) &:= \max\{\|\text{id}_\Gamma - \gamma_\star\|_{L^\infty(T^\Gamma)}^{1/2}, \|t_\Gamma - \partial_\Gamma \gamma_\star\|_{L^\infty(T^\Gamma)}\}.\end{aligned} \tag{6.2.5}$$

6.2.2. Adaptive geometry approximation. We propose a modified version of Algorithm 2.2.1 which includes also the geometric error (a similar algorithm can also be found in [**15**] for FEM). To that end, choose an initial approximation Γ_0 as well as the corresponding partition $\mathcal{T}_0$ of Γ_0 such that the requirements of Section 6.2.0.1 are satisfied.

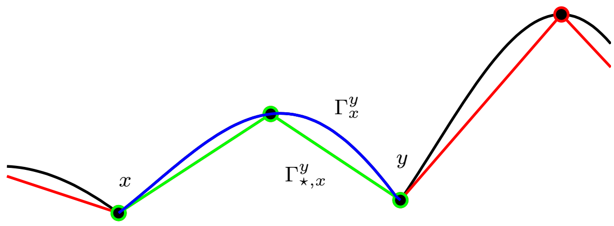

FIGURE 3. The curve segments Γ_x^y and $\Gamma_{\star,x}^y$.

ALGORITHM 6.2.2. INPUT: *Initial triangulation $\mathcal{T}_0$ and parameters $0 < \theta \leq 1$, $0 \leq \vartheta < 1$.*
Loop: *For $\ell = 0, 1, 2, \ldots$ do* (i) – (iv).

(i) *Compute solution $U(\mathcal{T}_\ell)$ of* (6.2.2).
(ii) *Compute error estimators $\rho_T(\mathcal{T}_\ell)$ and $\mathrm{geo}_T(\mathcal{T}_\ell)$ for all $T \in \mathcal{T}_\ell$.*
(iii) *Determine a set of marked elements $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ with minimal cardinality which satisfies the Dörfler marking*

$$\theta\rho(\mathcal{T}_\ell)^2 \leq \sum_{T\in\mathcal{M}_\ell} \rho_T(\mathcal{T}_\ell)^2 \tag{6.2.6a}$$

as well as

$$\mathcal{M}_\ell \supseteq \big\{T \in \mathcal{T}_\ell \,:\, \mathrm{geo}_T(\mathcal{T}_\ell) > \vartheta\mathrm{geo}(\mathcal{T}_\ell)\big\}. \tag{6.2.6b}$$

(iv) *Define the next partition $\mathcal{T}_{\ell+1} = \mathbb{T}(\mathcal{T}_\ell, \mathcal{M}_\ell)$ as detailed in Section 6.2.5 below.*

OUTPUT: *Error estimators $(\eta(\mathcal{T}_\ell))_{\ell\in\mathbb{N}_0}$ and approximations $(U(\mathcal{T}_\ell)^\Gamma)_{\ell\in\mathbb{N}_0}$.*

6.2.3. Some definitions. Below, we provide some definitions which are used throughout this chapter.

DEFINITION 6.2.3. *Given $x, y \in \Gamma$, define the compact and connected set $\Gamma_x^y \subseteq \Gamma$ with $x, y \in \Gamma_x^y$ as*

$$\int_{\Gamma_x^y} 1\,dx = \inf\Big\{\int_{\widetilde\Gamma} 1\,dx \,:\, \widetilde\Gamma \subseteq \Gamma \text{ compact and connected with } x, y \in \widetilde\Gamma\Big\}.$$

The set on the right-hand side is non-empty due to the fact that Γ is connected by assumption. Let $x_T, y_T \in T \cap \Gamma$ denote the endpoints of $T \in \mathcal{T}_\star$. Note that since $|T^\Gamma| < |\Gamma|/2$, there holds $T^\Gamma = \Gamma_{x_T}^{y_T}$. Given the approximate geometry $\Gamma_\star$ and $x, y \in \Gamma_\star$, define the compact and connected set $\Gamma_{\star,x}^y \subseteq \Gamma_\star$ with $x, y \in \Gamma_{\star,x}^y$ as

$$\int_{\Gamma_{\star,x}^y} 1\,dx = \inf\Big\{\int_{\widetilde\Gamma} 1\,dx \,:\, \widetilde\Gamma \subseteq \Gamma_\star \text{ compact and connected with } x, y \in \widetilde\Gamma\Big\}.$$

See also Figure 3 for an illustration.

DEFINITION 6.2.4. *For a boundary part $\omega \subseteq \Gamma \cup \Gamma_\star$ with a given approximate geometry $\Gamma_\star$, we denote by $|\omega| := \int_\omega 1\,dx$ the length of the curve. Moreover, given subsets $\omega, \omega' \subseteq \Gamma \cup \Gamma_\star$, define*

$$\mathrm{dist}(\omega, \omega') := \inf_{x\in\omega,\, y\in\omega'} |x - y| \geq 0.$$

Definition 6.2.5 (Arc-length derivative). *Given any approximate geometry $\Gamma_\star$ (also the exact geometry Γ is allowed here), $x \in \Gamma_\star$, and $g : \Gamma_\star \to \mathbb{R}^d$, $d \in \{1,2\}$, the arc-length derivative $\partial_{\Gamma_\star} g(x)$ (if exists) is defined as follows: Choose some $\delta > 0$ and some continuous one-to-one mapping $\gamma_x : (-\delta, \delta) \to \Gamma_\star$ with $\gamma_{\star,x}(0) = x$ and $\gamma'_{\star,x}(s) = t_{\Gamma_\star} \circ \gamma_{\star,x}(s)$ almost everywhere in $(-\delta,\delta)$. Then, define*

$$\partial_{\Gamma_\star} g(x) := (g \circ \gamma_{\star,x})'(0) \in \mathbb{R}^d. \tag{6.2.7}$$

The definition is unique since γ_x is uniquely defined locally around zero.

Given another approximate geometry $\Gamma_\bullet$ (also the exact geometry Γ is allowed here) and $g : \Gamma_\star \to \Gamma_\bullet$, the arc-length derivative $\partial_{\Gamma_\star} g(x)$ can be defined as in (6.2.7), or in the scalar version as

$$\partial^s_{\Gamma_\star} g(x) := (\gamma^{-1}_{\bullet,g(x)} \circ g \circ \gamma_{\star,x})'(0) \in \mathbb{R}. \tag{6.2.8}$$

There holds the identity

$$\partial_{\Gamma_\star} g(x) = (\gamma_{\bullet,g(x)} \circ \gamma^{-1}_{\bullet,g(x)} \circ g \circ \gamma_{\star,x})'(0) = \gamma'_{\bullet,g(x)}(0)\partial^s_{\Gamma_\star} g(x) = t_{\Gamma_\bullet} \circ g(x)\partial^s_{\Gamma_\star} g(x). \tag{6.2.9}$$

Finally, for a function $g\colon \mathbb{R}^d \to \Gamma_\star$, $d \geq 1$, and some $z \in \mathbb{R}^d$ define

$$\partial^s_z g(x) := \partial_z(\gamma^{-1}_{g(x)} \circ g)(x) \in \mathbb{R}.$$

There holds the identity

$$\partial_z g(x) = \gamma'_{g(x)}(0)\partial^s_z g(x) = t_\Gamma \circ g(x)\partial^s_z g(x). \tag{6.2.10}$$

Definition 6.2.6. *Given any approximate geometry $\Gamma_\star$ (also the exact geometry Γ is allowed here), choose a parametrization $\gamma_{\Gamma_\star}\colon [0, |\Gamma_\star|] \to \Gamma_\star$ with $\gamma_{\Gamma_\star}(0) = \gamma_{\Gamma_\star}(|\Gamma_\star|)$ and $\gamma'_{\Gamma_\star} = t_{\Gamma_\star} \circ \gamma_{\Gamma_\star}$. Then, there holds for smooth functions $g_1, g_2\colon \Gamma_\star \to \mathbb{R}$ that $\partial_{\Gamma_\star} g_i = (g_i \circ \gamma_{\Gamma_\star})' \circ \gamma^{-1}_{\Gamma_\star}$ and integration by parts*

$$\begin{aligned}\int_{\Gamma_\star} \partial_{\Gamma_\star} g_1\, g_2\, dx &= \int_0^{|\Gamma_\star|} (\partial_{\Gamma_\star} g_1) \circ \gamma_{\Gamma_\star}\, g_2 \circ \gamma_{\Gamma_\star}\, dx = \int_0^{|\Gamma_\star|} (g_1 \circ \gamma_{\Gamma_\star})'\, g_2 \circ \gamma_{\Gamma_\star}\, dx \\ &= -\int_0^{|\Gamma_\star|} g_1 \circ \gamma_{\Gamma_\star}\, (g_2 \circ \gamma_{\Gamma_\star})'\, dx = -\int_{\Gamma_\star} g_1\, \partial_{\Gamma_\star} g_2\, dx.\end{aligned}$$

With this, we define

$$H^1(\Gamma_\star) := \big\{g \in L^2(\Gamma_\star) \,:\, \partial_{\Gamma_\star} g \in L^2(\Gamma_\star) \text{ in the weak sense}\big\}.$$

The spaces $H^s(\Gamma_\star) := [L^2(\Gamma_\star), H^1(\Gamma_\star)]_{s,2}$ are defined by real interpolation for all $s \in (0,1)$. By $H^{-s}(\Gamma_\star)$ we denote the dual space of $H^s(\Gamma_\star)$ with respect to the extended $L^2(\Gamma_\star)$ scalar product.

Lemma 6.2.7 (Chain-rule). *Given the approximate geometries $\Gamma_\star$, $\Gamma_\bullet$, Γ_+ (also the exact geometry Γ is allowed instead of each of the approximate geometries) as well as $\mu : \Gamma_\star \to \Gamma_\bullet$, $\lambda : \Gamma_\bullet \to \Gamma_+$, and $g : \Gamma_\bullet \to \mathbb{R}^d$. Then, there holds almost everywhere in $\Gamma_\star$*

$$\partial_{\Gamma_\star}(g \circ \mu) = (\partial_{\Gamma_\bullet} g) \circ \mu\, \partial^s_{\Gamma_\star}\mu \quad \text{and} \quad \partial^s_{\Gamma_\star}(\lambda \circ \mu) = (\partial^s_{\Gamma_\bullet}\lambda) \circ \mu\, \partial^s_{\Gamma_\star}\mu \tag{6.2.11a}$$

in the sense that each side exists if and only if the other one does, too. Moreover, for $\mu\colon \mathbb{R}^2 \to \Gamma_\bullet$, there holds

$$\partial_z(\lambda \circ \mu) = (\partial_{\Gamma_\bullet}\lambda) \circ \mu \partial^s_z \mu. \tag{6.2.11b}$$

Proof. By definition, there holds

$$\partial_{\Gamma_\star}(g\circ\mu)(x) = (g\circ\mu\circ\gamma_{\star,x})'(0) = (g\circ\gamma_{\bullet,\mu(x)}\circ\gamma_{\bullet,\mu(x)}^{-1}\circ\mu\circ\gamma_{\star,x})'(0)$$
$$= (\partial_{\Gamma_\bullet} g)\circ\mu(x)\partial^s_{\Gamma_\star}\mu,$$

as well as

$$\partial^s_{\Gamma_\star}(\lambda\circ\mu) = (\gamma^{-1}_{+,\lambda\circ\mu(x)}\circ\lambda\circ\mu\circ\gamma_{\star,x})'(0) = (\gamma^{-1}_{+,\lambda\circ\mu(x)}\circ\lambda\circ\gamma_{\bullet,\mu(x)}\circ\gamma^{-1}_{\bullet,\mu(x)}\circ\mu\circ\gamma_{\star,x})'(0)$$
$$= (\partial^s_{\Gamma_\bullet}\lambda)\circ\mu(x)\,\partial^s_{\Gamma_\star}\mu(x).$$

The identity (6.2.11b) follows by

$$\partial_z(\lambda\circ\mu)(x) = \partial_z(\lambda\circ\gamma_{\bullet,\mu(x)}\circ\gamma^{-1}_{\bullet,\mu(x)}\circ\mu)(x) = (\partial_{\Gamma_\bullet}\lambda)\circ\mu\partial^s_z\mu.$$

□

Lemma 6.2.8. *Given an approximate geometry $\mathcal{T}_\star$ with* (Γ1)–(Γ2)*, there holds*

$$(\partial^s_{\Gamma_\star}\gamma_\star^{-1})\circ\gamma_\star = (\partial^s_\Gamma\gamma_\star)^{-1} \quad \text{and} \quad |(\partial_{\Gamma_\star}\gamma_\star^{-1})\circ\gamma_\star| = |\partial_\Gamma\gamma_\star|^{-1}. \tag{6.2.12}$$

Proof. The chain rule (6.2.11) shows

$$1 = \partial^s_\Gamma(\gamma_\star^{-1}\circ\gamma_\star) = (\partial^s_{\Gamma_\star}\gamma_\star^{-1})\circ\gamma_\star\,\partial^s_\Gamma\gamma_\star.$$

Since (Γ2) implies $\partial^s_\Gamma\gamma_\star \neq 0$, the first statement follows. The identity (6.2.9) proves the second statement. □

6.2.4. Sufficient conditions for approximate geometries. Below, we investigate the claimed properties of the exact and approximate geometries.

Lemma 6.2.9. *There exists a constant $C_\Gamma > 0$ which depends only on Γ, such that all $x, y \in \Gamma$ satisfy*

$$C_\Gamma^{-1}|x-y| \le |\Gamma_x^y| \le C_\Gamma|x-y|. \tag{6.2.13}$$

Under (Γ2) *all $x, y \in \Gamma_\star$ satisfy*

$$C_\Gamma^{-1}C_{\mathrm{Lip}}^{-1}|x-y| \le |\Gamma^y_{\star,x}| \le C_{\mathrm{Lip}}C_\Gamma|x-y| \tag{6.2.14}$$

and under (Γ1)*, there holds*

$$(\partial^s_\Gamma\gamma_\star)^{-1} = \partial^s_{\Gamma_\star}(\gamma_\star^{-1})\circ\gamma_\star > 0 \tag{6.2.15}$$

almost everywhere on Γ. Moreover, there exist constants $h_\Gamma > 0$ and $\varepsilon_\Gamma > 0$ such that for the approximate geometry $\mathcal{T}_\star$ holds

(i) *$h_\star \le C_\Gamma^{-1}\kappa_\Gamma^{-1}/2$ implies* (Γ3) *and* (Γ4) *with $C_\mu = 2C_\Gamma$,*
(ii) *$h_\star \le C_\Gamma^{-1}\kappa_\Gamma^{-1}/2$ and $\mathrm{geo}(\mathcal{T}_\star) \le \kappa_\Gamma^{-1}/2$ imply* (Γ1)*,*
(iii) *$\mathrm{geo}(\mathcal{T}_\star) \le C_\Gamma^{-1}/2$ implies* (Γ2)*.*

Proof of (6.2.13). Without loss of generality, assume that $\{0,1\} \notin \gamma^{-1}(\Gamma_x^y)$. The assumption that $|\gamma'|$ is constant and the minimality of Γ_x^y shows that $|\gamma^{-1}(\Gamma_x^y)| \le 1/2$ and hence $|\gamma^{-1}(x)-\gamma^{-1}(y)| = d(\gamma^{-1}(x),\gamma^{-1}(y))$ (where $d(\cdot,\cdot)$ defines the metric on $[0,1]$ from Section 6.2.0.1). With this, there holds

$$|\Gamma_x^y| = \int_{\Gamma_x^y} 1\,dx = \int_{\gamma^{-1}(x)}^{\gamma^{-1}(y)} |\gamma'(z)|\,dz \le \|\gamma'\|_{L^\infty([0,1])}|\gamma^{-1}(x)-\gamma^{-1}(y)| \lesssim |x-y|,$$

as well as

$$|x-y| \lesssim |\gamma^{-1}(x)-\gamma^{-1}(y)| = \Big|\int_{\gamma^{-1}(x)}^{\gamma^{-1}(y)} 1\,dz\Big| = \Big|\int_{\Gamma_x^y} |\partial_\Gamma \gamma^{-1}|\Big| \lesssim |\Gamma_x^y|.$$

□

Proof of (ii). To see (Γ1), we apply the implicit function theorem. Let $T\in\mathcal{T}_\star$ with endpoints $x_T, y_T \in T$, and let $\gamma_T : (0,1) \to T, \gamma_T(s) := (x_T - y_T)s + y_T$ be an affine parametrization of the interior of T. The implicit definition (6.2.1) rewrites as follows: Find $\widetilde{\gamma}_T : [0,1] \to [0,1]$ such that

$$F(t,\widetilde{\gamma}_T(t)) = 0 \quad \text{for all } t\in\gamma^{-1}(T^\Gamma), \quad \text{where } F(t,s) = (\gamma(t)-\gamma_T(s))\cdot t_\Gamma\circ\gamma(t). \tag{6.2.16}$$

Since Γ and γ are piecewise smooth, there holds that $F:\gamma^{-1}(T^\Gamma)\times(0,1)\to\mathbb{R}$ is smooth. If $\partial_s F(t_0,s_0)\neq 0$ for all $(t_0,s_0)\in\gamma^{-1}(T^\Gamma)\times[0,1]$, the implicit function theorem provides a unique map $\widetilde{\gamma}_T : \gamma^{-1}(T^\Gamma)\to(0,1)$ which is smooth and satisfies (6.2.16). With this, $\gamma_\star(x) := \gamma_T\circ\widetilde{\gamma}_T\circ\gamma^{-1}(x)$ for all $x\in T\setminus\{x_T,y_T\}$ satisfies (Γ1) up to injectiveness (which is shown below).

To prove $\partial_s F(t_0,s_0) = (x_T-y_T)\cdot t_\Gamma\circ\gamma(t_0)\neq 0$, assume

$$0 = \partial_s F(t_0,s_0) = (x_T-y_T)\cdot t_\Gamma\circ\gamma(t_0) = \int_{T^\Gamma} t_\Gamma(z)\cdot t_\Gamma\circ\gamma(t_0)\,dz. \tag{6.2.17}$$

The integrand $r(z) := t_\Gamma(z)\cdot t_\Gamma\circ\gamma(t_0)$ satisfies $r(\gamma(t_0)) = 1$. Due to (6.2.17), there exists at least one $z'\in T^\Gamma$ with $r(z') = 0$. This implies the existence of $z''\in T^\Gamma$ such that

$$\kappa_\Gamma \geq |(\partial_\Gamma t_\Gamma)(z'')| \geq |(\partial_\Gamma r)(z'')| \geq |T^\Gamma|^{-1} \geq C_\Gamma^{-1}|x_T-y_T|^{-1},$$

where we used $T^\Gamma = \Gamma_{x_T}^{y_T}$. This shows

$$\kappa_\Gamma^{-1}C_\Gamma^{-1} \leq |x_T-y_T| \leq \|h_\star\|_{L^\infty(\Gamma_\star)}.$$

This shows that for $h_\star \leq \kappa_\Gamma^{-1}C_\Gamma^{-1}$, $\partial_s F(t_0,s_0)\neq 0$ and hence (Γ1) up to injectiveness.

To prove that $\gamma_\star$ is injective, consider

$$0 = \partial_t F(t,\widetilde{\gamma}_T(t)) = (\partial_t F)(t,\widetilde{\gamma}_T(t)) + (\partial_s F)(t,\widetilde{\gamma}_T(t))\widetilde{\gamma}_T'(t),$$

which implies by use of $\gamma'(t) = |\Gamma| t_\Gamma\circ\gamma(t)$

$$\begin{aligned} |\widetilde{\gamma}_T'(t)| &= \Big|\frac{\partial_t F(t,\widetilde{\gamma}_T(t))}{(x_T-y_T)\cdot t_\Gamma\circ\gamma(t)}\Big| \\ &= \Big|\frac{\gamma'(t)\cdot t_\Gamma\circ\gamma(t) + (\gamma(t)-\gamma_T\circ\widetilde{\gamma}_T(t))\cdot(t_\Gamma\circ\gamma)'(t)}{(x_T-y_T)\cdot t_\Gamma\circ\gamma(t)}\Big| \\ &\geq \frac{|\gamma'(t)| - |(\gamma(t)-\gamma_T\circ\widetilde{\gamma}_T(t))||\partial_\Gamma t_\Gamma||\gamma'(t)|}{h_\star}. \end{aligned}$$

Hence, for $|(\gamma(t)-\gamma_T\circ\widetilde{\gamma}_T(t))| \leq \mathrm{geo}(\mathcal{T}_\star)^2 \leq \kappa_\Gamma^{-1}/2$, there holds with the Lipschitz continuity of γ

$$|\widetilde{\gamma}_T'(t)| \geq |\gamma'(t)|/2 \geq C_\gamma^{-1} > 0,$$

which implies that $\widetilde{\gamma}_T\colon [0,1]\to[0,1]$ is strictly monotone and hence injective. By definition, $\gamma_\star|_{T^\Gamma} := \gamma_T\circ\widetilde{\gamma}_T\circ\gamma^{-1}$ is also injective. □

Proof of (i) $\Longrightarrow$ (Γ3). The property (Γ3) can be seen as follows: Let $y_1, y_2 \in T^\Gamma$ such that $(y_1 - x) \cdot (x_T - y_T) = (y_2 - x) \cdot (x_T - y_T) = 0$ for some $x \in T$. Then, there holds

$$0 = (y_1 - y_2) \cdot (x_T - y_T) = \int_{\Gamma_{y_1}^{y_2}} t_\Gamma(z) \cdot (x_T - y_T)\, dz. \tag{6.2.18}$$

Rolle's theorem provides $z_0 \in T^\Gamma$ with $|t_\Gamma(z_0) \cdot (x_T - y_T)| = |x_T - y_T|$. Hence, the integrand $r(z) := t_\Gamma(z) \cdot (x_T - y_T)$ satisfies $|r(z_0)| = |x_T - y_T|$. Assume $y_1 \neq y_2$, then (6.2.18) shows $r(z_1) = 0$ for at least one $z_1 \in \Gamma_{y_1}^{y_2}$. This implies for some $z_2 \in T^\Gamma$

$$|T^\Gamma|^{-1}|x_T - y_T| \leq |\Gamma_{z_0}^{z_1}|^{-1}|x_T - y_T| \leq |\partial_\Gamma r(z_2)| \leq |x_T - y_T|\kappa_\Gamma.$$

Hence, $y_1 = y_2$ for $|T^\Gamma| \leq \kappa_\Gamma^{-1}/2$ or $h_\star \leq \kappa_\Gamma^{-1} C_\Gamma^{-1}/2$. This implies (Γ3). □

Proof of (iii). To see (Γ2) consider

$$\begin{aligned}
|\gamma_\star(x) - \gamma_\star(y)| &\leq |x - y| + |x - \gamma_\star(x) - (y - \gamma_\star(y))| \\
&\leq |x - y| + \Big| \int_{\Gamma_x^y} t_\Gamma(z) - \partial_\Gamma \gamma_\star(z)\, dz \Big| \\
&\leq |x - y| + \mathrm{geo}(\mathcal{T}_\star)|\Gamma_x^y| \\
&\leq (1 + C_\Gamma \mathrm{geo}(\mathcal{T}_\star))|x - y|,
\end{aligned}$$

as well as

$$\begin{aligned}
|\gamma_\star(x) - \gamma_\star(y)| &\geq |x - y| - |x - \gamma_\star(x) - (y - \gamma_\star(y))| \\
&\leq (1 - C_\Gamma \mathrm{geo}(\mathcal{T}_\star))|x - y|.
\end{aligned}$$

Therefore, (Γ2) holds for $\mathrm{geo}(\mathcal{T}_\star) \leq C_\Gamma^{-1}/2$. □

Proof of (6.2.14)–(6.2.15). To see (6.2.15), apply (6.2.11) to see

$$1 = \partial_\Gamma^s(\mathrm{id}_\Gamma) = \partial_\Gamma^s(\gamma_\star^{-1} \circ \gamma_\star) = \partial_{\Gamma_\star}^s(\gamma_\star^{-1}) \circ \gamma_\star\, \partial_\Gamma^s \gamma_\star.$$

This shows that $\partial_\Gamma^s \gamma_\star \neq 0$ almost everywhere on Γ. Moreover, since $\gamma_\star$ is piecewise smooth, $\partial_\Gamma^s \gamma_\star < 0$ is only possible if $\partial_\Gamma^s \gamma_\star < 0$ in the interior of some element T^Γ for $T \in \mathcal{T}_\star$ with endpoints x_T and y_T. However, this in combination with (6.2.9) and $t_{\Gamma_\star} = (y_T - x_T)(|y_T - x_T|)$ yields the contradiction

$$\begin{aligned}
y_T - x_T = \gamma_\star(y_T) - \gamma_\star(x_T) = \int_{T^\Gamma} \partial_\Gamma \gamma_\star(z)\, dz &\overset{(6.2.9)}{=} \int_{T^\Gamma} t_{\Gamma_\star} \circ \gamma_\star(z) \partial_\Gamma^s \gamma_\star(z)\, dz \\
&= \frac{y_T - x_T}{|y_T - x_T|} \int_{T^\Gamma} \partial_\Gamma^s \gamma_\star(z)\, dz.
\end{aligned}$$

This proves (6.2.15).

To see (6.2.14), assume (Γ2). Then there holds $\gamma_\star^{-1}(\Gamma_{x,\star}^y) = \Gamma_{\gamma_\star^{-1}(x)}^{\gamma_\star^{-1}(y)}$, since the bi-Lipschitz property (Γ2) ensures that endpoints are mapped to endpoints. This, however, implies

$$|\Gamma_{x,\star}^y| = \int_{\Gamma_{x,\star}^y} 1\, dx = \int_{\gamma_\star^{-1}(\Gamma_{x,\star}^y)} 1 |\partial_{\Gamma_\star} \gamma_\star^{-1}(x)|\, dx \simeq |\Gamma_{\gamma_\star^{-1}(x)}^{\gamma_\star^{-1}(y)}| \simeq |x - y|,$$

where we used $C_{\mathrm{Lip}}^{-1} \leq |\partial_{\Gamma_\star} \gamma_\star^{-1}| \leq C_{\mathrm{Lip}}$. □

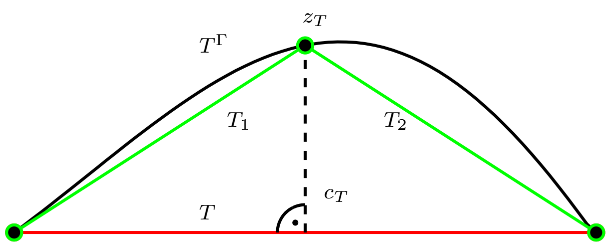

FIGURE 4. The bisection of an element $T \in \mathcal{T}_\star$ into its sons T_1, T_2 according to Algorithm 6.2.10.

PROOF OF (I) $\Longrightarrow$ (Γ4). Let $x \in T^\Gamma$ for some $T \in \mathcal{T}_\star$ and consider the right triangle with nodes $(x, \gamma_\star(x), \mu_\star(x))$ as depicted in Figure 2. Let $\alpha \geq 0$ denote the interior angle at the point x. By definition, the right-angle is at $\mu_\star(x)$. There holds by the Pythagoras theorem

$$|x - \gamma_\star(x)|^2 = |x - \mu_\star(x)|^2 + |\mu_\star(x) - \gamma_\star(x)|^2 = |x - \mu_\star(x)|^2 + |x - \gamma_\star(x)|^2 \sin^2(\alpha)$$

and hence

$$\cos^2(\alpha)|x - \gamma_\star(x)|^2 = |x - \mu_\star(x)|^2. \tag{6.2.19}$$

Obviously, α is also the angle between T and $t_\Gamma(x)$. Hence, one obtains with $x_T, y_T \in \Gamma \cap T$ denoting the endpoints of T

$$|\cos(\alpha)| = |t_\Gamma(x) \cdot \frac{x_T - y_T}{|x_T - y_T|}| = \Big||x_T - y_T|^{-1} \int_{\Gamma_{x_T}^{y_T}} t_\Gamma(x) \cdot t_\Gamma(z)\, dz\Big|.$$

The integrand $r(z) := t_\Gamma(x) \cdot t_\Gamma(z)$ satisfies $r(x) = 1$ and therefore also $|r(z) - r(x)| \leq \|\partial_\Gamma r\|_{L^\infty(\Gamma_{x_T}^{y_T})} |\Gamma_{x_T}^{y_T}| \leq \kappa_\Gamma |\Gamma_{x_T}^{y_T}|$. For $h_\star \leq C_\Gamma^{-1}\kappa_\Gamma^{-1}/2$, this implies $r(z) \geq 1/2$ for all $z \in \Gamma_{x_T}^{y_T}$ and hence

$$|\cos(\alpha)| \geq |x_T - y_T|^{-1} |\Gamma_{x_T}^{y_T}|/2 \geq C_\Gamma^{-1}/2 > 0. \tag{6.2.20}$$

Together with (6.2.19), this implies

$$\frac{1}{2C_\Gamma}|x - \gamma_\star(x)|^2 \leq |x - \mu_\star(x)|^2 \leq |x - \gamma_\star(x)|^2.$$

$\square$

6.2.5. Mesh refinement. Assume an approximate geometry $(\mathcal{T}_\star, \Gamma_\star)$ and define the convex hull of two points $x, y \in \mathbb{R}^2$ by $[x, y] := \big\{\lambda(x - y) + y \,:\, 0 \leq \lambda \leq 1\big\} \subset \mathbb{R}^2$. To bisect a given element $T \in \mathcal{T}_\star$, apply the following algorithm (see also Figure 4 for an illustration)

ALGORITHM 6.2.10. $\mathcal{T}_\star^+ := \mathrm{bisect}(\mathcal{T}_\star, T)$

(i) *Compute* $c_T := (x_T + y_T)/2$, *where* $x_T, y_T \in \mathcal{K}_\star \cap T$ *are the endpoints of* T.
(ii) *Find* $z_T \in T^\Gamma \subseteq \Gamma$ *with* $(z_T - c_T) \cdot (x_T - y_T) = 0$.
(iii) *Set* $\mathcal{T}_\star^+ = (\mathcal{T}_\star \setminus \{T\}) \cup \{T_1, T_2\}$ *with* $T_1 := [x_T, z_T]$ *and* $T_2 := [z_T, y_T]$.

LEMMA 6.2.11. *With* (Γ3), *Algorithm 6.2.10 is well-defined and satisfies*

$$\max\{|T_1|^2, |T_2|^2\} \leq \frac{|T|^2}{4} + \|\mathrm{id}_\Gamma - \mu_\star\|_{L^\infty(T)} \leq \big(\frac{1}{4} + C_\gamma^2 \|\gamma''\|^2_{L^\infty([0,1])}|T|^2\big)|T|^2, \tag{6.2.21}$$

as well as $|T|/2 \leq \min\{|T_1|, |T_2|\}$, *where* $\{T_1, T_2\} = \mathcal{T}_\star^+ \setminus \mathcal{T}_\star$ *denote the sons of* T *and* $\|\gamma''\|_{L^\infty([0,1])}$ *is understood piecewise.*

PROOF. Since y_T in Step (ii) of Algorithm 6.2.10 is unique due to (Γ3), the algorithm is well-defined. The Pythagoras theorem implies $|T_i|^2 = |T|^2/4 + |z_T - \mu_\star(z_T)|^2$. This implies $|T_i| \geq |T|/2$ and the first $\leq$ in (6.2.21). Since $\Gamma_\star$ is a nodal interpolation of Γ, a possible parametrization of $\Gamma_\star$ is given by $I_\star\gamma : [0,1] \to \Gamma_\star$, where $I_\star : C([0,1]) \to \mathcal{S}^1(\mathcal{T}_{[0,1]})$ is the affine nodal interpoland on the partition $\mathcal{T}_{[0,1]}$ which is induced by the nodes $\gamma^{-1}(\mathcal{K}_\star) \subseteq [0,1]$. By definition, $(I_\star\gamma) \circ \gamma^{-1}(x) \in T$ for all $x \in T^\Gamma$. There holds for $y \in T^\Gamma$

$$\begin{aligned}|y - \mu_\star(y)| &= \min_{x \in T} |y - x| \leq |x - (I_\star\gamma) \circ \gamma^{-1}(x)| = |(\gamma - I_\star\gamma) \circ \gamma^{-1}(x)| \\ &\leq |\gamma^{-1}(T)|^2 \|\gamma''\|_{L^\infty([0,1])} \leq C_\gamma^2 |T|^2 \|\gamma''\|_{L^\infty([0,1])},\end{aligned}$$

where the last norm on the right-hand side is understood piecewise. Thus, the above concludes (6.2.21). $\square$

Given a set of marked elements $\mathcal{M}_\star := \{T_1, \ldots, T_n\} \subseteq \mathcal{T}_\star$, we define the refinement $\mathbb{T}(\mathcal{T}_\star, \mathcal{M}_\star)$ by bisection from Section 3.2.8, where we use bisect$(\cdot,\cdot)$ to split the elements. Note that the assumptions of Section 3.2.1–3.2.7 are satisfied.

6.2.6. Auxiliary results. This section provides several results which are used for the a posteriori analysis of this chapter. Some of the techniques used in the proofs below are similar to the a priori analysis (with uniform partitions on smooth geometries) in [**75**, Chapter 8].

LEMMA 6.2.12. *Let $x, y \in \Gamma$ such that $\Gamma_x^y \cap \mathcal{P}_\Gamma = \emptyset$. Then, there holds for an approximate geometry $\mathcal{T}_\star \in \mathbb{T}$*

$$|(x - y) \cdot (\gamma_\star(x) - x)| \leq \kappa_\Gamma C_\Gamma^2 |x - y|^2 \|\mathrm{id}_\Gamma - \gamma_\star\|_{L^\infty(\Gamma)}.$$

PROOF. Define $r(z) := t_\Gamma(z) \cdot (\gamma_\star(x) - x)$. By definition of $\gamma_\star$, there holds $r(x) = 0$. This implies

$$\begin{aligned}|(x - y) \cdot (\gamma_\star(x) - x)| &= \Big| \int_{\Gamma_x^y} r(z)\, dz \Big| = \Big| \int_{\Gamma_x^y} \int_{\Gamma_x^z} \partial_\Gamma r(w)\, dw\, dz \Big| \\ &\leq |\Gamma_x^y|^2 \|\partial_\Gamma r(w)\|_{L^\infty(\Gamma_x^y)} \leq \kappa_\Gamma C_\Gamma^2 |x - y|^2 \|\mathrm{id}_\Gamma - \gamma_\star\|_{L^\infty(\Gamma)}.\end{aligned}$$

$\square$

LEMMA 6.2.13. *There exists a constant $C_{28} > 0$ such that all $x, y \in \Gamma$ satisfy* (i)–(iii).

(i) *If $\Gamma_x^y \cap \mathcal{P}_\Gamma = \emptyset$*

$$C_{28}^{-1} \Big| \log\Big(\frac{|x - y|^2}{|\gamma_\star(x) - \gamma_\star(y)|^2} \Big) \Big| \leq \|t_\Gamma - \partial_\Gamma \gamma_\star\|_{L^\infty(\Gamma)}^2 + \|\mathrm{id}_\Gamma - \gamma_\star\|_{L^\infty(\Gamma)}.$$

(ii) *If $\Gamma_x^y \cap \mathcal{P}_\Gamma = \{z_0\}$*

$$\begin{aligned}C_{28}^{-1} \Big| \log\Big(\frac{|x - y|^2}{|\gamma_\star(x) - \gamma_\star(y)|^2} \Big) \Big| &\leq \|t_\Gamma - \partial_\Gamma \gamma_\star\|_{L^\infty(\Gamma)}^2 \\ &\quad + \|\mathrm{id}_\Gamma - \gamma_\star\|_{L^\infty(\Gamma)} \big(1 + \frac{|z_0 - x| + |z_0 - y|}{|x - y|^2} \big)\end{aligned}$$

as well as

$$C_{28}^{-1} \Big| \log\Big(\frac{|x - y|^2}{|\gamma_\star(x) - \gamma_\star(y)|^2} \Big) \Big| \leq \|t_\Gamma - \partial_\Gamma \gamma_\star\|_{L^\infty(\Gamma)}.$$

(iii) *If* $x \neq y$

$$C_{28}^{-1}\Big|\log\Big(\frac{|x-y|^2}{|\gamma_\star(x)-\gamma_\star(y)|^2}\Big)\Big| \leq \|t_\Gamma - \partial_\Gamma\gamma_\star\|_{L^\infty(\Gamma)}^2 + \|\mathrm{id}_\Gamma - \gamma_\star\|_{L^\infty(\Gamma)}\big(1+\frac{1}{|x-y|^2}\big).$$

Proof. There holds for all $a \in \mathbb{R}$

$$1-\frac{1}{a} \leq \log(a) \leq a-1.$$

This implies

$$\frac{|x-y|^2 - |\gamma_\star(x)-\gamma_\star(y)|^2}{|x-y|^2} \leq \log\Big(\frac{|x-y|^2}{|\gamma_\star(x)-\gamma_\star(y)|^2}\Big) \leq \frac{|x-y|^2 - |\gamma_\star(x)-\gamma_\star(y)|^2}{|\gamma_\star(x)-\gamma_\star(y)|^2}$$

and hence

$$\begin{aligned}\Big|\log\Big(\frac{|x-y|^2}{|\gamma_\star(x)-\gamma_\star(y)|^2}\Big)\Big| &\leq C_{\mathrm{Lip}}^2\frac{\big||\gamma_\star(x)-\gamma_\star(y)|^2-|x-y|^2\big|}{|x-y|^2}\\ &= C_{\mathrm{Lip}}^2\frac{|x-\gamma_\star(x)-(y-\gamma_\star(y))|^2}{|x-y|^2}\\ &\quad + 2C_{\mathrm{Lip}}^2\frac{|(x-\gamma_\star(x)-(y-\gamma_\star(y)))\cdot(x-y)|}{|x-y|^2}.\end{aligned} \tag{6.2.22}$$

The first term on the right-hand side is estimated by

$$\begin{aligned}|x-\gamma_\star(x)-(y-\gamma_\star(y))|^2 &= |\int_{\Gamma_x^y}\partial_\Gamma(\mathrm{id}_\Gamma-\gamma_\star)(s)\,ds|^2 \leq \|t_\Gamma-\partial_\Gamma\gamma_\star\|_{L^\infty(\Gamma)}^2|\Gamma_x^y|^2\\ &\leq C_\Gamma^2\|t_\Gamma-\partial_\Gamma\gamma_\star\|_{L^\infty(\Gamma)}^2|x-y|^2.\end{aligned} \tag{6.2.23}$$

The second term on the right-hand side of (6.2.22) is treated separately for each case.

Case (i): There holds with Lemma 6.2.12

$$|(x-\gamma_\star(x)-(y-\gamma_\star(y)))\cdot(x-y)| \leq 2\kappa_\Gamma C_\Gamma^2|x-y|^2\|\mathrm{id}_\Gamma-\gamma_\star\|_{L^\infty(\Gamma)}.$$

Case (iii): There holds

$$|(x-\gamma_\star(x)-(y-\gamma_\star(y)))\cdot(x-y)| \leq 2\|\mathrm{id}_\Gamma-\gamma_\star\|_{L^\infty(\Gamma)}|x-y|.$$

Case (ii): Lemma 6.2.12 shows

$$\begin{aligned}&|(x-\gamma_\star(x)-(y-\gamma_\star(y)))\cdot(x-y)|\\ &\quad\leq |(x-\gamma_\star(x)-(y-\gamma_\star(y)))\cdot(x-z_0))| + |(x-\gamma_\star(x)-(y-\gamma_\star(y)))\cdot(z_0-y)|\\ &\quad\leq \|\mathrm{id}_\Gamma-\gamma_\star\|_{L^\infty(\Gamma)}\big(\kappa_\Gamma C_\Gamma^2|x-z_0|^2+|x-z_0|+\kappa_\Gamma C_\Gamma^2|y-z_0|^2+|y-z_0|\big)\\ &\quad\leq \|\mathrm{id}_\Gamma-\gamma_\star\|_{L^\infty(\Gamma)}\big(2\kappa_\Gamma C_\Gamma^6|x-y|^2+|x-z_0|+|y-z_0|\big),\end{aligned}$$

where we used $|x-z_0| \leq C_\Gamma|\Gamma_x^{z_0}| \leq C_\Gamma|\Gamma_x^y| \leq C_\Gamma^2|x-y|$. To see the second estimate in (ii), proceed as in (6.2.23) to obtain

$$\begin{aligned}|(x-\gamma_\star(x)-(y-\gamma_\star(y)))\cdot(x-y)| &\leq |x-y||x-\gamma_\star(x)-(y-\gamma_\star(y))|\\ &\lesssim |x-y|^2\|t_\Gamma-\partial_\Gamma\gamma_\star\|_{L^\infty(\Gamma)}.\end{aligned}$$

This concludes the proof. □

Lemma 6.2.14. *Let $\nu > 0$ and let the approximate geometry $\mathcal{T}_\star \in \mathbb{T}$ satisfy* (Γ1)–(Γ2). *Then, there holds $\partial_\Gamma^s \gamma_\star = |\partial_\Gamma \gamma_\star|$ and*

$$C_\nu^{-1} \| 1 - |\partial_\Gamma \gamma_\star| \|_{L^\infty(\Gamma)} \le \| 1 - |\partial_\Gamma \gamma_\star|^\nu \|_{L^\infty(\Gamma)} \le C_\nu \| 1 - |\partial_\Gamma \gamma_\star| \|_{L^\infty(\Gamma)},$$

as well as for all $T \in \mathcal{T}_\star$

$$\| 1 - |\partial_\Gamma \gamma_\star| \|_{L^\infty(T^\Gamma)} \le \| 1 - |\partial_\Gamma \gamma_\star|^2 \|_{L^\infty(T^\Gamma)} \le (1 + 2\kappa_\Gamma) \mathrm{geo}_T(\mathcal{T}_\star)^2.$$

The constant $C_\nu > 0$ depends only on C_{Lip} and ν.

Proof. The identity (6.2.9) and (6.2.15) show

$$|\partial_\Gamma \gamma_\star| = |t_{\Gamma_\star} \circ \gamma_\star| \partial_\Gamma^s \gamma_\star = \partial_\Gamma^s \gamma_\star.$$

Taylor expansion shows that for all $0 < \delta \le a \le \delta^{-1} < \infty$ exists $z_a > 0$ with $|1 - z_a| \le |1 - a|$ such that $a^\nu - 1 = \nu z_a^{\nu - 1}(a - 1)$. Since $a^\nu - 1$ and $a - 1$ have the same sign for all $\nu > 0$, this implies

$$C_\delta^{-1} |a^\nu - 1| \le |a - 1| \le C_\delta |a^\nu - 1|, \tag{6.2.24}$$

where $C_\delta > 0$ depends only on δ and ν. Due to (Γ2), there holds

$$C_{\mathrm{Lip}}^{-1} \le |\partial_\Gamma \gamma_\star| \le C_{\mathrm{Lip}} \quad \text{almost everywhere on } \Gamma.$$

This and (6.2.24) with $\delta = C_{\mathrm{Lip}}^{-1}$ and $a = |\partial_\Gamma \gamma_\star|$ show

$$\| 1 - |\partial_\Gamma \gamma_\star| \|_{L^\infty(\Gamma)} \simeq \| 1 - |\partial_\Gamma \gamma_\star|^\nu \|_{L^\infty(\Gamma)}.$$

Moreover, there holds for all $a \ge 0$ that $|1 - a| \le |1 - a^2|$. It remains to estimate $1 - |\partial_\Gamma \gamma_\star|^2$. To that end, calculate

$$1 - |\partial_\Gamma \gamma_\star|^2 = |\partial_\Gamma \gamma_\star - t_\Gamma|^2 - 2(\partial_\Gamma \gamma_\star - t_\Gamma) \cdot t_\Gamma.$$

By definition of $\gamma_\star$, there holds $(\gamma_\star - \mathrm{id}_\Gamma) \cdot t_\Gamma = 0$. This implies almost everywhere

$$0 = \partial_\Gamma \big((\gamma_\star - \mathrm{id}_\Gamma) \cdot t_\Gamma \big) = (\partial_\Gamma \gamma_\star - t_\Gamma) \cdot t_\Gamma + (\gamma_\star - \mathrm{id}_\Gamma) \cdot \partial_\Gamma t_\Gamma$$

and hence

$$|(\partial_\Gamma \gamma_\star - t_\Gamma) \cdot t_\Gamma| \le \|\partial_\Gamma t_\Gamma\|_{L^\infty(\Gamma)} \|\mathrm{id}_\Gamma - \gamma_\star\|_{L^\infty(T^\Gamma)} \le \kappa_\Gamma \|\mathrm{id}_\Gamma - \gamma_\star\|_{L^\infty(T^\Gamma)}.$$

The combination of the last estimates concludes the proof. □

Lemma 6.2.15. *Any $g \in L^2(\Gamma)$ with $\mathrm{supp}(g) \subseteq \Gamma_x^y$ for some $x, y \in \Gamma$ satisfies*

$$\| |g| \|_{H^{-1/2}(\Gamma)} \le C_{\mathrm{abs}} |\Gamma_x^y|^{1/2} (1 + |\log(|\Gamma_x^y|)|)^{1/2} \|g\|_{L^2(\Gamma)}.$$

The constant $C_{\mathrm{abs}} > 0$ depends only on Γ and C_γ.

Proof. Without loss of generality, assume $g \ge 0$. Construct a uniform partition $\mathcal{U}$ of Γ, with $h(\mathcal{U}) := |U| \simeq |\Gamma_x^y|^{1/2}$ for all $U \in \mathcal{U}$ and $\mathrm{supp}(g) \subset U_0$ for some $U_0 \in \mathcal{U}$. Let $\Pi^0 : L^2(\Gamma) \to \mathcal{P}^0(\mathcal{U})$ denote the corresponding L^2-orthogonal projection. There holds

$$\begin{aligned} \|g\|_{H^{-1/2}(\Gamma)} &\le \|\Pi^0 g\|_{H^{-1/2}(\Gamma)} + \|(1 - \Pi^0) g\|_{H^{-1/2}(\Gamma)} \\ &\lesssim \|\Pi^0 g\|_{H^{-1/2}(\Gamma)} + h(\mathcal{U})^{1/2} \|g\|_{L^2(\Gamma)}. \end{aligned} \tag{6.2.25}$$

By construction, there holds $\Pi^0 g = \alpha \chi_{U_0}$ for some $\alpha \ge 0$, where χ_{U_0} denotes the characteristic function with respect to U_0. Since $\langle \mathcal{V} \cdot\,, \cdot \rangle^{1/2}$ is an equivalent norm on $H^{-1/2}(\Gamma)$, there holds

$$\|\Pi^0 g\|_{H^{-1/2}(\Gamma)} = \alpha \|\chi_{U_0}\|_{H^{-1/2}(\Gamma)} \simeq \alpha \langle \mathcal{V} \chi_{U_0}\,, \chi_{U_0} \rangle_\Gamma^{1/2}.$$

Without loss of generality, assume $\{0,1\} \notin \gamma^{-1}(U_0)$. With the parametrization γ and $h := |\gamma^{-1}(U_0)|$, there holds

$$\begin{aligned}
2\pi\langle \mathcal{V}\chi_{U_0}\,,\,\chi_{U_0}\rangle_\Gamma &= \Big|\int_{U_0}\int_{U_0} \log|x-y|\,dx\,dy\Big| \\
&\leq \int_{\gamma^{-1}(U_0)}\int_{\gamma^{-1}(U_0)} \big|\log|\gamma(s)-\gamma(t)|\big||\gamma'(s)||\gamma'(t)|\,dt\,ds \\
&\leq C_\gamma^2 \int_{\gamma^{-1}(U_0)}\int_{\gamma^{-1}(U_0)} |\log(C_\gamma)| + \big|\log|s-t|\big|\,dt\,ds, \\
&= C_\gamma^2\Big(h^2|\log(C_\gamma)| + \int_0^h\int_0^h \big|\log|s-t|\big|\,dt\,ds\Big).
\end{aligned}$$

The integral term on the right-hand side is further estimated by

$$\begin{aligned}
\int_0^h\int_0^h \big|\log|s-t|\big|\,dt\,ds &= h^2\int_0^1\int_0^1 |\log(h)| + \big|\log|s-t|\big|\,dt\,ds \\
&\lesssim h^2(1+|\log(h)|),
\end{aligned}$$

since the remaining integral is finite. The Lipschitz continuity of γ shows $h \simeq h(\mathcal{U})$. Altogether, this proves

$$\|\Pi^0 g\|_{H^{-1/2}(\Gamma)} \simeq \alpha\langle \mathcal{V}\chi_{U_0}\,,\,\chi_{U_0}\rangle_\Gamma^{1/2} \lesssim \alpha h(\mathcal{U})(1+|\log(h(\mathcal{U}))|)^{1/2}.$$

The fact $\|\Pi^0 g\|_{L^2(\Gamma)} \simeq \alpha h(\mathcal{U})^{1/2}$ and $h(\mathcal{U}) \simeq |\Gamma_x^y|$ together with (6.2.25) conclude the proof. □

The following lemma is well-known and repeated here only for completeness.

Lemma 6.2.16. *Let $O_1,\ldots,O_N$ denote an open cover of some compact set $C \subseteq \mathbb{R}^d$, $d \in \mathbb{N}$. Then, there exists $\varepsilon > 0$ such that for all $x \in C$, there exists $i \in \{1,\ldots,N\}$ with $B_\varepsilon(x) \subseteq O_i$.*

Proof. Assume that the statement is wrong. Then, there exists a sequence $x_n \in C$ with $B_{1/n}(x_n) \not\subseteq O_i$ for all $i = 1,\ldots,N$ and all $n \in \mathbb{N}$. The compactness of C provides a subsequence $x_{n_k} \to x \in C$. By definition, there exists $i \in \{1,\ldots,N\}$ with $x \in O_i$. Hence, there also exists $k \in \mathbb{N}$ with $B_{1/n_k}(x_{n_k}) \subseteq O_i$, which contradicts the assumption. □

Lemma 6.2.17. *Given an approximate geometry $\mathcal{T}_\star \in \mathbb{T}$ with (Γ1)–(Γ3), there exists a continuous extension $\widehat{\gamma}_\star : \mathbb{R}^2 \to \mathbb{R}^2$ of $\gamma_\star$ such that*

$$\widehat{\gamma}_\star|_\Gamma = \gamma_\star, \tag{6.2.26}$$

$$\|\widehat{\gamma}_\star - \mathrm{id}_{\mathbb{R}^2}\|_{L^\infty(\mathbb{R}^2)} \leq \|\gamma_\star - \mathrm{id}_\Gamma\|_{L^\infty(\Gamma)}, \tag{6.2.27}$$

$$\|\nabla\widehat{\gamma}_\star - I\|_{L^\infty(\mathbb{R}^2)} \leq C_{\mathrm{ext}}\|\partial_\Gamma\gamma_\star - t_\Gamma\|_{L^\infty(\Gamma)}, \tag{6.2.28}$$

where $I \in \mathbb{R}^{2\times 2}$ denotes the identity matrix and $C_{\mathrm{ext}} > 0$ depends only on Γ. For $\mathrm{geo}(\mathcal{T}_\star) \leq C_{\mathrm{ext}}^{-1}/2$, $\widehat{\gamma}_\star$ is bijective and bi-Lipschitz such that

$$|x-y|/2 \leq |\widehat{\gamma}_\star(x) - \widehat{\gamma}_\star(y)| \leq (1 + C_{\mathrm{ext}}/2)|x-y|. \tag{6.2.29}$$

Particularly, there holds $\gamma_\star(\Omega) = \Omega_\star$ (with $\partial\Omega_\star = \Gamma_\star$ from Section 6.2.0.1) and

$$\|(\nabla\widehat{\gamma}_\star)^{-1}\|_{L^\infty(\mathbb{R}^2)} \leq 2. \tag{6.2.30}$$

Definition 6.2.18. *After the following proof and throughout this chapter, we will not distinguish between $\gamma_\star$ and its extension $\gamma_\star := \widehat{\gamma}_\star$. The meaning will be clear from the context.*

Proof. Without loss of generality, let the parametrization γ satisfy $\gamma'|\Gamma|^{-1} = t_\Gamma$. Approximate γ by some smooth $\gamma_\varepsilon\colon [0,1] \to \mathbb{R}^2$, $\partial_s^k \gamma_\varepsilon(0) = \partial_s^k \gamma_\varepsilon(1)$ for all $k \in \mathbb{N}_0$ such that $\|\gamma - \gamma_\varepsilon\|_{W^{1,\infty}([0,1])} \leq \varepsilon$. Let $M \in \mathbb{R}^{2\times 2}$ denote the orthogonal matrix which satisfies $Mt_\Gamma = n_\Gamma$. Then, define $n_\varepsilon := M(\gamma_\varepsilon' \circ \gamma^{-1})|\Gamma|^{-1} \in W^{1,\infty}(\Gamma, \mathbb{R}^2)$. With $n_\Gamma = M(\gamma' \circ \gamma^{-1})|\Gamma|^{-1}$, there holds

$$\|n_\Gamma - n_\varepsilon\|_{L^\infty(\Gamma)} \leq |\Gamma|^{-1} \|(\gamma_\varepsilon' - \gamma') \circ \gamma\|_{L^\infty(\Gamma)} \leq \varepsilon |\Gamma|^{-1}.$$

Define the function $\zeta\colon [0,1]\times\mathbb{R} \to \mathbb{R}^2$ by $\zeta(s,t) := \gamma(s) + tn_\varepsilon \circ \gamma(s)$. There holds with (6.2.11a)

$$\nabla\zeta(s,t) = \big(\partial_s\gamma(s) + t(\partial_\Gamma n_\varepsilon)\circ\gamma(s)\partial_s^s\gamma(s)\,,\, n_\varepsilon\circ\gamma(s)\big) \in \mathbb{R}^{2\times 2}.$$

By definition, there holds

$$\begin{aligned}
|\partial_s\gamma(s)\cdot M^{-1}(n_\varepsilon\circ\gamma(s))| &\geq |\partial_s\gamma(s)\cdot M^{-1}(n_\Gamma\circ\gamma(s))| - |\partial_s\gamma(s)|\|n_\Gamma - n_\varepsilon\|_{L^\infty(\Gamma)} \\
&\geq |\partial_s\gamma(s)\cdot\partial_s\gamma(s)||\Gamma|^{-1} - |\partial_s\gamma(s)|\varepsilon|\Gamma|^{-1} \\
&= |\partial_s\gamma(s)|^2|\Gamma|^{-1} - \varepsilon|\Gamma|^{-1}|\partial_s\gamma(s)|.
\end{aligned}$$

as well as

$$\begin{aligned}
|t(\partial_\Gamma n_\varepsilon)\circ\gamma(s)\partial_s^s\gamma(s)\cdot M^{-1}(n_\varepsilon\circ\gamma(s))| &\leq |t|\|\partial_\Gamma n_\varepsilon\|_{L^\infty(\Gamma)}|\partial_s^s\gamma(s)|\|n_\varepsilon\|_{L^\infty(\Gamma)} \\
&\leq |t|\|\partial_\Gamma n_\varepsilon\|_{L^\infty(\Gamma)}|\partial_s^s\gamma(s)|(1+\varepsilon|\Gamma|^{-1}).
\end{aligned}$$

Since M realizes a rotation by $\pi/2$, this shows

$$\begin{aligned}
|\det(\nabla\zeta(s,t))| &= |\partial_s\zeta(s,t)\cdot M^{-1}\partial_t\zeta(s,t)| \\
&\geq |\partial_s\gamma(s)|^2|\Gamma|^{-1} - \varepsilon|\Gamma|^{-1}|\partial_s\gamma(s)| - |t|\|\partial_\Gamma n_\varepsilon\|_{L^\infty(\Gamma)}|\partial_s^s\gamma(s)|(1+\varepsilon|\Gamma|^{-1}).
\end{aligned}$$

Since $|\partial_s\gamma(s)| = |\partial_s^s\gamma(s)| = |\Gamma|$, sufficiently small $\varepsilon, t_0 > 0$ with $|t| \leq t_0$ imply

$$|\det(\nabla\zeta(s,t))| \geq |\Gamma|/2.$$

Analogously, we bound for the Frobenius matrix norm $\|\cdot\|_F$ by

$$\|\nabla\zeta(s,t)\|_F^2 = (|\Gamma| + |t|\|\partial_\Gamma n_\varepsilon\|_{L^\infty(\Gamma)}|\Gamma|)^2 + (1+\varepsilon|\Gamma|^{-1})^2$$

and hence

$$\begin{aligned}
\|(\nabla\zeta(s,t))^{-1}\|_F &= \frac{1}{|\det(\nabla\zeta(s,t))|}\|(\nabla\zeta(s,t))\|_F \\
&\leq 2|\Gamma|^{-1}\sqrt{(|\Gamma| + |t|\|\partial_\Gamma n_\varepsilon\|_{L^\infty(\Gamma)}|\Gamma|)^2 + (1+\varepsilon|\Gamma|^{-1})^2} := C_\zeta,
\end{aligned} \tag{6.2.31}$$

where $C_\zeta > 0$ depends only on ε, t_0 and Γ. The inverse mapping theorem proves that ζ is a local diffeomorphism. The compactness of $[0,1]\times[-t_0,t_0]$ implies the existence of an open cover $O_1,\dots,O_N$ such that $\zeta|_{O_i}$ is a diffeomorphism onto its image. Let now $(s_i,t_i) \in [0,1]\times[-t_0,t_0]$, $i = 1,2$ with $\zeta(s_1,t_1) = \zeta(s_2,t_2)$. Then, there holds

$$|\gamma(s_1) - \gamma(s_2)| \leq 2\max\{t_1,t_2\}\|\partial_\Gamma n_\varepsilon\|_{L^\infty(\Gamma)}.$$

Lemma 6.2.16 shows that for $t_1, t_2 \leq t_0'$ and $t_0' > 0$ sufficiently small, there holds $(s_i,t_i) \in O_j$ for some $j \in \{1,\dots,N\}$ and $i = 1,2$. Since $\zeta|_{O_i}$ is a diffeomorphism, this shows $(s_1,t_1) = (s_2,t_2)$. Hence, $\zeta|_{[0,1]\times(-t_0',t_0')}$ is injective, and by the inverse mapping theorem also a diffeomorphism. Particularly, due to (6.2.31), ζ is a bi-Lipschitz, bijective function onto its image $O := \zeta([0,1]\times(-t_0',t_0')) \subseteq \mathbb{R}^2$, which is $[0,1]$-periodic with respect to its first argument. We prove that ζ is also bi-Lipschitz with respect to the metric $d(\cdot,\cdot)$ which identifies 0 and

1 of $[0,1]$ (as defined in Section 6.2.0.1). To that end, consider $s_1, s_2 \in [0,1]$, such that $|s_1 - 0| + |s_2 - 1| \le |s_1 - s_2|$. There holds

$$\begin{aligned}|\zeta(s_1,t_1) - \zeta(s_2,t_2)| &\le |\zeta(s_1,t_1) - \zeta(s_2,t_2)| \\ &\le |\zeta(s_1,t_1) - \zeta(0,t_1)| + |\zeta(1,t_1) - \zeta(s_2,t_2)| \\ &\lesssim |s_1 - 0| + |1 - s_2| + |t_1 - t_2| = d(s_1,s_2) + |t_1 - t_2|\end{aligned}$$

as well as with bi-Lipschitz continuity on $[0,1] \times [0,1]$ (without identification)

$$|\zeta(s_1,t_1) - \zeta(s_2,t_2)| \gtrsim |s_1 - s_2| + |t_1 - t_2| \ge d(s_1,s_2) + |t_1 - t_2|.$$

Since the set $[0,1] \times (-t_0', t_0')$ is open with respect to the product topology generated by $d(\cdot,\cdot)$ and the Euclidean topology, the set O is open by the bi-Lipschitz continuity above. Particularly, O is a neighborhood of Γ. With π_1 denoting the projection onto the first argument, the function

$$P := \gamma \circ \pi_1 \circ \zeta^{-1} \colon O \to \Gamma$$

is also Lipschitz continuous (where the periodicity of ζ is used) and satisfies $P(x) = x$ for all $x \in \Gamma$. Choose a smooth cut-off function $\chi : \mathbb{R}^2 \to [0,1]$ with $\chi|_\Gamma = 1$ and $\operatorname{supp}(\chi) \subseteq O$. Then, define

$$\widehat{\gamma}_\star(x) := x + \chi(x)(\gamma_\star \circ P(x) - P(x)).$$

There holds $\widehat{\gamma}_\star|_\Gamma = \gamma_\star$ as well as

$$|\widehat{\gamma}_\star(x) - x| \le \|\mathrm{id}_\gamma - \gamma_\star\|_{L^\infty(\Gamma)}.$$

This implies (6.2.27). Moreover, with the chain-rule (6.2.11b), we obtain for $z \in \mathbb{R}^2$

$$\partial_z(\widehat{\gamma}_\star - I) = (\partial_\Gamma \gamma_\star - t_\Gamma) \circ P(x)\, \partial_z^s P(x).$$

The identity (6.2.10) shows $|\partial_z^s P(x)| = |\partial_z P(x)|$ and hence proves (6.2.28) with $C_{\mathrm{ext}} := \|\nabla P\|_{L^\infty(O)}$. For $\operatorname{geo}(\mathcal{T}_\star) < C_{\mathrm{ext}}^{-1}/2$ and all $x, z \in \mathbb{R}^2$, there holds

$$x \cdot (\nabla \widehat{\gamma}_\star)(z) x \ge |x|^2 - |I - \nabla \widehat{\gamma}_\star(z)||x|^2 \ge |x|^2/2. \tag{6.2.32}$$

This implies (6.2.30). Assume that $\widehat{\gamma}_\star(x) = \widehat{\gamma}_\star(y)$ for some $x, y \in \mathbb{R}^2$. There holds with the convex hull $[x,y] := \big\{\lambda x + (1-\lambda) y \,:\, 0 \le \lambda \le 1\big\}$ and (6.2.32)

$$\begin{aligned}0 = \big|(x-y) \cdot (\widehat{\gamma}_\star(x) - \widehat{\gamma}_\star(y))\big| &= \Big|\int_{[x,y]} (x-y) \cdot (\nabla \widehat{\gamma}_\star(z)) \frac{x-y}{|x-y|}\, dz\Big| \\ &\ge |x-y| \int_{[x,y]} 1/2\, dz.\end{aligned}$$

This implies $x = y$. Hence $\widehat{\gamma}_\star$ is injective. The inverse mapping theorem shows that $\gamma_\star$ is a global diffeomorphism. The estimate (6.2.30) implies that $\widehat{\gamma}_\star$ is even bi-Lipschitz. The estimate (6.2.29) follows from (6.2.28) and(6.2.30). It remains to show that $\widehat{\gamma}_\star(\Omega) = \Omega_\star$. Assume that there exist $x, y \in \Omega$ such that $\widehat{\gamma}_\star(x) \in \Omega_\star$ and $\widehat{\gamma}_\star(y) \in \mathbb{R}^2 \setminus \overline{\Omega}_\star$. Then, there exists a compact path $G \subseteq \Omega$ which connects x and y. Since $\widehat{\gamma}_\star(G)$ is also a continuous and compact path, there exists $z \in G$ such that $\widehat{\gamma}_\star(z) \in \Gamma_\star$ and hence $z \in \Gamma$ by bijectivity of $\widehat{\gamma}_\star$ and $\gamma_\star$. This, however, contradicts $G \subseteq \Omega$. We showed that $\widehat{\gamma}_\star(\Omega) \subseteq \Omega_\star$ or $\widehat{\gamma}_\star(\Omega) \subseteq \mathbb{R}^2 \setminus \overline{\Omega}_\star$. The same arguments prove $\widehat{\gamma}_\star(\mathbb{R}^2 \setminus \overline{\Omega}) \subseteq \Omega_\star$ or $\widehat{\gamma}_\star(\mathbb{R}^2 \setminus \overline{\Omega}) \subseteq \mathbb{R}^2 \setminus \overline{\Omega}_\star$. However, the bi-Lipschitz continuity prohibits $\widehat{\gamma}_\star(\mathbb{R}^2 \setminus \overline{\Omega}) \subseteq \Omega_\star$, since $\mathbb{R}^2 \setminus \overline{\Omega}$ is unbounded. This shows $\widehat{\gamma}_\star(\Omega) = \Omega_\star$ and hence concludes the proof. □

By use of the chain-rule, there holds under the assumptions of Lemma 6.2.17 that

$$I = \nabla(\gamma_\star^{-1} \circ \gamma_\star) = (\nabla\gamma_\star^{-1}) \circ \gamma_\star \nabla\gamma_\star$$

and since $\nabla\gamma_\star$ is a regular matrix by (6.2.30), this shows

$$(\nabla\gamma_\star^{-1}) \circ \gamma_\star = (\nabla\gamma_\star)^{-1}. \tag{6.2.33}$$

Lemma 6.2.19. *Given an approximate geometry $\mathcal{T}_\star \in \mathbb{T}$ which satisfies* (Γ2)*, there holds for all $\psi \in H^{-1/2}(\Gamma)$ and all $v \in H^{1/2}(\Gamma)$*

$$C_{\rm Lip}^{-1/2}\|\psi\|_{H^{-1/2}(\Gamma)} \le \|\psi \circ \gamma_\star^{-1}|\partial_{\Gamma_\star}\gamma_\star^{-1}|\|_{H^{-1/2}(\Gamma_\star)} \le C_{\rm Lip}^{1/2}\|\psi\|_{H^{-1/2}(\Gamma)} \tag{6.2.34}$$

as well as

$$C_{\rm Lip}^{-1/2}\|v\|_{H^{1/2}(\Gamma)} \le \|v \circ \gamma_\star^{-1}\|_{H^{1/2}(\Gamma_\star)} \le C_{\rm Lip}^{1/2}\|v\|_{H^{1/2}(\Gamma)}. \tag{6.2.35}$$

Proof. There holds for $v \in H^1(\Gamma)$ with (6.2.12)

$$\begin{aligned}\|\partial_{\Gamma_\star}(v \circ \gamma_\star^{-1})\|_{L^2(\Gamma_\star)} &= \|(\partial_\Gamma v) \circ \gamma_\star^{-1}|\partial_{\Gamma_\star}\gamma_\star^{-1}|\|_{L^2(\Gamma_\star)} \\ &= \Big(\int_{\Gamma_\star}((\partial_\Gamma v) \circ \gamma_\star^{-1})^2|\partial_{\Gamma_\star}\gamma_\star^{-1}|^2\,dx\Big)^{1/2} \\ &= \Big(\int_\Gamma (\partial_\Gamma v)^2|(\partial_{\Gamma_\star}\gamma_\star^{-1}) \circ \gamma_\star|^2|\partial_\Gamma\gamma_\star|\,dx\Big)^{1/2} = \|\partial_\Gamma v|\partial_\Gamma\gamma_\star|^{-1/2}\|_{L^2(\Gamma)}\end{aligned}$$

as well as

$$\|v \circ \gamma_\star^{-1}\|_{L^2(\Gamma_\star)} = \|v|\partial_\Gamma\gamma_\star|^{1/2}\|_{L^2(\Gamma)}.$$

Due to (Γ2), there holds $C_{\rm Lip}^{-1} \le |\partial_\Gamma\gamma_\star| \le C_{\rm Lip}$ and hence

$$\begin{aligned}C_{\rm Lip}^{-1/2}\|\partial_\Gamma v\|_{L^2(\Gamma)} \le \|\partial_{\Gamma_\star}(v \circ \gamma_\star^{-1})\|_{L^2(\Gamma_\star)} \le C_{\rm Lip}^{1/2}\|\partial_\Gamma v\|_{L^2(\Gamma)}, \\ C_{\rm Lip}^{-1/2}\|v\|_{L^2(\Gamma)} \le \|v \circ \gamma_\star^{-1}\|_{L^2(\Gamma_\star)} \le C_{\rm Lip}^{1/2}\|v\|_{L^2(\Gamma)}\end{aligned}$$

Interpolation theory concludes (6.2.35).

On the other hand, there holds

$$\begin{aligned}\|\psi \circ \gamma_\star^{-1}|\partial_\Gamma\gamma_\star|^{-1}\|_{H^{-1/2}(\Gamma_\star)} &= \sup_{v \in H^{1/2}(\Gamma_\star)} \frac{\langle \psi \circ \gamma_\star^{-1}|\partial_\Gamma\gamma_\star|^{-1}\,,\, v\rangle_{\Gamma_\star}}{\|v\|_{H^{1/2}(\Gamma_\star)}} \\ &= \sup_{v \in H^{1/2}(\Gamma_\star)} \frac{\langle \psi\,,\, v \circ \gamma_\star\rangle_\Gamma}{\|v\|_{H^{1/2}(\Gamma_\star)}} \\ &= \sup_{v \in H^{1/2}(\Gamma)} \frac{\|v\|_{H^{1/2}(\Gamma)}}{\|v \circ \gamma_\star^{-1}\|_{H^{1/2}(\Gamma_\star)}}\frac{\langle \psi\,,\, v\rangle_\Gamma}{\|v\|_{H^{1/2}(\Gamma)}} \\ &\overset{(6.2.35)}{\simeq} \sup_{v \in H^{1/2}(\Gamma)} \frac{\langle \psi\,,\, v\rangle_\Gamma}{\|v\|_{H^{1/2}(\Gamma)}} = \|\psi\|_{H^{-1/2}(\Gamma)}.\end{aligned}$$

This concludes the proof. □

Lemma 6.2.20. *Given an approximate geometry $\mathcal{T}_\star \in \mathbb{T}$ with* (Γ1)–(Γ3) *and* $\mathrm{geo}(\mathcal{T}_\star) \le C_{\rm ext}^{-1}/2$*, there exists a lifting operator $\mathcal{L}_\star\colon H^{1/2}(\Gamma_\star) \to H^1(\mathbb{R}^2)$ with*

$$(\mathcal{L}_\star v)|_{\Gamma_\star} = v \quad \textit{and} \quad \|\mathcal{L}_\star v\|_{H^1(\mathbb{R}^2)} \le C_{\rm lift}\|v\|_{H^{1/2}(\Gamma_\star)} \quad \textit{for all } v \in H^{1/2}(\Gamma_\star).$$

The constant $C_{\rm lift} > 0$ depends only on Γ and $C_{\rm ext}$, $C_{\rm Lip}$.

PROOF. Let $\mathcal{L}\colon H^{1/2}(\Gamma) \to H^1(\mathbb{R}^2)$ denote a standard lifting operator. Define

$$\mathcal{L}_\star v := (\mathcal{L}(v \circ \gamma_\star)) \circ \gamma_\star^{-1}.$$

Then, there holds $(\mathcal{L}_\star v)|_{\Gamma_\star} = (\mathcal{L}v \circ \gamma_\star) \circ \gamma_\star^{-1}|_{\Gamma_\star} = v|_{\Gamma_\star}$. Moreover, we obtain

$$\begin{aligned}\|\mathcal{L}_\star v\|_{H^1(\mathbb{R}^2)}^2 &= \|\mathcal{L}_\star v\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla(\mathcal{L}_\star v)\|_{L^2(\mathbb{R}^2)}^2\\ &= \|\mathcal{L}_\star v\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla(\mathcal{L}v \circ \gamma_\star) \circ \gamma_\star^{-1}\nabla\gamma_\star^{-1}\|_{L^2(\mathbb{R}^2)}^2.\end{aligned}$$

The identity (6.2.33) implies

$$\begin{aligned}\|\nabla(\mathcal{L}v \circ \gamma_\star) \circ \gamma_\star^{-1}\nabla\gamma_\star^{-1}\|_{L^2(\mathbb{R}^2)}^2 &= \int_{\mathbb{R}^2} \left|\nabla(\mathcal{L}v \circ \gamma_\star) \circ \gamma_\star^{-1}\nabla\gamma_\star^{-1}\right|^2 dx\\ &\le \int_{\mathbb{R}^2} \left|\nabla(\mathcal{L}v \circ \gamma_\star)\right|^2 \left|(\nabla\gamma_\star^{-1}) \circ \gamma_\star\right|^2 |\nabla\gamma_\star|\, dx\\ &= \|\nabla(\mathcal{L}v \circ \gamma_\star)|\nabla\gamma_\star|^{-1/2}\|_{L^2(\mathbb{R}^2)}^2\\ &\le \||\nabla\gamma_\star|^{-1}\|_{L^\infty(\mathbb{R}^2)}\|\nabla(\mathcal{L}v \circ \gamma_\star)\|_{L^2(\mathbb{R}^2)}^2\end{aligned}$$

as well as

$$\begin{aligned}\|\mathcal{L}_\star v\|_{L^2(\mathbb{R}^2)}^2 &= \|\mathcal{L}(v \circ \gamma_\star)|\nabla\gamma_\star|^{1/2}\|_{L^2(\mathbb{R}^2)}^2\\ &\le \||\nabla\gamma_\star|\|_{L^\infty(\mathbb{R}^2)}\|\mathcal{L}(v \circ \gamma_\star)\|_{L^2(\mathbb{R}^2)}^2.\end{aligned}$$

With (6.2.30) and the continuity of $\mathcal{L}$, the last two inequalities prove

$$\begin{aligned}\|\mathcal{L}_\star v\|_{H^1(\mathbb{R}^2)}^2 &\lesssim (1 + \|\nabla\gamma_\star\|_{L^\infty(\mathbb{R}^2)})\|\mathcal{L}v \circ \gamma_\star\|_{H^1(\mathbb{R}^2)}^2\\ &\le \|v \circ \gamma_\star\|_{H^{1/2}(\Gamma)}.\end{aligned}$$

With (6.2.35), we see

$$\|v \circ \gamma_\star\|_{H^{1/2}(\Gamma)} \le C_{\mathrm{Lip}}^{1/2}\|v\|_{H^{1/2}(\Gamma_\star)}.$$

Moreover, (6.2.28) implies $\|\nabla\gamma_\star\|_{L^\infty(\mathbb{R}^2)} \le 1 + C_{\mathrm{ext}}\mathrm{geo}(\mathcal{T}_\star) \le 3/2$ and concludes the proof. □

The proofs of Lemma 6.2.21–6.2.22 and Proposition 6.2.23 are well-known in the literature. We repeat them for the sole purpose of ensuring the uniform boundedness of the constants appearing with respect to the domains $\Omega_\star$, as this is usually not found in the literature.

LEMMA 6.2.21. *Given an approximate geometry $\mathcal{T}_\star \in \mathbb{T}$ with* (Γ1)–(Γ3) *and* $\mathrm{geo}(\mathcal{T}_\star) \le C_{\mathrm{ext}}^{-1}/2$, *there holds*

$$\langle \mathcal{V}_\star v\,,\, v\rangle_{\Gamma_\star} \ge C_{\widetilde{\mathrm{V}}}^{-1}\|v\|_{H^{-1/2}(\Gamma_\star)} \quad \textit{for all } v \in H^{-1/2}(\Gamma) \textit{ with } \langle v\,,\,1\rangle_\Gamma = 0. \tag{6.2.36}$$

The constant $C_{\widetilde{\mathrm{V}}} > 0$ depends only on Γ and C_{ext}.

PROOF. Let $v \in L^2(\Gamma_\star)$ with $\langle v\,,\,1\rangle_{\Gamma_\star} = 0$. Define the interior and exterior normal derivatives $\partial_n^{\mathrm{int}}, \partial_n^{\mathrm{ext}}$. Then, there holds by Greens-identity, the fact $\Delta\mathcal{V}_\star v = 0$ in $\mathbb{R}^2 \setminus \Gamma_\star$, and $|(\mathcal{V}_\star v)(x)| \simeq |x|^{-1}$ as $|x| \to \infty$, that

$$\|\nabla\mathcal{V}_\star v\|_{L^2(\mathbb{R}^2)}^2 = \langle \partial_n^{\mathrm{int}}\mathcal{V}_\star v - \partial_n^{\mathrm{ext}}\mathcal{V}_\star v\,,\, \mathcal{V}_\star v\rangle_{\Gamma_\star}.$$

The jump property of $\mathcal{V}_\star$, i.e., $\partial_n^{\mathrm{int}}\mathcal{V}_\star v - \partial_n^{\mathrm{ext}}\mathcal{V}_\star v = v$, shows

$$\|\nabla\mathcal{V}_\star v\|_{L^2(\mathbb{R}^2)}^2 = \langle v\,,\, \mathcal{V}_\star v\rangle_{\Gamma_\star}.$$

On the other hand, the jump property implies

$$\|v\|_{H^{-1/2}(\Gamma_\star)} \le \|\partial_n^{\rm int}\mathcal{V}_\star v\|_{H^{-1/2}(\Gamma_\star)} + \|\partial_n^{\rm ext}\mathcal{V}_\star v\|_{H^{-1/2}(\Gamma_\star)}.$$

With the lifting $\mathcal{L}_\star$ from Lemma 6.2.20 and $\Delta\mathcal{V}_\star v = 0$ in $\mathbb{R}^2 \setminus \Gamma_\star$, we get

$$\begin{aligned}\|\partial_n^{\rm int}\mathcal{V}_\star v\|_{H^{-1/2}(\Gamma_\star)} &= \sup_{w\in H^{1/2}(\Gamma_\star)\setminus\{0\}} \frac{\langle \partial_n^{\rm int}\mathcal{V}_\star v\,,\, w\rangle_{\Gamma_\star}}{\|w\|_{H^{1/2}(\Gamma_\star)}} \\ &\le \sup_{w\in H^{1/2}(\Gamma_\star)\setminus\{0\}} \frac{|\langle \nabla\mathcal{V}_\star v\,,\, \nabla\mathcal{L}_\star w\rangle_{\mathbb{R}^2\setminus\Gamma_\star}|}{\|w\|_{H^{1/2}(\Gamma_\star)}} \lesssim \|\nabla\mathcal{V}_\star v\|_{L^2(\mathbb{R}^2)}.\end{aligned}$$

The analogous statement holds for $\partial_n^{\rm est}\mathcal{V}v$. Altogether, this concludes (6.2.36) $\square$

Lemma 6.2.22. *There exists $u_{\rm eq}(\mathcal{T}_\star) \in H^{-1/2}(\Gamma_\star)$ with $\mathcal{V}_\star u_{\rm eq}(\mathcal{T}_\star) = \lambda_{\rm eq}(\mathcal{T}_\star) \in \mathbb{R}$ and $\langle u_{\rm eq}(\mathcal{T}_\star)\,,\, 1\rangle_{\Gamma_\star} = 1$. All approximate geometries $\mathcal{T}_\star \in \mathbb{T}$ satisfy*

$$\lambda_{\rm eq}(\mathcal{T}_\star) \ge 2\pi|\log(\mathrm{diam}(\Omega_\star))| \ge 2\pi|\log(1-\varepsilon_{\rm scale})| > 0.$$

Proof. Let $(v_\star, \lambda_\star) \in H^{-1/2}(\Gamma_\star) \times \mathbb{R}$ solve the saddle-point problem

$$\begin{aligned}\langle \mathcal{V}_\star v_\star\,,\, v\rangle_{\Gamma_\star} - \langle v\,,\, \lambda_\star\rangle_{\Gamma_\star} &= 0,\\ -\langle v_\star\,,\, \mu\rangle_{\Gamma_\star} &= -\mu\end{aligned}$$

for all $(v,\mu) \in H^{-1/2}(\Gamma_\star) \times \mathbb{R}$. Since Lemma 6.2.21 proves that $\mathcal{V}_\star$ is elliptic on the kernel of $\langle \cdot\,,\, \mu\rangle_{\Gamma_\star}$, standard LBB theory shows

$$\|v_\star\|_{H^{-1/2}(\Gamma_\star)} + |\lambda_\star| \lesssim 1,$$

where the hidden constant depends only on $C_{\rm V}$ but not on the particular geometry $\mathcal{T}_\star$. There holds $u_{\rm eq}(\mathcal{T}_\star) = v_\star$ and $\lambda_{\rm eq}\mathcal{T}_\star = \lambda_\star$. Define Robins constant of the set $\Gamma_\star$ by

$$V_{\Gamma_\star} := -\inf_{\mu\in\mathcal{B}} \int_{\Gamma_\star}\int_{\Gamma_\star} \log|x-y|\, d\mu(x)\, d\mu(y),$$

where $\mathcal{B}$ denotes the set of all Borel probability measures on $\Gamma_\star$. A well-known result of potential theory (see, e.g., [**84**, Section 1] for the proof) is that the logarithmic capacity $\exp(-V_{\Gamma_\star})$ satisfies $\exp(-V_{\Gamma_\star}) \le \mathrm{diam}(\Gamma_\star) = \mathrm{diam}(\Omega_\star)$. The result [**84**, Theorem 1.2] shows that

$$\tfrac{1}{2\pi}\lambda_\star = \langle v_\star\,,\, 1\rangle_{\Gamma_\star} V_{\Gamma_\star} = V_{\Gamma_\star}.$$

Altogether, this implies by definition of $\Omega_\star$ in Section 6.2.0.1

$$\tfrac{1}{2\pi}\lambda_\star \ge -\log(\mathrm{diam}(\Omega_\star)) \ge -\log(1-\varepsilon_{\rm scale}) > 0.$$

This concludes the proof. $\square$

Proposition 6.2.23. *Given an approximate geometry $\mathcal{T}_\star \in \mathbb{T}$ with (Γ1)–(Γ3) with $\mathrm{geo}(\mathcal{T}_\star) \le C_{\rm ext}^{-1}/2$, there holds*

$$\|\mathcal{V}_\star v\|_{H^{1/2}(\Gamma_\star)} \le C_{\rm V}\|v\|_{H^{-1/2}(\Gamma_\star)} \quad \text{for all } v \in H^{-1/2}(\Gamma_\star) \tag{6.2.37}$$

as well as

$$\langle \mathcal{V}_\star v\,,\, v\rangle_{\Gamma_\star} \ge C_{\rm V}^{-1}\|v\|^2_{H^{-1/2}(\Gamma_\star)} \quad \text{for all } v \in H^{-1/2}(\Gamma_\star). \tag{6.2.38}$$

The constant $C_{\mathrm V}>0$ depends only on $\varepsilon_{\mathrm{scale}}$, $C_{\widetilde{\mathrm V}}$, Γ, C_{ext} and $\mathbb T$. This particularly implies for any closed subspace $\mathcal P^0(\mathcal T_\star)\subseteq\mathcal X\subseteq H^{-1/2}(\Gamma_\star)$ and the solution $U_{\mathcal X}\in\mathcal X$ of $\langle\mathcal V_\star U_{\mathcal X}\,,\,V\rangle_{\Gamma_\star}=\langle f_\star\,,\,V\rangle_{\Gamma_\star}$ for all $V\in\mathcal X$ that

$$\|U_{\mathcal X}-U(\mathcal T_\ell)\|_{H^{-1/2}(\Gamma_\star)}\le C_{\mathrm V}^2\min_{V\in\mathcal P^0(\mathcal T_\ell)}\|U_{\mathcal X}-V\|_{H^{-1/2}(\Gamma_\star)}. \tag{6.2.39}$$

Proof. To see (6.2.38), we use Lemma 6.2.21 and Lemma 6.2.22. Let $v\in H^{-1/2}(\Gamma_\star)$ and $v_0:=v-u_{\mathrm{eq}}(\mathcal T_\star)\langle v\,,\,1\rangle_{\Gamma_\star}$. Then, $\langle v_0\,,\,1\rangle_{\Gamma_\star}=0$ and with (6.2.36)

$$\begin{aligned}\langle\mathcal V_\star v\,,\,v\rangle_{\Gamma_\star}&=\langle\mathcal V_\star v_0\,,\,v_0\rangle_{\Gamma_\star}+2\langle v\,,\,1\rangle_{\Gamma_\star}\langle\mathcal V_\star u_{\mathrm{eq}}(\mathcal T_\star)\,,\,v_0\rangle_{\Gamma_\star}\\&\quad+\langle v\,,\,1\rangle_{\Gamma_\star}^2\langle\mathcal V_\star u_{\mathrm{eq}}(\mathcal T_\star)\,,\,u_{\mathrm{eq}}(\mathcal T_\star)\rangle_{\Gamma_\star}\\&=\langle\mathcal V_\star v_0\,,\,v_0\rangle_{\Gamma_\star}+\langle v\,,\,1\rangle_{\Gamma_\star}^2\langle\lambda_{\mathrm{eq}}(\mathcal T_\star)\,,\,u_{\mathrm{eq}}(\mathcal T_\star)\rangle_{\Gamma_\star}\\&\ge C_{\widetilde{\mathrm V}}^{-1}\|v_0\|_{H^{-1/2}(\Gamma_\star)}^2+\lambda_{\mathrm{eq}}(\mathcal T_\star)\langle v\,,\,1\rangle_{\Gamma_\star}^2\gtrsim\|v\|_{H^{-1/2}(\Gamma_\star)}^2,\end{aligned}$$

where the hidden constant depends only on $\varepsilon_{\mathrm{scale}}$ and on $C_{\widetilde{\mathrm V}}$.

To see (6.2.37), let $\Omega_\star\subset\mathbb R^2$ denote the domain enclosed by $\Gamma_\star$, i.e., $\Gamma_\star=\partial\Omega_\star$. Let $\widehat\Omega\subset\mathbb R^2$ denote a bounded Lipschitz domain such that $\Omega_\star\subseteq\widehat\Omega$ for all $\mathcal T_\star\in\mathbb T$ with $\mathrm{geo}(\mathcal T_\star)\le C_{\mathrm{ext}}^{-1}/2$ as well as $\Omega\subseteq\widehat\Omega$. There holds for $v\in H^{-1/2}(\Gamma_\star)$ and $g\in L^2(\Omega_\star)$

$$\begin{aligned}\langle\mathcal V_\star v\,,\,g\rangle_{\Omega_\star}&=\tfrac{1}{2\pi}\int_{\Gamma_\star}v(x)\int_{\Omega_\star}\log|x-y|g(y)\,dy\,dx\\&=\tfrac{1}{2\pi}\int_{\Gamma_\star}v(x)\int_{\widehat\Omega}\log|x-y|g(y)\,dy\,dx=\langle v\,,\,\mathcal N g\rangle_{\Gamma_\star},\end{aligned}$$

where $\mathcal N:\widetilde H^{-1}(\widehat\Omega)\to H^1(\widehat\Omega)$ denotes the Newton potential (see, e.g., **[75]** for the mapping properties). We obtain

$$\begin{aligned}\langle v\,,\,\mathcal N g\rangle_{\Gamma_\star}&=\langle v\circ\gamma_\star|\partial_\Gamma\gamma_\star|\,,\,(\mathcal N g)\circ\gamma_\star\rangle_\Gamma\\&\lesssim\|v\circ\gamma_\star|\partial_\Gamma\gamma_\star|\|_{H^{-1/2}(\Gamma)}\|(\mathcal N g)\circ\gamma_\star\|_{H^1(\Omega)}.\end{aligned}$$

Lemma 6.2.19 shows $\|v\circ\gamma_\star^{-1}|\partial_\Gamma\gamma_\star^{-1}|\|_{H^{-1/2}(\Gamma)}\simeq\|v\|_{H^{-1/2}(\Gamma_\star)}$ and Lemma 6.2.17 implies that $\gamma_\star$ is globally bi-Lipschitz and $\gamma_\star(\Omega)=\Omega_\star$. Hence, we have

$$\|(\mathcal N g)\circ\gamma_\star\|_{H^1(\Omega)}^2\lesssim\|\mathcal N g\|_{H^1(\gamma_\star(\Omega))}^2\le\|\mathcal N g\|_{H^1(\widehat\Omega)}^2.$$

Moreover, since $\mathrm{supp}(g)\subseteq\Omega_\star$, there holds

$$\begin{aligned}\|\mathcal N g\|_{H^1(\widehat\Omega)}\lesssim\|g\|_{\widetilde H^{-1}(\widehat\Omega)}&=\sup_{v\in H^1(\widehat\Omega)\setminus\{0\}}\frac{\langle g\,,\,v\rangle_{\widehat\Omega}}{\|v\|_{H^1(\widehat\Omega)}}\le\|g\|_{\widetilde H^{-1}(\Omega_\star)}\sup_{v\in H^1(\widehat\Omega)\setminus\{0\}}\frac{\|v\|_{H^1(\Omega_\star)}}{\|v\|_{H^1(\widehat\Omega)}}\\&\le\|g\|_{\widetilde H^{-1}(\Omega_\star)}.\end{aligned}$$

Altogether, this shows

$$\langle\mathcal V_\star v\,,\,g\rangle_{\Omega_\star}\lesssim\|v\|_{H^{-1/2}(\Gamma_\star)}\|g\|_{\widetilde H^{-1}(\Omega_\star)}.$$

Taking the supremum over all g shows $\|\mathcal V_\star v\|_{H^1(\Omega_\star)}\lesssim\|v\|_{H^{-1/2}(\Gamma_\star)}$. Finally, there holds with (6.2.35)

$$\begin{aligned}\|\mathcal V_\star v\|_{H^{1/2}(\Gamma_\star)}&\lesssim\|(\mathcal V_\star v)\circ\gamma_\star\|_{H^{1/2}(\Gamma)}\lesssim\|(\mathcal V_\star v)\circ\gamma_\star\|_{H^1(\Omega)}\\&\lesssim\|\mathcal V_\star v\|_{H^1(\Omega_\star)},\end{aligned}$$

where the hidden constant depends again on the bi-Lipschitz continuity of $\gamma_\star$ and $\gamma_\star(\Omega) = \Omega_\star$. This shows (6.2.37). The Céa Lemma (6.2.39) follows by standard arguments from (6.2.38)–(6.2.37). This concludes the proof. $\square$

Lemma 6.2.24. *Given $x, y \in \mathbb{R}^2$ and the approximate geometry $\mathcal{T}_\star \in \mathbb{T}$ with (Γ1)–(Γ3) and $\mathrm{geo}(\mathcal{T}_\star) \le C_{\mathrm{ext}}^{-1}/2$, the kernel*

$$\kappa_\star(x, y) := \log\Big(\frac{|x-y|^2}{|\gamma_\star(x) - \gamma_\star(y)|^2}\Big) \tag{6.2.40}$$

satisfies for $j = 1, 2$

$$\begin{aligned}\tfrac{1}{2}\partial_{x_j}\kappa_\star(x,y) = \frac{x-y}{|x-y|^2}\cdot\big(e_j - \partial_{x_j}\gamma_\star(x)\big) + \Big(\frac{x-y}{|x-y|^2}\frac{|\gamma_\star(x)-\gamma_\star(y)|^2 - |x-y|^2}{|\gamma_\star(x)-\gamma_\star(y)|^2}\\ + \frac{(x-y) - (\gamma_\star(x) - \gamma_\star(y))}{|\gamma_\star(x)-\gamma_\star(y)|^2}\Big)\cdot\partial_{x_j}\gamma_\star(x).\end{aligned} \tag{6.2.41}$$

This particularly implies

$$\big|\nabla_x\kappa_\star(x,y)\big| \le C(1 + \mathrm{geo}(\mathcal{T}_\star))\frac{1}{|x-y|}\|t_\Gamma - \partial_\Gamma\gamma_\star\|_{L^\infty(\Gamma)} \tag{6.2.42}$$

for all $x, y \in \mathbb{R}^2$, where $C > 0$ depends only on C_{Lip}, C_{ext}, and Γ. For $x, y \in \Gamma$, there holds even

$$\big|\nabla_x\kappa_\star(x,y)\big| \le CC_\Gamma(1 + \mathrm{geo}(\mathcal{T}_\star))\frac{1}{|x-y|}\max_{\substack{T\in\mathcal{T}_\star\\ |T^\Gamma\cap\Gamma_x^y|>0}}\|t_\Gamma - \partial_\Gamma\gamma_\star\|_{L^\infty(T^\Gamma)}. \tag{6.2.43}$$

as well as

$$C^{-1}\big|\partial_{\Gamma,x}\kappa_\star(x,y)\big| \le \frac{|(t_\Gamma - \partial_\Gamma\gamma_\star)(x)|}{|x-y|} + (1 + \mathrm{geo}(\mathcal{T}_\star))\mathrm{geo}(\mathcal{T}_\star)^2\frac{1}{|x-y|^2}. \tag{6.2.44}$$

Proof. The identity (6.2.41) follows from straightforward differentiation. Since $\nabla\gamma_\star \in L^\infty(\mathbb{R}^2)$, there holds with $[x,y] := \big\{\lambda(x-y) + y \,:\, 0 \le \lambda \le 1\big\}$

$$\begin{aligned}\big||\gamma_\star(x)-\gamma_\star(y)|^2 - |x-y|^2\big| &\le \big(|\gamma_\star(x)-\gamma_\star(y)| - |x-y|\big)\big(|\gamma_\star(x)-\gamma_\star(y)| + |x-y|\big)\\ &\le |\gamma_\star(x)-\gamma_\star(y) - (x-y)|\big(|\gamma_\star(x)-\gamma_\star(y)| + |x-y|\big)\\ &\le (1 + C_{\mathrm{Lip}})\Big|\int_{[x,y]}(I - \nabla\gamma_\star(z))\cdot\frac{x-y}{|x-y|}\,dz\Big||x-y|\\ &\le \|I - \nabla\gamma_\star(x)\|_{L^\infty(\mathbb{R}^2)}|x-y|^2.\end{aligned}$$

This and (6.2.28) show

$$\frac{|\gamma_\star(x)-\gamma_\star(y)|^2 - |x-y|^2}{|\gamma_\star(x)-\gamma_\star(y)|^2} \lesssim \|t_\Gamma - \partial_\Gamma\gamma_\star\|_{L^\infty(\Gamma)}.$$

Finally, the same argument shows

$$\frac{\gamma_\star(x)-\gamma_\star(y) - (x-y)}{|\gamma_\star(x)-\gamma_\star(y)|^2} \lesssim \frac{1}{|x-y|}\|t_\Gamma - \partial_\Gamma\gamma_\star\|_{L^\infty(\Gamma)}.$$

The bound (6.2.28) implies $|\partial_{x_j}\gamma_\star(x)| \leq 1 + \mathrm{geo}(\mathcal{T}_\star)$. This shows (6.2.42). The estimate (6.2.43) follows analogously by use of $[x,y] := \Gamma_x^y$ instead, i.e.,

$$\begin{aligned}\big||\gamma_\star(x)-\gamma_\star(y)|^2-|x-y|^2\big| &\leq \big(|\gamma_\star(x)-\gamma_\star(y)|-|x-y|\big)\big(|\gamma_\star(x)-\gamma_\star(y)|+|x-y|\big)\\ &\leq |\gamma_\star(x)-\gamma_\star(y)-(x-y)|\big(|\gamma_\star(x)-\gamma_\star(y)|+|x-y|\big)\\ &\leq (1+C_{\mathrm{Lip}})\Big|\int_{\Gamma_x^y} t_\Gamma(z)-\partial_\Gamma\gamma_\star(z)\,dz\Big||x-y|\\ &\leq \|t_\Gamma-\partial_\Gamma\gamma_\star\|_{L^\infty(\Gamma_x^y)}|x-y|^2.\end{aligned}$$

The estimate (6.2.44) follows from (6.2.41) and

$$\begin{aligned}\big||\gamma_\star(x)-\gamma_\star(y)|^2-|x-y|^2\big| &\leq \big(|\gamma_\star(x)-\gamma_\star(y)|-|x-y|\big)\big(|\gamma_\star(x)-\gamma_\star(y)|+|x-y|\big)\\ &\lesssim \mathrm{geo}(\mathcal{T}_\star)^2|x-y|.\end{aligned}$$

This concludes the proof. ∎

The following result can be found in **[43, 77, 83]** for real and complex interpolation. We include the proof for completeness and to underline the fact that the constant is independent of $\widetilde\Gamma$.

Lemma 6.2.25. *Let $\widetilde\Gamma = \partial\widetilde\Omega \subset \mathbb{R}^2$ denote a Lipschitz boundary. Let $f_1,\ldots,f_N \in H^1(\widetilde\Gamma)$ such that the supports $\mathrm{supp}(f_i)$ are connected and pairwise disjoint, i.e., $\mathrm{supp}(f_i)\cap\mathrm{supp}(f_j) = \emptyset$ for all $1 \leq i \neq j \leq N$. Then, there holds*

$$\|\sum_{i=1}^N f_i\|^2_{H^{1/2}(\widetilde\Gamma)} \leq 2\sum_{i=1}^N \|f_i\|^2_{H^{1/2}(\mathrm{supp}(f_i))}.$$

Proof. Define the auxiliary operators $T_0 : \prod_{i=1}^N L^2(\mathrm{supp}(f_i)) \to L^2(\widetilde\Gamma)$ as well as $T_1 : \prod_{i=1}^N H^1(\mathrm{supp}(f_i)) \to H^1(\widetilde\Gamma)$ by

$$T_\vartheta((f_1,\ldots,f_N)) := \sum_{i=1}^N f_i \quad \text{for } \vartheta \in \{0,1\}.$$

Obviously, there holds

$$\begin{aligned}\|T_0(f_1,\ldots,f_N)\|^2_{L^2(\widetilde\Gamma)} &\leq \sum_{i=1}^N \|f_i\|^2_{L^2(\mathrm{supp}(f_i))} = \|(f_1,\ldots,f_N)\|^2_{\prod_{i=1}^N L^2(\mathrm{supp}(f_i))},\\ \|T_1(f_1,\ldots,f_N)\|^2_{H^1(\widetilde\Gamma)} &\leq \sum_{i=1}^N \|f_i\|^2_{H^1(\mathrm{supp}(f_i))} = \|(f_1,\ldots,f_N)\|^2_{\prod_{i=1}^N H^1(\mathrm{supp}(f_i))}\end{aligned}$$

for all $(f_1,\ldots,f_N) \in \prod_{i=1}^N L^2(\mathrm{supp}(f_i))$ resp. all $(f_1,\ldots,f_N) \in \prod_{i=1}^N H^1(\mathrm{supp}(f_i))$. Real interpolation shows for $T_{1/2} : \mathcal{X} \to H^{1/2}(\widetilde\Gamma)$, $T_{1/2}(f_1,\ldots,f_N) := \sum_{i=1}^N f_i$ that

$$\|T_{1/2}(f_1,\ldots,f_N)\|^2_{H^{1/2}(\widetilde\Gamma)} \leq \|(f_1,\ldots,f_N)\|^2_{\mathcal{X}},$$

where $\mathcal{X} := [\prod_{i=1}^N L^2(\mathrm{supp}(f_i)), \prod_{i=1}^N H^1(\mathrm{supp}(f_i))]_{1/2}$ denotes the space defined with real interpolation. There holds $\mathcal{X} = \prod_{i=1}^N H^{1/2}(\mathrm{supp}(f_i))$ with equivalent norms. It remains to bound the equivalence constants. By definition of $\mathcal{X}$, there holds

$$\|(f_1,\ldots,f_N)\|^2_{\mathcal{X}} := \int_0^\infty t^{-2}K_t^2\,dt, \tag{6.2.45}$$

where

$$K_t := \inf \big\{ \big(\sum_{i=1}^N \|f_{0,i}\|^2_{L^2(\mathrm{supp}(f_i))} \big)^{1/2} + t \big(\sum_{i=1}^N \|f_{1,i}\|^2_{H^1(\mathrm{supp}(f_i))} \big)^{1/2} \,:\, f_i = f_{0,i} + f_{1,i},$$
$$f_{0,i} \in L^2(\mathrm{supp}(f_i)),\ f_{1,i} \in H^1(\mathrm{supp}(f_i)) \big\}.$$

Define

$$\widetilde{K}^2_{t,i} := \inf \big\{ \|f_{0,i}\|^2_{L^2(\mathrm{supp}(f_i))} + t^2 \|f_{1,i}\|^2_{H^1(\mathrm{supp}(f_i))} \,:\, f_i = f_{0,i} + f_{1,i},$$
$$f_{0,i} \in L^2(\mathrm{supp}(f_i)),\ f_{1,i} \in H^1(\mathrm{supp}(f_i)) \big\}.$$

Given $\varepsilon > 0$, let $g_{0,i} \in L^2(\mathrm{supp}(f_i))$ and $g_{1,i} \in H^1(\mathrm{supp}(f_i))$ such that $f_i = g_{0,i} + g_{1,i}$ and

$$\|g_{0,i}\|^2_{L^2(\mathrm{supp}(f_i))} + t^2 \|g_{1,i}\|^2_{H^1(\mathrm{supp}(f_i))} \le \frac{\varepsilon}{N} + \widetilde{K}^2_{t,i} \quad \text{for all } i = 1, \ldots, N.$$

Then, there holds

$$K_t^2/2 \le \sum_{i=1}^N \|g_{0,i}\|^2_{L^2(\mathrm{supp}(f_i))} + t^2 \sum_{i=1}^N \|g_{1,i}\|^2_{H^1(\mathrm{supp}(f_i))} \le \varepsilon + \sum_{i=1}^N \widetilde{K}^2_{t,i}.$$

Since $\varepsilon > 0$ is arbitrary and $a^2 + b^2 \le (a+b)^2$ for all $a, b \ge 0$, the above implies

$$K_t^2/2 \le \sum_{i=1}^N \widetilde{K}^2_{t,i} \le \sum_{i=1}^N K^2_{t,i},$$

where

$$K^2_{t,i} := \inf \big\{ \big(\|f_{0,i}\|_{L^2(\mathrm{supp}(f_i))} + t \|f_{1,i}\|_{H^1(\mathrm{supp}(f_i))} \big)^2 \,:\, f_i = f_{0,i} + f_{1,i},$$
$$f_{0,i} \in L^2(\mathrm{supp}(f_i)),\ f_{1,i} \in H^1(\mathrm{supp}(f_i)) \big\}.$$

Together with (6.2.45), this shows

$$\|(f_1, \ldots, f_N)\|^2_{\mathcal{X}} \le 2 \sum_{i=1}^N \int_0^\infty t^{-2} K^2_{t,i} \, dt = 4 \sum_{i=1}^N \|f_i\|^2_{H^{1/2}(\mathrm{supp}(f_i))}.$$

Altogether, this concludes the proof. □

Given $T \in \mathcal{T}_\star$, define the k-patch of T for all $k \ge 1$ as

$$\omega(T, \mathcal{T}_\star) := \omega^1(T, \mathcal{T}_\star) := \bigcup \big\{ T' \in \mathcal{T}_\star \,:\, T \cap T' \neq \emptyset \big\},$$
$$\omega^k(T, \mathcal{T}_\star) := \omega^{k-1}(\omega(T, \mathcal{T}_\star), \mathcal{T}_\star).$$

Note that $\omega(\cdot, \cdot)$ is a patch function in the sense of Section 4.5.1.

A similar result to the following is proved in **[43]** for certain residuals.

Lemma 6.2.26. *Let $\mathcal{T}$ denote a partition of Γ into connected curve segments. Define the weight-function $h(\mathcal{T})|_T := |T|$ for all $T \in \mathcal{T}$. Let $J(\mathcal{T}) : H^1(\Gamma) \to \mathcal{S}^1(\mathcal{T})$ denote the Scott-Zhang projection from Definition 3.3.2. Then, there exists a constant $C_{\mathrm{faer}} > 0$, such that all $v \in H^{1/2}(\Gamma)$ satisfy*

$$\|(1 - J(\mathcal{T}))v\|^2_{H^{1/2}(\Gamma)} \le C_{\mathrm{faer}} \sum_{T \in \mathcal{T}} \|(1 - J(\mathcal{T}))v\|^2_{H^{1/2}(\cup \omega^2(T, \mathcal{T}))}.$$

The constant C_{faer} depends only on Γ and $K(\mathcal{T})$ (where $K(\cdot)$ is defined in Section 3.2.2).

Proof. Let $\xi_1, \dots, \xi_N \in C(\Gamma)$ denote a $\mathcal{T}$-piecewise smooth partition of unity on Γ such that all $j = 1, \dots, N$ satisfy

$$\begin{aligned} \|\xi_j\|_{L^\infty(\Gamma)} &\leq 1, \\ \operatorname{supp}(\xi_j) &\subseteq T_{j,1} \cup T_{j,2} \quad \text{for some } T_{j,1}, T_{j,2} \in \mathcal{T} \text{with } T_{j,1} \cap T_{j,2} \neq \emptyset, \\ \|\partial_\Gamma \xi_j\|_{L^\infty(T_{j,i})} &\leq C h(\mathcal{T})|_{T_{j,i}}^{-1} \quad \text{for } i = 1, 2 \end{aligned}$$

for some constant $C > 1$. There holds

$$\|(1 - J(\mathcal{T}))v\|_{H^{1/2}(\Gamma)}^2 = \|\sum_{j=1}^N \xi_z (1 - J(\mathcal{T}))v\|_{H^{1/2}(\Gamma)}^2.$$

Let $\mathcal{K}_\mathcal{T}^1 \cup \mathcal{K}_\mathcal{T}^2 = \{1, \dots, N\}$ such that $|\operatorname{supp}(\xi_j) \cap \operatorname{supp}(\xi_k)| = 0$ for all $j \neq k$, $j, k \in \mathcal{K}_\mathcal{T}^1$ and for all $j \neq k$, $j, k \in \mathcal{K}_\mathcal{T}^2$. Lemma 6.2.25 shows

$$\begin{aligned} &\|\sum_{j=1}^N \xi_j (1 - J(\mathcal{T}))v\|_{H^{1/2}(\Gamma)}^2 \\ &\quad \leq 2\|\sum_{j \in \mathcal{K}_\mathcal{T}^1} \xi_j (1 - J(\mathcal{T}))v\|_{H^{1/2}(\Gamma)}^2 + 2\|\sum_{j \in \mathcal{K}_\mathcal{T}^2} \xi_j (1 - J(\mathcal{T}))v\|_{H^{1/2}(\Gamma)}^2 \\ &\quad \leq 4 \sum_{j \in \mathcal{K}_\mathcal{T}} \|\xi_j (1 - J(\mathcal{T}))v\|_{H^{1/2}(\operatorname{supp}(\xi_j))}^2. \end{aligned} \tag{6.2.46}$$

With $\omega_j := \operatorname{supp}(\xi_j)$, by definition of the $H^{1/2}$-norm by real interpolation, and with $w := (1 - J(\mathcal{T}))v$, there holds

$$\|\xi_j w\|_{H^{1/2}(\omega_j)}^2 = \int_0^\infty t^{-2} K_t^2 \, dt,$$

where

$$K_t := \inf \big\{ \|w_0\|_{L^2(\omega_j)} + t\|w_1\|_{H^1(\omega_j)} \, : \, \xi_j w = w_0 + w_1, \, w_0 \in L^2(\omega_j), \, w_1 \in H^1(\omega_j) \big\}.$$

Additionally, consider

$$\widetilde{K}_t := \inf \big\{ \|w_0\|_{L^2(\omega_j^2)} + t\|w_1\|_{H^1(\omega_j^2)} \, : \, w = w_0 + w_1, \, w_0 \in L^2(\omega_j^2), \, w_1 \in H^1(\omega_j^2) \big\}$$

with $\omega_j^2 := \bigcup \big\{ T \in \mathcal{T} \, : \, T \cap \omega_j \neq \emptyset \big\}$. Choose $\widetilde{w}_0$, $\widetilde{w}_1$ such that $\|\widetilde{w}_0\|_{L^2(\omega_j^2)} + t\|\widetilde{w}_1\|_{H^1(\omega_j^2)} \leq \widetilde{K}_t + \varepsilon$ for some $\varepsilon > 0$. Since $(1 - J(\mathcal{T}))w = w$, there holds $w = \widetilde{w}_0 + \widetilde{w}_1 = w_0 + w_1$ on ω_j with $w_i := (1 - J(\mathcal{T}))\widetilde{w}_i$ for $i = 1, 2$. With $\xi_j w = \xi_j w_1 + \xi_j w_2$ and $|\partial_\Gamma \xi_j| \simeq \operatorname{diam}(\omega_j)^{-1}$, this allows to estimate

$$\begin{aligned} K_t &\leq \|\xi_j w_0\|_{L^2(\omega_j)} + t\|\xi_j w_1\|_{H^1(\omega_j)} \\ &\lesssim \|w_0\|_{L^2(\omega_j)} + t\big(\|w_1\|_{L^2(\omega_j)} + \|\partial_\Gamma(\xi_j w_1)\|_{L^2(\omega_j)}\big) \\ &\lesssim \|w_0\|_{L^2(\omega_j)} + t\big(\|w_1\|_{L^2(\omega_j)} + \|\partial_\Gamma w_1\|_{L^2(\omega_j)} + \operatorname{diam}(\omega_j)^{-1}\|w_1\|_{L^2(\omega_j)}\big). \end{aligned}$$

The fact that $w_i = (1 - J(\mathcal{T}))\widetilde{w}_i$ for $i = 1, 2$ as well as the stability and approximation properties (3.3.2) of $J(\mathcal{T})$ lead to

$$\begin{aligned} K_t &\lesssim \|\widetilde{w}_0\|_{L^2(\omega_j^2)} + t\big(\|\widetilde{w}_1\|_{L^2(\omega_j^2)} + \|\partial_\Gamma \widetilde{w}_1\|_{L^2(\omega_j^2)}\big) \\ &\lesssim \|\widetilde{w}_0\|_{L^2(\omega_j^2)} + t\|\widetilde{w}_1\|_{H^1(\omega_j^2)} \lesssim \widetilde{K}_t + \varepsilon. \end{aligned}$$

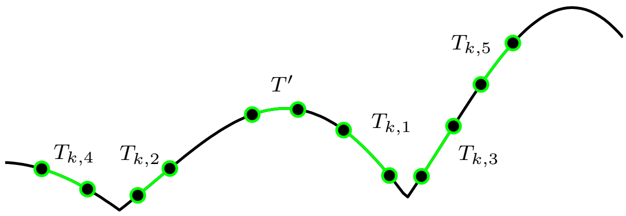

FIGURE 5. Illustration of the situation in the proof of Lemma 6.2.27.

Since $\varepsilon > 0$ is arbitrary and the hidden constants depend only on $K(\mathcal{T})$ (where $K(\cdot)$ is defined in Section 3.2.2) and Γ, there holds $K_t \lesssim \widetilde{K}_t$ and hence

$$\|\xi_j(1-J(\mathcal{T}))v\|^2_{H^{1/2}(\omega_j)} = \int_0^\infty t^{-2}K_t^2\,dt \lesssim \int_0^\infty t^{-2}\widetilde{K}_t^2\,dt = \|(1-J(\mathcal{T}))v\|^2_{H^{1/2}(\omega_j^2)}.$$

In combination with (6.2.46), this concludes the proof. □

LEMMA 6.2.27. *There exists a constant $C_\Sigma > 0$ such that each partition $\mathcal{T}$ of Γ satisfies for $\alpha \geq 1$*

$$\max_{T'\in\mathcal{T}} \sum_{\substack{T\in\mathcal{T}\\ \mathrm{dist}(T,T')>0}} \frac{|T|^\alpha}{\mathrm{dist}(T,T')^\alpha} \leq \begin{cases} C_\Sigma|\log\big(\frac{\max h(\mathcal{T})}{\min h(\mathcal{T})}\big)|(\log(|\mathcal{T}|)|+1) & \text{for } \alpha = 1,\\ C_\Sigma|\log\big(\frac{\max h(\mathcal{T})}{\min h(\mathcal{T})}\big)| & \text{for } \alpha > 1,\end{cases}$$

where $h(\mathcal{T})|_T := |T|$ for all $T\in\mathcal{T}$ and the constant C_Σ depends only on $K(\mathcal{T})$ (with $K(\cdot)$ from Section 3.2.2) and Γ.

PROOF. For $T, T' \in \mathcal{T}$, define $\Gamma_T^{T'} = \Gamma_x^y$ for some $x\in T$ and $y\in T'$ with $|\Gamma_T^{T'}| = \min_{x\in T, y\in T'}|\Gamma_x^y|$. Let $T'\in\mathcal{T}$. Define $P_k := \big\{T\in\mathcal{T} : |\Gamma|2^{-k} \leq |T| < |\Gamma|2^{-k+1}\big\}$ and choose a numbering $\{T_{k,1},\ldots,T_{k,n_k}\} = \big\{T\in P_k : \mathrm{dist}(T,T')>0\big\}$ such that $\Gamma_{T_{k,1}}^{T_{k,j}}$ contains $\lfloor\frac{j-2}{2}\rfloor$ elements from P_k and $\mathrm{dist}(T_{k,1},T')$ is minimal (see Figure 5 for an illustration of the concept). This implies

$$\mathrm{dist}(T',T_{k,j}) \overset{(6.2.13)}{\geq} C_\Gamma^{-1}|\Gamma_{T'}^{T_{k,j}}| \geq C_\Gamma^{-1}(\lfloor\frac{j-2}{2}\rfloor - 1)2^{-k}.$$

Moreover, for $1 \leq j < 4$, the K-mesh regularity and the fact that $\mathrm{dist}(T_{k,1},T')$ is minimal imply

$$\mathrm{dist}(T',T_{k,j}) \geq \mathrm{dist}(T',T_{k,1}) \geq K(\mathcal{T})^{-1}|T_{k,1}| \geq K(\mathcal{T})^{-1}2^{-k} \geq K(\mathcal{T})^{-1}/2|T_{k,j}|.$$

With this, compute

$$\begin{aligned}\sum_{\substack{T\in\mathcal{T}\\ \mathrm{dist}(T,T')>0}} \frac{|T|^\alpha}{\mathrm{dist}(T,T')^\alpha} &= \sum_{k=0}^\infty \sum_{\substack{T\in P_k\\ \mathrm{dist}(T,T')>0}} \frac{|T|^\alpha}{\mathrm{dist}(T,T')^\alpha} = \sum_{k=0}^\infty\sum_{j=1}^{n_k} \frac{|T_{k,j}|^\alpha}{\mathrm{dist}(T_{k,j},T')^\alpha}\\ &\lesssim \sum_{k=0}^{\max\{k\in\mathbb{N}\,:\,n_k>0\}} \Big(1 + \sum_{j=4}^{n_k} \frac{2^{\alpha(-k+1)}}{(\lfloor\frac{j-2}{2}\rfloor - 1)^\alpha 2^{-\alpha k}}\Big)\\ &\lesssim \sum_{k=0}^{\max\{k\in\mathbb{N}\,:\,n_k>0\}} \Big(1 + \sum_{j=1}^{n_k}\frac{1}{j^\alpha}\Big).\end{aligned}$$

There are at most $|\log_2(\max h(\mathcal{T})) - \log_2(\min h(\mathcal{T}))|$ numbers $k \in \mathbb{N}_0$ with $n_k > 0$. Hence, an asymptotic estimate for the harmonic series shows for $\alpha = 1$

$$\sum_{\substack{T\in\mathcal{T}_\ell\\ \operatorname{dist}(T,T')>0}} \frac{|T|}{\operatorname{dist}(T,T')} \lesssim \sum_{\substack{k\in\mathbb{N}_0\\ n_k>0}} \big(|\log(|P_k|))| + 1\big)$$
$$\lesssim |\log\big(\frac{\max h(\mathcal{T})}{\min h(\mathcal{T})}\big)|(\log(|\mathcal{T}|)| + 1).$$

For $\alpha > 1$, the Dirichlet series converges and hence

$$\sum_{\substack{T\in\mathcal{T}_\ell\\ \operatorname{dist}(T,T')>0}} \frac{|T|^\alpha}{\operatorname{dist}(T,T')^\alpha} \lesssim |\log\big(\frac{\max h(\mathcal{T})}{\min h(\mathcal{T})}\big)|.$$

This concludes the proof. □

6.2.7. Reliable error control. The following results prove the reliability of the error estimator.

Theorem 6.2.28. *There exists $C_{\mathrm{rel}} > 0$ such that all approximate geometries $\mathcal{T}_\star \in \mathbb{T}$ with $h_\star \le C_\Gamma^{-1}\kappa_\Gamma^{-1}/2$ and $\operatorname{geo}(\mathcal{T}_\star) \le \min\{C_{\mathrm{ext}}^{-1}/2, C_\Gamma^{-1}/2, C_\Gamma^{-1}\kappa_\Gamma^{-1}/2\}$ satisfy the reliable error estimate*

$$\|u - U(\mathcal{T}_\star)^\Gamma\|_{H^{-1/2}(\Gamma)} \le C_{\mathrm{rel}}\eta(\mathcal{T}_\star). \tag{6.2.47}$$

The proof is divided into several lemmas.

Lemma 6.2.29. *The approximate geometry $\mathcal{T}_\star \in \mathbb{T}$ defines the formal operator*

$$M_\star g(x) := \int_\Gamma \log\Big(\frac{|x-y|^2}{|\gamma_\star(x) - \gamma_\star(y)|^2}\Big) g(x)\,dx \quad \textit{for all } x \in \Omega\cup\Gamma. \tag{6.2.48}$$

If $\mathcal{T}_\star$ satisfies (Γ1)–(Γ3), there exists a constant $C_{\mathrm{res}} > 0$ such that all $v_\star \in L^2(\Gamma_\star)$ with $v_\star^\Gamma := v_\star \circ \gamma_\star|\partial_\Gamma\gamma_\star|$ satisfy

$$C_{\mathrm{res}}^{-1}\|u - v_\star^\Gamma\|_{H^{-1/2}(\Gamma)} \le \sup_{w\in H^{-1/2}(\Gamma_\star)} \frac{\langle f_\star - \mathcal{V}_\star v_\star\,,\, w\rangle_{\Gamma_\star}}{\|w\|_{H^{-1/2}(\Gamma_\star)}} + \|M_\star v_\star^\Gamma\|_{H^{1/2}(\Gamma)},$$

where we define $\|M_\star v_\star^\Gamma\|_{H^{1/2}(\Gamma)} := \infty$ for $M_\star v_\star^\Gamma \notin H^{1/2}(\Gamma)$. This holds particularly for $v_\star = U(\mathcal{T}_\star)$ and hence $v_\star^\Gamma = U(\mathcal{T}_\star)^\Gamma$.

Proof. The error $\|u - v_\star^\Gamma\|_{H^{-1/2}(\Gamma)}$ satisfies for $\widetilde{w} := w \circ \gamma_\star^{-1}|\partial_{\Gamma_\star}\gamma_\star^{-1}|$

$$\|u - v_\star^\Gamma\|_{H^{-1/2}(\Gamma)} \simeq \sup_{w\in H^{-1/2}(\Gamma)} \frac{\langle \mathcal{V}(u - v_\star^\Gamma)\,,\, w\rangle_\Gamma}{\|w\|_{H^{-1/2}(\Gamma)}} = \sup_{w\in H^{-1/2}(\Gamma)} \frac{\langle f - \mathcal{V}v_\star^\Gamma\,,\, w\rangle_\Gamma}{\|w\|_{H^{-1/2}(\Gamma)}}$$
$$= \sup_{w\in H^{-1/2}(\Gamma)} \frac{\langle f\,,\, w\rangle_\Gamma - \langle \mathcal{V}_\star v_\star\,,\, \widetilde{w}\rangle_{\Gamma_\star} + \langle \mathcal{V}_\star v_\star\,,\, \widetilde{w}\rangle_{\Gamma_\star} - \langle \mathcal{V}v_\star^\Gamma\,,\, w\rangle_\Gamma}{\|w\|_{H^{-1/2}(\Gamma)}}.$$

The identity $\langle f\,,\, w\rangle_\Gamma = \langle f_\star\,,\, w\circ\gamma_\star^{-1}|\partial_{\Gamma_\star}\gamma_\star^{-1}|\rangle_{\Gamma_\star} = \langle f_\star\,,\, \widetilde{w}\rangle_{\Gamma_\star}$ shows

$$\begin{aligned}\|u - v_\star^\Gamma\|_{H^{-1/2}(\Gamma)} &\simeq \sup_{w\in H^{-1/2}(\Gamma)} \frac{\langle f_\star - \mathcal{V}_\star v_\star\,,\, \widetilde{w}\rangle_{\Gamma_\star} + \langle \mathcal{V}_\star v_\star\,,\, \widetilde{w}\rangle_{\Gamma_\star} - \langle \mathcal{V}v_\star^\Gamma\,,\, w\rangle_\Gamma}{\|w\|_{H^{-1/2}(\Gamma)}}\\ &\lesssim \sup_{\widetilde{w}\in H^{-1/2}(\Gamma_\star)} \frac{\langle f_\star - \mathcal{V}_\star v_\star\,,\, \widetilde{w}\rangle_{\Gamma_\star}}{\|\widetilde{w}\|_{H^{-1/2}(\Gamma_\star)}} + \sup_{w\in H^{-1/2}(\Gamma)} \frac{\langle \mathcal{V}_\star v_\star\,,\, \widetilde{w}\rangle_{\Gamma_\star} - \langle \mathcal{V}v_\star^\Gamma\,,\, w\rangle_\Gamma}{\|w\|_{H^{-1/2}(\Gamma)}},\end{aligned} \tag{6.2.49}$$

where we used Lemma 6.2.19 to get $\|w\|_{H^{-1/2}(\Gamma)} \simeq \|\widetilde{w}\|_{H^{-1/2}(\Gamma_\star)}$. The numerator of the last term in (6.2.49) transforms to

$$
\begin{aligned}
-4\pi\big(\langle \mathcal{V}_\star v_\star\,,\, w\circ\gamma_\star^{-1}|\partial_\Gamma\gamma_\star^{-1}|\rangle_{\Gamma_\star} - \langle \mathcal{V} v_\star^\Gamma\,,\, w\rangle_\Gamma\big) &= -4\pi\big(\langle (\mathcal{V}_\star v_\star)\circ\gamma_\star - \mathcal{V} v_\star^\Gamma\,,\, w\rangle_\Gamma\big)\\
&= \int_\Gamma \Big(\int_{\Gamma_\star} \log\big(|\gamma_\star(x)-y|^2\big) v_\star(y)\,dy - \int_\Gamma \log\big(|x-y|^2\big) v_\star^\Gamma(y)\,dy\Big)\, w(x)\,dx\\
&= \int_\Gamma \Big(\int_{\Gamma} \log\big(|\gamma_\star(x)-\gamma_\star(y)|^2\big) v_\star^\Gamma(y)\,dy - \int_\Gamma \log\big(|x-y|^2\big) v_\star^\Gamma(y)\,dy\Big)\, w(x)\,dx\\
&= -\int_\Gamma\int_\Gamma \log\Big(\frac{|x-y|^2}{|\gamma_\star(x)-\gamma_\star(y)|^2}\Big) v_\star^\Gamma(y)\,dy\, w(x)\,dx = -\langle M_\star v_\star^\Gamma\,,\, w\rangle_\Gamma.
\end{aligned}
\tag{6.2.50}
$$

This concludes the proof. $\square$

The following result can also be found in **[34, 28]**. We refine the proof to ensure that the involved constants behave uniformly with respect to the approximate geometries $\Gamma_\star$.

Lemma 6.2.30. *Given the approximate geometry $\mathcal{T}_\star \in \mathbb{T}$, there holds*

$$
\sup_{w\in H^{-1/2}(\Gamma_\star)} \frac{\langle f_\star - \mathcal{V}_\star U(\mathcal{T}_\star)\,,\, w\rangle_{\Gamma_\star}}{\|w\|_{H^{-1/2}(\Gamma_\star)}} \le \sqrt{8} K(\mathcal{T}_\star)^{1/2} (5K(\mathcal{T}_\star)^2+3)^{1/4}\, \rho(\mathcal{T}_\star)
$$

with $K(\mathcal{T}_\star)$ from Section 3.2.2.

Proof. Let $\xi_1,\ldots,\xi_N \in C(\Gamma)$ denote a $\mathcal{T}_\star$-piecewise smooth partition of unity on $\Gamma_\star$ such that all $j = 1,\ldots,N$ satisfy

$$
\begin{aligned}
\|\xi_j\|_{L^\infty(\Gamma_\star)} &\le 1,\\
\mathrm{supp}(\xi_j) &\subseteq T_{j,1}\cup T_{j,2} \quad \text{for some } T_{j,1}, T_{j,2} \in \mathcal{T}_\star \text{with } T_{j,1}\cap T_{j,2} \neq \emptyset,\\
\|\partial_{\Gamma_\star}\xi_j\|_{L^\infty(T_{j,i})} &\le 2h_\star|_{T_{j,i}}^{-1} \quad \text{for } i = 1,2.
\end{aligned}
$$

There holds

$$
\begin{aligned}
\sup_{w\in H^{-1/2}(\Gamma_\star)} \frac{\langle f_\star - \mathcal{V}_\star U(\mathcal{T}_\star)\,,\, w\rangle_{\Gamma_\star}}{\|w\|_{H^{-1/2}(\Gamma_\star)}} &= \|f_\star - \mathcal{V}_\star U(\mathcal{T}_\star)\|_{H^{1/2}(\Gamma_\star)}\\
&= \|\sum_{j=1}^N \xi_j (f_\star - \mathcal{V}_\star U(\mathcal{T}_\star))\|_{H^{1/2}(\Gamma_\star)}.
\end{aligned}
$$

Let $\mathcal{K}^1\cup\mathcal{K}^2 = \{1,\ldots,N\}$ such that $|\mathrm{supp}(\xi_j)\cap\mathrm{supp}(\xi_k)| = 0$ for all $j\neq k$, $j,k\in\mathcal{K}^1$ and all $j\neq k$, $j,k\in\mathcal{K}^2$. Lemma 6.2.25 with $\widetilde{\Gamma} = \Gamma_\star$ shows

$$
\begin{aligned}
\|\sum_{j=1}^N \xi_j (f_\star - \mathcal{V}_\star U(\mathcal{T}_\star))\|_{H^{1/2}(\Gamma_\star)}^2 &\le 2\|\sum_{j\in\mathcal{K}^1} \xi_j (f_\star - \mathcal{V}_\star U(\mathcal{T}_\star))\|_{H^{1/2}(\Gamma_\star)}^2\\
&\quad + 2\|\sum_{j\in\mathcal{K}^2} \xi_j (f_\star - \mathcal{V}_\star U(\mathcal{T}_\star))\|_{H^{1/2}(\Gamma_\star)}^2\\
&\le 4\sum_{j=1}^N \|\xi_j (f_\star - \mathcal{V}_\star U(\mathcal{T}_\star))\|_{H^{1/2}(\mathrm{supp}(\xi_j))}^2.
\end{aligned}
$$

Real interpolation theory shows

$$\begin{aligned}\|\xi_j(f_\star - \mathcal{V}_\star U(\mathcal{T}_\star))\|^2_{H^{1/2}(\mathrm{supp}(\xi_j))}\\ &\lesssim \|\xi_j(f_\star - \mathcal{V}_\star U(\mathcal{T}_\star))\|_{L^2(\mathrm{supp}(\xi_j))}\|\xi_j(f_\star - \mathcal{V}_\star U(\mathcal{T}_\star))\|_{H^1(\mathrm{supp}(\xi_j))},\end{aligned}$$

where the hidden constant depends only on the scalar field of the involved Hilbert spaces, which is, in our case, $\mathbb{R}$. Hence, with $v_j := \xi_j(f_\star - \mathcal{V}_\star U(\mathcal{T}_\star))$, there holds

$$\|f_\star - \mathcal{V}_\star U(\mathcal{T}_\star)\|^2_{H^{1/2}(\Gamma)} \leq 4\sum_{j=1}^N \|v_j\|_{L^2(\mathrm{supp}(\xi_j))}\big(\|v_j\|^2_{L^2(\mathrm{supp}(\xi_j))} + \|\partial_{\Gamma_\star} v_j\|^2_{L^2(\mathrm{supp}(\xi_j))}\big)^{1/2}.$$

Elementary calculus and the definition of the ξ_j show

$$\begin{aligned}\|v_j\|_{L^2(\mathrm{supp}(\xi_j))} &\leq \|f_\star - \mathcal{V}_\star U(\mathcal{T}_\star)\|_{L^2(\mathrm{supp}(\xi_j))},\\ \|\partial_\Gamma v_j\|_{L^2(\mathrm{supp}(\xi_j))} &\leq 2\max_{i=1,2} h_\star|^{-1}_{T_{j,i}}\|f_\star - \mathcal{V}_\star U(\mathcal{T}_\star)\|_{L^2(\mathrm{supp}(\xi_j))}\\ &\qquad + \|\partial_\Gamma(f_\star - \mathcal{V}_\star U(\mathcal{T}_\star))\|_{L^2(\mathrm{supp}(\xi_j))}.\end{aligned}$$

Since $U(\mathcal{T}_\star)$ solves (6.2.2) and $f_\star - \mathcal{V}_\star U(\mathcal{T}_\star) \in H^1(\Gamma_\star)$, there exists at least one zero $z_T \in \Gamma_\star$ with $(f_\star - \mathcal{V}_\star U(\mathcal{T}_\star))(z_T) = 0$ for all $T \in \mathcal{T}_\star$. Hence, Friedrich's inequality proves

$$\|f_\star - \mathcal{V}_\star U(\mathcal{T}_\star)\|_{L^2(\mathrm{supp}(\xi_j))} \leq \max_{i=1,2} h_\star|_{T_{i,j}}\|\partial_{\Gamma_\star}(f_\star - \mathcal{V}_\star U(\mathcal{T}_\star))\|_{L^2(\mathrm{supp}(\xi_j))}.$$

The above together with the K-mesh property show

$$\begin{aligned}&\|f_\star - \mathcal{V}_\star U(\mathcal{T}_\star)\|^2_{H^{1/2}(\Gamma)}\\ &\leq 4\sum_{j=1}^N \|v_j\|_{L^2(\mathrm{supp}(\xi_j))}\big(\|v_j\|^2_{L^2(\mathrm{supp}(\xi_j))} + \|\partial_{\Gamma_\star} v_j\|^2_{L^2(\mathrm{supp}(\xi_j))}\big)^{1/2}\\ &\leq 4\sum_{j=1}^N \Big(\|f_\star - \mathcal{V}_\star U(\mathcal{T}_\star)\|_{L^2(\mathrm{supp}(\xi_j))}\\ &\quad\big(5\max_{i=1,2} h_\star|^{-2}_{T_{j,i}}\|f_\star - \mathcal{V}_\star U(\mathcal{T}_\star)\|^2_{L^2(\mathrm{supp}(\xi_j))} + 3\|\partial_{\Gamma_\star}(f_\star - \mathcal{V}_\star U(\mathcal{T}_\star))\|^2_{L^2(\mathrm{supp}(\xi_j))}\big)^{1/2}\Big)\\ &\leq 4\sum_{j=1}^N \Big(K(\mathcal{T}_\star)\|h_\star\partial_{\Gamma_\star}(f_\star - \mathcal{V}_\star U(\mathcal{T}_\star))\|_{L^2(\mathrm{supp}(\xi_j))}\\ &\quad(5K(\mathcal{T}_\star)^2 + 3)^{1/2}\|\partial_{\Gamma_\star}(f_\star - \mathcal{V}_\star U(\mathcal{T}_\star))\|_{L^2(\mathrm{supp}(\xi_j))}\Big)\\ &\leq 4K(\mathcal{T}_\star)(5K(\mathcal{T}_\star)^2 + 3)^{1/2}\sum_{j=1}^N \|h_\star^{1/2}\partial_\Gamma(f_\star - \mathcal{V}_\star U(\mathcal{T}_\star))\|^2_{L^2(\mathrm{supp}(\xi_j))}\\ &\leq 8K(\mathcal{T}_\star)(5K(\mathcal{T}_\star)^2 + 3)^{1/2}\|h_\star^{1/2}\partial_\Gamma(f_\star - \mathcal{V}_\star U(\mathcal{T}_\star))\|^2_{L^2(\Gamma_\star)}.\end{aligned}$$

This concludes the proof. □

Lemma 6.2.31. *Let the approximate geometry $\mathcal{T}_\star \in \mathbb{T}$ satisfy* (Γ1)–(Γ3). *Then, there exists $C_{\mathrm{L}^2} > 0$ such that all $g \in L^2(\Gamma)$ satisfy*

$$\|M_\star g\|_{L^2(\Gamma)} \leq C_{\mathrm{L}^2}\mathrm{geo}(\mathcal{T}_\star)^2(1 + |\log(\mathrm{geo}(\mathcal{T}_\star))|)\|g\|_{L^2(\Gamma)},$$

where $M_\star$ is defined in (6.2.48).

PROOF. By definition of $M_\star$, there holds

$$\begin{aligned}\|M_\star g\|_{L^2(\Gamma)}^2 &= \int_\Gamma \Big(\int_\Gamma \log\Big(\frac{|x-y|^2}{|\gamma_\star(x)-\gamma_\star(y)|^2}\Big) g(y)\,dy\Big)^2\,dx \\ &\le \int_\Gamma\int_\Gamma \log\Big(\frac{|x-y|^2}{|\gamma_\star(x)-\gamma_\star(y)|^2}\Big)^2\,dy\,dx \|g\|_{L^2(\Gamma)}^2.\end{aligned}$$

The remaining integral is split. Let $\Gamma_1,\dots,\Gamma_N$ denote the smooth and connected parts of Γ. There holds

$$\int_\Gamma\int_\Gamma \log\Big(\frac{|x-y|^2}{|\gamma_\star(x)-\gamma_\star(y)|^2}\Big)^2\,dy\,dx = \sum_{i=1}^N\sum_{j=1}^N \int_{\Gamma_i}\int_{\Gamma_j} \log\Big(\frac{|x-y|^2}{|\gamma_\star(x)-\gamma_\star(y)|^2}\Big)^2\,dy\,dx.$$

Case $\Gamma_i=\Gamma_j$: Lemma 6.2.13 (i) implies

$$\int_{\Gamma_i}\int_{\Gamma_i} \log\Big(\frac{|x-y|^2}{|\gamma_\star(x)-\gamma_\star(y)|^2}\Big)^2\,dy\,dx \lesssim |\Gamma|^2\mathrm{geo}(\mathcal{T}_\star)^4.$$

Case $\Gamma_i\cap\Gamma_j=\emptyset$: Lemma 6.2.13 (iii) implies

$$\begin{aligned}\int_{\Gamma_i}\int_{\Gamma_i} \log\Big(\frac{|x-y|^2}{|\gamma_\star(x)-\gamma_\star(y)|^2}\Big)^2\,dy\,dx &\lesssim |\Gamma|^2 \min_{\substack{x\in\Gamma_i\\ y\in\Gamma_j}} |x-y|^{-2}\mathrm{geo}(\mathcal{T}_\star)^4 \\ &\lesssim \mathrm{geo}(\mathcal{T}_\star)^4.\end{aligned}$$

Case $\Gamma_i\cap\Gamma_j=\{z\}\subseteq\mathcal{P}_\Gamma$: Given $\varepsilon>0$, define $B_\varepsilon := \big\{y\in\Gamma\,:\,|y-z|<\varepsilon\big\}$. There holds

$$\begin{aligned}&\int_{\Gamma_i}\int_{\Gamma_j} \log\Big(\frac{|x-y|^2}{|\gamma_\star(x)-\gamma_\star(y)|^2}\Big)^2\,dy\,dx \qquad (6.2.51)\\ &\quad= \int_{\Gamma_i\setminus B_\varepsilon}\int_{\Gamma_j} \log\Big(\frac{|x-y|^2}{|\gamma_\star(x)-\gamma_\star(y)|^2}\Big)^2\,dy\,dx + \int_{B_\varepsilon}\int_{\Gamma_j} \log\Big(\frac{|x-y|^2}{|\gamma_\star(x)-\gamma_\star(y)|^2}\Big)^2\,dy\,dx.\end{aligned}$$

For the first term, Lemma 6.2.13 (ii) and $|x-z|\le\Gamma_x^z\le\Gamma_x^y\lesssim|x-y|$ for all $x\in\Gamma_i$, $y\in\Gamma_j$ imply

$$\begin{aligned}&\int_{\Gamma_i\setminus B_\varepsilon}\int_{\Gamma_j} \log\Big(\frac{|x-y|^2}{|\gamma_\star(x)-\gamma_\star(y)|^2}\Big)^2\,dy\,dx \\ &\qquad\lesssim \mathrm{geo}(\mathcal{T}_\star)^4 \int_{\Gamma_i\setminus B_\varepsilon}\int_{\Gamma_j} \big(1+\frac{|z-x|+|z-y|}{|x-y|^2}\big)^2\,dx\,dy \\ &\qquad\lesssim \mathrm{geo}(\mathcal{T}_\star)^4 \int_{\Gamma_i\setminus B_\varepsilon}\int_{\Gamma_j} 1+|x-y|^{-2}\,dx\,dy.\end{aligned}$$

Without loss of generality, there holds $[a, b-\delta] = \gamma^{-1}(\Gamma_i \setminus B_\varepsilon)$ and $[b, c] = \gamma^{-1}(\Gamma_j)$ for some $a < b < c \in [0, 1]$ and $0 < \delta \simeq \varepsilon$. The Lipschitz continuity of γ shows

$$\begin{aligned}\int_{\Gamma_i \setminus B_\varepsilon} \int_{\Gamma_j} |x-y|^{-2} \, dx \, dy &= \int_{\gamma^{-1}(\Gamma_i \setminus B_\varepsilon)} \int_{\gamma^{-1}(\Gamma_j)} |\gamma(s) - \gamma(t)|^{-2} |\gamma'|^2 \, ds \, dt \\ &\lesssim \int_{\gamma^{-1}(\Gamma_i \setminus B_\varepsilon)} \int_{\gamma^{-1}(\Gamma_j)} |s-t|^{-2} \, ds \, dt \\ &= \int_a^{b-\delta} \int_b^c (s-t)^{-2} \, ds \, dt \\ &= \int_a^{b-\delta} (b-t)^{-1} - (c-t)^{-1} \, dt \\ &\lesssim 1 + |\log(\delta)| \simeq 1 + |\log(\varepsilon)|.\end{aligned}$$

For the second term of (6.2.51), Lemma 6.2.13 (ii) shows

$$\int_{B_\varepsilon} \int_{\Gamma_j} \log\Big(\frac{|x-y|^2}{|\gamma_\star(x) - \gamma_\star(y)|^2}\Big)^2 \, dy \, dx \lesssim \mathrm{geo}(\mathcal{T}_\star)^2 |B_\varepsilon| |\Gamma| \lesssim \varepsilon \mathrm{geo}(\mathcal{T}_\star)^2.$$

Altogether, this proves

$$\begin{aligned}\int_\Gamma \int_\Gamma \log\Big(\frac{|x-y|^2}{|\gamma_\star(x) - \gamma_\star(y)|^2}\Big)^2 \, dy \, dx &= \sum_{i=1}^N \sum_{j=1}^N \int_{\Gamma_i} \int_{\Gamma_j} \log\Big(\frac{|x-y|^2}{|\gamma_\star(x) - \gamma_\star(y)|^2}\Big)^2 \, dy \, dx \\ &\lesssim N^2 (\mathrm{geo}(\mathcal{T}_\star)^4 + \mathrm{geo}(\mathcal{T}_\star)^4 |\log(\varepsilon)| + \mathrm{geo}(\mathcal{T}_\star)^2 \varepsilon).\end{aligned}$$

Since N depends only on Γ, the choice $\varepsilon := \mathrm{geo}(\mathcal{T}_\star)^2$ concludes the proof. $\square$

Lemma 6.2.32. *Let the approximate geometry $\mathcal{T}_\star \in \mathbb{T}$ satisfy* (Γ1)–(Γ3) *and* $\mathrm{geo}(\mathcal{T}_\star) \leq C_{\mathrm{ext}}^{-1}/2$. *Given $g \in L^\infty(\Gamma)$ and with $M_\star$ from 6.2.48, there holds $M_\star g \in H^1(\Omega)$, whereas $g \in L^2(\Gamma)$ implies $M_\star g \in H^1(\Gamma)$.*

Proof. Given $x \in \Omega$, $\kappa_\star(x, y)$ is smooth and hence (6.2.42) shows

$$\begin{aligned}|\nabla_x M_\star g(x)| := \Big|\int_\Gamma \nabla_x \kappa_\star(x, y) g(y) \, dy\Big| &\lesssim \|g\|_{L^\infty(\Gamma)} \int_\Gamma |x-y|^{-1} \, dy \\ &\lesssim \|g\|_{L^\infty(\Gamma)} (1 + |\log(\mathrm{dist}(x, \Gamma))|),\end{aligned}$$

where the hidden constants depend only on C_γ and an upper bound of $\mathrm{geo}(\mathcal{T}_\star)$. This proves that $\nabla_x M_\star g(x) \in L^2(\Omega)$. Lemma 6.2.31 concludes $M_\star g \in H^1(\Omega)$. There holds

$$M_\star g(x) = \mathcal{V} g(x) - \int_\Gamma \log|\gamma_\star(x) - \gamma_\star(y)| g(y) \, dy = \mathcal{V} g(x) - \big(\mathcal{V}_\star(g \circ \gamma_\star^{-1} |\partial \gamma_\star^{-1}|)\big) \circ \gamma_\star(x).$$

Since $g \in L^2(\Gamma)$ and $g \circ \gamma_\star^{-1} |\partial \gamma_\star^{-1}| \in L^2(\Gamma_\star)$, the mapping properties of $\mathcal{V}$ and $\mathcal{V}_\star$ show $\mathcal{V} g \in H^1(\Gamma), \mathcal{V}_\star(g \circ \gamma_\star^{-1} |\partial \gamma_\star^{-1}|) \in H^1(\Gamma_\star)$. Since $\gamma_\star$ is continuous and piecewise smooth, this concludes the proof. $\square$

Lemma 6.2.33. *Let the approximate geometry $\mathcal{T}_\star \in \mathbb{T}$ satisfy* (Γ1)–(Γ3) *and* $\mathrm{geo}(\mathcal{T}_\star) \leq C_{\mathrm{ext}}^{-1}/2$. *Then, there exists a constant $C_{\widetilde{\mathrm{H}}^{1/2}} > 0$, such that all $g \in L^2(\Gamma)$ with* $\mathrm{supp}(g) \subseteq \Gamma_x^y$ *for some $x, y \in \Gamma$ satisfy*

$$\begin{aligned}\|M_\star g\|_{H^{1/2}(\Gamma)} \leq C_{\widetilde{\mathrm{H}}^{1/2}} \big(&\mathrm{geo}(\mathcal{T}_\star) |\Gamma_x^y|^{1/2} (1 + |\log(|\Gamma_x^y|)|)^{1/2} \\ &+ \mathrm{geo}(\mathcal{T}_\star)^2 (1 + |\log(\mathrm{geo}(\mathcal{T}_\star))|)\big) \|g\|_{L^2(\Gamma)},\end{aligned}$$

where $M_\star$ is defined in (6.2.48). *The constant $C_{\widetilde{\mathrm{H}}^{1/2}}$ depends only on C_{L^2}, C_{Lip}, C_{ext}, and on Ω.*

PROOF. Define the volume potential

$$Dv(x) := \int_\Omega \nabla_y \kappa_\star(x,y)v(y)\,dy.$$

Assume for the moment $g \in L^\infty(\Gamma)$. Lemma 6.2.32 shows $M_\star g \in H^1(\Omega)$. Given $v \in L^2(\Omega)$, there holds

$$\begin{aligned}\langle \nabla M_\star g\,,\, v\rangle_\Omega &= \int_\Omega \int_\Gamma \nabla_y \kappa_\star(x,y) g(x)\,dx\, v(y)\,dy \\ &= \int_\Gamma \int_\Omega \nabla_y \kappa_\star(x,y) v(y)\,dy\, g(x)\,dx = \langle g\,,\, Dv\rangle_\Gamma.\end{aligned} \tag{6.2.52}$$

Consider the simple-layer potential $\mathcal{V}_\Omega : \widetilde{H}^{-1/2}(\Omega) \to H^{1/2}(\Omega)$ on the 2D manifold Ω

$$\mathcal{V}_\Omega g(x) := \frac{1}{4\pi}\int_\Omega |x-y|^{-1} g(y)\,dy \quad \text{for all } x \in \mathbb{R}^3.$$

The identity (6.2.52) together with (6.2.42), shows

$$\begin{aligned}\langle g\,,\, D(v)\rangle_\Gamma &\lesssim \int_\Gamma |g(x)| \Big| \int_\Omega \nabla_y \kappa_\star(x,y) v(y)\,dy \Big|\,dx \\ &\lesssim \|t_\Gamma - \partial_\Gamma \gamma_\ell\|_{L^\infty(\Gamma)} \int_\Gamma |g(x)| \Big| \int_\Omega \frac{1}{|x-y|} |v(y)|\,dy \Big|\,dx \\ &\simeq \|t_\Gamma - \partial_\Gamma \gamma_\ell\|_{L^\infty(\Gamma)} \langle |g|\,,\, \mathcal{V}_\Omega(|v|)\rangle_\Gamma.\end{aligned}$$

With $|\Gamma_x^y| = h$, Lemma 6.2.15 shows $\||g|\|_{H^{-1/2}(\Gamma)} \lesssim h^{1/2}(1+|\log(h)|)^{1/2}\|g\|_{L^2(\Gamma)}$. This and the continuity $\mathcal{V}_\Omega : L^2(\Omega) \to H^1(\Omega)$ show

$$\begin{aligned}\sup_{v\in L^2(\Omega)} \langle g\,,\, D(v)\rangle_\Gamma &\lesssim \|t_\Gamma - \partial_\Gamma \gamma_\ell\|_{L^\infty(\Gamma)} \||g|\|_{H^{-1/2}(\Gamma)} \|\mathcal{V}_\Omega(|v|)\|_{H^{1/2}(\Gamma)} \\ &\lesssim \|t_\Gamma - \partial_\Gamma \gamma_\ell\|_{L^\infty(\Gamma)} h^{1/2}(1+|\log(h)|^{1/2})\|g\|_{L^2(\Gamma)} \|\mathcal{V}_\Omega(|v|)\|_{H^1(\Omega)} \\ &\lesssim \mathrm{geo}(\mathcal{T}_\star) h^{1/2}(1+|\log(h)|^{1/2})\|g\|_{L^2(\Gamma)}\|v\|_{L^2(\Omega)}.\end{aligned}$$

Altogether, this proves

$$\|\nabla M_\star g\|_{L^2(\Omega)} = \sup_{v\in L^2(\Omega)} \frac{\langle \nabla M_\star g\,,\, v\rangle_\Omega}{\|v\|_{L^2(\Omega)}} \lesssim \mathrm{geo}(\mathcal{T}_\star) h^{1/2}(1+|\log(h)|^{1/2})\|g\|_{L^2(\Gamma)}.$$

Continuous extension shows that the restriction $g \in L^\infty(\Gamma)$ is not necessary.

Let $M := |\Gamma|^{-1}\int_\Gamma M_\star g(x)\,dx$ denote the integral mean. Rellich's compactness theorem proves $\|M_\star g - M\|_{H^1(\Omega)} \lesssim \|\nabla M_\star g\|_{L^2(\Omega)}$. Altogether, this shows

$$\begin{aligned}\|Mg\|_{H^{1/2}(\Gamma)} &\le \|M\|_{H^{1/2}(\Gamma)} + \|M_\star g - M\|_{H^{1/2}(\Gamma)} \\ &\lesssim \|M\|_{L^2(\Gamma)} + \|\nabla M_\star g\|_{L^2(\Omega)} \\ &\lesssim \|M_\star g\|_{L^2(\Gamma)} + h^{1/2}|\log(h)|^{1/2}\mathrm{geo}(\mathcal{T}_\star)\|g\|_{L^2(\Gamma)}.\end{aligned} \tag{6.2.53}$$

Lemma 6.2.31 and (6.2.53) conclude the proof. □

Lemma 6.2.34. *Let the approximate geometry* $\mathcal{T}_\star \in \mathbb{T}$ *satisfy* (Γ1)–(Γ3) *and* $\operatorname{geo}(\mathcal{T}_\star) \le C_{\rm ext}^{-1}/2$. *There exists a constant* $C_{\rm H^{1/2}} > 0$ *such that all* $g \in L^2(\Gamma)$ *satisfy*

$$\|M_\star g\|_{H^{1/2}(\Gamma)} \le C_{H^{1/2}} \operatorname{geo}(\mathcal{T}_\star)^{3/2}(1 + |\log(\operatorname{geo}(\mathcal{T}_\star))|)\|g\|_{L^2(\Gamma)},$$

where $M_\star$ *is defined in* (6.2.48). *The constant* $C_{\rm H^{1/2}}$ *depends only on* $C_{\widetilde{\rm H}^{1/2}}$, $C_{\rm L^2}$, $C_{\rm faer}$, C_{28}, $C_{\rm ext}$, $C_{\rm Lip}$, C_Σ, *and on* Ω,

Proof. Construct a uniform partition $\mathcal{U}$ of Γ with element size $h(\mathcal{U}) \simeq \operatorname{geo}(\mathcal{T}_\star)$. With the Scott-Zhang projection $J(\mathcal{U}) : L^2(\Gamma) \to \mathcal{S}^1(\mathcal{U})$ from Definition 3.3.2, split

$$\begin{aligned}\|M_\star g\|_{H^{1/2}(\Gamma)} &\le \|J(\mathcal{U})M_\star g\|_{H^{1/2}(\Gamma)} + \|(1 - J(\mathcal{U}))M_\star g\|_{H^{1/2}(\Gamma)}\\ &\lesssim h(\mathcal{U})^{-1/2}\|M_\star g\|_{L^2(\Gamma)} + \|(1 - J(\mathcal{U}))M_\star g\|_{H^{1/2}(\Gamma)},\end{aligned}$$

where we applied the inverse estimate from **[57]**. The first term on the right-hand side is considered in Lemma 6.2.31. Lemma 6.2.26 applies for the second term to obtain

$$\|(1 - J(\mathcal{U}))M_\star g\|^2_{H^{1/2}(\Gamma)} \lesssim \sum_{U\in\mathcal{U}} \|(1 - J(\mathcal{U}))M_\star g\|^2_{H^{1/2}(\cup\omega^2(U,\mathcal{U}))}.$$

With $g_{U,1} := g|_{\cup\omega^4(U,\mathcal{U})}$ and $g_{U,2} := g - g_{U,1}$ and by use of the approximation properties (3.3.2) of $J(\mathcal{U})$, each term on the right-hand side is bounded by

$$\begin{aligned}\|(1-&J(\mathcal{U}))M_\star g\|^2_{H^{1/2}(\cup\omega^2(U,\mathcal{U}))}\\ &\lesssim \|M_\star g_{U,1}\|^2_{H^{1/2}(\cup\omega^3(U,\mathcal{U}))} + \|(1 - J(\mathcal{U}))M_\star g_{U,2}\|^2_{H^{1/2}(\cup\omega^2(U,\mathcal{U}))}\\ &\lesssim \|M_\star g_{U,1}\|^2_{H^{1/2}(\cup\omega^3(U,\mathcal{U}))}\\ &\qquad + \|(1 - J(\mathcal{U}))M_\star g_{U,2}\|_{L^2(\cup\omega^2(U,\mathcal{U}))}\|\partial_\Gamma(1 - J(\mathcal{U}))M_\star g_{U,2}\|_{L^2(\cup\omega^2(U,\mathcal{U}))}\\ &\lesssim \|M_\star g_{U,1}\|^2_{H^{1/2}(\cup\omega^3(U,\mathcal{U}))} + h(\mathcal{U})\|\partial_\Gamma M_\star g_{U,2}\|^2_{L^2(\cup\omega^3(U,\mathcal{U}))},\end{aligned} \tag{6.2.54}$$

where Lemma 6.2.32 shows that the right-hand side is well-defined. Since $|\operatorname{supp}(g_{U,1})| \simeq h(\mathcal{U})$, Lemma 6.2.33 applies for the first term and, with $h(\mathcal{U}) \simeq \operatorname{geo}(\mathcal{T}_\star)$, leads to

$$\begin{aligned}\sum_{U\in\mathcal{U}} \|M_\star g_{U,1}\|^2_{H^{1/2}(\cup\omega^3(U,\mathcal{U}))} &\lesssim \operatorname{geo}(\mathcal{T}_\star)^3(1 + |\log(\operatorname{geo}(\mathcal{T}_\star))|) \sum_{U\in\mathcal{U}} \|g_{U,1}\|^2_{L^2(\Gamma)}\\ &\lesssim \operatorname{geo}(\mathcal{T}_\star)^3(1 + |\log(\operatorname{geo}(\mathcal{T}_\star))|)\|g\|^2_{L^2(\Gamma)}.\end{aligned}$$

Given $U \in \mathcal{U}$, an explicit computation together with Lemma 6.2.24 shows

$$\begin{aligned}\int_{\cup\omega^3(U,\mathcal{U})} &\Big(\int_{\Gamma\setminus\cup\omega^4(U,\mathcal{U})} \partial_{\Gamma,x}\kappa_\star(x,y)g(y)\,dy\Big)^2\,dx\\ &\lesssim \operatorname{geo}(\mathcal{T}_\star)^2 \int_{\cup\omega^3(U,\mathcal{U})} \Big(\int_{\Gamma\setminus\cup\omega^4(U,\mathcal{U})} |x-y|^{-1}|g|(y)\,dy\Big)^2\,dx\\ &\le \operatorname{geo}(\mathcal{T}_\star)^2|\cup\omega^3(U,\mathcal{U})|\\ &\qquad \sup_{x\in\cup\omega^3(U,\mathcal{U})} \Big(\||x-\cdot|^{-1/2}\|^2_{L^2(\Gamma\setminus\cup\omega^4(U,\mathcal{U}))}\||x-\cdot|^{-1/2}g(\cdot)\|^2_{L^2(\Gamma\setminus\cup\omega^4(U,\mathcal{U}))}\Big).\end{aligned} \tag{6.2.55}$$

A computation in the parameter domain shows for $x \in \bigcup\omega^3_U$

$$\begin{aligned}\||x-\cdot|^{-1/2}\|^2_{L^2(\Gamma\setminus\cup\omega^4(U,\mathcal{U}))} &\lesssim \int_{\gamma^{-1}(\Gamma\setminus\cup\omega^4(U,\mathcal{U}))} |\gamma^{-1}(x) - t|^{-1}\,dt\\ &\lesssim (1 + |\log(h(\mathcal{U}))|),\end{aligned} \tag{6.2.56}$$

since $|\gamma^{-1}(x) - t| \gtrsim |x - \gamma(t)| \gtrsim h(\mathcal{U})$. With (6.2.13), there holds for all $U' \in \mathcal{U}$ with $U' \not\subseteq \omega^4(U,\mathcal{U})$

$$\mathrm{dist}(U,U') \lesssim \mathrm{dist}(\cup\omega^3(U,\mathcal{U}),U') + 2h(\mathcal{U}) \leq 3\mathrm{dist}(\cup\omega^3(U,\mathcal{U}),U')$$

and hence

$$\begin{aligned}\sup_{x\in\cup\omega^3(U,\mathcal{U})} \||x-\cdot|^{-1/2}g(\cdot)\|^2_{L^2(\Gamma\setminus\cup\omega^4(U,\mathcal{U}))} &= \sup_{x\in\cup\omega^3(U,\mathcal{U})} \sum_{U'\in\mathcal{U}\setminus\omega^4(U,\mathcal{U})} \int_{U'} |x-y|^{-1} g(y)^2\,dy \\ &\lesssim \sum_{U'\in\mathcal{U}\setminus\omega^4(U,\mathcal{U})} \frac{1}{\mathrm{dist}(U,U')}\|g\|^2_{L^2(U')}.\end{aligned}$$

Plugging the last two estimates into (6.2.55), we end up with

$$\begin{aligned}\|\partial_\Gamma M_\star g_{U,2}\|^2_{L^2(\cup\omega^3(U,\mathcal{U}))} &= \int_{\cup\omega^3(U,\mathcal{U})} \Big(\int_{\Gamma\setminus\cup\omega^4(U,\mathcal{U})} \partial_{\Gamma,x}\kappa_\star(x,y)g(y)\,dy\Big)^2\,dx \\ &\lesssim \mathrm{geo}(\mathcal{T}_\star)^2|\cup\omega^3(U,\mathcal{U})|(1+|\log(h(\mathcal{U}))|) \sum_{U'\in\mathcal{U}\setminus\omega^4(U,\mathcal{U})} \frac{1}{\mathrm{dist}(U,U')}\|g\|^2_{L^2(U')} \\ &\lesssim \mathrm{geo}(\mathcal{T}_\star)^2(1+|\log(\mathrm{geo}(\mathcal{T}_\star))|) \sum_{U'\in\mathcal{U}\setminus\omega^4(U,\mathcal{U})} \frac{|U|}{\mathrm{dist}(U,U')}\|g\|^2_{L^2(U')}.\end{aligned}$$

With the convention $\mathrm{dist}(U,U') = 1$ for $U\cap U' \neq \emptyset$ and $h(\mathcal{U}) \simeq \mathrm{geo}(\mathcal{T}_\star)$, this leads to

$$\begin{aligned}&\sum_{U\in\mathcal{U}} h(\mathcal{U})\|\partial_\Gamma M_\star g_{U,2}\|^2_{L^2(\cup\omega^3(U,\mathcal{U}))} \\ &\quad\lesssim (1+|\log(\mathrm{geo}(\mathcal{T}_\star))|)\mathrm{geo}(\mathcal{T}_\star)^3 \sum_{U\in\mathcal{U}} \sum_{U'\in\mathcal{U}\setminus\omega^4(U,\mathcal{U})} \|g\|^2_{L^2(U')} \frac{|U|}{\mathrm{dist}(U,U')} \\ &\quad\leq (1+|\log(\mathrm{geo}(\mathcal{T}_\star))|)\mathrm{geo}(\mathcal{T}_\star)^3 \sum_{U'\in\mathcal{U}} \|g\|^2_{L^2(U')} \sum_{U\in\mathcal{U}} \frac{|U|}{\mathrm{dist}(U,U')}.\end{aligned}$$

Lemma 6.2.27 provides an estimate for the last sum of the right-hand side. Altogether, this shows

$$\sum_{U\in\mathcal{U}} h(\mathcal{U})\|\partial_\Gamma M_\star g_{U,2}\|^2_{L^2(\cup\omega^3(U,\mathcal{U}))} \lesssim (1+|\log(\mathrm{geo}(\mathcal{T}_\star))|)\mathrm{geo}(\mathcal{T}_\star)^3\|g\|^2_{L^2(\Gamma)}(1+|\log(|\mathcal{U}|)|).$$

Since $|\mathcal{U}| \simeq |\Gamma|/h(\mathcal{U}) \simeq \mathrm{geo}(\mathcal{T}_\star)^{-1}$, the combination of the previous estimates concludes the proof. □

Proof of Theorem 6.2.28. Lemma 6.2.29–6.2.30, and Lemma 6.2.34 show the statement. □

6.3. Convergence

Throughout this section, we assume that Lemma 6.2.9 (i)–(iii) and $\mathrm{geo}(\mathcal{T}_\star) \leq C_{\mathrm{ext}}^{-1}/2$ hold for all approximate geometries $\mathcal{T}_\star \in \mathbb{T}$. Moreover, we assume that the exact boundary Γ satisfies the following: All approximate geometries $\mathcal{T}_\star \in \mathbb{T}$ and all elements $T \in \mathcal{T}_\star$ allow for a parametrization

$$\begin{aligned}
\gamma_T\colon [0,1] &\to T^\Gamma,\\
\gamma_T'(s) &\in \operatorname{span}\{t_\Gamma\circ\gamma_T(s)\}\quad\text{for all } s\in[0,1],\\
C_{\rm par}^{-1}|T^\Gamma| &\le |\gamma_T'| \le C_{\rm par}|T^\Gamma|,\\
\|\gamma_T''\|_{L^\infty([0,1])} + \|(\gamma_\star\circ\gamma_T)''\|_{L^\infty([0,1])} &\le C_{\rm par}|T^\Gamma|
\end{aligned}\tag{6.3.1a}$$

for some constant $C_{\rm par}>0$ which depends only on Γ. Moreover, there exists some $p\in\mathbb{N}$ such that for all $T\in\mathcal{T}_\star$ exist $\widetilde{\gamma}_T,\widetilde{\gamma}_{T,\star}\in\mathcal{P}^p([0,1])^2$ such that

$$\|\gamma_T-\widetilde{\gamma}_T\|_{W^{1,\infty}([0,1])} + \|\gamma_\star\circ\gamma_T-\widetilde{\gamma}_{T,\star}\|_{W^{1,\infty}([0,1])} \le C_{\rm par}\operatorname{geo}_T(\mathcal{T}_\star)^2. \tag{6.3.1b}$$

Remark 6.3.1. *The assumption* (6.3.1) *basically states that the Taylor expansion of the parametrization γ behaves nicely. Since $\gamma_\star$ is uniquely determined by γ,* (6.3.1b) *is an assumption on the Taylor expansion of γ, since $\inf_{\widetilde{\gamma}_T\in\mathcal{P}^p([0,1])}\|\gamma_T-\widetilde{\gamma}_T\|_{W^{1,\infty}([0,1])}\lesssim\|\gamma_T''\|_{L^\infty([0,1])}$ and $\operatorname{geo}_T(\mathcal{T}_\star)\gtrsim\min_{T^\Gamma}|\gamma''\circ\gamma^{-1}|$. Assumption* (6.3.1) *holds for example if Γ is parametrized by piecewise polynomials of arbitrary order, i.e., B-splines, or by NURBS.*

Lemma 6.3.2. *Under assumption 6.3.1 and with Lemma 6.2.9 (i)–(iii) as well as $\operatorname{geo}(\mathcal{T}_\star)\le C_{\rm ext}^{-1}/2$, there exists a constant $C_{\rm inv}>0$ such that the approximate geometry $\mathcal{T}_\star\in\mathbb{T}$ satisfies for all $T\in\mathcal{T}_\star$*

$$\|t_\Gamma-\partial_\Gamma\gamma_\star\|_{L^\infty(T^\Gamma)}\le C_{\rm inv}|T|^{-1}\operatorname{geo}_T(\mathcal{T}_\star)^2. \tag{6.3.2}$$

Given $x,y\in\Gamma$ with $x\in T_0^\Gamma$ for some $T_0\in\mathcal{T}_\star$, there holds additionally

$$|\partial_{\Gamma,x}\kappa_\star(x,y)|\le C_{\rm inv}\Big(\frac{|T_0|^{-1}}{|x-y|}+\frac{1}{|x-y|^2}\Big)\max_{\substack{T\in\mathcal{T}_\star\\ T^\Gamma\cap\Gamma_x^y\neq\emptyset}}\operatorname{geo}_T(\mathcal{T}_\star)^2 \tag{6.3.3}$$

as well as for $x,y\in\bigcup\omega(T_0^\Gamma,\mathcal{T}_\star^\Gamma)$

$$|\partial_{\Gamma,x}\kappa_\star(x,y)|\le C_{\rm inv}\frac{|T_0|^{-1/2}}{|x-y|}\max_{T\in\omega(T_0,\mathcal{T}_\star)}\operatorname{geo}_T(\mathcal{T}_\star)^{3/2}. \tag{6.3.4}$$

The constant $C_{\rm inv}$ depends only on $C_{\rm par}$, $K(\mathcal{T}_\star)$ (with $K(\cdot)$ from Section 3.2.2), and C_Γ.

Proof. Given $T\in\mathcal{T}_\star$, there holds with $(\gamma_T-\gamma_\star\circ\gamma_T)'=(t_\Gamma-\partial_\Gamma\gamma_\star)\circ\gamma_T\gamma_T'$ and (6.3.1a) that

$$\|t_\Gamma-\partial_\Gamma\gamma_\star\|_{L^\infty(T^\Gamma)}\simeq|T^\Gamma|^{-1}\|(\gamma_T-\gamma_\star\circ\gamma_T)'\|_{L^\infty([0,1])}. \tag{6.3.5}$$

Assumption (6.3.1b) and norm equivalence on $\mathcal{P}^p([0,1])$ imply

$$\begin{aligned}
\|(\gamma_T-\gamma_\star\circ\gamma_T)'\|_{L^\infty([0,1])} &\le \|(\widetilde{\gamma}_T-\widetilde{\gamma}_{T,\star})'\|_{L^\infty([0,1])}+\operatorname{geo}_T(\mathcal{T}_\star)^2\\
&\lesssim \|\widetilde{\gamma}_T-\widetilde{\gamma}_{T,\star}\|_{L^\infty([0,1])}+\operatorname{geo}_T(\mathcal{T}_\star)^2\\
&\lesssim \|\gamma_T-\gamma_\star\circ\gamma_T\|_{L^\infty([0,1])}+\operatorname{geo}_T(\mathcal{T}_\star)^2.
\end{aligned}$$

Finally, there holds

$$\|\gamma_T-\gamma_\star\circ\gamma_T\|_{L^\infty([0,1])}=\|\mathrm{id}_\Gamma-\gamma_\star\|_{L^\infty(T^\Gamma)}.$$

The combination of the last three estimates concludes the proof of (6.3.2). To see (6.3.3), combine (6.2.44) and (6.3.2). The estimate (6.3.4) follows from (6.3.2) and

$$\|t_\Gamma-\partial_\Gamma\gamma_\star\|_{L^\infty(T^\Gamma)}\lesssim|T|^{-1/2}\operatorname{geo}_T(\mathcal{T}_\star)\|t_\Gamma-\partial_\Gamma\gamma_\star\|_{L^\infty(T^\Gamma)}^{1/2}\le|T|^{-1/2}\operatorname{geo}_T(\mathcal{T}_\star)^{3/2}.$$

Together with the K-mesh property and (6.2.43), this implies (6.3.4) and concludes the proof. $\square$

Lemma 6.3.3. *Let assumption 6.3.1 hold and suppose Lemma 6.2.9 (i)–(iii) as well as* $\operatorname{geo}(\mathcal{T}_\star) \le C_{\mathrm{ext}}^{-1}/2$ *hold for* $\mathcal{T}_\star \in \mathbb{T}$*. Given* $T \in \mathcal{T}_\star$*, define*

$$g_T(s) := \int_0^1 \log\Big(\frac{|\gamma_T(s)-\gamma_T(t)|^2}{|\gamma_\star\circ\gamma_T(s)-\gamma_\star\circ\gamma_T(t)|^2}\Big)|\gamma_T'(t)||\partial_\Gamma\gamma_\star|\circ\gamma_T(t)\,dt.$$

There holds for all $\varepsilon > 0$

$$\|g_T'\|_{L^2([0,1])} \le C_{\mathrm{apx}}|T^\Gamma|(\varepsilon + (1+|\log(\varepsilon)|)\|t_\Gamma - \partial_\Gamma\gamma_\star\|_{L^\infty(T^\Gamma)}). \tag{6.3.6}$$

where the constant $C_{\mathrm{apx}} > 0$ *depends only on* C_{par}*,* C_{ext}*, and on* C_{Lip}*.*

Proof. Let $\kappa_T(s,t)$ denote the logarithmic kernel of g_T. Straightforward differentiation shows for $\gamma_{T,\star} := \gamma_\star\circ\gamma_T$

$$\begin{aligned}\frac{1}{2}&\partial_s\kappa_T(s,t)\\ &= \frac{(\gamma_T(s)-\gamma_T(t))\cdot\gamma_T'(s)|\gamma_{T,\star}(s)-\gamma_{T,\star}(t)|^2 - (\gamma_{T,\star}(s)-\gamma_{T,\star}(t))\cdot\gamma_{T,\star}'(s)|\gamma_T(s)-\gamma_T(t)|^2}{|\gamma_T(s)-\gamma_T(t)|^2|\gamma_{T,\star}(s)-\gamma_{T,\star}(t)|^2}.\end{aligned}$$

Taylor expansion shows for some $z_1, z_2, z_3, z_4 \in [0,1]$ and $s,t \in [0,1]$ that

$$\begin{aligned}|\partial_s&\kappa_T(s,t)||\gamma_T(s)-\gamma_T(t)|^2|\gamma_{T,\star}(s)-\gamma_{T,\star}(t)|^2/2\\ &= (s-t)|\gamma_T'(s)|^2|\gamma_{T,\star}(s)-\gamma_{T,\star}(t)|^2 + (s-t)^2\gamma_T''(z_1)\cdot\gamma_T'(s)|\gamma_{T,\star}(s)-\gamma_{T,\star}(t)|^2\\ &\qquad - (s-t)|\gamma_{T,\star}'(s)|^2|\gamma_T(s)-\gamma_T(t)|^2 - (s-t)^2\gamma_{T,\star}''(z_2)\cdot\gamma_{T,\star}'(s)|\gamma_T(s)-\gamma_T(t)|^2\\ &= (s-t)^3|\gamma_T'(s)|^2|\gamma_{T,\star}'(s)|^2 + (s-t)^5|\gamma_{T,\star}''(z_3)|^2|\gamma_T'(s)|^2\\ &\qquad - (s-t)^3|\gamma_T'(s)|^2|\gamma_{T,\star}'(s)|^2 + (s-t)^5|\gamma_T''(z_4)|^2|\gamma_{T,\star}'(s)|^2\\ &\qquad + (s-t)^2\gamma_T''(z_1)\cdot\gamma_T'(s)|\gamma_{T,\star}(s)-\gamma_{T,\star}(t)|^2\\ &\qquad - (s-t)^2\gamma_{T,\star}''(z_2)\cdot\gamma_{T,\star}'(s)|\gamma_T(s)-\gamma_T(t)|^2.\end{aligned}$$

Assumption (6.3.1a) bounds the above by

$$\begin{aligned}|\partial_s&\kappa_T(s,t)||\gamma_T(s)-\gamma_T(t)|^2|\gamma_{T,\star}(s)-\gamma_{T,\star}(t)|^2\\ &\lesssim (s-t)^5(\|\gamma_{T,\star}''\|_{L^\infty([0,1])}^2\|\gamma_T'\|_{L^\infty([0,1])}^2 + \|\gamma_T''\|_{L^\infty([0,1])}^2\|\gamma_{T,\star}'\|_{L^\infty([0,1])}^2)\\ &\qquad + (s-t)^2\|\gamma_T''\|_{L^\infty([0,1])}\|\gamma_T'\|_{L^\infty([0,1])}|\gamma_{T,\star}(s)-\gamma_{T,\star}(t)|^2\\ &\qquad + (s-t)^2\|\gamma_{T,\star}''\|_{L^\infty([0,1])}\|\gamma_{T,\star}'\|_{L^\infty([0,1])}|\gamma_T(s)-\gamma_T(t)|^2\\ &\lesssim (s-t)^5(\|\gamma_{T,\star}''\|_{L^\infty([0,1])}^2\|\gamma_T'\|_{L^\infty([0,1])}^2 + \|\gamma_T''\|_{L^\infty([0,1])}^2\|\gamma_{T,\star}'\|_{L^\infty([0,1])}^2)\\ &\qquad + (s-t)^4\big(\|\gamma_T''\|_{L^\infty([0,1])}\|\gamma_T'\|_{L^\infty([0,1])}\|\gamma_{T,\star}'\|_{L^\infty([0,1])}^2\\ &\qquad\qquad + \|\gamma_{T,\star}''\|_{L^\infty([0,1])}\|\gamma_{T,\star}'\|_{L^\infty([0,1])}\|\gamma_T'\|_{L^\infty([0,1])}^2\big)\\ &\lesssim |T^\Gamma|^4((s-t)^5 + (s-t)^4),\end{aligned}$$

where the hidden constants depend only on C_{par} and on C_{Lip}. Again with (6.3.1a), the above implies

$$|\partial_s\kappa_T(s,t)| \lesssim 1 + |s-t|, \tag{6.3.7}$$

where the hidden constant depends only on $C_{\rm par}$ and on $C_{\rm Lip}$. On the other hand, there holds $\kappa_T(s,t) = \kappa_\star(\gamma_T(s),\gamma_T(t))$ and hence by use of (6.2.43)

$$\begin{aligned}|\partial_s\kappa_T(s,t)| &\simeq |(\partial_1\kappa)(\gamma_T(s),\gamma_T(t))||T^\Gamma| \lesssim |T^\Gamma||\gamma_T(s)-\gamma_T(t)|^{-1}\|t_\Gamma - \partial_\Gamma\gamma_\star\|_{L^\infty(T^\Gamma)}\\ &\simeq |s-t|^{-1}\|t_\Gamma-\partial_\Gamma\gamma_\star\|_{L^\infty(T^\Gamma)}. \qquad (6.3.8)\end{aligned}$$

The estimates (6.3.7)–(6.3.8) and $|\partial_\Gamma\gamma_\star|\circ\gamma_T \le 1+\mathrm{geo}(\mathcal{T}_\star) \le 1 + C_{\rm ext}^{-1}/2$ show for $\varepsilon>0$

$$\begin{aligned}|g_T'(s)| &\lesssim \Big|\int_{[0,s-\varepsilon)\cup(s+\varepsilon,1]}\partial_s\kappa_T(s,t)|\gamma_T'(t)|\,dt\Big| + \Big|\int_{s-\varepsilon}^{s+\varepsilon}\partial_s\kappa_T(s,t)|\gamma_T'(t)|\,dt\Big|\\ &\lesssim \Big|\int_{[0,s-\varepsilon)\cup(s+\varepsilon,1]}|s-t|^{-1}\,dt\Big|\|t_\Gamma-\partial_\Gamma\gamma_\star\|_{L^\infty(T^\Gamma)}|T^\Gamma| + \varepsilon|T^\Gamma|\\ &\lesssim |T^\Gamma|(1+|\log(\varepsilon)|)\|t_\Gamma-\partial_\Gamma\gamma_\star\|_{L^\infty(T^\Gamma)} + |T^\Gamma|\varepsilon.\end{aligned}$$

This concludes the proof. □

Lemma 6.3.4. *Let assumption 6.3.1 hold and suppose Lemma 6.2.9 (i)–(iii) as well as* $\mathrm{geo}(\mathcal{T}_\star) \le C_{\rm ext}^{-1}/2$ *hold for* $\mathcal{T}_\star\in\mathbb{T}$. *Given* $G_\star \in \mathcal{P}^0(\mathcal{T}_\star^\Gamma)$, *there holds for all* $T\in\mathcal{T}_\star$

$$\begin{aligned}&|T|^{1/2}\|\partial_\Gamma M_\star((G_\star|\partial_\Gamma\gamma_\star|)|_{\cup\omega(T^\Gamma,\mathcal{T}_\star^\Gamma)})\|_{L^2(T^\Gamma)}\\ &\qquad\le C_{\rm M}\mathrm{geo}(\mathcal{T}_\star)^{3/2}(1+|\log(\mathrm{geo}(\mathcal{T}_\star))|)\|G_\star\|_{L^2(\cup\omega(T^\Gamma,\mathcal{T}_\star^\Gamma))},\end{aligned}$$

where $M_\star$ *is defined in* (6.2.48) *and the constant* $C_{\rm M}>0$ *depends only on* $C_{\rm inv}$, $C_{\rm Lip}$, $C_{\rm ext}$, C_Γ, $C_{\rm apx}$, $K(\mathcal{T}_\star)$ *(with* $K(\cdot)$ *from Section 3.2.2), and on* Γ.

Proof. We abbreviate $G := (G_\star|\partial_\Gamma\gamma_\star|)|_{\cup\omega(T^\Gamma,\mathcal{T}^\Gamma)}$ and get

$$\begin{aligned}\|\partial_\Gamma M_\star G\|^2_{L^2(T^\Gamma)} &= \int_{T^\Gamma}\Big(\int_{\cup\omega(T^\Gamma,\mathcal{T}_\star^\Gamma)}\partial_{\Gamma,x}\log\Big(\frac{|x-y|^2}{|\gamma_\star(x)-\gamma_\star(y)|^2}\Big)G(y)\,dy\Big)^2\,dx\\ &\lesssim \int_{T^\Gamma}\Big(\int_{\cup\omega(T^\Gamma,\mathcal{T}_\star^\Gamma)\setminus T^\Gamma}\partial_{\Gamma,x}\log\Big(\frac{|x-y|^2}{|\gamma_\star(x)-\gamma_\star(y)|^2}\Big)G(y)\,dy\Big)^2\,dx \qquad (6.3.9)\\ &\quad+\int_{T^\Gamma}\Big(\int_{T^\Gamma}\partial_{\Gamma,x}\log\Big(\frac{|x-y|^2}{|\gamma_\star(x)-\gamma_\star(y)|^2}\Big)G(y)\,dy\Big)^2\,dx.\end{aligned}$$

There holds with (6.3.4)

$$\begin{aligned}&\int_{T^\Gamma}\Big(\int_{\cup\omega(T^\Gamma,\mathcal{T}^\Gamma)\setminus T^\Gamma}\partial_{\Gamma,x}\log\Big(\frac{|x-y|^2}{|\gamma_\star(x)-\gamma_\star(y)|^2}\Big)G(y)\,dy\Big)^2\,dx\\ &\quad\lesssim (1+\mathrm{geo}(\mathcal{T}_\star))|T|^{-1}\max_{T'\in\omega(T,\mathcal{T}_\star)}\mathrm{geo}_{T'}(\mathcal{T}_\star)^3\int_{T^\Gamma}\Big(\int_{\cup\omega(T^\Gamma,\mathcal{T}_\star^\Gamma)\setminus T^\Gamma}|x-y|^{-1}|G(y)|\,dy\Big)^2\,dx.\end{aligned}$$

Let $T_1,T_2\in\mathcal{T}_\star^\Gamma$ such that $T_1\cup T_2 = \bigcup\omega(T^\Gamma,\mathcal{T}_\star^\Gamma)\setminus T^\Gamma$. Then, there holds for $i=1,2$

$$\begin{aligned}\int_{T^\Gamma}\Big(\int_{T_i}|x-y|^{-1}|G(y)|\,dy\Big)^2\,dx &\le |G_\star|_{T_i}|^2\|\partial_\Gamma\gamma_\star\|^2_{L^\infty(T_i)}\int_{T^\Gamma}\Big(\int_{T_i}|x-y|^{-1}\,dy\Big)^2\,dx\\ &\le (1+\mathrm{geo}(\mathcal{T}_\star)^2)|G_\star|_{T_i}|^2\int_{T^\Gamma}\log(\mathrm{dist}(x,T_i))^2\,dx.\end{aligned}$$

The Lipschitz continuity of γ and (6.2.13) show for $z_i := T_i\cap T\in\Gamma$

$$C^{-1}|x-z_i| \le \mathrm{dist}(x,T_i) \le C|x-z_i|$$

for some constant $C > 0$. This implies

$$\int_{T^\Gamma} \log(\operatorname{dist}(x,T_i))^2\,dx \lesssim \int_{T^\Gamma} (\log|x-z_i|)^2\,dx + \int_T (\log(C))^2\,dx \lesssim |T^\Gamma|(\log(|T^\Gamma|)^2+1).$$

Altogether, this shows

$$\begin{aligned}\int_{T^\Gamma}\Big(\int_{\cup\omega(T^\Gamma,\mathcal{T}^\Gamma)\setminus T^\Gamma} &\partial_{\Gamma,x}\log\Big(\frac{|x-y|^2}{|\gamma_\star(x)-\gamma_\star(y)|^2}\Big)G(y)\,dy\Big)^2\,dx\\ &\lesssim (1+\operatorname{geo}(\mathcal{T}_\star))|T|^{-1}(\log(|T|)^2+1)\max_{T'\in\omega(T^\Gamma,\mathcal{T}_\star^\Gamma)\setminus T^\Gamma}\operatorname{geo}_{T'}(\mathcal{T}_\star)^3\|G_\star\|^2_{L^2(\cup\omega(T^\Gamma,\mathcal{T}_\star^\Gamma))}\\ &\lesssim (1+\operatorname{geo}(\mathcal{T}_\star))|T|^{-1}(\log(|T|)^2+1)\operatorname{geo}(\mathcal{T}_\star)^3\|G_\star\|^2_{L^2(\cup\omega(T^\Gamma,\mathcal{T}_\star^\Gamma))},\end{aligned} \tag{6.3.10}$$

where we used the K-mesh property for the last estimate. The remaining term in (6.3.9) is bounded by use of Lemma 6.3.3 with $\varepsilon := \operatorname{geo}(\mathcal{T}_\star)^{3/2}$. Since $G_\star|_{T^\Gamma}$ is constant, consider

$$\begin{aligned}\|\partial_\Gamma M_\star G|_{T^\Gamma}\|^2_{L^2(T^\Gamma)} &= \int_{T^\Gamma}\Big(\partial_{\Gamma,x}\int_{T^\Gamma}\log\Big(\frac{|x-y|^2}{|\gamma_\star(x)-\gamma_\star(y)|^2}\Big)G(y)\,dy\Big)^2\,dx\\ &= |G_\star|_{T^\Gamma}|^2\|\partial_\Gamma(g\circ\gamma_T^{-1})\|^2_{L^2(T^\Gamma)} = |T^\Gamma|^{-1}|G_\star|_{T^\Gamma}|^2\|g'\|^2_{L^2([0,1])}\\ &\lesssim \big(\varepsilon^2+(1+|\log(\varepsilon)|)^2\|t_\Gamma-\partial_\Gamma\gamma_\star\|^2_{L^\infty(T^\Gamma)}\big)\|G_\star\|^2_{L^2(T^\Gamma)}\\ &\lesssim \big(\operatorname{geo}(\mathcal{T}_\star)^3+(1+|\log(\operatorname{geo}(\mathcal{T}_\star))|)^2\|t_\Gamma-\partial_\Gamma\gamma_\star\|^2_{L^\infty(T^\Gamma)}\big)\|G_\star\|^2_{L^2(T^\Gamma)}.\end{aligned}$$

Lemma 6.3.2 then shows $|T|\|t_\Gamma-\partial_\Gamma\gamma_\star\|^2_{L^\infty(\Gamma)} \lesssim \operatorname{geo}(\mathcal{T}_\star)^3$ and hence

$$\begin{aligned}&|T|\|\partial_\Gamma M_\star G|_{T^\Gamma}\|^2_{L^2(T^\Gamma)}\\ &\qquad\lesssim (1+|\log(\operatorname{geo}(\mathcal{T}_\star))|)^2\operatorname{geo}(\mathcal{T}_\star)^3\|G\|^2_{L^2(T^\Gamma)}.\end{aligned} \tag{6.3.11}$$

Putting together the estimates (6.3.9), (6.3.10), (6.3.11), we end up with

$$|T|\|\partial_\Gamma M(G|_{\cup\omega(T^\Gamma,\mathcal{T}_\star^\Gamma)})\|^2_{L^2(T)} \lesssim \operatorname{geo}(\mathcal{T}_\star)^3(\log(|T|)^2+\log(\operatorname{geo}(\mathcal{T}_\star))^2+1)\|G_\star\|^2_{L^2(\cup\omega(T^\Gamma,\mathcal{T}_\star^\Gamma))}.$$

This concludes the proof. $\square$

Lemma 6.3.5. *Let assumption 6.3.1 hold and suppose Lemma 6.2.9 (i)–(iii) as well as* $\operatorname{geo}(\mathcal{T}_\star) \le C_{\rm ext}^{-1}/2$ *hold for* $\mathcal{T}_\star \in \mathbb{T}$. *All* $G \in L^2(\Gamma)$ *satisfy*

$$\begin{aligned}\sum_{T\in\mathcal{T}_\star} &|T^\Gamma|\|\partial_\Gamma M_\star(G|_{\Gamma\setminus\cup\omega(T^\Gamma,\mathcal{T}_\star^\Gamma)})\|^2_{L^2(T^\Gamma)}\\ &\le C_{\rm M}\operatorname{geo}(\mathcal{T}_\star)^3|1+\log(\min h_\star)|^2(|\log(|\mathcal{T}_\star|)|+1)\|G\|^2_{L^2(\Gamma)},\end{aligned}$$

where $M_\star$ *is defined in* (6.2.48) *and the constant* $C_{\rm M} > 0$ *depends only on* $C_{\rm apx}$, $C_{\rm ext}$, $C_{\rm Lip}$, C_Σ, $K(\mathcal{T}_\star)$ *(with* $K(\cdot)$ *from Section 3.2.2), and on* Γ.

Proof. Let $x \in T^\Gamma$ for some $T \in \mathcal{T}_\star$. The estimate (6.3.3) shows

$$|\partial_{\Gamma,x}\kappa_\star(x,y)| \lesssim \Big(\frac{|T|^{-1}}{|x-y|}+\frac{1}{|x-y|^2}\Big)\operatorname{geo}(\mathcal{T}_\star)^2.$$

The estimate (6.2.42) shows also

$$|\partial_{\Gamma,x}\kappa_\star(x,y)| \lesssim |x-y|^{-1}\operatorname{geo}(\mathcal{T}_\star).$$

The combination of the last two estimates implies

$$\begin{aligned}|\partial_{\Gamma,x}\kappa_\star(x,y)| &\lesssim \frac{\mathrm{geo}(\mathcal{T}_\star)^{3/2}}{|x-y|^{1/2}}\Big(\frac{|T|^{-1}}{|x-y|}+\frac{1}{|x-y|^2}\Big)^{1/2}\\ &\lesssim \mathrm{geo}(\mathcal{T}_\star)^{3/2}\Big(\frac{|T|^{-1/2}}{|x-y|}+\frac{1}{|x-y|^{3/2}}\Big).\end{aligned}$$

We abbreviate $G := G|_{\Gamma\setminus\cup\omega(T^\Gamma,\mathcal{T}_\star^\Gamma)}$ and employ the above estimate to obtain

$$\begin{aligned}\|\partial_\Gamma M_\star G\|_{L^2(T)}^2 &= \int_T\Big(\int_{\Gamma\setminus\cup\omega(T^\Gamma,\mathcal{T}_\star^\Gamma)}\partial_{\Gamma,x}\kappa_\star(x,y)G(y)\,dy\Big)^2\,dx\\ &\lesssim |T|^{-1}\mathrm{geo}(\mathcal{T}_\star)^3\int_T\Big(\int_{\Gamma\setminus\cup\omega(T^\Gamma,\mathcal{T}_\star^\Gamma)}|x-y|^{-1}|G(y)|\,dy\Big)^2\,dx\\ &\quad+\mathrm{geo}(\mathcal{T}_\star)^3\int_T\Big(\int_{\Gamma\setminus\cup\omega(T^\Gamma,\mathcal{T}_\star^\Gamma)}|x-y|^{-3/2}|G(y)|\,dy\Big)^2\,dx.\end{aligned}$$

For $\alpha\in\{-1,-3/2\}$, there holds

$$\begin{aligned}&\int_T\Big(\int_{\Gamma\setminus\cup\omega(T^\Gamma,\mathcal{T}_\star^\Gamma)}|x-y|^\alpha|G(y)|\,dy\Big)^2\,dx\\ &\qquad\le |T|\sup_{x\in T}\||x-\cdot|^{-1/2}\|_{L^2(\Gamma\setminus\cup\omega(T^\Gamma,\mathcal{T}_\star^\Gamma)}^2\||x-\cdot|^{\alpha+1/2}|G(y)|\|_{L^2(\Gamma\setminus\cup\omega(T^\Gamma,\mathcal{T}_\star^\Gamma))}^2.\end{aligned}$$

The first term is estimated as in (6.2.56) to obtain

$$\begin{aligned}&\int_T\Big(\int_{\Gamma\setminus\cup\omega(T^\Gamma,\mathcal{T}_\star^\Gamma)}|x-y|^\alpha|G(y)|\,dy\Big)^2\,dx\\ &\quad\le |T|\sup_{x\in T}\||x-\cdot|^{-1/2}\|_{L^2(\Gamma\setminus\cup\omega(T^\Gamma,\mathcal{T}_\star^\Gamma))}^2\sum_{T_0\in\mathcal{T}_\star^\Gamma\setminus\omega(T^\Gamma,\mathcal{T}_\star^\Gamma)}\||x-\cdot|^{\alpha+1/2}|G(\cdot)|\|_{L^2(T_0)}^2\\ &\quad\lesssim |T|(1+|\log(|T|)|)\sum_{T_0\in\mathcal{T}_\star^\Gamma\setminus\omega(T^\Gamma,\mathcal{T}_\star^\Gamma)}\frac{1}{\mathrm{dist}(T,T_0)^{-2\alpha-1}}\|G\|_{L^2(T_0)}^2.\end{aligned}$$

Altogether, this yields

$$\begin{aligned}&\sum_{T\in\mathcal{T}_\star}|T|\|\partial_\Gamma M_\star G\|_{L^2(T)}^2\\ &\lesssim\sum_{T\in\mathcal{T}_\star}|T|^2\mathrm{geo}(\mathcal{T}_\star)^3(1+|\log(|T|)|)\sum_{T_0\in\mathcal{T}_\star^\Gamma\setminus\omega(T^\Gamma,\mathcal{T}_\star^\Gamma)}\Big(\frac{|T|^{-1}}{\mathrm{dist}(T,T_0)}+\frac{1}{\mathrm{dist}(T,T_0)^2}\Big)\|G\|_{L^2(T_0)}^2\\ &\le \mathrm{geo}(\mathcal{T}_\star)^3(1+|\log(\min h_\star)|)\sum_{T_0\in\mathcal{T}_\star^\Gamma}\|G\|_{L^2(T_0)}^2\sum_{T\in\mathcal{T}_\star^\Gamma\setminus\omega(T_0,\mathcal{T}_\star^\Gamma)}\Big(\frac{|T|}{\mathrm{dist}(T,T_0)}+\frac{|T|^2}{\mathrm{dist}(T,T_0)^2}\Big).\end{aligned}$$

Lemma 6.2.27 implies

$$\max_{T_0\in\mathcal{T}_\star}\sum_{T\in\mathcal{T}_\star^\Gamma\setminus\omega(T_0,\mathcal{T}_\star^\Gamma)}\Big(\frac{|T|}{\mathrm{dist}(T,T_0)}+\frac{|T|^2}{\mathrm{dist}(T,T_0)^2}\Big)\lesssim|\log\big(\frac{\max h_\star}{\min h_\star}\big)|(\log(|\mathcal{T}_\star|)|+1)$$

and thus concludes the proof. □

To formulate the next lemma, we define an auxiliary error estimator on the exact boundary. Of course, this in only a theoretical tool and does not have to be computed. For all $T^\Gamma \in \mathcal{T}_\star^\Gamma$, define

$$\rho_{T^\Gamma}(\mathcal{T}_\star^\Gamma) := \|h_\star^{1/2} \circ \gamma_\star \, \partial_\Gamma(\mathcal{V}U(\mathcal{T}_\star)^\Gamma - f)\|_{L^2(T^\Gamma)}. \tag{6.3.12}$$

Lemma 6.3.6. *Let assumption 6.3.1 hold and suppose Lemma 6.2.9 (i)–(iii) as well as* $\mathrm{geo}(\mathcal{T}_\star) \le C_{\mathrm{ext}}^{-1}/2$ *hold for* $\mathcal{T}_\star \in \mathbb{T}$. *Given some* $\mathcal{S}_\star \subseteq \mathcal{T}_\star$, *there holds*

$$\Big|\Big(\sum_{T^\Gamma \in \mathcal{S}_\star^\Gamma} \rho_{T^\Gamma}(\mathcal{T}_\star^\Gamma)^2\Big)^{1/2} - \Big(\sum_{T \in \mathcal{S}_\star} \rho_T(\mathcal{T}_\star)^2\Big)^{1/2}\Big| \le \alpha_\star, \tag{6.3.13}$$

where $\mathcal{S}_\star^\Gamma := \big\{T^\Gamma \,:\, T \in \mathcal{S}_\star\big\}$ *and*

$$\begin{aligned}\alpha_\star := \mathrm{geo}(\mathcal{T}_\star)^{3/2}\Big(&2\kappa_\Gamma C_\nu \rho(\mathcal{T}_\star)\mathrm{geo}(\mathcal{T}_\star)^{1/2} \\ &+ C_{\mathrm{M}} C_{\mathrm{Lip}}(1 + |\log(\mathrm{geo}(\mathcal{T}_\star))|)(1 + |\log(\min h_\star)|)(1 + |\log(|\mathcal{T}_\star|)|)^{1/2}\|U(\mathcal{T}_\star)\|_{L^2(\Gamma_\star)}\Big).\end{aligned}$$

Proof. There holds with $\omega_\star := \bigcup \mathcal{S}_\star$ and $\omega_\star^\Gamma := \bigcup \mathcal{S}_\star^\Gamma$

$$\begin{aligned}\Big(\sum_{T^\Gamma \in \mathcal{S}_\star^\Gamma} \rho_{T^\Gamma}(\mathcal{T}_\star^\Gamma)^2\Big)^{1/2} &= \|h_\star^{1/2} \circ \gamma_\star \partial_\Gamma(\mathcal{V}U(\mathcal{T}_\star)^\Gamma - f)\|_{L^2(\omega_\star^\Gamma)} \\ &\le \|h_\star^{1/2} \circ \gamma_\star\big(\partial_{\Gamma_\star}(\mathcal{V}_\star U(\mathcal{T}_\star) - f_\star)\big) \circ \gamma_\star |\partial_\Gamma \gamma_\star|^{1/2}\|_{L^2(\omega_\star^\Gamma)} \\ &\quad + \|h_\star^{1/2} \circ \gamma_\star\big(\partial_\Gamma(\mathcal{V}U(\mathcal{T}_\star)^\Gamma - f) - \big(\partial_{\Gamma_\star}(\mathcal{V}_\star U(\mathcal{T}_\star) - f_\star)\big) \circ \gamma_\star |\partial_\Gamma \gamma_\star|^{1/2}\big)\|_{L^2(\omega_\star^\Gamma)}.\end{aligned} \tag{6.3.14}$$

We introduce the notation

$$\begin{aligned}A &:= \partial_\Gamma(\mathcal{V}U(\mathcal{T}_\star)^\Gamma - f), \\ B &:= \big(\partial_{\Gamma_\star}(\mathcal{V}_\star U(\mathcal{T}_\star) - f_\star)\big) \circ \gamma_\star |\partial_\Gamma \gamma_\star|^{1/2}, \\ C &:= \partial_\Gamma((\mathcal{V}_\star U(\mathcal{T}_\star) - f_\star) \circ \gamma_\star).\end{aligned}$$

The first term on the right-hand side of (6.3.14) transforms to

$$\begin{aligned}\|h_\star^{1/2} \circ \gamma_\star B\|_{L^2(\omega_\star^\Gamma)}^2 &= \int_{\omega_\star^\Gamma} h_\star \circ \gamma_\star \big(\partial_{\Gamma_\star}(\mathcal{V}_\star U(\mathcal{T}_\star) - f_\star)\big)^2 \circ \gamma_\star |\partial_\Gamma \gamma_\star| \, dx \\ &= \int_{\omega_\star} h_\star \big(\partial_{\Gamma_\star}(\mathcal{V}_\star U(\mathcal{T}_\star) - f_\star)\big)^2 \, dx = \sum_{T \in \mathcal{S}_\star} \rho_T(\mathcal{T}_\star)^2.\end{aligned} \tag{6.3.15}$$

The second term on the right-hand side of (6.3.14) is further split into

$$\|h_\star^{1/2} \circ \gamma_\star(A - B)\|_{L^2(\omega_\star^\Gamma)} \le \|h_\star^{1/2} \circ \gamma_\star(A - C)\|_{L^2(\omega_\star^\Gamma)} + \|h_\star^{1/2} \circ \gamma_\star(C - B)\|_{L^2(\omega_\star^\Gamma)}.$$

The chain rule (6.2.11) implies

$$C = (\partial_{\Gamma_\star}(\mathcal{V}_\star U(\mathcal{T}_\star) - f_\star)) \circ \gamma_\star \, \partial_\Gamma^s \gamma_\star.$$

With (6.2.15) and (6.2.9), we get $\partial_\Gamma^s \gamma_\star = |\partial_\Gamma \gamma_\star|$. This shows together with (6.3.15) and Lemma 6.2.14

$$\begin{aligned}\|h_\star^{1/2} \circ \gamma_\star(C - B)\|_{L^2(\omega_\star^\Gamma)} &= \|h_\star^{1/2} \circ \gamma_\star(1 - |\partial_\Gamma \gamma_\star|^{1/2})B\|_{L^2(\omega_\star^\Gamma)} \\ &\le \|1 - |\partial_\Gamma \gamma_\star|^{1/2}\|_{L^\infty(\Gamma)}\|h_\star^{1/2} \circ \gamma_\star B\|_{L^2(\omega_\star^\Gamma)} \\ &\le 2\kappa_\Gamma C_\nu \mathrm{geo}(\mathcal{T}_\star)^2\Big(\sum_{T \in \mathcal{S}_\star} \rho_T(\mathcal{T}_\star)^2\Big)^{1/2}.\end{aligned}$$

Moreover, since $f_\star = f\circ\gamma_\star^{-1}$, there holds

$$\|h_\star^{1/2}\circ\gamma_\star(A-C)\|_{L^2(\omega_\star^\Gamma)} \le \|h_\star^{1/2}\circ\gamma_\star\partial_\Gamma\big(\mathcal{V}U(\mathcal{T}_\star)^\Gamma - (\mathcal{V}_\star U(\mathcal{T}_\star))\circ\gamma_\star\big)\|_{L^2(\omega_\star^\Gamma)}.$$

We obtain for $x\in\Gamma$

$$\begin{aligned}
&-2\pi(\mathcal{V}(U(\mathcal{T}_\star)^\Gamma) - (\mathcal{V}_\star U(\mathcal{T}_\star))\circ\gamma_\star)(x)\\
&\quad= \int_\Gamma \log|x-y|U(\mathcal{T}_\star)\circ\gamma_\star(y)|\partial_\Gamma\gamma_\star|\,dy - \int_\Gamma \log|\gamma_\star(x)-\gamma_\star(y)|U(\mathcal{T}_\star)\circ\gamma_\star(y)|\partial_\Gamma\gamma_\star|\,dy\\
&\quad= \frac{1}{2}M_\star(U(\mathcal{T}_\star)^\Gamma)(x).
\end{aligned}$$

We employ Lemma 6.3.4–6.3.5 to obtain

$$\begin{aligned}
&\frac{1}{2}\|h_\star^{1/2}\circ\gamma_\star\partial_\Gamma M_\star(U(\mathcal{T}_\star)^\Gamma)\|_{L^2(\Gamma)}^2\\
&\qquad\le \sum_{T\in\mathcal{T}_\star}\|h_\star^{1/2}\circ\gamma_\star\partial_\Gamma M_\star(U(\mathcal{T}_\star)^\Gamma|_{\cup\omega(T^\Gamma,\mathcal{T}_\star^\Gamma)})\|_{L^2(T^\Gamma)}^2\\
&\qquad\quad+\sum_{T\in\mathcal{T}_\star}\|h_\star^{1/2}\circ\gamma_\star\partial_\Gamma M_\star(U(\mathcal{T}_\star)^\Gamma|_{\Gamma\setminus\cup\omega(T^\Gamma,\mathcal{T}_\star^\Gamma)})\|_{L^2(T^\Gamma)}^2\\
&\qquad\le C_{\mathrm{M}}^2\mathrm{geo}(\mathcal{T}_\star)^3(1+|\log(\mathrm{geo}(\mathcal{T}_\star))|)^2\|U(\mathcal{T}_\star)\circ\gamma_\star\|_{L^2(\Gamma)}^2\\
&\qquad\quad+C_{\mathrm{M}}^2\mathrm{geo}(\mathcal{T}_\star)^3(1+|\log(\min h_\star)|)^2(|\log(|\mathcal{T}_\star|)|+1)\|U(\mathcal{T}_\star)^\Gamma\|_{L^2(\Gamma)}^2\\
&\qquad\le C_{\mathrm{M}}^2C_{\mathrm{Lip}}^2\mathrm{geo}(\mathcal{T}_\star)^3(1+|\log(\mathrm{geo}(\mathcal{T}_\star))|)^2\\
&\qquad\quad(1+|\log(\min h_\star)|)^2(|\log(|\mathcal{T}_\star|)|+1)\|U(\mathcal{T}_\star)\|_{L^2(\Gamma_\star)}^2.
\end{aligned}$$

This concludes

$$\Big(\sum_{T^\Gamma\in\mathcal{S}_\star^\Gamma}\rho_{T^\Gamma}(\mathcal{T}_\star^\Gamma)^2\Big)^{1/2} \le \Big(\sum_{T\in\mathcal{S}_\star}\rho_T(\mathcal{T}_\star)^2\Big)^{1/2} + \alpha_\star.$$

The converse inequality follows analogously by replacing all triangle inequalities with reverse triangle inequalities. This concludes the proof. □

Lemma 6.3.7. *Let assumption 6.3.1 hold and suppose Lemma 6.2.9 (i)–(iii) as well as* $\mathrm{geo}(\mathcal{T}_\star)\le C_{\mathrm{ext}}^{-1}/2$ *hold for* $\mathcal{T}_\star\in\mathbb{T}$. *With* $G_\star\in\mathcal{P}^0(\mathcal{T}_\star^\Gamma)$, *there holds*

$$\|h_\star^{1/2}\circ\gamma_\star G_\star|\partial_\Gamma\gamma_\star|\|_{L^2(\Gamma)} \le C_{\mathrm{inv}}\|G_\star|\partial_\Gamma\gamma_\star|\|_{H^{-1/2}(\Gamma)} + \mathrm{geo}(\mathcal{T}_\star)^2\|G_\star\|_{L^2(\Gamma)}.$$

The constant $C_{\mathrm{inv}}>0$ *depends only on* $K(\mathcal{T}_\star)$ *(with* $K(\cdot)$ *from Section 3.2.2),* (Γ2), *and on* Γ.

Proof. There holds with (Γ2) and the inverse estimate from **[57]**

$$\begin{aligned}
\|h_\star^{1/2}\circ\gamma_\star G_\star|\partial_\Gamma\gamma_\star|\|_{L^2(\Gamma)} &\le C_{\mathrm{Lip}}\|h_\star^{1/2}\circ\gamma_\star G_\star\|_{L^2(\Gamma)}\\
&\lesssim \|G_\star\|_{H^{-1/2}(\Gamma)}\\
&\le \|G_\star|\partial_\Gamma\gamma_\star|\|_{H^{-1/2}(\Gamma)} + \|G_\star(1-|\partial_\Gamma\gamma_\star|)\|_{H^{-1/2}(\Gamma)}.
\end{aligned}$$

Lemma 6.2.14 proves

$$\|G_\star(1-|\partial_\Gamma\gamma_\star|)\|_{H^{-1/2}(\Gamma)} \le \|1-|\partial_\Gamma\gamma_\star|\|_{L^\infty(\Gamma)}\|G_\star\|_{L^2(\Gamma)} \lesssim \mathrm{geo}(\mathcal{T}_\star)^2\|G_\star\|_{L^2(\Gamma)}.$$

This concludes the proof. □

THEOREM 6.3.8 (Stability and reduction (E1)). *Let assumption 6.3.1 hold. Given two approximate geometries $\mathcal{T}_\star \in \mathbb{T}$ and $\mathcal{T}_\bullet \in \mathbb{T}(\mathcal{T}_\star)$ such that Lemma 6.2.9 (i)–(iii) as well as* $\mathrm{geo}(\mathcal{T}_\star), \mathrm{geo}(\mathcal{T}_\bullet) \le C_{\mathrm{ext}}^{-1}/2$ *hold. Let* $q := \sqrt{1/4 + C_\gamma^2 \|\gamma''\|_{L^2([0,1])}^2 \max h_\star^2} < 1$. *Then, there holds* (E1) *for* $\rho(\cdot)$ *from* (6.2.5), *with*

$$\begin{aligned}\varrho(\mathcal{T}_\star, \mathcal{T}_\bullet) := C_{\mathrm{pert}}\big(&\|U(\mathcal{T}_\star)^\Gamma - U(\mathcal{T}_\bullet)^\Gamma\|_{H^{-1/2}(\Gamma)} + \alpha_\star + \alpha_\bullet \\ &+ (\mathrm{geo}(\mathcal{T}_\star)^2 + \mathrm{geo}(\mathcal{T}_\bullet)^2)(\|U(\mathcal{T}_\star)\|_{L^2(\Gamma_\star)} + \|U(\mathcal{T}_\bullet)\|_{L^2(\Gamma_\bullet)})\big),\end{aligned}$$

$\alpha_\star$, $\alpha_\bullet$ *from Lemma 6.3.6,* $\mathcal{S}(\mathcal{T}_\star, \mathcal{T}_\bullet) := \mathcal{T}_\star \setminus \mathcal{T}_\bullet$, $\widehat{\mathcal{S}}(\mathcal{T}_\star, \mathcal{T}_\bullet) := \mathcal{T}_\bullet \setminus \mathcal{T}_\star$, *and* $0 < \rho_{\mathrm{red}} < 1$ *depends only on* q, *whereas* $C_{\mathrm{pert}} > 0$ *depends additionally on* C_{inv}, C_{Lip}, Γ, *and* $K(\mathcal{T}_\bullet)$, $K(\mathcal{T}_\star)$ *(with* $K(\cdot)$ *from Section 3.2.2).*

PROOF. To see (E1a), we employ Lemma 6.3.6 two times with $\mathcal{S}_\star := \mathcal{S}_1 := \mathcal{T}_\star \setminus \mathcal{S}(\mathcal{T}_\star, \mathcal{T}_\bullet)$ and $\mathcal{S}_\star := \mathcal{S}_2 := \mathcal{T}_\bullet \setminus \widehat{\mathcal{S}}(\mathcal{T}_\star, \mathcal{T}_\bullet)$ to obtain

$$\begin{aligned}\Big|\Big(\sum_{T\in\mathcal{S}_1}&\rho_T(\mathcal{T}_\star)^2\Big)^{1/2} - \Big(\sum_{T\in\mathcal{S}_2}\rho_T(\mathcal{T}_\bullet)^2\Big)^{1/2}\Big| \\ &\le \Big|\Big(\sum_{T\in\mathcal{S}_1}\rho_T(\mathcal{T}_\star)^2\Big)^{1/2} - \Big(\sum_{T^\Gamma\in\mathcal{S}_1^\Gamma}\rho_{T^\Gamma}(\mathcal{T}_\star^\Gamma)^2\Big)^{1/2}\Big| \\ &\quad + \Big|\Big(\sum_{T^\Gamma\in\mathcal{S}_2^\Gamma}\rho_{T^\Gamma}(\mathcal{T}_\bullet^\Gamma)^2\Big)^{1/2} - \Big(\sum_{T\in\mathcal{S}_2}\rho_T(\mathcal{T}_\bullet)^2\Big)^{1/2}\Big| \\ &\quad + \Big|\Big(\sum_{T^\Gamma\in\mathcal{S}_2^\Gamma}\rho_{T^\Gamma}(\mathcal{T}_\bullet^\Gamma)^2\Big)^{1/2} - \Big(\sum_{T^\Gamma\in\mathcal{S}_1^\Gamma}\rho_{T^\Gamma}(\mathcal{T}_\star^\Gamma)^2\Big)^{1/2}\Big| \\ &\le \Big|\Big(\sum_{T^\Gamma\in\mathcal{S}_2^\Gamma}\rho_{T^\Gamma}(\mathcal{T}_\bullet^\Gamma)^2\Big)^{1/2} - \Big(\sum_{T^\Gamma\in\mathcal{S}_1^\Gamma}\rho_{T^\Gamma}(\mathcal{T}_\star^\Gamma)^2\Big)^{1/2}\Big| + \alpha_\star + \alpha_\bullet.\end{aligned} \tag{6.3.16}$$

By definition of the bisection rule in Algorithm 6.2.10, there holds $\bigcup \mathcal{S}_1^\Gamma = \bigcup \mathcal{S}_2^\Gamma$. Moreover, $h_\star \circ \gamma_\star = h_\bullet \circ \gamma_\bullet$ on $\bigcup \mathcal{S}_1^\Gamma$. Hence, the remaining term in the above estimate satisfies

$$\Big|\Big(\sum_{T^\Gamma\in\mathcal{S}_2^\Gamma}\rho_{T^\Gamma}(\mathcal{T}_\bullet^\Gamma)^2\Big)^{1/2} - \Big(\sum_{T^\Gamma\in\mathcal{S}_1^\Gamma}\rho_T(\mathcal{T}_\star)^2\Big)^{1/2}\Big| \le \|h_\bullet^{1/2}\circ\gamma_\bullet\partial_\Gamma\mathcal{V}(U(\mathcal{T}_\bullet)^\Gamma - U(\mathcal{T}_\star)^\Gamma)\|_{L^2(\Gamma)}.$$

The inverse estimate from **[2]** shows

$$\begin{aligned}\|h_\bullet^{1/2}\circ\gamma_\bullet\partial_\Gamma\mathcal{V}(U(\mathcal{T}_\bullet)^\Gamma - U(\mathcal{T}_\star)^\Gamma)\|_{L^2(\Gamma)} \lesssim \|h_\bullet^{1/2}&\circ\gamma_\bullet(U(\mathcal{T}_\bullet)^\Gamma - U(\mathcal{T}_\star)^\Gamma)\|_{L^2(\Gamma)} \\ &+ \|U(\mathcal{T}_\bullet)^\Gamma - U(\mathcal{T}_\star)^\Gamma\|_{H^{-1/2}(\Gamma)},\end{aligned} \tag{6.3.17}$$

where the hidden constant depends only on Γ as well as $K(\mathcal{T}_\star)$ and $K(\mathcal{T}_\bullet)$ (with $K(\cdot)$ from Section 3.2.2). Lemma 6.3.7 and Lemma 6.2.14 conclude

$$\begin{aligned}
&\|h_\bullet^{1/2}\circ\gamma_\bullet(U(\mathcal{T}_\bullet)^\Gamma-U(\mathcal{T}_\star)^\Gamma)\|_{L^2(\Gamma)}\\
&\qquad\lesssim\|h_\bullet^{1/2}\circ\gamma_\bullet(U(\mathcal{T}_\bullet)^\Gamma-U(\mathcal{T}_\star)\circ\gamma_\star|\partial_\Gamma\gamma_\bullet|)\|_{L^2(\Gamma)}\\
&\qquad\quad+\|1-|\partial_\Gamma\gamma_\bullet|\|_{L^\infty(\Gamma)}\|U(\mathcal{T}_\star)\|_{L^2(\Gamma_\star)}\\
&\qquad\lesssim\|(U(\mathcal{T}_\bullet)^\Gamma-U(\mathcal{T}_\star)\circ\gamma_\star)|\partial_\Gamma\gamma_\bullet|)\|_{H^{-1/2}(\Gamma)}\\
&\qquad\quad+\mathrm{geo}(\mathcal{T}_\bullet)^2\|U(\mathcal{T}_\bullet)\circ\gamma_\bullet-U(\mathcal{T}_\star)\circ\gamma_\star\|_{L^2(\Gamma)}\\
&\qquad\quad+\|1-|\partial_\Gamma\gamma_\bullet|\|_{L^\infty(\Gamma)}\|U(\mathcal{T}_\star)\|_{L^2(\Gamma_\star)}\\
&\qquad\lesssim\|U(\mathcal{T}_\bullet)^\Gamma-U(\mathcal{T}_\star)^\Gamma)\|_{H^{-1/2}(\Gamma)}\\
&\qquad\quad+(\mathrm{geo}(\mathcal{T}_\star)^2+\mathrm{geo}(\mathcal{T}_\bullet)^2)(\|U(\mathcal{T}_\star)\|_{L^2(\Gamma_\star)}+\|U(\mathcal{T}_\bullet)\|_{L^2(\Gamma_\bullet)}).
\end{aligned}\tag{6.3.18}$$

This concludes (E1a). To see (E1b), we use Lemma 6.3.6 two times with $\mathcal{S}_\star:=\mathcal{S}_1:=\mathcal{S}(\mathcal{T}_\star,\mathcal{T}_\bullet)$ and $\mathcal{S}_\star:=\mathcal{S}_2:=\backslash\widehat{\mathcal{S}}(\mathcal{T}_\star,\mathcal{T}_\bullet)$ to obtain for $\delta>0$

$$\begin{aligned}
\sum_{T\in\mathcal{S}_2}\rho_T(\mathcal{T}_\bullet)^2&\le(1+\delta)\sum_{T^\Gamma\in\mathcal{S}_2^\Gamma}\rho_{T^\Gamma}(\mathcal{T}_\bullet^\Gamma)^2+(1+\delta)^{-1}\alpha_\bullet^2\\
&\le(1+\delta)^2\|h_\bullet^{1/2}\circ\gamma_\bullet\partial_\Gamma\mathcal{V}(U(\mathcal{T}_\star)^\Gamma-f)\|^2_{L^2(\cup\mathcal{S}_1^\Gamma)}\\
&\quad+(1+\delta)^{-1}\alpha_\bullet^2+(1+\delta)(1+\delta^{-1})\|h_\bullet^{1/2}\circ\gamma_\bullet\partial_\Gamma\mathcal{V}(U(\mathcal{T}_\bullet)^\Gamma-U(\mathcal{T}_\star)^\Gamma)\|_{L^2(\Gamma)}\\
&\le(1+\delta)^2\|h_\star\circ\gamma_\star/h_\bullet\circ\gamma_\bullet\|_{L^\infty(\cup\mathcal{S}_1^\Gamma)}\|h_\star^{1/2}\circ\gamma_\star\partial_\Gamma\mathcal{V}(U(\mathcal{T}_\star)^\Gamma-f)\|^2_{L^2(\cup\mathcal{S}_1^\Gamma)}\\
&\quad+(1+\delta)^{-1}\alpha_\bullet^2+(1+\delta)(1+\delta^{-1})\|h_\bullet^{1/2}\circ\gamma_\bullet\partial_\Gamma\mathcal{V}(U(\mathcal{T}_\bullet)^\Gamma-U(\mathcal{T}_\star)^\Gamma)\|_{L^2(\Gamma)}\\
&\le(1+\delta)^3\|h_\star\circ\gamma_\star/h_\bullet\circ\gamma_\bullet\|_{L^\infty(\cup\mathcal{S}_1^\Gamma)}\sum_{T\in\mathcal{S}_1}\rho_T(\mathcal{T}_\star)^2+(1+\delta)^2(1+\delta^{-1})\alpha_\star^2\\
&\quad+(1+\delta)^{-1}\alpha_\bullet^2+(1+\delta)(1+\delta^{-1})\|h_\bullet^{1/2}\circ\gamma_\bullet\partial_\Gamma\mathcal{V}(U(\mathcal{T}_\bullet)^\Gamma-U(\mathcal{T}_\star)^\Gamma)\|_{L^2(\Gamma)}.
\end{aligned}$$

Lemma 6.2.11 implies that

$$\|h_\star\circ\gamma_\star/h_\bullet\circ\gamma_\bullet\|_{L^\infty(\cup\mathcal{S}_1^\Gamma)}\le q<1.$$

Hence, sufficiently small $\delta>0$ together with (6.3.17)–(6.3.18) conclude the proof. ☐

To prove convergence of Algorithm 6.2.2, we require the following assumption on the exact boundary Γ and the initial geometry $\mathcal{T}_0$: There exists $0<q_{\mathrm{geo}}<1$ such that all $\mathcal{T}_\star\in\mathbb{T}$ satisfy

$$\mathrm{geo}_{T'}(\mathcal{T}_\bullet)\le q_{\mathrm{geo}}\mathrm{geo}(\mathcal{T}_\star)\quad\text{for all }T'\in\mathcal{T}_\bullet\setminus\mathcal{T}_\star.\tag{6.3.19}$$

This assumption is met if, for example, the exact boundary Γ can be parametrized in terms of piecewise polynomials of arbitrary degree or NURBS and $h_\star$ is sufficiently small.

Moreover, we need the assume that there holds

$$\sup_{\ell\in\mathbb{N}_0}\max\{\vartheta,q_{\mathrm{geo}}\}^{(1-\varepsilon)3\ell/2}\|U(\mathcal{T}_\ell)\|_{L^2(\Gamma_\ell)}<\infty\tag{6.3.20}$$

for some $\varepsilon>0$.

Remark 6.3.9. *In case of quasi-uniform partitions with* $\min h_\ell\simeq\max h_\ell$*, assumption* (6.3.20) *is straightforward to prove even with* $\varepsilon=1$*, i.e.,* $\sup_{\ell\in\mathbb{N}}\|U(\mathcal{T}_\ell)\|_{L^2(\Gamma_\ell)}<\infty$.

However, we did not succeed in finding a proof for the general case of locally refined partitions. We conjecture that there exists $\nu > 0$ such that

$$\sup_{\ell\in\mathbb{N}_0} \ell^{-\nu}\|U(\mathcal{T}_\ell)\|_{L^2(\Gamma_\ell)} < \infty,$$

which would imply (6.3.20). *Note that not even uniform stability of the continuous problem $\mathcal{V}_\star^{-1}\colon H^1(\Gamma_\star) \to L^2(\Gamma_\star)$ for all $\mathcal{T}_\star \in \mathbb{T}$ is known in the literature. From the uniform case, we derive the (very conservative) worst case estimate $\|U(\mathcal{T}_\ell)\|_{L^2(\Gamma_\ell)} \lesssim 2^\ell$. Assumption* (6.3.20) *is easy to check numerically and in this sense, one should understand the convergence results of Lemma 6.3.10 and Theorem 6.4.1. If one numerically detects stability* (6.3.20), *Algorithm 6.2.2 leads to convergence towards the exact solution.*

Lemma 6.3.10. *Suppose Lemma 6.2.9 (i)–(iii) as well as $\mathrm{geo}(\mathcal{T}_\ell) \leq C_{\mathrm{ext}}^{-1}/2$ for all $\ell \in \mathbb{N}_0$. Under assumption* (6.3.1), (6.3.19), *and* (6.3.20) *there exists $U_\infty \in H^{-1/2}(\Gamma)$ such that there holds a priori convergence $\lim_{\ell\to\infty} \|U_\infty - U(\mathcal{T}_\ell)^\Gamma\|_{H^{-1/2}(\Gamma)} = 0$. Moreover, there holds $\lim_{\ell\to\infty} \varrho(\mathcal{T}_\ell, \mathcal{T}_{\ell+1}) = 0$, where $\varrho(\mathcal{T}_\ell, \mathcal{T}_{\ell+1})$ is defined in Theorem 6.3.8.*

Proof. There holds

$$\mathrm{geo}_T(\mathcal{T}_\ell) \overset{(6.2.6b)}{\leq} \vartheta\,\mathrm{geo}(\mathcal{T}_\ell) \quad \text{for } T \in \mathcal{T}_\ell \setminus \mathcal{M}_\ell$$

and

$$\mathrm{geo}_{T'}(\mathcal{T}_{\ell+1}) \overset{(6.3.19)}{\leq} q_{\mathrm{geo}}\mathrm{geo}(\mathcal{T}_\ell) \quad \text{for all } T' \in \mathcal{T}_{\ell+1} \setminus \mathcal{T}_\ell.$$

Since all $T \in \mathcal{T}_{\ell+1}$ satisfy either $T \in \mathcal{T}_{\ell+1} \setminus \mathcal{T}_\ell$ or $T \in \mathcal{T}_{\ell+1} \cap \mathcal{T}_\ell \subseteq \mathcal{T}_\ell \setminus \mathcal{M}_\ell$, the combination implies

$$\mathrm{geo}(\mathcal{T}_{\ell+1}) = \max_{T\in\mathcal{T}_{\ell+1}} \mathrm{geo}_T(\mathcal{T}_{\ell+1}) \leq \max\{q_{\mathrm{geo}}, \vartheta\}\mathrm{geo}(\mathcal{T}_\ell). \tag{6.3.21}$$

This implies $\mathrm{geo}(\mathcal{T}_\ell) \to 0$ as $\ell \to \infty$. Define $\mathcal{X}_\infty := \overline{\bigcup_{\ell\in\mathbb{N}_0} \big\{v|\partial_\Gamma\gamma_\ell| \,:\, v \in \mathcal{P}^0(\mathcal{T}_\ell^\Gamma)\big\}} \subseteq H^{-1/2}(\Gamma)$ and the a priori limit $U_\infty \in \mathcal{X}_\infty$ by

$$\langle \mathcal{V}U_\infty \,,\, v\rangle_\Gamma = \langle f \,,\, v\rangle_\Gamma \quad \text{for all } v \in \mathcal{X}_\infty.$$

For all $\ell \in \mathbb{N}_0$, define $\mathcal{X}_\infty(\mathcal{T}_\ell) := \overline{\bigcup_{k\in\mathbb{N}_0} \big\{v \circ \gamma_\ell^{-1}|\partial_{\Gamma_\ell}(\gamma_k \circ \gamma_\ell^{-1})| \,:\, v \in \mathcal{P}^0(\mathcal{T}_k^\Gamma)\big\}} \subseteq H^{-1/2}(\Gamma_\ell)$ and $U_\infty(\mathcal{T}_\ell) \in \mathcal{X}_\infty(\mathcal{T}_\ell)$ by

$$\langle \mathcal{V}_\ell U_\infty(\mathcal{T}_\ell) \,,\, v\rangle_{\Gamma_\ell} = \langle f_\ell \,,\, v\rangle_{\Gamma_\ell} \quad \text{for all } v \in \mathcal{X}_\infty(\mathcal{T}_\ell).$$

Then, there holds for all $v \in \mathcal{X}_\infty$

$$\langle \mathcal{V}U_\infty \,,\, v\rangle_\Gamma = \langle f \,,\, v\rangle_\Gamma = \langle f_\ell \,,\, v \circ \gamma_\ell^{-1}|\partial_{\Gamma_\ell}\gamma_\ell^{-1}|\rangle_{\Gamma_\ell}.$$

For $v \in \bigcup_{\ell\in\mathbb{N}_0} \big\{v|\partial_\Gamma\gamma_\ell| \,:\, v \in \mathcal{P}^0(\mathcal{T}_\ell^\Gamma)\big\}$ (which is a dense subset of $\mathcal{X}_\infty$), there holds $v = w|\partial_\Gamma\gamma_k|$ for some $w \in \mathcal{P}^0(\mathcal{T}_k^\Gamma)$ and $k \in \mathbb{N}_0$. In this case, we get with (6.2.12) that

$$\begin{aligned} v \circ \gamma_\ell^{-1}|\partial_{\Gamma_\ell}\gamma_\ell^{-1}| &= w \circ \gamma_\ell^{-1}|\partial_\Gamma\gamma_k| \circ \gamma_\ell^{-1}|\partial_{\Gamma_\ell}\gamma_\ell^{-1}| \\ &= w \circ \gamma_\ell^{-1}|\partial_{\Gamma_\ell}(\gamma_k \circ \gamma_\ell^{-1})| \in \mathcal{X}_\infty(\mathcal{T}_\ell). \end{aligned} \tag{6.3.22}$$

Together with $\|v\|_{H^{-1/2}(\Gamma)} \simeq \|v \circ \gamma_\ell^{-1}|\partial_{\Gamma_\ell}\gamma_\ell^{-1}|\|_{H^{-1/2}(\Gamma_\ell)}$ by Lemma 6.2.19, this implies

$$v \circ \gamma_\ell^{-1}|\partial_{\Gamma_\ell}\gamma_\ell^{-1}| \in \mathcal{X}_\infty(\mathcal{T}_\ell) \quad \text{for all } v \in \mathcal{X}_\infty. \tag{6.3.23}$$

Analogously, we obtain

$$w \circ \gamma_\ell|\partial_\Gamma\gamma_\ell| \in \mathcal{X}_\infty \quad \text{for all } w \in \mathcal{X}_\infty(\mathcal{T}_\ell). \tag{6.3.24}$$

This shows

$$\langle \mathcal{V}U_\infty\,,\, v\rangle_\Gamma = \langle f_\ell\,,\, v\circ\gamma_\ell^{-1}|\partial_\Gamma\gamma_\ell^{-1}|\rangle_{\Gamma_\ell} = \langle \mathcal{V}_\ell U_\infty(\mathcal{T}_\ell)\,,\, v\circ\gamma_\ell^{-1}|\partial_{\Gamma_\ell}\gamma_\ell^{-1}|\rangle_{\Gamma_\ell} \tag{6.3.25}$$

for all $v \in \mathcal{X}_\infty$.

With $U_\infty - U_\infty(\mathcal{T}_\ell)^\Gamma \in \mathcal{X}_\infty$ by (6.3.24), we obtain with $\widetilde{w} = w\circ\gamma_\ell^{-1}|\partial_\Gamma\gamma_\ell^{-1}|$

$$\begin{aligned}\|U_\infty - U(\mathcal{T}_\ell)^\Gamma\|_{H^{-1/2}(\Gamma)} &\simeq \sup_{w\in\mathcal{X}_\infty\setminus\{0\}} \frac{\langle \mathcal{V}(U_\infty - U(\mathcal{T}_\ell)^\Gamma)\,,\, w\rangle_\Gamma}{\|w\|_{H^{-1/2}(\Gamma)}}\\ &= \sup_{w\in\mathcal{X}_\infty\setminus\{0\}} \frac{\langle \mathcal{V}_\ell(U_\infty(\mathcal{T}_\ell) - U(\mathcal{T}_\ell))\,,\, \widetilde{w}\rangle_{\Gamma_\ell} + \langle \mathcal{V}_\ell U(\mathcal{T}_\ell)\,,\, \widetilde{w}\rangle_{\Gamma_\ell} - \langle \mathcal{V}U(\mathcal{T}_\ell)^\Gamma\,,\, w\rangle_\Gamma}{\|w\|_{H^{-1/2}(\Gamma)}}.\end{aligned} \tag{6.3.26}$$

As in (6.2.49), there holds with Lemma 6.2.19 and (6.2.37)

$$\|U_\infty - U(\mathcal{T}_\ell)^\Gamma\|_{H^{-1/2}(\Gamma)} \lesssim \|U_\infty(\mathcal{T}_\ell) - U(\mathcal{T}_\ell)\|_{H^{-1/2}(\Gamma_\ell)} + \|M_\ell U(\mathcal{T}_\ell)^\Gamma\|_{H^{1/2}(\Gamma)}.$$

The Céa Lemma 6.2.39 (since $\mathcal{P}^0(\mathcal{T}_\ell) \subseteq \mathcal{X}_\infty(\mathcal{T}_\ell)$ and Lemma 6.2.34 conclude

$$\begin{aligned}\|U_\infty - U(\mathcal{T}_\ell)^\Gamma\|_{H^{-1/2}(\Gamma)} \lesssim &\min_{V_\ell\in\mathcal{P}^0(\mathcal{T}_\ell)} \|U_\infty(\mathcal{T}_\ell) - V_\ell\|_{H^{-1/2}(\Gamma_\ell)}\\ &+ \mathrm{geo}(\mathcal{T}_\ell)^{3/2}(1 + |\log(\mathrm{geo}(\mathcal{T}_\ell))|)\|U(\mathcal{T}_\ell)\|_{L^2(\Gamma_\ell)}.\end{aligned} \tag{6.3.27}$$

As in (6.3.26), we get for $V_\ell \in \mathcal{P}^0(\mathcal{T}_\ell)$ and $V_\ell^\Gamma := V_\ell\circ\gamma_\ell|\partial_\Gamma\gamma_\ell|$ that

$$\begin{aligned}\|U_\infty - V_\ell^\Gamma\|_{H^{-1/2}(\Gamma)} &\simeq \sup_{w\in\mathcal{X}_\infty\setminus\{0\}} \frac{\langle \mathcal{V}(U_\infty - V_\ell^\Gamma)\,,\, w\rangle_\Gamma}{\|w\|_{H^{-1/2}(\Gamma)}}\\ &= \sup_{w\in\mathcal{X}_\infty\setminus\{0\}} \frac{\langle \mathcal{V}_\ell(U_\infty(\mathcal{T}_\ell) - V_\ell)\,,\, \widetilde{w}\rangle_{\Gamma_\ell} + \langle \mathcal{V}_\ell V_\ell\,,\, \widetilde{w}\rangle_{\Gamma_\ell} - \langle \mathcal{V}V_\ell^\Gamma\,,\, w\rangle_\Gamma}{\|w\|_{H^{-1/2}(\Gamma)}},\end{aligned} \tag{6.3.28}$$

which implies together with Lemma 6.2.34, Lemma 6.2.19, and the uniform ellipticity (6.2.38) that

$$\begin{aligned}\|U_\infty(\mathcal{T}_\ell) - V_\ell\|_{H^{-1/2}(\Gamma_\ell)} &\lesssim \sup_{\widetilde{w}\in\mathcal{X}_\infty(\mathcal{T}_\ell)\setminus\{0\}} \frac{\langle \mathcal{V}_\ell(U_\infty(\mathcal{T}_\ell) - V_\ell)\,,\, \widetilde{w}\rangle_{\Gamma_\ell}}{\|\widetilde{w}\|_{H^{-1/2}(\Gamma_\ell)}}\\ &\overset{(6.3.23)}{\simeq} \sup_{w\in\mathcal{X}_\infty\setminus\{0\}} \frac{\langle \mathcal{V}_\ell(U_\infty(\mathcal{T}_\ell) - V_\ell)\,,\, \widetilde{w}\rangle_{\Gamma_\ell}}{\|w\|_{H^{-1/2}(\Gamma)}}\\ &\overset{(6.3.28)}{\lesssim} \|U_\infty - V_\ell^\Gamma\|_{H^{-1/2}(\Gamma_\ell)}\\ &\qquad + \mathrm{geo}(\mathcal{T}_\ell)^{3/2}(1 + |\log(\mathrm{geo}(\mathcal{T}_\ell))|))\|V_\ell\|_{L^2(\Gamma_\ell)}.\end{aligned}$$

This and (6.3.27) imply

$$\begin{aligned}\|U_\infty - U(\mathcal{T}_\ell)^\Gamma\|_{H^{-1/2}(\Gamma)} \lesssim &\min_{V_\ell\in\mathcal{P}^0(\mathcal{T}_\ell)} \Big(\|U_\infty - V_\ell^\Gamma\|_{H^{-1/2}(\Gamma)}\\ &+ \mathrm{geo}(\mathcal{T}_\ell)^{3/2}(1 + |\log(\mathrm{geo}(\mathcal{T}_\ell))|)(\|V_\ell\|_{L^2(\Gamma_\ell)} + \|U(\mathcal{T}_\ell)\|_{L^2(\Gamma_\ell)})\Big).\end{aligned} \tag{6.3.29}$$

For all $k \in \mathbb{N}_0$, there holds with Lemma 6.2.14

$$\begin{aligned}\|U_\infty - V_\ell^\Gamma\|_{H^{-1/2}(\Gamma)} &\leq \|U_\infty - V_\ell\circ\gamma_\ell\|_{H^{-1/2}(\Gamma)} + \|1 - |\partial_\Gamma\gamma_\ell|\|_{L^\infty(\Gamma)}\|V_\ell\circ\gamma_\ell\|_{L^2(\Gamma)}\\ &\lesssim \|U_\infty - V_\ell\circ\gamma_\ell\|_{H^{-1/2}(\Gamma)} + \mathrm{geo}(\mathcal{T}_\ell)^2\|V_\ell\|_{L^2(\Gamma_\ell)}.\end{aligned}$$

With (6.3.29), this shows

$$\|U_\infty - U(\mathcal{T}_\ell)^\Gamma\|_{H^{-1/2}(\Gamma)} \lesssim \min_{V_\ell \in \mathcal{P}^0(\mathcal{T}_\ell)} \Big(\|U_\infty - V_\ell \circ \gamma_\ell\|_{H^{-1/2}(\Gamma)} \tag{6.3.30}$$
$$+ (\mathrm{geo}(\mathcal{T}_\ell)^2 + \mathrm{geo}(\mathcal{T}_\ell)^{3/2}(1 + |\log(\mathrm{geo}(\mathcal{T}_\ell))|))(\|V_\ell\|_{L^2(\Gamma_\ell)} + \|U(\mathcal{T}_\ell)\|_{L^2(\Gamma_\ell)}\Big).$$

The term $\mathrm{geo}(\mathcal{T}_\ell)^{3/2}(1 + |\log(\mathrm{geo}(\mathcal{T}_\ell))|)\|U(\mathcal{T}_\ell)\|_{L^2(\Gamma_\ell)}$ converges to zero by use of assumption (6.3.20) and (6.3.21).

It thus remains to prove that $U_\infty \in \overline{\bigcup_{\ell\in\mathbb{N}_0} \{V_\ell \circ \gamma_\ell : V_\ell \in \mathcal{P}^0(\mathcal{T}_\ell)\}} = \overline{\bigcup_{\ell\in\mathbb{N}_0} \mathcal{P}^0(\mathcal{T}_\ell^\Gamma)} \subseteq H^{-1/2}(\Gamma)$. To that end, we show that $\mathcal{X}_\infty \cap \bigcup_{\ell\in\mathbb{N}_0} \mathcal{P}^0(\mathcal{T}_\ell^\Gamma)$ is dense in $\mathcal{X}_\infty \cap L^2(\Gamma)$ with respect to the L^2-norm. Consider $\Gamma_0 := \{x \in \Gamma : \lim_{\ell\to\infty} h_\ell \circ \gamma_\ell(x) = 0\}$. Obviously, $\bigcup_{\ell\in\mathbb{N}_0} \mathcal{P}^0(\mathcal{T}_\ell^\Gamma)|_{\Gamma_0}$ is dense in $L^2(\Gamma_0)$ and thus also in $\mathcal{X}_\infty \cap L^2(\Gamma_0)$. For all $x \in \Gamma \setminus \Gamma_0$, there exists $\ell_0 \in \mathbb{N}$ such that $x \in T_x \in \mathcal{T}_{\ell_0}$ with $T_x \subseteq \Gamma$ and $T_x \in \mathcal{T}_\ell$ for all $\ell \geq \ell_0$. This implies $\partial_\Gamma \gamma_\ell|_{T_x} = t_\Gamma|_{T_x}$ and hence constant for all $\ell \geq \ell_0$. Moreover, $\partial_\Gamma \gamma_\ell|_{T_x} = c_\ell$ for all $\ell < \ell_0$, where $c_\ell \in \mathbb{R}^2$ depends only on $t_\Gamma|_{T_x}$ and the father element $T' \in \mathcal{T}_\ell$ of T_x. This shows that $\mathcal{X}_\infty|_{\Gamma\setminus\Gamma_0} = \bigcup_{\ell\in\mathbb{N}_0} \mathcal{P}^0(\mathcal{T}_\ell^\Gamma)|_{\Gamma\setminus\Gamma_0}$. Altogether, this implies that $\mathcal{X}_\infty \cap \bigcup_{\ell\in\mathbb{N}_0} \mathcal{P}^0(\mathcal{T}_\ell^\Gamma)$ is dense in $\mathcal{X}_\infty \cap L^2(\Gamma)$ with respect to the L^2-norm. Hence, $\bigcup_{\ell\in\mathbb{N}_0} \mathcal{P}^0(\mathcal{T}_\ell^\Gamma)$ is dense in $\mathcal{X}_\infty$ with respect to the $H^{-1/2}(\Gamma)$-norm and thus $U_\infty \in \overline{\bigcup_{\ell\in\mathbb{N}_0} \mathcal{P}^0(\mathcal{T}_\ell^\Gamma)}$.

Given $\varepsilon > 0$, this allows to choose $V_{\ell_0} \in \mathcal{P}^p(\mathcal{T}_{\ell_0})$ such that $\|U_\infty - V_{\ell_0} \circ \gamma_{\ell_0}\|_{H^{-1/2}(\Gamma)} \leq \varepsilon$. Then, choose $k \geq \ell_0$ such that all $\ell \geq k$ satisfy

$$(\mathrm{geo}(\mathcal{T}_\ell)^2 + \mathrm{geo}(\mathcal{T}_\ell)^{3/2}(1 + |\log(\mathrm{geo}(\mathcal{T}_\ell))|))\|V_{\ell_0}\|_{L^2(\Gamma)} \leq \varepsilon.$$

Since $V_{\ell_0} \circ \gamma_{\ell_0} \circ \gamma_\ell^{-1} \in \mathcal{P}^0(\mathcal{T}_\ell)$ and $V_{\ell_0} \circ \gamma_{\ell_0} \circ \gamma_\ell^{-1} \circ \gamma_\ell = V_{\ell_0} \circ \gamma_\ell$, (6.3.30) shows $\|U_\infty - U(\mathcal{T}_\ell)^\Gamma\|_{H^{-1/2}(\Gamma)} \lesssim 2\varepsilon$ for all $\ell \geq k$. This concludes $\|U_\infty - U(\mathcal{T}_\ell)^\Gamma\|_{H^{-1/2}(\Gamma)} \to 0$ as $\ell \to \infty$.

The above and the definition of $\varrho(\mathcal{T}_\ell, \mathcal{T}_{\ell+1})$ shows $\lim_{\ell\to\infty} \varrho(\mathcal{T}_\ell, \mathcal{T}_{\ell+1}) = 0$, where we use

$$(|\log(|\mathcal{T}_\ell|)| + |\log(\min h_\ell)|) \lesssim \ell \quad \text{for all } \ell \in \mathbb{N}_0,$$

which follows from the fact that each step maximally doubles the number of elements and approximately halves the size of the elements. This concludes the proof. □

6.4. Main result

Theorem 6.4.1. *Define $\mathbb{T}$ as in Section 6.2.5. Assume that all $\mathcal{T}_\star \in \mathbb{T}$ satisfy $h_\star \leq C_\Gamma^{-1}\kappa_\Gamma^{-1}/2$ and $\mathrm{geo}(\mathcal{T}_\star) \leq \min\{C_{\mathrm{ext}}^{-1}/2, C_\Gamma^{-1}/2, C_\Gamma^{-1}\kappa_\Gamma^{-1}/2\}$ (such that Lemma 6.2.9 (i)–(iii) hold). Then, the error estimator $\eta(\cdot)$ satisfies reliability* (6.2.47). *Under the assumption* (6.3.1), *the error estimator $\rho(\cdot)$ from* (6.2.5) *satisfies* (E1) *with $\varrho(\cdot,\cdot)$ as stated in Theorem 6.3.8. Moreover, under the assumptions* (6.3.19)–(6.3.20), *there holds convergence*

$$\|u - U(\mathcal{T}_\ell)^\Gamma\|_{H^{-1/2}(\Gamma)} \leq C_{\mathrm{rel}}\eta(\mathcal{T}_\ell) \to 0 \quad \textit{as } \ell \to \infty.$$

Proof. Since $\mathcal{T}_0$ satisfies (i)–(iii) from Lemma 6.2.9, all $\mathcal{T}_\star \in \mathbb{T}$ satisfy (i)–(iii), too. Therefore, Theorem 6.2.28 and Theorem 6.3.8 prove (6.2.47) and (E1). The estimator $\rho(\cdot)$ satisfies Dörfler marking (6.2.6a) in each step of Algorithm 6.2.2. Therefore, Lemma 2.3.5 proves estimator reduction 2.3.8 for $\rho(\cdot)$. Lemma 6.3.10 shows $\lim_{\ell\to\infty} \varrho(\mathcal{T}_\ell, \mathcal{T}_{\ell+1}) = 0$. Hence, Lemma 2.3.6 concludes the proof. □

Consequence 6.4.2. *Under the assumptions* (6.3.1)–(6.3.20), *Algorithm 6.2.2 leads to* $\lim_{\ell\to\infty} \varrho(\mathcal{T}_\ell, \mathcal{T}_{\ell+1}) = 0$ *and hence convergence in the sense of Theorem 2.3.3 (i).*

CHAPTER 7

General Quasi-Orthogonality (E2) For Non-Symmetric Problems

7.1. Introduction, state of the art & outline

The general quasi-orthogonality (E2) renders an important tool for the optimality proofs of the previous chapters. Section 2.6 shows that it is even necessary if the algorithm is R-linear convergent. The following investigations provide sufficient assumptions for (E2) to hold. Section 7.2–7.4 appear in similar manner in [**46**]. Figure 1 depicts a geometric view on the general quasi-orthogonality (E2).

7.2. General quasi-orthogonality (E2) for linear second-order elliptic equations

We stress that the quasi-orthogonality proof makes explicit use of the fact that we already have convergence $U(\mathcal{T}_\ell) \to u$ in $H^1_0(\Omega)$. We consider the setting of Section 3.6.1. The

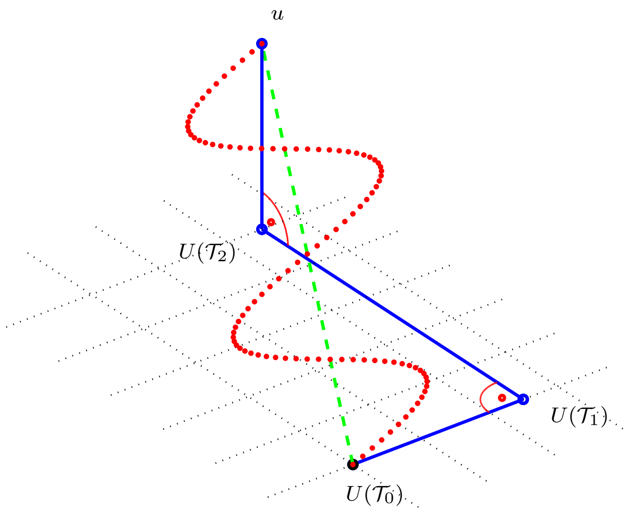

FIGURE 1. Geometric view on the general quasi-orthogonality (E2). For $\varrho(\mathcal{T}, \widehat{\mathcal{T}}) \simeq \|U(\mathcal{T}) - U(\widehat{\mathcal{T}})\|$, the general quasi-orthogonality bounds the ℓ_2-sum of the squared perturbations. Since the adaptive algorithm performs a step-by-step optimization of the triangulations without any foresight, it controls the perturbations $\varrho(\mathcal{T}_\ell, \mathcal{T}_{\ell+1})$ only. By Galerkin orthogonality, the solutions are in some sense orthogonal to each other. The general quasi-orthogonality (E2) ensures that the overall approximation (dashed green line), which is measured by $\eta(\mathcal{T}_\ell)$, is an upper bound for the sum of the individual steps. This would be automatically the case if $\eta(\mathcal{T}_\ell)$ is a Hilbert norm which corresponds to the orthogonality between the solutions. If (E2) is not satisfied, one has no argument that the individual steps approach the exact solution in an efficient way (dotted red line).

operator $\mathcal{L}$ is split as follows

$$\mathcal{A}u = -\mathrm{div}\boldsymbol{A}\nabla u,$$
$$\mathcal{C}u = \boldsymbol{b}\cdot\nabla u + cu.$$

The following observation is the key element of the proof of (E2).

Lemma 7.2.1. *The operators $\mathcal{A},\mathcal{C} : H^1_0(\Omega) \to H^{-1}(\Omega)$ are bounded. Moreover, $\mathcal{A}$ is symmetric, i.e., $\langle \mathcal{A}u\,,\,v\rangle = \langle \mathcal{A}v\,,\,u\rangle$ for all $u,v\in H^1_0(\Omega)$, and $\mathcal{C}$ is compact.*

Proof. The symmetry of $\mathcal{A}$ is obvious as $\boldsymbol{A}(x)$ is symmetric, and both operators $\mathcal{A}$ and $\mathcal{C}$ are also bounded, i.e.,

$$\|\mathcal{A}v\|_{H^{-1}(\Omega)} \le \|\boldsymbol{A}\|_{L^\infty(\Omega)}\|\nabla v\|_{L^2(\Omega)},$$
$$\|\mathcal{C}v\|_{H^{-1}(\Omega)} \le \|\mathcal{C}v\|_{L^2(\Omega)} \lesssim (\|\boldsymbol{b}\|_{L^\infty(\Omega)} + \|c\|_{L^\infty(\Omega)})\|\nabla v\|_{L^2(\Omega)},$$

for all $v\in H^1_0(\Omega)$. This implies that $\widetilde{\mathcal{C}} : H^1_0(\Omega)\to L^2(\Omega)$, $\widetilde{\mathcal{C}}v := \mathcal{C}v$ is well-defined and bounded. It remains to prove that $\mathcal{C}$ is compact. The Rellich compactness theorem shows that the embedding $\iota : H^1_0(\Omega)\hookrightarrow L^2(\Omega)$ is a compact operator. Therefore, according to Schauder's theorem, see e.g. [**73**, Theorem 4.19], the adjoint operator $\iota^\star : L^2(\Omega)\to H^{-1}(\Omega)$ is also compact. Obviously, $\iota^\star : L^2(\Omega)\to H^{-1}(\Omega)$ coincides with the natural embedding, and we may write

$$\mathcal{C} = \iota^\star\circ\widetilde{\mathcal{C}} : H^1_0(\Omega)\to L^2(\Omega)\to H^{-1}(\Omega).$$

Therefore, $\mathcal{C}$ is the composition of a bounded operator and a compact operator and hence compact. This concludes the proof. □

Lemma 7.2.2. *Let $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}_0}$ denote the output of Algorithm 2.2.1. Assume that there holds convergence $\lim_{\ell\to\infty}\|U(\mathcal{T}_\ell)-u\|_{H^1_0(\Omega)} = 0$ with u and $U(\mathcal{T}_\ell)$ from Section 3.6.1. The sequences $(e_\ell)_{\ell\in\mathbb{N}}$ and $(E_\ell)_{\ell\in\mathbb{N}}$ defined by*

$$e_\ell := \begin{cases}\frac{u-U(\mathcal{T}_\ell)}{\|\nabla(u-U(\mathcal{T}_\ell))\|_{L^2(\Omega)}}, & \text{for } u\neq U(\mathcal{T}_\ell),\\ 0, & \text{else,}\end{cases} \qquad \text{and}$$
$$E_\ell := \begin{cases}\frac{U(\mathcal{T}_{\ell+1})-U(\mathcal{T}_\ell)}{\|\nabla(U(\mathcal{T}_{\ell+1})-U(\mathcal{T}_\ell))\|_{L^2(\Omega)}}, & \text{for } U(\mathcal{T}_{\ell+1})\neq U(\mathcal{T}_\ell),\\ 0, & \text{else,}\end{cases}$$

converge to zero, weakly in $H^1_0(\Omega)$, i.e.,

$$\lim_{\ell\to\infty}\langle w\,,\,e_\ell\rangle = 0 = \lim_{\ell\to\infty}\langle w\,,\,E_\ell\rangle \quad \text{for all } w\in H^{-1}(\Omega), \tag{7.2.1}$$

where $\langle\cdot\,,\,\cdot\rangle$ denotes the extended $L^2(\Omega)$-scalar product.

Proof. We prove weak convergence of e_ℓ to zero. The weak convergence of E_ℓ follows with the same arguments. Let (e_{ℓ_j}) be a subsequence of (e_ℓ). Due to boundedness $\|\nabla e_{\ell_j}\|_{L^2(\Omega)}\le 1$ for all $j\in\mathbb{N}$, we may extract a weakly convergent subsequence (e_{ℓ_k}) of (e_{ℓ_j}) with

$$e_{\ell_k}\rightharpoonup w\in H^1_0(\Omega).$$

First, note that convergence $\lim_{\ell\to\infty}\|U(\mathcal{T}_\ell)-u\|_{H^1_0(\Omega)} = 0$ implies that $u, U(\mathcal{T}_\ell)\in\mathcal{X}_\infty := \overline{\bigcup_{\ell\in\mathbb{N}_0}\mathcal{S}^p_0(\mathcal{T}_\ell)}\subseteq H^1_0(\Omega)$ implies $e_\ell\in\mathcal{X}_\infty$ and hence $w\in\mathcal{X}_\infty$. Second, for all $\ell_k\ge\ell$ with $e_{\ell_k}\neq 0$ and all $V\in\mathcal{S}^p_0(\mathcal{T}_\ell)$, it holds

$$b(e_{\ell_k}\,,\,V) = \|\nabla(u-U_{\ell_k})\|^{-1}_{L^2(\Omega)}b(u-U_{\ell_k},V) = 0.$$

For all $\ell \in \mathbb{N}_0$, $V \in \mathcal{S}_0^p(\mathcal{T}_\ell)$, and $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that all $k \geq k_0$ satisfy

$$|b(w\,,\,V)| = |\langle w\,,\,\mathcal{L}^\star V\rangle| \leq \varepsilon + |\langle e_{\ell_k}\,,\,\mathcal{L}^\star V\rangle| = \varepsilon + |b(e_{\ell_k}\,,\,V)| = \varepsilon,$$

since k_0 is chosen large enough such that $\ell_k \geq \ell$. Therefore

$$b(w\,,\,V) = 0 \quad \text{for all } V \in \mathcal{S}_0^p(\mathcal{T}_\ell) \text{ and } \ell \in \mathbb{N}.$$

Due to definiteness of $b(\cdot,\cdot)$ and $w \in \mathcal{X}_\infty := \overline{\bigcup_{\ell\in\mathbb{N}} \mathcal{S}_0^p(\mathcal{T}_\ell)}$, this implies $w = 0$. Altogether, we have now shown that each subsequence of e_ℓ has a subsequence which converges weakly to zero. This immediately implies weak convergence $e_\ell \rightharpoonup 0$ as $\ell \to \infty$. □

The previous lemma shows that although $(E_\ell)_{\ell\in\mathbb{N}}$ is no orthonormal sequence, it shares the property of weak convergence to zero with orthonormal systems. Note that our proof already used convergence $U_\ell \to u$ as $\ell \to \infty$ in the sense that we required $u - U_\ell \in \mathcal{X}_\infty$. This suffices to prove the following quasi-Pythagoras theorem.

Proposition 7.2.3. *Define $\|\cdot\| := b(\cdot\,,\,\cdot)^{1/2}$ with $b(\cdot\,,\,\cdot)$ from Section 3.6.1. Assume that $\lim_{\ell\to\infty} \|U(\mathcal{T}_\ell) - u\|_{H_0^1(\Omega)} = 0$. Then, for all $0 < \varepsilon < 1$, there exists $\ell_{\mathrm{qo}} \in \mathbb{N}$ such that*

$$\|U(\mathcal{T}_{\ell+1}) - U(\mathcal{T}_\ell)\|^2 \leq \frac{1}{1-\varepsilon}\,\|u - U(\mathcal{T}_\ell)\|^2 - \|u - U(\mathcal{T}_{\ell+1})\|^2 \tag{7.2.2}$$

for all $\ell \geq \ell_{\mathrm{qo}}$, where u and $U(\mathcal{T}_\ell)$ are defined in Section 3.6.1.

Remark 7.2.4. *As in* [**36**, Theorem 5.1]*, the quasi-orthogonality* (7.2.2) *is an asymptotic statement. The advantage here is that* (7.2.2) *is automatically guaranteed after ℓ_0 steps of Algorithm 2.2.1. In contrast to that,* [**36**, Assumption 4.3] *used to prove* [**36**, Theorem 5.1]*, includes a element-size condition of the form $|T|^{1/d} \leq h_{\max} \ll 1$ for all $T \in \mathcal{T}_\ell$ which is not necessarily enforced by Algorithm 2.2.1, unless the initial triangulation is already sufficiently fine. Moreover, $h_{\max}$ is unknown in general and depends on the regularity of the dual problem.*

Proof. Lemma 7.2.2 shows that $e_\ell, E_\ell \rightharpoonup 0$ as $\ell \to \infty$. Due to Lemma 7.2.1, $\mathcal{C}$ is compact. Therefore, we have strong convergence $\mathcal{C}e_\ell, \mathcal{C}E_\ell \to 0$ in $H^{-1}(\Omega)$ as $\ell \to \infty$. With $\langle\cdot\,,\,\cdot\rangle := \langle\cdot\,,\,\cdot\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}$, this shows

$$\begin{aligned}\langle \mathcal{C}(u - U(\mathcal{T}_{\ell+1}))\,, U(\mathcal{T}_{\ell+1}) - U(\mathcal{T}_\ell)\rangle &= \langle \mathcal{C}e_{\ell+1}\,,\,U(\mathcal{T}_{\ell+1}) - U(\mathcal{T}_\ell)\rangle \|\nabla(u - U(\mathcal{T}_{\ell+1}))\|_{L^2(\Omega)}\\ &\leq \|\mathcal{C}e_{\ell+1}\|_{H^{-1}(\Omega)} \|\nabla(u - U(\mathcal{T}_{\ell+1}))\|_{L^2(\Omega)} \|\nabla(U(\mathcal{T}_{\ell+1}) - U(\mathcal{T}_\ell))\|_{L^2(\Omega)}\end{aligned}$$

as well as

$$\begin{aligned}&\langle \mathcal{C}(U(\mathcal{T}_{\ell+1}) - U(\mathcal{T}_\ell))\,,\,u - U(\mathcal{T}_{\ell+1})\rangle\\ &\quad= \langle \mathcal{C}E_\ell\,,\,u - U(\mathcal{T}_{\ell+1})\rangle \|\nabla(U(\mathcal{T}_{\ell+1}) - U(\mathcal{T}_\ell))\|_{L^2(\Omega)}\\ &\quad\leq \|\mathcal{C}E_\ell\|_{H^{-1}(\Omega)} \|\nabla(u - U(\mathcal{T}_{\ell+1}))\|_{L^2(\Omega)} \|\nabla(U(\mathcal{T}_{\ell+1}) - U(\mathcal{T}_\ell))\|_{L^2(\Omega)}.\end{aligned}$$

For any $\delta > 0$, this may be employed to obtain some $\ell_0 \in \mathbb{N}$ such that for all $\ell \geq \ell_0$, it holds

$$\begin{aligned}|\langle \mathcal{C}(U(\mathcal{T}_{\ell+1}) - U(\mathcal{T}_\ell))\,,\,u - U(\mathcal{T}_{\ell+1})\rangle| + |\langle \mathcal{C}(u - U(\mathcal{T}_{\ell+1}))\,,\,U(\mathcal{T}_{\ell+1}) - U(\mathcal{T}_\ell)\rangle|&\\ \leq \delta \|\nabla(u - U(\mathcal{T}_{\ell+1}))\|_{L^2(\Omega)} \|\nabla(U(\mathcal{T}_{\ell+1}) - U(\mathcal{T}_\ell))\|_{L^2(\Omega)}.&\end{aligned}$$

Together with Galerkin orthogonality

$$0 = b(u - U(\mathcal{T}_{\ell+1}), V_{\ell+1}) = \langle \mathcal{L}(u - U(\mathcal{T}_{\ell+1}))\,,\,V_{\ell+1}\rangle \quad \text{for all } V_{\ell+1} \in \mathcal{S}_0^p(\mathcal{T}_{\ell+1}),$$

we estimate

$$
\begin{aligned}
&|\langle \mathcal{L}(U(\mathcal{T}_{\ell+1}) - U(\mathcal{T}_\ell))\,,\, u - U(\mathcal{T}_{\ell+1})\rangle| \\
&\qquad = |\langle \mathcal{A}(u - U(\mathcal{T}_{\ell+1}))\,,\, U(\mathcal{T}_{\ell+1}) - U(\mathcal{T}_\ell)\rangle + \langle \mathcal{C}(U(\mathcal{T}_{\ell+1}) - U(\mathcal{T}_\ell))\,,\, u - U(\mathcal{T}_{\ell+1})\rangle| \\
&\qquad \le |\langle \mathcal{L}(u - U(\mathcal{T}_{\ell+1}))\,,\, U(\mathcal{T}_{\ell+1}) - U(\mathcal{T}_\ell)\rangle| + |\langle \mathcal{C}(U(\mathcal{T}_{\ell+1}) - U(\mathcal{T}_\ell))\,,\, u - U(\mathcal{T}_{\ell+1})\rangle| \\
&\qquad\qquad + |\langle \mathcal{C}(u - U(\mathcal{T}_{\ell+1}))\,,\, U(\mathcal{T}_{\ell+1}) - U(\mathcal{T}_\ell)\rangle| \\
&\qquad \le \delta \|\nabla(u - U(\mathcal{T}_{\ell+1}))\|_{L^2(\Omega)} \|\nabla(U(\mathcal{T}_{\ell+1}) - U(\mathcal{T}_\ell))\|_{L^2(\Omega)}.
\end{aligned} \tag{7.2.3}
$$

The definition of $\|\cdot\|$ and Galerkin orthogonality (2.7.3) yield

$$
\begin{aligned}
\|u - U(\mathcal{T}_{\ell+1})\|^2 + \|U(\mathcal{T}_{\ell+1}) - U(\mathcal{T}_\ell)\|^2 + 2\langle \mathcal{L}(U(\mathcal{T}_{\ell+1}) - U(\mathcal{T}_\ell))\,,\, u - U(\mathcal{T}_{\ell+1})\rangle& \\
= \|u - U(\mathcal{T}_\ell)\|^2,&
\end{aligned}
$$

whence

$$
\begin{aligned}
\|U(\mathcal{T}_{\ell+1}) - U(\mathcal{T}_\ell)\|^2 \le \|u - U(\mathcal{T}_\ell)\|^2 - \|u - U(\mathcal{T}_{\ell+1})\|^2& \\
+ 2\delta C_{\mathrm{norm}}^2 \|u - U(\mathcal{T}_{\ell+1})\| \|U(\mathcal{T}_{\ell+1}) - U(\mathcal{T}_\ell)\|,&
\end{aligned}
$$

where $C_{\mathrm{norm}} > 0$ is defined in Section 3.6.1. The application of Young's inequality $2ab \le a^2 + b^2$ and the choice $\varepsilon = \delta C_{\mathrm{norm}}^2$ conclude the proof. □

Theorem 7.2.5. *Assume that* $\lim_{\ell\to\infty} \|U(\mathcal{T}_\ell) - u\| = 0$ *with* u *and* $U(\mathcal{T}_\ell)$ *from Section 3.6.1. Then, for all* $\varepsilon_{\mathrm{qo}} > 0$, *there exists* $C_{\mathrm{qo}} > 0$ *such that* (E2) *holds with* $\varrho(\mathcal{T}_\ell, \mathcal{T}_{\ell+1}) := \|U(\mathcal{T}_\ell) - U(\mathcal{T}_{\ell+1})\| L^2(\Omega)$ *and each estimator* $\eta(\cdot)$ *which is reliable, i.e.,*

$$
\|u - U(\mathcal{T}_\ell)\| \le C_{\mathrm{rel}} \eta(\mathcal{T}_\ell) \quad \textit{for all } \ell \in \mathbb{N}_0.
$$

Particularly, this is satisfied by the error estimator $\eta(\cdot)$ *from Section 3.6.1.*

Proof. Proposition 7.2.3 proves the quasi-orthogonality (2.7.5) for all $\ell \ge \ell_0$ with $\varrho(\mathcal{T}_\ell, \mathcal{T}_{\ell+1}) = \|\nabla(U(\mathcal{T}_\ell) - U(\mathcal{T}_{\ell+1}))\|$ and $\alpha_\ell := \|u - U(\mathcal{T}_\ell)\|^2$. The Céa lemma 3.6.5 and reliability (in the setting of Section 3.6.1 from (2.4.1)) imply

$$
\varrho(\mathcal{T}_\ell, \mathcal{T}_{\ell+1}) \lesssim \|u - U(\mathcal{T}_\ell)\| \lesssim \eta(\mathcal{T}_\ell) \quad \text{for all } \ell \in \mathbb{N}_0.
$$

Therefore, Lemma 2.7.3 proves for all $\ell \ge \ell_0$.

$$
\sum_{k=\ell}^{\infty} \Big(\|U(\mathcal{T}_k) - U(\mathcal{T}_{k+1})\|^2 - \varepsilon_{\mathrm{qo}} \eta(\mathcal{T}_k)^2 \Big) \le C'_{\mathrm{qo}} \eta(\mathcal{T}_\ell)^2.
$$

For all $\ell < \ell_0$, there exists $C_\ell > 0$ with

$$
\sum_{k=\ell}^{\ell_0} \Big(\|U(\mathcal{T}_k) - U(\mathcal{T}_{k+1})\|^2 - \varepsilon_{\mathrm{qo}} \eta(\mathcal{T}_k)^2 \Big) \le C_\ell \eta(\mathcal{T}_\ell)^2,
$$

since both sides of the inequality are finite and if $\eta(\mathcal{T}_\ell) = 0$, then reliability (2.4.1) and the Céa lemma (3.6.5) imply

$$
\|U(\mathcal{T}_k) - U(\mathcal{T}_{k+1})\| \lesssim \|u - U(\mathcal{T}_\ell)\| \lesssim \eta(\mathcal{T}_\ell) = 0 \quad \text{for all } k \ge \ell.
$$

With $C_{\mathrm{qo}} := C'_{\mathrm{qo}} + \max_{\ell=0,\dots,\ell_0-1} C_\ell$, this concludes the proof. □

7.3. General quasi-orthogonality (E2) for problems with Gårding inequality

Lemma 7.3.1. *Let $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}_0}$ denote the output of Algorithm 3.6.3. Assume definiteness for all $v \in \mathcal{X}_\infty := \overline{\bigcup_{\ell=0}^\infty \mathcal{S}_0^p(\mathcal{T}_\ell)}$, i.e.,*

$$b(w\,,\,v) = 0 \quad \text{for all } v \in \mathcal{X}_\infty \quad \Longrightarrow \quad w = 0. \tag{7.3.1}$$

Then, the sequences $(e_\ell)_{\ell\in\mathbb{N}}$ and $(E_\ell)_{\ell\in\mathbb{N}}$ (with u and $U(\mathcal{T}_\ell)$ from Section 3.6.2) defined by

$$e_\ell := \begin{cases} \frac{u-U(\mathcal{T}_\ell)}{\|\nabla(u-U(\mathcal{T}_\ell))\|_{L^2(\Omega)}}, & \text{for } u \neq U(\mathcal{T}_\ell), \\ 0, & \text{else,} \end{cases} \quad \text{and}$$
$$E_\ell := \begin{cases} \frac{U(\mathcal{T}_{\ell+1})-U(\mathcal{T}_\ell)}{\|\nabla(U(\mathcal{T}_{\ell+1})-U(\mathcal{T}_\ell))\|_{L^2(\Omega)}}, & \text{for } U(\mathcal{T}_{\ell+1}) \neq U(\mathcal{T}_\ell), \\ 0, & \text{else,} \end{cases}$$

for all $\ell \geq \ell_0$ (from Lemma 3.6.10) converge to zero, weakly in $H_0^1(\Omega)$ in the sense (7.2.1).

Proof. We prove weak convergence of e_ℓ to zero. The weak convergence of E_ℓ follows with the same arguments. Let (e_{ℓ_j}) be a subsequence of (e_ℓ). Due to boundedness $\|\nabla e_{\ell_j}\|_{L^2(\Omega)} \leq 1$ for all $j \in \mathbb{N}$, we may extract a weakly convergent subsequence (e_{ℓ_k}) of (e_{ℓ_j}) with

$$e_{\ell_k} \rightharpoonup w \in H_0^1(\Omega).$$

Lemma 3.6.11 proves $\lim_{\ell\to\infty} \|\nabla(u-U(\mathcal{T}_\ell))\|_{L^2(\Omega)} = 0$ and particularly $u \in \mathcal{X}_\infty$. This implies $e_\ell \in \mathcal{X}_\infty$ and hence $w \in \mathcal{X}_\infty$. For all $\ell_k \geq \ell$ with $e_{\ell_k} \neq 0$ and all $V \in \mathcal{S}_0^p(\mathcal{T}_\ell)$, it holds

$$b(e_{\ell_k}\,,\,V) = \|\nabla(u-U_{\ell_k})\|_{L^2(\Omega)}^{-1} b(u-U_{\ell_k}, V) = 0.$$

For all $\ell \in \mathbb{N}$, $V \in \mathcal{S}_0^p(\mathcal{T}_\ell)$, and $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that all $k \geq k_0$ satisfy

$$|b(w\,,\,V)| = |\langle w\,,\,\mathcal{L}^\star V\rangle| \leq \varepsilon + |\langle e_{\ell_k}\,,\,\mathcal{L}^\star V\rangle| = \varepsilon + |b(e_{\ell_k}\,,\,V)| = \varepsilon,$$

since k_0 is chosen large enough such that $\ell_k \geq \ell$. Therefore

$$b(w\,,\,V) = 0 \quad \text{for all } V \in \mathcal{S}_0^p(\mathcal{T}_\ell) \text{ and } \ell \in \mathbb{N}.$$

Due to (7.3.1) and $w \in \mathcal{X}_\infty$, this implies $w = 0$. Altogether, we have now shown that each subsequence of e_ℓ has a subsequence which converges weakly to zero. This immediately implies weak convergence $e_\ell \rightharpoonup 0$ as $\ell \to \infty$. □

Lemma 7.3.2. *Assume definiteness* (7.3.1). *There exists an index $\ell_{\rm norm} \in \mathbb{N}$ such that for all $\ell \geq \ell_{\rm norm}$ there holds*

$$C_{\rm norm}^{-1}\|u-U_\ell\| \leq \|\nabla(u-U_\ell)\|_{L^2(\Omega)} \leq C_{\rm norm}\|u-U_\ell\| \quad \text{and}$$
$$C_{\rm norm}^{-1}\|U_{\ell+1}-U_\ell\| \leq \|\nabla(U_{\ell+1}-U_\ell)\|_{L^2(\Omega)} \leq C_{\rm norm}\|U_{\ell+1}-U_\ell\|$$

with u and $U(\mathcal{T}_\ell)$ from Section 3.6.2.

Proof. With (3.6.9) and $|b(\cdot,\cdot)| = \|\cdot\|^2$, we may estimate

$$\begin{aligned}\|\nabla(u-U_\ell)\|_{L^2(\Omega)}^2 &\lesssim \|u-U_\ell\|^2 + \|u-U_\ell\|_{L^2(\Omega)}^2 \\ &= \|u-U_\ell\|^2 + \|e_\ell\|_{L^2(\Omega)}^2 \|\nabla(u-U_\ell)\|_{L^2(\Omega)}^2.\end{aligned}$$

Lemma 7.3.1 shows weak convergence $e_\ell \rightharpoonup 0$ in $H^1_0(\Omega)$. The Rellich compactness theorem thus implies strong convergence $e_\ell \to 0$ in $L^2(\Omega)$. Therefore, there exists an index $\ell_{\rm norm} \in \mathbb{N}$ such that there holds

$$\|\nabla(u - U_\ell)\|^2_{L^2(\Omega)} \lesssim \|u - U_\ell\|^2 \quad \text{for all } \ell \geq \ell_{\rm norm}.$$

The remaining statements follow analogously. □

Proposition 7.3.3. *Assume definiteness* (7.3.1). *Then, for all* $0 < \varepsilon < 1$, *there exists* $\ell_{\rm qo} \in \mathbb{N}$ *with* $\ell_{\rm qo} \geq \ell_{\rm norm}$ *such that*

$$\|U(\mathcal{T}_{\ell+1}) - U(\mathcal{T}_\ell)\|^2 \leq \frac{1}{1-\varepsilon}\,\|u - U(\mathcal{T}_\ell)\|^2 - \|u - U(\mathcal{T}_{\ell+1})\|^2 \tag{7.3.2}$$

for all $\ell \geq \ell_{\rm qo}$ *with* u *and* $U(\mathcal{T}_\ell)$ *from Section 3.6.2.*

Proof. With Lemma 7.3.2 and Lemma 7.3.1, the proof follows analogously to the proof of Proposition 7.2.3. □

Theorem 7.3.4. *Assume definiteness* (7.3.1) *and the Céa lemma* (3.6.15) *for all* $\ell \geq \ell_1$ *and some* $\ell_1 \in \mathbb{N}$. *Then, for all* $\varepsilon_{\rm qo} > 0$, *there exists* $C_{\rm qo} > 0$ *such that* (E2) *holds with* $\varrho(\mathcal{T}_\ell, \mathcal{T}_{\ell+1}) := \|\nabla(U(\mathcal{T}_\ell) - U(\mathcal{T}_{\ell+1}))\|_{L^2(\Omega)}$ *for all* $\ell \geq \ell_0$ *with* ℓ_0 *from Lemma 3.6.10 and each estimator* $\eta(\cdot)$ *which is reliable, i.e.,*

$$\|\nabla(u - U(\mathcal{T}_\ell))\|_{L^2(\Omega)} \leq C_{\rm rel}\eta(\mathcal{T}_\ell) \quad \textit{for all } \ell \in \mathbb{N}_0.$$

The solutions u *and* $U(\mathcal{T}_\ell)$ *are defined Section 3.6.2. Particularly, this is satisfied by the error estimator* $\eta(\cdot)$ *from Section 3.6.2.*

Proof. Proposition 7.3.3 proves quasi-orthogonality (2.7.5) with $\varrho(\mathcal{T}_\ell, \mathcal{T}_{\ell+1}) = \|U(\mathcal{T}_\ell) - U(\mathcal{T}_{\ell+1})\|$ and $\alpha_\ell := \|u - U(\mathcal{T}_\ell)\|^2$ for all $\ell \geq \ell_{\rm qo}$. With the Céa lemma 3.6.15, Lemma 7.3.2, and reliability (in the setting of Section 3.6.2, reliability is proved in Lemma 3.6.6), this shows for all $\ell \geq \max\{\ell_{\rm qo}, \ell_1\}$

$$\varrho(\mathcal{T}_\ell, \mathcal{T}_{\ell+1}) \lesssim \|u - U(\mathcal{T}_\ell)\| \lesssim \eta(\mathcal{T}_\ell) \quad \text{for all } \ell \in \mathbb{N}_0.$$

Therefore, Lemma 2.7.3 proves for all $\ell \geq \max\{\ell_{\rm qo}, \ell_1\}$.

$$\sum_{k=\ell}^{\infty} \|U(\mathcal{T}_k) - U(\mathcal{T}_{k+1})\|^2 - \varepsilon_{\rm qo}\eta(\mathcal{T}_k)^2 \leq C'_{\rm qo}\eta(\mathcal{T}_\ell)^2.$$

For all $\ell_0 < \ell < \max\{\ell_{\rm qo}, \ell_1\}$, there exists $C_\ell > 0$ with

$$\sum_{k=\ell}^{\ell_0} \|\nabla(U(\mathcal{T}_k) - U(\mathcal{T}_{k+1}))\|^2_{L^2(\Omega)} - \varepsilon_{\rm qo}\eta(\mathcal{T}_k)^2 \leq C_\ell\eta(\mathcal{T}_\ell)^2,$$

since both sides of the inequality are finite and, by Remark 3.6.7, also $\eta(\mathcal{T}_\ell) > 0$. The combination of the last estimates with the norm equivalence from Lemma 7.3.2 concludes the proof. □

7.4. General quasi-orthogonality (E2) for nonlinear second-order elliptic equations

Similar to the proof in Section 7.2, we derive a corresponding result for the nonlinear case. We consider the setting of Section 3.7.

Lemma 7.4.1. *Recall* $\mathcal{X}_\infty := \overline{\bigcup_{\ell\in\mathbb{N}_0} \mathcal{S}_0^p(\mathcal{T}_\ell)} \subseteq H_0^1(\Omega)$. *The operator* $(D\mathcal{L})|_{\mathcal{X}_\infty}u\colon \mathcal{X}_\infty \to \mathcal{X}_\infty^\star$ *defined in Section 3.7 is injective and has closed range.*

Proof. With (3.7.6) and the definition of the Fréchet derivative, there holds for all $v \in \mathcal{X}_\infty$

$$\begin{aligned}\langle ((D\mathcal{L})|_{\mathcal{X}_\infty}u)(v)\,,\,v\rangle &= \lim_{\delta\to 0}\delta^{-1}\langle \mathcal{L}(u+\delta v) - \mathcal{L}u\,,\,v\rangle\\ &= \lim_{\delta\to 0}\delta^{-2}\langle \mathcal{L}(u+\delta v) - \mathcal{L}u\,,\,u+\delta v - u\rangle\\ &\gtrsim \lim_{\delta\to 0}\delta^{-2}\|\nabla(u+\delta v-u)\|_{L^2(\Omega)}^2 = \|\nabla v\|_{L^2(\Omega)}^2.\end{aligned}$$

Hence, we have $((D\mathcal{L})|_{\mathcal{X}_\infty}u)(v) \neq 0$ in $\mathcal{X}_\infty^\star$ for all $v \in \mathcal{X}_\infty \setminus \{0\}$. Let $w_n \in \mathrm{range}((D\mathcal{L})|_{\mathcal{X}_\infty}u)$ denote a Cauchy sequence. Then, the above estimate proves for $((D\mathcal{L})|_{\mathcal{X}_\infty}u)v_n = w_n$

$$\begin{aligned}\|\nabla(v_n - v_m)\|_{L^2(\Omega)}^2 &\lesssim \langle ((D\mathcal{L})|_{\mathcal{X}_\infty}u)(v_n-v_m)\,,\,v_n - v_m\rangle\\ &\le \|w_n - w_m\|_{\mathcal{X}_\infty^\star}\|\nabla(v_n-v_m)\|_{L^2(\Omega)},\end{aligned}$$

which concludes that $v_n \to v \in \mathcal{X}_\infty$ and hence $w_n \to ((D\mathcal{L})|_{\mathcal{X}_\infty}u)(v) \in \mathcal{X}_\infty^\star$ by continuity of $D\mathcal{L})|_{\mathcal{X}_\infty}u$. This concludes the proof. ☐

Lemma 7.4.2 (Taylor)**.** *For all* $v, w \in H_0^1(\Omega)$ *with* $\|\nabla(u-v)\|_{L^2(\Omega)} + \|\nabla(u-w)\|_{L^2(\Omega)} \le \varepsilon_{\ell oc}$*, there holds*

$$\|\mathcal{L}w - \mathcal{L}v - D\mathcal{L}(w)(w-v)\|_{H^{-1}(\Omega)} \le C_{17}\|\nabla(w-v)\|_{L^2(\Omega)}^2, \tag{7.4.1a}$$

$$\|\mathcal{A}w - \mathcal{A}v - D\mathcal{A}(w)(w-v)\|_{H^{-1}(\Omega)} \le C_{17}\|\nabla(w-v)\|_{L^2(\Omega)}^2, \tag{7.4.1b}$$

where $\mathcal{L}$ *and* $\mathcal{A}$ *are defined in Section 3.7.*

Proof. The local boundedness (3.7.10) together with [**37**, Theorem 6.5] applied to the operators $\mathcal{L}$ and $\mathcal{A}$ prove the statement. ☐

Lemma 7.4.3. *The sequence* $(e_\ell)_{\ell\in\mathbb{N}}$ *(with* u *and* $U(\mathcal{T}_\ell)$ *from Section 3.7) defined by*

$$e_\ell := \begin{cases}\frac{u-U(\mathcal{T}_\ell)}{\|\nabla(u-U(\mathcal{T}_\ell))\|_{L^2(\Omega)}}, & \text{for } u \neq U(\mathcal{T}_\ell),\\ 0, & \text{else}\end{cases}$$

converges to zero, weakly in $H_0^1(\Omega)$ *in the sense of* (7.2.1).

Proof. With Galerkin-orthogonality and the convention $\infty\cdot 0 = 0$, we obtain

$$\lim_{\ell\to\infty}\frac{\langle \mathcal{L}u - \mathcal{L}U(\mathcal{T}_\ell)\,,\,V_k\rangle}{\|\nabla(u-U(\mathcal{T}_\ell))\|_{L^2(\Omega)}} = 0 \quad \text{for all } V_k \in \mathcal{S}_0^p(\mathcal{T}_k) \text{ and } k \in \mathbb{N}.$$

By continuity of the duality brackets, this results in convergence for all $v \in \mathcal{X}_\infty$

$$\frac{\langle \mathcal{L}u - \mathcal{L}U(\mathcal{T}_\ell)\,,\,v\rangle}{\|\nabla(u-U(\mathcal{T}_\ell))\|_{L^2(\Omega)}} \to 0 \quad \text{as } \ell\to\infty.$$

By use of (7.4.1a) and convergence from (3.7.15), we observe for all $v \in \mathcal{X}_\infty$ and all sufficiently large $\ell \in \mathbb{N}$.

$$\frac{|\langle \mathcal{L}u - \mathcal{L}U(\mathcal{T}_\ell)\,,\,v\rangle|}{\|\nabla(u - U(\mathcal{T}_\ell))\|_{L^2(\Omega)}} \geq \frac{|\langle (D\mathcal{L}u)(u - U(\mathcal{T}_\ell))\,,\,v\rangle|}{\|\nabla(u - U(\mathcal{T}_\ell))\|_{L^2(\Omega)}} - C_{17}\|\nabla(u - U(\mathcal{T}_\ell))\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)}.$$

Again, with convergence $U(\mathcal{T}_\ell) \to u$ in $H^1_0(\Omega)$ from (3.7.15), this implies immediately for all $v \in \mathcal{X}_\infty$

$$\frac{|\langle u - U(\mathcal{T}_\ell)\,,\,((D\mathcal{L})|_{\mathcal{X}_\infty}u)^\star v\rangle|}{\|\nabla(u - U(\mathcal{T}_\ell))\|_{L^2(\Omega)}} = \frac{|\langle (D\mathcal{L}u)(u - U(\mathcal{T}_\ell))\,,\,v\rangle|}{\|\nabla(u - U(\mathcal{T}_\ell))\|_{L^2(\Omega)}} \to 0 \quad \text{as } \ell \to \infty. \tag{7.4.2}$$

According to Lemma 7.4.1, $(D\mathcal{L})|_{\mathcal{X}_\infty}u$ is injective and has closed range. Therefore, its adjoint operator $((D\mathcal{L})|_{\mathcal{X}_\infty}u)^\star$ has is surjective onto $\mathcal{X}_\infty^\star$ by the closed range theorem **[85]**. Convergence (3.7.15) implies that $e_\ell \in \mathcal{X}_\infty$. Hence, (7.4.2) is equivalent to $e_\ell \rightharpoonup 0$ as $\ell \to \infty$. This concludes the proof. □

To abbreviate notation, we define the quasi-metric (which is symmetric, definite, and satisfies the triangle inequality with a multiplicative constant)

$$\mathrm{d\!l}(w,v)^2 := \langle \mathcal{L}w - \mathcal{L}v\,,\,w - v\rangle \quad \text{for all } w, v \in H^1_0(\Omega).$$

Note that due to (3.7.5)–(3.7.6), there holds

$$C_{\mathrm{norm}}^{-1}\|\nabla(w - v)\|_{L^2(\Omega)} \leq \mathrm{d\!l}(w,v) \leq C_{\mathrm{norm}}\|\nabla(w - v)\|_{L^2(\Omega)} \quad \text{for all } w, v \in H^1_0(\Omega) \tag{7.4.3}$$

with $C_{\mathrm{norm}} = \max\{2C_{15}, C_{16}^{-1}\} > 0$.

Proposition 7.4.4. *For any $\varepsilon > 0$, there exists $\ell_{\mathrm{qo}} \in \mathbb{N}$ such that*

$$\mathrm{d\!l}(U_{\ell+1}, U(\mathcal{T}_\ell))^2 \leq \frac{1}{1-\varepsilon}\,\mathrm{d\!l}(u, U(\mathcal{T}_\ell))^2 - \mathrm{d\!l}(u, U(\mathcal{T}_{\ell+1}))^2 \tag{7.4.4}$$

for all $\ell \geq \ell_{\mathrm{qo}}$ and with u and $U(\mathcal{T}_\ell)$ from Section 3.7.

Proof. Due to convergence $U(\mathcal{T}_\ell) \to u$ in $H^1_0(\Omega)$ from (3.7.15), there exists $\ell_1 \in \mathbb{N}$ such that for all $\ell \geq \ell_1$ we may apply (7.4.1b), to obtain

$$\begin{aligned}
&|\langle \mathcal{A}U(\mathcal{T}_{\ell+1}) - \mathcal{A}U(\mathcal{T}_\ell)\,,\,u - U(\mathcal{T}_{\ell+1})\rangle| \\
&\qquad \leq |\langle D\mathcal{A}(U(\mathcal{T}_{\ell+1}))(U(\mathcal{T}_{\ell+1}) - U(\mathcal{T}_\ell))\,,\,u - U(\mathcal{T}_{\ell+1})\rangle| \\
&\qquad\qquad + C_{17}\|\nabla(U(\mathcal{T}_{\ell+1}) - U(\mathcal{T}_\ell))\|^2_{L^2(\Omega)}\|\nabla(u - U(\mathcal{T}_{\ell+1}))\|_{L^2(\Omega)}.
\end{aligned}$$

Using the symmetry of $D\mathcal{A}(U(\mathcal{T}_{\ell+1}))$, we conclude

$$\begin{aligned}
&|\langle \mathcal{A}U(\mathcal{T}_{\ell+1}) - \mathcal{A}U(\mathcal{T}_\ell)\,,\,u - U(\mathcal{T}_{\ell+1})\rangle| \\
&\qquad \leq |\langle D\mathcal{A}(U(\mathcal{T}_{\ell+1}))(u - U(\mathcal{T}_{\ell+1}))\,,\,U(\mathcal{T}_{\ell+1}) - U(\mathcal{T}_\ell)\rangle| \\
&\qquad\qquad + C_{17}\|\nabla(U(\mathcal{T}_{\ell+1}) - U(\mathcal{T}_\ell))\|^2_{L^2(\Omega)}\|\nabla(u - U(\mathcal{T}_{\ell+1}))\|_{L^2(\Omega)} \\
&\qquad \leq |\langle \mathcal{A}u - \mathcal{A}U(\mathcal{T}_{\ell+1})\,,\,U(\mathcal{T}_{\ell+1}) - U(\mathcal{T}_\ell)\rangle| \\
&\qquad\qquad + C_{17}\|\nabla(U(\mathcal{T}_{\ell+1}) - U(\mathcal{T}_\ell))\|^2_{L^2(\Omega)}\|\nabla(u - U(\mathcal{T}_{\ell+1}))\|_{L^2(\Omega)} \\
&\qquad\qquad + C_{17}\|\nabla(U(\mathcal{T}_{\ell+1}) - U(\mathcal{T}_\ell))\|_{L^2(\Omega)}\|\nabla(u - U(\mathcal{T}_{\ell+1}))\|^2_{L^2(\Omega)}.
\end{aligned} \tag{7.4.5}$$

Analogously to the estimate above but by use of the reverse triangle inequality, we obtain

$$\begin{aligned}|\langle \mathcal{A}U(\mathcal{T}_{\ell+1}) - \mathcal{A}U(\mathcal{T}_\ell)\,,\, u - U(\mathcal{T}_{\ell+1})\rangle| \\ \geq |\langle \mathcal{A}u - \mathcal{A}U(\mathcal{T}_{\ell+1})\,,\, U(\mathcal{T}_{\ell+1}) - U(\mathcal{T}_\ell)\rangle| \\ - C_{17}\|\nabla(U(\mathcal{T}_{\ell+1}) - U(\mathcal{T}_\ell))\|_{L^2(\Omega)}^2 \|\nabla(u - U(\mathcal{T}_{\ell+1}))\|_{L^2(\Omega)} \\ - C_{17}\|\nabla(U(\mathcal{T}_{\ell+1}) - U(\mathcal{T}_\ell))\|_{L^2(\Omega)} \|\nabla(u - U(\mathcal{T}_{\ell+1}))\|_{L^2(\Omega)}^2.\end{aligned} \tag{7.4.6}$$

Given $\delta > 0$, convergence $U(\mathcal{T}_\ell) \to u$ as $\ell \to \infty$ provides an index $\ell_0 \in \mathbb{N}$ such that $C_{17}(\|\nabla(u - U(\mathcal{T}_{\ell+1}))\|_{L^2(\Omega)} + \|\nabla(U(\mathcal{T}_{\ell+1}) - U(\mathcal{T}_\ell))\|_{L^2(\Omega)}) \leq \delta$. With (7.4.5)–(7.4.6) this implies

$$\begin{aligned}\big||\langle \mathcal{A}U(\mathcal{T}_{\ell+1}) - \mathcal{A}U(\mathcal{T}_\ell)\,,\, u - U(\mathcal{T}_{\ell+1})\rangle| - |\langle \mathcal{A}u - \mathcal{A}U(\mathcal{T}_{\ell+1})\,,\, U(\mathcal{T}_{\ell+1}) - U(\mathcal{T}_\ell)\rangle|\big| \\ \leq \delta\|\nabla(U(\mathcal{T}_{\ell+1}) - U(\mathcal{T}_\ell))\|_{L^2(\Omega)} \|\nabla(u - U(\mathcal{T}_{\ell+1}))\|_{L^2(\Omega)}\end{aligned}$$

for all $\ell \geq \ell_1$. Since e_ℓ converges to zero weakly in $H_0^1(\Omega)$, we have strong convergence $e_\ell \to 0$ as $\ell \to \infty$ in $L^2(\Omega)$. This together with Lipschitz continuity (3.7.5b) implies

$$\begin{aligned}|\langle \mathcal{C}U(\mathcal{T}_{\ell+1}) - \mathcal{C}U(\mathcal{T}_\ell)\,,\, u - U(\mathcal{T}_{\ell+1})\rangle| \\ \lesssim \|\nabla(U(\mathcal{T}_{\ell+1}) - U(\mathcal{T}_\ell))\|_{L^2(\Omega)} \|e_{\ell+1}\|_{L^2(\Omega)} \|\nabla(u - U(\mathcal{T}_{\ell+1}))\|_{L^2(\Omega)}\end{aligned}$$

and hence

$$\begin{aligned}|\langle \mathcal{C}U(\mathcal{T}_{\ell+1}) - \mathcal{C}U(\mathcal{T}_\ell)\,,\, u - U(\mathcal{T}_{\ell+1})\rangle| \\ \leq \delta\|\nabla(U(\mathcal{T}_{\ell+1}) - U(\mathcal{T}_\ell))\|_{L^2(\Omega)} \|\nabla(u - U(\mathcal{T}_{\ell+1}))\|_{L^2(\Omega)}\end{aligned}$$

for all $\ell \geq \ell_0$ with $\ell_0 \geq \ell_1$ sufficiently large. The adjoint term satisfies

$$\begin{aligned}&|\langle \mathcal{C}u - \mathcal{C}U(\mathcal{T}_{\ell+1})\,,\, U(\mathcal{T}_{\ell+1}) - U(\mathcal{T}_\ell)\rangle| \\ &\quad\leq |\langle \mathcal{C}u - \mathcal{C}U(\mathcal{T}_{\ell+1})\,,\, U(\mathcal{T}_{\ell+1}) - u\rangle| + |\langle \mathcal{C}u - \mathcal{C}U(\mathcal{T}_{\ell+1})\,,\, u - U(\mathcal{T}_\ell)\rangle| \\ &\quad\lesssim \|\nabla(u - U(\mathcal{T}_{\ell+1}))\|_{L^2(\Omega)}^2 \|e_{\ell+1}\|_{L^2(\Omega)} \\ &\qquad + \|\nabla(u - U(\mathcal{T}_\ell))\|_{L^2(\Omega)} \|e_\ell\|_{L^2(\Omega)} \|\nabla(u - U(\mathcal{T}_{\ell+1}))\|_{L^2(\Omega)} \\ &\quad\leq \delta\big(\|\nabla(u - U(\mathcal{T}_{\ell+1}))\|_{L^2(\Omega)}^2 \\ &\qquad + \|\nabla(u - U(\mathcal{T}_\ell))\|_{L^2(\Omega)} \|\nabla(u - U(\mathcal{T}_{\ell+1}))\|_{L^2(\Omega)}\big).\end{aligned}$$

So far, we end up with

$$\begin{aligned}&|\langle \mathcal{C}U(\mathcal{T}_{\ell+1}) - \mathcal{C}U(\mathcal{T}_\ell)\,,\, u - U(\mathcal{T}_{\ell+1})\rangle| + |\langle \mathcal{C}u - \mathcal{C}U(\mathcal{T}_{\ell+1})\,,\, U(\mathcal{T}_{\ell+1}) - U(\mathcal{T}_\ell)\rangle| \\ &\quad\leq \delta\big(\|\nabla(U(\mathcal{T}_{\ell+1}) - U(\mathcal{T}_\ell))\|_{L^2(\Omega)} \|\nabla(u - U(\mathcal{T}_{\ell+1}))\|_{L^2(\Omega)} \\ &\qquad + \|\nabla(u - U(\mathcal{T}_{\ell+1}))\|_{L^2(\Omega)}^2 \\ &\qquad + \|\nabla(u - U(\mathcal{T}_{\ell+1}))\|_{L^2(\Omega)} \|\nabla(u - U(\mathcal{T}_\ell))\|_{L^2(\Omega)}\big) \\ &\quad\leq \delta/2\|\nabla(U(\mathcal{T}_{\ell+1}) - U(\mathcal{T}_\ell))\|_{L^2(\Omega)}^2 + 2\delta\|\nabla(u - U(\mathcal{T}_{\ell+1}))\|_{L^2(\Omega)}^2 \\ &\qquad + \delta/2\|\nabla(u - U(\mathcal{T}_\ell))\|_{L^2(\Omega)}^2\end{aligned}$$

by use of Young's inequality. Putting everything together and by use of Galerkin orthogonality $\langle(\mathcal{A}+\mathcal{C})u - (\mathcal{A}+\mathcal{C})U(\mathcal{T}_{\ell+1})\,,\, U(\mathcal{T}_{\ell+1}) - U(\mathcal{T}_\ell)\rangle = 0$, we obtain

$$\begin{aligned}
&|\langle(\mathcal{A}+\mathcal{C})U(\mathcal{T}_{\ell+1}) - (\mathcal{A}+\mathcal{C})U(\mathcal{T}_\ell)\,,\, u - U(\mathcal{T}_{\ell+1})\rangle| \\
&\quad\le |\langle \mathcal{A}u - \mathcal{A}U(\mathcal{T}_{\ell+1})\,,\, U(\mathcal{T}_{\ell+1}) - U(\mathcal{T}_\ell)\rangle| \\
&\qquad + \delta\|\nabla(U(\mathcal{T}_{\ell+1}) - U(\mathcal{T}_\ell))\|_{L^2(\Omega)}\|\nabla(u - U(\mathcal{T}_{\ell+1}))\|_{L^2(\Omega)} \\
&\qquad + |\langle \mathcal{C}U(\mathcal{T}_{\ell+1}) - \mathcal{C}U(\mathcal{T}_\ell)\,,\, u - U(\mathcal{T}_{\ell+1})\rangle| \\
&\quad\le |\langle(\mathcal{A}+\mathcal{C})u - (\mathcal{A}+\mathcal{C})U(\mathcal{T}_{\ell+1})\,,\, U(\mathcal{T}_{\ell+1}) - U(\mathcal{T}_\ell)\rangle| \\
&\qquad + \delta\|\nabla(U(\mathcal{T}_{\ell+1}) - U(\mathcal{T}_\ell))\|_{L^2(\Omega)}\|\nabla(u - U(\mathcal{T}_{\ell+1}))\|_{L^2(\Omega)} \\
&\qquad + |\langle \mathcal{C}U(\mathcal{T}_{\ell+1}) - \mathcal{C}U(\mathcal{T}_\ell)\,,\, u - U(\mathcal{T}_{\ell+1})\rangle| + |\langle \mathcal{C}u - \mathcal{C}U(\mathcal{T}_{\ell+1})\,,\, U(\mathcal{T}_{\ell+1}) - U(\mathcal{T}_\ell)\rangle| \\
&\quad\le 3\delta\big(\|\nabla(U(\mathcal{T}_{\ell+1}) - U(\mathcal{T}_\ell))\|_{L^2(\Omega)}^2 + \|\nabla(u - U(\mathcal{T}_{\ell+1}))\|_{L^2(\Omega)}^2 + \|\nabla(u - U(\mathcal{T}_\ell))\|_{L^2(\Omega)}^2\big).
\end{aligned}$$

With that at hand, we obtain by definition of $\mathrm{d\!l}(\cdot,\cdot)$

$$\begin{aligned}
\mathrm{d\!l}(U(\mathcal{T}_{\ell+1}), U(\mathcal{T}_\ell))^2 &\le \mathrm{d\!l}(u, U(\mathcal{T}_\ell))^2 - \mathrm{d\!l}(u, U(\mathcal{T}_{\ell+1}))^2 \\
&\qquad + |\langle(\mathcal{A}+\mathcal{C})U(\mathcal{T}_{\ell+1}) - (\mathcal{A}+\mathcal{C})U(\mathcal{T}_\ell)\,,\, u - U(\mathcal{T}_{\ell+1})\rangle| \\
&\le \mathrm{d\!l}(u, U(\mathcal{T}_\ell))^2 - \mathrm{d\!l}(u, U(\mathcal{T}_{\ell+1}))^2 + 3\delta\big(\|\nabla(U(\mathcal{T}_{\ell+1}) - U(\mathcal{T}_\ell))\|_{L^2(\Omega)}^2 \\
&\qquad + \|\nabla(u - U(\mathcal{T}_{\ell+1}))\|_{L^2(\Omega)}^2 + \|\nabla(u - U(\mathcal{T}_\ell))\|_{L^2(\Omega)}^2\big).
\end{aligned}$$

With the equivalence (7.4.3), we conclude

$$\begin{aligned}
&(1 - 3C_{\mathrm{norm}}\delta)\mathrm{d\!l}(U(\mathcal{T}_{\ell+1}), U(\mathcal{T}_\ell))^2 \\
&\qquad\le (1 + 3C_{\mathrm{norm}}\delta)\mathrm{d\!l}(u, U(\mathcal{T}_\ell))^2 - (1 - 3C_{\mathrm{norm}}\delta)\mathrm{d\!l}(u, U(\mathcal{T}_{\ell+1}))^2
\end{aligned}$$

for all $\ell \ge \ell_0$. Finally, we choose $\delta > 0$ sufficiently small such that $(1 + 3C_{\mathrm{norm}}\delta)/(1 - 3C_{\mathrm{norm}}\delta) \le 1/(1-\varepsilon)$ and conclude the proof. □

Theorem 7.4.5. *Suppose the Céa lemma 3.7.8. For all $\varepsilon_{\mathrm{qo}} > 0$, there exists $C_{\mathrm{qo}} > 0$ such that* (E2) *holds with $\varrho(\mathcal{T}_\ell, \mathcal{T}_{\ell+1}) := \|\nabla(U(\mathcal{T}_\ell) - U(\mathcal{T}_{\ell+1}))\|_{L^2(\Omega)}$ (with u and $U(\mathcal{T}_\ell)$ from Section 3.7), and each estimator $\eta(\cdot)$ which is reliable, i.e.,*

$$\|\nabla(u - U(\mathcal{T}_\ell))\|_{L^2(\Omega)} \le C_{\mathrm{rel}}\eta(\mathcal{T}_\ell) \quad \textit{for all } \ell \in \mathbb{N}_0.$$

Particularly, this is satisfied by the error estimator $\eta(\cdot)$ from Section 3.7.

Proof. Proposition 7.4.4 proves the quasi-orthogonality (2.7.5) for all $\ell \ge \ell_0$ with $\varrho(\mathcal{T}_\ell, \mathcal{T}_{\ell+1}) = \mathrm{d\!l}(U(\mathcal{T}_\ell), U(\mathcal{T}_{\ell+1}))$ and $\alpha_\ell := \mathrm{d\!l}(u, U(\mathcal{T}_\ell))$. The Céa lemma 3.7.8, (7.4.3), and reliability (in the setting of Section 3.7 from (2.4.1)) imply

$$\varrho(\mathcal{T}_\ell, \mathcal{T}_{\ell+1}) \lesssim \|\nabla(u - U(\mathcal{T}_\ell))\|_{L^2(\Omega)} \lesssim \eta(\mathcal{T}_\ell) \quad \text{for all } \ell \in \mathbb{N}_0.$$

Therefore, Lemma 2.7.3 proves for all $\ell \ge \ell_{\mathrm{qo}}$.

$$\sum_{k=\ell}^{\infty} \mathrm{d\!l}(U(\mathcal{T}_k), U(\mathcal{T}_{k+1}))^2 - \varepsilon_{\mathrm{qo}}\eta(\mathcal{T}_k)^2 \le C'_{\mathrm{qo}}\eta(\mathcal{T}_\ell)^2.$$

For all $\ell < \ell_0$, there exists $C_\ell > 0$ with

$$\sum_{k=\ell}^{\ell_0} \mathrm{d\!l}(U(\mathcal{T}_k), U(\mathcal{T}_{k+1}))^2 - \varepsilon_{\mathrm{qo}}\eta(\mathcal{T}_k)^2 \le C_\ell\eta(\mathcal{T}_\ell)^2,$$

since both sides of the inequality are finite and if $\eta(\mathcal{T}_\ell) = 0$, then reliability (2.4.1) and the Céa lemma (3.7.8) imply

$$\mathrm{d\!l}(U(\mathcal{T}_k), U(\mathcal{T}_{k+1})) \lesssim \|\nabla(U(\mathcal{T}_k) - U(\mathcal{T}_{k+1}))\|_{L^2(\Omega)} \lesssim \|\nabla(u - U(\mathcal{T}_\ell))\|_{L^2(\Omega)} \lesssim \eta(\mathcal{T}_\ell) = 0.$$

With (7.4.3), the last two estimates conclude the proof. □

Bibliography

[1] Mark Ainsworth and J. Tinsley Oden. *A posteriori error estimation in finite element analysis.* Pure and Applied Mathematics (New York). Wiley-Interscience, New York, 2000.

[2] Markus Aurada, Michael Feischl, Thomas Führer, Michael Karkulik, Markus Melenk, and Dirk Praetorius. Local inverse estimates for non-local boundary integral operators. *work in progress*, 2015.

[3] Markus Aurada, Michael Feischl, Thomas Führer, Michael Karkulik, and Dirk Praetorius. Efficiency and optimality of some weighted-residual error estimator for adaptive 2D boundary element methods. *J. Comput. Appl. Math.*, 255:481–501, 2014.

[4] Markus Aurada, Michael Feischl, Josef Kemetmüller, Marcus Page, and Dirk Praetorius. Each $H^{1/2}$-stable projection yields convergence and quasi-optimality of adaptive FEM with inhomogeneous Dirichlet data in $\mathbb{R}^d$. *ESAIM Math. Model. Numer. Anal.*, 47:1207–1235, 2013.

[5] Markus Aurada, Samuel Ferraz-Leite, and Dirk Praetorius. Estimator reduction and convergence of adaptive BEM. *Appl. Numer. Math.*, 62(6):787–801, 2012.

[6] Ivo Babuska and Anthony Miller. A feedback finite element method with a posteriori error estimation. I. The finite element method and some basic properties of the a posteriori error estimator. *Comput. Methods Appl. Mech. Engrg.*, 61(1):1–40, 1987.

[7] Ivo Babuska and Werner C. Rheinboldt. Analysis of optimal finite-element meshes in $\mathbb{R}^1$. *Math. Comp.*, 33(146):435–463, 1979.

[8] Ivo Babuska and Michael Vogelius. Feedback and adaptive finite element solution of one-dimensional boundary value problems. *Numer. Math.*, 44(1):75–102, 1984.

[9] Eberhard Bänsch. Local mesh refinement in 2 and 3 dimensions. *Impact Comput. Sci. Engrg.*, 3(3):181–191, 1991.

[10] Sören Bartels and Carsten Carstensen. Each averaging technique yields reliable a posteriori error control in FEM on unstructured grids. II. Higher order FEM. *Math. Comp.*, 71(239):971–994, 2002.

[11] Sören Bartels, Carsten Carstensen, and Georg Dolzmann. Inhomogeneous Dirichlet conditions in a priori and a posteriori finite element error analysis. *Numer. Math.*, 99(1):1–24, 2004.

[12] Roland Becker, Shipeng Mao, and Zhongci Shi. A convergent nonconforming adaptive finite element method with quasi-optimal complexity. *SIAM J. Numer. Anal.*, 47(6):4639–4659, 2010.

[13] Liudmila Belenki, Lars Diening, and Christian Kreuzer. Optimality of an adaptive finite element method for the p-Laplacian equation. *IMA J. Numer. Anal.*, 32(2):484–510, 2012.

[14] Peter Binev, Wolfgang Dahmen, and Ronald DeVore. Adaptive finite element methods with convergence rates. *Numer. Math.*, 97(2):219–268, 2004.

[15] Andrea Bonito, J. Manuel Cascón, Pedro Morin, and Ricardo H. Nochetto. AFEM for geometric PDE: the Laplace-Beltrami operator. In *Analysis and numerics of partial differential equations*, volume 4 of *Springer INdAM Ser.*, pages 257–306. Springer, Milan, 2013.

[16] Andrea Bonito and Ricardo H. Nochetto. Quasi-optimal convergence rate of an adaptive discontinuous Galerkin method. *SIAM J. Numer. Anal.*, 48(2):734–771, 2010.

[17] Dietrich Braess, Carsten Carstensen, and Ronald H. W. Hoppe. Convergence analysis of a conforming adaptive finite element method for an obstacle problem. *Numer. Math.*, 107(3):455–471, 2007.

[18] Dietrich Braess, Carsten Carstensen, and Ronald H. W. Hoppe. Error reduction in adaptive finite element approximations of elliptic obstacle problems. *J. Comput. Math.*, 27(2-3):148–169, 2009.

[19] Carsten Carstensen. Efficiency of a posteriori BEM-error estimates for first-kind integral equations on quasi-uniform meshes. *Math. Comp.*, 65(213):69–84, 1996.

[20] Carsten Carstensen. An a posteriori error estimate for a first-kind integral equation. *Math. Comp.*, 66(217):139–155, 1997.

[21] Carsten Carstensen. A posteriori error estimate for the mixed finite element method. *Math. Comp.*, 66(218):465–476, 1997.

[22] Carsten Carstensen and Sören Bartels. Each averaging technique yields reliable a posteriori error control in FEM on unstructured grids. I. Low order conforming, nonconforming, and mixed FEM. *Math. Comp.*, 71(239):945–969, 2002.

[23] Carsten Carstensen, Martin Eigel, Ronald H. W. Hoppe, and Caroline Löbhard. A review of unified a posteriori finite element error control. *Numer. Math. Theory Methods Appl*, 5(4):509–558, 2012.

[24] Carsten Carstensen, Michael Feischl, Marcus Page, and Dirk Praetorius. Axioms of adaptivity. *Comput. Math. Appl.*, 67(6):1195–1253, 2014.

[25] Carsten Carstensen and Ronald H. W. Hoppe. Convergence analysis of an adaptive nonconforming finite element method. *Numer. Math.*, 103(2):251–266, 2006.

[26] Carsten Carstensen and Ronald H. W. Hoppe. Error reduction and convergence for an adaptive mixed finite element method. *Math. Comp.*, 75(255):1033–1042, 2006.

[27] Carsten Carstensen, Matthias Maischak, Dirk Praetorius, and Ernst P. Stephan. Residual-based a posteriori error estimate for hypersingular equation on surfaces. *Numer. Math.*, 97(3):397–425, 2004.

[28] Carsten Carstensen, Matthias Maischak, and Ernst P. Stephan. A posteriori error estimate and h-adaptive algorithm on surfaces for Symm's integral equation. *Numer. Math.*, 90(2):197–213, 2001.

[29] Carsten Carstensen, Daniel Peterseim, and Hella Rabus. Optimal adaptive nonconforming FEM for the Stokes problem. *Numer. Math.*, 123(2):291–308, 2013.

[30] Carsten Carstensen and Dirk Praetorius. Averaging techniques for the effective numerical solution of symm's integral equation of the first kind. *SIAM Journal on Scientific Computing*, 27(4):1226–1260, 2006.

[31] Carsten Carstensen and Hella Rabus. An optimal adaptive mixed finite element method. *Math. Comp.*, 80(274):649–667, 2011.

[32] Carsten Carstensen and Hella Rabus. The adaptive nonconforming FEM for the pure displacement problem in linear elasticity is optimal and robust. *SIAM J. Numer. Anal.*, 50(3):1264–1283, 2012.

[33] Carsten Carstensen and Ernst P. Stephan. A posteriori error estimates for boundary element methods. *Math. Comp.*, 64(210):483–500, 1995.

[34] Carsten Carstensen and Ernst P. Stephan. Adaptive boundary element methods for some first kind integral equations. *SIAM J. Numer. Anal.*, 33(6):2166–2183, 1996.

[35] J. Manuel Cascon, Christian Kreuzer, Ricardo H. Nochetto, and Kunibert G. Siebert. Quasi-optimal convergence rate for an adaptive finite element method. *SIAM J. Numer. Anal.*, 46(5):2524–2550, 2008.

[36] J. Manuel Cascon and Ricardo H. Nochetto. Quasioptimal cardinality of AFEM driven by nonresidual estimators. *IMA J. Numer. Anal.*, 32(1):1–29, 2012.

[37] Ruth F. Curtain and Anthony J. Pritchard. *Functional analysis in modern applied mathematics.* Academic Press [Harcourt Brace Jovanovich Publishers], London, 1977. Mathematics in Science and Engineering, Vol. 132.

[38] Alan Demlow and Gerhard Dziuk. An adaptive finite element method for the Laplace-Beltrami operator on implicitly defined surfaces. *SIAM J. Numer. Anal.*, 45(1):421–442 (electronic), 2007.

[39] Lars Diening and Christian Kreuzer. Linear convergence of an adaptive finite element method for the p-Laplacian equation. *SIAM J. Numer. Anal.*, 46(2):614–638, 2008.

[40] Willy Dörfler. A convergent adaptive algorithm for Poisson's equation. *SIAM J. Numer. Anal.*, 33(3):1106–1124, 1996.

[41] Todd Dupont and Ridgway Scott. Polynomial approximation of functions in Sobolev spaces. *Math. Comp.*, 34(150):441–463, 1980.

[42] Gerhard Dziuk. Finite elements for the Beltrami operator on arbitrary surfaces. In *Partial differential equations and calculus of variations*, volume 1357 of *Lecture Notes in Math.*, pages 142–155. Springer, Berlin, 1988.

[43] Birgit Faermann. Localization of the Aronszajn-Slobodeckij norm and application to adaptive boundary element methods. I. The two-dimensional case. *IMA J. Numer. Anal.*, 20(2):203–234, 2000.

[44] Michael Feischl, Thomas Führer, Michael Karkulik, Jens Markus Melenk, and Dirk Praetorius. Quasi-optimal convergence rates for adaptive boundary element methods with data approximation, part I: weakly-singular integral equation. *Calcolo*, 51(4):531–562, 2014.

[45] Michael Feischl, Thomas Führer, Michael Karkulik, Jens Markus Melenk, and Dirk Praetorius. Quasi-optimal convergence rates for adaptive boundary element methods with data approximation, part II: Hypersingular integral equation. *Electron. Trans. Numer. Anal.*, 44:153–176, 2015.

[46] Michael Feischl, Thomas Führer, and Dirk Praetorius. Adaptive FEM with optimal convergence rates for a certain class of nonsymmetric and possibly nonlinear problems. *SIAM J. Numer. Anal.*, 52(2):601–625, 2014.

[47] Michael Feischl, Michael Karkulik, J. Markus Melenk, and Dirk Praetorius. Quasi-optimal convergence rate for an adaptive boundary element method. *SIAM J. Numer. Anal.*, 51:1327–1348, 2013.

[48] Michael Feischl, Marcus Page, and Dirk Praetorius. Convergence and quasi-optimality of adaptive FEM with inhomogeneous Dirichlet data. *J. Comput. Appl. Math.*, 255:481–501, 2014.

[49] Michael Feischl, Marcus Page, and Dirk Praetorius. Convergence of adaptive FEM for elliptic obstacle problems with inhomogeneous Dirichlet data. *Int. J. Numer. Anal. Model.*, 11:229–253, 2014.

[50] Samuel Ferraz-Leite, Christof Ortner, and Dirk Praetorius. Convergence of simple adaptive Galerkin schemes based on $h - h/2$ error estimators. *Numer. Math.*, 116:291–316, 2010.

[51] Stefan Funken, Dirk Praetorius, and Philipp Wissgott. Efficient implementation of adaptive P1-FEM in Matlab. *Comput. Methods Appl. Math.*, 11(4):460–490, 2011.

[52] Dietmar Gallistl, Mira Schedensack, and Rob P. Stevenson. A remark on newest vertex bisection in any space dimension. *Comput. Methods Appl. Math.*, 14(3):317–320, 2014.

[53] Tsogtgerel Gantumur. Convergence rates of adaptive methods, Besov spaces, and multilevel approximation. *arXiv:1408.3889*, 2015.

[54] Eduardo M. Garau, Pedro Morin, and Carlos Zuppa. Quasi-optimal convergence rate of an AFEM for quasi-linear problems of monotone type. *Numer. Math. Theory Methods Appl.*, 5(2):131–156, 2012.

[55] Fernando D. Gaspoz and Pedro Morin. Convergence rates for adaptive finite elements. *IMA J. Numer. Anal.*, 29(4):917–936, 2009.

[56] Fernando D. Gaspoz and Pedro Morin. Approximation classes for adaptive higher order finite element approximation. *Math. Comp.*, 83(289):2127–2160, 2014.

[57] Ivan G. Graham, Wolfgang Hackbusch, and Stefan A. Sauter. Finite elements on degenerate meshes: inverse-type inequalities and applications. *IMA J. Numer. Anal.*, 25(2):379–407, 2005.

[58] George C. Hsiao and Wolfgang L. Wendland. *Boundary integral equations*, volume 164 of *Applied Mathematical Sciences*. Springer-Verlag, Berlin, 2008.

[59] Michael Karkulik, David Pavlicek, and Dirk Praetorius. On 2D newest vertex bisection: optimality of mesh-closure and H^1-stability of L_2-projection. *Constr. Approx.*, 38(2):213–234, 2013.

[60] Christian Kreuzer and Kunibert G. Siebert. Decay rates of adaptive finite elements with Dörfler marking. *Numer. Math.*, 117(4):679–716, 2011.

[61] Shipeng Mao, Xuying Zhao, and Zhongci Shi. Convergence of a standard adaptive nonconforming finite element method with optimal complexity. *Appl. Numer. Math.*, 60:673–688, July 2010.

[62] William McLean. *Strongly elliptic systems and boundary integral equations.* Cambridge University Press, Cambridge, 2000.

[63] Khamron Mekchay, Pedro Morin, and Ricardo H. Nochetto. AFEM for the Laplace-Beltrami operator on graphs: design and conditional contraction property. *Math. Comp.*, 80(274):625–648, 2011.

[64] Khamron Mekchay and Ricardo H. Nochetto. Convergence of adaptive finite element methods for general second order linear elliptic PDEs. *SIAM J. Numer. Anal.*, 43(5):1803–1827, 2005.

[65] Pedro Morin, Ricardo H. Nochetto, and Kunibert G. Siebert. Data oscillation and convergence of adaptive FEM. *SIAM J. Numer. Anal.*, 38(2):466–488, 2000.

[66] Pedro Morin, Ricardo H. Nochetto, and Kunibert G. Siebert. Local problems on stars: a posteriori error estimators, convergence, and performance. *Math. Comp.*, 72(243):1067–1097, 2003.

[67] Jean-Claude Nédélec. Curved finite element methods for the solution of singular integral equations on surfaces in R^3. *Comput. Methods Appl. Mech. Engrg.*, 8(1):61–80, 1976.

[68] Marcus Page. Schätzerreduktion und Konvergenz adaptiver FEM für Hindernisprobleme, Master thesis (in German). *Institute for Analysis and Scientific Computing, Vienna University of Technology*, 2010.

[69] Marcus Page and Dirk Praetorius. Convergence of adaptive FEM for some elliptic obstacle problem. *Appl. Anal.*, 92(3):595–615, 2013.

[70] David Pavlicek. Optimalität adaptiver FEM, Bachelor thesis (in German). *Institute for Analysis and Scientific Computing, Vienna University of Technology*, 2010.

[71] Hella Rabus. A natural adaptive nonconforming FEM of quasi-optimal complexity. *Comput. Methods Appl. Math.*, 10(3):315–325, 2010.

[72] Rodolfo Rodríguez. Some remarks on Zienkiewicz-Zhu estimator. *Numer. Methods Partial Differential Equations*, 10(5):625–635, 1994.

[73] Walter Rudin. *Functional analysis.* International Series in Pure and Applied Mathematics. McGraw-Hill Inc., New York, second edition, 1991.

[74] Roberta Sacchi and Andreas Veeser. Locally efficient and reliable a posteriori error estimators for Dirichlet problems. *Math. Models Methods Appl. Sci.*, 16(3):319–346, 2006.

[75] Stefan A. Sauter and Christoph Schwab. *Boundary element methods*, volume 39 of *Springer Series in Computational Mathematics.* Springer-Verlag, Berlin, 2011.

[76] L. Ridgway Scott and Shangyou Zhang. Finite element interpolation of nonsmooth functions satisfying boundary conditions. *Math. Comp.*, 54(190):483–493, 1990.

[77] Ernst P. Stephan and Manil Suri. The *h-p* version of the boundary element method on polygonal domains with quasiuniform meshes. *RAIRO Modél. Math. Anal. Numér.*, 25(6):783–807, 1991.

[78] Rob Stevenson. Optimality of a standard adaptive finite element method. *Found. Comput. Math.*, 7(2):245–269, 2007.

[79] Rob Stevenson. The completion of locally refined simplicial partitions created by bisection. *Math. Comp.*, 77(261):227–241, 2008.

[80] Gantumur Tsogtgerel. Adaptive boundary element methods with convergence rates. *Numerische Mathematik*, 124(3):471–516, 2013.

[81] Andreas Veeser. Approximating gradients with continuous piecewise polynomial functions. *arXiv:1402.3945*, 2014.

[82] Rüdiger Verfürth. *A posteriori error estimation techniques for finite element methods.* Numerical Mathematics and Scientific Computation. Oxford University Press, Oxford, 2013.

[83] Tobias von Petersdorff. *Randwertprobleme der Elastizitatstheorie für Polyeder - Singularitaten und Approximation mit Randelementmethoden (in German).* PhD thesis, Darmstadt, 1989.

[84] Yushan Yan and Ian. H. Sloan. On integral equations of the first kind with logarithmic kernels. *J. Integral Equations Appl.*, 1(4):549–579, 1988.

[85] Kosaku Yosida. *Functional analysis.* Springer-Verlag Berlin Heidelberg, 1980.

[86] Eberhard Zeidler. *Nonlinear functional analysis and its applications. II/B.* Springer-Verlag, New York, 1990.

[87] Olgierd C. Zienkiewicz and Jian Z. Zhu. A simple error estimator and adaptive procedure for practical engineering analysis. *Internat. J. Numer. Methods Engrg.*, 24(2):337–357, 1987.

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• Education

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• Top 3 publications

M. Feischl, T. Führer, D. Praetorius: *Adaptive FEM with optimal convergence rates for a certain class of non-symmetric and possibly non-linear problems*, SIAM J. Numer. Anal. 52 (2014), 601–625.
C. Carstensen, M. Feischl, M. Page, D. Praetorius: *Axioms of adaptivity*, Comput. Math. Appl. 67 (2014), 1195–1253.
M. Feischl, M. Karkulik, J. Melenk, D. Praetorius: *Quasi-optimal convergence rate for an adaptive boundary element method*, SIAM J. Numer. Anal. 51 (2013) 1327–1348.

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• Other publications

Altogether, 21 peer-reviewed journal publications since 2012, e.g., in *Appl. Numer. Math.* (to appear 2015), *Numer. Math.* (to appear 2015), *Arch. Comput. Methods Engrg.* (to appear 2015), *Electron. Trans. Numer. Anal.* (to appear 2015), *Math. Models Methods Appl. Sci.* (2014) *Calcolo* (2014), *Numer. Algorithms* (2014), *Comput. Math. Appl.* (2014), *Int. J. Numer. Anal. Model.* (2014), *J. Comput. Appl. Math.* (2014), *Comput. Methods Appl. Math.* (2013, 2014), *SIAM J. Numer. Anal.* (2013, 2014), *Eng. Anal. Bound. Elem.* (2012, 2014), *Comput. Mech.* (2013), *J. Magn. Magn. Mater.* (2012, 2013), *M2AN Math. Model. Numer. Anal.* (2012, 2013).
Currently, 15 publications are listed in MathSciNet and 19 in Scopus (state March 2015).

• Scientific talks

Workshop for Adaptive Wavelets and Frames for BEM in Acoustics (invited, 2014), *11th. World Congress on Computational Mechanics* (2014), *IABEM Symposium* (2013), *WONAPDE* (2013), *MAFELAP* (2013), *ECCOMAS* (2012), *Austrian Numerical Analysis Day* (2010–2013), *Workshop on Fast BEM in Industrial Applications* (2010–2013), *Colloquium of Institute for Applied Mathematics at Humboldt-University of Berlin* (invited, 2012–2013), *7th Zürich Summerschool* (2012).