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Optimal Adaptivity for Splines in Finite and Boundary Element Methods

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Univ.-Prof. Dipl.-Math. Dr.techn. Dirk Praetorius
E101 - Institut für Analysis und Scientific Computing, TU Wien

eingereicht an der Technischen Universität Wien
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von

Dipl.Ing. Gregor Gantner BSc.
Matrikelnummer: 0826963
Siedlung 36
2054 Haugsdorf

Diese Dissertation haben begutachtet:

1. **Prof. Dr. Dirk Praetorius**
   Institut für Analysis und Scientific Computing, TU Wien
2. **Prof. Dr. Helmut Harbrecht**
   Departement Mathematik und Informatik, Universität Basel
3. **Prof. Dr. Daniel Peterseim**
   Institut für Mathematik, Universität Augsburg

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Kurzfassung

Seit der Einführung der *isogeometrischen Analysis* (IGA) im Jahr 2005 sind die *Finite-Element-Methode* (FEM) und die *Randelementmethode* (BEM, engl. boundary element method) mit Splines zu einem aktiven Forschungsfeld geworden. Die zentrale Idee von IGA ist es, die gleichen Funktionen zur Approximation der Lösung der betrachteten partiellen Differentialgleichung (PDE, engl. partial differential equation) zu verwenden, die auch für die Darstellung der Problemgeometrie in Computer Aided Design (CAD) genutzt werden. Normalerweise basiert CAD auf Tensorprodukt-Splines. Um adaptive Verfeinerung zuzulassen, wurden einige Erweiterungen von diesen entwickelt, z.B. hierarchische Splines, T-Splines oder LR-Splines. Im Hinblick auf Geometrie induzierte Singularitäten und der Tatsache, dass isogeometrische Methoden Ansatzfunktionen mit hoher Ordnung verwenden, ist der Gewinn durch adaptive Verfeinerung (bzw. Verlust bei uniformer Verfeinerung) gewaltig.

In dieser Arbeit betrachten wir zuerst eine adaptive FEM mit hierarchischen Splines von beliebigem Grad für lineare elliptische PDE-Systeme zweiter Ordnung mit Dirichlet-Randbedingung in $\mathbb{R}^d$ für $d \geq 2$. Wir nehmen an, dass die Problemgeometrie über dem d-dimensionalen Einheitswürfel parametrisiert werden kann. Wir stellen eine Verfeinerungsstrategie vor, um eine Folge lokal verfeinerter Gitter und diskreter Lösungen zu erzeugen. Adaptivität wird hierbei von einem gewichteten *a posteriori* Residualfehlerschätzer gesteuert. Wir beweisen lineare Konvergenz des Fehlerschätzers (bzw. der Summe aus Fehler und Datenoszillationen) mit optimaler algebraischer Rate.

Danach betrachten wir eine adaptive Randelementmethode mit hierarchischen Splines von beliebigem Grad für schwach-singuläre Integralgleichungen erster Art, die bei der Lösung von linearen elliptischen PDE-Systemen zweiter Ordnung mit konstanten Koeffizienten und Dirichlet-Randbedingung auftreten. Wir nehmen an, dass der Geometrierand die Vereinigung von Oberflächen ist, die über dem $(d-1)$-dimensionalen Einheitswürfel parametrisiert werden können. Erneut stellen wir eine Verfeinerungsstrategie vor, um eine Folge lokal verfeinerter Gitter und diskreter Lösungen zu erzeugen, wobei Adaptivität durch einen gewichteten *a posteriori* Residualfehlerschätzer gesteuert wird. Wir beweisen lineare Konvergenz des Fehlerschätzers mit optimaler algebraischer Rate. Im Gegensatz zu früheren Arbeiten, welche auf das Laplace-Modellproblem beschränkt sind, lässt unsere Analysis beliebige elliptische PDE-Systeme zweiter Ordnung mit konstanten Koeffizienten zu.

Schließlich untersuchen wir für eindimensionale Ränder eine adaptive BEM mit Standardsplines statt hierarchischen Splines. Wir modifizieren den entsprechenden Algorithmus so, dass er zusätzlich die lokale Glattheit der Ansatzfunktionen steuert. Erneut beweisen wir lineare Konvergenz des Fehlerschätzers mit optimaler algebraischer Rate.

Um die genannten Resultate zu beweisen, entwickeln wir einen abstrakten Rahmen für adaptive konforme FEM und BEM. Insbesondere könnte dieser Rahmen auch für IGA mit T-Splines oder LR-Splines genutzt werden. Durchwegs belegen wir unsere theoretischen Ergebnisse mit numerischen Beispielen.

Abstract

Since the advent of *isogeometric analysis* (IGA) in 2005, the *finite element method* (FEM) and the *boundary element method* (BEM) with *splines* have become an active field of research. The central idea of IGA is to use the same functions for the approximation of the solution of the considered partial differential equation (PDE) as for the representation of the problem geometry in computer aided design (CAD). Usually, CAD is based on tensor-product splines. To allow for adaptive refinement, several extensions of these have emerged, e.g., hierarchical splines, T-splines, and LR-splines. In view of geometry induced generic singularities and the fact that isogeometric methods employ higher-order ansatz functions, the gain of adaptive refinement (resp. loss for uniform refinement) is huge.

In this work, we first consider an adaptive FEM with hierarchical splines of arbitrary degree for linear elliptic PDE systems of second order with Dirichlet boundary condition in $\mathbb{R}^d$ for $d \geq 2$. We assume that the problem geometry can be parametrized over the d-dimensional unit cube. We propose a refinement strategy to generate a sequence of locally refined meshes and corresponding discrete solutions. Adaptivity is driven by some weighted-residual *a posteriori* error estimator. We prove linear convergence of the error estimator (resp. the sum of error plus data oscillations) with optimal algebraic rate.

Next, we consider an adaptive BEM with hierarchical splines of arbitrary degree for weakly-singular integral equations of the first kind that arise from the solution of linear elliptic PDE systems of second order with constant coefficients and Dirichlet boundary condition. We assume that the boundary of the geometry is the union of surfaces that can be parametrized over the $(d-1)$-dimensional unit cube. Again, we propose a refinement strategy to generate a sequence of locally refined meshes and corresponding discrete solutions, where adaptivity is driven by some weighted-residual *a posteriori* error estimator. We prove linear convergence of the error estimator with optimal algebraic rate. In contrast to prior works, which are restricted to the Laplace model problem, our analysis allows for arbitrary elliptic PDE operators of second order with constant coefficients.

Finally, for one-dimensional boundaries, we investigate an adaptive BEM with standard splines instead of hierarchical splines. We modify the corresponding algorithm so that it additionally uses knot multiplicity increase which results in local smoothness reduction of the ansatz space. Again, we prove linear convergence of the employed weighted-residual error estimator with optimal algebraic rate.

In order to prove all these results, we provide an abstract framework for adaptive conforming FEM and BEM. In particular, this framework might also be applicable to IGA with T-splines or LR-splines. Throughout, we provide numerical evidence for our theoretical findings.

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Erklärung

Ich erkläre an Eides statt, dass ich die vorliegende Dissertation selbstständig und ohne fremde Hilfe verfasst, andere als die angegebenen Quellen und Hilfsmittel nicht benutzt bzw. die wörtlich oder sinngemäß entnommenen Stellen als solche kenntlich gemacht habe.

Wien, am 21. November 2017 Unterschrift: ____________________

Contents

1 Introduction

On a given rectangular mesh, *splines* are piecewise polynomials with certain smoothness properties across the boundaries of the mesh elements. Since the advent of *isogeometric analysis* (IGA) in 2005, the *finite element method* (FEM) based on *splines* has become an active field of research. The central idea of IGA is to use the same functions for the approximation of the solution of the considered partial differential equation (PDE) as for the representation of the problem geometry Ω in computer aided design (CAD); see [HCB05, CHB09, BBdVC$^+$06]. The CAD standard for geometry representation relies on splines resp. rational splines which are quotients of standard splines. IGA is of particular interest, if the solution of the PDE describes some geometric quantity, e.g., a deformation of Ω. In this case, one can directly use the approximate solution in the CAD program, since it is in the corresponding format.

Usually, CAD provides only a parametrization of the boundary $\partial\Omega$ instead of the domain Ω itself. Since FEM requires a mesh of Ω, the parametrization needs to be extended to the whole domain, which is non-trivial and still an open research topic, in particular, for CAD geometries consisting of multiple patches. The *boundary element method* (BEM), which can be seen as FEM on the boundary, circumvents this difficulty by working only on the CAD provided boundary mesh. However, compared to the literature on isogeometric analysis with FEM (IGAFEM), only little is found for isogeometric analysis with BEM (IGABEM). The latter was first considered in [PGK$^+$09] for 2D and in [SSE$^+$13] for 3D.

To obtain an accurate approximation of the PDE solution, the CAD provided boundary mesh has to be refined, since the initial mesh is often too coarse to resolve certain behavior of the solution. This can be achieved by uniformly bisecting all elements of the mesh. However, in general, this approach might be unnecessarily (or even prohibitively) expensive in terms of computational effort. Indeed, geometry or data induced singularities of the (unknown) exact solution might reduce the order of convergence significantly and hence spoil the accuracy of numerical simulations for uniform refinement. However, in many situations, local refinement at these singularities significantly improves the accuracy and is hence preferable. In order to automatically steer such a local refinement, one has to implement so-called *adaptive algorithms*. These algorithms estimate the approximation error on all current mesh elements and refine only those elements, where the error appears to be largest.

1.1 Goal of this work

The goal of this work is the development and the mathematical analysis of adaptive algorithms for IGAFEM and IGABEM. In particular, the emphasis is on the rigorous proof that the proposed algorithms lead to (optimal) convergence of the approximations towards the exact solution of the PDE. In order to numerically investigate these algorithms, implementations for adaptive IGABEM in 2D, adaptive IGAFEM in 2D, and adaptive IGABEM in 3D were developed. These implementations are used to underline the mathematical findings with numerical experiments.

1.2 Outline & Contributions

Chapter 2

This chapter is essentially a summary of the results from [CFPP14] and its slight generalization [Fei15]. Both works consider a standard adaptive algorithm from a very abstract point of view. They provide the so-called *axioms of adaptivity* for the error estimator and the mesh-refinement which imply convergence of the estimator at optimal algebraic rate; see **Theorem 2.3.1**.

Chapter 3

In this chapter, we introduce the so-called *splines*. On a given mesh of the unit interval, splines are piecewise polynomials with certain smoothness properties at the breakpoints of the mesh. A tensor-product approach provides a definition for the higher-dimensional case $d \geq 2$. Since the tensor-mesh structure has to be preserved in each refinement step, the standard splines are not suited for adaptive refinement if the dimension d is larger than one. To allow for adaptive refinement, several extensions of the standard model have recently emerged, e.g., *hierarchical splines* [Kra98, VGJS11], *analysis-suitable T-splines* [SLSH12, BdVBSV13], or *LR-splines* [DLP13, JKD14]. In this work, we focus on hierarchical splines which are defined in Section 3.4. We recall the definition of two well-known bases of the space of hierarchical splines, namely the *hierarchical B-splines* and the *truncated hierarchical B-splines* (THB-splines). In Section 3.4.5, we present a Scott–Zhang type projection onto hierarchical splines from the recent own work [GHP17]. Under additional assumptions on the underlying mesh, this operator is locally L^2-stable, locally H^1-stable, and has a local first-order approximation property.

Chapter 4

On a d-dimensional bounded Lipschitz domain Ω, we consider a general linear system of second-order PDEs with homogenous Dirichlet boundary condition

$$\begin{aligned} \mathfrak{P}u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma := \partial\Omega, \end{aligned} \tag{1.2.1}$$

where we seek the solution $u : \Omega \to \mathbb{R}^D$ with $D \geq 1$. Provided that $\mathfrak{P}$ is $H^1_0(\Omega)^D$-elliptic, (1.2.1) admits for arbitrary vector-valued $f \in L^2(\Omega)^D$ a unique weak solution $u \in H^1_0(\Omega)^D$ which can be approximated with the help of FEM.

For standard FEM with globally continuous piecewise polynomials, adaptive algorithms of the form

$$\boxed{\texttt{solve}} \longrightarrow \boxed{\texttt{estimate}} \longrightarrow \boxed{\texttt{mark}} \longrightarrow \boxed{\texttt{refine}}$$

are well understood; see, e.g., [Dör96, MNS00, BDD04, Ste07, CKNS08, FFP14] and [CFPP14] for milestones on convergence and optimal convergence rates.

In contrast, so far there exists only little literature concerning the thorough mathematical analysis of adaptive FEM with splines: [BG16a] investigates an estimator reduction for an IGAFEM with certain hierarchical splines. [BG16c] investigates linear convergence of an adaptive IGAFEM with truncated hierarchical B-splines. In the continuation of the latter work, [BGMP16] studies the corresponding mesh-refinement strategy together with some refinement-related properties for the proof of optimal convergence.

Chapter 4 builds upon the recent own work [GHP17] and proves that adaptive IGAFEM with hierarchical splines leads to linear convergence with optimal rate. At the time [GHP17] was written, the mathematical proof that the adaptive strategy of [BG16c] leads to optimal convergence rates, was still missing in the literature. Independently and during the review process of [GHP17], optimal convergence behavior for symmetric PDEs was proved in the preprint [BG17], but not underpinned by numerical experiments. Unlike our strategy from [GHP17], the algorithm of [BG16c] was designed for truncated hierarchical B-splines only and the use of hierarchical B-splines may lead to non-sparse Galerkin matrices. In general, truncated hierarchical B-splines have smaller but also more complicated supports, which are possibly not even connected. Further, the truncation procedure leads to an additional overhead that should not be neglected.

Sections 4.2–4.3

In Section 4.2, we give an abstract framework for adaptive mesh-refinement for conforming FEM for the model problem (1.2.1). In **Theorem 4.2.7**, we identify sufficient conditions for the underlying meshes, the local FEM spaces, as well as the employed (local) mesh-refinement rule which guarantee that the usual weighted-residual *a posteriori* error estimator satisfies the axioms of adaptivity from Chapter 2. In particular, we see that the corresponding adaptive algorithm (Algorithm 4.2.6) leads to linear convergence of the error estimator at optimal algebraic rate. Moreover, Theorem 4.2.7 states that under certain assumptions on the data approximation spaces, the employed error estimator is equivalent to the so-called *total error* $\inf_{V_\bullet \in \mathcal{X}_\bullet}(\|u - V_\bullet\|_{H^1(\Omega)} + \mathrm{osc}_\bullet(V_\bullet))$, where $\mathrm{osc}_\bullet(\cdot)$ denotes certain data oscillation terms. This implies that also the total error converges linearly at optimal rate. Section 4.3 is devoted to the proof of Theorem 4.2.7.

Sections 4.4–4.6

Section 4.4 defines hierarchical meshes and hierarchical splines on Ω, derives the canonical basis of the hierarchical spline space $\mathcal{X}_\bullet \subset H^1_0(\Omega)^D$ with Dirichlet boundary condition, and introduces some local mesh-refinement strategy (Algorithm 4.4.1) which preserves a certain admissibility property. This admissibility property particularly yields that the number of (truncated) hierarchical B-splines on each element as well as the number of elements contained in the support of each (truncated) hierarchical B-spline is uniformly bounded; see Proposition 3.4.3. If one uses the strategy of [BG16c, BGMP16, BG17] instead, this is not true for hierarchical B-splines, but only for truncated hierarchical B-splines.

The main result of this chapter is **Theorem 4.4.6** which states that hierarchical splines together with the proposed local mesh-refinement strategy satisfy all assumptions of Section 4.2, so that Theorem 4.2.7 applies and proves optimal convergence behavior of the adaptive algorithm. The proof is given in Section 4.5. Whereas the corresponding result of [BG16c, BG17] adapts the analysis of [CKNS08] and is thus restricted to symmetric problems, we exploit some recent ideas from [FFP14] in order to cover the non-symmetric case as well. Remark 4.4.7 extends Theorem 4.4.6 to rational hierarchical splines.

We conclude the chapter with three numerical experiments in Section 4.6 which underpin the theoretical results, but also demonstrate the limitations of hierarchical splines in the frame of adaptive FEM if the solution u exhibits edge singularities.

Chapter 5

We consider a general linear system of second-order PDEs on the d-dimensional bounded Lipschitz domain Ω with PDE operator $\mathfrak{P}$. We assume that the coefficients of $\mathfrak{P}$ are constant and that the induced bilinear form is $H^1_0(\Omega)^D$-elliptic up to some compact perturbation. Let $G : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}^{D\times D}$ be the corresponding (matrix-valued) fundamental solution in the sense of [McL00, page 198]. For $\psi \in L^\infty(\Gamma)^D$, we define the *single-layer operator* as boundary convolution with G, i.e.,

$$(\mathfrak{V}\psi)(x) := \int_\Gamma G(x-y)\psi(y)\,dy \quad \text{for all } x \in \Gamma. \tag{1.2.2}$$

This operator can be extended to a bounded linear operator

$$\mathfrak{V} : H^{-1/2}(\Gamma)^D \to H^{1/2}(\Gamma)^D, \tag{1.2.3}$$

where $H^{1/2}(\Gamma)$ denotes the space of traces of $H^1(\Omega)$ and $H^{-1/2}(\Gamma)$ is its dual space. $\mathfrak{V}$ is always elliptic up to some compact perturbation. We assume that it is elliptic even without perturbation. This is particularly satisfied for the Laplace problem or for the Lamé problem.

Given a right-hand side $f \in H^{1/2}(\Gamma)^D$, we investigate the boundary integral equation

$$\mathfrak{V}\phi = f. \tag{1.2.4}$$

Such integral equations arise from (and are even equivalent to) the solution of Dirichlet problems of the form

$$\begin{aligned} \mathfrak{P}u &= 0 \quad \text{in } \Omega \\ u &= g \quad \text{on } \Gamma \end{aligned} \tag{1.2.5}$$

for some $g \in H^{1/2}(\Gamma)^D$. Indeed, if $u \in H^1(\Omega)^D$ is a corresponding weak solution, then its conormal derivative $\phi := \mathfrak{D}_\nu u$ (i.e., the Neumann data) satisfies (1.2.4) with $f := (\mathfrak{K}+1/2)g$. Here, $\mathfrak{K} : H^{1/2}(\Gamma)^D \to H^{1/2}(\Gamma)^D$ denotes the *double-layer operator*. Conversely, if $\phi \in H^{-1/2}(\Gamma)^D$ satisfies (1.2.4) with $f := (\mathfrak{K}+1/2)g$, then a weak solution of (1.2.5) is given by the *representation formula* $u := \widetilde{\mathfrak{V}}\phi - \widetilde{\mathfrak{K}}g$, where $\widetilde{\mathfrak{V}} : H^{-1/2}(\Gamma)^D \to H^1(\Omega)^D$ denotes the *single-layer potential* and $\widetilde{\mathfrak{K}} : H^{1/2}(\Gamma)^D \to H^1(\Omega)^D$ denotes the *double-layer potential*.

The Lax–Milgram lemma provides existence and uniqueness of the solution $\phi \in H^{-1/2}(\Gamma)^D$ of the equivalent variational formulation of (1.2.4)

$$\langle \mathfrak{V}\phi \,,\, \psi\rangle = \langle f \,,\, \psi\rangle \quad \text{for all } \psi \in H^{-1/2}(\Gamma)^D. \tag{1.2.6}$$

In the Galerkin boundary element method, the test space $H^{-1/2}(\Gamma)^D$ is replaced by some discrete subspace $\mathcal{X}_\bullet \subset L^2(\Gamma)^D \subset H^{-1/2}(\Gamma)^D$ and the Lax–Milgram lemma guarantees the existence and uniqueness of a discrete solution $\Phi_\bullet \in \mathcal{X}_\bullet$.

For standard BEM with (dis)continuous piecewise polynomials, *a posteriori* error estimation and adaptive mesh-refinement are well understood. In particular, optimal convergence of mesh-refining adaptive algorithms has recently been proved for polyhedral boundaries [FFK+14, FFK+15, FKMP13] as well as smooth boundaries [Gan13]. The work [AFF+17] allows to transfer these results to piecewise smooth boundaries. However, *a posteriori* error estimation for IGABEM has only been considered for the two-dimensional Laplace problem in the recent own works [FGP15, FGHP16, FGHP17].

Sections 5.2–5.3

Similarly as in Section 4.2, in Section 5.2, we give an abstract framework for adaptive mesh-refinement for conforming BEM in 2D and 3D for the model problem (1.2.4). In **Theorem 5.2.5**, we identify sufficient conditions for the underlying meshes, the local BEM spaces, as well as the employed (local) mesh-refinement rule which guarantee that the standard weighted-residual *a posteriori* error estimator satisfies the axioms of adaptivity from Chapter 2. In particular, this implies that the corresponding adaptive algorithm (Algorithm 5.2.4) leads to linear convergence of the error estimator at optimal algebraic rate. In particular, Theorem 5.2.5 states that the employed error estimator is reliable, i.e., that it is an upper bound for the error $\|\phi - \Phi_\bullet\|_{H^{-1/2}(\Gamma)}$ (up to a multiplicative constant). The proof of Theorem 5.2.5 is given in Section 5.3.

To prove reliability of the weighted-residual estimator, we show that it is an upper bound for the so-called *Faermann estimator*, proposed and analyzed for standard BEM in [Fae00, Fae02]. For ansatz spaces that contain certain piecewise polynomials, Faermann proved reliability as well as efficiency of this estimator. We extend this result and prove that the Faermann estimator is reliable and efficient for arbitrary BEM spaces that satisfy certain assumptions; see Proposition 5.3.7 and Proposition 5.3.8. Moreover, Remark 5.3.10 states that one obtains at least plain convergence $\lim_{\ell\to\infty} \|\phi - \Phi_\ell\|_{H^{-1/2}(\Gamma)} = 0$ if Algorithm 5.2.4 is steered by the Faermann estimator. Such a result was first proved in [FFME+14] for piecewise constants on affine triangulations of Γ.

In contrast to [FFK+14, FKMP13] which only verify the axioms of adaptivity for a single model problem, namely the Laplace problem, our analysis allows for arbitrary linear second-order PDE operators $\mathfrak{P}$ with constant coefficients. The crucial step is the generalization (Proposition 5.3.15) of the inverse inequality from [AFF+17] with the help of a Caccioppoli type inequality (Lemma 5.3.13).

Sections 5.4–5.6

Section 5.4 defines hierarchical meshes and hierarchical splines on the boundary Γ and introduces some local mesh-refinement rule (Algorithm 5.4.2) which preserves a similar admissibility property as in Chapter 4. To the best of our knowledge, this thesis is the first work which investigates and analyzes IGABEM in 3D with hierarchical splines as ansatz space. The main result of Section 5.4 is **Theorem 5.4.5** which states that hierarchical splines together with the proposed local mesh-refinement strategy satisfy all assumptions of Section 5.2, so that Theorem 5.2.5 applies and proves optimal convergence behavior of the adaptive algorithm. The proof is given in Section 5.5. Remark 5.4.6 extends the result to rational hierarchical splines. In particular, in Section 5.5.9 and Section 5.5.15, we generalize an inverse inequality for piecewise polynomial ansatz functions from [GHS05] to rational hierarchical splines.

Two numerical experiments in Section 5.6 underpin the theoretical results, but also demonstrate the limitations of hierarchical splines in the frame of adaptive BEM if the solution ϕ exhibits edge singularities.

Sections 5.7–5.9

In Section 5.7, we consider IGABEM in 2D. We present an adaptive algorithm (Algorithm 5.7.3) from the recent own work [FGHP16] with one-dimensional splines as ansatz space. Whereas the adaptive algorithm of Section 5.2 resp. Section 5.4 only uses h-refinement, the latter additionally allows for knot multiplicity increase and thus for local smoothness control of the ansatz functions. **Theorem 5.7.4** states again reliability and linear convergence of the error estimator at optimal rate. The proof is given in Section 5.8. Remark 5.7.6 extends the result to rational splines. In particular, in Section 5.8.3 and Section 5.8.11, we generalize an inverse inequality for piecewise polynomials [GHS05] to rational splines. Again, we note in Remark 5.8.3 that the application of the Faermann estimator would at least lead to plain convergence. We conclude this chapter with three further numerical experiments in Section 5.9.

Implementations

During the PhD studies, implementations for adaptive IGABEM in 2D, adaptive IGAFEM in 2D, and adaptive IGABEM in 3D were developed. These implementations are used in the numerical experiments of Section 5.9, Section 4.6, resp. Section 5.6. The 2D IGABEM code was mainly written for the own master's thesis [Gan14], where the focus was on the Faermann error estimator. During the PhD studies, the weighted-residual error estimator was implemented. Moreover, the possibility of knot multiplicity increase instead of pure h-refinement was added. The MATLAB implementation for 2D IGAFEM was developed together with Daniel Haberlik within the framework of his bachelor's thesis [Hab] which was jointly supervised by the author of this work and Dirk Praetorius. The implementation of 3D IGABEM was developed for this thesis. As for the implementation of 2D IGABEM, the assembly of the Galerkin matrix, the right-hand side vector, and the estimator for 3D IGABEM is realized in C via MATLAB's MEX interface, whereas the refinement procedure is implemented in MATLAB.

Related own publications

Parts of this thesis are already found in the recent own works [FGP15, FGHP16, FGHP17, GHP17] that were written during the PhD studies. [FGP15, FGHP16] introduce the adaptive algorithm for one-dimensional rational splines steered by the Faermann estimator resp. the standard weighted-residual estimator and prove reliability of the employed estimator. There, the Laplace problem is chosen as model problem. [FGP15] additionally considers collocation IGABEM which is usually preferred by engineers. Both works are based on the own master's thesis [Gan14] which focuses on the *a posteriori* analysis and empirically investigates the corresponding adaptive algorithm without knot multiplicity increase. [FGHP17], which is also restricted to the two-dimensional Laplace problem, proves that the adaptive algorithm from [FGP15, FGHP16] leads to optimal convergence of the weighted-residual estimator and to plain convergence of the Faermann estimator. While the results of Section 5.7 and Section 5.8 go back to [FGP15, FGHP16, FGHP17], the current presentation differs and the results are generalized. [GHP17] treats optimal convergence for adaptive IGAFEM with hierarchical splines. The contents of Chapter 3 and Chapter 4 are found in [GHP17], while the present presentation provides more details. The results on adaptive 3D IGABEM with hierarchical splines have not been published yet. Besides the mentioned four publications [FGP15, FGHP16, FGHP17, GHP17] which are part of this PhD thesis, two further publications [FGH$^+$16, GHPS17] were written during the PhD studies.

1.3 General notation

Throughout, $|\cdot|$ denotes the absolute value of scalars, the Euclidean norm of vectors, and the Hausdorff measure of a set in $\mathbb{R}^n$ for $n \geq 1$, where the corresponding Hausdorff dimension is denoted by $\dim(\cdot)$. The respective meaning will be clear from the context. Moreover, $\#$ denotes the cardinality of a finite set.

For an arbitrary point $x \in \mathbb{R}^n$ and $r > 0$, we denote the corresponding open ball $B_r(x) := \{y \in \mathbb{R}^n : |x - y| < r\}$. If $S \subseteq \mathbb{R}^n$, we write $B_r(S) := \bigcup \{B_r(x) : x \in S\}$. Further, we define its characteristic function $\chi_S : \mathbb{R}^n \to \{0, 1\}$ via $\chi_S|_S = 1$ and $\chi_S|_{\mathbb{R}^n \setminus S} = 0$.

For real-valued quantities A, B, we write $A \lesssim B$ resp. $A \gtrsim B$ to abbreviate $A \leq cB$ resp. $A \geq cB$ with some generic constant $c > 0$ which is clear from the context. Moreover, $A \simeq B$ abbreviates $A \lesssim B \lesssim A$.

Mesh-related quantities have the same index, e.g., $\mathcal{X}_\bullet$ is the ansatz space corresponding to the mesh $\mathcal{T}_\bullet$. The analogous notation is used for meshes $\mathcal{T}_\circ$, $\mathcal{T}_\star$ or $\mathcal{T}_\ell$ etc. Moreover, we use $\widehat{\cdot}$ to transfer quantities in the physical domain to the parameter domain, e.g., we write $\widehat{\mathbb{T}}$ for the set of all admissible meshes in the parameter domain, whereas $\mathbb{T}$ denotes the set of all admissible meshes in the physical domain.

2 Axioms of Adaptivity

2.1 Introduction

In this chapter, we consider a standard adaptive algorithm from a very abstract point of view. We provide a set of sufficient properties for the error estimator as well as for the mesh-refinement which guarantee convergence of the estimator at optimal algebraic rate. These properties are known as *axioms of adaptivity*, and have been introduced in [CFPP14]. In one way or another, the axioms arose over the years in various works throughout the literature. In [CFPP14, Section 3.2], a historical overview on the their development can be found. This chapter is essentially a summary of the results from [CFPP14] and its slight generalization [Fei15]. The proofs are not new but included for the convenience of the reader. We fix the abstract framework in Section 2.2. In particular, the adaptive algorithm is given in Section 2.2.2. Then, in Section 2.3, we introduce the axioms of adaptivity and formulate the implied main results on convergence for the error estimator and for locally equivalent error estimators. Section 2.4 and Section 2.5 are devoted to the corresponding proofs. Note that, as in [CFPP14, Fei15], we focus on the error estimator instead of the error itself. This is in a certain sense natural, since the adaptive algorithm has no other information than the error estimator to steer the mesh-refinement. However, at least in Chapter 4, we will show that the corresponding concrete error estimator is equivalent to the so-called *total error* (which is the sum of error plus data oscillations).

2.2 Abstract framework

In this section, we introduce general abstract meshes and error estimators, and formulate the adaptive algorithm.

2.2.1 General meshes

Let $\mathbb{T}$ be a set of finite sets, which we refer to as *meshes*. Let $\texttt{refine}(\cdot,\cdot)$ be a fixed refinement strategy such that, for $\mathcal{T}_\bullet \in \mathbb{T}$ and marked $\mathcal{M}_\bullet \subseteq \mathcal{T}_\bullet$, there holds that $\mathcal{T}_\circ = \texttt{refine}(\mathcal{T}_\bullet, \mathcal{M}_\bullet) \in \mathbb{T}$ with $\mathcal{M}_\bullet \subseteq \mathcal{T}_\bullet \setminus \mathcal{T}_\circ$ and $\texttt{refine}(\mathcal{T}_\bullet, \emptyset) = \mathcal{T}_\bullet$. For arbitrary $\mathcal{T}_\bullet, \mathcal{T}_\circ \in \mathbb{T}$, we write $\mathcal{T}_\circ \in \texttt{refine}(\mathcal{T}_\bullet)$, if $\mathcal{T}_\circ$ is obtained by iterative application of $\texttt{refine}$, i.e., $\mathcal{T}_\circ = \mathcal{T}_{(J)} = \texttt{refine}(\mathcal{T}_{(J-1)}, \mathcal{M}_{(J-1)}), \mathcal{T}_{(J-1)} = \texttt{refine}(\mathcal{T}_{(J-2)}, \mathcal{M}_{(J-2)}), \dots, \mathcal{T}_{(1)} = \texttt{refine}(\mathcal{T}_{(0)}, \mathcal{M}_{(0)})$ with $\mathcal{T}_{(0)} = \mathcal{T}_\bullet$. Note that $\mathcal{T}_\bullet \in \texttt{refine}(\mathcal{T}_\bullet)$. We assume that $\texttt{refine}(\mathcal{T}_0) = \mathbb{T}$.

We suppose that we are given a function with integer values on the set of all possible elements $\mu : \bigcup_{\mathcal{T}_\bullet \in \mathbb{T}} \mathcal{T}_\bullet \to \mathbb{N}$ and define $\mu(\mathcal{S}) := \sum_{T \in \mathcal{S}} \mu(T)$ for $\mathcal{S} \subseteq \mathcal{T}_\bullet$ and $\mathcal{T}_\bullet \in \mathbb{T}$. We assume that $\mu(\mathcal{T}_\bullet) < \mu(\mathcal{T}_\circ)$ for all $\mathcal{T}_\bullet \in \mathbb{T}$ and all $\mathcal{T}_\circ \in \texttt{refine}(\mathcal{T}_\bullet)$ with $\mathcal{T}_\bullet \neq \mathcal{T}_\circ$. The subset of all refinements which have at most $N \in \mathbb{N}_0$ elements more than the initial mesh

$\mathcal{T}_0$, reads

$$\mathbb{T}(N) := \big\{\mathcal{T}_\bullet \in \mathbb{T} : \mu(\mathcal{T}_\bullet) - \mu(\mathcal{T}_0) \le N\big\}. \tag{2.2.1}$$

Note that the analysis of [CFPP14, Fei15] does not include such a general *measure* μ, but μ is just chosen as the cardinality # of a set. Nevertheless, the proofs work almost verbatim. We will make the same standard choice in Section 4.2 and in Section 5.2, whereas we will choose μ as *knot multiplicity* in Section 5.7.

2.2.2 Adaptive algorithm

We suppose that we are given an *error estimator* associated to each mesh $\mathcal{T}_\bullet \in \mathbb{T}$, i.e., a function $\eta_\bullet : \mathcal{T}_\bullet \to [0,\infty)$. By abuse of notation, we also write $\eta_\bullet := \eta_\bullet(\mathcal{T}_\bullet)$, where $\eta_\bullet(\mathcal{S}) := (\sum_{T\in\mathcal{S}} \eta_\bullet(T)^2)^{1/2}$ for all $\mathcal{S} \subseteq \mathcal{T}_\bullet$. Based on this error estimator, we consider the following adaptive algorithm.

Algorithm 2.2.1. ***Input:*** *Dörfler parameter $\theta \in (0,1]$ and marking constant $C_{\min} \in [1,\infty]$.* ***Loop:*** *For each $\ell = 0,1,2,\dots,$ iterate the following steps:*

(i) *Compute refinement indicators $\eta_\ell(T)$ for all elements $T \in \mathcal{T}_\ell$.*

(ii) *Determine a set of marked elements $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ which is up to the multiplicative constant $C_{\min}$ minimal*[1] *with respect to μ, such that the following Dörfler marking is satisfied*

$$\theta\, \eta_\ell^2 \le \eta_\ell(\mathcal{M}_\ell)^2. \tag{2.2.2}$$

(iii) *Generate refined mesh $\mathcal{T}_{\ell+1} := \mathtt{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$.*

Output: *Refined meshes $\mathcal{T}_\ell$ and error estimators η_ℓ for all $\ell \in \mathbb{N}_0$.*

2.3 The axioms

This section is devoted to the axioms of adaptivity from [CFPP14, Fei15]. They are sufficient to prove convergence of the error estimator sequence generated by the adaptive algorithm at optimal algebraic rate. Indeed, the axioms are even necessary in some sense; see [Fei15, Section 2.6].

2.3.1 Set of axioms

We suppose that we are given some fixed *perturbations*[2] $\varrho_{\bullet,\circ}$ for all $\mathcal{T}_\bullet \in \mathbb{T}$, $\mathcal{T}_\circ \in \mathtt{refine}(\mathcal{T}_\bullet)$, and constants $C_{\rm qo}, C_{\rm ref}, C_{\rm drel}, C_{\rm son}, C_{\rm clos}, C_{\rm over} \ge 1$, and $0 \le \rho_{\rm red}, \varepsilon_{\rm qo}, \varepsilon_{\rm drel} < 1$ such that for the sequence $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}_0}$ from Algorithm 2.2.1, there hold:

[1] This means that $\mu(\mathcal{M}_\ell) \le C_{\min}\mu(\mathcal{S})$ for all sets $\mathcal{S} \subseteq \mathcal{T}_\ell$ with $\theta\eta_\ell^2 \le \eta_\ell(\mathcal{S})^2$. If $C_{\min} = \infty$, this is always satisfied and allows for uniform refinement, where $\mathcal{M}_\ell = \mathcal{T}_\ell$.

[2] In the following chapters, $\varrho_{\bullet,\circ}$ will always be the error $\|U_\circ - U_\bullet\|$ between two approximations $U_\bullet$ and $U_\circ$ corresponding to the meshes $\mathcal{T}_\bullet$ and $\mathcal{T}_\circ$.

(E1) Stability on non-refined elements: For all $\mathcal{T}_\bullet \in \mathbb{T}$ and all $\mathcal{T}_\circ \in \texttt{refine}(\mathcal{T}_\bullet)$, it holds that

$$|\eta_\circ(\mathcal{T}_\bullet \cap \mathcal{T}_\circ) - \eta_\bullet(\mathcal{T}_\bullet \cap \mathcal{T}_\circ)| \leq \varrho_{\bullet,\circ}.$$

(E2) Reduction on refined elements: For all $\mathcal{T}_\bullet \in \mathbb{T}$ and all $\mathcal{T}_\circ \in \texttt{refine}(\mathcal{T}_\bullet)$, it holds that

$$\eta_\circ(\mathcal{T}_\circ \setminus \mathcal{T}_\bullet)^2 \leq \rho_{\rm red}\eta_\bullet(\mathcal{T}_\bullet \setminus \mathcal{T}_\circ)^2 + \varrho_{\bullet,\circ}^2.$$

(E3) General quasi-orthogonality: It holds that

$$0 \leq \varepsilon_{\rm qo} < \sup_{\delta>0} \frac{1-(1+\delta)(1-(1-\rho_{\rm red})\theta)}{2+\delta^{-1}},$$

and for all $\ell, N \in \mathbb{N}_0$ that

$$\sum_{j=\ell}^{\ell+N} (\varrho_{j,j+1}^2 - \varepsilon_{\rm qo}\eta_j^2) \leq C_{\rm qo}\eta_\ell^2.$$

(E4) Discrete reliability: For all $\mathcal{T}_\bullet \in \mathbb{T}$ and all $\mathcal{T}_\circ \in \texttt{refine}(\mathcal{T}_\bullet)$, there exists $\mathcal{T}_\bullet \setminus \mathcal{T}_\circ \subseteq \mathcal{R}_{\bullet,\circ} \subseteq \mathcal{T}_\bullet$ with $\mu(\mathcal{R}_{\bullet,\circ}) \leq C_{\rm ref}\big(\mu(\mathcal{T}_\circ) - \mu(\mathcal{T}_\bullet)\big)$ such that

$$\varrho_{\bullet,\circ}^2 \leq \varepsilon_{\rm drel}\eta_\bullet^2 + C_{\rm drel}^2\eta_\bullet(\mathcal{R}_{\bullet,\circ})^2.$$

(T1) Son estimate: For all $\ell \in \mathbb{N}_0$, it holds that

$$\mu(\mathcal{T}_{\ell+1}) \leq C_{\rm son}\mu(\mathcal{T}_\ell).$$

(T2) Closure estimate: For all $\ell \in \mathbb{N}_0$, it holds that

$$\mu(\mathcal{T}_\ell) - \mu(\mathcal{T}_0) \leq C_{\rm clos}\sum_{j=0}^{\ell-1}\mu(\mathcal{M}_j).$$

(T3) Overlay property: For all $\ell \in \mathbb{N}_0$ and $\mathcal{T}_\bullet \in \mathbb{T}$, there exists a common refinement $\mathcal{T}_\circ \in \texttt{refine}(\mathcal{T}_\ell) \cap \texttt{refine}(\mathcal{T}_\bullet)$ which satisfies the overlay estimate

$$\mu(\mathcal{T}_\circ) \leq C_{\rm over}\big(\mu(\mathcal{T}_\bullet) - \mu(\mathcal{T}_0)\big) + \mu(\mathcal{T}_\ell).$$

2.3.2 Optimal convergence for the error estimator

The following theorem is the main result of this chapter. It was already proved in [Fei15, Theorem 2.3.3], and, in a slightly weaker form, in [CFPP14, Theorem 4.1 and Corollary 4.8]. We include the proof in Section 2.4. For arbitrary $s > 0$, we set

$$C_{\rm approx}(s) := \sup_{N\in\mathbb{N}_0} \min_{\mathcal{T}_\bullet\in\mathbb{T}(N)} \big((N+1)^s\eta_\bullet)\big) \in [0,\infty]. \tag{2.3.1}$$

This definition characterizes the best possible algebraic convergence rate for the error estimator starting from $\mathcal{T}_0$.

Theorem 2.3.1. *Let $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}_0}$ be the meshes generated by Algorithm 2.2.1. Then, there hold:*

(i) *Suppose the axioms (E1)–(E2), where it suffices to consider $\mathcal{T}_{\ell+1} \in \texttt{refine}(\mathcal{T}_\ell)$ for all $\ell \in \mathbb{N}_0$, and assume that $\lim_{\ell\to\infty} \varrho_{\ell,\ell+1} = 0$. Then, for all $0 < \theta \le 1$ and all $C_{\min} \in [1,\infty]$, the estimator converges, i.e.,*

$$\lim_{\ell\to\infty} \eta_\ell = 0. \tag{2.3.2}$$

(ii) *Suppose the axioms (E1)–(E3), where it suffices to consider $\mathcal{T}_{\ell+1} \in \texttt{refine}(\mathcal{T}_\ell)$ in (E1)–(E2) for all $\ell \in \mathbb{N}_0$. Then, for all $0 < \theta \le 1$ and all $C_{\min} \in [1,\infty]$, the estimator converges linearly, i.e., there exist constants $0 < \rho_{\rm lin} < 1$ and $C_{\rm lin} \ge 1$ such that*

$$\eta_{\ell+j}^2 \le C_{\rm lin}\rho_{\rm lin}^j \eta_\ell^2 \quad \text{for all } j,\ell \in \mathbb{N}_0. \tag{2.3.3}$$

(iii) *Suppose the axioms (E1)–(E4) and (T1)–(T3). Then, for all $0 < \theta < \theta_{\rm opt} := (1-\varepsilon_{\rm drel})/(1+C_{\rm drel}^2)$ and all $C_{\min} \in [1,\infty)$, the estimator converges at optimal rate, i.e., for all $s > 0$ there exist constants $c_{\rm opt}, C_{\rm opt} > 0$ such that*

$$c_{\rm opt} C_{\rm approx}(s) \le \sup_{\ell\in\mathbb{N}_0} (\mu(\mathcal{T}_\ell) - \mu(\mathcal{T}_0) + 1)^s \eta_\ell \le C_{\rm opt} C_{\rm approx}(s), \tag{2.3.4}$$

where the lower bound relies only (T1).

The constants $C_{\rm lin}, \rho_{\rm lin}$ depend only on $\rho_{\rm red}, C_{\rm qo}, \varepsilon_{\rm qo}$ and on θ. The constant $C_{\rm opt}$ depends additionally on $C_{\min}, C_{\rm ref}, C_{\rm drel}, \varepsilon_{\rm drel}, C_{\rm clos}, C_{\rm over}$, and on s, while $c_{\rm opt}$ depends only on $C_{\rm son}, \mu(\mathcal{T}_0)$, s, and if there exists ℓ_0 with $\eta_{\ell_0} = 0$ also on ℓ_0 and η_0.

Remark 2.3.2. *The upper bound in (2.3.4) states that the estimator sequence η_ℓ of Algorithm 2.2.1 converges with algebraic rate s if $C_{\rm approx}(s) < \infty$. This means that if a decay with rate s is possible for optimally chosen meshes, the same decay is realized by the adaptive algorithm. The lower bound in (4.2.23) states that the convergence rate of the estimator sequence characterizes the theoretically optimal convergence rate.*

2.3.3 Optimal convergence for equivalent error estimators

We suppose that we are given a second locally equivalent error estimator associated to each $\mathcal{T}_\bullet \in \mathbb{T}$, i.e., a function $\widetilde{\eta}_\bullet : \mathcal{T}_\bullet \to [0,\infty)$ such that there exists a constant $C_{\rm eq} \ge 1$ with

$$C_{\rm eq}^{-1} \eta_\bullet(T)^2 \le \widetilde{\eta}_\bullet(T)^2 \le C_{\rm eq}\eta_\bullet(T)^2 \quad \text{for all } T \in \mathcal{T}_\bullet. \tag{2.3.5}$$

We also use the notation $\widetilde{\eta}_\bullet := \widetilde{\eta}_\bullet(\mathcal{T}_\bullet)$, where $\widetilde{\eta}_\bullet(\mathcal{S}) := (\sum_{T\in\mathcal{S}} \widetilde{\eta}_\bullet(T)^2)^{1/2}$ for all $\mathcal{S} \subseteq \mathcal{T}_\bullet$. Based on this error estimator, we consider Algorithm 2.2.1 with η replaced by $\widetilde{\eta}$.

Algorithm 2.3.3. ***Input:*** *Dörfler parameter $\widetilde{\theta} \in (0,1]$ and marking constant $\widetilde{C}_{\min} \in [1,\infty]$.*
Loop: *For each $\ell = 0,1,2,\dots$, iterate the following steps:*

(i) *Compute refinement indicators $\widetilde{\eta}_\ell(T)$ for all elements $T \in \mathcal{T}_\ell$.*

(ii) *Determine a set of marked elements $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ which is up to the multiplicative constant $\widetilde{C}_{\min}$ minimal*[3] *with respect to μ, such that the following Dörfler marking is satisfied*

$$\widetilde{\theta}\,\widetilde{\eta}_\ell^2 \leq \widetilde{\eta}_\ell(\mathcal{M}_\ell)^2. \tag{2.3.6}$$

(iii) *Generate refined mesh $\mathcal{T}_{\ell+1} := \texttt{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$.*

Output: *Refined meshes $\mathcal{T}_\ell$ and error estimators $\widetilde{\eta}_\ell$ for all $\ell \in \mathbb{N}_0$.*

A more general version of the next corollary is proved in [Fei15, Section 4.2]. It is an easy consequence of Theorem 2.3.1. We include the proof in Section 2.5.

Corollary 2.3.4. *Let $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}_0}$ be the meshes generated by Algorithm 2.3.3. Then, there hold:*

(i) *Suppose the axiom (E1)–(E2) (for the estimator η), where it suffices to consider $\mathcal{T}_{\ell+1} \in \texttt{refine}(\mathcal{T}_\ell)$ for all $\ell \in \mathbb{N}_0$, and assume that $\lim_{\ell\to\infty} \varrho_{\ell,\ell+1} = 0$. Then, for all $0 < \widetilde{\theta} \leq 1$, the equivalent estimator converges, i.e.,*

$$\lim_{\ell\to\infty} \widetilde{\eta}_\ell = 0. \tag{2.3.7}$$

(ii) *Suppose the axioms (E1)–(E3) (for the estimator η but the meshes generated by Algorithm 2.3.3), where it suffices to consider $\mathcal{T}_{\ell+1} \in \texttt{refine}(\mathcal{T}_\ell)$ for all $\ell \in \mathbb{N}_0$. Then, for all $0 < \theta \leq 1$ and all $C_{\min} \in [1,\infty]$, the equivalent estimator converges linearly, i.e., there exists $0 < \rho_{\rm lin} < 1$ and $C_{\rm lin} \geq 1$ such that*

$$\widetilde{\eta}_{\ell+j}^2 \leq C_{\rm eq} C_{\rm lin} \rho_{\rm lin}^j \widetilde{\eta}_\ell^2 \quad \textit{for all } j,\ell \in \mathbb{N}_0. \tag{2.3.8}$$

(iii) *Suppose the axioms (E1)–(E4) and (T1)–(T3) (for the estimator η but the meshes generated by Algorithm 2.3.3). Then, for all $0 < \widetilde{\theta} < C_{\rm eq}^{-2}\theta_{\rm opt} := C_{\rm eq}^{-2}(1-\varepsilon_{\rm drel})/(1+C_{\rm drel}^2)$ and all $\widetilde{C}_{\min} \in [1,\infty)$ the equivalent estimator converges at quasi-optimal rate, i.e., there exist $c_{\rm opt}, C_{\rm opt} > 0$ such that for all $s > 0$*

$$C_{\rm eq}^{-1} c_{\rm opt} C_{\rm approx}(s) \leq \sup_{\ell\in\mathbb{N}_0} (\mu(\mathcal{T}_\ell) - \mu(\mathcal{T}_0) + 1)^s \widetilde{\eta}_\ell \leq C_{\rm eq} C_{\rm opt} C_{\rm approx}(s), \tag{2.3.9}$$

where the lower bound requires only (T1) to hold.

The constants $C_{\rm lin}, \rho_{\rm lin}$ depend only on $\rho_{\rm red}, C_{\rm qo}, \varepsilon_{\rm qo}$ and on $C_{\rm eq}^{-2}\widetilde{\theta}$. The constant $C_{\rm opt}$ depends additionally on $C_{\min}, C_{\rm qo}, C_{\rm clos}, C_{\rm drel}, \varepsilon_{\rm drel}$, and on s, while $c_{\rm opt}$ depends only on $C_{\rm son}, \mu(\mathcal{T}_0)$, s, and if there exists ℓ_0 with $\eta_{\ell_0} = 0$ also on ℓ_0 and η_0. □

2.4 Proof of Theorem 2.3.1

In this section, we prove the main result stated in Theorem 2.3.1. The proof follows along the lines of [CFPP14, Section 4] or [Fei15, Chapter 2].

[3] This means that $\mu(\mathcal{M}_\ell) \leq \widetilde{C}_{\min}\mu(\mathcal{S})$ for all sets $\mathcal{S} \subseteq \mathcal{T}_\ell$ with $\widetilde{\theta}\eta_\ell^2 \leq \eta_\ell(\mathcal{S})^2$. If $\widetilde{C}_{\min} = \infty$, this is always satisfied.

2.4.1 Estimator reduction and convergence

In this section, we give the proof of Theorem 2.3.1 (i). The following lemma states uniform contraction of the error estimators η_ℓ of Algorithm 2.2.1 in each step up to some perturbation.

Lemma 2.4.1. *Let $0 < \theta \le 1$ and suppose the axioms (E1)–(E2), it suffices to consider $\mathcal{T}_{\ell+1} \in \texttt{refine}(\mathcal{T}_\ell)$. Then, there holds estimator reduction in the sense that*

$$\eta_{\ell+1}^2 \le \rho_{\rm est}\, \eta_\ell^2 + C_{\rm est}\, \varrho_{\ell,\ell+1}^2 \quad \text{for all } \ell \in \mathbb{N}_0, \tag{2.4.1a}$$

where

$$\rho_{\rm est} = (1+\delta)\big(1-(1-\rho_{\rm red})\theta\big) \quad \text{and} \quad C_{\rm est} = 2+\delta^{-1} \tag{2.4.1b}$$

for all sufficiently small $\delta > 0$ with $0 < \rho_{\rm est} < 1$.

Proof. First, we split the estimator $\eta_{\ell+1}^2 = \eta_{\ell+1}(\mathcal{T}_{\ell+1} \setminus \mathcal{T}_\ell)^2 + \eta_{\ell+1}(\mathcal{T}_\ell \cap \mathcal{T}_{\ell+1})^2$. We apply reduction (E2) and stability (E1) together with Young's inequality to obtain for arbitrary $\delta > 0$ that

$$\eta_{\ell+1}^2 \le \rho_{\rm red}\, \eta_\ell(\mathcal{T}_\ell \setminus \mathcal{T}_{\ell+1})^2 + (1+\delta)\, \eta_\ell(\mathcal{T}_\ell \cap \mathcal{T}_{\ell+1})^2 + C_{\rm est}\, \varrho_{\ell,\ell+1}^2.$$

Next, we rearrange this estimate as

$$\begin{aligned} \eta_{\ell+1}^2 &\le (1+\delta)\,\big(\eta_\ell^2 - \eta_\ell(\mathcal{T}_\ell \setminus \mathcal{T}_{\ell+1})^2\big) + \rho_{\rm red}\, \eta_\ell(\mathcal{T}_\ell \setminus \mathcal{T}_{\ell+1})^2 + C_{\rm est}\, \varrho_{\ell,\ell+1}^2 \\ &\le (1+\delta)\,\big(\eta_\ell^2 - (1-\rho_{\rm red})\, \eta_\ell(\mathcal{T}_\ell \setminus \mathcal{T}_{\ell+1})^2\big) + C_{\rm est}\, \varrho_{\ell,\ell+1}^2. \end{aligned}$$

Finally, $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell \setminus \mathcal{T}_{\ell+1}$ and Dörfler marking (2.2.2) show that $\eta_\ell(\mathcal{T}_\ell \setminus \mathcal{T}_{\ell+1})^2 \ge \eta_\ell(\mathcal{M}_\ell)^2 \ge \theta\eta_\ell^2$. Together with the latter estimate, this concludes the proof. □

Proof of Theorem 2.3.1 (i). First, Lemma 2.4.1 and our assumption $\lim_{\ell\to\infty} \varrho_{\ell,\ell+1} = 0$ show that

$$\limsup_{\ell\to\infty} \eta_{\ell+1}^2 \le \limsup_{\ell\to\infty} \big(\rho_{\rm est}\, \eta_\ell^2 + C_{\rm est}\, \varrho_{\ell,\ell+1}^2\big) = \rho_{\rm est} \limsup_{\ell\to\infty} \eta_{\ell+1}^2.$$

It remains to show that $\limsup_{\ell\to\infty} \eta_{\ell+1} < \infty$ to conclude that $0 = \liminf_{\ell\to\infty} \eta_{\ell+1}^2 = \limsup_{\ell\to\infty} \eta_{\ell+1}^2$. Induction on ℓ with the estimator reduction of Lemma 2.4.1 proves for all $\ell \in \mathbb{N}_0$ that

$$\eta_{\ell+1}^2 \le \rho_{\rm est}^{\ell+1} \eta_0^2 + C_{\rm est} \sum_{j=0}^{\ell} \rho_{\rm est}^{\ell-j}\, \varrho_{j,j+1}^2.$$

Since $\varrho_{j,j+1}$ converges by assumption, it is uniformly bounded from above by some constant $C > 0$. We obtain that

$$\eta_{\ell+1}^2 \le \rho_{\rm est}^{\ell+1} \eta_0^2 + C^2\, C_{\rm est} (1-\rho_{\rm est})^{-1},$$

and thus boundedness of $\limsup_{\ell\to\infty} \eta_\ell \le \sup_{\ell\in\mathbb{N}_0} \eta_\ell < \infty$. □

2.4.2 Linear convergence

Before we come to the proof of Theorem 2.3.1 (ii), we consider equivalent formulations of linear convergence.

Lemma 2.4.2. *Let $(a_\ell)_{\ell\in\mathbb{N}_0}$ be a sequence with $a_\ell \geq 0$ for all $\ell \in \mathbb{N}_0$. Then the following three statements are pairwise equivalent:*

(i) *There exists a constant $C_1 > 0$ such that*

$$\sum_{j=\ell+1}^{\infty} a_j^2 \leq C_1 a_\ell^2 \quad \textit{for all } \ell \in \mathbb{N}_0. \tag{2.4.2}$$

(ii) *For all $s > 0$, there exists a constant $C_2 > 0$ such that*

$$\sum_{j=0}^{\ell-1} a_j^{-1/s} \leq C_2 a_\ell^{-1/s} \quad \textit{for all } \ell \in \mathbb{N}, \tag{2.4.3}$$

where we use the convention $0^{-1/s} := \infty$.

(iii) *There exist constants $0 < \rho_3 < 1$ and $C_3 > 0$ such that*

$$a_{\ell+j}^2 \leq C_3 \rho_3^j a_\ell^2 \quad \textit{for all } j, \ell \in \mathbb{N}_0. \tag{2.4.4}$$

Proof. We show the equivalences (i) $\iff$ (iii) as well as (ii) $\iff$ (iii).
Step 1: We show the implication (iii) $\implies$ (i). There holds that

$$\sum_{j=\ell+1}^{\infty} a_j^2 \leq C_3 a_\ell^2 \sum_{j=\ell+1}^{\infty} \rho_3^{j-\ell} = C_3 \frac{\rho_3}{1-\rho_3} a_\ell^2 \quad \text{for all } \ell \in \mathbb{N}_0.$$

Step 2: We show the implication (iii) $\implies$ (ii). There holds that

$$a_\ell^{-1/s} \leq C_3^{1/(2s)} \rho_3^{j/(2s)} a_{\ell+j}^{-1/s} \quad \text{for all } j, \ell \in \mathbb{N}_0 \text{ with } a_{\ell+j} > 0,$$

Put differently, we get that

$$a_j^{-1/s} \leq C_3^{1/(2s)} \rho_3^{(\ell-j)/(2s)} a_\ell^{-1/s} \quad \text{for all } j, \ell \in \mathbb{N}_0 \text{ with } \ell \geq j \text{ and } a_\ell > 0.$$

Without loss of generality let $\ell \in \mathbb{N}$ with $a_\ell > 0$. This leads us to

$$\sum_{j=0}^{\ell-1} a_j^{-1/s} \leq C_3^{1/(2s)} a_\ell^{-1/s} \sum_{j=0}^{\ell-1} \rho_3^{(\ell-j)/(2s)} \leq \frac{C_3^{1/(2s)}}{1-\rho_3^{1/(2s)}} a_\ell^{-1/s}.$$

Step 3: We show the implication (i) $\implies$ (iii). There holds that

$$(1 + C_1^{-1}) \sum_{k=\ell+1}^{\infty} a_k^2 \leq \sum_{k=\ell+1}^{\infty} a_k^2 + a_\ell^2 = \sum_{k=\ell}^{\infty} a_k^2 \quad \text{for all } \ell \in \mathbb{N}_0.$$

We set $\rho_3 := (1 + C_1^{-1})^{-1}$. This shows that

$$\sum_{k=\ell+1}^{\infty} a_k^2 \leq \rho_3 \sum_{k=\ell}^{\infty} a_k^2 \quad \text{for all } \ell \in \mathbb{N}_0.$$

For all $j, \ell \in \mathbb{N}_0$, induction yields that

$$a_{\ell+j}^2 \leq \sum_{k=\ell+j}^{\infty} a_k^2 \leq \rho_3^j \sum_{k=\ell}^{\infty} a_k^2 = \rho_3^j \Big(\sum_{k=\ell+1}^{\infty} a_k^2 + a_\ell^2 \Big) \leq \rho_3^j (C_1 + 1) a_\ell^2.$$

Step 4: We show the implication (ii) $\Longrightarrow$ (iii). Essentially, the proof works as in Step 3. There holds that

$$(1 + C_2^{-1}) \sum_{k=0}^{\ell-1} a_k^{-1/s} \leq \sum_{k=0}^{\ell} a_k^{-1/s} \quad \text{for all } \ell \in \mathbb{N}.$$

We set $\widetilde{\rho}_3 := (1 + C_2^{-1})^{-1}$. This shows that

$$\sum_{k=0}^{\ell-1} a_k^{-1/s} \leq \widetilde{\rho}_3 \sum_{k=0}^{\ell} a_k^{-1/s} \quad \text{for all } \ell \in \mathbb{N}.$$

Without loss of generality, let $j, \ell \in \mathbb{N}_0$ with $a_{\ell+j} > 0$, which implies due to (2.4.3) that $a_k > 0$ for all $k \leq \ell + j$. Induction yields that

$$a_\ell^{-1/s} \leq \sum_{k=0}^{\ell} a_k^{-1/s} \leq \widetilde{\rho}_3^j \sum_{k=0}^{\ell+j} a_k^{-1/s} = \widetilde{\rho}_3^j \Big(\sum_{k=0}^{\ell+j-1} a_k^{-1/s} + a_{\ell+j}^{-1/s} \Big) \leq \widetilde{\rho}_3^j (C_2 + 1) a_{\ell+j}^{-1/s}.$$

Taking the equation to the power of $-2s$ shows (iii) with $\rho_3 = \widetilde{\rho}_3^{2s}$ and $C_3 = (C_2 + 1)^{2s}$. $\square$

Proof of Theorem 2.3.1 (i). We show that the estimator reduction (2.4.1) and general quasi-orthogonality (E3) imply linear convergence (2.3.3). Recall that the assumptions (E1)–(E2) imply (2.4.1) according to Lemma 2.4.1. By Lemma 2.4.2, it suffices to show (2.4.2) with $(a_\ell)_{\ell \in \mathbb{N}_0} = (\eta_\ell)_{\ell \in \mathbb{N}_0}$. For all $N \in \mathbb{N}_0$ and all $\widetilde{\delta} > 0$, estimator reduction yields that

$$\begin{aligned} \sum_{j=\ell+1}^{\ell+N+1} \eta_j^2 &\leq \sum_{j=\ell+1}^{\ell+N+1} \big(\rho_{\mathrm{est}} \eta_{j-1}^2 + C_{\mathrm{est}} \varrho_{j-1,j}^2 \big) \\ &= \sum_{j=\ell+1}^{\ell+N+1} \Big((\rho_{\mathrm{est}} + \widetilde{\delta}) \eta_{j-1}^2 + C_{\mathrm{est}} \big(\varrho_{j-1,j}^2 - \widetilde{\delta} C_{\mathrm{est}}^{-1} \eta_{j-1}^2 \big) \Big). \end{aligned} \tag{2.4.5}$$

Recall that

$$0 \leq \varepsilon_{\mathrm{qo}} < \sup_{\delta > 0} \frac{1 - (1+\delta)(1 - (1 - \rho_{\mathrm{red}})\theta)}{2 + \delta^{-1}}.$$

Thus, (2.4.1b) shows that one can choose $\delta > 0$ and hence $\rho_{\rm est}$ and $C_{\rm est}$ such that $\varepsilon_{\rm qo} < (1-\rho_{\rm est})/C_{\rm est}$. Therefore, we can choose $\widetilde\delta > 0$ with $\widetilde\delta < 1-\rho_{\rm est}$ and $\varepsilon_{\rm qo} \le \widetilde\delta C_{\rm est}^{-1}$. General quasi-orthogonality (E3) proves for the second term in (2.4.5) that

$$\sum_{j=\ell+1}^{\ell+N+1} (\varrho_{j-1,j}^2 - \widetilde\delta C_{\rm est}^{-1}\eta_{j-1}^2) = \sum_{j=\ell}^{\ell+N} (\varrho_{j,j+1}^2 - \widetilde\delta C_{\rm est}^{-1}\eta_j^2) \le C_{\rm qo}\eta_\ell^2.$$

We apply the latter estimate in (2.4.5) to obtain that

$$\sum_{j=\ell+1}^{\ell+N+1} \eta_j^2 \le \sum_{j=\ell+1}^{\ell+N+1} (\rho_{\rm est}+\widetilde\delta)\eta_{j-1}^2 + C_{\rm est}C_{\rm qo}\eta_\ell^2 \le \sum_{j=\ell+1}^{\ell+N+1} (\rho_{\rm est}+\widetilde\delta)\eta_j^2 + (\rho_{\rm est}+\widetilde\delta+C_{\rm est}C_{\rm qo})\eta_\ell^2.$$

Simplifying and passing to the limit $N \to \infty$ yields that

$$\sum_{j=\ell+1}^{\infty} \eta_j^2 \le \frac{\rho_{\rm est}+\widetilde\delta+C_{\rm est}C_{\rm qo}}{1-(\rho_{\rm est}+\widetilde\delta)}\,\eta_\ell^2.$$

This concludes the proof of (2.4.2) and by Lemma 2.4.2 also the proof of linear convergence (2.3.3). □

2.4.3 Optimal convergence

So far, we have seen that Dörfler marking (2.2.2) in the adaptive algorithm implies linear convergence (2.3.3) of η_ℓ. The next proposition essentially states the converse implication. In other words, Dörfler marking is not only sufficient for linear convergence, but in some sense even necessary.

Proposition 2.4.3. *Suppose stability* (E1) *and discrete reliability* (E4). Let $\mathcal{T}_\bullet \in \mathbb{T}$ and $\mathcal{T}_\circ \in \texttt{refine}(\mathcal{T}_\bullet)$. Then, for all $0 < \theta < \theta_{\rm opt} := (1-\varepsilon_{\rm drel})/(1+C_{\rm drel}^2)$, there exists some $0 < \rho_\theta < 1$ such that

$$\eta_\circ^2 \le \rho_\theta\eta_\bullet^2 \quad\Longrightarrow\quad \theta\eta_\bullet^2 \le \eta_\bullet(\mathcal{R}_{\bullet,\circ})^2. \tag{2.4.6}$$

The constant ρ_θ depends only on $C_{\rm drel}, \varepsilon_{\rm drel}$, and θ.

Proof. Throughout the proof, we work with a free variable $\rho_\theta > 0$, which will be fixed at the end. For all $\delta > 0$, the Young's inequality together with stability (E1) shows that

$$\eta_\bullet^2 = \eta_\bullet(\mathcal{T}_\bullet\setminus\mathcal{T}_\circ)^2 + \eta_\bullet(\mathcal{T}_\bullet\cap\mathcal{T}_\circ)^2 \le \eta_\bullet(\mathcal{T}_\bullet\setminus\mathcal{T}_\circ)^2 + (1+\delta^{-1})\eta_\circ(\mathcal{T}_\bullet\cap\mathcal{T}_\circ)^2 + (1+\delta)\varrho_{\bullet,\circ}^2.$$

With $\mathcal{R}_{\bullet,\circ} \supseteq \mathcal{T}_\bullet\setminus\mathcal{T}_\circ$, we get for the first term on the right-hand side that $\eta_\bullet(\mathcal{T}_\bullet\setminus\mathcal{T}_\circ)^2 \le \eta_\bullet(\mathcal{R}_{\bullet,\circ})^2$. The assumption (2.4.6) proves that $\eta_\circ(\mathcal{T}_\bullet\cap\mathcal{T}_\circ)^2 \le \eta_\circ^2 \le \rho_\theta\eta_\bullet^2$. Together with discrete reliability (E4), we obtain that

$$\eta_\bullet^2 \le \eta_\bullet(\mathcal{R}_{\bullet,\circ})^2 + (1+\delta^{-1})\rho_\theta\eta_\bullet^2 + (1+\delta)\big(\varepsilon_{\rm drel}\eta_\bullet^2 + C_{\rm drel}^2\eta_\bullet(\mathcal{R}_{\bullet,\circ})^2\big).$$

Put differently, we end up with

$$\frac{1-(1+\delta^{-1})\rho_\theta-(1+\delta)\varepsilon_{\mathrm{drel}}}{1+(1+\delta)C_{\mathrm{drel}}^2}\eta_\bullet^2 \le \eta_\bullet(\mathcal{R}_{\bullet,\circ})^2.$$

Finally, we choose $\delta > 0$ and then $0 < \rho_\theta < 1$ such that

$$\theta \le \frac{1-(1+\delta^{-1})\rho_\theta-(1+\delta)\varepsilon_{\mathrm{drel}}}{1+(1+\delta)C_{\mathrm{drel}}^2} < \frac{1-\varepsilon_{\mathrm{drel}}}{1+C_{\mathrm{drel}}^2} = \theta_{\mathrm{opt}}.$$

This concludes the proof. □

In the following lemma, we show that the estimator is monotone up to some multiplicative constant.

Lemma 2.4.4. *Suppose (E1)–(E2), where the restriction $\rho_{\mathrm{red}} < 1$ is not necessary, and (E4). Then, there exists a constant $C_{\mathrm{mon}} \ge 1$ such that there holds quasi-monotonicity in the sense that*

$$\eta_\circ^2 \le C_{\mathrm{mon}}\eta_\bullet^2 \quad \text{for all } \mathcal{T}_\bullet \in \mathbb{T}, \mathcal{T}_\circ \in \mathtt{refine}(\mathcal{T}_\bullet). \tag{2.4.7}$$

The constant C_{mon} depends only on $\rho_{\mathrm{red}}, \varepsilon_{\mathrm{drel}}$, and C_{drel}.

Proof. We split the estimator and apply stability (E1) in combination with Young's inequality, and reduction (E2). For all $\delta > 0$, we see that

$$\begin{aligned}\eta_\circ^2 = \eta_\circ(\mathcal{T}_\bullet\cap\mathcal{T}_\circ)^2 + \eta_\circ(\mathcal{T}_\circ\setminus\mathcal{T}_\bullet)^2 &\le (1+\delta)\eta_\bullet(\mathcal{T}_\bullet\cap\mathcal{T}_\circ)^2 + \rho_{\mathrm{red}}\eta_\bullet(\mathcal{T}_\bullet\setminus\mathcal{T}_\circ)^2 + (2+\delta^{-1})\varrho_{\bullet,\circ}^2\\ &\le (1+\delta+\rho_{\mathrm{red}})\eta_\bullet^2 + (2+\delta^{-1})\varrho_{\bullet,\circ}^2.\end{aligned}$$

The application of discrete reliability (E4) yields that

$$\begin{aligned}\eta_\circ^2 &\le \big(1+\delta+\rho_{\mathrm{red}}+(2+\delta^{-1})\varepsilon_{\mathrm{drel}}\big)\eta_\bullet^2 + (2+\delta^{-1})C_{\mathrm{drel}}^2\eta_\bullet(\mathcal{R}_{\bullet,\circ})^2\\ &\le \big(1+\delta+\rho_{\mathrm{red}}+(2+\delta^{-1})\varepsilon_{\mathrm{drel}}+(2+\delta^{-1})C_{\mathrm{drel}}^2\big)\eta_\bullet^2,\end{aligned}$$

which concludes the proof. □

The next lemma provides the key ingredient for the proof of optimal convergence of the error estimator.

Lemma 2.4.5. *Suppose the overlay property (T3) and quasi-monotonicity (2.4.7). Let $\ell \in \mathbb{N}_0$ such that $\eta_\ell > 0$ and let $0 < \rho < 1$. Then, for all $s > 0$ with $C_{\mathrm{approx}}(s) < \infty$ there exists a refinement $\mathcal{T}_\circ \in \mathtt{refine}(\mathcal{T}_\ell)$ with*

$$\eta_\circ^2 \le \rho\eta_\ell^2, \tag{2.4.8a}$$

$$\mu(\mathcal{T}_\circ) - \mu(\mathcal{T}_\ell) < C_{\mathrm{over}}\, C_{\mathrm{mon}}^{1/(2s)} C_{\mathrm{approx}}(s)^{1/s}\, \rho^{-1/(2s)}\eta_\ell^{-1/s}. \tag{2.4.8b}$$

Proof. We prove the assertion in two steps.
Step 1: We show a modified (2.4.8) for some $\mathcal{T}_\star \in \mathbb{T}$ instead of a refinement $\mathcal{T}_\circ \in$ `refine`$(\mathcal{T}_\bullet)$, i.e., we prove with $\widetilde{\rho} := \rho / C_{\mathrm{mon}}$ that

$$\eta_\star^2 \le \widetilde{\rho}\eta_\ell^2, \tag{2.4.9a}$$
$$\mu(\mathcal{T}_\star) - \mu(\mathcal{T}_0) < C_{\mathrm{approx}}(s)^{1/s}\,\widetilde{\rho}^{-1/(2s)}\eta_\ell^{-1/s}. \tag{2.4.9b}$$

Let $N \in \mathbb{N}_0$ be minimal such that $C_{\mathrm{approx}}(s)(N+1)^{-s} \le \widetilde{\rho}^{1/2}\eta_\ell$. Note that $N > 0$ by the fact that $\eta_\ell \le C_{\mathrm{mon}}^{1/2}\eta_0 \le C_{\mathrm{mon}}^{1/2}C_{\mathrm{approx}}(s)$ and $0 < \rho < 1$. Hence, minimality of N yields that

$$\widetilde{\rho}^{1/2}\eta_\ell < C_{\mathrm{approx}}(s)N^{-s}.$$

This leads us to

$$N < C_{\mathrm{approx}}(s)\widetilde{\rho}^{-1/(2s)}\eta_\ell^{-1/s}. \tag{2.4.10}$$

Next, we choose $\mathcal{T}_\star \in \mathbb{T}(N)$ with $\eta_\star = \min_{\mathcal{T}_\bullet \in \mathbb{T}(N)} \eta_\bullet$. By definition of $C_{\mathrm{approx}}(s)$ and the choice of N, this gives (2.4.9a). Moreover, (2.4.9b) follows at once from (2.4.10).
Step 2: We consider a common refinement $\mathcal{T}_\circ$ of $\mathcal{T}_\ell$ and $\mathcal{T}_\star$ as in (T3). (2.4.9a) and quasi-monotonicity (2.4.7) show (2.4.8a). Moreover, the overlay property from (T3) and (2.4.9b) prove that

$$\begin{aligned}\mu(\mathcal{T}_\circ) - \mu(\mathcal{T}_\ell) &\le C_{\mathrm{over}}\big(\mu(\mathcal{T}_\star) - \mu(\mathcal{T}_0)\big) + \mu(\mathcal{T}_\ell) - \mu(\mathcal{T}_\ell) = C_{\mathrm{over}}\big(\mu(\mathcal{T}_\star) - \mu(\mathcal{T}_0)\big)\\ &\le C_{\mathrm{over}}\,C_{\mathrm{approx}}(s)^{1/s}\,\widetilde{\rho}^{-1/(2s)}\eta_\ell^{-1/s},\end{aligned}$$

which is just (2.4.8b). ☐

Proof of Theorem 2.3.1 (iii). We split the proof into two steps.
Step 1: We show the first inequality of (2.3.4). Let $N \in \mathbb{N}_0$.

First, we suppose that there exists a minimal ℓ_0 with $\eta_{\ell_0} = 0$. Algorithm 2.2.1 implies that $\mathcal{M}_\ell = \emptyset$ and hence $\mathcal{T}_\ell = \mathcal{T}_{\ell_0}$ for all $\ell \ge \ell_0$. If $N \ge \mu(\mathcal{T}_{\ell_0}) - \mu(\mathcal{T}_0)$, we have that $\min_{\mathcal{T}_\bullet \in \mathbb{T}(N)}((N+1)^s\eta_\bullet) = 0$. If $N < \mu(\mathcal{T}_{\ell_0}) - \mu(\mathcal{T}_0)$, the son estimate (T1) yields that

$$\min_{\mathcal{T}_\bullet \in \mathbb{T}(N)}\big((N+1)^s\eta_\bullet\big) \le (\mu(\mathcal{T}_{\ell_0}) - \mu(\mathcal{T}_0))^s\eta_0 \le (C_{\mathrm{son}}^{\ell_0} - 1)^s\mu(\mathcal{T}_0)^s\eta_0.$$

Now, we suppose that $\eta_\ell > 0$ for all $\ell \in \mathbb{N}_0$. Due to Algorithm 2.2.1 this implies that $\mathcal{M}_\ell \ne \emptyset$ for all $\ell \in \mathbb{N}_0$ and thus $\lim_{\ell\to\infty}\mu(\mathcal{T}_\ell) = \infty$. Hence, there exists a maximal integer $\ell \in \mathbb{N}_0$ with $\mu(\mathcal{T}_\ell) - \mu(\mathcal{T}_0) \le N$, or equivalently $\mathcal{T}_\ell \in \mathbb{T}(N)$. This yields that

$$\min_{\mathcal{T}_\bullet \in \mathbb{T}(N)}\big((N+1)^s\eta_\bullet\big) \le (N+1)^s\eta_\ell. \tag{2.4.11}$$

Since ℓ is maximal, there holds that $N + 1 \le \mu(\mathcal{T}_{\ell+1}) - \mu(\mathcal{T}_0)$. Moreover, the son estimate (T1) implies that $N + 1 \le C_{\mathrm{son}}\mu(\mathcal{T}_\ell) - \mu(\mathcal{T}_0)$. There holds that $C_{\mathrm{son}}\mu(\mathcal{T}_\ell) - \mu(\mathcal{T}_0) \le$

$C(\mu(\mathcal{T}_\ell)-\mu(\mathcal{T}_0)+1)$, where the constant $C>0$ depends only on C_{son} and $\mu(\mathcal{T}_0)$. Together with (2.4.11), we see that

$$\min_{\mathcal{T}_\bullet\in\mathbb{T}(N)}\big((N+1)^s\eta_\bullet\big)\le C^s\big(\mu(\mathcal{T}_\ell)-\mu(\mathcal{T}_0)+1\big)^s\eta_\ell.$$

Step 2: We show the second inequality of (2.3.4). Without loss of generality, we assume that $C_{\mathrm{approx}}(s)<\infty$. If $\eta_{\ell_0}=0$ for some $\ell_0\in\mathbb{N}_0$, then, Algorithm 2.2.1 implies that $\eta_\ell=0$ for all $\ell\ge\ell_0$. Moreover $(\mu(\mathcal{T}_0)-\mu(\mathcal{T}_0)+1)^s\eta_0\le C_{\mathrm{approx}}(s)$ is trivially satisfied. Thus, it is sufficient to consider $0<\ell<\ell_0$ resp. $0<\ell$ if no such ℓ_0 exists. Now, let $j<\ell$. According to Lemma 2.4.4, we may apply Lemma 2.4.5 for the mesh $\mathcal{T}_j$, where we choose $\rho=\rho_\theta$ as in Proposition 2.4.3. In particular, (2.4.6) in combination with (2.4.8a) shows that $\mathcal{R}_{j,\circ}$ satisfies the Dörfler marking $\theta\eta_j^2\le\eta_j(\mathcal{R}_{j,\circ})^2$. Since, $\mathcal{M}_j$ is an essentially minimal set satisfying Dörfler marking, we get that

$$\mu(\mathcal{M}_j)\le C_{\min}\mu(\mathcal{R}_{j,\circ}).$$

Thus, discrete reliability (E3) and (2.4.8b) show that

$$\mu(\mathcal{M}_j)\le C_{\min}C_{\mathrm{ref}}\big(\mu(\mathcal{T}_\circ)-\mu(\mathcal{T}_j)\big)\le C_{\min}C_{\mathrm{ref}}C_{\mathrm{over}}C_{\mathrm{mon}}^{1/(2s)}C_{\mathrm{approx}}(s)^{1/s}\rho_\theta^{-1/(2s)}\eta_j^{-1/s}.$$

Together with the closure estimate (T2), this proves that

$$\begin{aligned}\mu(\mathcal{T}_\ell)-\mu(\mathcal{T}_0)+1&\le 2\big(\mu(\mathcal{T}_\ell)-\mu(\mathcal{T}_0)\big)\le 2C_{\mathrm{clos}}\sum_{j=0}^{\ell-1}\mu(\mathcal{M}_j)\\&\le 2C_{\mathrm{clos}}C_{\min}C_{\mathrm{ref}}C_{\mathrm{over}}C_{\mathrm{mon}}^{1/(2s)}C_{\mathrm{approx}}(s)^{1/s}\rho_\theta^{-1/(2s)}\sum_{j=0}^{\ell-1}\eta_j^{-1/s}.\end{aligned}$$

Finally, linear convergence (2.3.3) and Lemma 2.4.2 show that the term $\sum_{j=0}^{\ell-1}\eta_j^{-1/s}$ can be bounded from above by $C\eta_\ell^{-1/s}$ where $C>0$ depends only on $\rho_{\mathrm{lin}},C_{\mathrm{lin}}$, and s. Therefore, we end up with

$$\big(\mu(\mathcal{T}_\ell)-\mu(\mathcal{T}_0)+1\big)^s\eta_\ell\le 2^sC^sC_{\mathrm{clos}}^sC_{\min}^sC_{\mathrm{ref}}^sC_{\mathrm{over}}^sC_{\mathrm{mon}}^{1/2}C_{\mathrm{approx}}(s)\rho_\theta^{-1/2},$$

which concludes the proof. □

2.5 Proof of Corollary 2.3.4

Before we come to the proof of Corollary 2.3.4, note that Dörfler marking (2.3.6) for the equivalent estimator $\widetilde{\eta}$ implies by local equivalence (2.3.5) Dörfler marking for η, i.e., there holds that

$$\widetilde{\theta}\eta_\ell^2\le\widetilde{\theta}C_{\mathrm{eq}}\widetilde{\eta}_\ell^2\le C_{\mathrm{eq}}\widetilde{\eta}_\ell(\mathcal{M}_\ell)^2\le C_{\mathrm{eq}}^2\eta_\ell(\mathcal{M}_\ell)^2\quad\text{for all }\ell\in\mathbb{N}_0.\qquad(2.5.1)$$

Proof of Corollary 2.3.4 (i). (2.5.1) shows that $C_{\mathrm{eq}}^{-2}\widetilde{\theta}\eta_\ell^2\le\eta_\ell(\mathcal{M}_\ell)^2$ for all $\ell\in\mathbb{N}_0$. In particular, η_ℓ can be seen as the output of Algorithm 2.2.1 with $\theta=C_{\mathrm{eq}}^{-2}\widetilde{\theta}$ and $C_{\min}=\infty$. Therefore, Theorem 2.3.1 (i) implies the convergence $\lim_{\ell\to\infty}\eta_\ell=0$. The local equivalence (2.3.5) concludes the proof. □

Proof of Corollary 2.3.4 (ii). (2.5.1) shows that $C_{\rm eq}^{-2}\widetilde{\theta}\eta_\ell^2 \le \eta_\ell(\mathcal{M}_\ell)^2$ for all $\ell \in \mathbb{N}_0$. In particular, η_ℓ can be seen as the output of Algorithm 2.2.1 with $\theta = C_{\rm eq}^{-2}\widetilde{\theta}$ and $C_{\rm min} = \infty$. Therefore, Theorem 2.3.1 (ii) implies linear convergence $\eta_{\ell+j}^2 \le C_{\rm lin}\rho_{\rm lin}^j\eta_\ell^2$. The local equivalence (2.3.5) proves that

$$\widetilde{\eta}_{\ell+j}^2 \le C_{\rm eq}\eta_{\ell+j}^2 \le C_{\rm eq}C_{\rm lin}\rho_{\rm lin}^j\eta_\ell^2 \le C_{\rm eq}^2C_{\rm lin}\rho_{\rm lin}^j\widetilde{\eta}_\ell^2,$$

which concludes the proof. ☐

Proof of Corollary 2.3.4 (iii). The first inequality of (2.3.9) is satisfied due to local equivalence (2.3.5) and Theorem 2.3.1 (iii), since the lower bound in (2.3.4) requires only (T1) to hold.

To see the second inequality, we essentially copy Step 2 from the proof of Theorem 2.3.1 (iii). Without loss of generality, we assume that $C_{\rm approx}(s) < \infty$. If $\widetilde{\eta}_{\ell_0} = 0$ for some $\ell_0 \in \mathbb{N}_0$, then, Algorithm 2.3.3 implies that $\widetilde{\eta}_\ell = 0$ for all $\ell \ge \ell_0$. Moreover $(\mu(\mathcal{T}_0) - \mu(\mathcal{T}_0) + 1)^s\widetilde{\eta}_0 \le C_{\rm approx}(s)$ is trivially satisfied. Thus, it is sufficient to consider $0 < \ell < \ell_0$ resp. $0 < \ell$ if no such ℓ_0 exists. Now, let $j < \ell$. According to Lemma 2.4.4, we may apply Lemma 2.4.5 for the mesh $\mathcal{T}_j$, where we choose $\rho = \rho_\theta$ with $0 < \theta := C_{\rm eq}^2\widetilde{\theta} < \theta_{\rm opt}$ as in Proposition 2.4.3. In particular, (2.4.6) in combination with (2.4.8a) shows that $\mathcal{R}_{j,\circ}$ satisfies Dörfler marking $\theta\eta_j^2 \le \eta_j(\mathcal{R}_{j,\circ})^2$. Therefore, one sees the Dörfler marking $\widetilde{\theta}\,\widetilde{\eta}_j^2 \le \widetilde{\eta}_j(\mathcal{R}_{j,\circ})^2$ as in (2.5.1). Hence, minimality of $\mathcal{M}_j$ yields that $\mu(\mathcal{M}_j) \le \widetilde{C}_{\rm min}\,\mu(\mathcal{R}_{j,\circ})$. If one sets $C_{\rm min} := \widetilde{C}_{\rm min}$, the rest of the proof can be copied exactly from the proof of Theorem 2.3.1 (iii). ☐

3 Splines

3.1 Introduction

In this chapter, we introduce the so-called *splines*, which are piecewise polynomials on a given mesh with certain smoothness properties across the boundaries of the mesh elements. We will use these functions in the following chapters to approximate the solution of a PDE (Chapter 4) resp. the solution of an integral equation (Chapter 5). In order to do so, it is crucial to have a suitable basis at hand. Therefore, we introduce *B-splines* on $\mathbb{R}$ in Section 3.2. Next, in Section 3.3, we define splines on the d-dimensional unit cube. To this end, we consider first the one-dimensional case. A tensor-product approach provides a definition for the higher-dimensional case. Moreover, we consider a corresponding well-known quasi-interpolation projection. Since the tensor mesh structure has to be preserved in each refinement step, the standard splines are not suited for adaptive refinement if the dimension d is larger than one. However, to allow for adaptive refinement, several extensions of the standard model have recently emerged, e.g., *analysis-suitable T-splines* [SLSH12, BdVBSV13], *hierarchical splines* [Kra98, VGJS11], or *LR-splines* [DLP13, JKD14]. In this thesis, we focus on hierarchical splines which are defined in Section 3.4. We define two well-known bases of the space of hierarchical splines, namely the *hierarchical B-splines* and the *truncated hierarchical B-splines* (THB-splines). Under additional assumptions on the underlying mesh, and with the help of properly chosen dual basis functions, we constructed in a recent own work [GHP17] a locally L^2-stable projection onto hierarchical splines. This operator is finally presented in Section 3.4.5.

3.2 B-splines on $\mathbb{R}$

Throughout this section, let

$$\widehat{\mathcal{K}}_\bullet = (t_{\bullet,j})_{j\in\mathbb{Z}} \tag{3.2.1}$$

be a sequence of *knots* $t_{\bullet,j} \in \mathbb{R}$ such that $t_{\bullet,j-1} \le t_{\bullet,j}$ for $j \in \mathbb{Z}$ and $\lim_{j\to\pm\infty} t_{\bullet,j} = \pm\infty$. We introduce the *multiplicity* of an arbitrary real number $t \in \mathbb{R}$ as

$$\#_\bullet t := \#\{j \in \mathbb{Z} \,:\, t = t_{\bullet,j}\} \in \mathbb{N}_0. \tag{3.2.2}$$

Let

$$\widehat{\mathcal{N}}_\bullet := \{t_{\bullet,j} \,:\, j \in \mathbb{Z}\} = \{\widehat{z}_{\bullet,j} \,:\, j \in \mathbb{Z}\} \tag{3.2.3}$$

denote the corresponding set of *nodes* with $\widehat{z}_{\bullet,j-1} < \widehat{z}_{\bullet,j}$. Moreover, let

$$\widehat{\mathcal{T}}_\bullet := \{\widehat{T}_{\bullet,j} \,:\, j \in \mathbb{Z}\} \quad \text{with } \widehat{T}_{\bullet,j} := [\widehat{z}_{\bullet,j-1}, \widehat{z}_{\bullet,j}] \tag{3.2.4}$$

be the induced *mesh*. With the convention $(\cdot)/0 := 0$, we recursively define the j-th *B-spline* of degree $p \in \mathbb{N}_0$ for $j \in \mathbb{Z}$ as

$$\begin{aligned} \widehat{B}_{\bullet,j,0} &:= \chi_{[t_{\bullet,j-1}, t_{\bullet,j})}, \\ \widehat{B}_{\bullet,j,p} &:= \frac{t - t_{\bullet,j-1}}{t_{\bullet,j-1+p} - t_{\bullet,j-1}} \widehat{B}_{\bullet,j,p-1} + \frac{t_{\bullet,j+p} - t}{t_{\bullet,j+p} - t_{\bullet,j}} \widehat{B}_{\bullet,j+1,p-1} \quad \text{for } p \in \mathbb{N}, \end{aligned} \tag{3.2.5}$$

The following lemma collects essentially all properties of B-splines that will be needed throughout this thesis. All the results are well-known in the literature; see, e.g. [dB86]. A more detailed presentation of B-splines can be found in [dB86, BdVBSV14, Sch07, dB01].

Lemma 3.2.1. *Let $p \in \mathbb{N}_0$. Then, there hold the following points:*

(i) *For arbitrary finite intervals $I = [a, b)$, the set $\{\widehat{B}_{\bullet,j,p}|_I \,:\, j \in \mathbb{Z} \wedge \widehat{B}_{\bullet,j,p}|_I \neq 0\}$ is a basis for the space of all right-continuous $\widehat{\mathcal{T}}_\bullet$-piecewise polynomials of degree p on I which are, at each node $\widehat{z} \in \widehat{\mathcal{N}}_\bullet$, $p - \#_\bullet \widehat{z}$ times continuously differentiable if $p - \#_\bullet \widehat{z} \geq 0$.*

(ii) *For $j \in \mathbb{Z}$, the B-spline $\widehat{B}_{\bullet,j,p}$ vanishes outside the interval $[t_{\bullet,j-1}, t_{\bullet,j+p})$. It is positive on the open interval $(t_{\bullet,j-1}, t_{\bullet,j+p})$.*

(iii) *For $j \in \mathbb{Z}$, the B-spline $\widehat{B}_{\bullet,j,p}$ is completely determined by the $p+2$ knots $t_{\bullet,j-1}, \dots, t_{\bullet,j+p}$. Therefore, we sometimes use the notation*

$$\widehat{B}(\,\cdot\,| t_{\bullet,j-1}, \dots, t_{\bullet,j+p}) := \widehat{B}_{\bullet,j,p}. \tag{3.2.6}$$

(iv) *The B-splines of degree p form a (locally finite) partition of unity, i.e.,*

$$\sum_{j \in \mathbb{Z}} \widehat{B}_{\bullet,j,p} = 1. \tag{3.2.7}$$

(v) *For $j \in \mathbb{Z}$, $s \in \mathbb{R}$, and $c > 0$, we have with the transformed knots $\widehat{\mathcal{K}}_\star = (t_{\star,j})_{j \in \mathbb{Z}} := (s + c\, t_{\bullet,j})_{j \in \mathbb{Z}}$*

$$\widehat{B}_{\star,j,p} = \widehat{B}_{\bullet,j,p}\big((\cdot)/c - s\big). \tag{3.2.8}$$

(vi) *For $j \in \mathbb{Z}$ with $t_{\bullet,j-1} < t_{\bullet,j} = \dots = t_{\bullet,j+p} < t_{\bullet,j+p+1}$, it holds that*

$$\widehat{B}_{\bullet,j,p}(t_{\bullet,j}-) = 1 \quad \textit{and} \quad \widehat{B}_{\bullet,j+1,p}(t_{\bullet,j}) = 1. \tag{3.2.9}$$

(vii) *For $p \geq 1$ and $j \in \mathbb{Z}$, the right derivative satisfies that*

$$\widehat{B}'^{r}_{\bullet,j,p} = \frac{p}{t_{\bullet,j+p-1} - t_{\bullet,j-1}} \widehat{B}_{\bullet,j,p-1} - \frac{p}{t_{\bullet,j+p} - t_{\bullet,j}} \widehat{B}_{\bullet,j+1,p-1}, \tag{3.2.10}$$

where we suppose the convention $p/0 := 0$.

(viii) *Let $t' \in (t_{\ell-1}, t_\ell]$ for some $\ell \in \mathbb{Z}$ and let $\widehat{\mathcal{K}}_\circ$ be the refinement of $\widehat{\mathcal{K}}_\bullet$, obtained by adding t'. Then, for all coefficients $(a_{\bullet,j})_{j\in\mathbb{Z}}$, there exist coefficients $(a_{\circ,j})_{j\in\mathbb{Z}}$ such that*

$$\sum_{j\in\mathbb{Z}} a_{\bullet,j}\widehat{B}_{\bullet,j,p} = \sum_{j\in\mathbb{Z}} a_{\circ,j}\widehat{B}_{\circ,j,p}. \tag{3.2.11}$$

With the multiplicity $\#_\circ t'$ of t' in the knots $\widehat{\mathcal{K}}_\circ$, the new coefficients can be chosen as convex combinations of the old coefficients

$$a_{\circ,j} = \begin{cases} a_{\bullet,j} & \text{if } j \le \ell - p + \#_\circ t' - 1,\\ \frac{t_{\bullet,j-1+p}-t'}{t_{\bullet,j-1+p}-t_{\bullet,j-1}} a_{\bullet,j-1} + \frac{t'-t_{\bullet,j-1}}{t_{\bullet,j-1+p}-t_{\bullet,j-1}} a_{\bullet,j} & \text{if } \ell - p + \#_\circ t' \le j \le \ell,\\ a_{\bullet,j-1} & \text{if } \ell + 1 \le j. \end{cases} \tag{3.2.12}$$

If one assumes $\#_\bullet t_j \le p+1$ for all $j \in \mathbb{Z}$, these coefficients are unique. Note that the three cases are equivalent to $t_{\bullet,j-1+p} \le t'$, $t_{\bullet,j-1} < t' < t_{\bullet,j-1+p}$, resp. $t' \le t_{\bullet,j-1}$.

Proof. The proof of (i) is found, e.g., in [dB86, Theorem 6], and (ii)–(iii) are proved in [dB86, Section 2]. (iv) is proved in [dB86, Section 4]. (v) follows elementarily from the definition (3.2.5). The same holds for (vi); see, e.g., [Sch16, Lemma 2.1]. Finally, (vii)–(viii) are found in [dB86, Sections 10–11]. □

3.3 Standard splines

In this section, we introduce splines, first, in one dimension, and then also for higher dimensions via a tensor-product approach. Moreover, we consider a standard quasi-interpolation projection onto these functions. For more details on splines, see, e.g., [dB86, BdVBSV14, Sch07, dB01].

3.3.1 One-dimensional case

Let $p \in \mathbb{N}_0$ be a fixed polynomial degree. Let

$$\widehat{\mathcal{K}}_\bullet = (t_{\bullet,j})_{j=0}^{N_\bullet+p} \tag{3.3.1}$$

be a vector with $0 \le t_{\bullet,j-1} \le t_{\bullet,j} \le 1$ for $j \in \{1, \dots, N_\bullet+p\}$, where $t_{\bullet,0} = 0$ and $t_{\bullet,N_\bullet+p} = 1$. Again, we introduce for arbitrary $t \in [0,1]$ its *multiplicity* as

$$\#_\bullet t := \#\big\{j \in \{1, \dots, N_\bullet + p\} \,:\, t = t_{\bullet,j}\big\} \in \mathbb{N}_0. \tag{3.3.2}$$

We suppose that $\#_\bullet t_{\bullet,j} \le p+1$ for all $j \in \{1, \dots, N_\bullet + p\}$, where

$$\#t_{\bullet,0} = p+1 \quad \text{and} \quad \#t_{\bullet,N_\bullet+p} = p+1. \tag{3.3.3}$$

We call such a vector *p-open knot vector on* $[0,1]$. Let

$$\widehat{\mathcal{N}}_\bullet := \big\{t_{\bullet,j} \,:\, j \in \{0, \dots, N_\bullet + p\}\big\} = \big\{\widehat{z}_{\bullet,j} \,:\, j \in \{0, \dots, n_\bullet\}\big\} \tag{3.3.4}$$

denote the corresponding set of *nodes* with $\widehat{z}_{\bullet,j-1} < \widehat{z}_{\bullet,j}$ for $j \in \{0,\dots,n_\bullet\}$. Moreover, let

$$\widehat{\mathcal{T}}_\bullet := \big\{\widehat{T}_{\bullet,j} \,:\, j \in \{1,\dots,n_\bullet\}\big\} \quad \text{with } \widehat{T}_{\bullet,j} := [\widehat{z}_{\bullet,j-1}, \widehat{z}_{\bullet,j}] \tag{3.3.5}$$

be the induced *mesh* on $[0,1]$. For $\widehat{\omega} \subseteq [0,1]$, we introduce the *patches of order* $q \in \mathbb{N}_0$ inductively by

$$\pi_\bullet^0(\widehat{\omega}) := \widehat{\omega}, \quad \pi_\bullet^q(\widehat{\omega}) := \bigcup \big\{\widehat{T} \in \widehat{\mathcal{T}}_\bullet \,:\, \widehat{T} \cap \pi_\bullet^{q-1}(\widehat{\omega}) \neq \emptyset\big\}. \tag{3.3.6}$$

The corresponding set of elements is defined as

$$\Pi_\bullet^q(\widehat{\omega}) := \big\{\widehat{T} \in \widehat{\mathcal{T}}_\bullet \,:\, \widehat{T} \subseteq \pi_\bullet^q(\widehat{\omega})\big\}, \quad \text{i.e.,} \quad \pi_\bullet^q(\widehat{\omega}) = \bigcup \Pi_\bullet^q(\widehat{\omega}). \tag{3.3.7}$$

To abbreviate notation, we set $\pi_\bullet(\widehat{\omega}) := \pi_\bullet^1(\widehat{\omega})$ and $\Pi_\bullet(\widehat{\omega}) := \Pi_\bullet^1(\widehat{\omega})$.

For a p-open knot vector $\widehat{\mathcal{K}}_\bullet$, we define the corresponding *splines* $\widehat{\mathcal{S}}^p(\widehat{\mathcal{K}}_\bullet)$ *of degree* p as the set of all (right-continuous) $\widehat{\mathcal{T}}_\bullet$-piecewise polynomials of degree p on $[0,1)$ which are, at each node $\widehat{z} \in \widehat{\mathcal{N}}_\bullet$, $p - \#_\bullet\widehat{z}$ times continuously differentiable if $p - \#_\bullet\widehat{z} \geq 0$. In particular, if all nodes have the maximal multiplicity $p+1$, the corresponding spline space coincides with the space of all right-continuous $\widehat{\mathcal{T}}_\bullet$-piecewise polynomials of degree p.

In order to obtain a basis for $\widehat{\mathcal{S}}^p(\widehat{\mathcal{K}}_\bullet)$, we first extend the knot vector $\widehat{\mathcal{K}}_\bullet$ arbitrarily to a knot sequence $(t_j)_{j\in\mathbb{Z}}$ as in Section 3.2. For simplicity, we use the notation $\widehat{\mathcal{K}}_\bullet$ for both, the knot vector as well as the extended knot sequence. With this, we may apply Lemma 3.2.1 (i)–(ii) to see that

$$\widehat{\mathcal{S}}^p(\widehat{\mathcal{K}}_\bullet) = \operatorname{span}(\widehat{\mathcal{B}}_\bullet) \quad \text{with} \quad \widehat{\mathcal{B}}_\bullet := \big\{\widehat{B}_{\bullet,j,p}|_{[0,1)} \,:\, j \in \{1,\dots,N_\bullet\}\big\}, \tag{3.3.8}$$

where the set of B-splines $\widehat{\mathcal{B}}_\bullet$ even forms a basis; see Figure 3.1 for an illustration of some B-splines. Due to Lemma 3.2.1 (iii), the support of $\widehat{B}_{\bullet,j,p}$ is an interval in $[0,1]$ and the union of at most $p+1$ elements in $\widehat{\mathcal{T}}_\bullet$. It is well-known that the functions in $\widehat{\mathcal{B}}_\bullet$ are even *locally linearly independent*, i.e, for any open set $O \subseteq [0,1]^d$, the restricted B-splines $\big\{\widehat{B}_{\bullet,j,p}|_O \,:\, j \in \{1,\dots,N_\bullet\} \wedge \operatorname{supp}(\widehat{B}_{\bullet,j,p}) \cap O \neq \emptyset\big\}$ are linearly independent. This follows easily from Lemma 3.2.1 (i): Suppose the assertion is false, then there exists a non-trivial linear combination of 0. Let $\widehat{B}_{\bullet,j,p}$ have a corresponding non-zero coefficient, and let $[a,b) \subset \operatorname{supp}(\widehat{B}_{\bullet,j,p}) \cap O$. If we restrict the non-trivial linear combination to $[a,b)$, we get a contradiction to Lemma 3.2.1 (i).

If $\widehat{\mathcal{K}}_\circ$ is a *finer* p-open knot vector, which means that $\widehat{\mathcal{K}}_\bullet$ is a subsequence of $\widehat{\mathcal{K}}_\circ$, the corresponding spline spaces are nested

$$\widehat{\mathcal{S}}^p(\widehat{\mathcal{K}}_\bullet) \subseteq \widehat{\mathcal{S}}^p(\widehat{\mathcal{K}}_\circ). \tag{3.3.9}$$

3.3.2 Higher-dimensional case

For $d \geq 1$, let $(p_1,\dots,p_d)$ be a vector of fixed polynomial degrees in $\mathbb{N}_0$. Let

$$\widehat{\mathcal{K}}_\bullet = (\widehat{\mathcal{K}}_{1(\bullet)},\dots,\widehat{\mathcal{K}}_{d(\bullet)}) \tag{3.3.10}$$

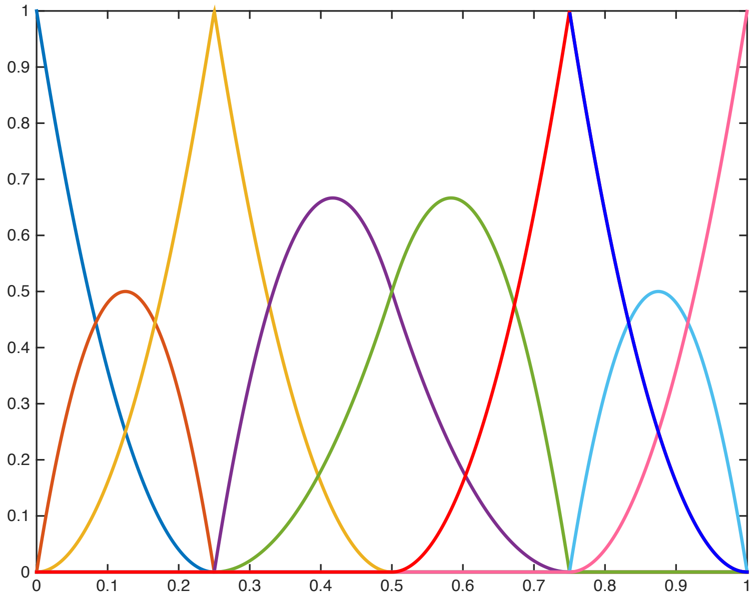


Figure 3.1: The B-splines $\widehat{\mathcal{B}}_\bullet$ for the polynomial degree $p = 2$ and the p-open knot vector $\widehat{\mathcal{K}}_\bullet = (0, 0, 0, 0.25, 0.25, 0.5, 0.75, 0.75, 0.75, 1, 1, 1)$ are depicted.

be a d-dimensional vector, where the i-th entry $\widehat{\mathcal{K}}_{i(\bullet)}$ is a p_i-open knot vector as in the previous Section 3.3.1 for all $i \in \{1, \dots, d\}$. In particular, this induces the *tensor mesh*

$$\widehat{\mathcal{T}}_\bullet := \big\{\widehat{T}_1 \times \cdots \times \widehat{T}_d \,:\, \widehat{T}_i \in \widehat{\mathcal{T}}_{i(\bullet)} \text{ for } i \in \{1, \dots, d\}\big\} \tag{3.3.11}$$

We define the corresponding *tensor-product splines* as

$$\begin{aligned} \widehat{\mathcal{S}}^{(p_1,\dots,p_d)}(\widehat{\mathcal{K}}_\bullet) &:= \widehat{\mathcal{S}}_{1(\bullet)} \otimes \cdots \otimes \widehat{\mathcal{S}}_{d(\bullet)} \\ &:= \big\{\widehat{S}_1 \otimes \cdots \otimes \widehat{S}_d \,:\, \widehat{S}_i \in \widehat{\mathcal{S}}^{p_i}(\widehat{\mathcal{K}}_{i(\bullet)}) \text{ for } i \in \{1, \dots, d\}\big\}, \end{aligned} \tag{3.3.12}$$

where we define the *tensor-product* of one-dimensional splines as

$$(\widehat{S}_1 \otimes \cdots \otimes \widehat{S}_d)(t) := \prod_{i=1}^{d} \widehat{S}_i(t_i) \quad \text{for all } t = (t_1, \dots, t_d) \in [0, 1)^d. \tag{3.3.13}$$

According to Section 3.3.1, the functions in $\widehat{\mathcal{S}}^{(p_1,\dots,p_d)}(\widehat{\mathcal{K}}_\bullet)$ are $\widehat{\mathcal{T}}_\bullet$-piecewise tensor-product polynomials with certain smoothness properties across the boundaries of the mesh elements. In particular, if the one-dimensional knots $\widehat{\mathcal{K}}_{i(\bullet)}$ have the maximal multiplicity $p_i + 1$ for all $i \in \{1, \dots, d\}$, then the corresponding tensor-product spline space just coincides with the space of all $\widehat{\mathcal{T}}_\bullet$-piecewise tensor-product polynomials of degree $(p_1, \dots, p_d)$, which are right-continuous in each component.

With (3.3.8), we see that

$$\widehat{\mathcal{S}}^{(p_1,\dots,p_d)}(\widehat{\mathcal{K}}_\bullet) = \operatorname{span}(\widehat{\mathcal{B}}_\bullet) \quad \text{with} \quad \widehat{\mathcal{B}}_\bullet := \big\{\widehat{\beta}_1 \otimes \cdots \otimes \widehat{\beta}_d : \widehat{\beta}_i \in \widehat{\mathcal{B}}_{i(\bullet)} \text{ for } i \in \{1,\dots,d\}\big\}, \tag{3.3.14}$$

where the set of tensor-product B-splines $\widehat{\mathcal{B}}_\bullet$ even forms a basis. Due to Lemma 3.2.1 (iii), their support is a d-dimensional rectangle which is the union of at most $\prod_{i=1}^d (p_i+1)$ elements in $\widehat{\mathcal{T}}_\bullet$. It is well-known that these functions are even locally linearly independent, i.e, for any open set $O \subseteq [0,1]^d$, the restricted B-splines $\big\{\widehat{\beta}|_O : \widehat{\beta} \in \widehat{\mathcal{B}}_\bullet \text{ with } \operatorname{supp}(\widehat{\beta}) \cap O \neq \emptyset\big\}$ are linearly independent. This follows as for the one-dimensional case.

If $\widehat{\mathcal{K}}_\circ$ is a *finer* vector of p_i-open knot vectors, which means that $\widehat{\mathcal{K}}_{i(\bullet)}$ is a subsequence of $\widehat{\mathcal{K}}_{i(\circ)}$ for all $i \in \{1,\dots,d\}$, the corresponding tensor-product spline spaces are nested

$$\widehat{\mathcal{S}}^{(p_1,\dots,p_d)}(\widehat{\mathcal{K}}_\bullet) \subseteq \widehat{\mathcal{S}}^{(p_1,\dots,p_d)}(\widehat{\mathcal{K}}_\circ). \tag{3.3.15}$$

3.3.3 Quasi-interpolation projection

In this section, we introduce a well-known quasi-interpolation projection onto the space of splines as in [BdVBSV14, Sections 2.1.5 and 2.2.2]. We only consider the one-dimensional case $d = 1$, since we will not need such an operator for standard tensor-product splines. However, a similar operator can also be defined for $d > 1$. Moreover, we will introduce a different quasi-interpolation operator for hierarchical splines in Section 3.4.5. Let $p \in \mathbb{N}_0$ and $\widehat{\mathcal{K}}_\bullet$ be a p-open knot vector on $[0,1]$. In [Sch07, Section 4.6], it is shown that for all $j \in \{1,\dots,N_\bullet\}$ there exists a dual basis function $\widehat{B}^*_{\bullet,j,p} \in L^2(0,1)$ with $\operatorname{supp}(\widehat{B}^*_{\bullet,j,p}) \subseteq \operatorname{supp}(\widehat{B}_{\bullet,j,p})$ such that

$$\int_0^1 \widehat{B}^*_{\bullet,j,p}(t)\widehat{B}_{\bullet,j',p}(t)\,dt = \delta_{jj'} = \begin{cases} 1 & \text{if } j = j', \\ 0 & \text{else,} \end{cases} \tag{3.3.16}$$

and

$$\|\widehat{B}^*_{\bullet,j,p}\|_{L^2(0,1)} \le (2p+3)9^p |\operatorname{supp}(\widehat{B}_{\bullet,j,p})|^{-1/2}. \tag{3.3.17}$$

Note that there holds $\operatorname{supp}(\widehat{B}_{\bullet,j,p}) = [t_{\bullet,j-1}, t_{\bullet,j+p}] \subseteq [0,1]$ according to Lemma 3.2.1 (ii). For higher dimensions, such dual basis functions can be defined as tensor-product. We define the operator

$$\widehat{I}_\bullet : L^2(0,1) \to \widehat{\mathcal{S}}^p(\widehat{\mathcal{K}}_\bullet), \quad \widehat{v} \mapsto \sum_{j=1}^{N_\bullet} \int_0^1 \widehat{B}^*_{\bullet,j,p}(t)\,\widehat{v}(t)\,dt\,\widehat{B}_{\bullet,j,p}|_{[0,1)}. \tag{3.3.18}$$

With the properties of the dual basis functions, one easily proves that $\widehat{I}_\bullet$ is a local L^2-stable projection. Formally, this is stated in the following proposition which is taken from [BdVBSV14, Proposition 2.2]. The proof is included for completeness.

Proposition 3.3.1. *The operator $\widehat{I}_\bullet$ from (3.3.18) satisfies the following two properties:*

(i) *For* $\widehat{T} \in \widehat{\mathcal{T}}_\bullet$ *and* $\widehat{v} \in L^2(0,1)$*, the inclusion* $\widehat{v}|_{\pi_\bullet^p(\widehat{T})} \in \widehat{\mathcal{S}}^p(\widehat{\mathcal{K}}_\bullet)|_{\pi_\bullet^p(\widehat{T})} := \{\widehat{S}|_{\pi_\bullet^p(\widehat{T})} : \widehat{S} \in \widehat{\mathcal{S}}^p(\widehat{\mathcal{K}}_\bullet)\}$ *implies that* $\widehat{v}|_{\widehat{T}} = (\widehat{I}_\bullet \widehat{v})|_{\widehat{T}}$.

(ii) *Let* $\widehat{C}_{\rm locuni} > 0$ *be an upper bound for the quotient of lengths of neighboring elements, i.e.,*

$$\max\Big\{\frac{|\widehat{T}|}{|\widehat{T}'|} : \widehat{T}, \widehat{T}' \in \widehat{\mathcal{T}}_\bullet \text{ with } \widehat{T} \cap \widehat{T}' \neq \emptyset\Big\} \leq \widehat{C}_{\rm locuni}. \tag{3.3.19}$$

There exists a constant $C > 0$ *such that for all* $\widehat{v} \in L^2(0,1)$ *and all* $\widehat{T} \in \widehat{\mathcal{T}}_\bullet$*, there holds that*

$$\|\widehat{I}_\bullet \widehat{v}\|_{L^2(\widehat{T})} \leq C \|\widehat{v}\|_{L^2(\pi_\bullet^p(\widehat{T}))}, \tag{3.3.20}$$

where C *depends only on* p *and* $\widehat{C}_{\rm locuni}$.

Proof. We prove the assertion in two steps.
Step 1: Lemma 3.2.1 (ii) shows that all B-splines $\widehat{B}_{\bullet,j,p}$ which are non-zero on $\widehat{T}$, have support in $\pi_\bullet^p(\widehat{T})$. Let $\widehat{S} \in \widehat{\mathcal{S}}^p(\widehat{\mathcal{K}}_\bullet)$ with $\widehat{v}|_{\pi_\bullet^p((\widehat{T}))} = \widehat{S}|_{\pi_\bullet^p((\widehat{T}))}$. Due to (3.3.8), there exist coefficients $(a_j)_{j=1}^{N_\bullet}$ with $\widehat{S} = \sum_{j=1}^{N_\bullet} a_j \widehat{B}_{\bullet,j,p}|_{[0,1)}$. Altogether, we see with duality (3.3.16) that

$$\begin{aligned}
(\widehat{I}_\bullet \widehat{v})|_{\widehat{T}} &= \sum_{\substack{j=1\\ \mathrm{supp}(\widehat{B}_{\bullet,j,p}) \subseteq \pi_\bullet^p(\widehat{T})}}^{N_\bullet} \int_{\mathrm{supp}(\widehat{B}_{\bullet,j,p})} \widehat{B}^*_{\bullet,j,p}(t)\, \widehat{v}(t)\, dt\, \widehat{B}_{\bullet,j,p}|_{\widehat{T}} \\
&= \sum_{\substack{j=1\\ \mathrm{supp}(\widehat{B}_{\bullet,j,p}) \subseteq \pi_\bullet^p(\widehat{T})}}^{N_\bullet} a_j\, \widehat{B}_{\bullet,j,p}|_{\widehat{T}} = \widehat{S}|_{\widehat{T}}.
\end{aligned}$$

Step 2: To see (ii), we apply two times the Cauchy–Schwarz inequality

$$\begin{aligned}
\|\widehat{I}_\bullet \widehat{v}\|_{L^2(\widehat{T})} &= \Big\| \sum_{\substack{j=1\\ \mathrm{supp}(\widehat{B}_{\bullet,j,p}) \subseteq \pi_\bullet^p(\widehat{T})}}^{N_\bullet} \Big(\int_{\mathrm{supp}(\widehat{B}_{\bullet,j,p})} \widehat{B}^*_{\bullet,j,p}(t) \widehat{v}(t)\, dt\Big) \widehat{B}_{\bullet,j,p} \Big\|_{L^2(\widehat{T})} \\
&\leq \sum_{\substack{j=1\\ \mathrm{supp}(\widehat{B}_{\bullet,j,p}) \subseteq \pi_\bullet^p(\widehat{T})}}^{N_\bullet} \|\widehat{B}^*_{\bullet,j,p}\|_{L^2(\mathrm{supp}(\widehat{B}_{\bullet,j,p}))} \|\widehat{v}\|_{L^2(\mathrm{supp}(\widehat{B}_{\bullet,j,p}))} \|\widehat{B}_{\bullet,j,p}\|_{L^2(\widehat{T})}
\end{aligned}$$

Next, we use the fact that $0 \leq \widehat{B}_{\bullet,j,p} \leq 1$ which follows from Lemma 3.2.1 (ii) and (iv). With (3.3.17), this gives that

$$\|\widehat{I}_\bullet \widehat{v}\|_{L^2(\widehat{T})} \lesssim \sum_{\substack{j=1\\ \mathrm{supp}(\widehat{B}_{\bullet,j,p}) \subseteq \pi_\bullet^p(\widehat{T})}}^{N_\bullet} |\mathrm{supp}(\widehat{B}_{\bullet,j,p})|^{-1/2} \|\widehat{v}\|_{L^2(\mathrm{supp}(\widehat{B}_{\bullet,j,p}))} |\widehat{T}|^{1/2}.$$

Finally, we apply (3.3.19) and Lemma 3.2.1 (ii) to see that

$$\|\widehat{I}_\bullet \widehat{v}\|_{L^2(\widehat{T})} \lesssim \sum_{\substack{j=1 \\ \mathrm{supp}(\widehat{B}_{\bullet,j,p}) \subseteq \pi_\bullet^p(\widehat{T})}}^{N_\bullet} \|\widehat{v}\|_{L^2(\pi_\bullet^p(\widehat{T}))} \lesssim \|\widehat{v}\|_{L^2(\pi_\bullet^p(\widehat{T}))}.$$

This concludes the proof. □

3.4 Hierarchical splines

We use the notation from Section 3.3 to define hierarchical meshes and splines. For a more detailed introduction, we refer to, e.g., [VGJS11, GJS12, BG16b, SM16].

3.4.1 Nested tensor-product splines

For $d \geq 1$, let $(p_1, \ldots, p_d)$ be a vector of fixed polynomial degrees in $\mathbb{N}$, and set

$$p_{\max} := \max_{i=1,\ldots,d} p_i. \tag{3.4.1}$$

Let

$$\widehat{\mathcal{K}}_0 = (\widehat{\mathcal{K}}_{1(0)}, \ldots, \widehat{\mathcal{K}}_{d(0)}) \tag{3.4.2}$$

be a fixed initial d-dimensional vector of p_i-open knot vectors as in Section 3.3.2, where we additionally suppose that all interior knots $t_{i(0),j} \in (0,1)$ satisfy that

$$\#_{i(0)} t_{i(0),j} \leq p_i \quad \text{for all } i \in \{1, \ldots, d\}, j \in \{2 + p_i, \ldots, N_{i(0)} - 1\}. \tag{3.4.3}$$

Note that this ensures at least continuity of the corresponding spline functions. We set $\widehat{\mathcal{K}}_{\mathrm{uni}(0)} := \widehat{\mathcal{K}}_0$ and recursively define $\widehat{\mathcal{K}}_{\mathrm{uni}(k+1)}$ for $k \in \mathbb{N}_0$ as the uniform h-refinement of $\widehat{\mathcal{K}}_{\mathrm{uni}(k)}$, i.e., it is obtained by inserting the knot $(\widehat{z}_{i(\mathrm{uni}(k)),j-1} + \widehat{z}_{i(\mathrm{uni}(k)),j})/2$ with multiplicity one of each one-dimensional element $[\widehat{z}_{i(\mathrm{uni}(k)),j-1}, \widehat{z}_{i(\mathrm{uni}(k)),j}] \in \widehat{\mathcal{T}}_{i(\mathrm{uni}(k))}$ to the knots $\widehat{\mathcal{K}}_{i(\mathrm{uni}(k))}$, where $i \in \{1, \ldots, d\}$ and $j \in \{1, \ldots, n_{i(\mathrm{uni}(k))}\}$. This yields a nested sequence of tensor-product spline spaces

$$\widehat{\mathcal{S}}^{(p_1,\ldots,p_d)}(\widehat{\mathcal{K}}_{\mathrm{uni}(k)}) \subset \widehat{\mathcal{S}}^{(p_1,\ldots,p_d)}(\widehat{\mathcal{K}}_{\mathrm{uni}(k+1)}) \subset C^0([0,1)^d), \tag{3.4.4}$$

where the last relation follows from the assumption for the multiplicity of the interior knots. In particular, each $\widehat{\mathcal{B}}_{\mathrm{uni}(k)}$ can be written as linear combination of functions in $\widehat{\mathcal{B}}_{\mathrm{uni}(k')}$ if $k' > k$. Due to Lemma 3.2.1 (viii), the corresponding coefficients are non-negative. In [SM16], this property is referred to as *two-scale relation with only non-negative coefficients between bases of consecutive levels*. Finally, we remark that $\widehat{\mathcal{B}}_{\mathrm{uni}(k)} \cap \widehat{\mathcal{B}}_{\mathrm{uni}(k')} = \emptyset$ for all $k \neq k'$. This follows easily from Lemma 3.2.1 (viii) and the fact that both sets are bases; see [BdVBSV14, page 167].

Remark 3.4.1. *We define the nested sequence of tensor-product spline spaces in the most simple way. Another natural approach is to use uniform h-refinement with knots of some fixed multiplicities* $1 \le q_i \le p_i$ *for* $i \in \{1, \dots, d\}$*: With* $q := (q_1, \dots, q_d)$*, we set* $\widehat{\mathcal{K}}_{\mathrm{uni}(0,q)} := \widehat{\mathcal{K}}_0$*, and define* $\widehat{\mathcal{K}}_{\mathrm{uni}(k+1,q)}$ *recursively for* $k \in \mathbb{N}_0$ *as the d-dimensional knot vector, that results from inserting the knot* $(\widehat{z}_{i(\mathrm{uni}(k,q)),j-1} + \widehat{z}_{i(\mathrm{uni}(k,q)),j})/2$ *with multiplicity* q_i *to the knots* $\widehat{\mathcal{K}}_{i(\mathrm{uni}(k))}$*, where* $i \in \{1, \dots, d\}$ *and* $j \in \{1, \dots, n_{i(\mathrm{uni}(k,q))}\}$*. If* $q_i = 1$ *for all* $i \in \{1, \dots, d\}$*, we have that* $\widehat{\mathcal{K}}_{\mathrm{uni}(k)} = \widehat{\mathcal{K}}_{\mathrm{uni}(k,q)}$*. This choice leads to the highest possible regularity of the splines at newly inserted mesh lines, whereas the maximal choice* $q_i = p_i$ *only leads to continuity at new mesh lines. In particular, if all interior knots of the initial knots* $\widehat{\mathcal{K}}_{i(0)}$ *already have multiplicity* p_i*, the latter choice leads to the space of all continuous* $\widehat{\mathcal{T}}_{\mathrm{uni}(k,q)}$*-piecewise tensor-product polynomials of degree* $(p_1, \dots, p_d)$*. Note that all following definitions of this chapter can be made similarly if* $\mathrm{uni}(k)$ *is replaced by* $\mathrm{uni}(k,q)$*. Also the corresponding results remain valid.*

3.4.2 Hierarchical meshes and splines

We say that a set

$$\widehat{\mathcal{T}}_\bullet \subseteq \bigcup_{k \in \mathbb{N}_0} \widehat{\mathcal{T}}_{\mathrm{uni}(k)} \tag{3.4.5}$$

is a *hierarchical mesh* if it is a partition of $[0,1]^d$ in the sense that $\bigcup \widehat{\mathcal{T}}_\bullet = [0,1]^d$, where the intersection of two different elements $\widehat{T} \neq \widehat{T}'$ with $\widehat{T}, \widehat{T}' \in \widehat{\mathcal{T}}_\bullet$ has (d-dimensional) measure zero. Since $\widehat{\mathcal{T}}_{\mathrm{uni}(k)} \cap \widehat{\mathcal{T}}_{\mathrm{uni}(k')} = \emptyset$ for $k, k' \in \mathbb{N}_0$ with $k \neq k'$, we can define for an element $\widehat{T} \in \widehat{\mathcal{T}}_\bullet$

$$\mathrm{level}(\widehat{T}) := k \in \mathbb{N}_0 \quad \text{with } \widehat{T} \in \widehat{\mathcal{T}}_{\mathrm{uni}(k)}. \tag{3.4.6}$$

For an illustrative example of a hierarchical mesh, see Figure 3.2. In particular, any uniformly refined tensor mesh $\widehat{\mathcal{T}}_{\mathrm{uni}(k)}$ with $k \in \mathbb{N}_0$ is a hierarchical mesh. For $\widehat{\omega} \subseteq [0,1]^d$, we introduce the *patches of order* $q \in \mathbb{N}_0$ inductively by

$$\pi_\bullet^0(\widehat{\omega}) := \widehat{\omega}, \quad \pi_\bullet^q(\widehat{\omega}) := \bigcup \big\{ \widehat{T} \in \widehat{\mathcal{T}}_\bullet \, : \, \widehat{T} \cap \pi_\bullet^{q-1}(\widehat{\omega}) \neq \emptyset \big\}. \tag{3.4.7}$$

The corresponding set of elements is defined as

$$\Pi_\bullet^q(\widehat{\omega}) := \big\{ \widehat{T} \in \widehat{\mathcal{T}}_\bullet \, : \, \widehat{T} \subseteq \pi_\bullet^q(\widehat{\omega}) \big\}, \quad \text{i.e.,} \quad \pi_\bullet^q(\widehat{\omega}) = \bigcup \Pi_\bullet^q(\widehat{\omega}). \tag{3.4.8}$$

To abbreviate notation, we set $\pi_\bullet(\widehat{\omega}) := \pi_\bullet^1(\widehat{\omega})$ and $\Pi_\bullet(\widehat{\omega}) := \Pi_\bullet^1(\widehat{\omega})$.

For a hierarchical mesh $\widehat{\mathcal{T}}_\bullet$, we define a corresponding nested sequence $(\widehat{\Omega}_\bullet^k)_{k \in \mathbb{N}_0}$ of closed subsets of $[0,1]^d$ as

$$\widehat{\Omega}_\bullet^k := \bigcup_{k' \ge k} \big(\widehat{\mathcal{T}}_\bullet \cap \widehat{\mathcal{T}}_{\mathrm{uni}(k')} \big). \tag{3.4.9}$$

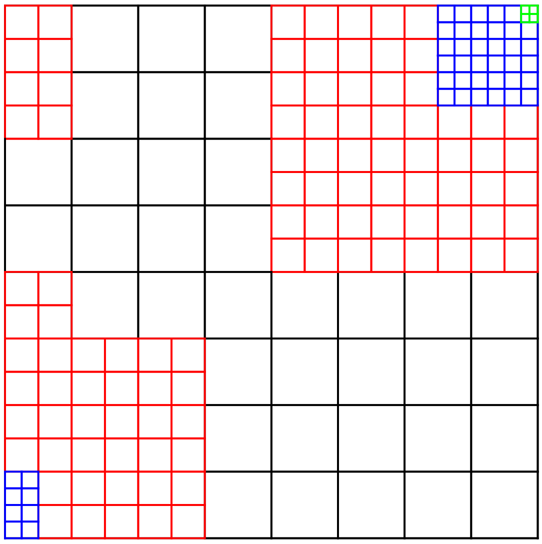


Figure 3.2: A two-dimensional hierarchical mesh $\widehat{\mathcal{T}}_\bullet$ is depicted, where $\widehat{\Omega}_\bullet^4 = \emptyset$. The corresponding domains $\widehat{\Omega}_\bullet^0 \supseteq \widehat{\Omega}_\bullet^1 \supseteq \widehat{\Omega}_\bullet^2 \supseteq \widehat{\Omega}_\bullet^3$ are highlighted in black, red, blue and green.

With $K_\bullet \in \mathbb{N}$, we denote the minimal integer such that $\widehat{\Omega}_\bullet^{K_\bullet} = \emptyset$. There holds that

$$\widehat{\mathcal{T}}_\bullet = \bigcup_{k \in \mathbb{N}_0} \{\widehat{T} \in \widehat{\mathcal{T}}_{\mathrm{uni}(k)} : \widehat{T} \subseteq \widehat{\Omega}_\bullet^k \wedge \widehat{T} \not\subseteq \widehat{\Omega}_\bullet^{k+1}\}. \tag{3.4.10}$$

Indeed, in the literature, one usually assumes that one is given the sequence $(\widehat{\Omega}_\bullet^k)_{k\in\mathbb{N}_0}$ and defines the corresponding hierarchical mesh via (3.4.10). Note that, for $\widehat{T} \in \widehat{\mathcal{T}}_\bullet$, level$(\widehat{T})$ is also the unique integer $k \in \mathbb{N}_0$ with $\widehat{T} \subseteq \widehat{\Omega}_\bullet^k$ and $\widehat{T} \not\subseteq \widehat{\Omega}_\bullet^{k+1}$.

With the notation $\widehat{\mathcal{S}}^{(p_1,\dots,p_d)}(\widehat{\mathcal{K}}_{\mathrm{uni}(k)})|_{\widehat{\omega}} := \{\widehat{S}|_{\widehat{\omega}} : \widehat{S} \in \widehat{\mathcal{S}}^{(p_1,\dots,p_d)}(\widehat{\mathcal{K}}_{\mathrm{uni}(k)})\}$ for any set $\widehat{\omega} \subseteq [0,1)^d$, we define the space of *hierarchical splines* as in [SM16, Section 3] by

$$\begin{aligned}\widehat{\mathcal{S}}^{(p_1,\dots,p_d)}(\widehat{\mathcal{K}}_0, \widehat{\mathcal{T}}_\bullet) := \{\widehat{S} : [0,1)^d \to \mathbb{R} :\\ \widehat{S}|_{[0,1)^d \setminus \widehat{\Omega}_\bullet^{k+1}} \in \widehat{\mathcal{S}}^{(p_1,\dots,p_d)}(\widehat{\mathcal{K}}_{\mathrm{uni}(k)})|_{[0,1)^d\setminus\widehat{\Omega}_\bullet^{k+1}} \text{ for all } k \in \mathbb{N}_0\}.\end{aligned} \tag{3.4.11}$$

In particular, each hierarchical spline is a $\widehat{\mathcal{T}}_\bullet$-piecewise tensor product polynomial of degree $(p_1,\dots,p_d)$. Put into words, hierarchical splines are coarse splines on coarse mesh elements, and they are fine splines on fine mesh elements.

Throughout the whole thesis, we will rarely make direct use of this definition, but rather consider the space of hierarchical splines as the span of the set of all *hierarchical B-splines* $\widehat{\mathcal{B}}_\bullet := \widehat{\mathcal{B}}_\bullet^{K_\bullet - 1}$:

(i) Define $\widehat{\mathcal{B}}_\bullet^0 := \widehat{\mathcal{B}}_{\mathrm{uni}(0)}$.

(ii) For $k = 0, \dots, K_\bullet - 2$, define $\widehat{\mathcal{B}}_\bullet^{k+1} := \mathrm{old}(\widehat{\mathcal{B}}_\bullet^{k+1}) \cup \mathrm{new}(\widehat{\mathcal{B}}_\bullet^{k+1})$, where

$$\begin{aligned}\mathrm{old}(\widehat{\mathcal{B}}_\bullet^{k+1}) &:= \big\{\widehat{\beta} \in \widehat{\mathcal{B}}_\bullet^k \,:\, \mathrm{supp}(\widehat{\beta}) \not\subseteq \widehat{\Omega}_\bullet^{k+1}\big\},\\ \mathrm{new}(\widehat{\mathcal{B}}_\bullet^{k+1}) &:= \big\{\widehat{\beta} \in \widehat{\mathcal{B}}_{\mathrm{uni}(k+1)} \,:\, \mathrm{supp}(\widehat{\beta}) \subseteq \widehat{\Omega}_\bullet^{k+1}\big\}.\end{aligned} \tag{3.4.12}$$

It is easy to check, that if $\widehat{\mathcal{T}}_\bullet$ is a tensor mesh, and hence coincides with some $\widehat{\mathcal{T}}_{\mathrm{uni}(k)}$, then the hierarchical basis and the standard tensor product B-spline basis are the same. Thus, the notation is consistent with the notation from (3.3.14). One can prove that the *hierarchical basis* $\widehat{\mathcal{B}}_\bullet$ is linearly independent (see [VGJS11, Lemma 2]) and spans the space $\widehat{\mathcal{S}}^{(p_1,\dots,p_d)}(\widehat{\mathcal{K}}_0, \widehat{\mathcal{T}}_\bullet)$ (see [SM16, Theorem 2]). By definition, it holds that

$$\widehat{\mathcal{B}}_\bullet = \bigcup_{k\in\mathbb{N}_0} \big\{\widehat{\beta} \in \widehat{\mathcal{B}}_{\mathrm{uni}(k)} \,:\, \mathrm{supp}(\widehat{\beta}) \subseteq \widehat{\Omega}_\bullet^k \wedge \mathrm{supp}(\widehat{\beta}) \not\subseteq \widehat{\Omega}_\bullet^{k+1}\big\}. \tag{3.4.13}$$

Since $\widehat{\mathcal{B}}_{\mathrm{uni}(k)} \cap \widehat{\mathcal{B}}_{\mathrm{uni}(k')} = \emptyset$ for $k \neq k' \in \mathbb{N}_0$, we can define for a basis function $\widehat{\beta} \in \widehat{\mathcal{B}}_\bullet$

$$\mathrm{level}(\widehat{\beta}) := k \in \mathbb{N}_0 \quad \text{with } \widehat{\beta} \in \widehat{\mathcal{B}}_{\mathrm{uni}(k)}. \tag{3.4.14}$$

Note that $\mathrm{level}(\widehat{\beta})$ is also the unique integer $k \in \mathbb{N}_0$ with $\mathrm{supp}(\widehat{\beta}) \subseteq \widehat{\Omega}_\bullet^k$ and $\mathrm{supp}(\widehat{\beta}) \not\subseteq \widehat{\Omega}_\bullet^{k+1}$.

The hierarchical basis $\widehat{\mathcal{B}}_\bullet$ and the mesh $\widehat{\mathcal{T}}_\bullet$ are compatible in the following sense: For all $\widehat{\beta} \in \widehat{\mathcal{B}}_\bullet$, the corresponding support can be written as union of elements in $\widehat{\mathcal{T}}_{\mathrm{uni}(\mathrm{level}(\widehat{\beta}))}$, i.e.,

$$\mathrm{supp}(\widehat{\beta}) = \bigcup \big\{\widehat{T} \in \widehat{\mathcal{T}}_{\mathrm{uni}(\mathrm{level}(\widehat{\beta}))} \,:\, \widehat{T} \subseteq \mathrm{supp}(\widehat{\beta})\big\}. \tag{3.4.15}$$

Each such element $\widehat{T} \in \widehat{\mathcal{T}}_{\mathrm{uni}(\mathrm{level}(\widehat{\beta}))}$ with $\widehat{T} \subseteq \mathrm{supp}(\widehat{\beta}) \subseteq \widehat{\Omega}_\bullet^{\mathrm{level}(\widehat{\beta})}$ satisfies that $\widehat{T} \in \widehat{\mathcal{T}}_\bullet$ or $\widehat{T} \subseteq \widehat{\Omega}_\bullet^{\mathrm{level}(\widehat{\beta})+1}$. In either case, we see that $\widehat{T}$ can be written as union of elements in $\widehat{\mathcal{T}}_\bullet$ with level greater or equal than $\mathrm{level}(\widehat{\beta})$. Altogether, we have that

$$\mathrm{supp}(\widehat{\beta}) = \bigcup_{k\geq \mathrm{level}(\widehat{\beta})} \big\{\widehat{T} \in \widehat{\mathcal{T}}_\bullet \cap \widehat{\mathcal{T}}_{\mathrm{uni}(k)} \,:\, \widehat{T} \subseteq \mathrm{supp}(\widehat{\beta})\big\}. \tag{3.4.16}$$

Moreover, $\mathrm{supp}(\widehat{\beta})$ must contain at least one element of level $\mathrm{level}(\widehat{\beta})$ Otherwise one would get the contradiction $\mathrm{supp}(\widehat{\beta}) \subseteq \widehat{\Omega}_\bullet^{\mathrm{level}(\widehat{\beta})+1}$. In particular, this shows that

$$\mathrm{level}(\widehat{\beta}) = \min_{\substack{\widehat{T}\in\widehat{\mathcal{T}}_\bullet\\ \widehat{T}\subseteq \mathrm{supp}(\widehat{\beta})}} \mathrm{level}(\widehat{T}) \quad \text{for all } \widehat{\beta} \in \widehat{\mathcal{B}}_\bullet. \tag{3.4.17}$$

Finally, we say that a hierarchical mesh $\widehat{\mathcal{T}}_\circ$ is *finer* than $\widehat{\mathcal{T}}_\bullet$ if $\widehat{\mathcal{T}}_\circ$ is obtained from $\widehat{\mathcal{T}}_\bullet$ via iterative dyadic bisection. Formally, this can be stated as $\widehat{\Omega}_\bullet^k \subseteq \widehat{\Omega}_\circ^k$ for all $k \in \mathbb{N}_0$. In this case, the corresponding hierarchical spline spaces are nested, i.e.,

$$\widehat{\mathcal{S}}^{(p_1,\dots,p_d)}(\widehat{\mathcal{K}}_0, \widehat{\mathcal{T}}_\bullet) \subseteq \widehat{\mathcal{S}}^{(p_1,\dots,p_d)}(\widehat{\mathcal{K}}_0, \widehat{\mathcal{T}}_\circ). \tag{3.4.18}$$

This follows immediately from (3.4.11). In particular, this implies that

$$\widehat{\mathcal{S}}^{(p_1,\dots,p_d)}(\widehat{\mathcal{K}}_0, \widehat{\mathcal{T}}_{\mathrm{uni}(0)}) \subseteq \widehat{\mathcal{S}}^{(p_1,\dots,p_d)}(\widehat{\mathcal{K}}_0, \widehat{\mathcal{T}}_\bullet) \subseteq \widehat{\mathcal{S}}^{(p_1,\dots,p_d)}(\widehat{\mathcal{K}}_0, \widehat{\mathcal{T}}_{\mathrm{uni}(K_\bullet-1)}) \tag{3.4.19}$$

In the following chapters, we will need the following auxiliary lemma.

Lemma 3.4.2. *Let $\widehat{\mathcal{T}}_\bullet$ and $\widehat{\mathcal{T}}_\circ$ be hierarchical meshes such that $\widehat{\mathcal{T}}_\circ$ is finer than $\widehat{\mathcal{T}}_\bullet$, i.e., $\widehat{\Omega}_\bullet^k \subseteq \widehat{\Omega}_\circ^k$ for all $k \in \mathbb{N}_0$. Then, for all $\widehat{\beta}_\circ \in \widehat{\mathcal{B}}_\circ$ there exists $\widehat{\beta}_\bullet \in \widehat{\mathcal{B}}_\bullet$ with* $\mathrm{supp}(\widehat{\beta}_\circ) \subseteq \mathrm{supp}(\widehat{\beta}_\bullet)$.

Proof. Clearly, we may assume that $\widehat{\beta}_\circ \in \widehat{\mathcal{B}}_\circ \setminus \widehat{\mathcal{B}}_\bullet$. Let $k := \mathrm{level}(\widehat{\beta}_\circ)$ and define $\widehat{\beta}_k := \widehat{\beta}_\circ$. Since $\widehat{\beta}_k \in \widehat{\mathcal{B}}_\circ$, (3.4.13) implies that $\mathrm{supp}(\widehat{\beta}_k)\setminus\widehat{\Omega}_\circ^{k+1} \neq \emptyset$ and $\mathrm{supp}(\widehat{\beta}_k) \subseteq \widehat{\Omega}_\circ^k$. Since $\widehat{\beta}_k \notin \widehat{\mathcal{B}}_\bullet$, (3.4.13) implies that $\mathrm{supp}(\widehat{\beta}_k) \setminus \widehat{\Omega}_\bullet^{k+1} = \emptyset$ or $\mathrm{supp}(\widehat{\beta}_k) \not\subseteq \widehat{\Omega}_\bullet^k$. However, $\widehat{\Omega}_\bullet^{k+1} \subseteq \widehat{\Omega}_\circ^{k+1}$ and $\mathrm{supp}(\widehat{\beta}_k) \setminus \widehat{\Omega}_\circ^{k+1} \neq \emptyset$ imply that $\mathrm{supp}(\widehat{\beta}_k) \setminus \widehat{\Omega}_\bullet^{k+1} \neq \emptyset$. Hence, we have $\mathrm{supp}(\widehat{\beta}_k) \not\subseteq \widehat{\Omega}_\bullet^k$, which especially implies that $k > 0$. This is equivalent to $\mathrm{supp}(\widehat{\beta}_k)\setminus\widehat{\Omega}_\bullet^k \neq \emptyset$. Clearly, there exists $\widehat{\beta}_{k-1} \in \widehat{\mathcal{B}}_{\mathrm{uni}(k-1)}$ with $\mathrm{supp}(\widehat{\beta}_k) \subseteq \mathrm{supp}(\widehat{\beta}_{k-1})$. If $\widehat{\beta}_{k-1} \in \widehat{\mathcal{B}}_\bullet$, we are done. Otherwise, (3.4.13) implies that $\mathrm{supp}(\widehat{\beta}_{k-1}) \setminus \widehat{\Omega}_\bullet^k = \emptyset$ or $\mathrm{supp}(\widehat{\beta}_{k-1}) \not\subseteq \widehat{\Omega}_\bullet^{k-1}$. Again, the first case is not possible because

$$\mathrm{supp}(\widehat{\beta}_{k-1}) \setminus \widehat{\Omega}_\bullet^k \supseteq \mathrm{supp}(\widehat{\beta}_k) \setminus \widehat{\Omega}_\bullet^k \neq \emptyset.$$

Hence, we have that $\mathrm{supp}(\widehat{\beta}_{k-1}) \not\subseteq \widehat{\Omega}_\bullet^{k-1}$ which especially implies that $k-1 > 0$. This is equivalent to $\mathrm{supp}(\widehat{\beta}_{k-1}) \setminus \widehat{\Omega}_\bullet^{k-1} \neq \emptyset$. Inductively, we obtain a sequence $\widehat{\beta}_k, \dots, \widehat{\beta}_J$ with $\widehat{\beta}_j \in \widehat{\mathcal{B}}_{\mathrm{uni}(j)}$ and $\mathrm{supp}(\widehat{\beta}_J) \supseteq \cdots \supseteq \mathrm{supp}(\widehat{\beta}_k)$, where $\widehat{\beta}_J \in \widehat{\mathcal{B}}_\bullet$ for some $J \geq 0$. ☐

3.4.3 Truncated hierarchical B-splines

For a hierarchical mesh $\widehat{\mathcal{T}}_\bullet$, we present a second basis for the corresponding hierarchical splines $\widehat{\mathcal{S}}^{(p_1,\dots,p_d)}(\widehat{\mathcal{K}}_0, \widehat{\mathcal{T}}_\bullet)$, namely the *truncated hierarchical B-splines* (THB-splines) introduced in [GJS12]. In general, they have a smaller but also more complicated support than the hierarchical B-splines.

For $k \in \mathbb{N}_0$, we define the *truncation* $\mathrm{trunc}_\bullet^{k+1} : \widehat{\mathcal{S}}^{(p_1,\dots,p_d)}(\widehat{\mathcal{K}}_{\mathrm{uni}(k)}) \to \widehat{\mathcal{S}}^{(p_1,\dots,p_d)}(\widehat{\mathcal{K}}_{\mathrm{uni}(k+1)})$ as follows:

$$\mathrm{trunc}_\bullet^{k+1}(\widehat{S}) := \sum_{\substack{\widehat{\beta}\in\widehat{\mathcal{B}}_{\mathrm{uni}(k+1)} \\ \mathrm{supp}(\widehat{\beta})\not\subseteq\widehat{\Omega}_\bullet^{k+1}}} a_{\widehat{\beta}}\widehat{\beta} \quad \text{for } \widehat{S} = \sum_{\widehat{\beta}\in\widehat{\mathcal{B}}_{\mathrm{uni}(k+1)}} a_{\widehat{\beta}}\widehat{\beta} \in \widehat{\mathcal{S}}^{(p_1,\dots,p_d)}(\widehat{\mathcal{K}}_{\mathrm{uni}(k)}), \tag{3.4.20}$$

i.e., truncation is defined via the (unique) basis representation of $\widehat{S} \in \widehat{\mathcal{S}}^{(p_1,\dots,p_d)}(\widehat{\mathcal{K}}_{\mathrm{uni}(k)})$ with respect to the consecutive basis $\widehat{\mathcal{B}}_{\mathrm{uni}(k+1)}$. Recall that $K_\bullet \in \mathbb{N}$ is the minimal integer such that $\widehat{\Omega}_\bullet^{K_\bullet} = \emptyset$. For all $\widehat{\beta} \in \widehat{\mathcal{B}}_\bullet$, the corresponding *truncated hierarchical B-spline* (THB-spline) reads

$$\mathrm{Trunc}_\bullet(\widehat{\beta}) := \mathrm{trunc}_\bullet^{K_\bullet-1}\Big(\mathrm{trunc}_\bullet^{K_\bullet-2}\Big(\dots\Big(\mathrm{trunc}_\bullet^{\mathrm{level}(\widehat{\beta})+1}(\widehat{\beta})\Big)\dots\Big)\Big), \tag{3.4.21}$$

As the set $\widehat{\mathcal{B}}_\bullet$, the set of THB-splines $\{\mathrm{Trunc}_\bullet(\widehat{\beta}) : \widehat{\beta} \in \widehat{\mathcal{B}}_\bullet\}$ forms a basis of the space of hierarchical splines $\widehat{\mathcal{S}}^{(p_1,\dots,p_d)}(\widehat{\mathcal{K}}_0, \widehat{\mathcal{T}}_\bullet)$; see, e.g., [GJS12, Theorem 6 and 9]. In Section 3.4.1, we mentioned the two-scale relation with only non-negative coefficients between bases of consecutive levels, i.e., the fact that each basis function in $\widehat{\mathcal{B}}_{\mathrm{uni}(k)}$ is the linear combination of basis functions $\widehat{\mathcal{B}}_{\mathrm{uni}(k+1)}$, where the corresponding coefficients are non-negative. For $\widehat{\beta} \in \widehat{\mathcal{B}}_\bullet$, this proves that

$$0 \leq \mathrm{Trunc}_\bullet(\widehat{\beta}) \leq \widehat{\beta}, \tag{3.4.22}$$

and in particular $\mathrm{supp}(\mathrm{Trunc}_\bullet(\widehat{\beta})) \subseteq \mathrm{supp}(\widehat{\beta})$. Moreover, [GJS12, Theorem 10] states that, in contrast to hierarchical B-splines, THB-splines form a partition of unity, i.e.,

$$\sum_{\widehat{\beta}\in\widehat{\mathcal{B}}_\bullet} \mathrm{Trunc}_\bullet(\widehat{\beta}) = 1. \tag{3.4.23}$$

3.4.4 Admissible hierarchical meshes

The results of this section stem from the recent own work [GHP17]. Let $\widehat{\mathcal{T}}_\bullet$ be an arbitrary hierarchical mesh. We define the set of all *neighbors* of an element $\widehat{T} \in \widehat{\mathcal{T}}_\bullet$ as

$$\mathbf{N}_\bullet(\widehat{T}) := \big\{\widehat{T}' \in \widehat{\mathcal{T}}_\bullet : \exists \widehat{\beta} \in \widehat{\mathcal{B}}_\bullet \quad \widehat{T}, \widehat{T}' \subseteq \mathrm{supp}(\widehat{\beta})\big\}, \tag{3.4.24}$$

According to (3.4.16), the condition $\widehat{T}, \widehat{T}' \subseteq \mathrm{supp}(\widehat{\beta})$ is equivalent to $|\widehat{T} \cap \mathrm{supp}(\widehat{\beta})| \neq 0 \neq |\widehat{T}' \cap \mathrm{supp}(\widehat{\beta})|$. As in [GHP17], we call $\widehat{\mathcal{T}}_\bullet$ *admissible* if

$$|\mathrm{level}(\widehat{T}) - \mathrm{level}(\widehat{T}')| \leq 1 \quad \text{for all } \widehat{T}, \widehat{T}' \in \widehat{\mathcal{T}}_\bullet \text{ with } \widehat{T}' \in \mathbf{N}_\bullet(\widehat{T}). \tag{3.4.25}$$

Let $\widehat{\mathbb{T}}$ be the set of all admissible hierarchical meshes. Clearly, all tensor meshes $\widehat{\mathcal{T}}_{\mathrm{uni}(k)}$, $k \in \mathbb{N}_0$, belong to $\widehat{\mathbb{T}}$. Moreover, admissible meshes satisfy the following interesting properties which are also important for an efficient implementation of finite or boundary element methods with hierarchical splines.

Proposition 3.4.3. *Let $\widehat{\mathcal{T}}_\bullet \in \widehat{\mathbb{T}}$. The support of any hierarchical B-spline $\widehat{\beta} \in \widehat{\mathcal{B}}_\bullet$ is the union of at most $2^d(p_{\max}+1)^d$ elements $\widehat{T}' \in \widehat{\mathcal{T}}_\bullet$. Moreover, for any $\widehat{T} \in \widehat{\mathcal{T}}_\bullet$, there are at most $2(p_{\max}+1)^d$ basis functions $\widehat{\beta}' \in \widehat{\mathcal{B}}_\bullet$ that have support on $\widehat{T}$, i.e., $|\mathrm{supp}(\widehat{\beta}') \cap \widehat{T}| > 0$.*

Proof. We abbreviate $k := \mathrm{level}(\widehat{\beta})$. By (3.4.17), there exists $\widehat{T}'' \subseteq \mathrm{supp}(\widehat{\beta})$ with $\mathrm{level}(\widehat{T}'') = k$. Admissibility of $\widehat{\mathcal{T}}_\bullet$ together with (3.4.16) shows that $\mathrm{level}(\widehat{T}') \in \{k, k+1\}$ for all $\widehat{T}' \in \widehat{\mathcal{T}}_\bullet$ with $\widehat{T}' \subseteq \mathrm{supp}(\widehat{\beta})$. Since $\widehat{\beta}$ is an element of $\widehat{\mathcal{B}}_{\mathrm{uni}(k)}$, its support is the union of at most $2^d(p_{\max}+1)^d$ elements in $\widehat{\mathcal{T}}_{\mathrm{uni}(k+1)}$. This proves the first assertion. For $\widehat{\beta}' \in \widehat{\mathcal{B}}_\bullet$ and $\widehat{T} \in \widehat{\mathcal{T}}_\bullet$ with $|\mathrm{supp}(\widehat{\beta}') \cap \widehat{T}| > 0$, the characterization (3.4.16) proves that $\widehat{T} \subseteq \mathrm{supp}(\widehat{\beta}')$. Hence, (3.4.17) together with admissibility of $\widehat{\mathbb{T}}$ proves that $\mathrm{level}(\widehat{\beta}') = \widetilde{k} := \mathrm{level}(\widehat{T})$ or $\mathrm{level}(\widehat{\beta}') = \widetilde{k} - 1$. With $\widehat{\mathcal{B}}_{\mathrm{uni}(-1)} := \widehat{\mathcal{B}}_{\mathrm{uni}(0)}$, there are at most $(p_{\max}+1)^d$ basis functions in $\widehat{\mathcal{B}}_{\mathrm{uni}(\widetilde{k}-1)}$ and $(p_{\max}+1)^d$ basis functions in $\widehat{\mathcal{B}}_{\mathrm{uni}(\widetilde{k})}$ that have support on the element $\widehat{T}$. This concludes the proof. □

Remark 3.4.4. *Since the support of any $\widehat{\beta} \in \widehat{\mathcal{B}}_\bullet$ is connected, Proposition 3.4.3 particularly shows that $\widehat{T}' \subseteq \mathrm{supp}(\widehat{\beta})$ for an element $\widehat{T}' \in \widehat{\mathcal{T}}_\bullet$ implies that $\mathrm{supp}(\widehat{\beta}) \subseteq \pi_\bullet^{2(p_{\max}+1)}(\widehat{T}')$. By (3.4.16), $\widehat{T}' \subseteq \mathrm{supp}(\widehat{\beta})$ is equivalent to $|\widehat{T}' \cap \mathrm{supp}(\widehat{\beta})| > 0$.*

The following lemma provides a relation between the set of neighbors and the patch of an element $T \in \mathcal{T}_\bullet$.

Lemma 3.4.5. *Let $\widehat{\mathcal{T}}_\bullet$ be an arbitrary hierarchical mesh. Then, there holds that*

$$\Pi_\bullet(\widehat{T}) \subseteq \mathbf{N}_\bullet(\widehat{T}) \quad \textit{for all } \widehat{T} \in \widehat{\mathcal{T}}_\bullet. \tag{3.4.26}$$

Proof. Let $\widehat{T}' \in \Pi_\bullet(\widehat{T})$, i.e., $\widehat{T}' \in \widehat{\mathcal{T}}_\bullet$ with $\widehat{T} \cap \widehat{T}' \neq \emptyset$. We abbreviate $k := \mathrm{level}(\widehat{T})$. Since all multiplicities of interior knots of $\widehat{\mathcal{K}}_{i(\bullet)}$ are smaller than $p_i + 1$ for all $i \in \{1, \dots, d\}$, Lemma 3.2.1 (ii) implies the existence of some $\widehat{\beta}_k \in \widehat{\mathcal{B}}_{\mathrm{uni}(k)}$ such that $|\widehat{T} \cap \mathrm{supp}(\widehat{\beta}_k)| \neq 0 \neq |\widehat{T}' \cap \mathrm{supp}(\widehat{\beta}_k)|$. If $\widehat{\beta}_k \in \widehat{\mathcal{B}}_\bullet$, then $\widehat{T}' \in \widehat{\mathbf{N}}_\bullet(\widehat{T})$. If $\widehat{\beta}_k \notin \widehat{\mathcal{B}}_\bullet$, the characterization (3.4.13) shows that $\mathrm{supp}(\widehat{\beta}_k) \not\subseteq \widehat{\Omega}^k_\bullet$ or $\mathrm{supp}(\widehat{\beta}_k) \subseteq \widehat{\Omega}^{k+1}_\bullet$. By choice of k, it holds that $\widehat{T} \subseteq \mathrm{supp}(\widehat{\beta}_k)$. In view of (3.4.10), $\widehat{T} \in \widehat{\mathcal{T}}_\bullet$ implies that $\widehat{T} \not\subseteq \widehat{\Omega}^{k+1}_\bullet$. Hence, $\mathrm{supp}(\widehat{\beta}_k) \not\subseteq \widehat{\Omega}^k_\bullet$ and, in particular, $k > 0$. Next, there exists $\widehat{\beta}_{k-1} \in \widehat{\mathcal{B}}_{\mathrm{uni}(k-1)}$ such that $\mathrm{supp}(\widehat{\beta}_k) \subseteq \mathrm{supp}(\widehat{\beta}_{k-1})$. If $\widehat{\beta}_{k-1} \in \widehat{\mathcal{B}}_\bullet$, then $\widehat{T}' \in \widehat{\mathbf{N}}_\bullet(\widehat{T})$. If $\widehat{\beta}_{k-1} \notin \widehat{\mathcal{B}}_\bullet$, there holds again that either $\mathrm{supp}(\widehat{\beta}_{k-1}) \not\subseteq \widehat{\Omega}^{k-1}_\bullet$ or $\mathrm{supp}(\widehat{\beta}_{k-1}) \subseteq \widehat{\Omega}^k_\bullet$. Due to $\mathrm{supp}(\widehat{\beta}_k) \not\subseteq \widehat{\Omega}^k_\bullet$, the second case is not possible. Hence, $\mathrm{supp}(\widehat{\beta}_{k-1}) \not\subseteq \widehat{\Omega}^{k-1}_\bullet$ and, in particular, $k - 1 > 0$. We proceed in the same way to get a sequence $\widehat{\beta}_k, \dots, \widehat{\beta}_J$ with $\widehat{\beta}_j \in \widehat{\mathcal{B}}_{\mathrm{uni}(j)}$ and $\mathrm{supp}(\widehat{\beta}_J) \supseteq \dots \supseteq \mathrm{supp}(\widehat{\beta}_k)$, where $\widehat{\beta}_J \in \widehat{\mathcal{B}}_\bullet$ for some $J \geq 0$. □

Remark 3.4.6. *In the proof of Lemma 3.4.5, we used that all interior knot multiplicities are smaller or equal than the corresponding polynomial degree p_i. Actually, this is the only place, where we need this assumption. However, this lemma is of course essential as it implies for example local quasi-uniformity for admissible meshes. If one drops the additional assumption on the knot multiplicities and allows multiplicities up to $p_i + 1$ as well as lowest-order polynomial degrees $p_i = 0$, one could define the neighbors of an element $\widehat{T} \in \widehat{\mathcal{T}}_\bullet$ differently as*

$$\mathbf{N}_\bullet(\widehat{T}) := \big\{\widehat{T}' \in \widehat{\mathcal{T}}_\bullet \,:\, \big(\exists \widehat{\beta} \in \widehat{\mathcal{B}}_\bullet \quad \widehat{T}, \widehat{T}' \subseteq \mathrm{supp}(\widehat{\beta})\big) \vee \big(\widehat{T} \cap \widehat{T}' \neq \emptyset\big)\big\}. \tag{3.4.27}$$

Then, all results of the current Section 3.4 remain valid. Moreover, newly inserted knots can also have multiplicity $p_i + 1$, i.e., the choice $q_i = p_i + 1$ in Remark 3.4.1 is possible.

The next proposition shows that for an admissible mesh $\widehat{\mathcal{T}}_\bullet \in \widehat{\mathcal{T}}_\bullet$, the full truncation $\mathrm{Trunc}_\bullet$ reduces to $\mathrm{trunc}_\bullet^{\mathrm{level}(\widehat{\beta})+1}$.

Proposition 3.4.7. *Let $\widehat{\mathcal{T}}_\bullet \in \widehat{\mathbb{T}}$ and $\widehat{\beta} \in \widehat{\mathcal{B}}_\bullet$. Then, it holds that*

$$\mathrm{Trunc}_\bullet(\widehat{\beta}) = \mathrm{trunc}_\bullet^{\mathrm{level}(\widehat{\beta})+1}(\widehat{\beta}). \tag{3.4.28}$$

Proof. We prove the assertion in two steps.
Step 1: Let $k' < k'' \in \mathbb{N}_0$ and $\widehat{\beta}' \in \widehat{\mathcal{B}}_{\mathrm{uni}(k')}$ with representation $\widehat{\beta}' = \sum_{\widehat{\beta}'' \in \widehat{\mathcal{B}}_{\mathrm{uni}(k'')}} a_{\widehat{\beta}''} \widehat{\beta}''$. Let $\widehat{\beta}'' \in \widehat{\mathcal{B}}_{\mathrm{uni}(k'')}$ such that $a_{\widehat{\beta}''} \neq 0$. Then, local linear independence (with the open set $O := (0,1)^d \setminus \mathrm{supp}(\widehat{\beta}')$) of $\widehat{\mathcal{B}}_{\mathrm{uni}(k'')}$ implies that $\mathrm{supp}(\widehat{\beta}'') \subseteq \mathrm{supp}(\widehat{\beta}')$.
Step 2: We prove (3.4.28). We abbreviate $k := \mathrm{level}(\widehat{\beta})$. Let $\widehat{\beta} = \sum_{\widehat{\beta}' \in \widehat{\mathcal{B}}_{\mathrm{uni}(k+1)}} a_{\widehat{\beta}'} \widehat{\beta}'$. Let $\widehat{\beta}' \in \widehat{\mathcal{B}}_{\mathrm{uni}(k+1)}$ with $\mathrm{supp}(\widehat{\beta}') \not\subseteq \widehat{\Omega}^{k+1}_\bullet$ and $a_{\widehat{\beta}'} \neq 0$. By Step 1, this proves that $\mathrm{supp}(\widehat{\beta}') \subseteq \mathrm{supp}(\widehat{\beta})$. For $k'' > k + 1$, we consider the representation

$$\mathrm{trunc}_\bullet^{k''}(\widehat{\beta}') = \sum_{\substack{\widehat{\beta}'' \in \widehat{\mathcal{B}}_{\mathrm{uni}(k'')} \\ \mathrm{supp}(\widehat{\beta}'') \not\subseteq \widehat{\Omega}^{k''}_\bullet}} a_{\widehat{\beta}''} \widehat{\beta}'', \quad \text{where } \widehat{\beta}' = \sum_{\widehat{\beta}'' \in \widehat{\mathcal{B}}_{\mathrm{uni}(k'')}} a_{\widehat{\beta}''} \widehat{\beta}''.$$

For $\widehat{\beta}'' \in \widehat{\mathcal{B}}_{\mathrm{uni}(k'')}$ with $\mathrm{supp}(\widehat{\beta}'') \subseteq \widehat{\Omega}_\bullet^{k''}$, let $\widehat{T}'' \in \widehat{\mathcal{T}}_{\mathrm{uni}(k'')}$ with $\widehat{T}'' \subseteq \mathrm{supp}(\widehat{\beta}'')$. (3.4.10) shows the existence of an element $\widehat{T} \in \widehat{\mathcal{T}}_\bullet$ with $\mathrm{level}(\widehat{T}) \geq k''$ such that $\widehat{T} \subseteq \widehat{T}''$. To see that $a_{\widehat{\beta}''} = 0$, we argue by contradiction and assume that $a_{\widehat{\beta}''} \neq 0$. By Step 1, this implies that $\widehat{T} \subseteq \mathrm{supp}(\widehat{\beta}'') \subseteq \mathrm{supp}(\widehat{\beta}') \subseteq \mathrm{supp}(\widehat{\beta})$. Due to $\mathrm{level}(\widehat{T}) > k+1$ and (3.4.17), this contradicts admissibility of $\widehat{\mathcal{T}}_\bullet$. This proves that $a_{\widehat{\beta}''} = 0$. Overall, we conclude that $\mathrm{trunc}_\bullet^{k''}(\widehat{\beta}') = \widehat{\beta}'$, and thus $\mathrm{trunc}_\bullet^{k''}(\mathrm{trunc}_\bullet^{k+1}(\widehat{\beta})) = \mathrm{trunc}_\bullet^{k+1}(\widehat{\beta})$ as well as (3.4.28).

□

3.4.5 Quasi-interpolation projection

In this section, we introduce a quasi-interpolation projection onto the space of hierarchical splines which was developed in the recent own work [GHP17]. Let $\widehat{\mathcal{T}}_\bullet \in \widehat{\mathbb{T}}$ be a given admissible hierarchical mesh. First, we define certain dual basis functions for the tensor-product B-splines. Recall that $\widehat{\mathcal{B}}_{\mathrm{uni}(k)} \cap \widehat{\mathcal{B}}_{\mathrm{uni}(k')} = \emptyset$ for $k \neq k'$. For $k \in \mathbb{N}_0$ and $\widehat{\beta} \in \widehat{\mathcal{B}}_{\mathrm{uni}(k)}$, let $\widehat{T}_{\widehat{\beta}} \in \widehat{\mathcal{T}}_{\mathrm{uni}(k)}$ be an arbitrary but fixed element with $\widehat{T}_{\widehat{\beta}} \subseteq \mathrm{supp}(\widehat{\beta})$. If $\widehat{\beta} \in \widehat{\mathcal{B}}_\bullet$, we additionally require that[1] $\widehat{T}_{\widehat{\beta}} \in \widehat{\mathcal{T}}_\bullet$, which is possible due to (3.4.17). Let $\widehat{T}_{\widehat{\beta}}^\circ$ denote the interior of $\widehat{T}_{\widehat{\beta}}$. By local linear independence of $\widehat{\mathcal{B}}_{\mathrm{uni}(k)}$ (see Section 3.4.1), also the restricted basis functions $\{\widehat{\beta}|_{\widehat{T}_{\widehat{\beta}}^\circ} : \widehat{\beta} \in \widehat{\mathcal{B}}_{\mathrm{uni}(k)} \wedge \mathrm{supp}(\widehat{\beta}) \cap \widehat{T}_{\widehat{\beta}}^\circ \neq \emptyset\}$ are linearly independent. Hence, the Riesz theorem guarantees the existence and uniqueness of some $\widehat{\beta}^* \in \{\widehat{S}|_{\widehat{T}_{\widehat{\beta}}} : \widehat{S} \in \widehat{\mathcal{S}}^{(p_1,\dots,p_d)}(\widehat{\mathcal{K}}_{\mathrm{uni}(k)})\}$ such that

$$\int_{\widehat{T}_{\widehat{\beta}}} \widehat{\beta}^* \widehat{\beta}' \, dt = \delta_{\widehat{\beta},\widehat{\beta}'} \quad \text{for all } \widehat{\beta}' \in \widehat{\mathcal{B}}_{\mathrm{uni}(k)}. \tag{3.4.29}$$

In contrast to the one-dimensional dual functions presented in Section 3.3.3, the support of $\widehat{\beta}^*$ consists only of one single element $\widehat{T}_{\widehat{\beta}}$. These dual basis functions $\widehat{\beta}^*$ satisfy the following scaling property.

Lemma 3.4.8. *There exists a constant $C > 0$ such that for all $k \in \mathbb{N}_0$ and all $\widehat{\beta} \in \widehat{\mathcal{B}}_{\mathrm{uni}(k)}$, it holds that*

$$\|\widehat{\beta}^*\|_{L^\infty(\widehat{T}_{\widehat{\beta}})} \leq C|\widehat{T}_{\widehat{\beta}}|^{-1}. \tag{3.4.30}$$

The constant C depends only on d, $\widehat{\mathcal{T}}_0$, and $(p_1,\dots,p_d)$.

Proof. Recall that the element $\widehat{T}_{\widehat{\beta}}$ is a rectangle of the form

$$[t_{1(\mathrm{uni}(k)),\ell_1-1}, t_{1(\mathrm{uni}(k)),\ell_1}] \times \dots \times [t_{d(\mathrm{uni}(k)),\ell_d-1}, t_{d(\mathrm{uni}(k)),\ell_d}].$$

We use the abbreviations $C_1 := |\widehat{T}_{\widehat{\beta}}|^{1/d}$ and $(a_1,\dots,a_d) := (t_{1(\mathrm{uni}(k)),\ell_1-1},\dots,t_{d(\mathrm{uni}(k)),\ell_d-1})$. We define the normalized element $\widetilde{T}_{\widehat{\beta}} := (\widehat{T}_{\widehat{\beta}} - (a_1,\dots,a_d))/C_1$ and the corresponding affine

[1]Therefore, the elements $\widehat{T}_{\widehat{\beta}}$ depend additionally on the considered mesh $\widehat{\mathcal{T}}_\bullet$.

transformation $\Phi : \widetilde{T}_{\widehat{\beta}} \to \widehat{T}_{\widehat{\beta}}$. We apply the transformation formula to see that

$$\int_{\widehat{T}_{\widehat{\beta}}} \widehat{\beta}^* \widehat{\beta}' \, dt = C_1^d \int_{\widetilde{T}_{\widehat{\beta}}} (\widehat{\beta}^* \circ \Phi)(\widehat{\beta} \circ \Phi) \, dt.$$

Therefore, the Riesz theorem implies that $\widehat{\beta}^* = (\widetilde{\beta}^* \circ \Phi^{-1})/C_1^d$, where $\widetilde{\beta}^*$ is the unique element in $\{\widehat{S} \circ \Phi : \widehat{S} \in \widehat{\mathcal{S}}^{(p_1,\dots,p_d)}(\widehat{\mathcal{K}}_{\mathrm{uni}(k)})\}$ such that

$$\int_{\widetilde{T}_{\widehat{\beta}}} \widetilde{\beta}^* \widetilde{\beta}' \, dt = \delta_{\widetilde{\beta},\widetilde{\beta}'} \quad \text{for all } \widetilde{\beta}' \in \widetilde{\mathcal{B}}_{\mathrm{uni}(k)} := \{\widehat{\beta}' \circ \Phi \, : \, \widehat{\beta}' \in \widehat{\mathcal{B}}_{\mathrm{uni}(k)}\}.$$

By definition (3.3.14) and Lemma 3.2.1 (iii), each $\widetilde{\beta}' \in \widetilde{\mathcal{B}}_{\mathrm{uni}(k)}$ is at any point $(s_1, \dots, s_d) = (\widetilde{s}_1, \dots, \widetilde{s}_d)C + (a_1, \dots, a_d)$ of the form

$$\widetilde{\beta}'(\widetilde{s}_1, \dots, \widetilde{s}_d) = \prod_{i=1}^{d} \widehat{B}(s_i | t_{i(\mathrm{uni}(k)),j_i-1}, \dots, t_{i(\mathrm{uni}(k)),j_i+p_i}).$$

We only have to consider $\widetilde{\beta}'$ that are supported on $\widetilde{T}_{\widehat{\beta}}$. Since the support of any B-spline $\widehat{B}(\cdot | t_{i(\mathrm{uni}(k)),j_i-1}, \dots, t_{i(\mathrm{uni}(k)),j_i+p_i})$ is just $[t_{i(\mathrm{uni}(k)),j_i-1}, \dots, t_{i(\mathrm{uni}(k)),j_i+p_i})]$ (see Lemma 3.2.1 (ii)), it is sufficient to consider $j_i = \ell_i - p_i, \dots, \ell_i$. According to Lemma 3.2.1 (v), an affine transformation in the parameter domain can just be passed to the knots, i.e.,

$$\widehat{B}(s_i | t_{i(\mathrm{uni}(k)),j_i-1}, \dots, t_{i(\mathrm{uni}(k)),j_i+p_i}) = \widehat{B}\big(\widetilde{s}_i \big| (t_{i(\mathrm{uni}(k)),j_i-1} - a_i)/C_1, \dots, (t_{i(\mathrm{uni}(k)),j_i+p_i} - a_i)/C_1\big).$$

Altogether, we see that $\widetilde{\beta}^*$ depends only on the knots

$$\Big(\frac{t_{i(\mathrm{uni}(k)),j_i-1} - a_i}{C_1}, \dots, \frac{t_{i(\mathrm{uni}(k)),j_i+p_i} - a_i}{C_1} : i = 1, \dots, d \wedge j_i = \ell_i - p_i, \dots, \ell_i\Big).$$

Since we only use global dyadic bisection between two consecutive levels, we see that these knots depend only on d, $\widehat{\mathcal{T}}_0$ and $(p_1, \dots, p_d)$ but not on the level k. This shows that $\|\widetilde{\beta}\|_{L^\infty(\widetilde{T}_{\widehat{\beta}})} \lesssim 1$, where the hidden constant depends only on d, $\widehat{\mathcal{T}}_0$, and $(p_1, \dots, p_d)$. □

We use the approach of [SM16] with our concrete dual functions, and define an operator which maps to the space of hierarchical splines

$$\widehat{I}_\bullet : L^2([0,1]^d) \to \widehat{\mathcal{S}}^{(p_1,\dots,p_d)}(\widehat{\mathcal{K}}_0, \widehat{\mathcal{T}}_\bullet), \quad \widehat{v} \mapsto \sum_{\widehat{\beta} \in \widehat{\mathcal{B}}_\bullet} \int_{\widehat{T}_{\widehat{\beta}}} \widehat{\beta}^* \, \widehat{v} \, dt \ \mathrm{Trunc}_\bullet(\widehat{\beta}). \tag{3.4.31}$$

Note that, for $d = 1$, the operator for standard splines $\widehat{I}_\bullet$ from (3.3.1) does not necessarily coincide with the currently considered operator $\widehat{I}_\bullet$ from (3.4.9). Still, the latter satisfies the same properties as in Proposition 3.3.1, which is stated in the next proposition.

Proposition 3.4.9. *With the abbreviation $q := 2(p_{\max} + 1)$, the operator $\widehat{I}_\bullet$ from (3.4.31) satisfies the following two properties:*

(i) *For all* $\widehat{T} \in \widehat{\mathcal{T}}_\bullet$ *and all* $\widehat{v} \in L^2([0,1]^d)$*, the inclusion* $\widehat{v}|_{\pi_\bullet^q(\widehat{T})} \in \widehat{\mathcal{S}}^{(p_1,\dots,p_d)}(\widehat{\mathcal{K}}_0, \widehat{\mathcal{T}}_\bullet)|_{\pi_\bullet^q(\widehat{T})} := \{\widehat{S}|_{\pi_\bullet^q(\widehat{T})} : \widehat{S} \in \widehat{\mathcal{S}}^{(p_1,\dots,p_d)}(\widehat{\mathcal{K}}_0, \widehat{\mathcal{T}}_\bullet)\}$ *implies that* $\widehat{v}|_{\widehat{T}} = (\widehat{I}_\bullet \widehat{v})|_{\widehat{T}}$.

(ii) *There exists a constant* $C > 0$ *such that for all* $\widehat{v} \in L^2([0,1]^d)$ *and all* $\widehat{T} \in \widehat{\mathcal{T}}_\bullet$*, there holds that*

$$\|\widehat{I}_\bullet \widehat{v}\|_{L^2(\widehat{T})} \leq C \|\widehat{v}\|_{L^2(\pi_\bullet^q(\widehat{T}))}, \tag{3.4.32}$$

where C *depends only on* d*,* $\widehat{\mathcal{T}}_0$*, and* $(p_1, \dots, p_d)$*.*

Proof. We prove the assertions in three steps.

Step 1: Remark 3.4.4 shows that for $\widehat{\beta} \in \widehat{\mathcal{B}}_\bullet$ with $|\mathrm{supp}(\widehat{\beta}) \cap \widehat{T}| > 0$, it holds that $\mathrm{supp}(\widehat{\beta}) \subseteq \pi_\bullet^q(\widehat{T})$. By (3.4.22), the same holds true for $\mathrm{Trunc}_\bullet(\widehat{\beta})$, i.e., $|\mathrm{supp}(\mathrm{Trunc}_\bullet(\widehat{\beta})) \cap \widehat{T}| > 0$ implies that $\mathrm{supp}(\widehat{\beta}) \subseteq \pi_\bullet^q(\widehat{T})$. This yields the identity

$$(\widehat{I}_\bullet \widehat{v})|_{\widehat{T}} = \sum_{\substack{\widehat{\beta} \in \widehat{\mathcal{B}}_\bullet \\ \mathrm{supp}(\widehat{\beta}) \subseteq \pi_\bullet^q(\widehat{T})}} \int_{\widehat{T}_{\widehat{\beta}}} \widehat{\beta}^* \, \widehat{v} \, dt \; \mathrm{Trunc}_\bullet(\widehat{\beta})|_{\widehat{T}}.$$

Step 2: We prove (i). Let $\widehat{S} \in \widehat{\mathcal{S}}^{(p_1,\dots,p_d)}(\widehat{\mathcal{K}}_0, \widehat{\mathcal{T}}_\bullet)$ such that $\widehat{v}|_{\pi_\bullet^q(\widehat{T})} = \widehat{S}|_{\pi_\bullet^q(\widehat{T})}$. With Step 1 and the fact that $\widehat{T}_{\widehat{\beta}} \subseteq \mathrm{supp}(\widehat{\beta})$, we see that

$$(\widehat{I}_\bullet \widehat{v})|_{\widehat{T}} = \sum_{\substack{\widehat{\beta} \in \widehat{\mathcal{B}}_\bullet \\ \mathrm{supp}(\widehat{\beta}) \subseteq \pi_\bullet^q(\widehat{T})}} \int_{\widehat{T}_{\widehat{\beta}}} \widehat{\beta}^* \, \widehat{S} \, dt \; \mathrm{Trunc}_\bullet(\widehat{\beta})|_{\widehat{T}} = (\widehat{I}_\bullet \widehat{S})|_{\widehat{T}}.$$

According to [SM16, Theorem 4], $\widehat{I}_\bullet$ is a global projection in the sense that $\widehat{S} \in \widehat{\mathcal{S}}^{(p_1,\dots,p_d)}(\widehat{\mathcal{K}}_0, \widehat{\mathcal{T}}_\bullet)$ implies that $\widehat{I}_\bullet \widehat{S} = \widehat{S}$. Thus, we conclude that

$$(\widehat{I}_\bullet \widehat{v})|_{\widehat{T}} = (\widehat{I}_\bullet \widehat{S})|_{\widehat{T}} = \widehat{S}|_{\widehat{T}} = \widehat{v}|_{\widehat{T}}.$$

Step 3: We prove (ii). Step 1 and the triangle inequality prove that

$$\|\widehat{I}_\bullet \widehat{v}\|_{L^2(\widehat{T})} \leq \sum_{\substack{\widehat{\beta} \in \widehat{\mathcal{B}}_\bullet \\ \mathrm{supp}(\widehat{\beta}) \subseteq \pi_\bullet^q(\widehat{T})}} \|\widehat{\beta}^*\|_{L^2(\widehat{T}_{\widehat{\beta}})} \|\widehat{v}\|_{L^2(\widehat{T}_{\widehat{\beta}})} \|\mathrm{Trunc}_\bullet(\widehat{\beta})\|_{L^2(\widehat{T})}.$$

This and the fact that $\widehat{T}_{\widehat{\beta}} \subseteq \mathrm{supp}(\widehat{\beta})$ yield that

$$\|\widehat{I}_\bullet \widehat{v}\|_{L^2(\widehat{T})} \leq \|\widehat{v}\|_{L^2(\pi_\bullet^q(\widehat{T}))} \sum_{\substack{\widehat{\beta} \in \widehat{\mathcal{B}}_\bullet \\ \mathrm{supp}(\widehat{\beta}) \subseteq \pi_\bullet^q(\widehat{T})}} \|\widehat{\beta}^*\|_{L^2(\widehat{T}_{\widehat{\beta}})} \|\mathrm{Trunc}_\bullet(\widehat{\beta})\|_{L^2(\widehat{T})}. \tag{3.4.33}$$

We consider the set $\{\widehat{\beta} \in \widehat{\mathcal{B}}_\bullet : \mathrm{supp}(\widehat{\beta}) \subseteq \pi_\bullet^q(\widehat{T})\}$. Since the support of each basis function in $\widehat{\mathcal{B}}_\bullet$ consists of elements in $\widehat{\mathcal{T}}_\bullet$ (see (3.4.16)), this set is a subset of $\{\widehat{\beta} \in \widehat{\mathcal{B}}_\bullet : \exists \widehat{T}' \in \Pi_\bullet^q(\widehat{T}) \text{ with } \widehat{T}' \subseteq \mathrm{supp}(\widehat{\beta})\}$. Lemma 3.4.5 and admissibility of $\widehat{\mathcal{T}}_\bullet$ show that the number of elements in $\Pi_\bullet^q(\widehat{T})$ is uniformly bounded by a constant which depends only on d and $(p_1, \ldots, p_d)$. Therefore, Proposition 3.4.3 proves that also the cardinality of the latter set is uniformly bounded. Now, let $\widehat{\beta} \in \widehat{\mathcal{B}}_\bullet$ with $\operatorname{supp}(\widehat{\beta}) \subseteq \pi_\bullet^q(\widehat{T})$. (3.4.22), (3.4.30), and Lemma 3.2.1 (iv) prove that

$$\|\widehat{\beta}^*\|_{L^2(\widehat{T}_{\widehat{\beta}})} \|\mathrm{Trunc}_\bullet(\widehat{\beta})\|_{L^2(\widehat{T})} \leq |\widehat{T}_{\widehat{\beta}}|^{1/2}\, \|\widehat{\beta}^*\|_{L^\infty(\widehat{T}_{\widehat{\beta}})} |\widehat{T}|^{1/2} \lesssim |\widehat{T}_{\widehat{\beta}}|^{-1/2} |\widehat{T}|^{1/2}.$$

Note that $\widehat{T}_{\widehat{\beta}} \in \Pi_\bullet^q(\widehat{T})$, wherefore Lemma 3.4.5 in combination with admissibility yields that $|\widehat{T}_{\widehat{\beta}}|^{-1/2} |\widehat{T}|^{1/2} \lesssim 1$. Plugging everything into (3.4.33) concludes the proof. □

4 Finite Element Method

4.1 Introduction

In this chapter, we propose and investigate an adaptive finite element method with (rational) hierarchical splines for general second-order elliptic systems of partial differential equations (PDEs) in arbitrary dimension $d \geq 2$. We essentially present the results from the recent own work [GHP17].

4.1.1 State of the art

Due to the advent of *isogeometric analysis* (IGA), the spline-based *finite element method* (FEM) has become an active field of research in the last decade. The central idea of IGA is to use the same ansatz functions for the discretization of the PDE as for the representation of the problem geometry Ω in computer aided design (CAD); see [HCB05, CHB09, BBdVC$^+$06]. The CAD standard for spline representation in a multivariate setting relies on tensor-product splines. However, to allow for adaptive refinement, several extensions of the standard model have recently emerged, e.g., analysis-suitable T-splines [SLSH12, BdVBSV13], hierarchical splines [VGJS11, GJS12, KVVdZvB14], or LR-splines [DLP13, JKD14]; see also [JRK15, HKMP17] for a comparison of these approaches in the frame of FEM. All these concepts have been studied via numerical experiments. However, so far there exists only little literature concerning the thorough mathematical analysis of adaptive isogeometric finite element methods (IGAFEM): [BG16a] investigates an estimator reduction of an IGAFEM with certain hierarchical splines introduced in [BG16b]. [BG16c] investigates linear convergence of an IGAFEM with truncated hierarchical B-splines introduced in [GJS12]. In the continuation of the latter work [BG16c], [BGMP16] studies the corresponding mesh-refinement strategy together with some refinement related properties for the proof of optimal convergence. At the time the recent own work [GHP17], which will be treated in the current chapter, was written, the mathematical proof that the adaptive strategy of [BG16c] leads to optimal convergence rates, was still missing in the literature. During the review process of [GHP17], the preprint [BG17] filled this gap. Unlike our strategy from [GHP17], the algorithm of [BG16c] was designed for truncated hierarchical B-splines only and the use of hierarchical B-splines may lead to non-sparse Galerkin matrices. It is important to note that the procedure of truncation requires a specific construction that entails complicated supports of the basis functions, which are in general not even connected, and their use may produce an overhead with an adaptive strategy that cannot be neglected. So far, the adaptive algorithm of [BG16c] has not been investigated numerically. Further, their analysis is restricted to symmetric partial differential operators. For standard FEM with globally continuous piecewise polynomials, adaptivity is well understood; see, e.g., [Dör96, MNS00, BDD04, Ste07, CKNS08, FFP14] and [CFPP14] for milestones on convergence and optimal convergence rates.

4.1.2 Model problem

Let $\Omega \subset \mathbb{R}^d$ with $d \geq 2$ be a bounded Lipschitz domain as in [McL00, Definition 3.28]. We consider a general second-order linear system of PDEs with homogenous Dirichlet boundary condition

$$\begin{aligned} \mathfrak{P}u := -\sum_{i=1}^{d}\sum_{i'=1}^{d} \partial_i(A_{ii'}\partial_{i'}u) + \sum_{i=1}^{d} b_i\partial_i u + cu = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \Gamma := \partial\Omega, \end{aligned} \tag{4.1.1}$$

where the coefficients $A_{ii'}, b_i, c$ are functions from Ω into $\mathbb{R}^{D\times D}$ with some fixed dimension $D \geq 1$. We pose the following regularity assumptions on the coefficients: $A_{ii'}$ is Lipschitz continuous, i.e, $A_{ii'} \in W^{1,\infty}(\Omega)^{D\times D}$, and b_i as well as c are bounded, i.e., $b_i, c \in L^\infty(\Omega)^{D\times D}$. We use the abbreviations $\|A\|_{L^\infty(\Omega)} := \max_{i,i'\in\{1,\dots,d\}} \|A_{ii'}\|_{L^\infty(\Omega)}$, $\|A\|_{W^{1,\infty}(\Omega)} := \max_{i,i'\in\{1,\dots,d\}} \|A_{ii'}\|_{W^{1,\infty}(\Omega)}$ and $\|b\|_{L^\infty(\Omega)} := \max_{i\in\{1,\dots,d\}} \|b_i\|_{L^\infty(\Omega)}$. Moreover, we suppose that $A_{ii'}^\top = A_{i'i}$. We interpret $\mathfrak{P}$ in its weak form and define the corresponding bilinear form

$$\langle w\,,\,v\rangle_{\mathfrak{P}} := \int_\Omega \sum_{i=1}^{d}\sum_{i'=1}^{d}(A_{ii'}\partial_{i'}v)\cdot\partial_i w + \sum_{i=1}^{d}(b_i\partial_i v)\cdot w + (cv)\cdot w\,dx. \tag{4.1.2}$$

The bilinear form is continuous, i.e., it holds with $C_{\mathrm{cont}} := \|A\|_{L^\infty(\Omega)} + \|b\|_{L^\infty(\Omega)} + \|c\|_{L^\infty(\Omega)}$ that

$$\langle w\,,\,v\rangle_{\mathfrak{P}} \leq C_{\mathrm{cont}}\|w\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)} \text{ for all } v, w \in H^1(\Omega)^D. \tag{4.1.3}$$

Additionally, we suppose ellipticity of $\langle\cdot\,,\,\cdot\rangle_{\mathfrak{P}}$ on $H^1_0(\Omega)^D$, i.e.,

$$\langle v\,,\,v\rangle_{\mathfrak{P}} \geq C_{\mathrm{ell}}\|v\|^2_{H^1(\Omega)} \quad \text{for all } v \in H^1_0(\Omega)^D. \tag{4.1.4}$$

Note that, for scalar PDEs with $D = 1$, (4.1.4) is for instance satisfied if the matrix $A := (A_{ii'})_{i,i'=1}^d$ is uniformly positive definite and if the vector $b := (b_1, \dots, b_d) \in H(\mathrm{div}, \Omega)$ satisfies that $-\frac{1}{2}\,\mathrm{div}\, b + c \geq 0$ almost everywhere in Ω.

Overall, the boundary value problem (4.1.1) fits into the setting of the Lax–Milgram theorem. For arbitrary vector-valued $f \in L^2(\Omega)^D$, it therefore admits a unique solution $u \in H^1_0(\Omega)^D$ to the weak formulation

$$\langle u\,,\,v\rangle_{\mathfrak{P}} = \int_\Omega f\cdot v\,dx \quad \text{for all } v \in H^1_0(\Omega)^D. \tag{4.1.5}$$

We note that the additional regularity $A_{ii'} \in W^{1,\infty}(\Omega)^{D\times D}$ (instead of $A_{ii'} \in L^\infty(\Omega)^{D\times D}$) is only required for the well-posedness of the residual *a posteriori* error estimator; see Section 4.2.

4.1.3 Outline & Contributions

The remainder of this chapter is roughly organized as follows: Section 4.2 provides an abstract framework for adaptive mesh-refinement for conforming FEM for the model problem (4.1.1). Its main result is Theorem 4.2.7 which states optimal convergence behavior of the standard adaptive Algorithm 2.2.1 applied to the model problem at hand. In Section 4.4, a conforming FEM based on hierarchical splines is presented. Its main result is Theorem 4.4.6 which states that hierarchical splines fit into the framework of Section 4.2. The proofs of Theorem 4.2.7 and Theorem 4.4.6 are given in Section 4.3 and Section 4.5, respectively. Three numerical experiments in Section 4.6 underpin the theoretical results, but also demonstrate the limitations of hierarchical splines in the frame of adaptive FEM when the solution u exhibits edge singularities.

Sections 4.2–4.3

In more detail, the contribution of Section 4.2 can be paraphrased as follows: We formulate a concrete realization (Algorithm 4.2.6) of the abstract adaptive Algorithm 2.2.1 driven by some weighted-residual *a posteriori* error estimator (4.2.13) in the frame of conforming FEM. We formulate four assumptions (M1)–(M4) on the underlying meshes (Section 4.2.1), five assumptions (R1)–(R5) on the mesh-refinement (Section 4.2.2), six assumptions (S1)–(S6) on the FEM spaces (Section 4.2.3), and four assumptions (O1)–(O4) on the data approximation spaces (Section 4.2.5). First, these assumptions are sufficient to guarantee that the error estimator $\eta_\bullet$ associated with the FEM solution $U_\bullet \in \mathcal{X}_\bullet \subset H^1_0(\Omega)^D$ is efficient and reliable, i.e., there exist $C_{\rm eff}$, $C_{\rm rel} > 0$ such that

$$C_{\rm eff}^{-1}\,\eta_\bullet \le \inf_{V_\bullet \in \mathcal{X}_\bullet} \big(\|u - V_\bullet\|_{H^1(\Omega)} + {\rm osc}_\bullet(V_\bullet)\big) \le \|u - U_\bullet\|_{H^1(\Omega)} + {\rm osc}_\bullet(U_\bullet) \le C_{\rm rel}\,\eta_\bullet, \quad (4.1.6)$$

where ${\rm osc}_\bullet(\cdot)$ denotes certain data oscillation terms. Second, Theorem 4.2.7 states that Algorithm 4.2.6 leads to linear convergence with optimal rate as in Theorem 2.3.1. Section 4.3 is devoted to the proof of Theorem 4.2.7.

In explicit terms, we identify sufficient conditions of the underlying meshes, the local FEM spaces, as well as the employed (local) mesh-refinement rule which guarantee that the related residual *a posteriori* error estimator satisfies the axioms of adaptivity from Chapter 2. Although this framework is only exploited for IGAFEM with hierarchical splines, it is likely that it serves as a promising starting point to analyze different technologies for adaptive IGAFEM like (analysis-suitable) T-splines or LR-splines, as well as for other conforming discretizations like the virtual element method (VEM) from [BdVBC+13]. Indeed, for analysis-suitable T-splines, the refinement properties are already found in [MP15] for 2D resp. in [Mor16] for 3D.

Sections 4.4–4.6

Based on the definitions from Section 3.4, Section 4.4 defines hierarchical meshes and hierarchical splines on the physical domain Ω (Section 4.4.2), derives the canonical basis of the hierarchical spline space $\mathcal{X}_\bullet \subset H^1_0(\Omega)^D$ with Dirichlet boundary condition (Section 4.5.8), and introduces some local mesh-refinement rule which preserves admissibility

(Section 4.4.3). One crucial observation is that the new mesh-refinement strategy for hierarchical meshes (Algorithm 4.4.1) guarantees that the number of (truncated) hierarchical B-splines on each element as well as the number of active elements contained in the support of each (truncated) hierarchical B-spline is uniformly bounded; see Proposition 3.4.3. If one uses the strategy of [BG16c, BGMP16, BG17] instead, this property is not satisfied for hierarchical B-splines, but only for truncated hierarchical B-splines. In general, the latter have a smaller, but also more complicated and not necessarily connected support.

The main result of Section 4.4 is Theorem 4.4.6 which states that hierarchical splines together with the proposed local mesh-refinement strategy satisfy all assumptions of Section 4.2, so that Theorem 4.2.7 applies. Whereas the corresponding result of [BG16c, BG17] adapts the analysis of [CKNS08] and is thus restricted to symmetric problems, we exploit some recent ideas from [FFP14] in order to cover the non-symmetric case as well. Finally, Remark 4.4.7 extends Theorem 4.4.6 to rational hierarchical splines.

Technical contributions of general interest in Section 4.5, which is devoted to the proof of Theorem 4.4.6, include the following: We prove that a hierarchical mesh is admissible if and only if it can be obtained by the mesh-refinement strategy of Algorithm 4.4.1 (Proposition 4.4.2). Recall that admissible meshes also allow a simpler computation of truncated hierarchical B-splines in the sense that truncation simplifies considerably (Proposition 3.4.7). Together with some ideas from [SM16], we use this observation to define a Scott–Zhang type projector $J_\bullet : L^2(\Omega)^D \to \mathcal{X}_\bullet$ which is locally L^2- and H^1-stable and has a first-order approximation property (Section 4.5.10).

We conclude this part with three numerical examples in Section 4.6, where we also give a heuristic explanation for the observed rates for solutions with edge-singularity.

4.2 Axioms of adaptivity (revisited)

A similar version of the current section is already found in the recent own work [GHP17, Section 2]. The aim of it is to formulate an adaptive algorithm (Algorithm 4.2.6) for conforming FEM discretizations of our model problem (4.1.1), where adaptivity is driven by the weighted-residual *a posteriori* error estimator (4.2.13). We identify the crucial properties of the underlying meshes, the mesh-refinement, as well as the finite element spaces which ensure that the residual error estimator fits into the general framework of Chapter 2 and which hence guarantee optimal convergence behavior of the adaptive algorithm. The main result of this section is Theorem 4.2.7 which is proved in Section 4.3.

4.2.1 Meshes

Throughout, $\mathcal{T}_\bullet$ is a *mesh* of the bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ in the following sense:

- $\mathcal{T}_\bullet$ is a finite set of compact[1] Lipschitz domains;
- for all $T, T' \in \mathcal{T}_\bullet$ with $T \neq T'$, the intersection $T \cap T'$ has measure zero;
- $\overline{\Omega} = \bigcup_{T\in\mathcal{T}_\bullet} T$, i.e., $\mathcal{T}_\bullet$ is a partition of Ω.

[1] A compact Lipschitz domain is the closure of a bounded Lipschitz domain.

We suppose that there is a countably infinite set $\mathbb{T}$ of *admissible* meshes. In order to ease notation, we introduce for $\mathcal{T}_\bullet \in \mathbb{T}$ the corresponding *mesh-width function*

$$h_\bullet \in L^\infty(\Omega) \quad \text{with} \quad h_\bullet|_T = h_T := |T|^{1/d} \text{ for all } T \in \mathcal{T}_\bullet. \tag{4.2.1}$$

For $\omega \subseteq \overline{\Omega}$, we define the *patches of order* $q \in \mathbb{N}_0$ inductively by

$$\pi_\bullet^0(\omega) := \omega, \quad \pi_\bullet^q(\omega) := \bigcup \big\{ T \in \mathcal{T}_\bullet \,:\, T \cap \pi_\bullet^{q-1}(\omega) \neq \emptyset \big\}. \tag{4.2.2}$$

The corresponding set of elements is defined as

$$\Pi_\bullet^q(\omega) := \big\{ T \in \mathcal{T}_\bullet \,:\, T \subseteq \pi_\bullet^q(\omega) \big\}, \quad \text{i.e.,} \quad \pi_\bullet^q(\omega) = \bigcup \Pi_\bullet^q(\omega). \tag{4.2.3}$$

To abbreviate notation, we set $\pi_\bullet(\omega) := \pi_\bullet^1(\omega)$ and $\Pi_\bullet(\omega) := \Pi_\bullet^1(\omega)$. For $\mathcal{S} \subseteq \mathcal{T}_\bullet$, we define $\pi_\bullet^q(\mathcal{S}) := \pi_\bullet^q(\bigcup \mathcal{S})$ and $\Pi_\bullet^q(\mathcal{S}) := \Pi_\bullet^q(\bigcup \mathcal{S})$.

We suppose that there exist constants $C_{\rm locuni}, C_{\rm patch}, C_{\rm trace}, C_{\rm dual} > 0$ such that all meshes $\mathcal{T}_\bullet \in \mathbb{T}$ satisfy the following four properties (M1)–(M4):

(M1) Bounded element patch: For all $T \in \mathcal{T}_\bullet$, it holds that

$$\#\Pi_\bullet(T) \le C_{\rm patch},$$

i.e., the number of elements in a patch is uniformly bounded.

(M2) Local quasi-uniformity: For all $T \in \mathcal{T}_\bullet$, it holds that

$$h_T/h_{T'} \le C_{\rm locuni} \quad \text{for all } T' \in \Pi_\bullet(T),$$

i.e., neighboring elements have comparable size.

(M3) Trace inequality: For all $T \in \mathcal{T}_\bullet$ and all $v \in H^1(\Omega)$, it holds that

$$\|v\|_{L^2(\partial T)}^2 \le C_{\rm trace}\big(h_T^{-1}\|v\|_{L^2(T)}^2 + \|v\|_{L^2(T)}\|\nabla v\|_{L^2(T)}\big).$$

(M4) Local estimate in dual norm: For all $T \in \mathcal{T}_\bullet$ and all $w \in L^2(T)$, it holds that

$$h_T^{-1}\|w\|_{H^{-1}(T)} \le C_{\rm dual}\|w\|_{L^2(T)},$$

where $\|w\|_{H^{-1}(T)} = \sup\big\{ \int_T wv\,dx \,:\, v \in H_0^1(T) \wedge \|v\|_{H^1(T)} = 1 \big\}$.

Remark 4.2.1. *Actually, we will apply* (M3)–(M4) *for vector-valued* $v \in H^1(\Omega)^D$ *resp.* $w \in L^2(\Omega)^D$. *Indeed,* (M3)–(M4) *easily imply the corresponding higher-dimensional versions. Moreover, note that* (M4) *is only needed for the proof of efficiency for the estimator; see Theorem 2.3.1* (ii).

The following two propositions show that (M3)–(M4) are actually always satisfied. However, in general the multiplicative constants depend on the shape of the elements.

Proposition 4.2.2. *Let ω be an arbitrary d-dimensional bounded Lipschitz domain. Then, there exists a constant $C_{\text{trace}}(\omega) > 0$ such that for all $v \in H^1(\omega)$, it holds that*

$$\|v\|^2_{L^2(\partial\omega)} \leq C_{\text{trace}}(\omega)\big(|\omega|^{-1/d}\|v\|^2_{L^2(\omega)} + \|v\|_{L^2(\omega)}\|\nabla v\|_{L^2(\omega)}\big). \tag{4.2.4}$$

The constant $C_{\text{trace}}(\omega) > 0$ depends only on the shape of ω.

Proof. Without loss of generality, we assume that $|\omega| = 1$. The general case follows with a simple scaling argument. We prove the assertion in three steps.

Step 1: By definition of Lipschitz domains [McL00, Definition 3.28], the boundary $\partial\omega$ is locally the graph of Lipschitz functions. Formally, this means that there exist a finite set of open and bounded sets $\big\{W_j \,:\, j \in \{1,\dots,J\}\big\}$ with $\partial\omega \subset \bigcup_{j=1}^J W_j$ such that for all $j \in \{1,\dots,J\}$, $\omega \cap W_j = \omega_j \cap W_j$ for some set ω_j which is up to some rigid motion a Lipschitz hypograph. Neglecting the rigid motion, ω_j has the form $\omega_j = \big\{x \in \mathbb{R}^d \,:\, x_d < \zeta_j(x_1,\dots,x_{d-1})\big\}$ for some Lipschitz continuous mapping $\zeta_j : \mathbb{R}^{d-1} \to \mathbb{R}$. Note that Rademacher's theorem shows that each ζ_j is almost everywhere differentiable with bounded gradient $\nabla\zeta_j \in L^\infty(\mathbb{R}^{d-1})$. According to [McL00, page 97], the outer normal vector ν_j satisfies for almost all $x \in \partial\omega_j$ that

$$\nu_j(x) = (|\nabla\zeta_j(x_1,\dots,x_{d-1})|^2 + 1)^{-1/2}\begin{pmatrix}-\nabla\zeta_j(x_1,\dots,x_{d-1})\\ 1\end{pmatrix}.$$

In particular, this shows for arbitrary $o_j \in \mathbb{R}^d$ of the form $o_j = (0\dots,0,o_{j,d})$ that

$$\nu_j(x)\cdot(x-o_j) = (|\nabla\zeta_j(x_1,\dots,x_{d-1})|^2 + 1)^{-1/2}\left(-\nabla\zeta_j(x_1,\dots,x_{d-1})\cdot\begin{pmatrix}x_1\\ \vdots\\ x_{d-1}\end{pmatrix} + (x_d - o_{j,d})\right)$$

Hence, we can fix a sufficiently small $o_{j,d} < 0$ such that for almost all $x \in \partial\omega_j \cap W_j$, it holds that

$$\nu_j(x)\cdot(x-o_j) \geq \frac{-(\|\nabla\zeta_j\|_{L^\infty(\mathbb{R}^{d-1})}+1)\sup_{x\in\partial\omega_j\cap W_j}|x| - o_{j,d}}{(\|\nabla\zeta_j\|^2_{L^\infty(\mathbb{R}^{d-1})}+1)^{1/2}} := \varepsilon_j > 0.$$

Note that $o_{j,d}$ as well as ε_j depend only on ω. We set $\varepsilon := \min_{j\in\{1,\dots,J\}}\varepsilon_j > 0$.

Step 2: We come to the proof of (4.2.4). Let $\big\{\varphi_j \,:\, j \in \{1,\dots,J\}\big\}$ be a smooth partition of unity on $\partial\omega$ subject to $\big\{W_j \,:\, j\in\{1,\dots,J\}\big\}$, i.e., $\varphi_j \in C^\infty(\mathbb{R}^d)$, $\sum_{j=1}^J \varphi_j = 1$, and $\operatorname{supp}(\varphi_j) \subset W_j$ for all $j \in \{1,\dots,J\}$.

Step 1 shows that

$$\|v\|_{L^2(\partial\omega)} \leq \sum_{j=1}^J \|\varphi_j v\|_{L^2(\partial\omega_j)} \leq \varepsilon^{-1/2}\sum_{j=1}^J\Big(\int_{\partial\omega_j}\varphi_j(x)^2 v(x)^2\,(x-o_j)\cdot\nu_j(x)\,dx\Big)^{1/2}.$$

Next, we apply the divergence theorem [McL00, Theorem 3.34]

$$\begin{aligned}\int_{\partial\omega_j}\varphi_j(x)^2v(x)^2\,(x-o_j)\cdot\nu_j(x)\,dx &= \int_{\omega_j}\operatorname{div}(\varphi_j(x)^2v(x)^2(x-o_j))\,dx\\ &= 2\int_{\omega_j\cap W_j}\varphi_j(x)v(x)\nabla(\varphi_j(x)v(x))\cdot(x-o_j)\,dx + d\,\|\varphi_j v\|^2_{L^2(\omega_j\cap W_j)}\\ &\lesssim \|v\|_{L^2(\omega\cap W_j)}\|\nabla v\|_{L^2(\omega\cap W_j)} + \|v\|^2_{L^2(\omega\cap W_j)}.\end{aligned}$$

Altogether, we obtain that

$$\begin{aligned}\|v\|_{L^2(\partial\omega)}^2 &\lesssim \sum_{j=1}^{J}\Big(\|v\|_{L^2(\omega\cap W_j)}\|\nabla v\|_{L^2(\omega\cap W_j)} + \|v\|_{L^2(\omega\cap W_j)}^2\Big)\\ &\lesssim \|v\|_{L^2(\omega)}\|\nabla v\|_{L^2(\omega)} + \|v\|_{L^2(\omega)}^2,\end{aligned}$$

which concludes the proof. □

Proposition 4.2.3. *Let ω be an arbitrary d-dimensional bounded Lipschitz domain. Then, there exists a constant $C_{\rm dual}(\omega) > 0$ such that for all $w \in L^2(\omega)$, it holds that*

$$|\omega|^{-1/d}\|w\|_{H^{-1}(\omega)} \le C_{\rm dual}(\omega)\|w\|_{L^2(\omega)}. \tag{4.2.5}$$

The constant $C_{\rm dual}(\omega) > 0$ depends only on the shape of ω.

Proof. Without loss of generality, we assume that $|\omega| = 1$. The general case follows with a simple scaling argument. Let $v \in H_0^1(\omega)$. The Cauchy–Schwarz inequality proves that

$$\int_\omega wv\,dx \le \|w\|_{L^2(\omega)}\|v\|_{L^2(\omega)} \le \|w\|_{L^2(\omega)}\|v\|_{H^1(\omega)},$$

which concludes the proof. □

4.2.2 Mesh-refinement

For $\mathcal{T}_\bullet \in \mathbb{T}$ and an arbitrary set of marked elements $\mathcal{M}_\bullet \subseteq \mathcal{T}_\bullet$, we associate a corresponding *refinement* $\mathcal{T}_\circ := \texttt{refine}(\mathcal{T}_\bullet, \mathcal{M}_\bullet) \in \mathbb{T}$ with $\mathcal{M}_\bullet \subseteq \mathcal{T}_\bullet \setminus \mathcal{T}_\circ$, i.e., at least the marked elements are refined. Moreover, we suppose for the cardinalities that $\#\mathcal{T}_\bullet < \#\mathcal{T}_\circ$ if $\mathcal{M}_\bullet \neq \emptyset$ and $\mathcal{T}_\circ = \mathcal{T}_\bullet$ else. We define $\texttt{refine}(\mathcal{T}_\bullet)$ as the set of all $\mathcal{T}_\circ$ such that there exist meshes $\mathcal{T}_{(0)}, \dots, \mathcal{T}_{(J)}$ and marked elements $\mathcal{M}_{(0)}, \dots, \mathcal{M}_{(J-1)}$ with $\mathcal{T}_\circ = \mathcal{T}_{(J)} = \texttt{refine}(\mathcal{T}_{(J-1)}, \mathcal{M}_{(J-1)}), \dots, \mathcal{T}_{(1)} = \texttt{refine}(\mathcal{T}_{(0)}, \mathcal{M}_{(0)})$ and $\mathcal{T}_{(0)} = \mathcal{T}_\bullet$. We assume that there exists a fixed initial mesh $\mathcal{T}_0 \in \mathbb{T}$ with $\mathbb{T} = \texttt{refine}(\mathcal{T}_0)$.

We suppose that there exist $C_{\rm son} \ge 2$ and $0 < \rho_{\rm son} < 1$ such that all meshes $\mathcal{T}_\bullet \in \mathbb{T}$ satisfy for arbitrary marked elements $\mathcal{M}_\bullet \subseteq \mathcal{T}_\bullet$ with corresponding refinement $\mathcal{T}_\circ := \texttt{refine}(\mathcal{T}_\bullet, \mathcal{M}_\bullet)$, the following elementary properties (R1)–(R3):

(R1) Son estimate: It holds that

$$\#\mathcal{T}_\circ \le C_{\rm son}\,\#\mathcal{T}_\bullet,$$

i.e., one step of refinement leads to a bounded increase of elements.

(R2) Father is union of sons: For all $T \in \mathcal{T}_\bullet$, it holds that

$$T = \bigcup\{T' \in \mathcal{T}_\circ \,:\, T' \subseteq T\},$$

i.e., each element T is the union of its successors.

(R3) Reduction of sons: For all $T \in \mathcal{T}_\bullet$, it holds that

$$|T'| \le \rho_{\rm son}\,|T| \quad \text{for all } T' \in \mathcal{T}_\circ \text{ with } T' \subsetneqq T,$$

i.e., successors are uniformly smaller than their father.

By induction and the definition of $\mathtt{refine}(\mathcal{T}_\bullet)$, one easily sees that (R2)–(R3) remain valid if $\mathcal{T}_\circ$ is an arbitrary mesh in $\mathtt{refine}(\mathcal{T}_\bullet)$. In particular, (R2)–(R3) imply that each refined element $T \in \mathcal{T}_\bullet \setminus \mathcal{T}_\circ$ is split into at least two sons, wherefore

$$\#(\mathcal{T}_\bullet \setminus \mathcal{T}_\circ) \le \#\mathcal{T}_\circ - \#\mathcal{T}_\bullet \quad \text{for all } \mathcal{T}_\circ \in \mathtt{refine}(\mathcal{T}_\bullet). \tag{4.2.6}$$

Besides (R1)–(R3), we suppose the following non-trivial requirements (R4)–(R5) with generic constants $C_{\rm clos}, C_{\rm over} > 0$:

(R4) Closure estimate: Let $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}_0}$ be an arbitrary sequence in $\mathbb{T}$ such that $\mathcal{T}_{\ell+1} = \mathtt{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$ with some $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ for all $\ell \in \mathbb{N}_0$. Then, for all $\ell \in \mathbb{N}_0$, there holds that

$$\#\mathcal{T}_\ell - \#\mathcal{T}_0 \le C_{\rm clos} \sum_{j=0}^{\ell-1} \#\mathcal{M}_j.$$

(R5) Overlay property: For all $\mathcal{T}_\bullet, \mathcal{T}_\star \in \mathbb{T}$, there exists a common refinement $\mathcal{T}_\circ \in \mathtt{refine}(\mathcal{T}_\bullet) \cap \mathtt{refine}(\mathcal{T}_\star)$ which satisfies the overlay estimate

$$\#\mathcal{T}_\circ \le C_{\rm over}(\#\mathcal{T}_\star - \#\mathcal{T}_0) + \#\mathcal{T}_\bullet.$$

4.2.3 Finite element space

With each $\mathcal{T}_\bullet \in \mathbb{T}$, we associate a finite dimensional space of vector valued functions

$$\mathcal{X}_\bullet \subset \big\{v \in H^1_0(\Omega)^D \,:\, v|_T \in H^2(T)^D \text{ for all } T \in \mathcal{T}_\bullet\big\}. \tag{4.2.7}$$

Let $U_\bullet \in \mathcal{X}_\bullet$ be the corresponding Galerkin approximation to the solution $u \in H^1_0(\Omega)^D$, i.e.,

$$\langle U_\bullet\,,\, V_\bullet\rangle_{\mathfrak{P}} = \int_\Omega f \cdot V_\bullet \, dx \quad \text{for all } V_\bullet \in \mathcal{X}_\bullet. \tag{4.2.8}$$

We note the Galerkin orthogonality

$$\langle u - U_\bullet\,,\, V_\bullet\rangle_{\mathfrak{P}} = 0 \quad \text{for all } V_\bullet \in \mathcal{X}_\bullet, \tag{4.2.9}$$

as well as the resulting Céa type quasi-optimality

$$\|u - U_\bullet\|_{H^1(\Omega)} \le C_{\text{Céa}} \min_{V_\bullet \in \mathcal{X}_\bullet} \|u - V_\bullet\|_{H^1(\Omega)} \quad \text{with} \quad C_{\text{Céa}} := \frac{C_{\rm cont}}{C_{\rm ell}}. \tag{4.2.10}$$

We suppose that there exist constants $C_{\rm inv} > 0$ and $q_{\rm loc}, q_{\rm proj} \in \mathbb{N}_0$ such that the following properties (S1)–(S3) hold for all $\mathcal{T}_\bullet \in \mathbb{T}$:

(S1) Inverse inequality: For all $j,k \in \{0,1,2\}$ with $k \le j$, all $V_\bullet \in \mathcal{X}_\bullet$ and all $T \in \mathcal{T}_\bullet$, it holds that

$$h_T^{(j-k)} \|V_\bullet\|_{H^j(T)} \le C_{\rm inv} \, \|V_\bullet\|_{H^k(T)}.$$

(S2) Nestedness: For all $\mathcal{T}_\circ \in \texttt{refine}(\mathcal{T}_\bullet)$, it holds that

$$\mathcal{X}_\bullet \subseteq \mathcal{X}_\circ.$$

(S3) Local domain of definition: For all $\mathcal{T}_\circ \in \texttt{refine}(\mathcal{T}_\bullet)$, $T \in \mathcal{T}_\bullet \setminus \Pi_\bullet^{q_{\rm loc}}(\mathcal{T}_\bullet \setminus \mathcal{T}_\circ) \subseteq \mathcal{T}_\bullet \cap \mathcal{T}_\circ$, and $V_\circ \in \mathcal{X}_\circ$, it holds that

$$V_\circ|_{\pi_\bullet^{q_{\rm proj}}(T)} \in \big\{ V_\bullet|_{\pi_\bullet^{q_{\rm proj}}(T)} \,:\, V_\bullet \in \mathcal{X}_\bullet \big\}.$$

Besides (S1)–(S3), we suppose that there exist $C_{\rm sz} > 0$ as well as $q_{\rm sz} \in \mathbb{N}_0$ such that for all $\mathcal{T}_\bullet \in \mathbb{T}$, there exists a Scott–Zhang type projector $J_\bullet : H_0^1(\Omega)^D \to \mathcal{X}_\bullet$ with the following properties (S4)–(S6):

(S4) Local projection property: Let $q_{\rm proj} \in \mathbb{N}_0$ from (S3). For all $v \in H_0^1(\Omega)^D$ and $T \in \mathcal{T}_\bullet$, it holds that

$$(J_\bullet v)|_T = v|_T \quad \text{if } v|_{\pi_\bullet^{q_{\rm proj}}(T)} \in \big\{ V_\bullet|_{\pi_\bullet^{q_{\rm proj}}(T)} \,:\, V_\bullet \in \mathcal{X}_\bullet \big\}.$$

(S5) Local $\boldsymbol{L^2}$-approximation property: For all $T \in \mathcal{T}_\bullet$ and all $v \in H_0^1(\Omega)^D$, it holds that

$$\|(1 - J_\bullet)v\|_{L^2(T)} \le C_{\rm sz} \, h_T \, \|v\|_{H^1(\pi_\bullet^{q_{\rm sz}}(T))}.$$

(S6) Local $\boldsymbol{H^1}$-stability: For all $T \in \mathcal{T}_\bullet$ and $v \in H_0^1(\Omega)^D$, it holds that

$$\|\nabla J_\bullet v\|_{L^2(T)} \le C_{\rm sz} \|v\|_{H^1(\pi_\bullet^{q_{\rm sz}}(T))}.$$

4.2.4 Error estimator

Let $\mathcal{T}_\bullet \in \mathbb{T}$ and $T_1 \in \mathcal{T}_\bullet$. For almost every $x \in \partial T_1 \cap \Omega$, there exists a unique element $T_2 \in \mathcal{T}_\bullet$ with $x \in T_1 \cap T_2$. We denote the corresponding outer normal vectors by $\nu_1 = (\nu_{1,1}, \dots, \nu_{1,d})$ resp. $\nu_2 = (\nu_{2,1}, \dots, \nu_{2,d})$. With the notation

$$\mathfrak{D}_{\nu_1}(\cdot) := \sum_{i=1}^{d} \sum_{i'=1}^{d} \nu_{1,i} \, A_{ii'} \, \partial_{i'}(\cdot) \quad \text{resp.} \quad \mathfrak{D}_{\nu_2}(\cdot) := \sum_{i=1}^{d} \sum_{i'=1}^{d} \nu_{2,i} \, A_{ii'} \, \partial_{i'}(\cdot), \tag{4.2.11}$$

we define the *normal jump* as

$$[\mathfrak{D}_\nu U_\bullet](x) := (\mathfrak{D}_{\nu_1} U_\bullet|_{T_1})(x) + (\mathfrak{D}_{\nu_2} U_\bullet|_{T_2})(x). \tag{4.2.12}$$

With this definition, we employ the weighted-residual *a posteriori* error estimator

$$\eta_\bullet := \eta_\bullet(\mathcal{T}_\bullet) \quad \text{with} \quad \eta_\bullet(\mathcal{S})^2 := \sum_{T\in\mathcal{S}} \eta_\bullet(T)^2 \text{ for all } \mathcal{S} \subseteq \mathcal{T}_\bullet, \tag{4.2.13a}$$

where, for all $T \in \mathcal{T}_\bullet$, the local refinement indicators read

$$\eta_\bullet(T)^2 := h_T^2 \|f - \mathfrak{P}U_\bullet\|_{L^2(T)}^2 + h_T \|[\mathfrak{D}_\nu U_\bullet]\|_{L^2(\partial T\cap\Omega)}^2. \tag{4.2.13b}$$

We refer, e.g., to the monographs [AO00, Ver13] for the analysis of the residual *a posteriori* error estimator (4.2.13) in the frame of standard FEM with piecewise polynomials of fixed order.

Remark 4.2.4. *If $\mathcal{X}_\bullet \subset C^1(\overline{\Omega})$, then the jump contributions in* (4.2.13) *vanish and $\eta_\bullet(T)$ consists only of the volume residual; see [BG16c] in the frame of IGAFEM.*

4.2.5 Data oscillations

The definition of the data oscillations corresponding to the residual error estimator (4.2.13) requires some further notation. Let $\mathcal{P}(\Omega) \subset H^1(\Omega)^D$ be a fixed discrete subspace. We suppose that there exists $C'_{\rm inv} > 0$ such that the following property (O1) holds for all $\mathcal{T}_\bullet \in \mathbb{T}$:

(O1) Inverse inequality in dual norm: For all $W \in \mathcal{P}(\Omega)$, and $T \in \mathcal{T}_\bullet$, it holds that

$$h_T\|W\|_{L^2(T)} \le C'_{\rm inv}\,\|W\|_{H^{-1}(T)},$$

where $\|W\|_{H^{-1}(T)}^2 = \sum_{j=1}^D \|W_j\|_{H^{-1}(T)}^2$ and $\|W_j\|_{H^{-1}(T)} = \sup\big\{\int_T W_j v\,dx \;:\; v \in H_0^1(T) \wedge \|v\|_{H^1(T)} = 1\big\}$.

Besides (O1), we suppose that there exists $C_{\rm lift} > 0$ such that for all $\mathcal{T}_\bullet \in \mathbb{T}$ and all $T, T' \in \mathcal{T}_\bullet$ with $(d-1)$-dimensional intersection $E := T \cap T'$, there exists an operator $L_{\bullet,E} : \big\{W|_E \;:\; W \in \mathcal{P}(\Omega)\big\} \to H_0^1(T\cup T')^D$ with the following properties (O2)–(O4):

(O2) Lifting inequality: For all $W \in \mathcal{P}(\Omega)$, it holds that

$$\int_E W\cdot W\,dx \le C_{\rm lift}\int_E L_{\bullet,E}(W|_E)\cdot W\,dx.$$

(O3) L^2-control: For all $W \in \mathcal{P}(\Omega)$, it holds that

$$\|L_{\bullet,E}(W|_E)\|_{L^2(T\cup T')} \le C_{\rm lift}(h_T^{1/2} + h_{T'}^{1/2})\|W\|_{L^2(E)}.$$

(O4) H^1-control: For all $W \in \mathcal{P}(\Omega)$, it holds that

$$\|\nabla L_{\bullet,E}(W|_E)\|_{L^2(T\cup T')} \le C_{\rm lift}(h_T^{-1/2} + h_{T'}^{-1/2})\|W\|_{L^2(E)}.$$

Let $\mathcal{T}_\bullet \in \mathbb{T}$. For $T \in \mathcal{T}_\bullet$, we define the L^2-orthogonal projection $P_{\bullet,T} : L^2(T) \to \{W|_T : W \in \mathcal{P}(\Omega)\}$. For an interior edge $E \in \mathcal{E}_{\bullet,T} := \{T \cap T' : T' \in \mathcal{T}_\bullet \wedge \dim(T \cap T') = d-1\}$, where $\dim(\cdot)$ denotes the dimension, we define the L^2-orthogonal projection $P_{\bullet,E} : L^2(E) \to \{W|_E : W \in \mathcal{P}(\Omega)\}$. For $V_\bullet \in \mathcal{X}_\bullet$, we define the corresponding *oscillations*

$$\mathrm{osc}_\bullet(V_\bullet) := \mathrm{osc}_\bullet(V_\bullet, \mathcal{T}_\bullet) \quad \text{with} \quad \mathrm{osc}_\bullet(V_\bullet, \mathcal{S})^2 := \sum_{T\in\mathcal{S}} \mathrm{osc}_\bullet(V_\bullet, T)^2 \text{ for all } \mathcal{S} \subseteq \mathcal{T}_\bullet, \tag{4.2.14a}$$

where, for all $T \in \mathcal{T}_\bullet$, the local oscillations read

$$\begin{aligned}\mathrm{osc}_\bullet(V_\bullet, T)^2 &:= h_T^2 \|(1 - P_{\bullet,T})(f - \mathfrak{P}U_\bullet)\|_{L^2(T)}^2 \\ &\quad + \sum_{E\in\mathcal{E}_{\bullet,T}} h_T \|(1 - P_{\bullet,E})[\mathfrak{D}_\nu U_\bullet]\|_{L^2(E)}^2.\end{aligned} \tag{4.2.14b}$$

We refer, e.g., to [NV11] for the analysis of oscillations in the frame of standard FEM with piecewise polynomials of fixed order.

Remark 4.2.5. *If $\mathcal{X}_\bullet \subset C^1(\overline{\Omega})$, then the jump contributions in (4.2.14) vanish and $\mathrm{osc}_\bullet(V_\bullet, T)$ consists only of the volume oscillations; see [BG16c] in the frame of IGAFEM.*

4.2.6 Adaptive algorithm

We consider the following concrete realization of the abstract Algorithm 2.2.1.

Algorithm 4.2.6. ***Input:*** *Dörfler parameter $\theta \in (0, 1]$ and marking constant $C_{\min} \in [1, \infty]$.*
Loop: *For each $\ell = 0, 1, 2, \ldots,$ iterate the following steps:*

(i) *Compute Galerkin approximation $U_\ell \in \mathcal{X}_\ell$.*

(ii) *Compute refinement indicators $\eta_\ell(T)$ for all elements $T \in \mathcal{T}_\ell$.*

(iii) *Determine a set of marked elements $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ which has up to the multiplicative constant $C_{\min}$ minimal cardinality, such that the following Dörfler marking is satisfied*

$$\theta\, \eta_\ell^2 \le \eta_\ell(\mathcal{M}_\ell)^2. \tag{4.2.15}$$

(iv) *Generate refined mesh $\mathcal{T}_{\ell+1} := \mathtt{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$.*

Output: *Refined meshes $\mathcal{T}_\ell$ and corresponding Galerkin approximations U_ℓ with error estimators η_ℓ for all $\ell \in \mathbb{N}_0$.*

4.2.7 Optimal convergence

With μ from Section 2.2.1 defined as cardinality #, we recall the definitions of Chapter 2

$$\mathbb{T}(N) := \{\mathcal{T}_\bullet \in \mathbb{T} : \#\mathcal{T}_\bullet - \#\mathcal{T}_0 \le N\} \quad \text{for all } N \in \mathbb{N}_0, \tag{4.2.16}$$

and for all $s > 0$

$$C_{\mathrm{approx}}(s) := \sup_{N\in\mathbb{N}_0} \min_{\mathcal{T}_\bullet\in\mathbb{T}(N)} (N+1)^s\, \eta_\bullet \in [0, \infty]. \tag{4.2.17}$$

We say that the solution $u \in H^1_0(\Omega)^D$ lies in the *approximation class s with respect to the estimator* if

$$\|u\|_{\mathbb{A}_s^{\mathrm{est}}} := C_{\mathrm{approx}}(s) < \infty. \tag{4.2.18}$$

Further, we say that it lies in the *approximation class s with respect to the minimal total error* if

$$\|u\|_{\mathbb{A}_s^{\mathrm{tot}}} := \sup_{N\in\mathbb{N}_0} \Big(\min_{\mathcal{T}_\bullet\in\mathbb{T}(N)} (N+1)^s \inf_{V_\bullet\in\mathcal{X}_\bullet} \big(\|u-V_\bullet\|_{H^1(\Omega)} + \mathrm{osc}_\bullet(V_\bullet)\big)\Big) < \infty. \tag{4.2.19}$$

By definition, $\|u\|_{\mathbb{A}_s^{\mathrm{est}}} < \infty$ resp. $\|u\|_{\mathbb{A}_s^{\mathrm{tot}}} < \infty$ implies that the error estimator $\eta_\bullet$ resp. the *minimal total error* on the optimal meshes $\mathcal{T}_\bullet$ decays at least with rate $\mathcal{O}\big((\#\mathcal{T}_\bullet)^{-s}\big)$. The following main theorem states that each possible rate $s > 0$ is in fact realized by Algorithm 4.2.6. The proof is given in Section 4.3 and is also found in [GHP17, Section 4]. It essentially follows from its abstract counterpart Theorem 2.3.1 by verifying the axioms of Section 2.3. For piecewise polynomials on shape-regular triangulations of a polyhedral domain Ω, optimal convergence was already proved in [CKNS08] for symmetric $\mathfrak{P}$ resp. in [FFP14] for non-symmetric $\mathfrak{P}$.

Theorem 4.2.7. *Let $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}_0}$ be the sequence of meshes generated by Algorithm 4.2.6. Then, there hold:*

(i) *Suppose (M1), (M3) and (S5)–(S6). Then, the residual error estimator satisfies reliability, i.e., there exists a constant $C_{\mathrm{rel}} > 0$ such that*

$$\|u-U_\bullet\|_{H^1(\Omega)} + \mathrm{osc}_\bullet \le C_{\mathrm{rel}}\eta_\bullet \quad \textit{for all } \mathcal{T}_\bullet \in \mathbb{T}. \tag{4.2.20}$$

(ii) *Suppose (M1)–(M4), (S1), and (O1)–(O4). Then, the residual error estimator satisfies efficiency, i.e., there exists a constant $C_{\mathrm{eff}} > 0$ such that*

$$C_{\mathrm{eff}}^{-1}\eta_\bullet \le \inf_{V_\bullet\in\mathcal{X}_\bullet} \big(\|u-V_\bullet\|_{H^1(\Omega)} + \mathrm{osc}_\bullet(V_\bullet)\big) \quad \textit{for all } \mathcal{T}_\bullet \in \mathbb{T}. \tag{4.2.21}$$

(iii) *Suppose (M1)–(M3), (R2)–(R3), (S1)–(S2), and (S5)–(S6). Then, for arbitrary $0 < \theta \le 1$ and $C_{\mathrm{min}} \in [1,\infty]$, the residual error estimator converges linearly, i.e., there exist constants $0 < \rho_{\mathrm{lin}} < 1$ and $C_{\mathrm{lin}} \ge 1$ such that*

$$\eta_{\ell+j}^2 \le C_{\mathrm{lin}}\rho_{\mathrm{lin}}^j \eta_\ell^2 \quad \textit{for all } j,\ell \in \mathbb{N}_0. \tag{4.2.22}$$

(iv) *Suppose (M1)–(M3), (R1)–(R5), and (S1)–(S6). Then, there exists a constant $0 < \theta_{\mathrm{opt}} \le 1$ such that for all $0 < \theta < \theta_{\mathrm{opt}}$ and $C_{\mathrm{min}} \in [1,\infty)$, the estimator converges at optimal rate, i.e., for all $s > 0$ there exist constants $c_{\mathrm{opt}}, C_{\mathrm{opt}} > 0$ such that*

$$c_{\mathrm{opt}}\|u\|_{\mathbb{A}_s^{\mathrm{est}}} \le \sup_{\ell\in\mathbb{N}_0} (\#\mathcal{T}_\ell - \#\mathcal{T}_0 + 1)^s \,\eta_\ell \le C_{\mathrm{opt}}\|u\|_{\mathbb{A}_s^{\mathrm{est}}}, \tag{4.2.23}$$

where the lower bound requires only (R1) to hold.

All involved constants $C_{\rm rel}, C_{\rm eff}, C_{\rm lin}, q_{\rm lin}, \theta_{\rm opt}$*, and* $C_{\rm opt}$ *depend only on the assumptions made as well as the dimensions* d, D*, the coefficients of the differential operator* $\mathfrak{P}$ *and* $\operatorname{diam}(\Omega)$*, where* $C_{\rm lin}, \rho_{\rm lin}$ *depend additionally on* θ *and the sequence* $(U_\ell)_{\ell\in\mathbb{N}_0}$*, and* $C_{\rm opt}$ *depends furthermore on* $C_{\rm min}$ *and* s*. The constant* $c_{\rm opt}$ *depends only on* $C_{\rm son}, \#\mathcal{T}_0$*,* s*, and if there exists* ℓ_0 *with* $\eta_{\ell_0} = 0$ *also on* ℓ_0 *and* η_0*.*

Remark 4.2.8. *If the assumptions of Theorem 4.2.7 (i)–(ii) are satisfied, there holds in particular that*

$$C_{\rm eff}^{-1}\|u\|_{\mathbb{A}_s^{\rm est}} \le \|u\|_{\mathbb{A}_s^{\rm tot}} \le C_{\rm rel}\|u\|_{\mathbb{A}_s^{\rm est}} \quad \textit{for all } s>0. \tag{4.2.24}$$

Remark 4.2.9. *If the bilinear form* $\langle\cdot\,,\,\cdot\rangle_{\mathfrak{P}}$ *is symmetric, then* $C_{\rm lin}$*,* $\rho_{\rm lin}$*, and* $C_{\rm opt}$ *are independent of* $(U_\ell)_{\ell\in\mathbb{N}_0}$*; see Remark 4.3.3 below.*

Remark 4.2.10. *If* $\mathcal{X}_\bullet \subset C^1(\overline{\Omega})$*, all jump contributions vanish; see Remark 4.2.4 and Remark 4.2.5. In this case, the assumptions (O2)–(O4) are not necessary for the proof of (4.2.21).*

Remark 4.2.11. (a) *Under the assumption that* $\|h_\ell\|_{L^\infty(\Omega)} \to 0$ *as* $\ell\to\infty$*, one can show that* $\mathcal{X}_\infty := \overline{\bigcup_{\ell\in\mathbb{N}_0}\mathcal{X}_\ell} = H_0^1(\Omega)^D$*. To see this, recall that nestedness (S2) ensures that* $\bigcup_{\ell\in\mathbb{N}_0}\mathcal{X}_\ell$ *is a vector space and, in particular, convex. By Mazur's lemma (see, e.g., [Rud91, Theorem 3.12]), it is thus sufficient to show that* $\bigcup_{\ell\in\mathbb{N}_0}\mathcal{X}_\ell$ *is weakly dense in* $H_0^1(\Omega)^D$*. Let* $v\in H_0^1(\Omega)^D$*. The Banach–Alaoglu theorem (see, e.g., [Rud91, Theorem 3.15]) together with (M1) and (S5)–(S6) proves that each subsequence* $(J_{\ell_m}v)_{m\in\mathbb{N}_0}$ *admits a further subsequence* $(J_{\ell_{m_n}}v)_{n\in\mathbb{N}_0}$ *which is weakly convergent in* $H_0^1(\Omega)^D$ *towards some limit* $w\in H_0^1(\Omega)^D$*. The Rellich compactness theorem hence implies that* $\|w - J_{\ell_{m_n}}v\|_{L^2(\Omega)} \to 0$ *as* $n\to\infty$*. On the other hand, (S5) together with (M1), (R2)–(R3), and* $\|h_\ell\|_{L^\infty(\Omega)}\to 0$ *shows that* $\|v - J_\ell v\|_{L^2(\Omega)} \lesssim \|h_\ell\|_{L^\infty(\Omega)}\|v\|_{H^1(\Omega)} \to 0$ *as* $\ell\to\infty$*. Together with the uniqueness of limits, these two observations imply that* $v = w$*. Overall, each subsequence* $(J_{\ell_m}v)_{m\in\mathbb{N}_0}$ *of* $(J_\ell v)_{\ell\in\mathbb{N}}$ *admits a further subsequence* $(J_{\ell_{m_n}}v)_{n\in\mathbb{N}_0}$ *which converges weakly in* $H_0^1(\Omega)^D$ *to* v*. Basic calculus thus yields that* $J_\ell v \rightharpoonup v$ *weakly in* $H_0^1(\Omega)^D$ *as* $\ell\to\infty$*. This concludes the proof.*

(b) *The latter observation allows to follow the ideas of [BHP17] and to show that the adaptive algorithm yields convergence even if the bilinear form* $\langle\cdot\,,\,\cdot\rangle_{\mathfrak{P}}$ *is only elliptic up to some compact perturbation, provided that the continuous problem is well-posed. This includes, e.g., adaptive FEM for the Helmholtz equation. For details, the reader is referred to [BHP17].*

4.3 Proof of Theorem 4.2.7

In the following subsections, we prove Theorem 4.2.7. To prove (iii)–(iv), we just verify the abstract axioms from Section 2.3, which allows to apply Theorem 2.3.1. The perturbation $\varrho_{\bullet,\circ}$ is chosen as

$$\varrho_{\bullet,\circ} := C_\varrho\|U_\circ - U_\bullet\|_{H^1(\Omega)} \quad \text{for all } \mathcal{T}_\bullet\in\mathbb{T}, \mathcal{T}_\circ\in\mathtt{refine}(\mathcal{T}_\bullet), \tag{4.3.1}$$

with some constant $C_\varrho > 0$ which is fixed later in Section 4.3.3. To apply Theorem 2.3.1 (i), we additionally have to show that $\lim_{\ell\to\infty}\varrho_{\ell,\ell+1} = 0$. Finally, reliability (4.2.20) resp. efficiency (4.2.21) are treated explicitly in Section 4.3.7 resp. Section 4.3.8.

4.3.1 Convergence of perturbations

Nestedness (S2) ensures that $\mathcal{X}_\infty := \overline{\bigcup_{\ell\in\mathbb{N}_0} \mathcal{X}_\ell}$ is a closed subspace of $H^1_0(\Omega)^D$ and hence admits a unique Galerkin solution $U_\infty \in \mathcal{X}_\infty$. Note that U_ℓ is also a Galerkin approximation of U_∞. Hence, the Céa lemma (4.2.10) with u replaced by U_∞ proves that $\|U_\infty - U_\ell\|_{H^1(\Omega)} \to 0$ as $\ell \to \infty$. In particular, we obtain that $\lim_{\ell\to\infty} \|U_{\ell+1} - U_\ell\|_{H^1(\Omega)} = 0$.

4.3.2 Stability on non-refined elements (E1)

Similarly as in [CKNS08, Corollary 3.4], we show that the assumptions (M1)–(M3) and (S1)–(S2) imply stability (E1), i.e., the existence of $C_{\rm stab} \ge 1$ such that for all $\mathcal{T}_\bullet \in \mathbb{T}$, and all $\mathcal{T}_\circ \in \texttt{refine}(\mathcal{T}_\bullet)$, it holds that

$$|\eta_\circ(\mathcal{T}_\bullet \cap \mathcal{T}_\circ) - \eta_\bullet(\mathcal{T}_\bullet \cap \mathcal{T}_\circ)| \le C_{\rm stab} \|U_\circ - U_\bullet\|_{H^1(\Omega)}.$$

In Section 4.3.3, we will fix the constant $C_\varrho > 0$ for the perturbations (4.3.1) such that $C_{\rm stab} \le C_\varrho$. For $\mathcal{S} \subseteq \mathcal{T}_\circ$, we abbreviate

$$\alpha_{\bullet,\circ}(\mathcal{S})^2 := \Big(\sum_{T\in\mathcal{S}} h_T^2 \|\mathfrak{P}(U_\circ - U_\bullet)\|_{L^2(T)}^2 \Big)^{1/2} + \Big(\sum_{T\in\mathcal{S}} h_T \|[\mathfrak{D}_\nu(U_\circ - U_\bullet)]\|_{L^2(\partial T\cap\Omega)}^2 \Big)^{1/2}. \tag{4.3.2}$$

The inverse triangle inequality shows that

$$\begin{aligned}
&|\eta_\circ(\mathcal{T}_\bullet \cap \mathcal{T}_\circ) - \eta_\bullet(\mathcal{T}_\bullet \cap \mathcal{T}_\circ)|^2 \\
&\le \Big(\Big(\sum_{T\in\mathcal{T}_\bullet\cap\mathcal{T}_\circ} h_T^2 \|f - \mathfrak{P}U_\circ\|_{L^2(T)}^2 \Big)^{1/2} - \Big(\sum_{T\in\mathcal{T}_\bullet\cap\mathcal{T}_\circ} h_T^2 \|f - \mathfrak{P}U_\bullet\|_{L^2(T)}^2 \Big)^{1/2} \Big)^2 \\
&\quad + \Big(\Big(\sum_{T\in\mathcal{T}_\bullet\cap\mathcal{T}_\circ} h_T \|[\mathfrak{D}_\nu U_\circ]\|_{L^2(\partial T\cap\Omega)}^2 \Big)^{1/2} - \Big(\sum_{T\in\mathcal{T}_\bullet\cap\mathcal{T}_\circ} h_T \|[\mathfrak{D}_\nu U_\bullet]\|_{L^2(\partial T\cap\Omega)}^2 \Big)^{1/2} \Big)^2 \\
&\le \alpha_{\bullet,\circ}(\mathcal{T}_\bullet \cap \mathcal{T}_\circ)^2.
\end{aligned} \tag{4.3.3}$$

It remains to control the term $\alpha_{\bullet,\circ}(\mathcal{T}_\bullet \cap \mathcal{T}_\circ)$. We consider an arbitrary set $\mathcal{S} \subseteq \mathcal{T}_\circ$. We use the inverse triangle inequality and the inverse inequality (S1) in combination with nestedness (S2), to see that the volume residual part satisfies that

$$\Big(\sum_{T\in\mathcal{S}} h_T^2 \|\mathfrak{P}(U_\circ - U_\bullet)\|_{L^2(T)}^2 \Big)^{1/2} \lesssim \|U_\circ - U_\bullet\|_{H^1(\Omega)}.$$

To deal with the jump part of $\alpha_{\bullet,\circ}(\mathcal{S})$, let $T_1 \in \mathcal{S}$ be arbitrary, and $T_2 \in \mathcal{T}_\circ$ with $\dim(T_1 \cap T_2) = d-1$, where $\dim(\cdot)$ denotes the dimension. We set $E := T_1 \cap T_2 \in \mathcal{E}_{\circ,T_1}$. Note that the number of such elements T_2 is uniformly bounded due to (M1). Then, the definition of $[\cdot]$ and local quasi-uniformity (M2) show that

$$\begin{aligned}
h_{T_1} \|[\mathfrak{D}_\nu(U_\circ - U_\bullet)]\|_{L^2(E)}^2 &= h_{T_1} \|\mathfrak{D}_{\nu_1}(U_\circ - U_\bullet)|_{T_1} + \mathfrak{D}_{\nu_2}(U_\circ - U_\bullet)|_{T_2}\|_{L^2(E)}^2 \\
&\lesssim h_{T_1} \sum_{i=1}^d \|\partial_i(U_\circ - U_\bullet)|_{T_1}\|_{L^2(\partial T_1)}^2 + h_{T_2} \sum_{i=1}^d \|\partial_i(U_\circ - U_\bullet)|_{T_2}\|_{L^2(\partial T_2)}^2.
\end{aligned} \tag{4.3.4}$$

For $v = (v_1, \dots, v_D) \in H^1(\Omega)^D$, we abbreviate $\nabla v := (\nabla v_1, \dots, \nabla v_D) \in L^2(\Omega)^{D^2}$. To estimate the first summand, we apply the trace inequality (M3) and the inverse inequality (S1) in combination with nestedness (S2), and see for $i \in \{1, \dots, d\}$ that

$$\begin{aligned} h_{T_1} \|\partial_i (U_\circ - U_\bullet)|_{T_1}\|_{L^2(\partial T_1)}^2 &\lesssim \|\partial_i (U_\circ - U_\bullet)\|_{L^2(T_1)}^2 \\ &+ h_{T_1} \|\partial_i (U_\circ - U_\bullet)\|_{L^2(T_1)} \|\nabla (\partial_i (U_\circ - U_\bullet))\|_{L^2(T_1)} \lesssim \|U_\circ - U_\bullet\|_{H^1(T_1)}^2. \end{aligned} \tag{4.3.5}$$

The second summand of (4.3.4) is estimated similarly. Altogether, we have deduced that

$$\alpha_{\bullet,\circ}(\mathcal{S}) \leq C_{\rm stab} \, \|U_\circ - U_\bullet\|_{H^1(\Omega)}. \tag{4.3.6}$$

where the constant $C_{\rm stab}$ depends only on (M1)–(M3), (S1), as well as on $d, D, \|A\|_{W^{1,\infty}(\Omega)}$, $\|b\|_{L^\infty(\Omega)}$, $\|c\|_{L^\infty(\Omega)}$, and $\mathrm{diam}(\Omega)$. Thus, (4.3.3) concludes the proof of (E1).

4.3.3 Reduction on refined elements (E2)

Similarly as in [CKNS08, Corollary 3.4], we show that the assumptions (M1)–(M3), (R2)–(R3), and (S1)–(S2) imply reduction on refined elements (E2), i.e., the existence of $C_{\rm red} \geq 1$ and $0 < \rho_{\rm red} < 1$ such that all $\mathcal{T}_\bullet \in \mathbb{T}$ and all $\mathcal{T}_\circ \in \texttt{refine}(\mathcal{T}_\bullet)$ satisfy that

$$\eta_\circ(\mathcal{T}_\circ \setminus \mathcal{T}_\bullet)^2 \leq \rho_{\rm red} \, \eta_\bullet(\mathcal{T}_\bullet \setminus \mathcal{T}_\circ)^2 + C_{\rm red} \|U_\circ - U_\bullet\|_{H^1(\Omega)}^2.$$

With this, we can fix the constant $C_\varrho > 0$ for the perturbations (4.3.1) as

$$C_\varrho := \max(C_{\rm stab}, C_{\rm red}^{1/2}). \tag{4.3.7}$$

First, we apply the triangle inequality and the Young's inequality, and use the definition of $\alpha_{\bullet,\circ}(\mathcal{T}_\circ \setminus \mathcal{T}_\bullet)$ from (4.3.2) to see for arbitrary $\delta > 0$ that

$$\begin{aligned} \eta_\circ(\mathcal{T}_\circ \setminus \mathcal{T}_\bullet)^2 &\leq (1 + \delta^{-1}) \alpha_{\bullet,\circ}(\mathcal{T}_\circ \setminus \mathcal{T}_\bullet)^2 \\ &+ (1 + \delta) \Big(\sum_{T \in \mathcal{T}_\circ \setminus \mathcal{T}_\bullet} h_T^2 \|f - \mathfrak{P} U_\bullet\|_{L^2(T)}^2 + h_T \|[\mathfrak{D}_\nu U_\bullet]\|_{L^2(\partial T \cap \Omega)}^2 \Big). \end{aligned}$$

According to (4.3.6), there holds that $\alpha_\circ(U_\circ, U_\bullet) \lesssim \|U_\circ - U_\bullet\|_{H^1(\Omega)}$. To control the volume residual term, we use (R2)–(R3)

$$\begin{aligned} \sum_{T \in \mathcal{T}_\circ \setminus \mathcal{T}_\bullet} h_T^2 \|f - \mathfrak{P} U_\bullet\|_{L^2(T)}^2 &\leq \sum_{T' \in \mathcal{T}_\bullet \setminus \mathcal{T}_\circ} \rho_{\rm son}^{2/d} \, h_{T'}^2 \sum_{\substack{T \in \mathcal{T}_\circ \\ T \subset T'}} \|f - \mathfrak{P} U_\bullet\|_{L^2(T)}^2 \\ &= \rho_{\rm son}^{2/d} \sum_{T' \in \mathcal{T}_\bullet \setminus \mathcal{T}_\circ} h_{T'}^2 \|f - \mathfrak{P} U_\bullet\|_{L^2(T')}^2. \end{aligned}$$

With the same arguments, we can also estimate the jump term. Here, we additionally use the fact that $[\mathfrak{D}_\nu U_\bullet] = 0$ on $(\partial T \setminus \partial T') \cap \Omega$ for all sons $T \subsetneqq T'$ of an element $T' \in \mathcal{T}_\bullet$, which follows from $U_\bullet|_{T'} \in H^2(T')^D$. This gives

$$\begin{aligned} \sum_{T \in \mathcal{T}_\circ \setminus \mathcal{T}_\bullet} h_T \|[\mathfrak{D}_\nu U_\bullet]\|_{L^2(\partial T \cap \Omega)}^2 &\leq \sum_{T' \in \mathcal{T}_\bullet \setminus \mathcal{T}_\circ} \rho_{\rm son}^{1/d} h_{T'} \sum_{\substack{T \in \mathcal{T}_\circ \\ T \subset T'}} \|[\mathfrak{D}_\nu U_\bullet]\|_{L^2(\partial T \cap \Omega)}^2 \\ &= \rho_{\rm son}^{1/d} \sum_{T' \in \mathcal{T}_\bullet \setminus \mathcal{T}_\circ} \|[\mathfrak{D}_\nu U_\bullet]\|_{L^2(\partial T' \cap \Omega)}^2. \end{aligned}$$

Choosing $\delta > 0$ sufficiently small, we conclude the proof of (E2), where the constants $C_{\rm red}$ and $\rho_{\rm red}$ depend only on (M1)–(M3), (R2)–(R3), (S1) as well as on d, D, $\|A\|_{W^{1,\infty}(\Omega)}$, $\|b\|_{L^\infty(\Omega)}$, $\|c\|_{L^\infty(\Omega)}$, and $\mathrm{diam}(\Omega)$.

4.3.4 General quasi-orthogonality (E3)

According to Theorem 2.3.1 (i), Section 4.3.1, Section 4.3.2 and Section 4.3.3 already imply estimator convergence $\lim_{\ell\to\infty}\eta_\ell = 0$. Therefore, reliability (4.2.20), which will be proved in Section 4.3.7 below, implies error convergence $\lim_{\ell\to\infty}\|u - U_\ell\|_{H^1(\Omega)} = 0$. In particular, we obtain that $u \in \mathcal{X}_\infty = \overline{\bigcup_{\ell\in\mathbb{N}_0}\mathcal{X}_\ell}$. Similarly as in [FFP14, Proof of Theorem 4.1], we show that the latter inclusion $u \in \mathcal{X}_\infty$, reliability (4.2.20), and (S2) imply general quasi-orthogonality (E3), i.e., the existence of

$$0 \le \varepsilon_{\rm qo} < \sup_{\delta>0} \frac{1-(1+\delta)(1-(1-\rho_{\rm red})\theta)}{2+\delta^{-1}}, \tag{4.3.8}$$

and $C_{\rm qo} \ge 1$ such that

$$\sum_{j=\ell}^{\ell+N}(C_\varrho\|U_{j+1}-U_j\|_{H^1(\Omega)}^2 - \varepsilon_{\rm qo}\eta_j^2) \le C_{\rm qo}\eta_\ell^2 \quad \text{for all } \ell, N \in \mathbb{N}_0. \tag{4.3.9}$$

Recall that we already fixed the constant C_ϱ in (4.3.7). The key ingredient is provided by the following Lemma 4.3.2 which stems from [FFP14, Proposition 3.6]. There, the assertion is formulated in a more concrete setting. However, the generalization to Hilbert spaces is straightforward and is also found in [BHP17, Lemma 18]. The proof is only given for completeness and requires the assertion of the next lemma.

Lemma 4.3.1. *Let $\mathcal{H}$ be a Hilbert space with dual space $\mathcal{H}^*$, and let $(\mathcal{H}_\ell)_{\ell\in\mathbb{N}_0}$ be a nested sequence of subspaces with $\mathcal{H}_\ell \subseteq \mathcal{H}_{\ell+1} \subseteq \mathcal{H}$ for all $\ell \in \mathbb{N}_0$. Further, let $\mathfrak{B} : \mathcal{H} \to \mathcal{H}^*$ be a continuous linear operator which is elliptic, i.e., there exists a constant $C > 0$ such that*

$$\|y\|_{\mathcal{H}}^2 \le C\,\langle \mathfrak{B}y\,,\,y\rangle \quad \text{for all } y \in \mathcal{H}. \tag{4.3.10}$$

For given $F \in \mathcal{H}^*$, let $x \in \mathcal{H}$ denote the unique solution to

$$\mathfrak{B}x = F. \tag{4.3.11}$$

Suppose that $x \in \mathcal{H}_\infty := \overline{\bigcup_{\ell\in\mathbb{N}_0}\mathcal{H}_\ell}$, and let X_ℓ be the corresponding Galerkin approximations in $\mathcal{H}_\ell$. Then, the sequences $(e_\ell)_{\ell\in\mathbb{N}_0}$ and $(E_\ell)_{\ell\in\mathbb{N}_0}$ defined by

$$e_\ell := \begin{cases} \frac{x-X_\ell}{\|x-X_\ell\|_{\mathcal{H}}} & \text{for } x \ne X_\ell, \\ 0 & \text{else,} \end{cases} \quad \text{and} \quad E_\ell := \begin{cases} \frac{X_{\ell+1}-X_\ell}{\|X_{\ell+1}-X_\ell\|_{\mathcal{H}}} & \text{for } X_{\ell+1} \ne X_\ell, \\ 0 & \text{else.} \end{cases} \tag{4.3.12}$$

converge weakly to zero.

Proof. We only prove the assertion for the sequence $(e_\ell)_{\ell\in\mathbb{N}_0}$, the weak convergence of $(E_\ell)_{\ell\in\mathbb{N}_0}$ follows along the same lines. We show that every subsequence $(e_{\ell_j})_{j\in\mathbb{N}_0}$ has itself a subsequence $(e_{\ell_{j_k}})_{k\in\mathbb{N}_0}$ which converges weakly to zero, which concludes the proof. Boundedness $\|e_\ell\|_{\mathcal{H}} \le 1$ implies by the Banach–Alaoglu theorem the existence of $(e_{\ell_{j_k}})_{k\in\mathbb{N}_0}$ such that $e_{\ell_{j_k}} \rightharpoonup y$ for some $y \in \mathcal{H}$. It remains to show that $y = 0$. We apply Mazur's lemma, which states that closed convex sets are weakly closed, and obtain that $y \in \mathcal{H}_\infty$. Now, let $n \in \mathbb{N}_0$ and $Y_n \in \mathcal{H}_n$ arbitrary but fixed. Due to Galerkin orthogonality there holds that

$$0 = \langle \mathfrak{B}(x - X_{\ell_{j_k}})\,,\, Y_n\rangle \quad \text{for } \ell_{j_k} \ge n.$$

In particular, we derive that $\langle \mathfrak{B}e_{\ell_{j_k}}\,,\, Y_n\rangle = 0$. Therefore, weak convergence leads to $\langle \mathfrak{B}y\,,\, Y_n\rangle = 0$. By definition of $\mathcal{H}_\infty \ni y$ and since Y_n was arbitrary, this yields that $\langle \mathfrak{B}y\,,\, y\rangle = 0$. With ellipticity (4.3.10), we conclude that $y = 0$ and thus the proof. ☐

Lemma 4.3.2. *Under the assumptions of Lemma 4.3.1, we additionally suppose that the corresponding operator $\mathfrak{B}$ can be written as $\mathfrak{B} = \mathfrak{A} + \mathfrak{C}$ with continuous linear operators $\mathfrak{A}, \mathfrak{C} : \mathcal{H} \to \mathcal{H}^*$, where $\mathfrak{A}$ is symmetric, i.e.,*

$$\langle \mathfrak{A}y\,,\, z\rangle = \langle \mathfrak{A}z\,,\, y\rangle \quad \text{for all } y, z \in \mathcal{H}, \tag{4.3.13}$$

and $\mathfrak{C}$ is compact. We define

$$\|y\|_{\mathfrak{B}} := \langle \mathfrak{B}y\,,\, y\rangle^{1/2} \quad \text{for all } y \in \mathcal{H}. \tag{4.3.14}$$

Then, for all $0 < \delta < 1$, there exists an index $\ell_0 \in \mathbb{N}_0$ such that

$$\|x - X_{\ell+1}\|_{\mathfrak{B}}^2 + \|X_{\ell+1} - X_\ell\|_{\mathfrak{B}}^2 \le \frac{1}{1-\delta}\|x - X_\ell\|_{\mathfrak{B}}^2 \quad \text{for all } \ell \ge \ell_0. \tag{4.3.15}$$

Proof. Elementary algebra shows for all $\ell \in \mathbb{N}_0$ that

$$\begin{aligned}\|x - X_{\ell+1}\|_{\mathfrak{B}}^2 &+ \|X_{\ell+1} - X_\ell\|_{\mathfrak{B}}^2 + \langle \mathfrak{B}(x - X_{\ell+1})\,,\, X_{\ell+1} - X_\ell\rangle \\ &= \|x - X_\ell\|_{\mathfrak{B}}^2 - \langle \mathfrak{B}(X_{\ell+1} - X_\ell)\,,\, x - X_{\ell+1}\rangle.\end{aligned}$$

Note that the third summand vanishes due to Galerkin orthogonality and nestedness of the ansatz spaces, i.e.,

$$\|x - X_{\ell+1}\|_{\mathfrak{B}}^2 + \|X_{\ell+1} - X_\ell\|_{\mathfrak{B}}^2 = \|x - X_\ell\|_{\mathfrak{B}}^2 - \langle \mathfrak{B}(X_{\ell+1} - X_\ell)\,,\, x - X_{\ell+1}\rangle. \tag{4.3.16}$$

Note that, if $\mathfrak{B}$ was symmetric, the last term would vanish too. The difficulty comes with the non-symmetric part $\mathfrak{C}$. Exploiting the symmetry of $\mathfrak{A}$, we see that

$$\begin{aligned}\big|\langle \mathfrak{B}(X_{\ell+1} - X_\ell)\,,\, x - X_{\ell+1}\rangle\big| &= \big|\langle \mathfrak{A}(x - X_{\ell+1})\,,\, X_{\ell+1} - X_\ell\rangle + \langle \mathfrak{C}(X_{\ell+1} - X_\ell)\,,\, x - X_{\ell+1}\rangle\big| \\ &\le \big|\langle \mathfrak{B}(x - X_{\ell+1})\,,\, X_{\ell+1} - X_\ell\rangle\big| + \big|\langle \mathfrak{C}(x - X_{\ell+1}\,,\, X_{\ell+1} - X_\ell\rangle\big| \\ &\quad + \big|\langle \mathfrak{C}(X_{\ell+1} - X_\ell)\,,\, x - X_{\ell+1}\rangle\big|.\end{aligned}$$

Using again Galerkin orthogonality, we derive that

$$\begin{aligned}
\big|\langle\mathfrak{B}(X_{\ell+1}-X_\ell)\,,\,x-X_{\ell+1}\rangle\big| &\le \big|\langle\mathfrak{C}(x-X_{\ell+1}\,,\,X_{\ell+1}-X_\ell\rangle\big| + \big|\langle\mathfrak{C}(X_{\ell+1}-X_\ell)\,,\,x-X_{\ell+1}\rangle\big| \\
&\le \|\mathfrak{C}e_{\ell+1}\|_{\mathcal{H}^*}\|x-X_{\ell+1}\|_{\mathcal{H}}\|X_{\ell+1}-X_\ell\|_{\mathcal{H}} + \|\mathfrak{C}E_\ell\|_{\mathcal{H}^*}\|X_{\ell+1}-X_\ell\|_{\mathcal{H}}\|x-X_{\ell+1}\|_{\mathcal{H}} \\
&= \big(\|\mathfrak{C}e_{\ell+1}\|_{\mathcal{H}^*}+\|\mathfrak{C}E_\ell\|_{\mathcal{H}^*}\big)\,\|x-X_{\ell+1}\|_{\mathcal{H}}\|X_{\ell+1}-X_\ell\|_{\mathcal{H}}.
\end{aligned}$$

Recall that compact operators transfer weak convergence into strong convergence. Since $e_{\ell+1}, E_\ell \rightharpoonup 0$ as $\ell\to\infty$, we see that $\mathfrak{C}e_{\ell+1}, \mathfrak{C}E_\ell \to 0$ in $\mathcal{H}^*$ as $\ell\to\infty$. In particular, for all $\epsilon>0$, this provides some $\ell_0\in\mathbb{N}_0$ such that

$$\|\mathfrak{C}e_{\ell+1}\|_{\mathcal{H}^*}+\|\mathfrak{C}E_\ell\|_{\mathcal{H}^*} \le \epsilon \quad \text{for all } \ell\ge\ell_0. \tag{4.3.17}$$

From now on, let $\ell\ge\ell_0$. We obtain that

$$\big|\langle\mathfrak{B}(X_{\ell+1}-X_\ell)\,,\,x-X_{\ell+1}\rangle\big| \le \epsilon\,\|x-X_{\ell+1}\|_{\mathcal{H}}\|X_{\ell+1}-X_\ell\|_{\mathcal{H}}.$$

We plug this into (4.3.16) and use ellipticity (4.3.10) to see that

$$\begin{aligned}
\|x-X_{\ell+1}\|_{\mathfrak{B}}^2+\|X_{\ell+1}-X_\ell\|_{\mathfrak{B}}^2 &\le \|x-X_\ell\|_{\mathfrak{B}}^2+\epsilon\,\|x-X_{\ell+1}\|_{\mathcal{H}}\|X_{\ell+1}-X_\ell\|_{\mathcal{H}} \\
&\le \|x-X_\ell\|_{\mathfrak{B}}^2 + C\epsilon\,\big(\|x-X_{\ell+1}\|_{\mathfrak{B}}^2+\|X_{\ell+1}-X_\ell\|_{\mathfrak{B}}^2\big).
\end{aligned}$$

Choosing $\epsilon = C^{-1}\delta$ concludes the proof. □

We come to the proof of (E3) itself.

Step 1: We show that our concrete setting fits into the framework of Lemma 4.3.2. We choose $\mathcal{H} := H_0^1(\Omega)^D$ with $\mathcal{H}^* = (H_0^1(\Omega)^D)^*$ and $\mathcal{H}_\ell := \mathcal{X}_\ell$ for all $\ell\in\mathbb{N}_0$. Note that, with $H^{-1}(\Omega) := H_0^1(\Omega)^*$, $H^{-1}(\Omega)^D$ is a realization of $(H_0^1(\Omega)^D)^*$ with equivalent norms. The involved operators are defined as

$$\langle\mathfrak{A}v\,,\,w\rangle := \int_\Omega \sum_{i=1}^d\sum_{i'=1}^d (A_{ii'}\partial_{i'}v)\cdot\partial_i w\,dx \quad \text{for all } v,w\in H_0^1(\Omega)^D. \tag{4.3.18}$$

and

$$\langle\mathfrak{C}v\,,\,w\rangle := \int_\Omega \sum_{i=1}^d (b_i\partial_i v)\cdot w + (cv)\cdot w\,dx \quad \text{for all } v,w\in H_0^1(\Omega)^D. \tag{4.3.19}$$

which gives $\langle\mathfrak{B}\,\cdot\,,\,\cdot\rangle = \langle\cdot\,,\,\cdot\rangle_{\mathfrak{B}}$. Due to our assumption $A_{ii'}^\top = A_{i'i}$, $\mathfrak{A}$ is symmetric. Rellich's compactness theorem easily implies that $\mathfrak{C}$ is compact; see, e.g., [FFP14, Lemma 3.4]. Finally, we fix the right-hand side as

$$F(v) := \int_\Omega f\cdot v\,dx \quad \text{for all } v\in H_0^1(\Omega)^D. \tag{4.3.20}$$

Recall that we already observed at the beginning of the current subsection that $u\in\mathcal{X}_\infty$.

Step 2: Let $\varepsilon_{\rm qo}>0$ with (4.3.8) and $0<\delta<1$ be sufficiently small such that

$$0 < \frac{C_\varrho C_{\rm ell}}{1-\delta} - \varepsilon_{\rm qo}(C_{\rm rel}C_{\rm cont})^{-2} \le C_\varrho C_{\rm ell}. \tag{4.3.21}$$

Further, let $\ell_0 \in \mathbb{N}_0$ be the corresponding index with (4.3.15). With this, reliability (4.2.20), and ellipticity (4.1.4), we get for all $\ell \geq \ell_0$ and $N \in \mathbb{N}_0$ that

$$\sum_{j=\ell}^{\ell+N}(C_\varrho\|U_{j+1}-U_j\|_{H^1(\Omega)}^2-\varepsilon_{\rm qo}\eta_j^2) \leq \sum_{j=\ell}^{\ell+N}(C_\varrho\|U_{j+1}-U_j\|_{H^1(\Omega)}^2-\varepsilon_{\rm qo}C_{\rm rel}^{-2}\|u-U_j\|_{H^1(\Omega)}^2)$$
$$\leq \sum_{j=\ell}^{\ell+N}\Big(\frac{C_\varrho C_{\rm ell}}{1-\delta}\|u-U_j\|_{\mathfrak{B}}^2 - C_\varrho C_{\rm ell}\|u-U_{j+1}\|_{\mathfrak{B}}^2 - \varepsilon_{\rm qo}C_{\rm rel}^{-2}\|u-U_j\|_{H^1(\Omega)}^2\Big).$$

With continuity (4.1.3) and (4.3.21), we proceed

$$\leq \sum_{j=\ell}^{\ell+N}\Big(\big(\frac{C_\varrho C_{\rm ell}}{1-\delta}-\varepsilon_{\rm qo}(C_{\rm rel}C_{\rm cont})^{-2}\big)\|u-U_j\|_{\mathfrak{B}}^2 - C_\varrho C_{\rm ell}\|u-U_{j+1}\|_{\mathfrak{B}}^2\Big)$$
$$\leq \big(\frac{C_\varrho C_{\rm ell}}{1-\delta}-\varepsilon_{\rm qo}(C_{\rm rel}C_{\rm cont})^{-2}\big)\sum_{j=\ell}^{\ell+N}\big(\|u-U_j\|_{\mathfrak{B}}^2-\|u-U_{j+1}\|_{\mathfrak{B}}^2\big)$$
$$\leq \big(\frac{C_\varrho C_{\rm ell}}{1-\delta}-\varepsilon_{\rm qo}(C_{\rm rel}C_{\rm cont})^{-2}\big)\|u-U_\ell\|_{\mathfrak{B}}^2 \leq \big(\frac{C_\varrho C_{\rm ell}}{1-\delta}-\varepsilon_{\rm qo}(C_{\rm rel}C_{\rm cont})^{-2}\big)C_{\rm cont}\|u-U_\ell\|_{H^1(\Omega)}^2.$$

Reliability (4.2.20) concludes the proof for $\ell \geq \ell_0$. It remains to consider $0 \leq \ell < \ell_0$. To that end, we define

$$C_{\max} := \max_{\ell\in\{0,\dots\ell_0-1\}}\|u-U_\ell\|_{H^1(\Omega)}^{-2}\sum_{j=\ell}^{\ell_0-1}C_\varrho\|U_{j+1}-U_j\|_{H^1(\Omega)}^2.$$

With the convention $\infty\cdot 0 = 0$, this term is well-defined, since $\|u-U_\ell\|_{H^1(\Omega)} = 0$ implies due to nestedness (S2) of the ansatz spaces that $u = U_\ell = U_j$ for all $j \geq \ell$ and therefore also $\|U_{j+1}-U_j\|_{H^1(\Omega)} = 0$. The previous estimate and reliability (4.2.20) imply that

$$\sum_{j=\ell}^{\ell+N}(C_\varrho\|U_{j+1}-U_j\|_{H^1(\Omega)}^2-\varepsilon_{\rm qo}\eta_j^2)$$
$$\leq \sum_{j=\ell}^{\ell_0-1}C_\varrho\|U_{j+1}-U_j\|_{H^1(\Omega)}^2+\sum_{j=\ell_0}^{\ell+N}(C_\varrho\|U_{j+1}-U_j\|_{H^1(\Omega)}^2-\varepsilon_{\rm qo}\eta_j^2) \lesssim \|u-U_\ell\|_{H^1(\Omega)}^2 \lesssim \eta_\ell^2.$$

Altogether, this concludes the proof of (4.3.9), where $C_{\rm qo}$ depends only on the dimension D, the chosen $\varepsilon_{\rm qo}$, the perturbation constant C_ϱ, the reliability constant $C_{\rm rel}$, the coefficients of $\mathfrak{P}$, and the sequence $(U_\ell)_{\ell\in\mathbb{N}_0}$.

Remark 4.3.3. *If the bilinear form $\langle\cdot\,,\,\cdot\rangle_{\mathfrak{P}}$ is symmetric, (4.3.9) follows from the Pythagoras theorem $\|u-U_j\|_{\mathfrak{P}}^2+\|U_{j+1}-U_j\|_{\mathfrak{P}}^2 = \|u-U_j\|_{\mathfrak{P}}^2$ in the $\mathfrak{P}$-induced energy norm $\|v\|_{\mathfrak{P}}^2 := \langle v\,,\,v\rangle_{\mathfrak{P}}$ and norm equivalence*

$$\sum_{j=\ell}^{\ell+N}\|U_{j+1}-U_j\|_{H^1(\Omega)}^2 \simeq \sum_{j=\ell}^{\ell+N}\|U_{j+1}-U_j\|_{\mathfrak{P}}^2 = \|u-U_\ell\|_{\mathfrak{P}}^2-\|u-U_{\ell+N}\|_{\mathfrak{P}}^2 \lesssim \|u-U_\ell\|_{H^1(\Omega)}^2.$$

Together with reliability (4.2.20), this proves (4.3.9) even for $\varepsilon_{\rm qo} = 0$, and $C_{\rm qo}$ is independent of the sequence $(U_\ell)_{\ell\in\mathbb{N}_0}$.

4.3.5 Discrete reliability (E4)

Under the assumptions (M1), (M3), (4.2.6), and (S2)–(S6), we show that there exist $C_{\rm drel}, C_{\rm ref} \geq 1$ such that for all $\mathcal{T}_\bullet \in \mathbb{T}$ and all $\mathcal{T}_\circ \in \texttt{refine}(\mathcal{T}_\bullet)$, the subset $\mathcal{R}_{\bullet,\circ} := \Pi_\bullet^{q_{\rm loc}}(\mathcal{T}_\bullet \setminus \mathcal{T}_\circ) \subseteq \mathcal{T}_\bullet$ satisfies that

$$C_\varrho \|U_\circ - U_\bullet\|_{H^1(\Omega)} \leq C_{\rm drel}\, \eta_\bullet(\mathcal{R}_{\bullet,\circ}), \quad \mathcal{T}_\bullet \setminus \mathcal{T}_\circ \subseteq \mathcal{R}_{\bullet,\circ}, \quad \text{and} \quad \#\mathcal{R}_{\bullet,\circ} \leq C_{\rm ref}(\#\mathcal{T}_\circ - \#\mathcal{T}_\bullet).$$

The last two properties are obvious with $C_{\rm ref} = C_{\rm patch}^{q_{\rm loc}}$ by validity of (M1), (S3), and (4.2.6). For the first property, we argue as in [Ste07, Theorem 4.1]: Ellipticity (4.1.4), $e_\circ := U_\circ - U_\bullet \in \mathcal{X}_\circ$ (which follows from (S2)), and Galerkin orthogonality (4.2.9) with $V_\bullet := J_\bullet e_\circ \in \mathcal{X}_\bullet$ prove that

$$\|U_\circ - U_\bullet\|_{H^1(\Omega)}^2 \lesssim \langle e_\circ\,,\, e_\circ\rangle_{\mathfrak{P}} = \langle e_\circ\,,\, (1-J_\bullet)e_\circ\rangle_{\mathfrak{P}}.$$

The Galerkin formulation (4.2.8) and nestedness (S2) yield that

$$= \int_\Omega f\cdot(1-J_\bullet)e_\circ\,dx - \langle U_\bullet\,,\,(1-J_\bullet)e_\circ\rangle_{\mathfrak{P}}.$$

We split Ω into elements $T \in \mathcal{T}_\bullet$ and apply elementwise integration by parts, where we denote the conormal derivative by $\mathfrak{D}_\nu(\cdot)$ (see (4.2.11)). With $U_\bullet|_T \in H^2(T)^D$, this leads to

$$= \sum_{T\in\mathcal{T}_\bullet} \Big(\int_T (f - \mathfrak{P}U_\bullet)\cdot(1-J_\bullet)e_\circ\,dx + \int_{\partial T} (\mathfrak{D}_\nu U_\bullet)\,(1-J_\bullet)e_\circ\,ds \Big). \tag{4.3.22}$$

The properties (S3)–(S4) immediately prove for any $V_\circ \in \mathcal{X}_\circ$ that

$$J_\bullet V_\circ = V_\circ \quad \text{on } \overline{\Omega \setminus \pi_\bullet^{q_{\rm loc}}(\mathcal{T}_\bullet \setminus \mathcal{T}_\circ)} = \overline{\Omega \setminus \bigcup \mathcal{R}_{\bullet,\circ}} = \bigcup(\mathcal{T}_\bullet \setminus \mathcal{R}_{\bullet,\circ}).$$

Hence, the sum in (4.3.22) reduces from $T \in \mathcal{T}_\bullet$ to $T \in \mathcal{R}_{\bullet,\circ}$. Since $(1-J_\bullet)e_\circ \in H_0^1(\Omega)^D$, we have that $(1-J_\bullet)e_\circ = 0$ on $\partial(\bigcup\mathcal{R}_{\bullet,\circ})$ in the sense of traces. We define the set of facets $\mathcal{E}_{\bullet,\circ} := \big\{T_1 \cap T_2 \,:\, T_1, T_2 \in \mathcal{R}_{\bullet,\circ} \wedge \dim(T_1\cap T_2) = d-1\big\}$. Almost all $x \in \bigcup\mathcal{E}_{\bullet,\circ}$ belong to precisely two elements with opposite normal vectors. Hence,

$$\begin{aligned}\sum_{T\in\mathcal{R}_{\bullet,\circ}} \int_{\partial T} (\mathfrak{D}_\nu U_\bullet)\,(1-J_\bullet)e_\circ\,ds &= \sum_{T\in\mathcal{R}_{\bullet,\circ}} \int_{\partial T\cap\Omega} (\mathfrak{D}_\nu U_\bullet)\,(1-J_\bullet)e_\circ\,ds \\ &\leq \sum_{E\in\mathcal{E}_{\bullet,\circ}} \int_E \big|[\mathfrak{D}_\nu U_\bullet]\,(1-J_\bullet)e_\circ\big|\,ds = \frac{1}{2}\sum_{T\in\mathcal{R}_{\bullet,\circ}} \int_{\partial T\cap\Omega} \big|[\mathfrak{D}_\nu U_\bullet](1-J_\bullet)e_\circ\big|\,ds.\end{aligned}$$

Altogether, we have derived that

$$\begin{aligned}\|U_\circ - U_\bullet\|_{H^1(\Omega)}^2 &\lesssim \sum_{T\in\mathcal{R}_{\bullet,\circ}} \Big(\int_T (f-\mathfrak{P}U_\bullet)(1-J_\bullet)e_\circ\,dx + \int_{\partial T\cap\Omega} \big|[\mathfrak{D}_\nu U_\bullet](1-J_\bullet)e_\circ\big|\,ds \Big) \\ &\leq \sum_{T\in\mathcal{R}_{\bullet,\circ}} \Big(h_T\|f-\mathfrak{P}U_\bullet\|_{L^2(T)}\, h_T^{-1}\|(1-J_\bullet)e_\circ\|_{L^2(T)} \\ &\qquad + h_T^{1/2}\|[\mathfrak{D}_\nu U_\bullet]\|_{L^2(\partial T\cap\Omega)}\, h_T^{-1/2}\|(1-J_\bullet)e_\circ\|_{L^2(\partial T\cap\Omega)} \Big).\end{aligned} \tag{4.3.23}$$

By (M3), (S5), and (S6), we have that

$$h_T^{-1}\|(1-J_\bullet)e_\circ\|_{L^2(T)} + h_T^{-1/2}\|(1-J_\bullet)e_\circ\|_{L^2(\partial T\cap\Omega)} \lesssim \|U_\circ - U_\bullet\|_{H^1(\pi_\bullet^{q_{\rm sz}}(T))}.$$

Plugging this into (4.3.23) and using the Cauchy–Schwarz inequality, we obtain that

$$\|U_\circ - U_\bullet\|_{H^1(\Omega)}^2 \lesssim \Big(\sum_{T\in\mathcal{R}_{\bullet,\circ}} \eta_\bullet(T)^2\Big)^{1/2}\Big(\sum_{T\in\mathcal{R}_{\bullet,\circ}} \|U_\circ - U_\bullet\|_{H^1(\pi_\bullet^{q_{\rm sz}}(T))}^2\Big)^{1/2}.$$

With (M1), the second factor is controlled by $\|U_\circ - U_\bullet\|_{H^1(\Omega)}$. This concludes the current section, and $C_{\rm drel}$ depends only on $C_\varrho, d, D, C_{\rm ell}$, (M1), (M3), and (S2)–(S6).

4.3.6 Refinement axioms (T1)–(T3).

Clearly, the properties (R1), (R4), and (R5) are even slightly stronger versions of the axioms (T1)–(T3).

4.3.7 Reliability (4.2.20)

Note that $\mathrm{osc}_\bullet \le \eta_\bullet$ follows immediately from their definitions (4.2.13)–(4.2.14). If one replaces $U_\circ \in \mathcal{X}_\circ$ by the exact solution $u \in H_0^1(\Omega)^D$, $\mathcal{R}_{\bullet,\circ}$ by $\mathcal{T}_\bullet$, and C_ϱ by 1, reliability (4.2.20) follows along the lines of Section 4.3.5, but now, (S2)–(S4) are not needed for the proof.

4.3.8 Efficiency (4.2.21)

We prove efficiency in three steps.
Step 1: As in [NV11, Theorem 7], we show that the assumptions (M1)–(M4) and (O1)–(O4) imply that

$$\eta_\bullet \lesssim \|u - U_\bullet\|_{H^1(\Omega)} + \mathrm{osc}_\bullet(U_\bullet). \tag{4.3.24}$$

First, we bound the volume residual part of $\eta_\bullet$. We abbreviate $r_\bullet := f - \mathfrak{P}U_\bullet$. For all $T \in \mathcal{T}_\bullet$, there holds with the triangle inequality and (O1) that

$$\begin{aligned} h_T\|r_\bullet\|_{L^2(T)} &\le h_T\|P_{\bullet,T} r_\bullet\|_{L^2(T)} + h_T\|(1-P_{\bullet,T})r_\bullet\|_{L^2(T)} \\ &\lesssim \|P_{\bullet,T} r_\bullet\|_{H^{-1}(T)} + h_T\|(1-P_{\bullet,T})r_\bullet\|_{L^2(T)} \\ &\le \|r_\bullet\|_{H^{-1}(T)} + \|(1-P_{\bullet,T})r_\bullet\|_{H^{-1}(T)} + h_T\|(1-P_{\bullet,T})r_\bullet\|_{L^2(T)}. \end{aligned} \tag{4.3.25}$$

Elementary calculations show that

$$\|r_\bullet\|_{H^{-1}(T)} \simeq \sup\Big\{\int_T r_\bullet\cdot v\,dx : v\in H_0^1(T)^D \wedge \|v\|_{H^1(T)} = 1\Big\}, \tag{4.3.26}$$

where the hidden constants depend only on the dimension D. Moreover, the definition (4.1.5) of the weak solution u as well as partial integration yield that

$$= \sup\big\{\langle u - U_\bullet\,,\, v\rangle_{\mathfrak{P}} \,:\, v\in H_0^1(T)^D \wedge \|v\|_{H^1(T)} = 1\big\} \tag{4.3.27}$$

Since $\langle \cdot\,,\,\cdot\rangle_{\mathfrak{P}}$ is continuous (see (4.1.3)), the latter term can be bounded up to some multiplicative constant by $\|u - U_\bullet\|_{H^1(T)}$. Due to (M4), the second summand of (4.3.25) can be bounded by $h_T\|(1 - P_{\bullet,T})r_\bullet\|_{L^2(T)}$. Thus, we conclude that

$$h_T^2\|r_\bullet\|_{L^2(T)}^2 \lesssim \|u - U_\bullet\|_{H^1(T)}^2 + h_T^2\|(1 - P_{\bullet,T})r_\bullet\|_{L^2(T)}^2. \tag{4.3.28}$$

Now, we come to the jump part of $\eta_\bullet$. Let $T, T' \in \mathcal{T}_\bullet$ and $E := T \cap T' \in \mathcal{E}_{\bullet,T}$ an interior $(d-1)$-dimensional edge of T. We abbreviate $j_\bullet := [\mathfrak{D}_\nu U_\bullet]$ and $\widetilde{j}_\bullet := L_{\bullet,E}(P_{\bullet,E}\, j_\bullet) \in H_0^1(T \cup T')$. We start with the simple observation that

$$h_T^{1/2}\|j_\bullet\|_{L^2(E)} \leq h_T^{1/2}\|P_{\bullet,E}\, j_\bullet\|_{L^2(E)} + h_T^{1/2}\|(1 - P_{\bullet,E})j_\bullet\|_{L^2(E)}. \tag{4.3.29}$$

It remains to estimate the first summand. (O2) shows that

$$\begin{aligned}\|P_{\bullet,E}\, j_\bullet\|_{L^2(E)}^2 &\lesssim \int_E \widetilde{j}_\bullet \cdot P_{\bullet,E}\, j_\bullet \, dx \\ &= \int_E \widetilde{j}_\bullet \cdot (P_{\bullet,E} - 1)j_\bullet \, dx + \int_E \widetilde{j}_\bullet \cdot j_\bullet \, dx\end{aligned} \tag{4.3.30}$$

To control the first summand of (4.3.30), we apply the Cauchy–Schwarz inequality as well as the trace inequality (M3) in combination with (O3)–(O4)

$$\Big|\int_E \widetilde{j}_\bullet \cdot (P_{\bullet,E} - 1)j_\bullet \, dx\Big| \leq \|\widetilde{j}_\bullet\|_{L^2(E)}\|(1 - P_{\bullet,E})j_\bullet\|_{L^2(E)} \lesssim \|P_{\bullet,E}\, j_\bullet\|_{L^2(E)}\|(1 - P_{\bullet,E})j_\bullet\|_{L^2(E)}$$

To control the second summand of (4.3.30), we note that the definition (4.1.5) of the weak solution u as well as partial integration on T and T' yield that

$$\int_E \widetilde{j}_\bullet \cdot j_\bullet \, dx = -\int_{T \cup T'} r_\bullet \cdot \widetilde{j}_\bullet \, dx + \langle u - U_\bullet\,,\, \widetilde{j}_\bullet\rangle_{\mathfrak{P}}.$$

The Cauchy–Schwarz inequality and continuity (4.1.3) of $\langle \cdot\,,\,\cdot\rangle_{\mathfrak{P}}$ imply that

$$\Big|\int_E \widetilde{j}_\bullet \cdot j_\bullet \, dx\Big| \lesssim \|r_\bullet\|_{L^2(T\cup T')}\|\widetilde{j}_\bullet\|_{L^2(T\cup T')} + \|u - U_\bullet\|_{H^1(T\cup T')}\|\widetilde{j}_\bullet\|_{H^1(T\cup T')}.$$

Note that the Friedrichs inequality applied on a ball of diameter $\mathrm{diam}(\Omega)$ which covers Ω, shows that $\|\widetilde{j}_\bullet\|_{H^1(T\cup T')} \lesssim \|\nabla \widetilde{j}_\bullet\|_{L^2(T\cup T')}$, where the hidden constant depends only on the dimensions d, D and $\mathrm{diam}(\Omega)$. Therefore, we obtain with the stability assumptions (O3)–(O4) and the shape-regularity (M2) that

$$\Big|\int_E \widetilde{j}_\bullet \cdot j_\bullet \, dx\Big| \leq \Big(h_T^{1/2}\|r_\bullet\|_{L^2(T\cup T')} + h_T^{-1/2}\|u - U_\bullet\|_{H^1(T\cup T')}\Big)\|P_{\bullet,E}\, j_\bullet\|_{L^2(E)}.$$

Plugging everything into (4.3.30) and dividing by $\|P_{\bullet,E}\, j_\bullet\|_{L^2(E)}$, we end up with

$$\|P_{\bullet,E}\, j_\bullet\|_{L^2(E)} \lesssim \|(1 - P_{\bullet,E})j_\bullet\|_{L^2(E)} + h_T^{1/2}\|r_\bullet\|_{L^2(T\cup T')} + h_T^{-1/2}\|u - U_\bullet\|_{H^1(T\cup T')}.$$

Together with (4.3.29), we obtain that

$$h_T\|j_\bullet\|^2_{L^2(E)} \lesssim h_T\|(1-P_{\bullet,E})j_\bullet\|^2_{L^2(E)} + \|u-U_\bullet\|^2_{H^1(T\cup T')} + h_T^2\|r_\bullet\|^2_{L^2(T\cup T')}. \tag{4.3.31}$$

To obtain (4.3.24), we finally combine (4.3.28) and (4.3.31), sum over all elements, and apply the property (M1)

$$\begin{aligned}\eta_\bullet^2 = \sum_{T\in\mathcal{T}_\bullet} h_T^2\|r_\bullet\|^2_{L^2(T)} + h_T\|j_\bullet\|^2_{L^2(\partial T\cap\Omega)} &\lesssim \sum_{T\in\mathcal{T}_\bullet}\sum_{\substack{T'\in\mathcal{T}_\bullet\\ \dim(T\cap T')=d-1}} \Big(\|u-U_\bullet\|^2_{H^1(T\cup T')}\\ &+ h_T^2\|(1-P_{\bullet,T})r_\bullet\|^2_{L^2(T)} + h_{T'}^2\|(1-P_{\bullet,T'})r_\bullet\|^2_{L^2(T')} + h_T\|(1-P_{\bullet,E})j_\bullet\|^2_{L^2(E)}\Big)\\ &\lesssim \|u-U_\bullet\|^2_{H^1(\Omega)} + \mathrm{osc}_\bullet(U_\bullet)^2.\end{aligned}$$

Step 2: As in [CKNS08, Proposition 3.3], we show that the assumptions (M1)–(M3) and (S1) imply for all $V_\bullet\in\mathcal{X}_\bullet$ that

$$\mathrm{osc}(U_\bullet) \lesssim \mathrm{osc}_\bullet(V_\bullet) + \|U_\bullet - V_\bullet\|_{H^1(\Omega)}. \tag{4.3.32}$$

Let $T\in\mathcal{T}_\bullet$. With the triangle inequality, we see that

$$\begin{aligned}\mathrm{osc}_\bullet(U_\bullet,T)^2 &= h_T^2\|(1-P_{\bullet,T})(f-\mathfrak{P}U_\bullet)\|^2_{L^2(T)} + \sum_{E\in\mathcal{E}_{\bullet,T}} h_T\|(1-P_{\bullet,E})[\mathfrak{D}_\nu U_\bullet]\|^2_{L^2(E)}\\ &\lesssim h_T^2\|(1-P_{\bullet,T})(f-\mathfrak{P}V_\bullet)\|^2_{L^2(T)} + \sum_{E\in\mathcal{E}_{\bullet,T}} h_T\|(1-P_{\bullet,E})[\mathfrak{D}_\nu V_\bullet]\|^2_{L^2(E)}\\ &\quad + h_T^2\|(1-P_{\bullet,T})\mathfrak{P}(U_\bullet-V_\bullet)\|^2_{L^2(T)} + \sum_{E\in\mathcal{E}_{\bullet,T}} h_T\|(1-P_{\bullet,E})[\mathfrak{D}_\nu(U_\bullet-V_\bullet)])\|^2_{L^2(E)}\end{aligned}$$

Hence, stability of orthogonal projections (with constant one) yields that

$$\mathrm{osc}_\bullet(U_\bullet,T)^2 \lesssim \mathrm{osc}_\bullet(V_\bullet,T)^2 + h_T^2\|\mathfrak{P}(U_\bullet-V_\bullet)\|^2_{L^2(T)} + \sum_{E\in\mathcal{E}_{\bullet,T}} h_T\|[\mathfrak{D}_\nu(U_\bullet-V_\bullet)])\|^2_{L^2(E)}.$$

Due to the inverse estimate (S1), the second summand can be bounded up to some multiplicative constant by $\|U_\bullet - V_\bullet\|^2_{H^1(T)}$. The third one can be bounded up to some multiplicative constant as in (4.3.4)–(4.3.5) by $\|U_\bullet - V_\bullet\|^2_{H^1(\pi_\bullet(T))}$. Summing over all elements and taking into account (M1), we conclude (4.3.32).
Step 3: Step 2 in combination with the triangle inequality and the Céa lemma (4.2.10) show for all $V_\bullet\in\mathcal{X}_\bullet$ that

$$\begin{aligned}\|u-U_\bullet\|_{H^1(\Omega)} + \mathrm{osc}_\bullet(U_\bullet) &\overset{(4.3.32)}{\lesssim} \|u-U_\bullet\|_{H^1(\Omega)} + \mathrm{osc}_\bullet(V_\bullet) + \|U_\bullet-V_\bullet\|_{H^1(\Omega)}\\ &\overset{(4.2.10)}{\lesssim} \|u-V_\bullet\|_{H^1(\Omega)} + \mathrm{osc}_\bullet(V_\bullet).\end{aligned}$$

This proves that $\|u-U_\bullet\|_{H^1(\Omega)}+\mathrm{osc}_\bullet(U_\bullet) \simeq \inf_{V_\bullet\in\mathcal{X}_\bullet}\big(\|u-V_\bullet\|_{H^1(\Omega)}+\mathrm{osc}_\bullet(V_\bullet)\big)$. Combining this observation with Step 1, we conclude efficiency (4.2.21), where C_{eff} depends only on (M1)–(M4), (S1) and (O1)–(O4), as well as on d, D, $\|A\|_{W^{1,\infty}(\Omega)}$, $\|b\|_{L^\infty(\Omega)}$, $\|c\|_{L^\infty(\Omega)}$, C_{ell}, and $\mathrm{diam}(\Omega)$.

4.4 Finite element method with hierarchical splines

A similar version of the current section is already found in the recent own work [GHP17, Section 3]. We introduce hierarchical splines in the *physical domain* Ω and propose a local mesh-refinement strategy. We use the notation from Section 3.4, where we already introduced hierarchical splines in the *parameter domain* $\widehat{\Omega} := (0,1)^d$. The main result of this section is Theorem 4.4.6 which states that hierarchical splines together with the proposed mesh-refinement strategy fit into the abstract setting of Section 4.2 and are hence covered by Theorem 4.2.7. The proof of Theorem 4.4.6 is given in Section 4.5.

4.4.1 Parametrization of the physical domain

We assume that Ω can be parametrized via

$$\gamma : \overline{\widehat{\Omega}} \to \overline{\Omega} \quad \text{with} \quad \gamma \in W^{1,\infty}(\widehat{\Omega}) \cap C^2(\widehat{\mathcal{T}}_0) \quad \text{and} \quad \gamma^{-1} \in W^{1,\infty}(\Omega) \cap C^2(\mathcal{T}_0), \tag{4.4.1}$$

where $C^2(\widehat{\mathcal{T}}_0) := \big\{v : \overline{\widehat{\Omega}} \to \mathbb{R} \,:\, \widehat{v}|_{\widehat{T}} \in C^2(\widehat{T}) \text{ for all } \widehat{T} \in \widehat{\mathcal{T}}_0\big\}$ resp. $C^2(\mathcal{T}_0) := \big\{v : \overline{\Omega} \to \mathbb{R} \,:\, v|_T \in C^2(T) \text{ for all } T \in \mathcal{T}_0\big\}$. Consequently, there exists $C_\gamma > 0$ such that for all $i,j,k \in \{1,\dots,d\}$

$$\begin{aligned} \Big\|\frac{\partial}{\partial t_j}\gamma_i\Big\|_{L^\infty(\widehat{\Omega})} \le C_\gamma, \quad \Big\|\frac{\partial}{\partial x_j}(\gamma^{-1})_i\Big\|_{L^\infty(\Omega)} \le C_\gamma, \\ \Big\|\frac{\partial^2}{\partial t_j \partial t_k}\gamma_i\Big\|_{L^\infty(\widehat{\Omega})} \le C_\gamma, \quad \Big\|\frac{\partial^2}{\partial x_j \partial x_k}(\gamma^{-1})_i\Big\|_{L^\infty(\Omega)} \le C_\gamma, \end{aligned} \tag{4.4.2}$$

where γ_i resp. $(\gamma^{-1})_i$ denote the i-th component of γ resp. γ^{-1} and the second derivatives are defined elementwise.

4.4.2 Hierarchical meshes and splines in the physical domain

Let $(p_1,\dots,p_d)$ be a vector of fixed positive polynomial degrees in $\mathbb{N}$, and set

$$p_{\max} := \max_{i \in \{1,\dots,d\}} p_i. \tag{4.4.3}$$

Let

$$\widehat{\mathcal{K}}_0 = (\widehat{\mathcal{K}}_{1(0)}, \dots, \widehat{\mathcal{K}}_{d(0)}) \tag{4.4.4}$$

be a fixed initial d-dimensional vector of p_i-open knot vectors as in Section 3.3.2, where we additionally suppose that all interior knots $t_{i(0),j} \in (0,1)$ even satisfy that

$$\#_{i(0)} t_{i(0),j} \le p_i \quad \text{for all } i \in \{1,\dots,d\}, j \in \{2+p_i,\dots,N_{i(0)}-1\}. \tag{4.4.5}$$

For an arbitrary hierarchical mesh $\widehat{\mathcal{T}}_\bullet$, we define the space of all hierarchical splines which vanish (in the sense of traces) at the boundary as

$$\begin{aligned} \widehat{\mathcal{X}}_\bullet := \widehat{\mathcal{S}}_0^{(p_1,\dots,p_d)}(\widehat{\mathcal{K}}_0, \widehat{\mathcal{T}}_\bullet)^D := \big\{\widehat{V}_\bullet \in \widehat{\mathcal{S}}^{(p_1,\dots,p_d)}(\widehat{\mathcal{K}}_0, \widehat{\mathcal{T}}_\bullet)^D \,:\, \widehat{V}_\bullet|_{\partial\widehat{\Omega}} = 0\big\} \\ \subset \big\{\widehat{v} \in W_0^{1,\infty}(\widehat{\Omega})^D \,:\, \widehat{v}|_T \in C^\infty(\widehat{T})^D \text{ for all } \widehat{T} \in \widehat{\mathcal{T}}_\bullet\big\}. \end{aligned} \tag{4.4.6}$$

In order to transform the definitions from the parameter domain $\widehat{\Omega}$ to the physical domain Ω, we use the parametrization from Section 4.4.1. All definitions can now also be made in the physical domain, just by pulling them from the parameter domain via the diffeomorphism γ from Section 4.4.1. For these definitions, we drop the symbol $\widehat{\cdot}$. If $\widehat{\mathcal{T}}_\bullet$ is a hierarchical mesh, we define the corresponding mesh in the physical domain as $\mathcal{T}_\bullet := \{\gamma(\widehat{T}) \,:\, \widehat{T} \in \widehat{\mathcal{T}}_\bullet\}$. In particular, we have that $\mathcal{T}_0 = \{\gamma(\widehat{T}) \,:\, \widehat{T} \in \widehat{\mathcal{T}}_0\}$. Moreover, let $\mathbb{T} := \{\mathcal{T}_\bullet \,:\, \widehat{\mathcal{T}}_\bullet \in \widehat{\mathbb{T}}\}$ denote the set of all admissible meshes in the physical domain, where $\widehat{\mathbb{T}}$ is the set of all admissible meshes in the sense of Section 3.4.4. For a hierarchical mesh $\mathcal{T}_\bullet$, let $\mathcal{X}_\bullet := \{\widehat{V}_\bullet \circ \gamma^{-1} \,:\, \widehat{V}_\bullet \in \widehat{\mathcal{X}}_\bullet\}$ be the the corresponding hierarchical spline space. By regularity of γ, we particularly obtain that

$$\begin{aligned} \mathcal{X}_\bullet &\subset \{v \in W_0^{1,\infty}(\Omega)^D \,:\, v|_T \in C^2(T)^D \text{ for all } T \in \mathcal{T}_\bullet\} \\ &\subset \{v \in H_0^1(\Omega)^D \,:\, v|_T \in H^2(T)^D \text{ for all } T \in \mathcal{T}_\bullet\}. \end{aligned} \tag{4.4.7}$$

4.4.3 Refinement of hierarchical meshes

In this section, we present a concrete refinement algorithm to specify the setting of Section 4.2.2. We start in the parameter domain. Recall that we call a hierarchical mesh $\widehat{\mathcal{T}}_\circ$ finer than another hierarchical mesh $\widehat{\mathcal{T}}_\bullet$ if $\widehat{\Omega}_\bullet^k \subseteq \widehat{\Omega}_\circ^k$ for all $k \in \mathbb{N}_0$; see (3.4.9). This just means that $\widehat{\mathcal{T}}_\circ$ is obtained from $\widehat{\mathcal{T}}_\bullet$ by iterative dyadic bisections of the elements in $\widehat{\mathcal{T}}_\bullet$. To bisect an element $\widehat{T} \in \widehat{\mathcal{T}}_\bullet$, one just has to add it to the set $\widehat{\Omega}_\bullet^{\mathrm{level}(\widehat{T})+1}$; see (4.4.10) below. In this case, the corresponding spaces are nested according to (3.4.18), i.e.,

$$\widehat{\mathcal{X}}_\bullet \subseteq \widehat{\mathcal{X}}_\circ. \tag{4.4.8}$$

Let $\widehat{\mathcal{T}}_\bullet$ be a hierarchical mesh and $\widehat{T} \in \widehat{\mathcal{T}}_\bullet$. With the set of neighbors $\mathbf{N}_\bullet(\widehat{T}) = \{\widehat{T}' \in \widehat{\mathcal{T}}_\bullet \,:\, \exists \widehat{\beta} \in \widehat{\mathcal{B}}_\bullet \quad \widehat{T}, \widehat{T}' \subseteq \mathrm{supp}(\widehat{\beta})\}$ from (3.4.24), we define the set of *bad neighbors*

$$\mathbf{N}_\bullet^{\mathrm{bad}}(\widehat{T}) := \{\widehat{T}' \in \mathbf{N}_\bullet(\widehat{T}) \,:\, \mathrm{level}(\widehat{T}') = \mathrm{level}(\widehat{T}) - 1\}. \tag{4.4.9}$$

Algorithm 4.4.1. ***Input:*** *Hierarchical mesh* $\widehat{\mathcal{T}}_\bullet$ *, marked elements* $\widehat{\mathcal{M}}_\bullet =: \widehat{\mathcal{M}}_\bullet^{(0)} \subseteq \widehat{\mathcal{T}}_\bullet$.

(i) *Iterate the following steps* (a)–(b) *for* $i = 0, 1, 2, \dots$ *until* $\widehat{\mathcal{U}}_\bullet^{(i)} = \emptyset$*:*

  (a) *Define* $\widehat{\mathcal{U}}_\bullet^{(i)} := \bigcup_{\widehat{T} \in \widehat{\mathcal{M}}_\bullet^{(i)}} \{\widehat{T}' \in \widehat{\mathcal{T}}_\bullet \setminus \widehat{\mathcal{M}}_\bullet^{(i)} \,:\, \widehat{T}' \in \mathbf{N}_\bullet^{\mathrm{bad}}(\widehat{T})\}$.

  (b) *Define* $\widehat{\mathcal{M}}_\bullet^{(i+1)} := \widehat{\mathcal{M}}_\bullet^{(i)} \cup \widehat{\mathcal{U}}_\bullet^{(i)}$.

(ii) *Dyadically bisect all* $\widehat{T} \in \widehat{\mathcal{M}}_\bullet^{(i)}$ *by adding* $\widehat{T}$ *to the set* $\widehat{\Omega}_\bullet^{\mathrm{level}(\widehat{T})+1}$ *and obtain a finer hierarchical mesh* $\widehat{\mathcal{T}}_\circ$*, where*

$$\widehat{\Omega}_\circ^k = \widehat{\Omega}_\bullet^k \cup \bigcup \{\widehat{T} \in \widehat{\mathcal{M}}_\bullet^{(i)} \,:\, \mathrm{level}(\widehat{T}) = k - 1\} \quad \textit{for all } k \in \mathbb{N}. \tag{4.4.10}$$

Output: *Refined mesh* $\widehat{\mathcal{T}}_\circ = \texttt{refine}(\widehat{\mathcal{T}}_\bullet, \widehat{\mathcal{M}}_\bullet)$.

Clearly, $\texttt{refine}(\widehat{\mathcal{T}}_\bullet, \widehat{\mathcal{M}}_\bullet)$ is finer than $\widehat{\mathcal{T}}_\bullet$. For any hierarchical mesh $\widehat{\mathcal{T}}_\bullet$, we define $\texttt{refine}(\widehat{\mathcal{T}}_\bullet)$ as the set of all hierarchical meshes $\widehat{\mathcal{T}}_\circ$ such that there exist hierarchical meshes $\widehat{\mathcal{T}}_{(0)}, \dots, \widehat{\mathcal{T}}_{(J)}$ and marked elements $\widehat{\mathcal{M}}_{(0)}, \dots, \widehat{\mathcal{M}}_{(J-1)}$ with $\widehat{\mathcal{T}}_\circ = \widehat{\mathcal{T}}_{(J)} = \texttt{refine}(\widehat{\mathcal{T}}_{(J-1)}, \widehat{\mathcal{M}}_{(J-1)})$, $\dots, \widehat{\mathcal{T}}_{(1)} = \texttt{refine}(\widehat{\mathcal{T}}_{(0)}, \widehat{\mathcal{M}}_{(0)})$, and $\widehat{\mathcal{T}}_{(0)} = \widehat{\mathcal{T}}_\bullet$. Note that $\texttt{refine}(\widehat{\mathcal{T}}_\bullet, \emptyset) = \widehat{\mathcal{T}}_\bullet$, wherefore $\widehat{\mathcal{T}}_\bullet \in \texttt{refine}(\widehat{\mathcal{T}}_\bullet)$. The following proposition characterizes the set $\texttt{refine}(\widehat{\mathcal{T}}_\bullet)$.

Proposition 4.4.2. *If $\widehat{\mathcal{T}}_\bullet \in \widehat{\mathbb{T}}$, then $\texttt{refine}(\widehat{\mathcal{T}}_\bullet)$ coincides with the set of all admissible hierarchical meshes $\widehat{\mathcal{T}}_\circ \in \widehat{\mathbb{T}}$ (see Section 3.4.4) that are finer than $\widehat{\mathcal{T}}_\bullet$.*

Proof. We prove the assertion in four steps.
Step 1: We show that $\widehat{\mathcal{T}}_\circ := \texttt{refine}(\widehat{\mathcal{T}}_\bullet, \widehat{\mathcal{M}}_\bullet) \in \widehat{\mathbb{T}}$ for any $\widehat{\mathcal{M}}_\bullet \subseteq \widehat{\mathcal{T}}_\bullet$. Let $\widehat{T}, \widehat{T}' \in \widehat{\mathcal{T}}_\circ$ with $\widehat{T}' \in \mathbf{N}_\circ(\widehat{T})$, i.e., there exists $\widehat{\beta}_\circ \in \widehat{\mathcal{B}}_\circ$ with $|\widehat{T} \cap \operatorname{supp}(\widehat{\beta}_\circ)| \neq 0 \neq |\widehat{T}' \cap \operatorname{supp}(\widehat{\beta}_\circ)|$; see (3.4.24). By Lemma 3.4.2, there exists some (not necessarily unique) $\widehat{\beta}_\bullet \in \widehat{\mathcal{B}}_\bullet$ with $\operatorname{supp}(\widehat{\beta}_\circ) \subseteq \operatorname{supp}(\widehat{\beta}_\bullet)$. We consider four different cases.

(i) Let $\widehat{T}, \widehat{T}' \in \widehat{\mathcal{T}}_\bullet$. Then, $|\widehat{T} \cap \operatorname{supp}(\widehat{\beta}_\bullet)| \neq 0 \neq |\widehat{T}' \cap \operatorname{supp}(\widehat{\beta}_\bullet)|$, i.e., $\widehat{T}' \in \mathbf{N}_\bullet(\widehat{T})$ and hence $|\operatorname{level}(\widehat{T}) - \operatorname{level}(\widehat{T}')| \leq 1$ by $\widehat{\mathcal{T}}_\bullet \in \widehat{\mathbb{T}}$.

(ii) Let $\widehat{T}, \widehat{T}' \in \widehat{\mathcal{T}}_\circ \setminus \widehat{\mathcal{T}}_\bullet$. Let $\widehat{T}_\bullet, \widehat{T}'_\bullet \in \widehat{\mathcal{T}}_\bullet$ with $\widehat{T} \subsetneqq \widehat{T}_\bullet$, $\widehat{T}' \subsetneqq \widehat{T}'_\bullet$. Then, it holds that $\operatorname{level}(\widehat{T}) = \operatorname{level}(\widehat{T}_\bullet) + 1$, $\operatorname{level}(\widehat{T}') = \operatorname{level}(\widehat{T}'_\bullet) + 1$ as well as $|\widehat{T}_\bullet \cap \operatorname{supp}(\widehat{\beta}_\bullet)| \neq 0 \neq |\widehat{T}'_\bullet \cap \operatorname{supp}(\widehat{\beta}_\bullet)|$. By definition, it follows that $\widehat{T}'_\bullet \in \mathbf{N}_\bullet(\widehat{T}_\bullet)$ and hence $|\operatorname{level}(\widehat{T}) - \operatorname{level}(\widehat{T}')| = |\operatorname{level}(\widehat{T}_\bullet) - \operatorname{level}(\widehat{T}'_\bullet)| \leq 1$ by $\widehat{\mathcal{T}}_\bullet \in \widehat{\mathbb{T}}$.

(iii) Let $\widehat{T} \in \widehat{\mathcal{T}}_\circ \setminus \widehat{\mathcal{T}}_\bullet, \widehat{T}' \in \widehat{\mathcal{T}}_\bullet$. Let $\widehat{T}_\bullet \in \widehat{\mathcal{T}}_\bullet$ with $\widehat{T} \subsetneqq \widehat{T}_\bullet$. Then, $|\widehat{T}_\bullet \cap \operatorname{supp}(\widehat{\beta}_\bullet)| \neq 0 \neq |\widehat{T}' \cap \operatorname{supp}(\widehat{\beta}_\bullet)|$, and $|\operatorname{level}(\widehat{T}_\bullet) - \operatorname{level}(\widehat{T}')| \leq 1$ by $\widehat{\mathcal{T}}_\bullet \in \widehat{\mathbb{T}}$. We argue by contradiction and assume that $|\operatorname{level}(\widehat{T}) - \operatorname{level}(\widehat{T}')| > 1$. Together with $\operatorname{level}(\widehat{T}_\bullet) + 1 = \operatorname{level}(\widehat{T})$, this yields that $\operatorname{level}(\widehat{T}_\bullet) - 1 = \operatorname{level}(\widehat{T}')$. Hence, $\widehat{T}' \in \mathbf{N}^{\mathrm{bad}}_\bullet(\widehat{T}_\bullet)$ with $\widehat{T}_\bullet \in \widehat{\mathcal{M}}^{(\mathrm{end})}_\bullet$. By Algorithm 4.4.1 (i), we get that $\widehat{T}' \in \widehat{\mathcal{M}}^{(\mathrm{end})}_\bullet$. This contradicts $\widehat{T}' \in \widehat{\mathcal{T}}_\bullet$ and hence proves that $|\operatorname{level}(\widehat{T}) - \operatorname{level}(\widehat{T}')| \leq 1$.

(iv) Let $\widehat{T} \in \widehat{\mathcal{T}}_\bullet, \widehat{T}' \in \widehat{\mathcal{T}}_\circ \setminus \widehat{\mathcal{T}}_\bullet$. Since $\widehat{T}' \in \mathbf{N}_\bullet(\widehat{T})$ is equivalent to $\widehat{T} \in \mathbf{N}_\bullet(\widehat{T}')$, we argue as in (iii) to conclude that $|\operatorname{level}(\widehat{T}) - \operatorname{level}(\widehat{T}')| \leq 1$.

Step 2: It is clear that an arbitrary $\widehat{\mathcal{T}}_\circ \in \texttt{refine}(\widehat{\mathcal{T}}_\bullet)$ is finer than $\widehat{\mathcal{T}}_\bullet$. By induction, Step 1 proves the inclusion $\texttt{refine}(\widehat{\mathcal{T}}_\bullet) \subseteq \widehat{\mathbb{T}}$.
Step 3: To prove the converse inclusion, let $\widehat{\mathcal{T}}_\circ \in \widehat{\mathbb{T}}$ be an admissible mesh that is finer than $\widehat{\mathcal{T}}_\bullet$. Moreover, let $\widehat{T} \in \widehat{\mathcal{T}}_\bullet \backslash \widehat{\mathcal{T}}_\circ$. We show that $\widehat{\mathcal{T}}_\circ$ is also finer than $\widehat{\mathcal{T}}_\star := \texttt{refine}(\widehat{\mathcal{T}}_\bullet, \{\widehat{T}\})$. We argue by contradiction and suppose that $\widehat{\mathcal{T}}_\circ$ is not finer than $\widehat{\mathcal{T}}_\star$. Since $\texttt{refine}$ bisects each element of $\widehat{\mathcal{T}}_\bullet$ at most once, there exists a refined element $\widehat{T}^{(0)} \in \widehat{\mathcal{T}}_\bullet \backslash \widehat{\mathcal{T}}_\star$ which is also in $\widehat{\mathcal{T}}_\circ$, i.e., $\widehat{T}^{(0)} \in (\widehat{\mathcal{T}}_\bullet \backslash \widehat{\mathcal{T}}_\star) \cap \widehat{\mathcal{T}}_\circ$. In particular, $\widehat{T}^{(0)} \neq \widehat{T} \in \widehat{\mathcal{T}}_\bullet \backslash \widehat{\mathcal{T}}_\circ$. Thus, Algorithm 4.4.1 shows that $\widehat{T}^{(0)} \in \mathbf{N}^{\mathrm{bad}}_\bullet(\widehat{T}^{(1)})$ for some $\widehat{T}^{(1)} \in \widehat{\mathcal{T}}_\bullet \setminus \widehat{\mathcal{T}}_\star$. If $\widehat{T}^{(1)} \in \widehat{\mathcal{T}}_\circ$ and hence $\widehat{T}^{(1)} \in (\widehat{\mathcal{T}}_\bullet \setminus \widehat{\mathcal{T}}_\star) \cap \widehat{\mathcal{T}}_\circ$, we have again that $\widehat{T}^{(1)} \neq \widehat{T}$ as well as $\widehat{T}^{(1)} \in \mathbf{N}^{\mathrm{bad}}_\bullet(\widehat{T}^{(2)})$ for some $\widehat{T}^{(2)} \in \widehat{\mathcal{T}}_\bullet \setminus \widehat{\mathcal{T}}_\star$. Inductively, we see the existence of $\widehat{T}^{(J-1)} \in (\widehat{\mathcal{T}}_\bullet \setminus \widehat{\mathcal{T}}_\star) \cap \widehat{\mathcal{T}}_\circ$ such that $\widehat{T}^{(J-1)} \in \mathbf{N}^{\mathrm{bad}}_\bullet(\widehat{T}^{(J)})$ for some $\widehat{T}^{(J)} \in \widehat{\mathcal{T}}_\bullet \setminus \widehat{\mathcal{T}}_\star$ with $\widehat{T}^{(J)} \notin \widehat{\mathcal{T}}_\circ$. In particular, this implies the existence of $\widehat{T}^{(J)}_\circ \in \widehat{\mathcal{T}}_\circ$ with $\widehat{T}^{(J)}_\circ \subsetneqq \widehat{T}^{(J)}$.

By definition of $\mathbf{N}^{\mathrm{bad}}_\bullet(\cdot)$, we have that $\widehat{T}^{(J)}, \widehat{T}^{(J-1)} \subseteq \operatorname{supp}(\widehat{\beta})$ for some $\widehat{\beta} \in \widehat{\mathcal{B}}_\bullet$ as well as $\operatorname{level}(\widehat{T}^{(J-1)}) = \operatorname{level}(\widehat{T}^{(J)}) - 1$. Hence, (3.4.17) and $\widehat{\mathcal{T}}_\bullet \in \widehat{\mathbb{T}}$ show that $k := \operatorname{level}(\widehat{\beta}) =$

$\text{level}(\widehat{T}^{(J-1)})$. Since $\widehat{T}^{(J-1)} \in \widehat{\mathcal{T}}_\circ$, (3.4.10) implies that $\widehat{T}^{(J-1)} \not\subseteq \widehat{\Omega}_\circ^{k+1}$ and hence $\text{supp}(\widehat{\beta}) \not\subseteq \widehat{\Omega}_\circ^{k+1}$. Moreover, (3.4.13) shows that $\text{supp}(\widehat{\beta}) \subseteq \widehat{\Omega}_\bullet^k \subseteq \widehat{\Omega}_\circ^k$. Using (3.4.13) again, we see that $\widehat{\beta} \in \widehat{\mathcal{B}}_\circ$. Together with $\widehat{T}_\circ^{(J)}, \widehat{T}^{(J-1)} \subseteq \text{supp}(\widehat{\beta})$ and $\text{level}(\widehat{T}_\circ^{(J)}) \geq \text{level}(\widehat{T}^{(J)}) + 1 = \text{level}(\widehat{T}^{(J-1)}) + 2$, this contradicts admissibility of $\widehat{\mathcal{T}}_\circ \in \widehat{\mathbb{T}}$, and concludes the proof.
Step 4: Let again $\widehat{\mathcal{T}}_\circ \in \widehat{\mathbb{T}}$ be an arbitrary admissible mesh that is finer than $\widehat{\mathcal{T}}_\bullet$. Step 3 together with Step 2 shows that we can iteratively refine $\widehat{\mathcal{T}}_\bullet$ and obtain a sequence $\widehat{\mathcal{T}}_{(0)}, \dots, \widehat{\mathcal{T}}_{(J)}$ with $\widehat{\mathcal{T}}_\bullet = \widehat{\mathcal{T}}_{(0)}$, $\widehat{\mathcal{T}}_{(j+1)} = \texttt{refine}(\widehat{\mathcal{T}}_{(j)}, \{\widehat{T}_{(j)}\})$ with some $\widehat{T}_{(j)} \in \widehat{\mathcal{T}}_{(j)} \setminus \widehat{\mathcal{T}}_{(j+1)}$ for $j = 1, \dots, J-1$ and $\widehat{\mathcal{T}}_{(J)} = \widehat{\mathcal{T}}_\circ$. By definition, this proves that $\widehat{\mathcal{T}}_\circ \in \texttt{refine}(\widehat{\mathcal{T}}_\bullet)$. □

Finally, we transfer the definitions and results of this section to the physical domain Ω. We say that a hierarchical mesh $\mathcal{T}_\circ$ is *finer* than another hierarchical mesh $\mathcal{T}_\bullet$ if the corresponding meshes in the parameter domain satisfy this relation, i.e., if $\widehat{\mathcal{T}}_\circ$ is finer than $\widehat{\mathcal{T}}_\bullet$. In this case, there holds that

$$\mathcal{X}_\bullet \subseteq \mathcal{X}_\circ. \tag{4.4.11}$$

If now $\mathcal{M}_\bullet \subseteq \mathcal{T}_\bullet$ with a hierarchical mesh $\mathcal{T}_\bullet$, we abbreviate $\widehat{\mathcal{M}_\bullet} := \{\gamma^{-1}(T) : T \in \mathcal{M}_\bullet\}$ and define $\texttt{refine}(\mathcal{T}_\bullet, \mathcal{M}_\bullet) := \{\gamma(\widehat{T}) : \widehat{T} \in \texttt{refine}(\widehat{\mathcal{T}}_\bullet, \widehat{\mathcal{M}_\bullet})\}$. For an admissible $\mathcal{T}_\bullet \in \mathbb{T}$, we define $\texttt{refine}(\mathcal{T}_\bullet)$ similarly as in Section 4.2.2. According to Proposition 4.4.2, $\texttt{refine}(\mathcal{T}_\bullet)$ coincides with the set of all $\mathcal{T}_\circ \in \mathbb{T}$ that are finer than $\mathcal{T}_\bullet$. In particular, we have that $\texttt{refine}(\mathcal{T}_0) = \mathbb{T}$.

Remark 4.4.3. *The works [BG16c, BGMP16] studied a related refinement strategy, where* $\mathbf{N}_\bullet(\widehat{T})$ *from* (3.4.24) *and* $\mathbf{N}_\bullet^{\text{bad}}(\widehat{T})$ *from* (4.4.9) *are replaced by*

$$\begin{aligned}\widetilde{\mathbf{N}}_\bullet(\widehat{T}) &:= \{\widehat{T}' \in \widehat{\mathcal{T}}_\bullet : \exists \widehat{\beta} \in \widehat{\mathcal{B}}_{\text{uni(level}(\widehat{T}))} \text{ with } |\widehat{T} \cap \text{supp}(\widehat{\beta})| \neq 0 \neq |\widehat{T}' \cap \text{supp}(\widehat{\beta})|\},\\ \widetilde{\mathbf{N}}_\bullet^{\text{bad}}(\widehat{T}) &:= \{\widehat{T}' \in \widetilde{\mathbf{N}}_\bullet : \text{level}(\widehat{T}') = \text{level}(\widehat{T}) - 1\}.\end{aligned} \tag{4.4.12}$$

There, the refinement strategy was designed for truncated hierarchical B-splines; see Section 3.4.3. Compared to the hierarchical B-splines $\widehat{\mathcal{B}}_\bullet$*, those have generically a smaller, but also more complicated and not necessarily connected support. Then, [BG16c, Corollary 17] shows that the generated meshes are strictly admissible in the sense of [BG16c, BGMP16], i.e., for all* $k \in \mathbb{N}$*, it holds that*

$$\widehat{\Omega}_\bullet^k \subseteq \bigcup \{\widehat{T} \in \widehat{\mathcal{T}}_{\text{uni}(k-1)} : \forall \widehat{\beta} \in \widehat{\mathcal{B}}_{\text{uni}(k-1)} \quad (\widehat{T} \subseteq \text{supp}(\widehat{\beta}) \implies \text{supp}(\widehat{\beta}) \subseteq \widehat{\Omega}_\bullet^{k-1})\}. \tag{4.4.13}$$

This definition actually goes back to [GJS14, Appendix A]. According to [BG16c, Section 2.4], strictly admissible meshes satisfy a similar version of Proposition 3.4.3 for truncated hierarchical B-splines. However, the example from Figure 4.1 shows that the proposition fails for hierarchical B-splines and the refinement strategy from [BG16c]. In particular, strictly admissible meshes are not necessarily admissible in the sense of Section 3.4.4.

Remark 4.4.4. *Actually, the proposed refinement strategy of Algorithm 4.4.1 was designed for hierarchical B-splines; see also Proposition 3.4.3. However,* (3.4.22) *implies that Proposition 3.4.3 holds accordingly for truncated hierarchical B-splines. Moreover, if one applies the refinement strategy of Algorithm 4.4.1, Proposition 3.4.7 shows that the computation of the truncated hierarchical B-splines simplifies significantly.*

4.4.4 Optimal convergence for hierarchical splines

Before we come to the main result of this section, we fix positive polynomial orders $(p_1', \dots, p_d')$ and define for $\mathcal{T}_\bullet \in \mathbb{T}$ the space of transformed polynomials

$$\mathcal{P}(\Omega) := \big\{\widehat{W} \circ \gamma : \widehat{W} \text{ is a tensor-product polynomial of order } (p_1', \dots, p_d')\big\} \tag{4.4.14}$$

Remark 4.4.5. *In order to obtain higher-order oscillations, the natural choice of the polynomial orders is $p_i' \geq 2p_i - 1$; see, e.g., [NV11, Section 3.1]. If $\mathcal{X}_\bullet \subset C^1(\overline{\Omega})$, it is sufficient to choose $p_i' \geq 2p_i - 2$; see Remark 4.2.5.*

Altogether, we have specified the abstract framework of Section 4.2 to hierarchical meshes and splines. The following theorem is the second main result of this chapter. It shows that all assumptions of Theorem 4.2.7 are satisfied for the present hierarchical approach. The proof is given in Section 4.5 and is already found in the recent own work [GHP17, Theorem 3.1].

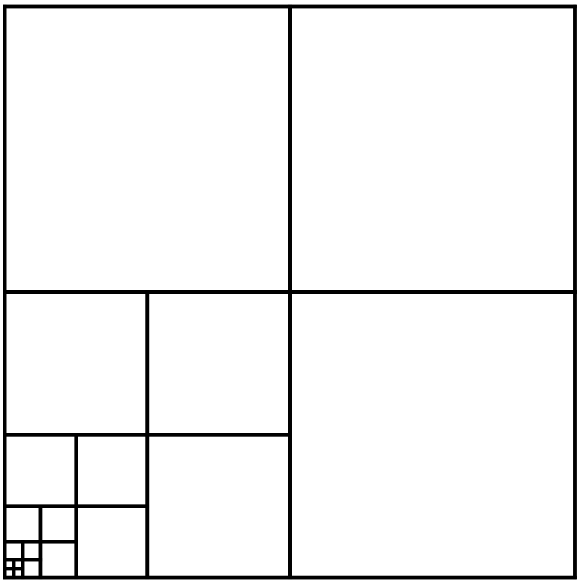
Theorem 4.4.6. *Hierarchical splines on admissible meshes satisfy the abstract assumptions* (M1)–(M4), (R1)–(R5), *and* (S1)–(S6) *from Section 4.2, where the constants depend only on d, D, C_γ, $\widehat{\mathcal{T}}_0$, and $(p_1, \dots, p_d)$. Moreover, the piecewise polynomials $\mathcal{P}(\Omega)$ from* (4.4.14) *on admissible meshes satisfy the abstract assumptions* (O1)–(O4), *where the constants depend only on d, D, C_γ, $\widehat{\mathcal{T}}_0$, and $(p_1', \dots, p_d')$. By Theorem 4.2.7, this implies reliability* (4.2.20) *as well as efficiency* (4.2.21) *of the error estimator, and linear convergence* (4.2.22) *at optimal rate* (4.2.23) *for the adaptive strategy from Algorithm 4.2.6.*

Remark 4.4.7. *Theorem 4.4.6 is still valid if one replaces the ansatz space $\mathcal{X}_\bullet$ by* rational hierarchical splines, *i.e., by the set*

$$\mathcal{X}_\bullet^{W_0} := \Big\{W_0^{-1} V_\bullet : V_\bullet \in \mathcal{X}_\bullet\Big\}, \tag{4.4.15}$$

where $\widehat{W}_0 := W_0 \circ \gamma$ is a fixed positive weight function in the initial space of hierarchical splines $\widehat{\mathcal{S}}^{(p_1,\dots,p_d)}(\widehat{\mathcal{K}}_0, \widehat{\mathcal{T}}_0)$. We will prove this version in Section 4.5.13. The constants depend additionally on $\widehat{W}_0$. Moreover, Theorem 4.4.6 still holds true if newly inserted knots have a multiplicity higher than one, i.e., if one uses as in Remark 3.4.1 the uniformly refined knots $\widehat{\mathcal{K}}_{\mathrm{uni}(k,q)}$ with $1 \leq q_i \leq p_i$ instead of $\widehat{\mathcal{K}}_{\mathrm{uni}(k)}$ to define (rational) hierarchical splines. The corresponding proof works verbatim.

Algorithm [BG16c]



Algorithm 4.4.1

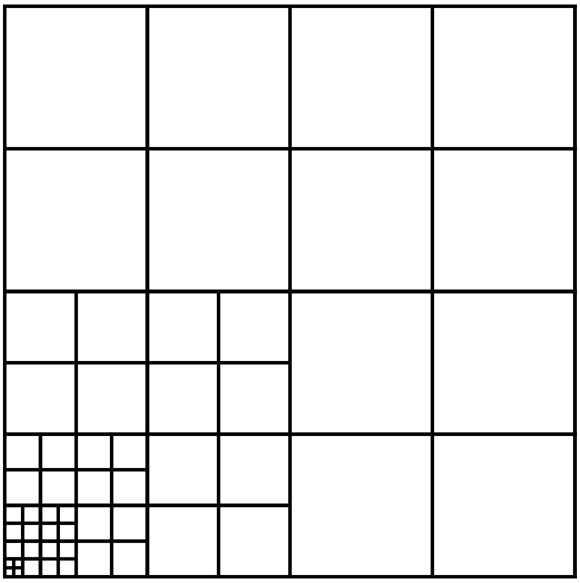


Figure 4.1: An initial mesh $\widehat{\mathcal{T}}_0$ with only one element $[0,1]^2$ is locally refined in the lower left corner using the refinement of [BG16c] (above) resp. the refinement of Algorithm 4.4.1 (below); see Remark 4.4.3 . The lowest-order case $(p_1, p_2) = (1,1)$ is considered. By repetitive refinement via [BG16c], the number of elements in the support of the hierarchical B-spline $\widehat{B}(s_1|0,0.5,1)\widehat{B}(s_2|0,0.5,1)$ grows to infinity. Moreover, the number of hierarchical B-splines with support on the element in the lower left corner grows to infinity. This is not the case if one uses Algorithm 4.4.1; see Proposition 3.4.3.

4.5 Proof of Theorem 4.4.6

This section is devoted to the proof of Theorem 4.4.6, i.e., to the verification of the mesh properties (M1)–(M4), the refinement properties (R1)–(R5), the finite element space properties (S1)–(S6) as well as the data approximation properties (O1)–(O4). In Section 4.5.8, we characterize the restriction of hierarchical splines to the boundary. With this, we are able to give a basis for the used finite element spaces, which will be used for the verification of (S3)–(S6).

4.5.1 Verification of (M1)–(M2)

With Lemma 3.4.5, one can easily verify that $\mathbb{T}$ satisfies (M1)–(M2): Let $\mathcal{T}_\bullet \in \mathbb{T}$. We start with (M1). Let $T \in \mathcal{T}_\bullet$ and $T' \in \Pi_\bullet(T)$. Lemma 3.4.5 and admissibility show for the corresponding elements $\widehat{T}, \widehat{T}'$ in the parameter domain that $|\mathrm{level}(\widehat{T}) - \mathrm{level}(\widehat{T}')| \le 1$. With this, one easily sees that $\#\Pi_\bullet(T) \le C_{\mathrm{patch}}$ with a constant $C_{\mathrm{patch}} > 0$ that depends only on the dimension d. To prove local quasi-uniformity (M2), let $T \in \mathcal{T}_\bullet$ and $T' \in \Pi_\bullet(T)$. As before, we have that $|\mathrm{level}(\widehat{T}) - \mathrm{level}(\widehat{T}')| \le 1$ for the corresponding elements in the parameter domain. Regularity (4.4.2) of the transformation γ yields that $|T| \simeq |T'|$. The constant $C_{\mathrm{locuni}} > 0$ depends only on d, C_γ, and $\widehat{\mathcal{T}}_0$.

4.5.2 Verification of (M3)–(M4)

Proposition 4.2.2 implies that the trace inequality (M3) holds in the parameter domain, where the constant depends only on the shape of the elements. Since we only use dyadic bisection, the number of different shapes is uniformly bounded. Regularity (4.4.2) of γ yields (M3), where the constant C_{trace} depends only on d, C_γ, and $\widehat{\mathcal{T}}_0$.

Proposition 4.2.3 implies that (M4) holds in the parameter domain, where the constant depends only on the shape of the elements. We have just seen that the number of different shapes is uniformly bounded. Regularity (4.4.2) of γ shows that $\|w\|_{L^2(T)} \simeq \|\widehat{w}\|_{L^2(\widehat{T})}$ for all $T \in \mathcal{T}_\bullet \in \mathbb{T}$ and $w \in L^2(T)$ with $\widehat{T} := \gamma^{-1}(T)$ and $\widehat{w} := w \circ \gamma|_{\widehat{T}}$. Further, we show that

$$\|w\|_{H^{-1}(T)} \simeq \|\widehat{w}\|_{H^{-1}(\widehat{T})}. \tag{4.5.1}$$

To see this, let $v \in H^1_0(T)$ and $\widehat{v} := v \circ \gamma|_{\widehat{T}}$. Due to the assumptions on γ, we can assume without loss of generality that $\det(D\gamma) > 0$ on $\widehat{T}$. In particular, we have that $|\det(D\gamma)|_{\widehat{T}}| \in C^1(\widehat{T})$ and $\widehat{v}\,|\det D\gamma|_{\widehat{T}}| \in H^1_0(\widehat{T})$. Therefore, regularity (4.4.2) of γ proves that

$$\int_T wv\,dx = \int_{\widehat{T}} \widehat{w}\;\widehat{v}\,|\det(D\gamma)|\,dt \lesssim \|\widehat{w}\|_{H^{-1}(\widehat{T})} \|\widehat{v}\,|\det D\gamma|\|_{H^1(\widehat{T})} \simeq \|\widehat{w}\|_{H^{-1}(\widehat{T})} \|v\|_{H^1(T)}.$$

We conclude that $\|w\|_{H^{-1}(T)} \lesssim \|\widehat{w}\|_{H^{-1}(\widehat{T})}$. The proof of the converse inequality works analogously. This concludes (M4), where C_{dual} depends only on d, C_γ, and $\widehat{\mathcal{T}}_0$.

4.5.3 Verification of (R1)–(R3)

Let $\mathcal{T}_\bullet \in \mathbb{T}$, $\mathcal{T}_\circ \in \texttt{refine}(\mathcal{T}_\bullet)$, and $T \in \mathcal{T}_\bullet$. The son estimate (R1) is trivially satisfied with $C_{\rm son} = 2^d$, since each refined element is split into exactly 2^d elements. Moreover, the union of sons property (R2) holds by definition.

To see the reduction property (R3), let $T' \in \mathcal{T}_\circ$ with $T' \subsetneqq T$. Since each refined element is split into 2^d elements, we have for the corresponding elements in the parameter domain that $|\widehat{T}'| = 2^{-d}|\widehat{T}|$. Next, we prove $|T'| \le \rho_{\rm son}|T|$ with a constant $0 < \rho_{\rm son} < 1$ which depends only on d and C_γ. Indeed, we even prove for arbitrary measurable sets $\widehat{\omega}' \subseteq \widehat{\omega} \subseteq \overline{\widehat{\Omega}}$ and $\omega := \gamma(\widehat{\omega})$, $\omega' := \gamma(\widehat{\omega}')$ that $0 < |\widehat{\omega}'| \le 2^{-d}|\widehat{\omega}|$ implies that $|\omega'| \le \rho_{\rm son}|\omega|$. To see this, we argue by contradiction and assume that there are two sequences of such sets $(\widehat{\omega}_n)_{n\in\mathbb{N}}$ and $(\widehat{\omega}_n')_{n\in\mathbb{N}}$ with $|\omega_n'|/|\omega_n| \to 1$. This implies that $|\omega_n \setminus \omega_n'|/|\omega_n| \to 0$ and yields the contradiction

$$1 - 2^{-d} \le \frac{|\widehat{\omega}_n \setminus \widehat{\omega}_n'|}{|\widehat{\omega}_n|} \simeq \frac{\int_{\widehat{\omega}_n \setminus \widehat{\omega}_n'} |\det D\gamma(t)|\, dt}{\int_{\widehat{\omega}_n} |\det D\gamma(t)|\, dt} = \frac{|\omega_n \setminus \omega_n'|}{|\omega_n|} \xrightarrow{n\to\infty} 0.$$

4.5.4 Verification of (R4)

The closure estimate (R4) was first showed in the seminal works [BDD04, Ste08b]. Our analysis builds on [BGMP16, Section 3] which proves (R4) for the refinement strategy of [BG16c]; see also Remark 4.4.3. The following auxiliary result states that $\texttt{refine}(\cdot,\cdot)$ is equivalent to iterative refinement of one single element. For a mesh in the parameter domain $\widehat{\mathcal{T}}_\bullet \in \widehat{\mathbb{T}}$ and an arbitrary set $\widehat{\mathcal{M}}_\bullet$, we define $\texttt{refine}(\widehat{\mathcal{T}}_\bullet, \widehat{\mathcal{M}}_\bullet) := \texttt{refine}(\widehat{\mathcal{T}}_\bullet, \widehat{\mathcal{M}}_\bullet \cap \widehat{\mathcal{T}}_\bullet)$ and note that $\texttt{refine}(\widehat{\mathcal{T}}_\bullet, \emptyset) = \widehat{\mathcal{T}}_\bullet$.

Lemma 4.5.1. *Let $\widehat{\mathcal{T}}_\bullet \in \widehat{\mathbb{T}}$ and $\widehat{\mathcal{M}}_\bullet = \{\widehat{T}_1, \dots, \widehat{T}_n\} \subseteq \widehat{\mathcal{T}}_\bullet$, where $n \in \mathbb{N}$. Then, it holds that*

$$\texttt{refine}(\widehat{\mathcal{T}}_\bullet, \widehat{\mathcal{M}}_\bullet) = \texttt{refine}(\texttt{refine}(\dots\texttt{refine}(\widehat{\mathcal{T}}_\bullet, \{\widehat{T}_1\})\dots, \{\widehat{T}_{n-1}\}), \{\widehat{T}_n\}). \tag{4.5.2}$$

Proof. We only show that $\texttt{refine}(\widehat{\mathcal{T}}_\bullet, \widehat{\mathcal{M}}_\bullet) = \texttt{refine}(\texttt{refine}(\widehat{\mathcal{T}}_\bullet, \{\widehat{T}_1\}), \widehat{\mathcal{M}}_\bullet \setminus \{\widehat{T}_1\})$, and then (4.5.2) follows by induction. We define

$$\widehat{\mathcal{T}}_{(1)} := \texttt{refine}(\widehat{\mathcal{T}}_\bullet, \{\widehat{T}_1\}), \quad \widehat{\mathcal{T}}_{(2)} := \texttt{refine}(\widehat{\mathcal{T}}_{(1)}, \widehat{\mathcal{M}}_\bullet \setminus \{\widehat{T}_1\}),$$
$$\widehat{\mathcal{M}}_{(0)} := \widehat{\mathcal{M}}_{(0)}^{(0)} := \{\widehat{T}_1\}, \quad \widehat{\mathcal{M}}_{(1)} := \widehat{\mathcal{M}}_{(1)}^{(0)} := \widetilde{\mathcal{M}}_{(1)} := \widetilde{\mathcal{M}}_{(1)}^{(0)} := \widehat{\mathcal{M}}_\bullet \setminus \{\widehat{T}_1\}.$$

For $i \in \mathbb{N}_0$, we introduce the following notation which is conform with that of Algorithm 4.4.1:

$$\widehat{\mathcal{M}}_{(0)}^{(i+1)} := \widehat{\mathcal{M}}_{(0)}^{(i)} \cup \bigcup_{\widehat{T} \in \widehat{\mathcal{M}}_{(0)}^{(i)}} \mathbf{N}_\bullet^{\rm bad}(\widehat{T}), \quad \widehat{\mathcal{M}}_{(1)}^{(i+1)} := \widehat{\mathcal{M}}_{(1)}^{(i)} \cup \bigcup_{T \in \widehat{\mathcal{M}}_{(1)}^{(i)}} \mathbf{N}_{(1)}^{\rm bad}(\widehat{T}),$$
$$\widetilde{\mathcal{M}}_{(1)}^{(i+1)} := \widetilde{\mathcal{M}}_{(1)}^{(i)} \cup \bigcup_{\widehat{T} \in \widetilde{\mathcal{M}}_{(1)}^{(i)}} \mathbf{N}_\bullet^{\rm bad}(\widehat{T}).$$

Finally, we set

$$\widehat{\mathcal{M}}_{(0)}^{(\mathrm{end})} := \bigcup_{i\in\mathbb{N}_0} \widehat{\mathcal{M}}_{(0)}^{(i)}, \quad \widehat{\mathcal{M}}_{(1)}^{(\mathrm{end})} := \bigcup_{i\in\mathbb{N}_0} \widehat{\mathcal{M}}_{(1)}^{(i)}, \quad \widetilde{\mathcal{M}}_{(1)}^{(\mathrm{end})} := \bigcup_{i\in\mathbb{N}_0} \widetilde{\mathcal{M}}_{(1)}^{(i)}.$$

With these notations, we have that

$$\widehat{\mathcal{T}}_\bullet \setminus \widehat{\mathcal{T}}_{(1)} = \widehat{\mathcal{M}}_{(0)}^{(\mathrm{end})}, \quad \widehat{\mathcal{T}}_{(1)} \setminus \widehat{\mathcal{T}}_{(2)} = \widehat{\mathcal{M}}_{(1)}^{(\mathrm{end})},$$
$$\widehat{\mathcal{T}}_\bullet \setminus \mathtt{refine}(\widehat{\mathcal{T}}_\bullet, \widehat{\mathcal{M}}_\bullet) = \widehat{\mathcal{M}}_{(0)}^{(\mathrm{end})} \cup \widetilde{\mathcal{M}}_{(1)}^{(\mathrm{end})}.$$

To conclude the proof, we will prove that $\widehat{\mathcal{M}}_{(0)}^{(\mathrm{end})} \cup \widehat{\mathcal{M}}_{(1)}^{(\mathrm{end})} = \widehat{\mathcal{M}}_{(0)}^{(\mathrm{end})} \cup \widetilde{\mathcal{M}}_{(1)}^{(\mathrm{end})}$. To this end, we split the proof into three steps.

Step 1: We first prove that $\widehat{\mathcal{M}}_{(1)}^{(\mathrm{end})} \subseteq \widehat{\mathcal{T}}_\bullet$ by induction. Clearly, we have that $\widehat{\mathcal{M}}_{(1)}^{(0)} \subseteq \widehat{\mathcal{T}}_\bullet$. Now, let $i \in \mathbb{N}_0$ and suppose that $\widehat{\mathcal{M}}_{(1)}^{(i)} \subseteq \widehat{\mathcal{T}}_\bullet$. To see that $\widehat{\mathcal{M}}_{(1)}^{(i+1)} \subseteq \widehat{\mathcal{T}}_\bullet$, we argue by contradiction and assume that there exists $\widehat{T} \in \widehat{\mathcal{M}}_{(1)}^{(i)}$ and $\widehat{T}' \in \mathbf{N}_{(1)}^{\mathrm{bad}}(\widehat{T}) \setminus \widehat{\mathcal{T}}_\bullet$. By Lemma 3.4.2, the unique father element $\widehat{T}'_\bullet \in \widehat{\mathcal{T}}_\bullet$ with $\widehat{T}' \subsetneqq \widehat{T}'_\bullet$ satisfies that $\widehat{T}'_\bullet \in \mathbf{N}_\bullet(\widehat{T})$. Therefore, admissibility of $\widehat{\mathcal{T}}_\bullet$ proves that $|\mathrm{level}(\widehat{T}) - \mathrm{level}(\widehat{T}'_\bullet)| \le 1$, which contradicts $\mathrm{level}(\widehat{T}'_\bullet) = \mathrm{level}(\widehat{T}') - 1 = \mathrm{level}(\widehat{T}) - 2$.

Step 2: Let $\widehat{T} \in \widehat{\mathcal{M}}_{(1)}^{(\mathrm{end})}$. In this step, we prove that

$$\widehat{\mathcal{M}}_0^{(\mathrm{end})} \cup \mathbf{N}_{(1)}^{\mathrm{bad}}(\widehat{T}) = \widehat{\mathcal{M}}_{(0)}^{(\mathrm{end})} \cup \mathbf{N}_\bullet^{\mathrm{bad}}(\widehat{T}). \tag{4.5.3}$$

By Step 1, we have that $\widehat{T} \in \widehat{\mathcal{T}}_\bullet$. Lemma 3.4.2 proves that $\mathbf{N}_{(1)}^{\mathrm{bad}}(\widehat{T}) \cap \widehat{\mathcal{T}}_\bullet \subseteq \mathbf{N}_\bullet^{\mathrm{bad}}(\widehat{T})$. Using Step 1 again, we see that $\mathbf{N}_{(1)}^{\mathrm{bad}}(\widehat{T}) \subseteq \widehat{\mathcal{M}}_{(1)}^{(\mathrm{end})} \subseteq \widehat{\mathcal{T}}_\bullet$ and conclude "$\subseteq$" in (4.5.3). To see "$\supseteq$", let $\widehat{T}' \in \mathbf{N}_\bullet^{\mathrm{bad}}(\widehat{T}) \setminus \widehat{\mathcal{M}}_{(0)}^{(\mathrm{end})}$. Note that $\widehat{T}' \in \widehat{\mathcal{T}}_\bullet \cap \widehat{\mathcal{T}}_{(1)}$ since $\widehat{\mathcal{T}}_\bullet \setminus \widehat{\mathcal{T}}_{(1)} = \widehat{\mathcal{M}}_{(0)}^{(\mathrm{end})}$. There exists $\widehat{\beta} \in \widehat{\mathcal{B}}_\bullet$ with $\widehat{T}, \widehat{T}' \subseteq \mathrm{supp}(\widehat{\beta})$. By admissibility of $\widehat{\mathcal{T}}_\bullet \in \widehat{\mathbb{T}}$, $\mathrm{level}(\widehat{T}') = \mathrm{level}(\widehat{T}) - 1$, and (3.4.17), we see that $\mathrm{level}(\widehat{\beta}) = \mathrm{level}(\widehat{T}') =: k'$. Hence, (3.4.13) yields that $\mathrm{supp}(\widehat{\beta}) \subseteq \widehat{\Omega}_\bullet^{k'}$ as well as $\mathrm{supp}(\widehat{\beta}) \not\subseteq \widehat{\Omega}_\bullet^{k'+1}$. The definition of k' and (3.4.10) show that $\widehat{T}' \not\subseteq \widehat{\Omega}_{(1)}^{k'+1}$. We conclude that $\mathrm{supp}(\widehat{\beta}) \subseteq \widehat{\Omega}_\bullet^{k'} \subseteq \widehat{\Omega}_{(1)}^{k'}$ and $\mathrm{supp}(\widehat{\beta}) \not\subseteq \widehat{\Omega}_{(1)}^{k'+1}$, since $\widehat{\mathcal{T}}_{(1)} \ni \widehat{T}' \subseteq \mathrm{supp}(\widehat{\beta})$. Therefore, (3.4.13) shows that $\widehat{\beta} \in \widehat{\mathcal{B}}_{(1)}$. Altogether, we have that $\widehat{T}' \in \mathbf{N}_{(1)}^{\mathrm{bad}}(\widehat{T})$.

Step 3: Finally, we prove that $\widehat{\mathcal{M}}_{(0)}^{(\mathrm{end})} \cup \widehat{\mathcal{M}}_{(1)}^{(i)} = \widehat{\mathcal{M}}_{(0)}^{(\mathrm{end})} \cup \widetilde{\mathcal{M}}_{(1)}^{(i)}$ by induction on $i \in \mathbb{N}_0$. In particular, this will imply that $\widehat{\mathcal{M}}_{(0)}^{(\mathrm{end})} \cup \widehat{\mathcal{M}}_{(1)}^{(\mathrm{end})} = \widehat{\mathcal{M}}_{(0)}^{(\mathrm{end})} \cup \widetilde{\mathcal{M}}_{(1)}^{(\mathrm{end})}$. For $i = 0$, the claim follows from $\widehat{\mathcal{M}}_{(1)}^{(0)} = \widetilde{\mathcal{M}}_{(1)}^{(0)}$. By Step 2, the induction step works as follows:

$$\begin{aligned}
\widehat{\mathcal{M}}_{(0)}^{(\mathrm{end})} \cup \widehat{\mathcal{M}}_{(1)}^{(i+1)} &= \widehat{\mathcal{M}}_{(0)}^{(\mathrm{end})} \cup \widehat{\mathcal{M}}_{(1)}^{(i)} \cup \bigcup_{\widehat{T} \in \widehat{\mathcal{M}}_{(1)}^{(i)}} \mathbf{N}_{(1)}^{\mathrm{bad}}(\widehat{T}) \\
&\overset{(4.5.3)}{=} \widehat{\mathcal{M}}_{(0)}^{(\mathrm{end})} \cup \widehat{\mathcal{M}}_{(1)}^{(i)} \cup \bigcup_{\widehat{T} \in \widehat{\mathcal{M}}_{(0)}^{(\mathrm{end})} \cup \widehat{\mathcal{M}}_{(1)}^{(i)}} \mathbf{N}_{\bullet}^{\mathrm{bad}}(\widehat{T}) \\
&= \widehat{\mathcal{M}}_{(0)}^{(\mathrm{end})} \cup \widetilde{\mathcal{M}}_{(1)}^{(i)} \cup \bigcup_{\widehat{T} \in \widehat{\mathcal{M}}_{(0)}^{(\mathrm{end})} \cup \widetilde{\mathcal{M}}_{(1)}^{(i)}} \mathbf{N}_{\bullet}^{\mathrm{bad}}(\widehat{T}) \\
&= \widehat{\mathcal{M}}_{(0)}^{(\mathrm{end})} \cup \widetilde{\mathcal{M}}_{(1)}^{(i+1)}.
\end{aligned}$$

This concludes the proof. □

Let $\widehat{\mathcal{T}}_{\bullet} \in \widehat{\mathbb{T}}$. For $\widehat{T} \in \widehat{\mathcal{T}}_{\bullet}$, let $\mathrm{mid}(\widehat{T})$ denote its midpoint in the parameter domain. Let $\widehat{T} \in \widehat{\mathcal{T}}_{\bullet}$ and $\widehat{T}' \in \mathbf{N}_{\bullet}(\widehat{T})$. Hence, there is $\widehat{\beta} \in \widehat{\mathcal{B}}_{\bullet}$ such that $\widehat{T}, \widehat{T}' \subseteq \mathrm{supp}(\widehat{\beta})$. In particular, it holds that $|\mathrm{mid}(\widehat{T}) - \mathrm{mid}(\widehat{T}')| \le \mathrm{diam}(\mathrm{supp}(\widehat{\beta}))$. By admissibility of $\widehat{\mathcal{T}}_{\bullet}$ and (3.4.17), we see that $|\mathrm{level}(\widehat{\beta}) - \mathrm{level}(\widehat{T})| \le 1$. This proves that

$$|\mathrm{mid}(\widehat{T}) - \mathrm{mid}(\widehat{T}')| \le C\, 2^{-\mathrm{level}(\widehat{T})} \quad \text{for all } \widehat{T} \in \widehat{\mathcal{T}}_{\bullet}, \widehat{T}' \in \mathbf{N}_{\bullet}(\widehat{T}), \tag{4.5.4}$$

where $C > 0$ depends only on d, $\widehat{\mathcal{T}}_0$ and $(p_1, \dots, p_d)$. With this observation, we can prove the following lemma. The proof follows along the lines of [BGMP16, Lemma 11], but is included here for completeness.

Lemma 4.5.2. *There exists a constant $C > 0$ such that for all $\widehat{\mathcal{T}}_{\bullet} \in \widehat{\mathbb{T}}$, $\widehat{T}' \in \widehat{\mathcal{T}}_{\bullet}$, and $\widehat{\mathcal{T}}_{\circ} = \mathtt{refine}(\widehat{\mathcal{T}}_{\bullet}, \{\widehat{T}'\})$, it holds that*

$$|\mathrm{mid}(\widehat{T}) - \mathrm{mid}(\widehat{T}')| \le C\, 2^{-\mathrm{level}(\widehat{T})} \quad \text{for all } \widehat{T} \in \widehat{\mathcal{T}}_{\circ} \setminus \widehat{\mathcal{T}}_{\bullet}, \tag{4.5.5}$$

where $C > 0$ depends only on d, $\widehat{\mathcal{T}}_0$, and $(p_1, \dots, p_d)$.

Proof. $\widehat{T} \in \widehat{\mathcal{T}}_{\circ} \setminus \widehat{\mathcal{T}}_{\bullet}$ implies the existence of a sequence $\widehat{T}' = \widehat{T}_J, \widehat{T}_{J-1}, \dots, \widehat{T}_0$ such that $\widehat{T}_{j-1} \in \mathbf{N}_{\bullet}^{\mathrm{bad}}(\widehat{T}_j)$ for $j = 1, \dots, J$ and $\widehat{T}$ is a child of $\widehat{T}_0$, i.e., $\widehat{T} \subsetneqq \widehat{T}_0$ and $\mathrm{level}(\widehat{T}) = \mathrm{level}(\widehat{T}_0) + 1$. Since $\mathrm{level}(\widehat{T}_{j-1}) = \mathrm{level}(\widehat{T}_j) - 1$, it follows that

$$\mathrm{level}(\widehat{T}_j) = \mathrm{level}(\widehat{T}_0) + j. \tag{4.5.6}$$

The triangle inequality proves that

$$\begin{aligned}
|\mathrm{mid}(\widehat{T}) - \mathrm{mid}(\widehat{T}')| &\le |\mathrm{mid}(\widehat{T}) - \mathrm{mid}(\widehat{T}_0)| + |\mathrm{mid}(\widehat{T}_0) - \mathrm{mid}(\widehat{T}')| \\
&\le |\mathrm{mid}(\widehat{T}) - \mathrm{mid}(\widehat{T}_0)| + \sum_{j=1}^{J} |\mathrm{mid}(\widehat{T}_j) - \mathrm{mid}(\widehat{T}_{j-1})|
\end{aligned}$$

Further, there holds that

$$|\mathrm{mid}(\widehat{T}) - \mathrm{mid}(\widehat{T}_0)| \lesssim 2^{-\mathrm{level}(\widehat{T})},$$

where the hidden constant depends only on $\widehat{\mathcal{T}}_0$ and d. With (4.5.4) and (4.5.6), we see that

$$\sum_{j=1}^{J} |\mathrm{mid}(\widehat{T}_j) - \mathrm{mid}(\widehat{T}_{j-1})| \overset{(4.5.4)}{\lesssim} \sum_{j=1}^{J} 2^{-\mathrm{level}(\widehat{T}_j)} \overset{(4.5.6)}{=} \sum_{j=1}^{J} 2^{-\mathrm{level}(\widehat{T}_0)-j} \leq 2^{-\mathrm{level}(\widehat{T})-1},$$

which concludes the proof. □

Finally, let $\widehat{\mathcal{T}}_\bullet \in \widehat{\mathbb{T}}$ and $\widehat{T} \in \widehat{\mathcal{T}}_\bullet$. We abbreviate $\widehat{\mathcal{T}}_\circ = \mathtt{refine}(\widehat{\mathcal{T}}_\bullet, \{\widehat{T}\})$. Then, there holds that

$$\mathrm{level}(\widehat{T}') \leq \mathrm{level}(\widehat{T}) + 1 \quad \text{for all refined elements } \widehat{T}' \in \widehat{\mathcal{T}}_\circ \setminus \widehat{\mathcal{T}}_\bullet. \tag{4.5.7}$$

To see this, note that all elements $\widehat{T}'' \in \widehat{\mathcal{T}}_\bullet \setminus \widehat{\mathcal{T}}_\circ$, which are refined, satisfy that $\widehat{T}'' = \widehat{T}$ or $\mathrm{level}(\widehat{T}'') \leq \mathrm{level}(\widehat{T}) - 1$. Therefore, their children satisfy that $\mathrm{level}(\widehat{T}') \leq \mathrm{level}(\widehat{T}) + 1$. With Lemma 4.5.1 and Lemma 4.5.2 and this last observation, we can argue as in the proof of [BGMP16, Theorem 12] to show the closure estimate (R4). The constant $C_{\mathrm{clos}} > 0$ depends only on $d, \widehat{\mathcal{T}}_0$, and $(p_1, \dots, p_d)$.

4.5.5 Verification of (R5)

We prove (R5) in the parameter domain $\widehat{\Omega}$. Let $\widehat{\mathcal{T}}_\bullet, \widehat{\mathcal{T}}_\star \in \widehat{\mathbb{T}}$ be two admissible hierarchical meshes. We define the overlay

$$\widehat{\mathcal{T}}_\circ := \big\{\widehat{T}_\bullet \in \widehat{\mathcal{T}}_\bullet \,:\, \exists \widehat{T}_\star \in \widehat{\mathcal{T}}_\star \quad \widehat{T}_\bullet \subseteq \widehat{T}_\star\big\} \cup \big\{\widehat{T}_\star \in \widehat{\mathcal{T}}_\star \,:\, \exists \widehat{T}_\bullet \in \widehat{\mathcal{T}}_\bullet \quad \widehat{T}_\star \subseteq \widehat{T}_\bullet\big\}. \tag{4.5.8}$$

Note that $\widehat{\mathcal{T}}_\circ$ is a hierarchical mesh with hierarchical domains $\widehat{\Omega}_\circ^k = \widehat{\Omega}_\bullet^k \cup \widehat{\Omega}_\star^k$ for $k \in \mathbb{N}_0$. In particular, $\widehat{\mathcal{T}}_\circ$ is finer than $\widehat{\mathcal{T}}_\bullet$ and $\widehat{\mathcal{T}}_\star$. Moreover, the overlay estimate easily follows from the definition of $\widehat{\mathcal{T}}_\circ$. It remains to prove that $\widehat{\mathcal{T}}_\circ$ is admissible. To see this, let $\widehat{T}, \widehat{T}' \in \widehat{\mathcal{T}}_\circ$ with $\widehat{T}' \in \mathbf{N}_\circ(\widehat{T})$, i.e., there exists $\widehat{\beta}_\circ \in \widehat{\mathcal{B}}_\circ$ such that $|\widehat{T} \cap \mathrm{supp}(\widehat{\beta}_\circ)| \neq 0 \neq |\widehat{T}' \cap \mathrm{supp}(\widehat{\beta}_\circ)|$. Without loss of generality, we suppose that $\mathrm{level}(\widehat{T}) \geq \mathrm{level}(\widehat{T}')$ and $\widehat{T} \in \widehat{\mathcal{T}}_\bullet$. If $\widehat{T}' \in \widehat{\mathcal{T}}_\bullet$, Lemma 3.4.2 shows that $\widehat{T}' \in \mathbf{N}_\bullet(\widehat{T})$, and admissibility of $\widehat{\mathcal{T}}_\bullet$ implies that $|\mathrm{level}(\widehat{T}) - \mathrm{level}(\widehat{T}')| \leq 1$. Now, let $\widehat{T}' \in \widehat{\mathcal{T}}_\star$. By definition of the overlay, there exists $\widehat{T}_\bullet' \in \widehat{\mathcal{T}}_\bullet$ with $\widehat{T}' \subseteq \widehat{T}_\bullet'$ and $\mathrm{level}(\widehat{T}_\bullet') \leq \mathrm{level}(\widehat{T}')$. Further, Lemma 3.4.2 provides some (not necessarily unique) $\widehat{\beta}_\bullet \in \widehat{\mathcal{B}}_\bullet$ such that $\mathrm{supp}(\widehat{\beta}_\circ) \subseteq \mathrm{supp}(\widehat{\beta}_\bullet)$. Hence, $|\widehat{T} \cap \mathrm{supp}(\widehat{\beta}_\bullet)| \neq 0 \neq |\widehat{T}_\bullet' \cap \mathrm{supp}(\widehat{\beta}_\bullet)|$, i.e., $\widehat{T}_\bullet' \in \mathbf{N}_\bullet(\widehat{T})$. Since $\widehat{\mathcal{T}}_\bullet \in \widehat{\mathbb{T}}$, it follows that $|\mathrm{level}(\widehat{T}) - \mathrm{level}(\widehat{T}_\bullet')| \leq 1$. Altogether, we see that

$$|\mathrm{level}(\widehat{T}) - \mathrm{level}(\widehat{T}')| = \mathrm{level}(\widehat{T}) - \mathrm{level}(\widehat{T}') \leq \mathrm{level}(\widehat{T}) - \mathrm{level}(\widehat{T}_\bullet') \leq 1.$$

This concludes the proof of (R5).

4.5.6 Verification of (S1)

Let $T \in \mathcal{T}_\bullet \in \mathbb{T}$ and $V_\bullet \in \mathcal{X}_\bullet$. Define $\widehat{V}_\bullet := V_\bullet \circ \gamma \in \widehat{\mathcal{X}}_\bullet$ and $\widehat{T} := \gamma^{-1}(T) \in \widehat{\mathcal{T}}_\bullet$. Regularity (4.4.2) of γ proves for $j \in \{0,1,2\}$ that

$$\|V_\bullet\|_{H^j(T)} \simeq \|\widehat{V}_\bullet\|_{H^j(\widehat{T})}, \tag{4.5.9}$$

where the hidden constants depend only on d, D and C_γ. Since $\widehat{V}_\bullet$ is a $\widehat{\mathcal{T}}_\bullet$-piecewise tensor-product polynomial, a standard inverse estimate shows for $j,k \in \{0,1,2\}$ with $k \le j$ that

$$|\widehat{T}|^{(j-k)/d} \|\widehat{V}_\bullet\|_{H^j(\widehat{T})} \lesssim \|\widehat{V}_\bullet\|_{H^k(\widehat{T})}, \tag{4.5.10}$$

where the hidden constant depends only on d, D, $\widehat{\mathcal{T}}_0$, and $(p_1,\dots,p_d)$. Together, (4.5.9)–(4.5.10) conclude the proof of (S1), where $C_{\rm inv}$ depends only on d, D, C_γ, $\widehat{\mathcal{T}}_0$, and $(p_1,\dots,p_d)$.

4.5.7 Verification of (S2)

In (4.4.11), we already saw that $\mathcal{T}_\circ \in \texttt{refine}(\mathcal{T}_\bullet)$ implies nestedness of the corresponding ansatz spaces $\mathcal{X}_\bullet \subseteq \mathcal{X}_\circ$.

4.5.8 Basis of hierarchical splines which vanish on the boundary

In this section, we characterize a basis of the hierarchical splines $\widehat{\mathcal{X}}_\bullet$ that vanish on the boundary. Clearly, this provides a corresponding basis of $\mathcal{X}_\bullet$ in the physical domain as well. First, we determine the restriction (in the sense of traces) of the hierarchical basis $\widehat{\mathcal{B}}_\bullet$ to a facet of the boundary. It turns out that this restriction coincides with the set of $(d-1)$-dimensional hierarchical B-splines.

Proposition 4.5.3. *Let $\widehat{\mathcal{T}}_\bullet$ be an arbitrary hierarchical mesh on the parameter domain $\widehat{\Omega}$. For $\widehat{E} := [0,1]^{i'-1} \times \{e\} \times [0,1]^{d-i'}$ with some $i' \in \{1,\dots,d\}$ and some $e \in \{0,1\}$, set*

$$\widehat{\mathcal{K}}_{{\rm uni}(0),\widehat{E}} := (\widehat{\mathcal{K}}_{1({\rm uni}(0))}, \dots, \widehat{\mathcal{K}}_{(i'-1)({\rm uni}(0))}, \widehat{\mathcal{K}}_{(i'+1)({\rm uni}(0))}, \dots, \widehat{\mathcal{K}}_{d({\rm uni}(0))}),$$

and for $k \in \mathbb{N}_0$

$$\widehat{\Omega}^k_{\bullet,\widehat{E}} := \big\{(s_1,\dots,s_{i'-1},s_{i'+1},\dots,s_d) \,:\, (s_1,\dots,s_d) \in \widehat{\Omega}^k_\bullet \cap \widehat{E}\big\}.$$

Moreover, let $\widehat{\mathcal{T}}_{\bullet,\widehat{E}}$ be the corresponding hierarchical mesh and $\widehat{\mathcal{B}}_{\bullet,\widehat{E}}$ the corresponding hierarchical basis. Then, there holds[2] that $\widehat{\mathcal{B}}_{\bullet,\widehat{E}} = \big\{\widehat{\beta}|_{\widehat{E}} \,:\, \widehat{\beta} \in \widehat{\mathcal{B}}_\bullet \wedge \widehat{\beta}|_{\widehat{E}} \neq 0\big\}$. Moreover, the restriction $(\cdot)|_{\widehat{E}} : \widehat{\mathcal{B}}_\bullet \to \widehat{\mathcal{B}}_{\bullet,\widehat{E}}$ is essentially injective, i.e., for $\widehat{\beta}_1, \widehat{\beta}_2 \in \widehat{\mathcal{B}}_\bullet$ with $\widehat{\beta}_1 \neq \widehat{\beta}_2$ and $\widehat{\beta}_1|_{\widehat{E}} \neq 0$, it follows that $\widehat{\beta}_1|_{\widehat{E}} \neq \widehat{\beta}_2|_{\widehat{E}}$.

[2] Actually, the set on left-hand side consists of functions defined on $[0,1]^{d-1}$, whereas the right-hand side functions are defined on $\widehat{E}$. However, clearly these functions can be identified.

Proof. We prove the assertion in two steps.
Step 1: Let $k \in \mathbb{N}_0$. We recall that the knot vectors $\widehat{\mathcal{K}}_{i(\mathrm{uni}(k))}$ are p_i-open for all $i \in \{1, \dots, d\}$. In particular, this implies that the corresponding one-dimensional B-splines $\widehat{\mathcal{B}}_{i(\mathrm{uni}(k))}$ are interpolatoric at the end points $e \in \{0, 1\}$: This means that the first B-spline (i.e., $\widehat{B}_{i(\mathrm{uni}(k)),1,p_i}$) is equal to one at 0 and vanishes at 1; the last B-spline (i.e., $\widehat{B}_{i(\mathrm{uni}(k)),N_{i(\mathrm{uni}(k))},p_i}$) is equal to one at 1 and vanishes at 0; all other B-splines of $\widehat{\mathcal{B}}_{i(\mathrm{uni}(k))}$ vanish at 0 and 1; see Lemma 3.2.1 (vi) and (iv).
Step 2: For $k \in \mathbb{N}_0$, let $\widehat{\mathcal{B}}_{\mathrm{uni}(k),\widehat{E}}$ be the set of tensor product B-splines induced by the reduced knots $\widehat{\mathcal{K}}_{\mathrm{uni}(k),\widehat{E}}$ which are defined analogously as $\widehat{\mathcal{K}}_{\mathrm{uni}(0),\widehat{E}}$. Since $\widehat{\mathcal{K}}_{i(\mathrm{uni}(k))}$ is p_i-open, it holds that $\widehat{\mathcal{B}}_{\mathrm{uni}(k),\widehat{E}} = \{\widehat{\beta}|_{\widehat{E}} : \widehat{\beta} \in \widehat{\mathcal{B}}_{\mathrm{uni}(k)} \wedge \widehat{\beta}|_{\widehat{E}} \neq 0\}$; see also Step 1. Then, the identity (3.4.13) shows that

$$\widehat{\mathcal{B}}_{\bullet,\widehat{E}} = \bigcup_{k \in \mathbb{N}_0} \Big\{\widehat{\beta}|_{\widehat{E}} : \widehat{\beta} \in \widehat{\mathcal{B}}_{\mathrm{uni}(k)} \wedge \widehat{\beta}|_{\widehat{E}} \neq 0 \wedge \mathrm{supp}(\widehat{\beta}|_{\widehat{E}}) \subseteq \widehat{\Omega}^k_{\bullet,\widehat{E}} \wedge \mathrm{supp}(\widehat{\beta}|_{\widehat{E}}) \not\subseteq \widehat{\Omega}^{k+1}_{\bullet,\widehat{E}}\Big\}. \tag{4.5.11}$$

Let $\widehat{\beta} \in \widehat{\mathcal{B}}_{\mathrm{uni}(k)}$ for some $k \in \mathbb{N}_0$ with $\widehat{\beta}|_{\widehat{E}} \neq 0$. We set $J := 1$ for $e = 0$ resp. $J := N_{i'(\mathrm{uni}(k))}$ for $e = 1$. Since the trace of $\widehat{B}_{i'(\mathrm{uni}(k)),j_{i'},p_{i'}}$ at e does not vanish only if $j_{i'} = J$ (see Step 1), $\widehat{\beta}$ must be of the form

$$\widehat{\beta}(s_1, \dots, s_d) = \widehat{B}_{i'(\mathrm{uni}(k)),J,p_{i'}}(s_{i'}) \prod_{\substack{i=1 \\ i \neq i'}}^{d} \widehat{B}_{i(\mathrm{uni}(k)),j_i,p_i}(s_i), \tag{4.5.12}$$

where the first factor is one if $s_{i'} = e$ and satisfies that

$$\mathrm{supp}(\widehat{B}_{i'(\mathrm{uni}(k)),J,p_{i'}}) = [t_{i'(\mathrm{uni}(k)),J-1}, t_{i'(\mathrm{uni}(k)),J+p_{i'}+1}].$$

In particular, this shows that $\mathrm{supp}(\widehat{\beta})$ is the union of elements $\widehat{T} \in \widehat{\mathcal{T}}_{\mathrm{uni}(k)}$ with non-empty intersection with $\widehat{E}$. Hence $\mathrm{supp}(\widehat{\beta}|_{\widehat{E}}) \subseteq \widehat{\Omega}^k_{\bullet,\widehat{E}}$ is equivalent to $\mathrm{supp}(\widehat{\beta}) \subseteq \widehat{\Omega}^k_\bullet$, and $\mathrm{supp}(\widehat{\beta}|_{\widehat{E}}) \not\subseteq \widehat{\Omega}^{k+1}_{\bullet,\widehat{E}}$ is equivalent to $\mathrm{supp}(\widehat{\beta}) \not\subseteq \widehat{\Omega}^{k+1}_\bullet$. Therefore, (4.5.11) becomes

$$\widehat{\mathcal{B}}_{\bullet,\widehat{E}} = \bigcup_{k \in \mathbb{N}_0} \Big\{\widehat{\beta}|_{\widehat{E}} : \widehat{\beta} \in \widehat{\mathcal{B}}_{\mathrm{uni}(k)} \wedge \widehat{\beta}|_{\widehat{E}} \neq 0 \wedge \mathrm{supp}(\widehat{\beta}) \subseteq \widehat{\Omega}^k_\bullet \wedge \mathrm{supp}(\widehat{\beta}) \not\subseteq \widehat{\Omega}^{k+1}_\bullet\Big\}.$$

Together with (3.4.13), this shows that $\widehat{\mathcal{B}}_{\bullet,\widehat{E}} = \{\widehat{\beta}|_{\widehat{E}} : \widehat{\beta} \in \widehat{\mathcal{B}}_\bullet \wedge \widehat{\beta}|_{\widehat{E}} \neq 0\}$. Finally, let $\widehat{\beta}_1, \widehat{\beta}_2 \in \widehat{\mathcal{B}}_\bullet$ with $\widehat{\beta}_1|_{\widehat{E}} \neq 0$. If $\widehat{\beta}_1|_{\widehat{E}} = \widehat{\beta}_2|_{\widehat{E}}$, then (4.5.12) already implies that $\widehat{\beta}_1 = \widehat{\beta}_2$. This concludes the proof. □

Corollary 4.5.4. *Let $\widehat{\mathcal{T}}_\bullet$ be an arbitrary hierarchical mesh on the parameter domain $\widehat{\Omega}$. Then, $\{\widehat{\beta} \in \widehat{\mathcal{B}}^D_\bullet : \widehat{\beta}|_{\partial\widehat{\Omega}} = 0\}$ and $\{\mathrm{Trunc}_\bullet(\widehat{\beta}) : \widehat{\beta} \in \widehat{\mathcal{B}}^D_\bullet \wedge \widehat{\beta}|_{\partial\widehat{\Omega}} = 0\}$ are bases of $\widehat{\mathcal{X}}_\bullet$, where* $\mathrm{Trunc}_\bullet(\cdot)$ *denotes the componentwise truncation operator from Section 3.4.3.*

Proof. We prove the assertion in two steps. Without loss of generality, we can assume that $D = 1$.

Step 1: Linear independence as well as $\{\widehat{\beta} \in \widehat{\mathcal{B}}_\bullet : \widehat{\beta}|_{\partial\widehat{\Omega}} = 0\} \subseteq \widehat{\mathcal{X}}_\bullet$ are obvious. To see $\widehat{\mathcal{X}}_\bullet \subseteq \operatorname{span}\{\widehat{\beta} \in \widehat{\mathcal{B}}_\bullet : \widehat{\beta}|_{\partial\widehat{\Omega}} = 0\}$, let $\widehat{V}_\bullet \in \widehat{\mathcal{X}}_\bullet$. Consider the unique representation $\widehat{V}_\bullet = \sum_{\widehat{\beta}\in\widehat{\mathcal{B}}_\bullet} a_{\widehat{\beta}}\widehat{\beta}$ with $a_{\widehat{\beta}} \in \mathbb{R}$. For arbitrary $\widehat{\beta} \in \widehat{\mathcal{B}}_\bullet$ with $\widehat{\beta}|_{\partial\widehat{\Omega}} \neq 0$, we have to prove $a_{\widehat{\beta}} = 0$, i.e., we have to show the implication

$$\sum_{\substack{\widehat{\beta}\in\widehat{\mathcal{B}}_\bullet \\ \widehat{\beta}|_{\partial\widehat{\Omega}}\neq 0}} a_{\widehat{\beta}}\widehat{\beta}|_{\partial\widehat{\Omega}} = 0 \quad\Longrightarrow\quad \Big(\forall \widehat{\beta} \in \widehat{\mathcal{B}}_\bullet \text{ with } \widehat{\beta}|_{\partial\widehat{\Omega}} \neq 0 \quad a_{\widehat{\beta}} = 0\Big).$$

Let $\widehat{E} = [0,1]^{i'-1} \times \{e\} \times [0,1]^{d-i'}$ with $i' \in \{1,\dots,d\}$ and $\sum_{\widehat{\beta}\in\widehat{\mathcal{B}}_\bullet\wedge\widehat{\beta}|_{\widehat{E}}\neq 0} a_{\widehat{\beta}}\widehat{\beta}|_{\widehat{E}} = 0$. According to Proposition 4.5.3, the family $\big(\widehat{\beta}|_{\widehat{E}} : \widehat{\beta} \in \widehat{\mathcal{B}}_\bullet \wedge \widehat{\beta}|_{\widehat{E}} \neq 0\big)$ is linearly independent. Hence, $a_{\widehat{\beta}} = 0$ for $\widehat{\beta} \in \widehat{\mathcal{B}}_\bullet$ with $\widehat{\beta}|_{\widehat{E}} \neq 0$. Since $\partial\widehat{\Omega}$ is the union of such facets $\widehat{E}$, this concludes Step 1.

Step 2: We show that the second set is a basis. In Section 3.4.3, we saw that $\{\mathrm{Trunc}_\bullet\widehat{\beta} : \widehat{\beta} \in \widehat{\mathcal{B}}_\bullet\}$ is a basis of the space of all hierarchical splines $\widehat{\mathcal{S}}^{(p_1,\dots,p_d)}(\widehat{\mathcal{K}}_0, \widehat{\mathcal{T}}_\bullet)$. Moreover, (3.4.22) states that $0 \le \mathrm{Trunc}_\bullet(\widehat{\beta}) \le \widehat{\beta}$. Thus, we see that $\{\mathrm{Trunc}_\bullet(\widehat{\beta}) : \widehat{\beta} \in \widehat{\mathcal{B}}_\bullet \wedge \widehat{\beta}|_{\partial\widehat{\Omega}} = 0\}$ is a subset of $\widehat{\mathcal{X}}_\bullet$, and has the same cardinality as $\{\widehat{\beta} \in \widehat{\mathcal{B}}_\bullet : \widehat{\beta}|_{\partial\widehat{\Omega}} = 0\}$. With Step 1, we conclude the proof. □

4.5.9 Verification of (S3)

We show the assertion in the parameter domain. Without loss of generality, we may assume that $D = 1$. The higher-dimensional case follows immediately from the one-dimensional case, since $\widehat{\mathcal{X}}_\bullet$ is the D-dimensional product of one-dimensional hierarchical spline spaces. For arbitrary $q_{\mathrm{proj}} \in \mathbb{N}_0$ (which will be fixed later in Section 4.5.10 to be $q_{\mathrm{proj}} := 2(p_{\max}+1)$), we set $q_{\mathrm{loc}} := q_{\mathrm{proj}} + 2(p_{\max} + 1)$. Let $\widehat{\mathcal{T}}_\bullet \in \widehat{\mathbb{T}}$, $\widehat{\mathcal{T}}_\circ \in \mathtt{refine}(\widehat{\mathcal{T}}_\bullet)$, and $\widehat{V}_\circ \in \widehat{\mathcal{X}}_\circ$. First, we show that

$$\Pi_\bullet^{q_{\mathrm{loc}}}(\widehat{T}) \subseteq \widehat{\mathcal{T}}_\bullet \cap \widehat{\mathcal{T}}_\circ \quad \text{for all } \widehat{T} \in \widehat{\mathcal{T}}_\bullet \setminus \Pi_\bullet^{q_{\mathrm{loc}}}(\widehat{\mathcal{T}}_\bullet \setminus \widehat{\mathcal{T}}_\circ) \tag{4.5.13}$$

To this end, we argue by contradiction and assume that there exists $\widehat{T}' \in \Pi_\bullet^{q_{\mathrm{loc}}}(\widehat{T})$ with $\widehat{T}' \notin \widehat{\mathcal{T}}_\bullet \cap \widehat{\mathcal{T}}_\circ$. This is equivalent to $\widehat{T} \in \Pi_\bullet^{q_{\mathrm{loc}}}(\widehat{T}')$ and $\widehat{T}' \in \widehat{\mathcal{T}}_\bullet \setminus \widehat{\mathcal{T}}_\circ$. This implies that $\widehat{T} \in \Pi_\bullet^{q_{\mathrm{loc}}}(\widehat{\mathcal{T}}_\bullet \setminus \widehat{\mathcal{T}}_\circ)$, contradicts $\widehat{T} \in \widehat{\mathcal{T}}_\bullet \setminus \Pi_\bullet^{q_{\mathrm{loc}}}(\widehat{\mathcal{T}}_\bullet \setminus \widehat{\mathcal{T}}_\circ)$, and hence proves (4.5.13). According to Corollary 4.5.4, it holds that

$$\big\{\widehat{V}_\bullet|_{\pi_\bullet^{q_{\mathrm{proj}}}(\widehat{T})} : \widehat{V}_\bullet \in \widehat{\mathcal{X}}_\bullet\big\} = \operatorname{span}\big\{\widehat{\beta}|_{\pi_\bullet^{q_{\mathrm{proj}}}(\widehat{T})} : \widehat{\beta} \in \widehat{\mathcal{B}}_\bullet \wedge \widehat{\beta}|_{\partial\widehat{\Omega}} = 0 \wedge |\operatorname{supp}(\widehat{\beta}) \cap \pi_\bullet^{q_{\mathrm{proj}}}(\widehat{T})| > 0\big\},$$

as well as

$$\big\{\widehat{V}_\circ|_{\pi_\bullet^{q_{\mathrm{proj}}}(\widehat{T})} : \widehat{V}_\circ \in \widehat{\mathcal{X}}_\circ\big\} = \operatorname{span}\big\{\widehat{\beta}|_{\pi_\bullet^{q_{\mathrm{proj}}}(\widehat{T})} : \widehat{\beta} \in \widehat{\mathcal{B}}_\circ \wedge \widehat{\beta}|_{\partial\widehat{\Omega}} = 0 \wedge |\operatorname{supp}(\widehat{\beta}) \cap \pi_\bullet^{q_{\mathrm{proj}}}(\widehat{T})| > 0\big\}.$$

We prove that

$$\big\{\widehat{\beta} \in \widehat{\mathcal{B}}_\bullet : |\operatorname{supp}(\widehat{\beta}) \cap \pi_\bullet^{q_{\mathrm{proj}}}(\widehat{T})| > 0\big\} = \big\{\widehat{\beta} \in \widehat{\mathcal{B}}_\circ : |\operatorname{supp}(\widehat{\beta}) \cap \pi_\bullet^{q_{\mathrm{proj}}}(\widehat{T})| > 0\big\}, \tag{4.5.14}$$

which concludes (S3). First, let $\widehat{\beta}$ be an element of the left set. By Remark 3.4.4, this implies that $\operatorname{supp}(\widehat{\beta}) \subseteq \pi_\bullet^{q_{\rm loc}}(\widehat{T})$. Together with (4.5.13), we see that $\operatorname{supp}(\widehat{\beta}) \subseteq \bigcup(\widehat{\mathcal{T}}_\bullet \cap \widehat{\mathcal{T}}_\circ)$. This proves that no element within $\operatorname{supp}(\widehat{\beta})$ is changed during refinement, i.e., $\widehat{\Omega}_\bullet^k \cap \operatorname{supp}(\widehat{\beta}) = \widehat{\Omega}_\circ^k \cap \operatorname{supp}(\widehat{\beta})$ for all $k \in \mathbb{N}_0$. Thus, (3.4.13) proves that $\widehat{\beta} \in \widehat{\mathcal{B}}_\circ$. The proof works the same if we start with some $\widehat{\beta}$ in the right set. This proves (4.5.14) and therefore (S3).

4.5.10 Verification of (S4)–(S6)

Given $\mathcal{T}_\bullet \in \mathbb{T}$, we introduce a suitable Scott–Zhang type operator $J_\bullet : H_0^1(\Omega)^D \to \mathcal{X}_\bullet$ which satisfies (S4)–(S6). To this end, it is sufficient to construct a corresponding operator $\widehat{J}_\bullet : H_0^1(\widehat{\Omega}) \to \widehat{\mathcal{X}}_\bullet$ in the parameter domain, and to define

$$J_\bullet v := \big(\widehat{J}_\bullet(v \circ \gamma)\big) \circ \gamma^{-1} \quad \text{for all } v \in H_0^1(\Omega). \tag{4.5.15}$$

By regularity (4.4.2) of γ, the properties (S4)–(S6) immediately transfer from the parameter domain $\widehat{\Omega}$ to the physical domain Ω. Since $\widehat{\mathcal{X}}_\bullet$ is the D-dimensional product of one-dimensional hierarchical spline spaces, $\widehat{J}_\bullet$ can be defined componentwise for the higher-dimensional case. Hence, we may assume without loss of generality that $D = 1$. We define $\widehat{J}_\bullet$ essentially as the operator $\widehat{I}_\bullet$ from Section 3.4.5, where we drop the basis functions which does not vanish at the boundary

$$\widehat{J}_\bullet : H_0^1(\widehat{\Omega}) \to \widehat{\mathcal{X}}_\bullet, \quad \widehat{v} \mapsto \sum_{\substack{\widehat{\beta} \in \widehat{\mathcal{B}}_\bullet \\ \widehat{\beta}|_{\partial\widehat{\Omega}} = 0}} \int_{\widehat{T}_{\widehat{\beta}}} \widehat{\beta}^* \, \widehat{v} \, dt \; \operatorname{Trunc}_\bullet(\widehat{\beta}). \tag{4.5.16}$$

Recall that $\big\{\operatorname{Trunc}_\bullet(\widehat{\beta}) : \widehat{\beta} \in \widehat{\mathcal{B}}_\bullet \wedge \widehat{\beta}|_{\partial\widehat{\Omega}} = 0\big\}$ is a basis of $\widehat{\mathcal{X}}_\bullet$ according to Corollary 4.5.4.

To verify (S4) in the parameter domain with $q_{\rm proj} := 2(p_{\max} + 1)$, one can copy the proof of Proposition 3.4.9 (i) , which states the same for the operator $\widehat{I}_\bullet$. Note that the required [SM16, Theorem 4] is only proved for $\widehat{I}_\bullet$. However, with Corollary 4.5.4, the proof works exactly the same. Also the next proposition can be proved verbatim to Proposition 3.4.9 (ii).

Proposition 4.5.5. *There exists a constant $C > 0$ such that for all $\widehat{v} \in L^2(\widehat{\Omega})$ and $\widehat{T} \in \widehat{\mathcal{T}}_\bullet$, there holds that*

$$\|\widehat{J}_\bullet \widehat{v}\|_{L^2(\widehat{T})} \le C \|\widehat{v}\|_{L^2(\pi_\bullet^{2(p_{\max}+1)}(\widehat{T}))}, \tag{4.5.17}$$

where C depends only on d, $\widehat{\mathcal{T}}_0$, and $(p_1, \dots, p_d)$. □

Next, we prove (S5) with $q_{\rm sz} := 4(p_{\max} + 1)$. Let $\widehat{T} \in \widehat{\mathcal{T}}_\bullet$, $\widehat{v} \in H_0^1(\widehat{\Omega})$, and $\widehat{V}_\bullet \in \widehat{\mathcal{X}}_\bullet$. By (S4) and (4.5.17), it holds that

$$\begin{aligned} \|(1 - \widehat{J}_\bullet)\widehat{v}\|_{L^2(\widehat{T})} &\overset{\text{(S4)}}{=} \|(1 - \widehat{J}_\bullet)(\widehat{v} - \widehat{V}_\bullet)\|_{L^2(\widehat{T})} \le \|\widehat{v} - \widehat{V}_\bullet\|_{L^2(\widehat{T})} + \|\widehat{J}_\bullet(\widehat{v} - \widehat{V}_\bullet)\|_{L^2(\widehat{T})} \\ &\overset{(4.5.17)}{\lesssim} \|\widehat{v} - \widehat{V}_\bullet\|_{L^2(\pi_\bullet^{2(p_{\max}+1)}(\widehat{T}))}. \end{aligned}$$

To proceed, we distinguish between two cases, first, $\pi_\bullet^{4(p_{\max}+1)}(\widehat{T}) \cap \partial\widehat{\Omega} = \emptyset$ and, second, $\pi_\bullet^{4(p_{\max}+1)}(\widehat{T}) \cap \partial\widehat{\Omega} \neq \emptyset$, i.e., if $\widehat{T}$ is far away from the boundary or not. Note that these cases are equivalent to $|\pi_\bullet^{4(p_{\max}+1)}(\widehat{T}) \cap \partial\widehat{\Omega}| = 0$ resp. $|\pi_\bullet^{4(p_{\max}+1)}(\widehat{T}) \cap \partial\widehat{\Omega}| > 0$, since the elements in the parameter domain are rectangular. In the first case, we proceed as follows: (3.4.19) especially proves that $1 \in \widehat{\mathcal{S}}^{(p_1,\dots,p_d)}(\widehat{\mathcal{K}}_0, \widehat{\mathcal{T}}_\bullet)$ with $1 = \sum_{\widehat{\beta}\in\widehat{\mathcal{B}}_\bullet} a_{\widehat{\beta}}\widehat{\beta}$ on $\widehat{\Omega}$ for some coefficients $a_{\widehat{\beta}}$. With Remark 3.4.4, we see that $|\mathrm{supp}(\widehat{\beta}) \cap \pi_\bullet^{2(p_{\max}+1)}(\widehat{T})| > 0$ implies that $\mathrm{supp}(\widehat{\beta}) \subseteq \pi_\bullet^{4(p_{\max}+1)}(\widehat{T})$. Therefore, the restriction onto $\pi_\bullet^{2(p_{\max}+1)}(\widehat{T})$ satisfies that

$$
\begin{aligned}
1 = \sum_{\widehat{\beta}\in\widehat{\mathcal{B}}_\bullet} a_{\widehat{\beta}}\widehat{\beta}|_{\pi_\bullet^{2(p_{\max}+1)}(\widehat{T})} &= \sum_{\substack{\widehat{\beta}\in\widehat{\mathcal{B}}_\bullet \\ |\mathrm{supp}(\widehat{\beta})\cap\pi_\bullet^{2(p_{\max}+1)}(\widehat{T})|>0}} a_{\widehat{\beta}}\widehat{\beta}|_{\pi_\bullet^{2(p_{\max}+1)}(\widehat{T})} \\
&= \sum_{\substack{\widehat{\beta}\in\widehat{\mathcal{B}}_\bullet \\ \mathrm{supp}(\widehat{\beta})\subseteq\pi_\bullet^{4(p_{\max}+1)}(\widehat{T})}} a_{\widehat{\beta}}\widehat{\beta}|_{\pi_\bullet^{2(p_{\max}+1)}(\widehat{T})}.
\end{aligned}
$$

In the first case, we define

$$
\widehat{V}_\bullet := \widehat{v}_{\pi_\bullet^{2(p_{\max}+1)}(\widehat{T})} \sum_{\substack{\widehat{\beta}\in\widehat{\mathcal{B}}_\bullet \\ \mathrm{supp}(\widehat{\beta})\subseteq\pi_\bullet^{4(p_{\max}+1)}(\widehat{T})}} a_{\widehat{\beta}}\widehat{\beta} \quad \in \widehat{\mathcal{X}}_\bullet,
$$

$$
\text{where} \quad \widehat{v}_{\pi_\bullet^{2(p_{\max}+1)}(\widehat{T})} := |\pi_\bullet^{2(p_{\max}+1)}(\widehat{T})|^{-1} \int_{\pi_\bullet^{2(p_{\max}+1)}(\widehat{T})} \widehat{v}\,dt.
$$

In the second case, we set $\widehat{V}_\bullet := 0$. For the first case, we apply the Poincaré inequality, whereas we use the Friedrichs inequality in the second case. In either case, we obtain that $\widehat{V}_\bullet \in \widehat{\mathcal{X}}_\bullet$ and

$$
\|\widehat{v} - \widehat{V}_\bullet\|_{L^2(\pi_\bullet^{2(p_{\max}+1)}(\widehat{T}))} \lesssim \mathrm{diam}(\pi_\bullet^{4(p_{\max}+1)}(\widehat{T}))\|\nabla\widehat{v}\|_{L^2(\pi_\bullet^{4(p_{\max}+1)}(\widehat{T}))}, \tag{4.5.18}
$$

where the hidden constant depends only on the shape of the patch $\pi_\bullet^{2(p_{\max}+1)}(\widehat{T})$ resp. the shape of $\pi_\bullet^{4(p_{\max}+1)}(\widehat{T})$ and of $\pi_\bullet^{2(p_{\max}+1)}(\widehat{T}) \cap \partial\widehat{\Omega}$. However, Lemma 3.4.5 and admissibility show that $|\mathrm{level}(\widehat{T}') - \mathrm{level}(\widehat{T}'')| \le 1$ for all $\widehat{T}', \widehat{T}'' \in \widehat{\mathcal{T}}_\bullet$ with $\widehat{T}' \cap \widehat{T}'' \neq \emptyset$. This shows that the number of such patch shapes is bounded itself by a constant which depends only on d, $\widehat{\mathcal{T}}_0$ and $(p_1,\dots,p_d)$. Moreover, this yields that $\mathrm{diam}(\pi_\bullet^{4(p_{\max}+1)}(\widehat{T})) \simeq |\widehat{T}|^{1/d}$, which concludes the proof of (S5) in the parameter domain for $D = 1$.

Finally, we prove (S6) with $q_{\mathrm{sz}} = 4(p_{\max}+1)$. Let again $\widehat{T} \in \widehat{\mathcal{T}}_\bullet$ and $\widehat{v} \in H_0^1(\widehat{\Omega})$. For all $\widehat{V}_\bullet \in \widehat{\mathcal{X}}_\bullet$ which are constant on $\widehat{T}$, the projection property (S4), the inverse estimate (4.5.10) in the parameter domain as well as the local L^2-stability (4.5.17) imply that

$$
\begin{aligned}
\|\nabla\widehat{J}_\bullet\widehat{v}\|_{L^2(\widehat{T})} \overset{\text{(S4)}}{=} \|\nabla\widehat{J}_\bullet(\widehat{v} - \widehat{V}_\bullet)\|_{L^2(\widehat{T})} &\overset{(4.5.10)}{\lesssim} |\widehat{T}|^{-1/d}\|\widehat{J}_\bullet(v - \widehat{V}_\bullet)\|_{L^2(\widehat{T})} \\
&\overset{(4.5.17)}{\lesssim} |\widehat{T}|^{-1/d}\|\widehat{v} - \widehat{V}_\bullet\|_{L^2(\pi_\bullet^{2(p_{\max}+1)}(\widehat{T}))}.
\end{aligned}
$$

Arguing as before and using (4.5.18), we conclude the proof.

4.5.11 Verification of (O1)

An analogous inverse estimate in the parameter domain, i.e., $|\widehat{T}|^{1/d}\|W\circ\gamma\|_{L^2(\widehat{T})}\lesssim\|W\circ\gamma\|_{H^{-1}(\widehat{T})}$ for all $W\in\mathcal{P}(\Omega)$ and all $\widehat{T}\in\widehat{\mathcal{T}}_\bullet\in\widehat{\mathbb{T}}$, follows by a standard scaling argument. With $T:=\gamma(\widehat{T})$, the regularity (4.4.2) of γ immediately shows that $\|W\circ\gamma\|_{L^2(\widehat{T})}\simeq\|W\|_{L^2(T)}$. Moreover, $\|W\circ\gamma\|_{H^{-1}(\widehat{T})}\simeq\|W\|_{H^{-1}(T)}$ follows from the corresponding one-dimensional equality (4.5.1). This concludes (O1), where the constant $C'_{\rm inv}$ depends only on d, D, C_γ, $\widehat{\mathcal{T}}_0$, and $(p'_1,\dots,p'_d)$.

4.5.12 Verification of (O2)–(O4)

This section adapts [NV11, Section 3.4], where similar assertions are proved on regular triangulations. Let $W\in\mathcal{P}(\Omega)$, $\mathcal{T}_\bullet\in\mathbb{T}$, and $T,T'\in\mathcal{T}_\bullet$ with $(d-1)$-dimensional intersection $E:=T\cap T'$. We set $\widehat{W}:=W\circ\gamma$, $\widehat{T}:=\gamma^{-1}(T)$, $\widehat{T}':=\gamma^{-1}(T')$, and $\widehat{E}:=\gamma^{-1}(E)$. Let $\gamma_{\widehat{T}}:\mathbb{R}^d\to\mathbb{R}^d$ be the affine transformation with the reference element $\widetilde{T}:=\gamma_{\widehat{T}}^{-1}(\widehat{T})=[0,1]^d$. Due to admissibility of $\widehat{\mathcal{T}}_\bullet$, the number of different configurations for the set $\widetilde{T}':=\gamma_{\widehat{T}}^{-1}(\widehat{T}')$ is uniformly bounded by a constant that depends only on d and $\widehat{\mathcal{T}}_0$. We fix some smooth cut-off function $\widetilde{\varphi}\in C^\infty(\widetilde{T}\cup\widetilde{T}')\cap H^1_0(\widetilde{T}\cup\widetilde{T}')$ with $\widetilde{\varphi}>0$ almost everywhere on $\widetilde{E}:=\gamma_{\widehat{T}}^{-1}(\widehat{E})$. We define $\varphi:=\widetilde{\varphi}\circ\gamma_{\widehat{T}}^{-1}\circ\gamma^{-1}$, and $\widetilde{W}:=(W\circ\gamma\circ\gamma_{\widehat{T}})|_{\widetilde{T}\cup\widetilde{T}'}$. We denote $\mathcal{P}(\widetilde{T}\cup\widetilde{T}')$ as the space of all D-dimensional (non-piecewise) tensor-product polynomials of degree $(p'_1,\dots,p'_d)$ on $\widetilde{T}\cup\widetilde{T}'$, and $\mathcal{P}(\widetilde{T}\cup\widetilde{T}')|_{\widetilde{E}}$ as the corresponding space of restrictions onto $\widetilde{E}$. Note that $\widetilde{W}\in\mathcal{P}(\widetilde{T}\cup\widetilde{T}')$. Equipping $\mathcal{P}(\widetilde{T}\cup\widetilde{T}')|_{\widetilde{E}}$ with the norm $\widetilde{V}|_{\widetilde{E}}\mapsto\|\widetilde{V}\|_{L^2(\widetilde{E})}$ or with the quotient norm $\widetilde{V}|_{\widetilde{E}}\mapsto\inf\big\{\|\widetilde{\varphi}\widetilde{V}'\|_{H^1(\widetilde{T}\cup\widetilde{T}')}:\widetilde{V}'\in\mathcal{P}(\widetilde{T}\cup\widetilde{T}')\wedge\widetilde{V}'|_E=\widetilde{V}|_E\big\}$, and exploiting its finite dimension, proves the existence of $\widetilde{W}'\in\mathcal{P}(\widetilde{T}\cup\widetilde{T}')$ with $\widetilde{W}|_{\widetilde{E}}=\widetilde{W}'|_{\widetilde{E}}$ and

$$\|\widetilde{\varphi}\widetilde{W}'\|_{H^1(\widetilde{T}\cup\widetilde{T}')}\lesssim\|\widetilde{W}\|_{L^2(\widetilde{E})}. \tag{4.5.19}$$

Finally, we set $W':=\widetilde{W}'\circ\gamma_{\widehat{T}'}^{-1}\circ\gamma^{-1}$, and $L_{\bullet,E}(W|_E):=\varphi W'$. Finite dimension of $\mathcal{P}(\widetilde{T}\cup\widetilde{T}')$ shows that

$$\int_{\widetilde{E}}\widetilde{W}\cdot\widetilde{W}\,dt\lesssim\int_{\widetilde{E}}\widetilde{W}\cdot(\varphi\widetilde{W})\,dt=\int_{\widetilde{E}}\widetilde{W}\cdot(L_{\bullet,E}(W|_E))\,dt \tag{4.5.20}$$

Standard scaling arguments together with the regularity (4.4.2) of γ applied to (4.5.19)–(4.5.20), prove that (O2)–(O4) are satisfied, where the constants depend only on d, D, C_γ, $\widehat{\mathcal{T}}_0$, and $(p'_1,\dots,p'_d)$.

4.5.13 Proof of Theorem 4.4.6 for rational hierarchical splines

As mentioned in Remark 4.4.7 , Theorem 4.4.6 is still valid if one replaces the ansatz space $\mathcal{X}_\bullet$ for $\mathcal{T}_\bullet\in\mathbb{T}$ by rational hierarchical splines, i.e., by the set

$$\mathcal{X}_\bullet^{W_0}:=\Big\{W_0^{-1}V_\bullet:V_\bullet\in\mathcal{X}_\bullet\Big\}, \tag{4.5.21}$$

where $\widehat{W}_0 := W_0 \circ \gamma$ is a fixed positive weight function in the initial space of hierarchical splines $\widehat{\mathcal{S}}^{(p_1,\dots,p_d)}(\widehat{\mathcal{K}}_0, \widehat{\mathcal{T}}_0)$. Indeed, the mesh properties (M1)–(M4) as well as the refinement properties (R1)–(R5) from Section 4.2 are independent of the discrete spaces. To verify the validity of Theorem 4.4.6 in the rational setting, it thus only remains to verify the properties (S1)–(S6) for the rational finite element spaces. The inverse estimate (S1) follows from the analogous version for standard hierarchical splines, since for all $v \in H^2(T)^D$ and $j \in \{0,1,2\}$, it holds that $\|v\|_{H^j(T)} \simeq \|W_0^{-1} v\|_{H^j(T)}$, where the hidden constants depend only on d, D, C_γ, and $\widehat{W}_0$. The properties (S2)–(S3) depend only on the numerator of rational hierarchical splines and thus transfer. To see (S4)–(S6), one can use again the corresponding results for standard hierarchical splines. The Scott–Zhang type operator $J_\bullet^{W_0} : H_0^1(\Omega)^D \to \mathcal{X}_\bullet^{W_0}$ now reads

$$J_\bullet^{W_0} v := {W_0}^{-1} J_\bullet(W_0 v) \quad \text{for all } v \in H_0^1(\Omega)^D. \tag{4.5.22}$$

With this definition, (S4) follows immediately from the version of (S4) for hierarchical splines. Next, we prove (S5). For all $v \in H_0^1(\Omega)^D$ and $T \in \mathcal{T}_\bullet \in \mathbb{T}$, (S5) for hierarchical splines implies that

$$\|(1 - J_\bullet^{W_0})v\|_{L^2(T)} \lesssim \|(1 - J_\bullet)(W_0 v)\|_{L^2(T)} \overset{\text{(S5)}}{\lesssim} h_T \|W_0 v\|_{H^1(\pi_\bullet^{q_{\rm sz}}(T))} \lesssim h_T \|v\|_{H^1(\pi_\bullet^{q_{\rm sz}}(T))}.$$

We abbreviate $q := \max(2(p_{\max}+1), q_{\rm sz})$. To see (S6), we use Proposition 4.5.5 as well as (S6) for hierarchical splines

$$\begin{aligned}
\|\nabla J_\bullet^{W_0} v\|_{L^2(T)} &= \|\nabla (W_0^{-1} J_\bullet(W_0 v))\|_{L^2(T)} \lesssim \|J_\bullet(W_0 v)\|_{L^2(T)} + \|\nabla J_\bullet(W_0 v)\|_{L^2(T)} \\
&\overset{(4.5.17)+(\text{S6})}{\lesssim} \|v\|_{L^2(\pi_\bullet^{2(p_{\max}+1)}(T))} + \|W_0 v\|_{H^1(\pi_\bullet^{q_{\rm sz}}(T))} \lesssim \|v\|_{H^1(\pi_\bullet^{q}(T))}.
\end{aligned}$$

This concludes the proof of (S6), and hence of Theorem 4.4.6 for rational hierarchical splines.

4.6 Numerical examples

In this section, we apply Algorithm 4.2.6 to the two-dimensional Poisson problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{4.6.1}$$

on different domains $\Omega \subset \mathbb{R}^2$. In Section 4.6.1, we consider a solution with edge singularity, and give at least a heuristic explanation for the observed adaptive convergence rates in Section 4.6.2. In Section 4.6.3, the solution exhibits a generic (i.e., geometry induced) singularity. In Section 4.6.4, we prescribe a piecewise constant right-hand side f, which leads to less regularity of the solution. Similar examples are also considered in the recent own work [GHP17].

In all examples, the geometry Ω can be parametrized via rational splines, i.e., there exist polynomial orders $p_{1(\gamma)}, p_{2(\gamma)} \in \mathbb{N}$, a two-dimensional vector $\widehat{\mathcal{K}}_\gamma = (\widehat{\mathcal{K}}_{1(\gamma)}, \widehat{\mathcal{K}}_{2(\gamma)})$ of

$p_{i(\gamma)}$-open knot vectors with multiplicity smaller or equal to $p_{i(\gamma)}$ for the interior knots, and a positive spline weight function $\widehat{W}_\gamma \in \widehat{\mathcal{S}}^{(p_{1(\gamma)},p_{2(\gamma)})}(\widehat{\mathcal{K}}_\gamma)$ such that the parametrization satisfies that

$$\gamma \in \big\{\widehat{W}_\gamma^{-1}\,\widehat{S} \,:\, \widehat{S} \in \widehat{\mathcal{S}}^{(p_{1(\gamma)},p_{2(\gamma)})}(\widehat{\mathcal{K}}_\gamma)^2\big\}. \tag{4.6.2}$$

Based on the knots $\widehat{\mathcal{K}}_\gamma$ for the geometry, we choose the initial knots $\widehat{\mathcal{K}}_0$ for the discretization. As basis for the considered ansatz spaces of (non-rational) hierarchical splines, we use the (non-truncated) basis given in Corollary 4.5.4. To (approximately) calculate the Galerkin matrix, the right-hand side vector, and the weighted-residual error estimator (4.2.13), we use tensor Gauss quadrature. Recall that Lemma 3.2.1 (vii) provides a formula for the derivative of B-splines. The MATLAB-implementation, which is used for the following experiments, was developed together with Daniel Haberlik within the framework of his bachelor's thesis [Hab] supervised by Dirk Praetorius.

To (approximately) calculate, the energy error, we proceed as follows: Let $U_\ell \in \mathcal{X}_\ell$ be the Galerkin approximation of the ℓ-th step with the corresponding coefficient vector $\boldsymbol{c_\ell}$. Further, let $\boldsymbol{A_\ell}$ be the Galerkin matrix. With the Galerkin orthogonality (4.2.9) and the energy norm $\|\nabla u\|^2_{L^2(\Omega)}$, we can compute the energy error as

$$\|\nabla u - \nabla U_\ell\|^2_{L^2(\Omega)} = \|\nabla u\|^2_{L^2(\Omega)} - \|\nabla U_\ell\|^2_{L^2(\Omega)} = \|\nabla u\|^2_{L^2(\Omega)} - \boldsymbol{A_\ell c_\ell} \cdot \boldsymbol{c_\ell}. \tag{4.6.3}$$

In Section 4.6.1, where the solution u is known, the term $\|\nabla u\|^2_{L^2(\Omega)}$ is computed exactly, whereas it is obtained by Aitken's Δ^2-extrapolation in Section 4.6.3 and Section 4.6.4.

4.6.1 Solution with edge singularity on square

In the first experiment, we consider the unit square

$$\Omega := (0,1)^2, \tag{4.6.4}$$

where we choose $p_{1(\gamma)} := p_{2(\gamma)} := 1$, $\widehat{\mathcal{K}}_{1(\gamma)} := \widehat{\mathcal{K}}_{2(\gamma)} := (0,0,1,1)$, and $\widehat{W}_\gamma := 1$; see [GHP17, Section 6.1]. We choose f such that the exact solution of (4.6.1) is given by

$$u(x_1,x_2) = x_1^\tau(1-x_1)x_2(1-x_2) \tag{4.6.5}$$

with a parameter $\tau > 1/2$ with $\tau \notin \mathbb{N}$. The solution is singular at the edge $\{0\} \times [0,1]$. Elementary calculations show for all $j \in \mathbb{N}_0$ that $u \in H^j(\Omega)$ if and only if $\tau + 1/2 > j$. Assuming that this property is also satisfied for arbitrary $\sigma > 0$ instead of j, we see that $u \in H^{\tau+1/2-\epsilon}(\Omega)$ for all $\epsilon > 0$.

For the initial ansatz space with spline degrees $p_1 := p_2 \in \{2,3,4,5\}$, we choose the initial knot vectors $\widehat{\mathcal{K}}_{1(0)} := \widehat{\mathcal{K}}_{2(0)} := (0,\dots 0,1,\dots,1)$, where the multiplicity of 0 and 1 is $p_1 + 1 = p_2 + 1$. We choose the parameters of Algorithm 4.2.6 as $\theta = 0.5$ and $C_{\min} = 1$. For comparison, we also consider uniform refinement, where we mark all elements in each step, i.e., $\mathcal{M}_\ell = \mathcal{T}_\ell$ for all $\ell \in \mathbb{N}_0$. This leads to uniform bisection of all elements. Note that in both cases the resulting ansatz functions in $\mathcal{X}_\ell$ are even in $C^1(\overline{\Omega})$. In particular, the jump terms of the error estimator η_ℓ vanish; see Remark 4.2.4. First, we consider $\tau := 9/4$, i.e., $u \in H^{9/4+1/2-\epsilon}(\Omega) = H^{1+7/4-\epsilon}(\Omega)$. For uniform mesh-refinement, one may expect a convergence rate of $\mathcal{O}(h^{7/4}) = \mathcal{O}(N^{-7/8})$ with respect to the uniform mesh-size h resp. the number of elements N. In Figure 4.2, one can see some adaptively generated hierarchical meshes. In Figure 4.3 and Figure 4.4, we plot the energy error $\|\nabla u - \nabla U_\ell\|_{L^2(\Omega)}$ and the error estimator η_ℓ against the number of elements $\#\mathcal{T}_\ell$. All values are plotted in a double logarithmic scale such that the experimental convergence rates are visible as the slope of the corresponding curves. In all cases, the lines of the error and the error estimator are parallel, which numerically indicates reliability (4.2.20) and efficiency (4.2.21). With $p := p_1 = p_2$, the uniform approach leads to the suboptimal convergence rate $\mathcal{O}((\#\mathcal{T}_\ell)^{-\min(\tau-1/2,p)/2})$ due to the edge singularity at $\{0\}\times[0,1]$. However, it seems that the adaptive strategy converges at rate $\mathcal{O}((\#\mathcal{T}_\ell)^{-\min(\tau-1/2,p/2)})$, i.e., (if possible) at double rate. The speed of convergence remains unchanged if one decreases the adaptivity parameter θ; see, e.g., Figure 4.4 for $\theta = 0.1$. For smooth solutions u, one would expect a rate of $\mathcal{O}((\#\mathcal{T}_\ell)^{-p/2})$. However, according to Theorem 4.4.6, the achieved rate is optimal if one uses the proposed refinement strategy and the resulting hierarchical splines. The reduced optimal convergence rate is probably due to the edge singularity which would actually require anisotropic refinement. In Figure 4.6, we consider $\tau \in \{5/4, 7/4\}$ with $\theta = 0.5$. Then, $u \in H^{1+3/4}(\Omega)$ resp. $u \in H^{1+5/4}(\Omega)$, and we expect a rate of $\mathcal{O}(N^{-3/8})$ resp. $\mathcal{O}(N^{-5/8})$ for uniform refinement. In the following Section 4.6.2, we give a heuristic argumentation which suggests that the optimal convergence rate with respect to the number of elements for isotropic refinement is bounded by $\min(\tau - 1/2, p/2)$. In our numerical examples, it seems that this rate is attained exactly.

4.6.2 Convergence rate for solutions with edge singularity

In this section, we try to understand the observed adaptive convergence rates of the previous Section 4.6.1. We essentially follow the heuristic argumentation of [CMPS04, Section 7.3], where a similar reduced convergence rate is witnessed for boundary integral equations on screens and continuous piecewise affine ansatz functions.

Let $\mathcal{T}$ be an anisotropic rectangular mesh graded towards the singular edge $\{0\} \times [0,1]$ with grading parameter $\beta \geq 1$, i.e.,

$$\mathcal{T} = \left\{ \left[\left(\frac{j_1 - 1}{n}\right)^\beta, \left(\frac{j_1}{n}\right)^\beta \right] \times \left[\frac{j_2 - 1}{n}, \frac{j_2}{n}\right] : j_1, j_2 \in \{1, \dots, n\} \right\}.$$

We abbreviate the width of the elements as $h := n^{-1}$, and the number of elements $N := n^2$. Note that there are n elements at the singular edge having a length h^β. To obtain the same accuracy, a corresponding isotropic rectangular mesh $\mathcal{T}_{\rm iso}$ requires elements at this edge of the same length. However, in order to limit the aspect ratio, the width must be of the same order $\mathcal{O}(h^\beta)$. We see that, while the anisotropic mesh has only N elements, the corresponding isotropic mesh must have solely at the singular edge more than $1/h^\beta = N^{\beta/2}$ elements, i.e., $N_{\rm iso} \geq N^{\beta/2}$.

The mesh grading procedure yields a convergence rate $\mathcal{O}(N^{-\alpha})$ for some $\alpha > 0$. With the initial number of elements N_0, this implies for the corresponding errors that

$$\alpha = \frac{\log(e_{N_0}/e_N)}{\log(N/N_0)}.$$

With $N_{\text{iso}} \geq N^{\beta/2}$, and assuming the same error $e_{\text{iso},N_{\text{iso}}} = e_N$, we obtain the isotropic convergence rate

$$\alpha_{\text{iso}} = \frac{\log(e_{N_0}/e_{\text{iso},N_{\text{iso}}})}{\log(N_{\text{iso}}/N_0)} \leq \frac{\log(e_{N_0}/e_N)}{\log(N^{\beta/2}/N_0)} = \alpha\frac{\log(N/N_0)}{\log(N^{\beta/2}/N_0)} = \alpha\frac{1-\log(N_0)/\log(N)}{\beta/2-\log(N_0)/\log(N)}.$$

For $N \to \infty$, the last term converges to $2\alpha/\beta$. To obtain the optimal convergence rate $\alpha = \alpha_{\text{opt}}$ for graded meshes, [Ape99, Theorem 4.2 and Remark 4.4] suggests the choice $\beta := \alpha_{\text{opt}}/\alpha_{\text{uni}}$ for the grading parameter β, where α_{uni} is the convergence rate for uniform refinement. With this, we conclude that $\alpha_{\text{iso}} \leq 2\alpha_{\text{opt}}/\beta = 2\alpha_{\text{uni}}$ and altogether

$$\alpha_{\text{iso}} \leq \max(2\alpha_{\text{uni}}, \alpha_{\text{opt}}). \tag{4.6.6}$$

Note that, in particular, α_{iso} is generically bounded independently of p, while α_{opt} depends on p.

4.6.3 Generically singular solution on L-shape

To obtain the L-shaped domain

$$\Omega := (0,1)^2 \setminus ([0,1/2] \times [0,1/2]), \tag{4.6.7}$$

we choose $p_{1(\gamma_1)} := p_{2(\gamma)} = 1$ and $\widehat{\mathcal{K}}_{1(\gamma)} := (0,0,0.5,1,1)$, $\widehat{\mathcal{K}}_{2(\gamma)} := (0,0,1,1)$, and $\widehat{W}_\gamma := 1$; see [GHP17, Section 6.2]. We consider the Poisson problem (4.6.1) with

$$f := 1. \tag{4.6.8}$$

For the initial ansatz space with spline degrees $p_1 := p_2 \in \{2,3,4,5\}$, we choose the initial knot vectors $\widehat{\mathcal{K}}_{1(0)} := (0,\dots,0,0.5,\dots,0.5,1,\dots,1)$ and $\widehat{\mathcal{K}}_{2(0)} := (0,\dots,0,1\dots,1)$, where the multiplicity of 0 and 1 is $p_1 + 1 = p_2 + 1$, whereas the multiplicity of 0.5 is p_1. We choose the parameters of Algorithm 4.2.6 as $\theta = 0.4$ and $C_{\min} = 1$. For comparison, we also consider uniform refinement, where we mark all elements in each step, i.e., $\mathcal{M}_\ell = \mathcal{T}_\ell$ for all $\ell \in \mathbb{N}_0$. This leads to uniform bisection of all elements. Note that in both cases the resulting ansatz functions in $\mathcal{X}_\ell$ are differentiable except at the line $\gamma(\{0.5\} \times [0,1])$, where they are only continuous due to the higher multiplicity of 0.5. In particular, the jump terms of the error estimator η_ℓ only have to be calculated at this line; see Remark 4.2.4. In Figure 4.7, one can see some adaptively generated hierarchical meshes. In Figure 4.8 and Figure 4.9, we plot the energy error $\|\nabla u - \nabla U_\ell\|_{L^2(\Omega)}$ and the error estimator η_ℓ against the number of elements $\#\mathcal{T}_\ell$. All values are plotted in a double logarithmic scale such that the experimental convergence rates are visible as the slope of the corresponding curves. In all cases, the lines of the error and the error estimator are parallel, which numerically indicates reliability (4.2.20) and efficiency (4.2.21). The uniform approach leads to the suboptimal convergence rate $\mathcal{O}((\#\mathcal{T}_\ell)^{-1/3})$, since the reentrant corner at $(0.5, 0.5)$ causes a generic singularity of the solution u. However, the adaptive strategy recovers the optimal convergence rate $\mathcal{O}((\#\mathcal{T}_\ell)^{-p/2})$ with $p := p_1 = p_2$.

4.6.4 Piecewise constant right-hand side on quarter ring

We construct the rational spline surface given in polar coordinates

$$\Omega := \{r(\cos(\varphi), \sin(\varphi)) : r \in (1/2, 1) \wedge \varphi \in (0, \pi/2)\} \tag{4.6.9}$$

by choosing $p_{1(\gamma)} := 2, p_{2(\gamma)} := 1$ and $\widehat{\mathcal{K}}_{1(\gamma)} := (0,0,0,1,1,1)$, $\widehat{\mathcal{K}}_{2(\gamma)} := (0,0,1,1)$; see [GHP17, Section 6.3]. As right-hand side in (4.6.1), we choose the indicator function

$$f := \chi_S \quad \text{with} \quad S := \{r(\cos(\varphi), \sin(\varphi)) : r \in (1/2, 3/4) \wedge \varphi \in (0, \pi/4)\}. \tag{4.6.10}$$

There holds that $S = \gamma((0.5, 1) \times (0, 0.5))$.

For the (non-rational) initial ansatz space with spline degrees $p_1 := p_2 \in \{2, 3, 4, 5\}$, we choose the initial knot vectors $\widehat{\mathcal{K}}_{1(0)} := \widehat{\mathcal{K}}_{2(0)} := (0, \dots, 0, 0.5, \dots, 0.5, 1 \dots, 1)$, where the multiplicity of 0 and 1 is $p_1 + 1 = p_2 + 1$, whereas the multiplicity of 0.5 is $p_1 = p_2$. In particular, this implies that f is smooth on each element which allows for standard tensor Gauss quadrature. We choose the parameters of Algorithm 4.2.6 as $\theta = 0.8$ and $C_{\min} = 1$. For comparison, we also consider uniform refinement, where we mark all elements in each step, i.e., $\mathcal{M}_\ell = \mathcal{T}_\ell$ for all $\ell \in \mathbb{N}_0$. This leads to uniform bisection of all elements. In Figure 4.10, some adaptively generated hierarchical meshes are illustrated. In Figure 4.11 and Figure 4.12, we plot the energy error $\|\nabla u - \nabla U_\ell\|_{L^2(\Omega)}$ and the error estimator η_ℓ against the number of elements $\#\mathcal{T}_\ell$. All values are plotted in a double logarithmic scale such that the experimental convergence rates are visible as the slope of the corresponding curves. In all cases, the lines of the error and the error estimator are parallel, which numerically indicates reliability (4.2.20) and efficiency (4.2.21). The uniform approach leads to the suboptimal convergence rate $\mathcal{O}((\#\mathcal{T}_\ell)^{-1})$ due to the lack of regularity of the right-hand side f. However, the adaptive strategy recovers the optimal convergence rate $\mathcal{O}((\#\mathcal{T}_\ell)^{-p/2})$ with $p := p_1 = p_2$.

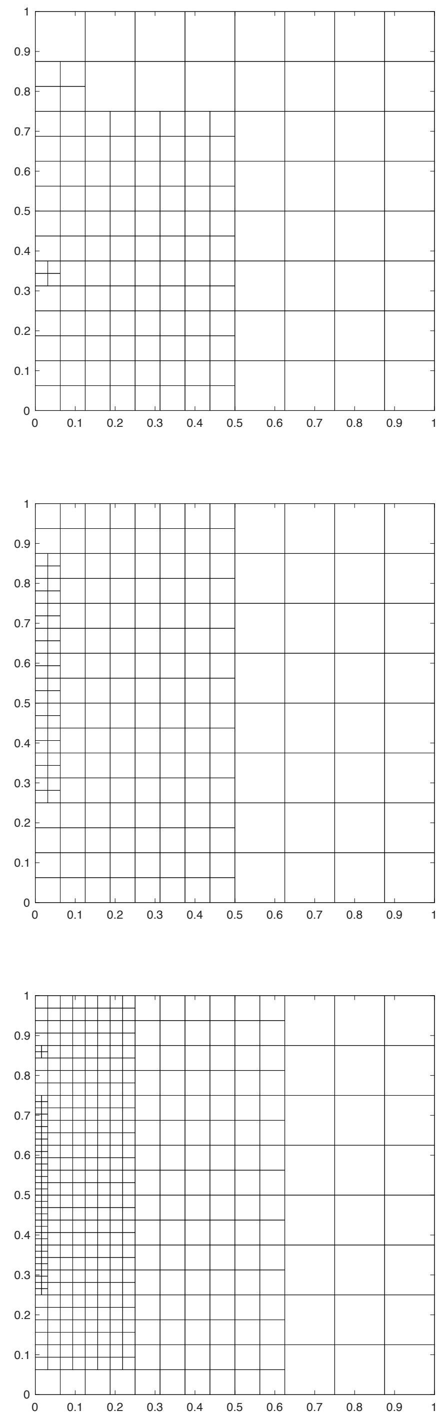


Figure 4.2: Experiment with solution with edge singularity ($\tau = 9/4$) on square of Section 4.6.1. Hierarchical meshes $\mathcal{T}_5, \mathcal{T}_6, \mathcal{T}_7$ generated by Algorithm 4.2.6 (with $\theta = 0.5$) for hierarchical splines of degree $p_1 = p_2 = 3$.

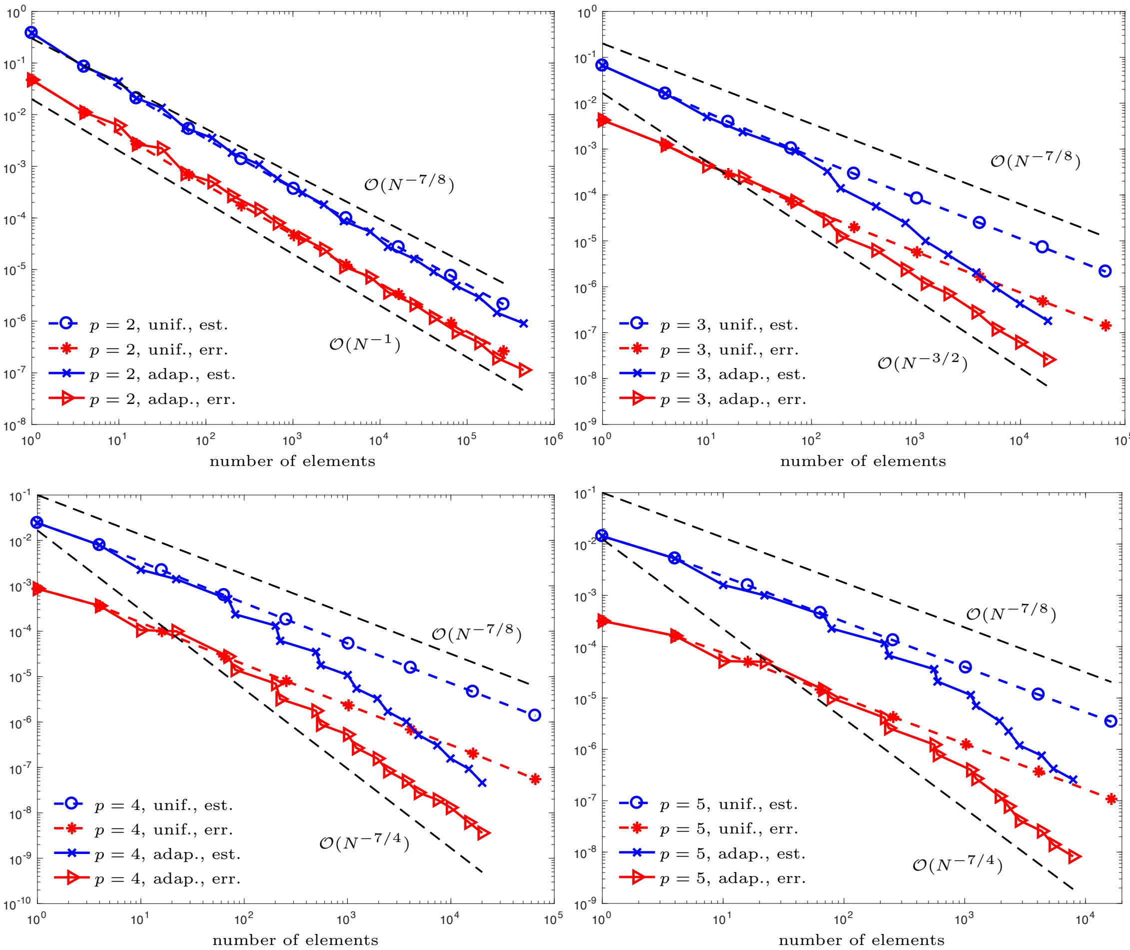


Figure 4.3: Experiment with solution with edge singularity ($\tau = 9/4$) on square of Section 4.6.1. Energy error $\|\nabla u - \nabla U_\ell\|_{L^2(\Omega)}$ and estimator η_ℓ of Algorithm 4.2.6 for hierarchical splines of degree $p_1 = p_2 \in \{2, 3, 4, 5\}$ are plotted versus the number of elements $\#\mathcal{T}_\ell$. Uniform and adaptive ($\theta = 0.5$) refinement is considered.

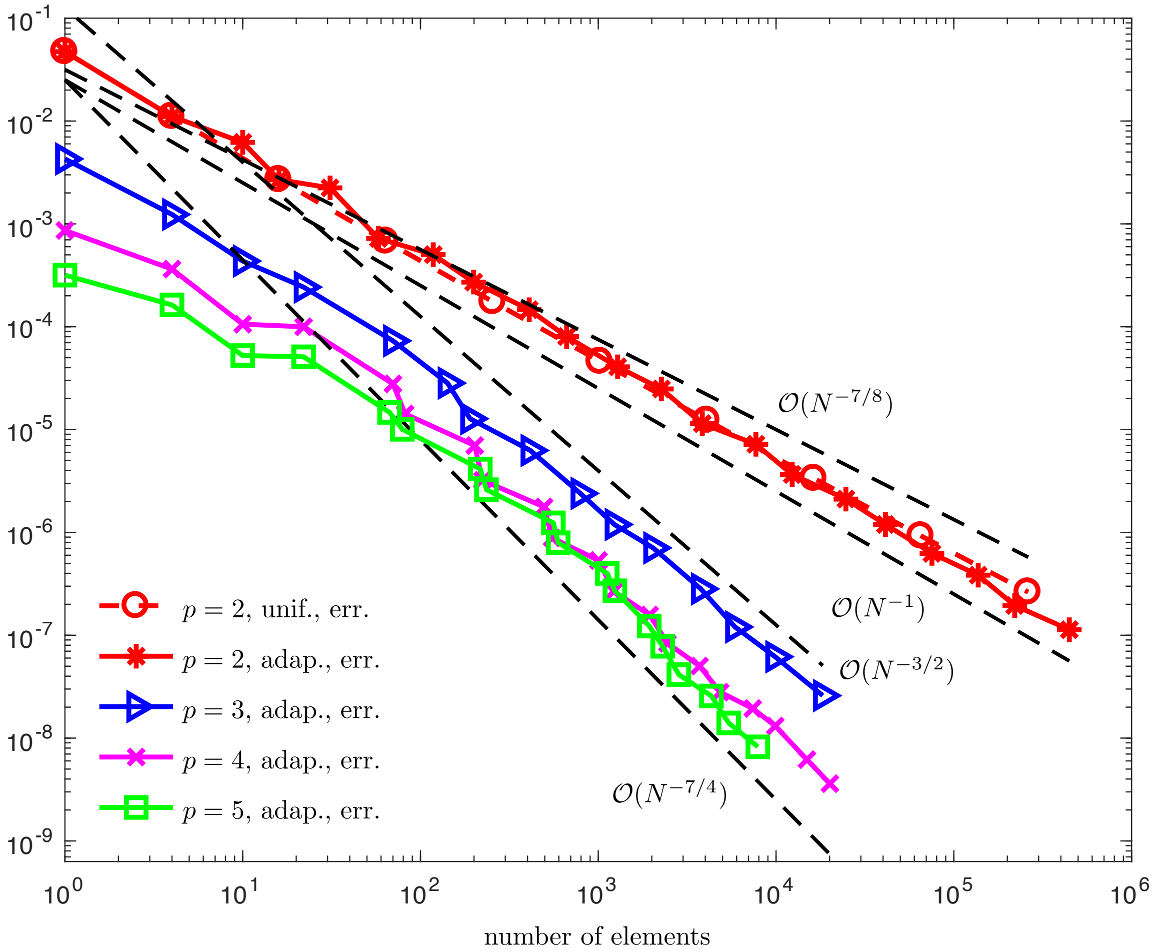


Figure 4.4: Experiment with solution with edge singularity ($\tau = 9/4$) on square of Section 4.6.1. The energy errors $\|\nabla u - \nabla U_\ell\|_{L^2(\Omega)}$ of Algorithm 4.2.6 for hierarchical splines of degree $p_1 = p_2 \in \{2, 3, 4, 5\}$ are plotted versus the number of elements $\#\mathcal{T}_\ell$. Uniform (for $p_1 = p_2 = 2$) and adaptive ($\theta = 0.5$ for $p_1 = p_2 \in \{2, 3, 4, 5\}$) refinement is considered.

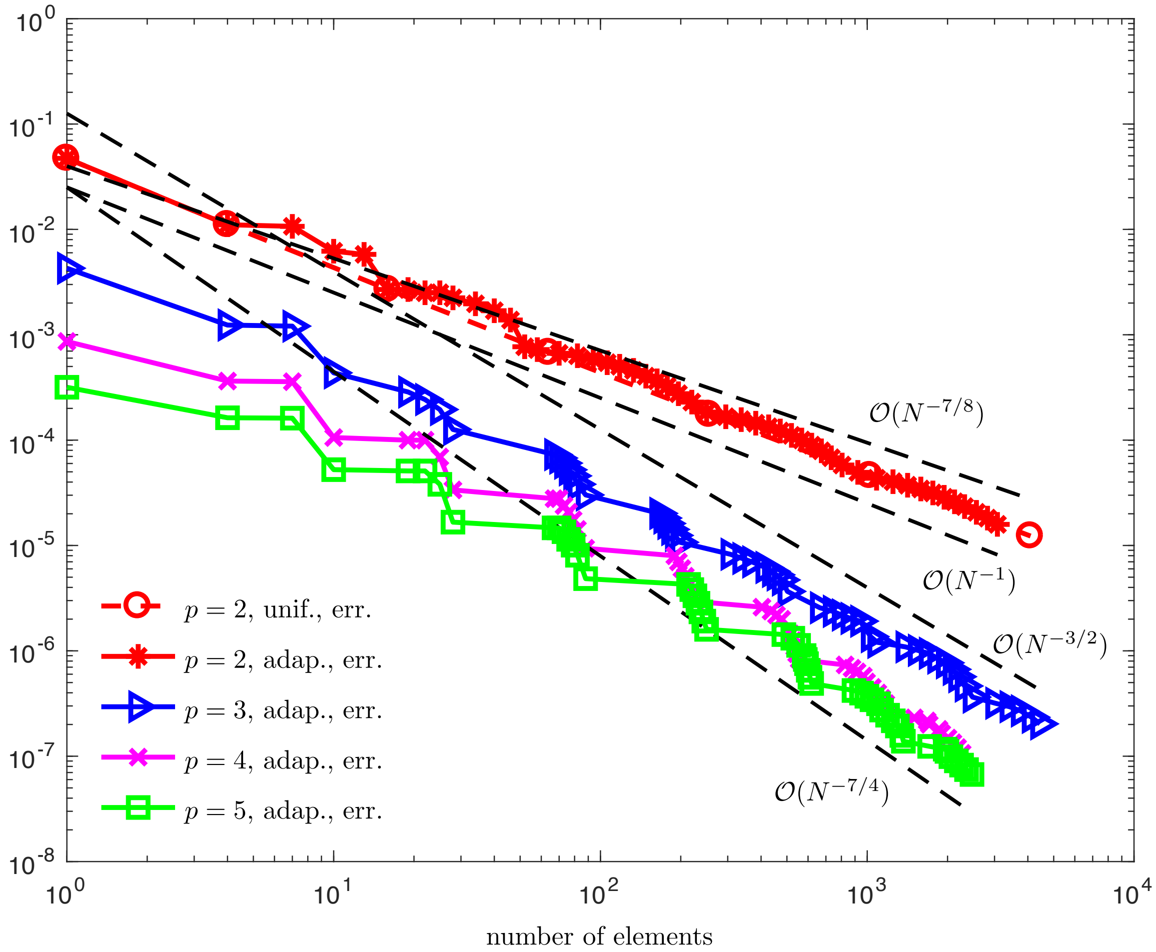


Figure 4.5: Experiment with solution with edge singularity ($\tau = 9/4$) on square of Section 4.6.1. The energy errors $\|\nabla u - \nabla U_\ell\|_{L^2(\Omega)}$ of Algorithm 4.2.6 for hierarchical splines of degree $p_1 = p_2 \in \{2, 3, 4, 5\}$ are plotted versus the number of elements $\#\mathcal{T}_\ell$. Uniform (for $p_1 = p_2 = 2$) and adaptive ($\theta = 0.1$ for $p_1 = p_2 \in \{2, 3, 4, 5\}$) refinement is considered.

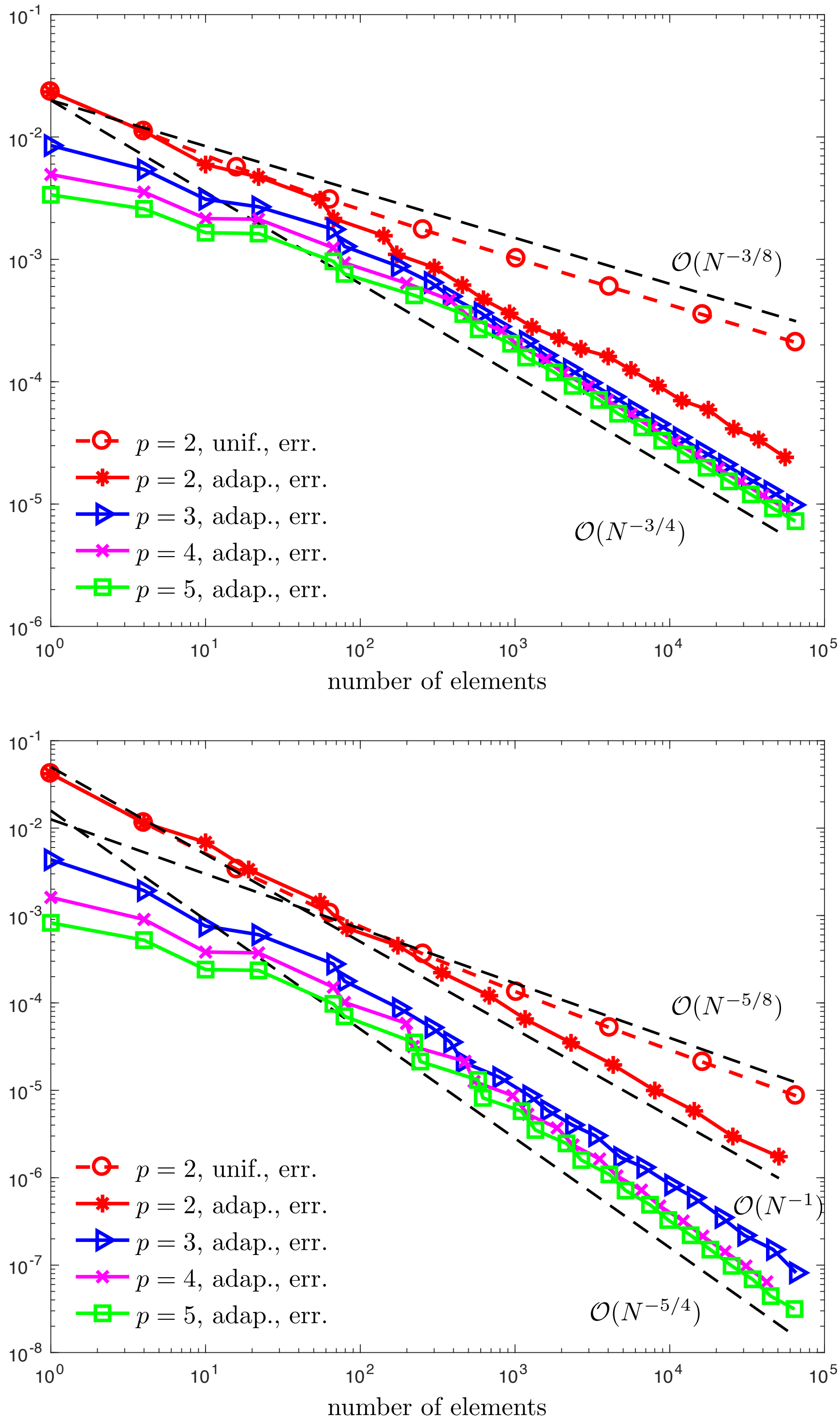


Figure 4.6: Experiment with solution with edge singularity ($\tau = 5/4$ above and $\tau = 7/4$ below) on square of Section 4.6.1. The energy errors $\|\nabla u - \nabla U_\ell\|_{L^2(\Omega)}$ of Algorithm 4.2.6 for hierarchical splines of degree $p_1 = p_2 \in \{2, 3, 4, 5\}$ are plotted versus the number of elements $\#\mathcal{T}_\ell$. Uniform (for $p_1 = p_2 = 2$) and adaptive ($\theta = 0.5$ for $p_1 = p_2 \in \{2, 3, 4, 5\}$) refinement is considered.

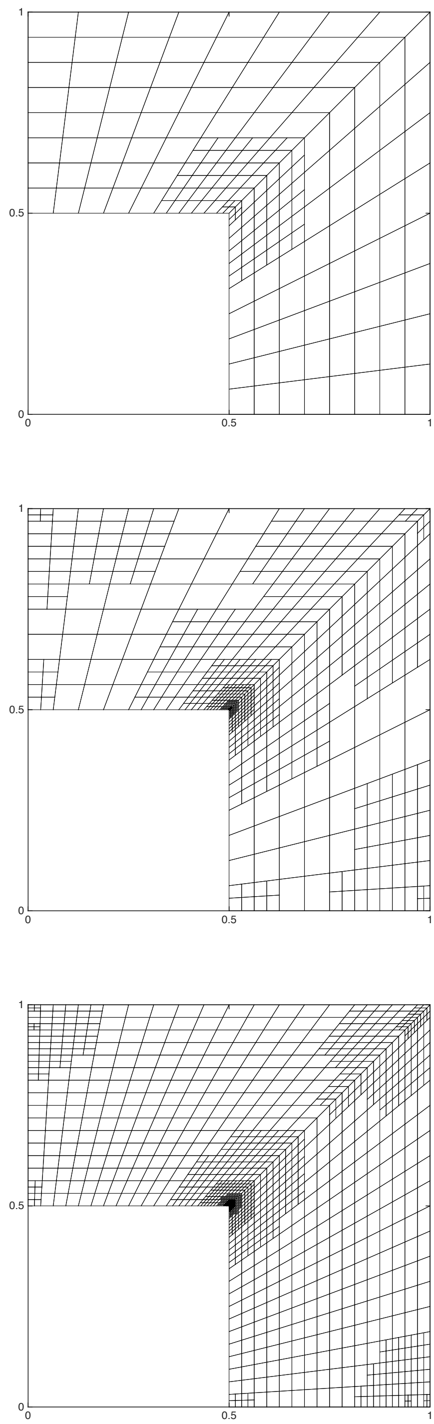


Figure 4.7: Experiment with generically singular solution on L-shape of Section 4.6.3. Hierarchical meshes $\mathcal{T}_6, \mathcal{T}_9, \mathcal{T}_{11}$ generated by Algorithm 4.2.6 (with $\theta = 0.4$) for hierarchical splines of degree $p_1 = p_2 = 2$.

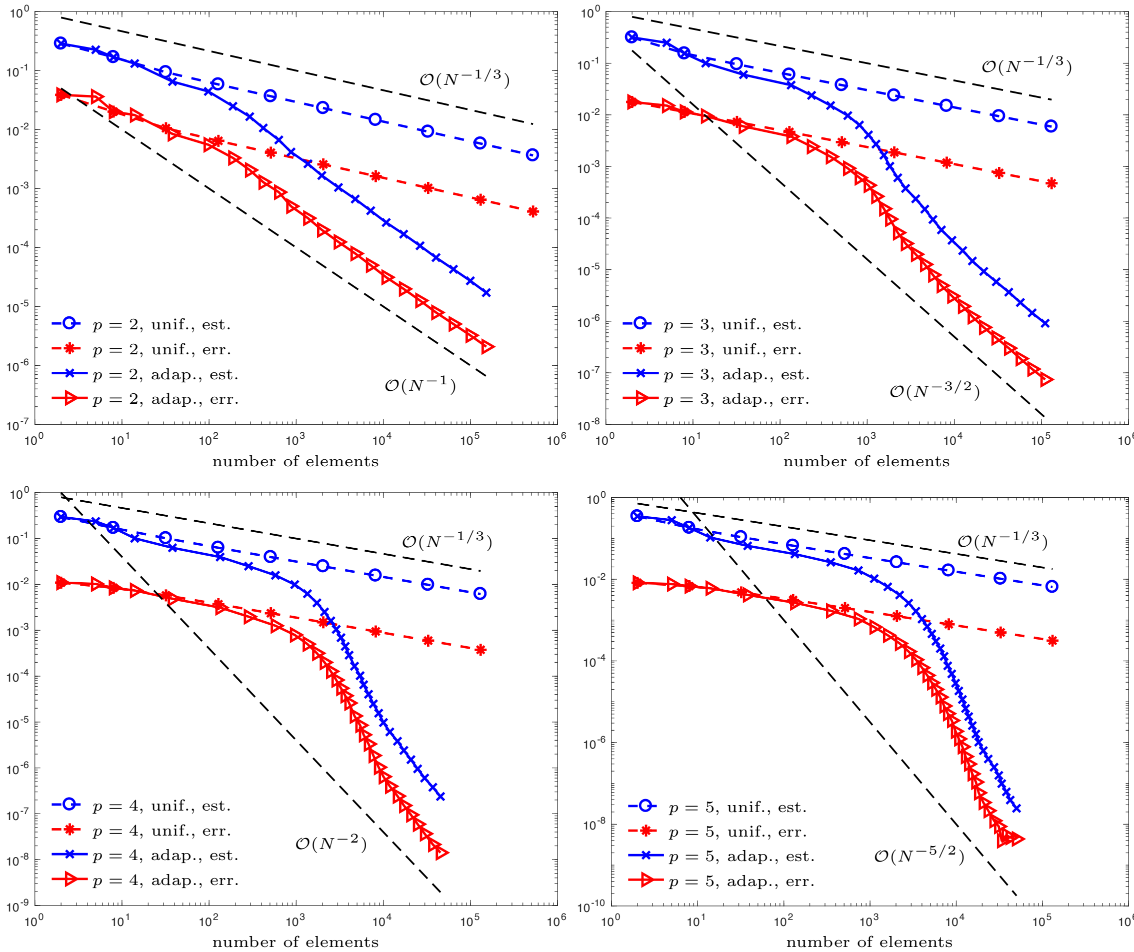


Figure 4.8: Experiment with generically singular solution on L-shape of Section 4.6.3. Energy error $\|\nabla u - \nabla U_\ell\|_{L^2(\Omega)}$ and estimator η_ℓ of Algorithm 4.2.6 for hierarchical splines of degree $p_1 = p_2 \in \{2, 3, 4, 5\}$ are plotted versus the number of elements $\#\mathcal{T}_\ell$. Uniform and adaptive ($\theta = 0.4$) refinement is considered.

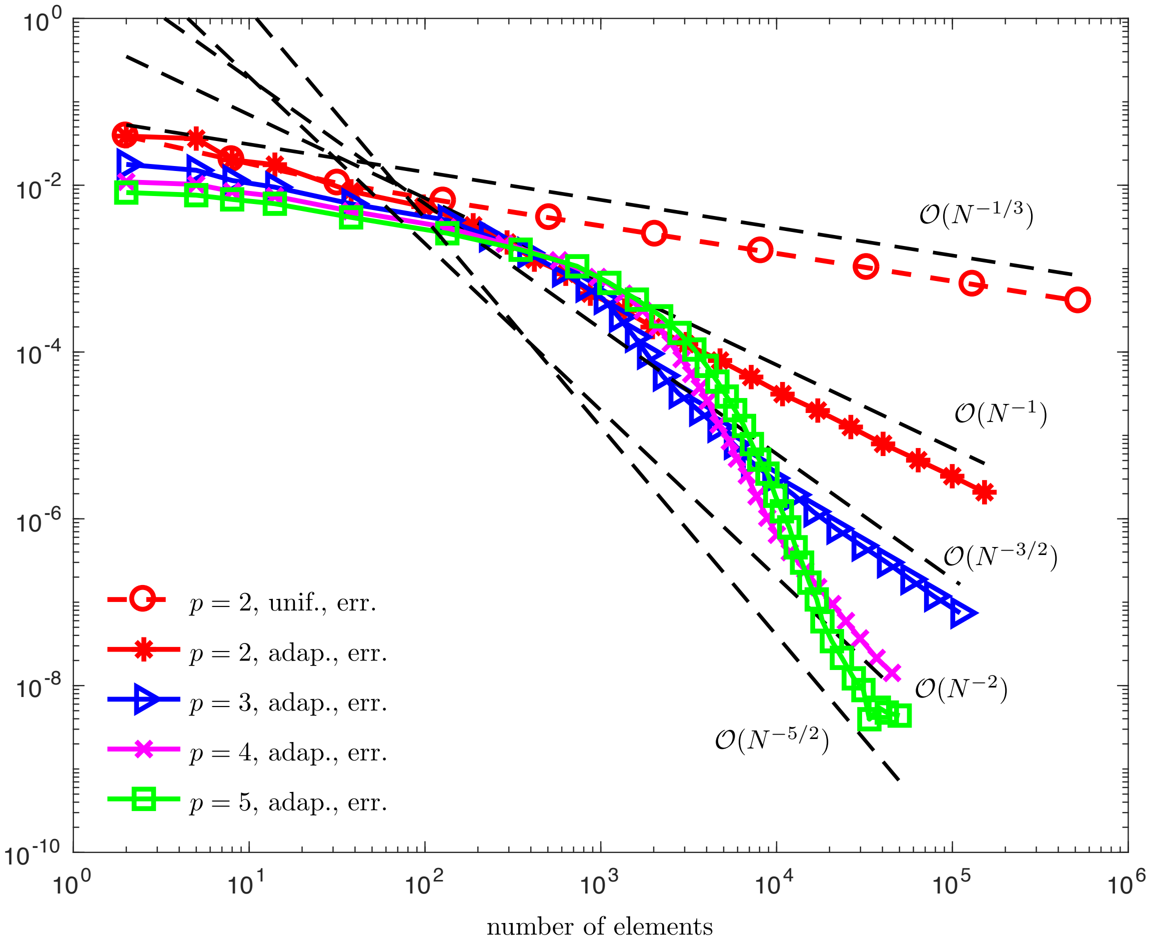


Figure 4.9: Experiment with generically singular solution on L-shape of Section 4.6.3. The energy errors $\|\nabla u - \nabla U_\ell\|_{L^2(\Omega)}$ of Algorithm 4.2.6 for hierarchical splines of degree $p_1 = p_2 \in \{2, 3, 4, 5\}$ are plotted versus the number of elements $\#\mathcal{T}_\ell$. Uniform (for $p_1 = p_2 = 2$) and adaptive ($\theta = 0.4$ for $p_1 = p_2 \in \{2, 3, 4, 5\}$) refinement is considered.

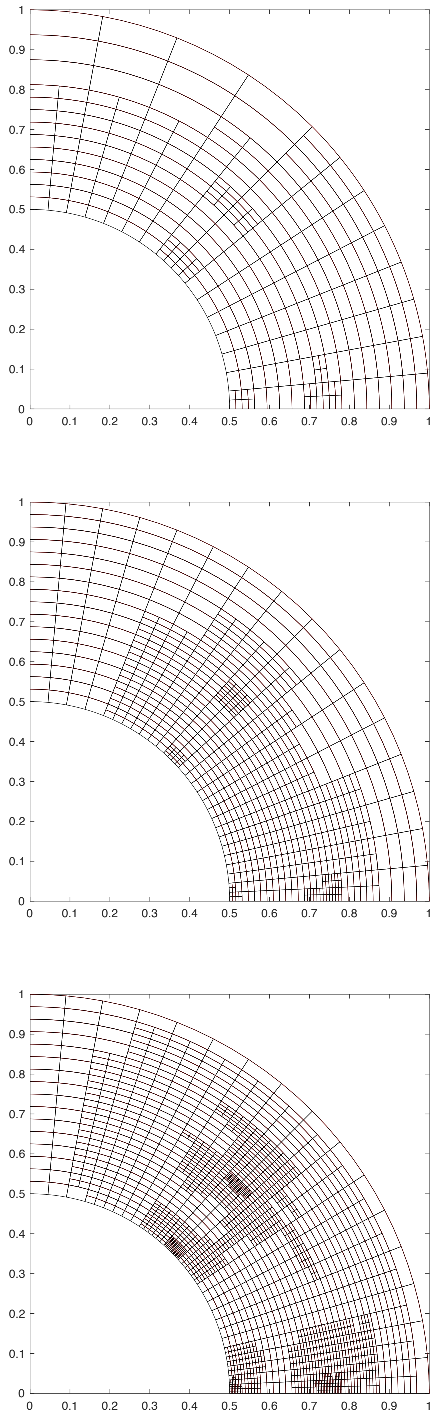


Figure 4.10: Experiment with piecewise constant right-hand side on quarter ring of Section 4.6.4. Hierarchical meshes $\mathcal{T}_4, \mathcal{T}_5, \mathcal{T}_6$ generated by Algorithm 4.2.6 (with $\theta = 0.8$) for hierarchical splines of degree $p_1 = p_2 = 3$.

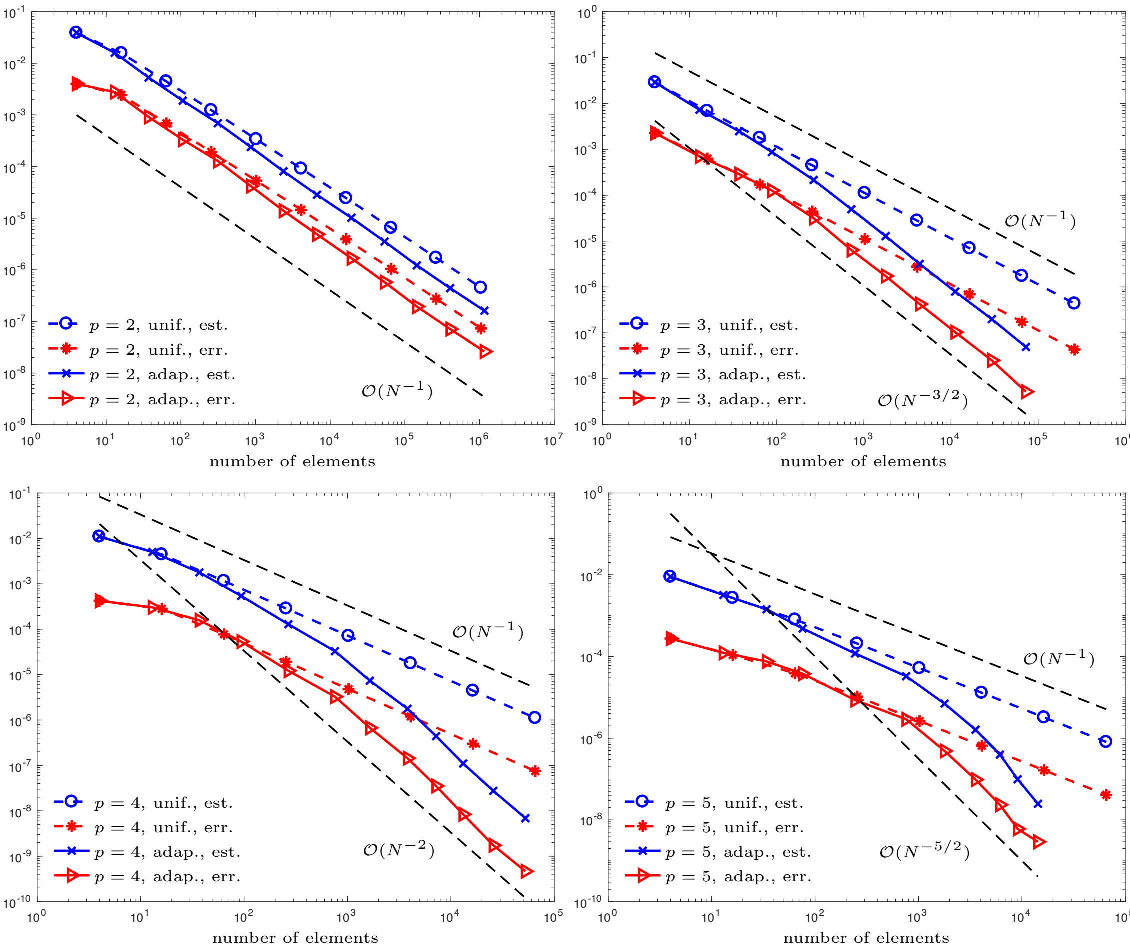


Figure 4.11: Experiment with generically singular solution on quarter ring of Section 4.6.4. Energy error $\|\nabla u - \nabla U_\ell\|_{L^2(\Omega)}$ and estimator η_ℓ of Algorithm 4.2.6 for hierarchical splines of degree $p_1 = p_2 \in \{2,3,4,5\}$ are plotted versus the number of elements $\#\mathcal{T}_\ell$. Uniform and adaptive ($\theta = 0.8$) refinement is considered.

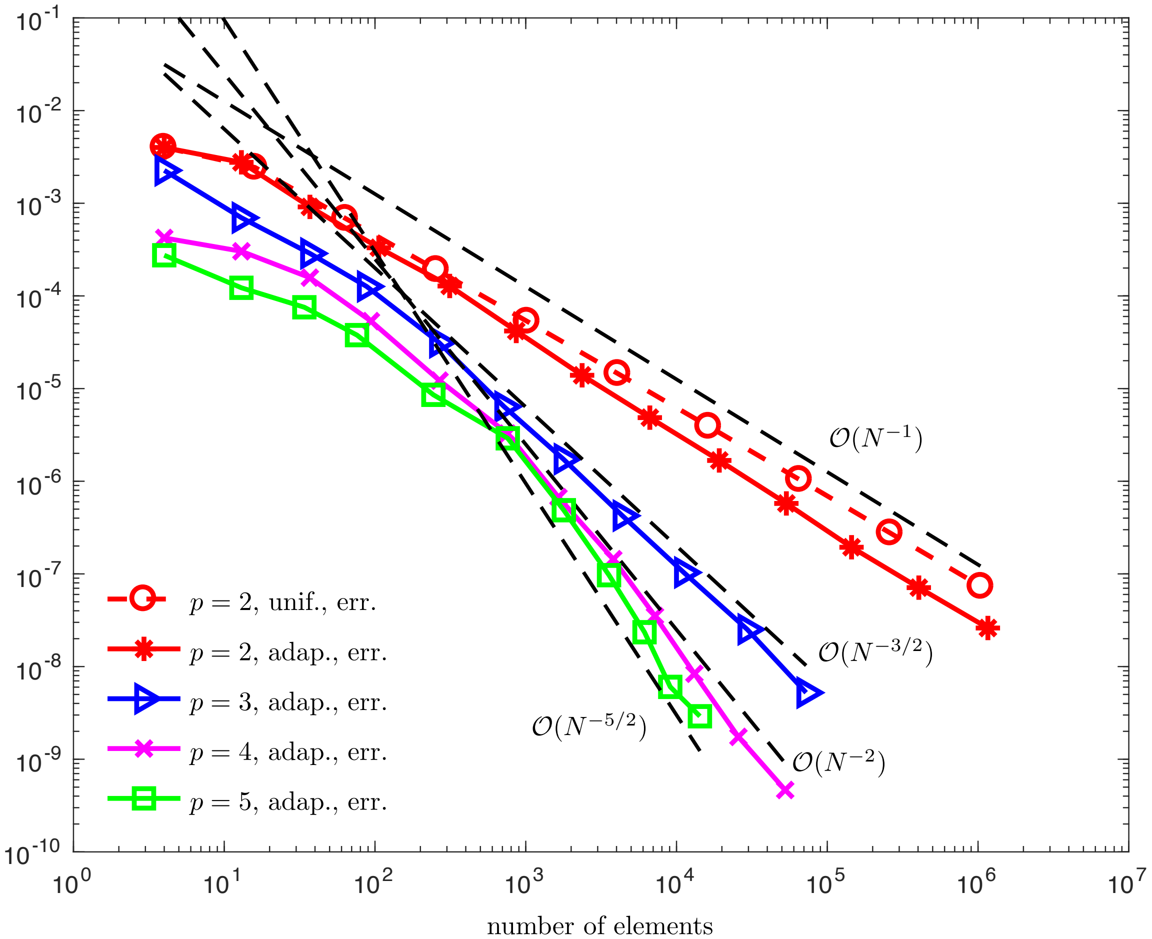


Figure 4.12: Experiment with generically singular solution on quarter ring of Section 4.6.4. The energy errors $\|\nabla u - \nabla U_\ell\|_{L^2(\Omega)}$ of Algorithm 4.2.6 for hierarchical splines of degree $p_1 = p_2 \in \{2, 3, 4, 5\}$ are plotted versus the number of elements $\#\mathcal{T}_\ell$. Uniform (for $p_1 = p_2 = 2$) and adaptive ($\theta = 0.8$ for $p_1 = p_2 \in \{2, 3, 4, 5\}$) refinement is considered.

5 Boundary Element Method

5.1 Introduction

In this chapter, we propose and investigate an adaptive boundary element method with (rational) hierarchical splines for general second-order elliptic systems of partial differential equations (PDEs) in arbitrary dimension $d \geq 3$. For $d = 2$, we study an adaptive boundary element method with one-dimensional (rational) splines which allows for knot multiplicity increase. Whereas the method for hierarchical splines has not been published yet, the results for the latter method are essentially collected from the recent own works [FGP15, FGHP16, FGHP17] which treat the two-dimensional Laplace problem.

5.1.1 State of the art

Usually, CAD programs only provide a parametrization of the boundary $\partial\Omega$ instead of the domain Ω itself. In particular, for isogeometric FEM, the parametrization needs to be extended to the whole domain Ω, which is non-trivial and still an open research topic [MCK08, AHJ+09, XMDG13, XKW17]. The *boundary element method* (BEM) circumvents this difficulty by working only on the CAD provided boundary $\partial\Omega$. However, compared to the IGAFEM literature, only little is found for isogeometric BEM (IGABEM). The latter was first considered in [PGK+09] for 2D and in [SSE+13] for 3D. Unlike standard BEM with piecewise polynomials which is well-studied in the literature, cf. the monographs [SS11, Ste08a] and the references therein, the numerical analysis of IGABEM is widely open. We refer to [SBTR12, PTC13, SBLT13, NZW+17] for numerical experiments, to [HR10, TM12, DHK+17, MZBF15, DHP16] for fast IGABEM based on wavelets, fast multipole, $\mathcal{H}$-matrices resp. $\mathcal{H}^2$-matrices, and to [HAD14, KHZvE17, GD17, ACD+17] for some quadrature analysis. However, to the best of our knowledge, *a posteriori* error estimation for IGABEM, has only been considered for simple 2D model problems in the recent own works [FGP15, FGHP16, FGHP17].

For standard BEM with (dis)continuous piecewise polynomials, *a posteriori* error estimation and adaptive mesh-refinement are well understood. We refer to [CMPS04, CMS01, AFF+13] for weighted-residual error estimators and to [FFH+15, FFKP14] for recent overviews on available *a posteriori* error estimation strategies. Moreover, optimal convergence of mesh-refining adaptive algorithms has recently been proved for polyhedral boundaries [FFK+14, FFK+15, FKMP13] as well as smooth boundaries [Gan13]. The work [AFF+17] allows to transfer these results to piecewise smooth boundaries; see also the discussion in the review article [CFPP14].

5.1.2 Sobolev spaces

For arbitrary $d \geq 2$, let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain as in [McL00, Definition 3.28] and $\Gamma := \partial\Omega$ its boundary. For $\sigma \in [0,1]$, we define the Hilbert spaces $H^{\pm\sigma}(\Gamma)$ as in [McL00, page 99] by use of Bessel potentials on $\mathbb{R}^{d-1}$ and liftings via bi-Lipschitz mappings[1] that describe Γ. For $\sigma = 0$, there holds that $H^0(\Gamma) = L^2(\Gamma)$ with equivalent norms. We set $\|\cdot\|_{H^0(\Gamma)} := \|\cdot\|_{L^2(\Gamma)}$.

For $\sigma \in (0,1]$, any measurable subset $\omega \subseteq \Gamma$, and all $v \in H^\sigma(\Gamma)$, we define the associated Sobolev–Slobodeckij norm

$$\|v\|_{H^\sigma(\omega)}^2 := \|v\|_{L^2(\omega)}^2 + |v|_{H^\sigma(\omega)}^2 \text{ with } |v|_{H^\sigma(\omega)}^2 := \begin{cases} \int_\omega \int_\omega \frac{|v(x)-v(y)|^2}{|x-y|^{d-1+2\sigma}}\, dx dy & \text{if } \sigma \in (0,1), \\ \|\nabla_\Gamma v\|_{L^2(\omega)}^2 & \text{if } \sigma = 1. \end{cases} \tag{5.1.1}$$

It is well-known that $\|\cdot\|_{H^\sigma(\Gamma)}$ provides an equivalent norm on $H^\sigma(\Gamma)$; see, e.g., [Ste08a, Lemma 2.19] and [McL00, Theorem 3.30 and page 99] for $\sigma \in (0,1)$ and [ME14, Theorem 2.28] for $\sigma = 1$. Here, $\nabla_\Gamma(\cdot)$ denotes the usual (weak) surface gradient which can be defined for almost all $x \in \Gamma$ as follows: Since Γ is a Lipschitz boundary, there exist an open cover $(O_j)_{j=1}^J$ in $\mathbb{R}^d$ of Γ such that each $\omega_j := O_j \cap \Gamma$ can be parametrized by a bi-Lipschitz mapping $\gamma_{\omega_j} : \widehat{\omega}_j \to \omega_j$, where $\widehat{\omega}_j \subset \mathbb{R}^{d-1}$ is an open set. By Rademacher's theorem, γ_{ω_j} is almost everywhere differentiable. The corresponding Gram determinant $\det(D\gamma_{\omega_j}^\top D\gamma_{\omega_j})$ is almost everywhere positive; see Lemma 5.2.1 below. Moreover, by definition of the space $H^1(\Gamma)$, $v \in H^1(\Gamma)$ implies that $v \circ \gamma_{\omega_j} \in H^1(\widehat{\omega}_j)$. With the weak derivative $\nabla(v \circ \gamma_{\omega_j}) \in L^2(\widehat{\omega}_j)^d$, we can hence define

$$(\nabla_\Gamma v)|_{\omega_j} := \big(D\gamma_{\omega_j}(D\gamma_{\omega_j}^\top D\gamma_{\omega_j})^{-1}\nabla(v \circ \gamma_{\omega_j})\big) \circ \gamma_{\omega_j}^{-1} \quad \text{for all } v \in H^1(\Gamma). \tag{5.1.2}$$

One can show that this definition does not depend on the particular choice of the open sets $(O_j)_{j=1}^J$ and the corresponding parametrizations $(\gamma_{\omega_j})_{j=1}^J$; see [ME14, Theorem 2.28]. With (5.1.2), we immediately obtain the chain rule

$$\nabla(v \circ \gamma_{\omega_j}) = D\gamma_{\omega_j}^\top((\nabla_\Gamma v) \circ \gamma_{\omega_j}) \quad \text{for all } v \in H^1(\Gamma). \tag{5.1.3}$$

For $\sigma \in (0,1]$, $H^{-\sigma}(\Gamma)$ is a realization of the dual space of $H^\sigma(\Gamma)$ according to [McL00, Theorem 3.30 and page 99]. With the dual bracket $\langle \cdot\,,\,\cdot\rangle$, we define an equivalent norm

$$\|\psi\|_{H^{-\sigma}(\Gamma)} := \sup\big\{\langle v\,,\,\psi\rangle \,:\, v \in H^\sigma(\Gamma) \wedge \|v\|_{H^\sigma(\Gamma)} = 1\big\} \quad \text{for all } \psi \in H^{-\sigma}(\Gamma). \tag{5.1.4}$$

[McL00, page 76] states that $H^{\sigma_1}(\Gamma) \subseteq H^{\sigma_2}(\Gamma)$ for $-1 \leq \sigma_1 \leq \sigma_2 \leq 1$, where the inclusion is continuous and dense. In particular, $H^\sigma(\Gamma) \subset L^2(\Gamma) \subset H^{-\sigma}(\Gamma)$ forms a Gelfand triple in the sense of [SS11, Section 2.1.2.4] for all $\sigma \in (0,1]$, where $\psi \in L^2(\Gamma)$ is interpreted as function in $H^{-\sigma}(\Gamma)$ via

$$\langle v\,,\,\psi\rangle := \langle v\,,\,\psi\rangle_{L^2(\Gamma)} = \int_\Gamma v\,\psi\,dx \quad \text{for all } v \in H^\sigma(\Gamma), \psi \in L^2(\Gamma). \tag{5.1.5}$$

[1] For $\widehat{\omega} \subseteq \mathbb{R}^{d-1}$ and $\omega \subseteq \mathbb{R}^d$, a mapping $\gamma : \widehat{\omega} \to \omega$ is bi-Lipschitz if it is bijective and γ as well as its inverse γ^{-1} are Lipschitz continuous.

So far, we have only dealt with scalar-valued functions. For $D \geq 1$, $\sigma \in [0,1]$, $v = (v_1, \dots, v_D) \in H^\sigma(\Gamma)^D$, we define $\|v\|^2_{H^{\pm\sigma}(\Gamma)} := \sum_{j=1}^D \|v_j\|^2_{H^{\pm\sigma}(\Gamma)}$. If $\sigma > 0$, and $\omega \subseteq \Gamma$ is an arbitrary measurable set, we define $\|v\|_{H^\sigma(\omega)}$ and $|v|_{H^\sigma(\omega)}$ similarly. With the definition

$$\nabla_\Gamma v := \begin{pmatrix} \nabla_\Gamma v_1 \\ \vdots \\ \nabla_\Gamma v_D \end{pmatrix} \in L^2(\Gamma)^{D^2} \quad \text{for all } v \in H^1(\Gamma)^D, \tag{5.1.6}$$

there holds that $|v|_{H^1(\omega)} = \|\nabla_\Gamma v\|_{L^2(\omega)}$. Note that $H^{-\sigma}(\Gamma)^D$ with $\sigma \in (0,1]$ can be identified with the dual space of $H^\sigma(\Gamma)^D$, where we set

$$\langle v\,,\,\psi\rangle := \sum_{j=1}^D \langle v_j\,,\,\psi_j\rangle \quad \text{for all } v \in H^\sigma(\Gamma)^D, \psi \in H^{-\sigma}(\Gamma)^D. \tag{5.1.7}$$

Moreover, we set

$$\langle v\,,\,\psi\rangle := \sum_{j=1}^D \langle v_j\,,\,\psi_j\rangle = \int_\Gamma v \cdot \psi \, dx \quad \text{for all } v \in H^\sigma(\Gamma)^D, \psi \in L^2(\Gamma)^D. \tag{5.1.8}$$

The spaces $H^\sigma(\Gamma)$ can be also defined as trace spaces or via interpolation, where the resulting norms are always equivalent with constants which depend only on the dimension d and the boundary Γ. For a more detailed introduction to Sobolev spaces on the boundary, the reader is referred to [McL00, SS11, Ste08a].

5.1.3 Model problem

Again, we consider a general second-order linear system of PDEs on the d-dimensional bounded Lipschitz domain Ω with partial differential operator

$$\mathfrak{P}u := -\sum_{i=1}^d \sum_{i'=1}^d \partial_i (A_{ii'} \partial_{i'} u) + \sum_{i=1}^d b_i \partial_i u + cu, \tag{5.1.9}$$

where the coefficients $A_{ii'}, b_i, c \in \mathbb{R}^{D\times D}$ are constant for some fixed dimension $D \geq 1$. We suppose that $A_{ii'}^\top = A_{i'i}$. Moreover, we assume that $\mathfrak{P}$ is coercive on $H^1_0(\Omega)^D$, i.e., the bilinear form $\langle \cdot\,,\,\cdot\rangle_{\mathfrak{P}}$ of (4.1.2) is elliptic up to some compact perturbation. This is equivalent to *strong ellipticity*[2] of the matrices $A_{ii'}$ in the sense of [McL00, page 119].

Let $G : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}^{D\times D}$ be a corresponding (matrix-valued) fundamental solution in the sense of [McL00, page 198], i.e., a distributional solution of $\mathfrak{P}G = \delta$, where δ denotes the Dirac delta function. For $\psi \in L^\infty(\Gamma)^D$, we define the *single-layer operator* as

$$(\mathfrak{V}\psi)(x) := \int_\Gamma G(x-y)\psi(y)\, dy \quad \text{for all } x \in \Gamma. \tag{5.1.10}$$

[2]Unfortunately, this name might be misleading. Indeed, strong ellipticity in the sense of [McL00] does not necessarily imply ellipticity as in (4.1.4).

According to [McL00, pages 209 and 219–220] and [HMT09, Corollary 3.38], this operator can be extended for arbitrary $\sigma \in (-1/2, 1/2]$ to a bounded linear operator

$$\mathfrak{V} : H^{-1/2+\sigma}(\Gamma)^D \to H^{1/2+\sigma}(\Gamma)^D. \tag{5.1.11}$$

[McL00, Theorem 7.6] states that $\mathfrak{V}$ is always elliptic up to some compact perturbation. We assume that it is elliptic even without perturbation, i.e.,

$$\langle \mathfrak{V}\psi \,,\, \psi\rangle \geq C_{\rm ell}\|\psi\|^2_{H^{-1/2}(\Gamma)} \quad \text{for all } \psi \in H^{-1/2}(\Gamma)^D. \tag{5.1.12}$$

This is particularly satisfied for the Laplace problem or for the Lamé problem, where the case $d = 2$ requires an additional scaling of the geometry Ω; see, e.g., [Ste08a, Chapter 6]. Moreover, the bilinear form $\langle \mathfrak{V}\,\cdot\,,\,\cdot\rangle$ is continuous due to (5.1.11), i.e., it holds with $C_{\rm cont} := \|\mathfrak{V}\|_{H^{-1/2}(\Gamma)^D\to H^{1/2}(\Gamma)^D}$ that

$$\langle \mathfrak{V}\psi \,,\, \xi\rangle \leq C_{\rm cont}\|\psi\|_{H^{-1/2}(\Gamma)}\|\xi\|_{H^{-1/2}(\Gamma)} \quad \text{for all } \psi, \xi \in H^{-1/2}(\Gamma)^D. \tag{5.1.13}$$

Given a right-hand side $f \in H^1(\Gamma)^D$, we consider the boundary integral equation

$$\mathfrak{V}\phi = f. \tag{5.1.14}$$

Such equations arise from the solution of Dirichlet problems of the form $\mathfrak{P}u = 0$ in Ω with $u = g$ on Γ for some $g \in H^{1/2}(\Gamma)^D$; see, e.g., Section 5.6 or [McL00, pages 226–229] for more details. The Lax–Milgram lemma provides existence and uniqueness of the solution $\phi \in H^{-1/2}(\Gamma)^D$ of the equivalent variational formulation of (5.1.14)

$$\langle \mathfrak{V}\phi \,,\, \psi\rangle = \langle f \,,\, \psi\rangle \quad \text{for all } \psi \in H^{-1/2}(\Gamma)^D. \tag{5.1.15}$$

In particular, we see that $\mathfrak{V} : H^{-1/2}(\Gamma)^D \to H^{1/2}(\Gamma)^D$ is an isomorphism. In the Galerkin boundary element method, the test space $H^{-1/2}(\Gamma)^D$ is replaced by some discrete subspace $\mathcal{X}_\bullet \subset L^2(\Gamma)^D \subset H^{-1/2}(\Gamma)^D$. Again, the Lax–Milgram lemma guarantees existence and uniqueness of the solution $\Phi_\bullet \in \mathcal{X}_\bullet$ of the discrete variational formulation

$$\langle \mathfrak{V}\Phi_\bullet \,,\, \Psi_\bullet\rangle = \langle f \,,\, \Psi_\bullet\rangle \quad \text{for all } \Psi_\bullet \in \mathcal{X}_\bullet, \tag{5.1.16}$$

and $\Phi_\bullet$ can in fact be computed by solving a linear system of equations. Note that (5.1.11) implies that $\mathfrak{V}\Psi_\bullet \in H^1(\Gamma)^D$ for arbitrary $\Psi_\bullet \in \mathcal{X}_\bullet$. The additional regularity $f \in H^1(\Gamma)^D$ instead of $f \in H^{1/2}(\Gamma)^D$ is only needed to define the residual error estimator (5.2.17) below. For a more detailed introduction to boundary integral equations, the reader is referred to the monographs [McL00, SS11, Ste08a].

5.1.4 Outline & Contributions

The remainder of this chapter is roughly organized as follows: Section 5.2 provides an abstract framework for adaptive mesh-refinement for conforming BEM for the model problem (5.1.14). Its main result is Theorem 5.2.5 which states optimal convergence behavior of the standard adaptive Algorithm 2.2.1 applied to the model problem at hand. Its proof is given in Section 5.3. In Section 5.4, a conforming BEM for $d \geq 3$ based on hierarchical splines is presented. Its main result is Theorem 5.4.5 which states that hierarchical splines fit into the framework of Section 5.2. Section 5.5 is devoted to the proof of Theorem 5.4.5. Two numerical experiments in Section 5.6 underpin the theoretical results, but also demonstrate the limitations of hierarchical splines in the frame of adaptive BEM when the solution ϕ exhibits edge singularities. In Section 5.7, we introduce a new adaptive algorithm (Algorithm 5.7.3) for $d = 2$ with one-dimensional splines as ansatz space. Whereas the adaptive algorithm of Section 5.2 resp. Section 5.4 only uses h-refinement, the latter additionally allows for knot multiplicity increase and thus for local smoothness control of the ansatz functions. Theorem 5.7.4 states optimal convergence behavior of Algorithm 5.7.3, which is proved in Section 5.8. We conclude this chapter with three further numerical experiments in Section 5.9.

Sections 5.2–5.3

In more detail, the contribution of Section 5.2 can be paraphrased as follows: Similarly as in Section 4.2, we formulate a concrete realization (Algorithm 5.2.4) of the abstract adaptive Algorithm 2.2.1 driven by some weighted-residual *a posteriori* error estimator (5.2.17) in the frame of conforming BEM. We formulate five assumptions (M1)–(M5) on the underlying meshes (Section 5.2.1), five assumptions (R1)–(R5) on the mesh-refinement (Section 5.2.2), and six assumptions (S1)–(S6) on the BEM spaces (Section 5.2.3). First, these assumptions are sufficient to guarantee that the error estimator $\eta_\bullet$ associated with the BEM solution $\Phi_\bullet \in \mathcal{X}_\bullet \subset L^2(\Gamma)^D \subset H^{-1/2}(\Gamma)^D$ is reliable, i.e., there exists $C_{\mathrm{rel}} > 0$ such that

$$\|\phi - \Phi_\bullet\|_{H^{-1/2}(\Gamma)} \leq C_{\mathrm{rel}}\, \eta_\bullet. \tag{5.1.17}$$

Second, Theorem 5.2.5 states that Algorithm 5.2.4 leads to linear convergence with optimal rate as in Theorem 2.3.1. In explicit terms, we identify sufficient conditions of the underlying meshes, the local BEM spaces, as well as the employed (local) mesh-refinement rule which guarantee that the related residual *a posteriori* error estimator is reliable and satisfies the axioms of adaptivity from Chapter 2.

Section 5.3 is devoted to the proof of Theorem 5.2.5. To prove reliability (5.1.17), we use a localization argument (Proposition 5.3.7), i.e.,

$$\|v\|^2_{H^{1/2}(\Gamma)} \leq C_{\mathrm{split}} \sum_{T \in \mathcal{T}_\bullet} \sum_{T' \in \Pi_\bullet(T)} |v|^2_{H^{1/2}(T \cup T')} \tag{5.1.18}$$

for all $v \in H^{1/2}(\Gamma)^D$ that are L^2-orthogonal onto the ansatz space $\mathcal{X}_\bullet$ corresponding to some mesh $\mathcal{T}_\bullet$, where $C_{\mathrm{split}} > 0$ is independent of v. Here, $\Pi_\bullet(T)$ denotes the patch of T. For certain piecewise polynomial ansatz functions, this result goes back to [Fae00, Fae02]. In Remark 5.3.10, we note that one obtains at least plain convergence $\lim_{\ell\to\infty} \|\phi - \Phi_\ell\|_{H^{-1/2}(\Gamma)} = 0$ if Algorithm 5.2.4 is steered by the so-called *Faermann estimator* which is reliable and efficient. This result was first proved in [FFME+14] for piecewise constants on affine triangulations of Γ. In contrast to [FFK+14, FKMP13] which only verify the axioms of adaptivity for the Laplace problem, our analysis allows for arbitrary strongly-elliptic partial differential operators $\mathfrak{P}$ with constant coefficients as in Section 5.1.3. The crucial step is the generalization (Proposition 5.3.15) of the inverse inequality from [AFF+17], i.e.,

$$\|h_\bullet^{1/2}\nabla_\Gamma \mathfrak{V}\psi\|_{L^2(\Gamma)} \le C_{\mathrm{inv},\mathfrak{V}}\big(\|\psi\|_{H^{-1/2}(\Gamma)} + \|h_\bullet^{1/2}\psi\|_{L^2(\Gamma)}\big) \quad \text{for all } \psi \in L^2(\Gamma)^D, \tag{5.1.19}$$

where $C_{\mathrm{inv},\mathfrak{V}} > 0$, with the help of a Caccioppoli-type inequality (Lemma 5.3.13). Here, $h_\bullet \in L^\infty(\Gamma)$ denotes the local mesh-size function. Moreover, to cover the non-symmetric PDEs, we apply some ideas from [FFP14].

Sections 5.4–5.6

Based on the definitions from Section 3.4, Section 5.4 defines hierarchical meshes and hierarchical splines on the boundary Γ and introduces some local mesh-refinement rule (Algorithm 5.4.2) which preserves admissibility. To the best of our knowledge, this work is the first one which investigates BEM in 3D with hierarchical splines as ansatz space. The main result of Section 5.4 is Theorem 5.4.5 which states that hierarchical splines together with the proposed local mesh-refinement strategy satisfy all assumptions of Section 5.2, so that Theorem 5.2.5 applies. Remark 5.4.6 extends the result to rational hierarchical splines.

To prove this result in Section 5.5, we verify the properties from Section 5.2 for (rational) hierarchical splines. In particular, we derive the following inverse inequality (Sections 5.5.9 and 5.5.15)

$$\|h_\bullet^{1/2}\Psi_\bullet\|_{L^2(\Gamma)} \le C_{\mathrm{inv}}\|\Psi_\bullet\|_{H^{-1/2}(\Gamma)} \quad \text{for all } \Psi_\bullet \in \mathcal{X}_\bullet, \tag{5.1.20}$$

where $C_{\mathrm{inv}} > 0$ and $\mathcal{X}_\bullet$ denotes the space of all (rational) hierarchical splines on some mesh $\mathcal{T}_\bullet$. For piecewise polynomial ansatz functions, this result goes back to [DFG+04, GHS05, Geo08]. Further, we construct a quasi-interpolation projection $J_{\bullet,\mathcal{T}_\bullet} : L^2(\Gamma)^D \to \mathcal{X}_\bullet$ which is locally L^2-stable (Section 5.5.14 and Section 5.5.15).

We conclude this part with two numerical examples in Section 5.6.

Sections 5.7–5.9

Section 5.7 defines knot vectors and corresponding univariate splines on the boundary Γ of a two-dimensional domain $\Omega \subset \mathbb{R}^2$. We formulate an adaptive algorithm (Algorithm 5.7.3) which is driven by a node-based version of the weighted-residual *a posteriori* error estimator (5.2.17). Instead of marking elements, it marks nodes. Given these nodes, we apply a refinement strategy (Algorithm 5.7.2) which uses classical bisection as well as knot multiplicity increase to obtain a finer knot vector. Theorem 5.7.4 states again reliability (5.1.17) and linear convergence of the error estimator at optimal algebraic rate. Remark 5.7.6 extends the result to rational splines.

To prove this result in Section 5.8, we consider an equivalent reformulation of Algorithm 5.7.3. We prove slightly adapted versions of the properties from Section 5.2 to see that the reformulation fits into the abstract framework of Chapter 2. In particular, Theorem 5.7.4 follows from Corollary 2.3.4. The adapted properties from Section 5.2 include the inverse estimate (5.1.20) (Section 5.8.3 and Section 5.8.11) for the space of (rational) splines $\mathcal{X}_\bullet$ and the existence of a quasi-interpolation projection $J_{\bullet,\mathcal{T}_\bullet} : L^2(\Gamma)^D \to \mathcal{X}_\bullet$ which is locally L^2-stable (Section 5.8.7 and Section 5.8.11). Again, we note in Remark 5.8.3 that the application of the Faermann estimator would lead at least to plain convergence.

We conclude this part with three numerical examples in Section 5.9.

5.2 Axioms of adaptivity (revisited)

The aim of this section is to formulate an adaptive algorithm (Algorithm 5.2.4) for conforming BEM discretizations of our model problem (5.1.14), where adaptivity is driven by the *residual a posteriori error estimator* (see (5.2.17) below). We identify the crucial properties of the underlying meshes, the mesh-refinement, as well as the boundary element spaces which ensure that the residual error estimator fits into the general framework of Chapter 2 and which hence guarantee optimal convergence behavior of the adaptive algorithm. The main result of this section is Theorem 5.2.5 which is proved in Section 5.3.

5.2.1 Meshes

Throughout, $\mathcal{T}_\bullet$ is a *mesh* of the boundary $\Gamma = \partial\Omega$ of the bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ in the following sense:

- $\mathcal{T}_\bullet$ is a finite set of compact Lipschitz domains on Γ, i.e., each element T has the form $T = \gamma_T(\widehat{T})$, where $\widehat{T}$ is a compact[3] Lipschitz domain in $\mathbb{R}^{d-1}$ and $\gamma_T : \widehat{T} \to T$ is bi-Lipschitz;
- for all $T, T' \in \mathcal{T}_\bullet$ with $T \neq T'$, the intersection $T \cap T'$ has $(d-1)$-dimensional Hausdorff measure zero;
- $\mathcal{T}_\bullet$ is a partition of Γ, i.e., $\Gamma = \bigcup_{T \in \mathcal{T}_\bullet} T$.

We suppose that there is a countably infinite set $\mathbb{T}$ of *admissible* meshes. In order to ease notation, we introduce for $\mathcal{T}_\bullet \in \mathbb{T}$ the corresponding *mesh-width function*

$$h_\bullet \in L^\infty(\Gamma) \quad \text{with} \quad h_\bullet|_T = h_T := |T|^{1/(d-1)} \text{ for all } T \in \mathcal{T}_\bullet. \tag{5.2.1}$$

For $\omega \subseteq \Gamma$, we define the patches of order $q \in \mathbb{N}_0$ inductively by

$$\pi_\bullet^0(\omega) := \omega, \quad \pi_\bullet^q(\omega) := \bigcup \big\{T \in \mathcal{T}_\bullet \,:\, T \cap \pi_\bullet^{q-1}(\omega) \neq \emptyset\big\}. \tag{5.2.2}$$

The corresponding set of elements is

$$\Pi_\bullet^q(\omega) := \big\{T \in \mathcal{T}_\bullet \,:\, T \subseteq \pi_\bullet^q(\omega)\big\}, \quad \text{i.e.,} \quad \pi_\bullet^q(\omega) = \bigcup \Pi_\bullet^q(\omega). \tag{5.2.3}$$

To abbreviate notation, we set $\pi_\bullet(\omega) := \pi_\bullet^1(\omega)$ and $\Pi_\bullet(\omega) := \Pi_\bullet^1(\omega)$. If $\omega = \{z\}$ for some $z \in \Gamma$, we write $\pi_\bullet^q(z) := \pi_\bullet^q(\{z\})$ and $\Pi_\bullet^q(z) := \Pi_\bullet^q(\{z\})$, where we skip the index for

[3] A compact Lipschitz domain is the closure of a bounded Lipschitz domain. For $d = 2$, it is the finite union of compact intervals with non-empty interior.

$q = 1$ as before. For $\mathcal{S} \subseteq \mathcal{T}_\bullet$, we define $\pi_\bullet^q(\mathcal{S}) := \pi_\bullet^q(\bigcup \mathcal{S})$ and $\Pi_\bullet^q(\mathcal{S}) := \Pi_\bullet^q(\bigcup \mathcal{S})$, and the superscript is omitted for $q = 1$.

We assume the existence of constants $C_{\rm patch}, C_{\rm locuni}, C_{\rm shape}, C_{\rm cent}, C_{\rm semi} > 0$ such that the following assumptions are satisfied for all $\mathcal{T}_\bullet \in \mathbb{T}$:

(M1) Bounded element patch: For all $T \in \mathcal{T}_\bullet$, there holds that

$$\#\Pi_\bullet(T) \leq C_{\rm patch},$$

i.e., the number of elements in a patch is uniformly bounded.

(M2) Local quasi-uniformity: For all $T \in \mathcal{T}_\bullet$, there holds that

$$\operatorname{diam}(T)/\operatorname{diam}(T') \leq C_{\rm locuni} \quad \text{for all } T' \in \Pi_\bullet(T),$$

i.e., neighboring elements have comparable diameter.

(M3) Shape-regularity: For all $T \in \mathcal{T}_\bullet$, there holds that

$$\operatorname{diam}(T)/h_T \leq C_{\rm shape}.$$

Since there always holds that $h_T \leq \operatorname{diam}(T)$, this implies that $h_T \simeq \operatorname{diam}(T)$.

(M4) Patch centered elements: For all $T \in \mathcal{T}_\bullet$, there holds[4] that

$$\operatorname{diam}(T) \leq C_{\rm cent} \operatorname{dist}(T, \Gamma \setminus \pi_\bullet(T)),$$

i.e., each element lies essentially in the center of its patch.

(M5) Local seminorm estimate: For all $z \in \Gamma$ and $v \in H^1(\Gamma)$, there holds that

$$|v|_{H^{1/2}(\pi_\bullet(z))} \leq C_{\rm semi} \operatorname{diam}(\pi_\bullet(z))^{1/2} |v|_{H^1(\pi_\bullet(z))}.$$

The following proposition shows that (M5) is actually always satisfied. However, in general the multiplicative constant depends on the shape of the point patches. The proof is inspired by [DNPV12, Proposition 2.2], where an analogous assertion for norms instead of seminorms is found. For $\sigma = 1/2$ and $d = 2$, we already showed the assertion in the recent own work [FGHP16, Lemma 4.5]. For polyhedral domains Ω, it is proved in [FFME+14, Proposition 3.3] via interpolation techniques. First, we need the following elementary lemma, which is stated, e.g., in [ME14, Lemma 2.14]. We include the proof for completeness.

Lemma 5.2.1. *Let $\widehat{\omega}$ be an open set in $\mathbb{R}^{d-1}$, $\omega \subseteq \mathbb{R}^d$ and $\gamma : \widehat{\omega} \to \omega$ bi-Lipschitz, i.e., there exists a constant $C > 0$ such that*

$$C^{-1}|s - t| \leq |\gamma(s) - \gamma(t)| \leq C|s - t| \quad \text{for all } s, t \in \widehat{\omega}. \tag{5.2.4}$$

Then, γ is differentiable almost everywhere, and it holds for almost all $t \in \widehat{\omega}$ that

$$C^{-1}|r| \leq |D\gamma(t)r| \leq C|r| \quad \text{for all } r \in \mathbb{R}^{d-1}. \tag{5.2.5}$$

For the corresponding Gram determinant, this implies that almost all $t \in \widehat{\omega}$ satisfy that

$$C^{-(d-1)} \leq \sqrt{\det(D\gamma^\top(t) D\gamma(t))} \leq C^{d-1} \tag{5.2.6}$$

[4] We use the convention $\operatorname{dist}(T, \emptyset) := \operatorname{diam}(\Gamma)$.

Proof. By Rademacher's theorem γ is differentiable almost everywhere. If γ is differentiable at $t \in \widehat{\omega}$, there holds that

$$D\gamma(t)r = \lim_{\epsilon \to 0} \frac{\gamma(t + \epsilon r) - \gamma(t)}{\epsilon},$$

and bi-Lipschitz continuity of γ immediately implies (5.2.5). If r is even an eigenvector of $D\gamma(t)^\top D\gamma(t)$ with eigenvalue λ, we derive that

$$C^{-2} \le \lambda = \frac{|D\gamma(t)r|^2}{|r|^2} \le C^2.$$

Since $\det(D\gamma^\top(t)D\gamma(t))$ is the product of the $d-1$ eigenvalues of $D\gamma^\top(t)D\gamma(t)$, we conclude (5.2.6), and thus the proof. ☐

Proposition 5.2.2. *Let $\widehat{\omega} \subset \mathbb{R}^{d-1}$ be a bounded and connected Lipschitz domain and $\gamma_\omega : \widehat{\omega} \to \omega \subseteq \Gamma$ bi-Lipschitz. In particular, there exists a constant $C_{\rm lipref} > 0$ such that*

$$C_{\rm lipref}^{-1}|s-t| \le \frac{|\gamma_\omega(s) - \gamma_\omega(t)|}{\operatorname{diam}(\omega)} \le C_{\rm lipref}|s-t| \quad \textit{for all } s, t \in \widehat{\omega}. \tag{5.2.7}$$

Then, for arbitrary $\sigma \in (0,1)$ there exists a constant $C_{\rm semi}(\widehat{\omega}) > 0$ such that

$$|v|_{H^\sigma(\omega)} \le C_{\rm semi}(\widehat{\omega}) \operatorname{diam}(\omega)^{1-\sigma} |v|_{H^1(\omega)} \quad \textit{for all } v \in H^1(\Gamma). \tag{5.2.8}$$

The constant $C_{\rm semi}(\widehat{\omega}) > 0$ depends only on the dimension d, σ, the set $\widehat{\omega}$, and $C_{\rm lipref}$.

Proof. We split the proof into three steps.
Step 1: According to [McL00, Theorem A.4], there exists a continuous linear extension operator $\mathfrak{E}_1 : H^1(\widehat{\omega}) \to H^1(\mathbb{R}^{d-1})$ with $(\mathfrak{E}_1 v)|_{\widehat{\omega}} = v|_{\widehat{\omega}}$. We define the operator

$$\mathfrak{E} : H^1(\widehat{\omega}) \to H^1(\mathbb{R}^{d-1}), \quad v \mapsto \mathfrak{E}_1\Big(v - \frac{1}{|\widehat{\omega}|}\int_{\widehat{\omega}} v\,dx\Big) + \frac{1}{|\widehat{\omega}|}\int_{\widehat{\omega}} v\,dx.$$

Then, $\mathfrak{E}$ is also a continuous linear extension operator with $(\mathfrak{E}v)|_{\widehat{\omega}} = v|_{\widehat{\omega}}$. The Poincaré inequality proves the existence of a constant $C_{\rm ext}(\widehat{\omega}) > 0$ depending only on $\widehat{\omega}$ and the operator norm $\|\mathfrak{E}_1\|$ such that

$$\|\nabla \mathfrak{E} v\|_{L^2(\mathbb{R}^{d-1})} \le \|\mathfrak{E}_1\| \Big\| v - \frac{1}{|\widehat{\omega}|}\int_{\widehat{\omega}} v\,dx \Big\|_{H^1(\widehat{\omega})} \le C_{\rm ext}(\widehat{\omega}) \|\nabla v\|_{L^2(\widehat{\omega})}. \tag{5.2.9}$$

Step 2: We prove (5.2.8) in the parameter domain, where the corresponding seminorms are defined analogously. Let $\mathfrak{E}$ be the extension operator of Step 1. Note that $\mathfrak{E}$ particularly extends any function on $\widehat{\omega}$ to its convex hull $\operatorname{co}(\widehat{\omega})$. Let $\widehat{v} \in H^1(\widehat{\omega})$. First, we assume that $\mathfrak{E}\widehat{v} \in C^\infty(\mathbb{R}^{d-1}) \cap H^1(\mathbb{R}^{d-1})$. Then, the (higher-dimensional) fundamental theorem of calculus and the Cauchy–Schwarz inequality prove that

$$\begin{aligned}|\widehat{v}|^2_{H^\sigma(\widehat{\omega})} &= \int_{\widehat{\omega}}\int_{\widehat{\omega}} \frac{|\widehat{v}(s)-\widehat{v}(t)|^2}{|s-t|^{d-1+2\sigma}}\,dtds \\ &= \int_{\widehat{\omega}}\int_{\widehat{\omega}} \frac{\left(\int_0^1 \nabla\mathfrak{E}\widehat{v}(s+\tau(t-s))\cdot(t-s)\,d\tau\right)^2}{|s-t|^{d-1+2\sigma}}\,dtds \\ &\le \int_{\widehat{\omega}}\int_{\widehat{\omega}}\int_0^1 \frac{|\nabla\mathfrak{E}\widehat{v}(s+\tau(t-s))|^2}{|s-t|^{d-3+2\sigma}}\,d\tau dtds. \\ &= \int_{\widehat{\omega}}\int_{\widehat{\omega}-s}\int_0^1 \frac{|\nabla\mathfrak{E}\widehat{v}(s+\tau r)|^2}{|r|^{d-3+2\sigma}}\,d\tau drds.\end{aligned}$$

Next, we enlarge the integration domains and apply the Fubini theorem

$$\begin{aligned}|\widehat{v}|^2_{H^\sigma(\widehat{\omega})} &\le \int_{\mathbb{R}^{d-1}}\int_{\widehat{\omega}-\widehat{\omega}}\int_0^1 \frac{|\nabla\mathfrak{E}\widehat{v}(s+\tau r)|^2}{|r|^{d-3+2\sigma}}\,d\tau drds \\ &= \int_{\widehat{\omega}-\widehat{\omega}}\int_{\mathbb{R}^{d-1}} \frac{|\nabla\mathfrak{E}\widehat{v}(s)|^2}{|r|^{d-3+2\sigma}}\,dsdr.\end{aligned}$$

Note that $\widehat{\omega}-\widehat{\omega}\subseteq B_{2\mathrm{diam}(\widehat{\omega})}(0)$, where $B_{2\mathrm{diam}(\widehat{\omega})}(0)$ denotes the open ball in $\mathbb{R}^{d-1}$ with center 0 and radius $2\,\mathrm{diam}(\widehat{\omega})$. With this and (5.2.9) from Step 1, we conclude that

$$\begin{aligned}|\widehat{v}|^2_{H^\sigma(\widehat{\omega})} &\le \|\nabla\mathfrak{E}\widehat{v}\|^2_{L^2(\mathbb{R}^{d-1})}\int_{B_{2\mathrm{diam}(\widehat{\omega})}(0)} |r|^{-d+3-2\sigma}dr \\ &\le C_{\mathrm{ext}}(\widehat{\omega})^2\|\nabla\widehat{v}\|^2_{L^2(\widehat{\omega})}\int_{B_{2\mathrm{diam}(\widehat{\omega})}(0)} |r|^{-d+3-2\sigma}dr.\end{aligned} \tag{5.2.10}$$

Transforming to polar coordinates, shows that the integral in (5.2.10) is finite. By density of $C^\infty(\mathbb{R}^{d-1})\cap H^1(\mathbb{R}^{d-1})$ in $H^1(\mathbb{R}^{d-1})$ (see, e.g. [McL00, page 76]), (5.2.10) is also valid if $\mathfrak{E}\widehat{v}\in H^1(\mathbb{R}^{d-1})$.

Step 3: Now, we prove (5.2.8). Lemma 5.2.1 shows that

$$C_{\mathrm{lipref}}^{-(d-1)} \le \frac{\sqrt{\det(D\gamma_\omega^\top D\gamma_\omega)(s)}}{\mathrm{diam}(\omega)^{d-1}} \le C_{\mathrm{lipref}}^{d-1} \quad \text{for almost all } s\in\widehat{\omega}. \tag{5.2.11}$$

With (5.2.7), it hence holds that

$$\begin{aligned}|v|^2_{H^\sigma(\omega)} &= \int_\omega\int_\omega \frac{|v(x)-v(y)|^2}{|x-y|^{d-1+2\sigma}}\,dxdy \\ &\le (\mathrm{diam}(\omega)^{-1}C_{\mathrm{lipref}})^{d-1+2\sigma}(\mathrm{diam}(\omega)C_{\mathrm{lipref}})^{2(d-1)}\int_{\widehat{\omega}}\int_{\widehat{\omega}} \frac{|v(\gamma_\omega(s))-v(\gamma_\omega(t))|^2}{|s-t|^{d-1+2\sigma}}\,dsdt \\ &\simeq \mathrm{diam}(\omega)^{d-1-2\sigma}\int_{\widehat{\omega}}\int_{\widehat{\omega}} \frac{|v(\gamma_\omega(s))-v(\gamma_\omega(t))|^2}{|s-t|^{d-1+2\sigma}}\,dsdt \\ &= \mathrm{diam}(\omega)^{d-1-2\sigma}|v\circ\gamma_\omega|^2_{H^\sigma(\widehat{\omega})}.\end{aligned}$$

Next, we apply Step 2 with the chain rule (5.1.3) to see that

$$
\begin{aligned}
|v|^2_{H^\sigma(\omega)} &\lesssim \operatorname{diam}(\omega)^{d-1-2\sigma} \|\nabla(v \circ \gamma_\omega)\|^2_{L^2(\widehat{\omega})} \\
&= \operatorname{diam}(\omega)^{d-1-2\sigma} \int_{\widehat{\omega}} |D\gamma_\omega(t)^\top \nabla_\Gamma v(\gamma_\omega(t))|^2 \, dt.
\end{aligned}
$$

Note that $\|D\gamma_\omega\|_{L^\infty(\widehat{\omega})} \le C_{\rm lipref} \operatorname{diam}(\omega)$ due to Lemma 5.2.1. Together with (5.2.11), we derive that

$$
|v|^2_{H^\sigma(\omega)} \lesssim \operatorname{diam}(\omega)^{d-1-2\sigma} \operatorname{diam}(\omega)^2 \operatorname{diam}(\omega)^{-(d-1)} \int_\omega |\nabla_\Gamma v(x)|^2 \, dx.
$$

This concludes the proof. □

5.2.2 Mesh-refinement

We make exactly the same assumptions as in Section 4.2.2. For convenience of the reader, we state them again in this section.

For $\mathcal{T}_\bullet \in \mathbb{T}$ and an arbitrary set of marked elements $\mathcal{M}_\bullet \subseteq \mathcal{T}_\bullet$, we associate a corresponding *refinement* $\mathcal{T}_\circ := \texttt{refine}(\mathcal{T}_\bullet, \mathcal{M}_\bullet) \in \mathbb{T}$ with $\mathcal{M}_\bullet \subseteq \mathcal{T}_\bullet \setminus \mathcal{T}_\circ$, i.e., at least the marked elements are refined. Moreover, we suppose for the cardinalities that $\#\mathcal{T}_\bullet < \#\mathcal{T}_\circ$ if $\mathcal{M}_\bullet \neq \emptyset$ and $\mathcal{T}_\circ = \mathcal{T}_\bullet$ else. We define $\texttt{refine}(\mathcal{T}_\bullet)$ as the set of all $\mathcal{T}_\circ$ such that there exist meshes $\mathcal{T}_{(0)}, \dots, \mathcal{T}_{(J)}$ and marked elements $\mathcal{M}_{(0)}, \dots, \mathcal{M}_{(J-1)}$ with $\mathcal{T}_\circ = \mathcal{T}_{(J)} = \texttt{refine}(\mathcal{T}_{(J-1)}, \mathcal{M}_{(J-1)}), \dots, \mathcal{T}_{(1)} = \texttt{refine}(\mathcal{T}_{(0)}, \mathcal{M}_{(0)})$ and $\mathcal{T}_{(0)} = \mathcal{T}_\bullet$. We assume that there exists a fixed initial mesh $\mathcal{T}_0 \in \mathbb{T}$ with $\mathbb{T} = \texttt{refine}(\mathcal{T}_0)$.

We suppose that there exist $C_{\rm son} \ge 2$ and $0 < \rho_{\rm son} < 1$ such that all meshes $\mathcal{T}_\bullet \in \mathbb{T}$ satisfy for arbitrary marked elements $\mathcal{M}_\bullet \subseteq \mathcal{T}_\bullet$ with corresponding refinement $\mathcal{T}_\circ := \texttt{refine}(\mathcal{T}_\bullet, \mathcal{M}_\bullet)$, the following elementary properties (R1)–(R3):

(R1) Son estimate: It holds that

$$
\#\mathcal{T}_\circ \le C_{\rm son} \, \#\mathcal{T}_\bullet,
$$

i.e., one step of refinement leads to a bounded increase of elements.

(R2) Father is union of sons: For all $T \in \mathcal{T}_\bullet$, it holds that

$$
T = \bigcup \{ T' \in \mathcal{T}_\circ \, : \, T' \subseteq T \},
$$

i.e., each element T is the union of its successors.

(R3) Reduction of sons: For all $T \in \mathcal{T}_\bullet$, it holds that

$$
|T'| \le \rho_{\rm son} \, |T| \quad \text{for all } T' \in \mathcal{T}_\circ \text{ with } T' \subsetneqq T,
$$

i.e., successors are uniformly smaller than their father.

By induction and the definition of $\texttt{refine}(\mathcal{T}_\bullet)$, one easily sees that (R2)–(R3) remain valid if $\mathcal{T}_\circ$ is an arbitrary mesh in $\texttt{refine}(\mathcal{T}_\bullet)$. In particular, (R2)–(R3) imply that each refined element $T \in \mathcal{T}_\bullet \setminus \mathcal{T}_\circ$ is split into at least two sons, wherefore

$$\#(\mathcal{T}_\bullet \setminus \mathcal{T}_\circ) \le \#\mathcal{T}_\circ - \#\mathcal{T}_\bullet \quad \text{for all } \mathcal{T}_\circ \in \texttt{refine}(\mathcal{T}_\bullet). \tag{5.2.12}$$

Besides (R1)–(R3), we suppose the following less trivial requirements (R4)–(R5) with generic constants $C_{\rm clos}, C_{\rm over} > 0$:

(R4) Closure estimate: Let $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}_0}$ be an arbitrary sequence in $\mathbb{T}$ such that $\mathcal{T}_{\ell+1} = \texttt{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$ with some $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ for all $\ell \in \mathbb{N}_0$. Then, for all $\ell \in \mathbb{N}_0$, there holds that

$$\#\mathcal{T}_\ell - \#\mathcal{T}_0 \le C_{\rm clos} \sum_{j=0}^{\ell-1} \#\mathcal{M}_j.$$

(R5) Overlay property: For all $\mathcal{T}_\bullet, \mathcal{T}_\star \in \mathbb{T}$, there exists a common refinement $\mathcal{T}_\circ \in \texttt{refine}(\mathcal{T}_\bullet) \cap \texttt{refine}(\mathcal{T}_\star)$ which satisfies the overlay estimate

$$\#\mathcal{T}_\circ \le C_{\rm over}(\#\mathcal{T}_\star - \#\mathcal{T}_0) + \#\mathcal{T}_\bullet.$$

5.2.3 Boundary element space

With each $\mathcal{T}_\bullet \in \mathbb{T}$, we associate a finite dimensional space of vector valued functions

$$\mathcal{X}_\bullet \subset L^2(\Gamma)^D \subset H^{-1/2}(\Gamma)^D. \tag{5.2.13}$$

Let $\Phi_\bullet \in \mathcal{X}_\bullet$ be the corresponding Galerkin approximation to the solution $\phi \in H^{-1/2}(\Gamma)^D$ of (5.1.14), i.e.,

$$\langle \mathfrak{V}\Phi_\bullet\,,\,\Psi_\bullet\rangle = \langle f\,,\,\Psi_\bullet\rangle \quad \text{for all } \Psi_\bullet \in \mathcal{X}_\bullet. \tag{5.2.14}$$

We note the Galerkin orthogonality

$$\langle f - \mathfrak{V}\Phi_\bullet\,,\,\Psi_\bullet\rangle = 0 \quad \text{for all } \Psi_\bullet \in \mathcal{X}_\bullet, \tag{5.2.15}$$

as well as the resulting Céa type quasi-optimality

$$\|\phi - \Phi_\bullet\|_{H^{-1/2}(\Gamma)} \le C_{\text{Céa}} \min_{\Psi_\bullet \in \mathcal{X}_\bullet} \|\phi - \Psi_\bullet\|_{H^{-1/2}(\Gamma)} \quad \text{with} \quad C_{\text{Céa}} := \tfrac{C_{\rm cont}}{C_{\rm ell}}. \tag{5.2.16}$$

We assume the existence of constants $C_{\rm inv} > 0$, $q_{\rm loc}, q_{\rm proj}, q_{\rm supp} \in \mathbb{N}_0$, and $0 < \rho_{\rm unity} < 1$, such that the following properties (S1)–(S4) hold for all $\mathcal{T}_\bullet \in \mathbb{T}$:

(S1) Inverse inequality: For all $\Psi_\bullet \in \mathcal{X}_\bullet$, it holds that

$$\|h_\bullet^{1/2}\Psi_\bullet\|_{L^2(\Gamma)} \le C_{\rm inv}\,\|\Psi_\bullet\|_{H^{-1/2}(\Gamma)}.$$

(S2) Nestedness: For all $\mathcal{T}_\circ \in \texttt{refine}(\mathcal{T}_\bullet)$, it holds that

$$\mathcal{X}_\bullet \subseteq \mathcal{X}_\circ.$$

(S3) Local domain of definition: For all $\mathcal{T}_\circ \in \texttt{refine}(\mathcal{T}_\bullet)$, $T \in \mathcal{T}_\bullet \setminus \Pi_\bullet^{q_{\rm loc}}(\mathcal{T}_\bullet \setminus \mathcal{T}_\circ) \subseteq \mathcal{T}_\bullet \cap \mathcal{T}_\circ$, and $\Psi_\circ \in \mathcal{X}_\circ$, it holds that

$$\Psi_\circ|_{\pi_\bullet^{q_{\rm proj}}(T)} \in \big\{\Psi_\bullet|_{\pi_\bullet^{q_{\rm proj}}(T)} \,:\, \Psi_\bullet \in \mathcal{X}_\bullet\big\}.$$

(S4) Componentwise local approximation of unity: For all $T \in \mathcal{T}_\bullet$ and all $j \in \{1,\dots,D\}$, there exists some $\Psi_{\bullet,T,j} \in \mathcal{X}_\bullet$ with

$$T \subseteq \operatorname{supp}(\Psi_{\bullet,T,j}) \subseteq \pi_\bullet^{q_{\rm supp}}(T),$$

such that only the j-th component does not vanish, i.e.,

$$(\Psi_{\bullet,T,j})_{j'} = 0 \quad \text{for} \quad j' \neq j,$$

and

$$\|1 - (\Psi_{\bullet,T,j})_j\|_{L^2(\operatorname{supp}(\Psi_{\bullet,T,j}))} \le \rho_{\rm unity} |\operatorname{supp}(\Psi_{\bullet,T,j})|^{1/2}.$$

Remark 5.2.3. *Clearly, (S4) is in particular satisfied if $\mathcal{X}_\bullet$ is a product space, i.e., $\mathcal{X}_\bullet = \prod_{j=1}^D (\mathcal{X}_\bullet)_j$, and each component $(\mathcal{X}_\bullet)_j \subset L^2(\Gamma)$ satisfies (S4).*

Besides (S1)–(S4), we suppose that there exist constants $C_{\rm sz} > 0$ as well as $q_{\rm sz} \in \mathbb{N}_0$ such that for all $\mathcal{T}_\bullet \in \mathbb{T}$ and $\mathcal{S} \subseteq \mathcal{T}_\bullet$, there exists a linear operator $J_{\bullet,\mathcal{S}} : L^2(\Gamma)^D \to \big\{\Psi_\bullet \in \mathcal{X}_\bullet : \Psi_\bullet|_{\bigcup(\mathcal{T}_\bullet \setminus \mathcal{S})} = 0\big\}$ with the following properties (S5)–(S6):

(S5) Local projection property. Let $q_{\rm loc}, q_{\rm proj} \in \mathbb{N}_0$ from (S3). For all $\psi \in L^2(\Gamma)^D$ and $T \in \mathcal{T}_\bullet$ with $\Pi_\bullet^{q_{\rm loc}}(T) \subseteq \mathcal{S}$, it holds that

$$(J_{\bullet,\mathcal{S}}\psi)|_T = \psi|_T, \quad \text{if } \psi|_{\pi_\bullet^{q_{\rm proj}}(T)} \in \big\{\Psi_\bullet|_{\pi_\bullet^{q_{\rm proj}}(T)} \,:\, \Psi_\bullet \in \mathcal{X}_\bullet\big\}.$$

(S6) Local L^2-stability. For all $\psi \in L^2(\Gamma)^D$ and $T \in \mathcal{T}_\bullet$, it holds that

$$\|J_\bullet \psi\|_{L^2(T)} \le C_{\rm sz} \|\psi\|_{L^2(\pi_\bullet^{q_{\rm sz}}(T))}.$$

5.2.4 Error estimator

Let $\mathcal{T}_\bullet \in \mathbb{T}$. Due to the regularity assumption $f \in H^1(\Gamma)^D$, the mapping property (5.1.11), and $\mathcal{X}_\bullet \subset L^2(\Gamma)^D$, there holds that $f - \mathfrak{V}\Psi_\bullet \in H^1(\Gamma)^D$ for all $\Psi_\bullet \in \mathcal{X}_\bullet$. This allows to employ the weighted-residual *a posteriori* error estimator

$$\eta_\bullet := \eta_\bullet(\mathcal{T}_\bullet) \quad \text{with} \quad \eta_\bullet(\mathcal{S})^2 := \sum_{T\in\mathcal{S}} \eta_\bullet(T)^2 \text{ for all } \mathcal{S} \subseteq \mathcal{T}_\bullet, \tag{5.2.17a}$$

where, for all $T \in \mathcal{T}_\bullet$, the local refinement indicators read

$$\eta_\bullet(T)^2 := h_T |f - \mathfrak{V}\Phi_\bullet|^2_{H^1(T)}. \tag{5.2.17b}$$

This estimator goes back to the works [CS96, Car97], where reliability (5.2.22) is proved for standard 2D BEM with piecewise polynomials on polygonal geometries, while the corresponding result for 3D BEM is found in [CMS01].

5.2.5 Adaptive algorithm

We consider the following concrete realization of the abstract Algorithm 2.2.1.

Algorithm 5.2.4. ***Input:*** *Dörfler parameter* $\theta \in (0,1]$ *and marking constant* $C_{\min} \in [1,\infty]$.
Loop: *For each* $\ell = 0,1,2,\dots$, *iterate the following steps:*

(i) *Compute Galerkin approximation* $\Phi_\ell \in \mathcal{X}_\ell$.

(ii) *Compute refinement indicators* $\eta_\ell(T)$ *for all elements* $T \in \mathcal{T}_\ell$.

(iii) *Determine a set of marked elements* $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ *which has up to the multiplicative constant* $C_{\min}$ *minimal cardinality, such that the following Dörfler marking is satisfied*

$$\theta\,\eta_\ell^2 \le \eta_\ell(\mathcal{M}_\ell)^2. \tag{5.2.18}$$

(iv) *Generate refined mesh* $\mathcal{T}_{\ell+1} := \texttt{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$.

Output: *Refined meshes* $\mathcal{T}_\ell$ *and corresponding Galerkin approximations* Φ_ℓ *with error estimators* η_ℓ *for all* $\ell \in \mathbb{N}_0$.

5.2.6 Optimal convergence

With μ from Section 2.2.1 defined as cardinality $\#(\cdot)$, we recall the definitions of Chapter 2

$$\mathbb{T}(N) := \big\{\mathcal{T}_\bullet \in \mathbb{T} \,:\, \#\mathcal{T}_\bullet - \#\mathcal{T}_0 \le N\big\} \quad \text{for all } N \in \mathbb{N}_0 \tag{5.2.19}$$

and for all $s > 0$

$$C_{\rm approx}(s) := \sup_{N\in\mathbb{N}_0} \min_{\mathcal{T}_\bullet\in\mathbb{T}(N)} (N+1)^s\,\eta_\bullet \in [0,\infty]. \tag{5.2.20}$$

We say that the solution $\phi \in H^{-1/2}(\Gamma)^D$ lies in the *approximation class* s *with respect to the estimator* if

$$\|\phi\|_{\mathbb{A}_s^{\rm est}} := C_{\rm approx}(s) < \infty. \tag{5.2.21}$$

By definition, $\|\phi\|_{\mathbb{A}_s^{\rm est}} < \infty$ implies that the error estimator $\eta_\bullet$ on the optimal meshes $\mathcal{T}_\bullet$ decays at least with rate $\mathcal{O}\big((\#\mathcal{T}_\bullet)^{-s}\big)$. The following main theorem states that each possible rate $s > 0$ is in fact realized by Algorithm 5.2.4. The proof is given in Section 5.3. It essentially follows from its abstract counterpart Theorem 2.3.1 by verifying the axioms of Section 2.3. Such an optimality result was first proved in [FKMP13] for the Laplace operator $\mathfrak{P} = -\Delta$ on a polyhedral domain Ω. As ansatz space, they considered piecewise constants on shape-regular triangulations. [FFK+14] in combination with [AFF+17] extends the assertion to piecewise polynomials on shape-regular curvilinear triangulations of some piecewise smooth boundary Γ. Independently, [Gan13] proved the same result for globally smooth Γ and arbitrary symmetric and elliptic boundary integral operators.

Theorem 5.2.5. *Let* $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}_0}$ *be the sequence of meshes generated by Algorithm 5.2.4. Then, there hold:*

(i) *Suppose* (M1)–(M5), *and* (S4)*. Then, the residual error estimator satisfies reliability, i.e., there exists a constant* $C_{\rm rel} > 0$ *such that*

$$\|\phi - \Phi_\bullet\|_{H^{-1/2}(\Gamma)} \le C_{\rm rel}\eta_\bullet \quad \text{for all } \mathcal{T}_\bullet \in \mathbb{T}. \tag{5.2.22}$$

(ii) *Suppose* (M1)–(M5), (R2)–(R3), *and* (S1)–(S2)*. Then, for arbitrary* $0 < \theta \le 1$ *and* $C_{\rm min} \in [1, \infty]$*, the estimator converges linearly, i.e., there exist constants* $0 < \rho_{\rm lin} < 1$ *and* $C_{\rm lin} \ge 1$ *such that*

$$\eta_{\ell+j}^2 \le C_{\rm lin}\rho_{\rm lin}^j \eta_\ell^2 \quad \text{for all } j, \ell \in \mathbb{N}_0. \tag{5.2.23}$$

(iii) *Suppose* (M1)–(M5), (R1)–(R5), *and* (S1)–(S6)*. Then, there exists a constant* $0 < \theta_{\rm opt} \le 1$ *such that for all* $0 < \theta < \theta_{\rm opt}$ *and* $C_{\rm min} \in [1, \infty)$*, the estimator converges at optimal rate, i.e., for all* $s > 0$ *there exist constants* $c_{\rm opt}, C_{\rm opt} > 0$ *such that*

$$c_{\rm opt}\|\phi\|_{\mathbb{A}_s^{\rm est}} \le \sup_{\ell\in\mathbb{N}_0} \left(\#\mathcal{T}_\ell - \#\mathcal{T}_0 + 1\right)^s \eta_\ell \le C_{\rm opt}\|\phi\|_{\mathbb{A}_s^{\rm est}}, \tag{5.2.24}$$

where the lower bound requires only (R1) *to hold.*

All involved constants $C_{\rm rel}, C_{\rm lin}, q_{\rm lin}, \theta_{\rm opt}$*, and* $C_{\rm opt}$ *depend only on the assumptions made as well as the dimensions* d, D*, the coefficients of the differential operator* $\mathfrak{P}$*, and* Γ*, while* $C_{\rm lin}, \rho_{\rm lin}$ *depend additionally on* θ *and the sequence* $(\Phi_\ell)_{\ell\in\mathbb{N}_0}$*, and* $C_{\rm opt}$ *depends furthermore on* $C_{\rm min}$*, and* $s > 0$*. The constant* $c_{\rm opt}$ *depends only on* $C_{\rm son}, \#\mathcal{T}_0$*,* s*, and if there exists* ℓ_0 *with* $\eta_{\ell_0} = 0$*, then also on* ℓ_0 *and* η_0*.*

Remark 5.2.6. *In contrast to FEM, efficiency* (4.2.21) *for the weighted-residual error estimator* $\eta_\bullet$ *is an open question. Indeed, [AFF+13] is the only available result in the literature. However, [AFF+13] is restricted to the two dimensional case* $\Omega \subset \mathbb{R}^2$ *with piecewise constant ansatz functions. Moreover, additional (regularity) assumptions on the right-hand side* f *are required.*

Remark 5.2.7. *If the bilinear form* $\langle \mathfrak{V}\,\cdot\,, \cdot\rangle$ *is symmetric, then* $C_{\rm lin}$*,* $\rho_{\rm lin}$*, and* $C_{\rm opt}$ *are independent of* $(\Phi_\ell)_{\ell\in\mathbb{N}_0}$*; see Remark 5.3.17 below.*

Remark 5.2.8. *Let* $\Gamma_0 \subsetneqq \Gamma$ *be an open subset of* $\Gamma = \partial\Omega$ *and let* $\mathfrak{E}_0 : L^2(\Gamma_0)^D \to L^2(\Gamma)^D$ *denote the operator that extends a function defined on* Γ_0 *to a function on* Γ *by zero. We define the space of restrictions* $H^{1/2}(\Gamma_0) := \left\{v|_{\Gamma_0} \,:\, v \in H^{1/2}(\Gamma)\right\}$ *endowed with the quotient norm* $v_0 \mapsto \inf\left\{\|v\|_{H^{1/2}(\Gamma)} \,:\, v|_{\Gamma_0} = v_0\right\}$ *and its dual space* $\widetilde{H}^{-1/2}(\Gamma_0) := H^{1/2}(\Gamma_0)^*$*. According to [AFF+17, Section 2.1],* $\mathfrak{E}_0$ *can be extended to an isometric operator* $\mathfrak{E}_0 : \widetilde{H}^{-1/2}(\Gamma_0)^D \to H^{-1/2}(\Gamma)^D$*. Then, one can consider the integral equation*

$$(\mathfrak{V}\mathfrak{E}_0\phi)|_{\Gamma_0} = f|_{\Gamma_0}, \tag{5.2.25}$$

where $(\mathfrak{V}\mathfrak{E}_0(\cdot))|_{\Gamma_0} : \widetilde{H}^{-1/2}(\Gamma_0)^D \to H^{1/2}(\Gamma_0)^D$*. In the literature, such problems are known as* screen problems*; see, e.g., [SS11, Section 3.5.3]. Theorem 5.2.5 should hold analogously for the screen problem* (5.2.25)*. Indeed, the works [FKMP13, FFK+14, AFF+17, Gan13]* *cover this case as well. However, the literature on restricted Sobolev spaces and their equivalent definitions is often not very thorough. In particular, the corresponding proof requires the fact that for all $v_0 \in H^\sigma(\Gamma_0)$, the norm $\|v_0\|_{H^{1/2}(\Gamma_0)}$ defined in (5.1.1) is equivalent to the quotient norm $\inf\big\{\|v\|_{H^{1/2}(\Gamma)} : v|_{\Gamma_0} = v_0\big\}$ (which is considered in [McL00]). To ease the presentation, we focus on closed boundaries $\Gamma = \partial\Omega$.*

Remark 5.2.9. (a) *We additionally assume that $\mathcal{X}_\bullet$ contains all componentwise constant functions, i.e.,*

$$x \in \mathcal{X}_\bullet \quad \textit{for all } x \in \mathbb{R}^D. \tag{5.2.26}$$

Then, under the assumption that $\|h_\ell\|_{L^\infty(\Omega)} \to 0$ as $\ell \to \infty$, one can show that $\mathcal{X}_\infty := \overline{\bigcup_{\ell\in\mathbb{N}_0} \mathcal{X}_\ell} = H^{-1/2}(\Gamma)^D$. To see this, recall that $H^{1/2}(\Gamma)^D$ is continuously and densely embedded in $L^2(\Gamma)^D$ which is itself continuously and densely embedded in $H^{-1/2}(\Gamma)^D$. For $\psi \in H^{-1/2}(\Gamma)^D$ and arbitrary $\epsilon > 0$, let $\psi_\epsilon \in H^{1/2}(\Gamma)^D$ with $\|\psi - \psi_\epsilon\|_{H^{-1/2}(\Gamma)} < \epsilon$. We abbreviate the projection operator $J_\ell := J_{\ell,\mathcal{T}_\ell}$ for all $\ell \in \mathbb{N}_0$. For all $T \in \mathcal{T}_\ell$, the projection property (S5) in combination with our additional assumption (5.2.26) as well as the local L^2-stability (S6) show that

$$\begin{aligned}\|(1-J_\ell)\psi_\epsilon\|_{L^2(T)} &= \Big\|(1-J_\ell)\Big(\psi_\epsilon - \frac{1}{|\pi_\bullet^{q_{\rm sz}}(T)|}\int_{\pi_\bullet^{q_{\rm sz}}(T)} \psi_\epsilon\,dx\Big)\Big\|_{L^2(T)} \\ &\lesssim \Big\|\psi_\epsilon - \frac{1}{|\pi_\bullet^{q_{\rm sz}}(T)|}\int_{\pi_\bullet^{q_{\rm sz}}(T)} \psi_\epsilon\,dx\Big\|_{L^2(\pi_\bullet^{q_{\rm sz}}(T))}.\end{aligned}$$

With this, the Poincaré–type inequality from Lemma 5.3.3 below, and (M1)–(M3), we see that

$$\|(1-J_\ell)\psi_\epsilon\|_{L^2(T)} \le h_T^{1/2}|\psi_\epsilon|_{H^{1/2}(T)} \le \|h_\ell\|_{L^\infty(\Gamma)}^{1/2}|\psi_\epsilon|_{H^{1/2}(T)}.$$

Squaring and summing over all elements yields that

$$\|(1-J_\ell)\psi_\epsilon\|_{H^{-1/2}(\Gamma)}^2 \lesssim \|(1-J_\ell)\psi_\epsilon\|_{L^2(\Gamma)}^2 \lesssim \|h_\ell\|_{L^\infty(\Gamma)} \sum_{T\in\mathcal{T}_\bullet} |\psi_\epsilon|_{H^{1/2}(T)}^2.$$

Elementary calculations prove that $\sum_{T\in\mathcal{T}_\bullet} |\psi_\epsilon|_{H^{1/2}(T)}^2 \le |\psi_\epsilon|_{H^{1/2}(\Gamma)}^2$; see also Proposition 5.3.8. Since $\lim_{\ell\to\infty} \|h_\ell\|_{L^\infty(\Gamma)} = 0$ and ϵ was arbitrary, this concludes the proof.

(b) *The latter observation allows to follow the ideas of [BHP17] and to show that the adaptive algorithm yields convergence even if the bilinear form $\langle\mathfrak{V}\,\cdot\,,\cdot\rangle$ is only elliptic up to some compact perturbation, provided that the continuous problem is well-posed. This includes, e.g., adaptive BEM for the Helmhotz equation; see [Ste08a, Section 6.9]. For details, the reader is referred to [BHP17].*

5.3 Proof of Theorem 5.2.5

In the following subsections, we prove Theorem 5.2.5. Reliability (5.2.22) is treated explicitly in Section 5.3.2. It follows immediately from an auxiliary result on the localization of the Sobolev–Slobodeckij norm which is investigated in Section 5.3.1. To prove (ii)–(iii), we just verify the abstract axioms of Section 2.3 following the ideas of the seminal works [FKMP13, FFK+14]. This allows to apply Theorem 2.3.1. The perturbations $\varrho_{\bullet,\circ}$ are chosen as

$$\varrho_{\bullet,\circ} := C_\varrho \|\Phi_\circ - \Phi_\bullet\|_{H^{-1/2}(\Gamma)} \quad \text{for all } \mathcal{T}_\bullet \in \mathbb{T}, \mathcal{T}_\circ \in \texttt{refine}(\mathcal{T}_\bullet), \tag{5.3.1}$$

with some constant $C_\varrho > 0$ which is fixed later in Section 5.3.6. To apply Theorem 2.3.1 (i), we additionally have to show that $\lim_{\ell\to\infty} \varrho_{\ell,\ell+1} = 0$.

5.3.1 Localization of the Sobolev–Slobodeckij norm

Let $\mathcal{T}_\bullet \in \mathbb{T}$. In contrast to the integer-case, for $\sigma \in (0,1)$, the norm $\|\cdot\|_{H^\sigma(\Gamma)}$ is not additive in the sense that

$$\|v\|^2_{H^\sigma(\Gamma)} \simeq \sum_{T\in\mathcal{T}_\bullet} \|v\|^2_{H^\sigma(T)} \quad \text{for all } v \in H^\sigma(\Gamma)^D.$$

Whereas the lower bound "$\gtrsim$" can be proved elementarily for arbitrary $v \in H^\sigma(\Gamma)^D$ (see Proposition 5.3.8), the upper bound "$\lesssim$" is in general false; see [CF01, Section 3]. The main result of this section is Proposition 5.3.7. It states that, if one replaces the elements T by some overlapping patches, then the upper bound is satisfied for functions $v \in H^\sigma(\Gamma)^D$ which are L^2-orthogonal to the ansatz space $\mathcal{X}_\bullet$. With this, one can immediately construct a reliable and efficient error estimator, namely the so-called *Faermann estimator*; see Remark 5.3.10. For $d = 2$, the result of the proposition goes back to [Fae00], where $\mathcal{X}_\bullet$ is chosen as space of splines transformed via the arc length parametrization $\gamma : [a,b] \to \Gamma$ onto the one-dimensional boundary. In the recent own works [Gan14, FGP15], we generalized the assertion to rational splines, where we could also drop the restriction that γ is the arc length parametrization. For $d = 3$, [Fae02] proved the result for certain (transformed) polynomials of degree $p \in \{0,1,5,6\}$ on a curvilinear triangulation of Γ. The proof of our extended version was essentially inspired by [Fae02]. The key ingredient is the assumption (S4) which is exploited in Lemma 5.3.4. We start with the following basic estimate, which is also proved in [Hac95, Lemma 8.2.4] for a piecewise smooth boundary Γ.

Lemma 5.3.1. *For all $\lambda > 0$, there is a constant $C(\lambda) > 0$ such that for all $x \in \mathbb{R}^d$ and all $\epsilon > 0$, there holds that*

$$\int_{\Gamma\setminus B_\epsilon(x)} |x-y|^{-d+1-\lambda} dy \le C(\lambda)\epsilon^{-\lambda}. \tag{5.3.2}$$

The constant $C(\lambda)$ depends only on the parameter λ, the dimension d, and Γ.

Proof. We only prove the assertion if Γ is the graph of a Lipschitz mapping $\zeta : \mathbb{R}^{d-1} \to \mathbb{R}$. The general case then follows easily from the definition of Lipschitz domains [McL00, Definition 3.28]. Note that the mapping $\gamma : \mathbb{R}^{d-1} \to \mathbb{R}^d$ with $\gamma(s) = (s, \zeta(s))$ is bi-Lipschitz. Let C_1 be an upper bound for the Lipschitz constants of γ and γ^{-1}. Lemma 5.2.1 shows that

$$C_1^{-(d-1)} \le \sqrt{\det(D\gamma^\top D\gamma)(s)} \le C_1^{d-1} \quad \text{for almost all } s \in \mathbb{R}^{d-1}. \tag{5.3.3}$$

Step 1: First, we consider $x = \gamma(s) \in \Gamma$ with some $s \in \mathbb{R}^{d-1}$. By Lipschitz continuity of γ, there holds that $B_{\epsilon/C_1}(s) \subseteq \gamma^{-1}(B_\epsilon(x))$. This together with Lipschitz continuity of γ^{-1} and the boundedness (5.3.3) of the Gram determinant shows that

$$\begin{aligned}\int_{\Gamma\setminus B_\epsilon(x)} |x-y|^{-d+1-\lambda} dy &= \int_{\mathbb{R}^{d-1}\setminus\gamma^{-1}(B_\epsilon(x))} |\gamma(s)-\gamma(t)|^{-d+1-\lambda}\sqrt{\det(D\gamma^\top D\gamma)(t)}dt \\ &\leq \int_{\mathbb{R}^{d-1}\setminus B_{(\epsilon/C_1)}(s)} C_1^{d-1+\lambda}|s-t|^{-d+1-\lambda} C_1^{d-1} dt.\end{aligned}$$

Transforming to polar coordinates, we conclude that

$$\int_{\Gamma\setminus B_\epsilon(x)} |x-y|^{-d+1-\lambda} dy \lesssim \int_{\epsilon/C_1}^\infty r^{-d+1-\lambda} r^{d-2} dr = \frac{C_1^\lambda}{\lambda}\epsilon^{-\lambda}.$$

Step 2: Let $x \in \Gamma^c$ and let $x_0 = \gamma(s_0) \in \Gamma$ with $|x-x_0| = \operatorname{dist}(x,\Gamma)$ and some $s_0 \in \mathbb{R}^{d-1}$. First, let $|x-x_0| \geq \epsilon/2$. The triangle inequality shows that $|x_0-x|+|x_0-y| \leq |y-x| + |x_0-x|+|x-y| \leq 3|x-y|$ for all $y \in \Gamma$. This and (5.3.3) imply that

$$\begin{aligned}\int_{\Gamma\setminus B_\epsilon(x)} |x-y|^{-d+1-\lambda} dy \leq \int_\Gamma |x-y|^{-d+1-\lambda} dy &\lesssim \int_\Gamma (|x_0-x|+|x_0-y|)^{-d+1-\lambda} dy \\ &\lesssim \int_{\mathbb{R}^{d-1}} (|x_0-x|+|\gamma(s_0)-\gamma(t)|)^{-d+1-\lambda} dt \\ &\lesssim \int_{\mathbb{R}^{d-1}} (\epsilon/2 + C_1^{-1}|s_0-t|)^{-d+1-\lambda} dt.\end{aligned}$$

By transforming to polar coordinates, we conclude that

$$\int_{\Gamma\setminus B_\epsilon(x)} |x-y|^{-d+1-\lambda} dy \lesssim \int_0^\infty (\epsilon+r)^{-d+1-\lambda} r^{d-2} dr \leq \int_\epsilon^\infty r^{-\lambda-1} dr = \epsilon^{-\lambda}/\lambda.$$

Now, let $|x-x_0| < \epsilon/2$. We use again $|x_0-x|+|x_0-y| \leq 3|x-y|$ and $B_{(\epsilon/2)}(x_0) \subseteq B_\epsilon(x)$ to see that

$$\begin{aligned}\int_{\Gamma\setminus B_\epsilon(x)} |x-y|^{-d+1-\lambda} dy &\lesssim \int_{\Gamma\setminus B_\epsilon(x)} (|x_0-x|+|x_0-y|)^{-d+1-\lambda} dy \\ &\lesssim \int_{\Gamma\setminus B_{(\epsilon/2)(x_0)}} (|x_0-x|+|x_0-y|)^{-d+1-\lambda} dy \\ &\leq \int_{\Gamma\setminus B_{(\epsilon/2)}(x_0)} |x_0-y|^{-d+1-\lambda} dy.\end{aligned}$$

We already proved in Step 1 that the last term satisfies that

$$\int_{\Gamma\setminus B_{(\epsilon/2)(x_0)}} |x_0-y|^{-d+1-\lambda} dy \lesssim \epsilon^{-\lambda}/2^\lambda,$$

which concludes the proof. □

The following lemma is the first step towards the localization of the norm $\|v\|_{H^\sigma(\Gamma)}$ for certain functions $v \in H^\sigma(\Gamma)^D$.

Lemma 5.3.2. *Let $\sigma \in (0,1)$ and $\mathcal{T}_\bullet \in \mathbb{T}$. Then, (M4) implies the existence of a constant $C > 0$ such that for all $v \in H^\sigma(\Gamma)^D$ there holds that*

$$\|v\|_{H^\sigma(\Gamma)}^2 \leq \sum_{T\in\mathcal{T}_\bullet} \sum_{T'\in\Pi_\bullet(T)} |v|_{H^\sigma(T\cup T')}^2 + C \sum_{T\in\mathcal{T}_\bullet} \operatorname{diam}(T)^{-2\sigma} \|v\|_{L^2(T)}^2. \tag{5.3.4}$$

The constant C depends only on the dimension d, σ, Γ, and the constant from (M4).

Proof. Without loss of generality, we may assume that $D = 1$. Since $\operatorname{diam}(T) \leq \operatorname{diam}(\Gamma)$, we can immediately bound

$$\|v\|_{L^2(T)}^2 \leq \operatorname{diam}(\Gamma)^{2\sigma} \sum_{T\in\mathcal{T}_\bullet} \operatorname{diam}(T)^{-2\sigma} \|v\|_{L^2(T)}^2.$$

It remains to estimate the seminorm $|v|_{H^\sigma(\Gamma)}$. To this end, we introduce the abbreviation

$$V(x,y) := \frac{|v(x)-v(y)|^2}{|x-y|^{d-1+2\sigma}} \quad \text{for all } x,y \in \Gamma, x \neq y. \tag{5.3.5}$$

There holds that

$$|v|_{H^\sigma(\Gamma)}^2 = \sum_{T\in\mathcal{T}_\bullet} \left(\int_T \int_{\pi_\bullet(T)} V(x,y)dxdy + \int_T \int_{\Gamma\setminus\pi_\bullet(T)} V(x,y)dxdy \right). \tag{5.3.6}$$

The first summand in (5.3.6) can be estimated from above by $\sum_{T'\in\Pi_\bullet(T)} |v|_{H^\sigma(T\cup T')}^2$. Hence, we only need to estimate the second one

$$\begin{aligned} \sum_{T\in\mathcal{T}_\bullet} \int_T \int_{\Gamma\setminus\pi_\bullet(T)} V(x,y)dxdy \leq{}& 2 \sum_{T\in\mathcal{T}_\bullet} \int_T |v(y)|^2 \int_{\Gamma\setminus\pi_\bullet(T)} |x-y|^{-d+1-2\sigma} dxdy \\ &+ 2 \sum_{T\in\mathcal{T}_\bullet} \int_{\Gamma\setminus\pi_\bullet(T)} |v(x)|^2 \int_T |x-y|^{-d+1-2\sigma} dydx. \end{aligned} \tag{5.3.7}$$

Next, we show that the two sums of the last term in (5.3.7) are equal. Indeed, we have with the characteristic function $\chi_{\Gamma\setminus\pi_\bullet(T)}$ of $\Gamma \setminus \pi_\bullet(T)$ that

$$\begin{aligned} &\sum_{T\in\mathcal{T}_\bullet} \int_{\Gamma\setminus\pi_\bullet(T)} |v(x)|^2 \int_T |x-y|^{-d+1-2\sigma} dydx \\ &\quad = \int_\Gamma |v(x)|^2 \underbrace{\sum_{T\in\mathcal{T}_\bullet} \chi_{\Gamma\setminus\pi_\bullet(T)}(x) \int_T |x-y|^{-d+1-2\sigma} dy}_{=:H(x)}\, dx = \sum_{T'\in\mathcal{T}_\bullet} \int_{T'} |v(x)|^2 H(x) dx. \end{aligned}$$

Let $T' \in \mathcal{T}_\bullet$ and x in the interior of T'. We show that $H(x) = \int_{\Gamma\setminus\pi_\bullet(T')} |x-y|^{-d+1-2\sigma} dy$, which proves equality of the two summands in (5.3.7). We see for any $T \in \mathcal{T}_\bullet$ that

$$\chi_{\Gamma\setminus\pi_\bullet(T')}(x) = 1 \iff x \in \Gamma \setminus \pi_\bullet(T') \iff T \cap T' = \emptyset$$

This implies for almost every $x \in T'$ that

$$H(x) = \sum_{\substack{T\in\mathcal{T}_\bullet \\ T\cap T'=\emptyset}} \int_T |x-y|^{-d+1-2\sigma} dy = \int_{\Gamma\setminus\pi_\bullet(T')} |x-y|^{-d+1-2\sigma} dy.$$

(5.3.7) now becomes

$$\sum_{T\in\mathcal{T}_\bullet} \int_T \int_{\Gamma\setminus\pi_\bullet(T)} V(x,y)dxdy \leq 4 \sum_{T\in\mathcal{T}_\bullet} \int_T |v(y)|^2 \int_{\Gamma\setminus\pi_\bullet(T)} |x-y|^{-d+1-2\sigma} dxdy.$$

For $y \in T$, it holds that $B_{\mathrm{dist}(T,\Gamma\setminus\pi_\bullet(T))}(y) \cap \Gamma \subseteq \pi_\bullet(T)$, which yields together with Lemma 5.3.1 that

$$\int_{\Gamma\setminus\pi_\bullet(T)} |x-y|^{-d+1-2\sigma} dx \leq \int_{\Gamma\setminus B_{\mathrm{dist}(T,\Gamma\setminus\pi_\bullet(T))}(y)} |x-y|^{-d+1-2\sigma} dx \lesssim \mathrm{dist}(T,\Gamma\setminus\pi_\bullet(T))^{-2\sigma}.$$

(M4) shows that $\mathrm{dist}(T,\Gamma\setminus\pi_\bullet(T))^{-2\sigma} \lesssim \mathrm{diam}(T)^{-2\sigma}$ and concludes the proof. □

It remains to control the second summand in (5.3.4). To this end, we need the following elementary Poincaré type inequality of [Fae00, Lemma 2.5].

Lemma 5.3.3. *For any $\sigma \in (0,1)$ and any measurable $\omega \subseteq \Gamma$, there holds for all $v \in H^\sigma(\omega)$ that*

$$\|v\|_{L^2(\omega)}^2 \leq \frac{\mathrm{diam}(\omega)^{d-1+2\sigma}}{2|\omega|} |v|_{H^\sigma(\omega)}^2 + \frac{1}{|\omega|} \left| \int_\omega v(x)dx \right|^2. \tag{5.3.8}$$

Proof. We have that

$$\begin{aligned}
&2|\omega| \int_\omega |v(x)|^2 dx - 2\left|\int_\omega v(x)dx\right|^2 \\
&= \int_\omega\int_\omega |v(x)|^2 dxdy + \int_\omega\int_\omega |v(y)|^2 dxdy - 2\int_\omega\int_\omega v(x)\cdot v(y)dxdy \\
&= \int_\omega\int_\omega |v(x)-v(y)|^2 dxdy \\
&= \int_\omega\int_\omega \frac{|v(x)-v(y)|^2}{|x-y|^{d-1+2\sigma}} |x-y|^{d-1+2\sigma} dxdy \\
&\leq |v|_{H^\sigma(\omega)}^2 \mathrm{diam}(\omega)^{d-1+2\sigma}.
\end{aligned}$$

This is just the assertion of the lemma. □

We start to estimate the second summand in (5.3.4).

Lemma 5.3.4. *Let $\sigma \in (0,1)$, $\mathcal{T}_\bullet \in \mathbb{T}$ and $T \in \mathcal{T}_\bullet$. Then, (M1)–(M3) and (S4) imply the existence of a constant $C > 0$ such that for all $v \in H^\sigma(\Gamma)^D$ which satisfy that $\langle v\,,\, \Psi_{\bullet,T,j}\rangle_{L^2(\Gamma)} = 0$ for all $j \in \{1,\dots,D\}$, where $\Psi_{\bullet,T,j}$ are the functions from (S4), there holds that*

$$\|h_\bullet^{-\sigma} v\|_{L^2(T)} \leq C|v|_{H^\sigma(\pi_\bullet^{q_{\mathrm{supp}}}(T))}, \tag{5.3.9}$$

where q_{supp} is the constant from (S4). The constant C depends only on the dimension d, σ, Γ, and the constants from (M1)–(M3) and (S4).

Proof. We prove (5.3.9) for each component v_j of v, where $j \in \{1, \dots, D\}$. Then, squaring and summing up all components concludes the proof. (S4) and Lemma 5.3.3 show that

$$\begin{aligned}\|v_j\|^2_{L^2(T)} &\le \|v_j\|^2_{L^2(\mathrm{supp}(\Psi_{\bullet,T,j}))} \\ &\le \frac{\mathrm{diam}(\mathrm{supp}(\Psi_{\bullet,T,j}))^{d-1+2\sigma}}{2|\mathrm{supp}(\Psi_{\bullet,T,j})|}|v_j|^2_{H^\sigma(\mathrm{supp}(\Psi_{\bullet,T,j}))} + \frac{1}{|\mathrm{supp}(\Psi_{\bullet,T,j})|}\left|\int_{\mathrm{supp}(\Psi_{\bullet,T,j})} v_j(x)dx\right|^2.\end{aligned} \tag{5.3.10}$$

Now, we apply the orthogonality and (S4) to get for the second summand that

$$\begin{aligned}\frac{1}{|\mathrm{supp}(\Psi_{\bullet,T,j})|}\left|\int_{\mathrm{supp}(\Psi_{\bullet,T,j})} v_j(x)dx\right|^2 &= \frac{1}{|\mathrm{supp}(\Psi_{\bullet,T,j})|}\left|\int_{\mathrm{supp}(\Psi_{\bullet,T,j})} v_j(x)(1-\Psi_{\bullet,T,j}(x))dx\right|^2 \\ \le \frac{1}{|\mathrm{supp}(\Psi_{\bullet,T,j})|}\|v_j\|^2_{L^2(\mathrm{supp}(\Psi_{\bullet,T,j}))}\|1-(\Psi_{\bullet,T,j})_j\|^2_{L^2(\mathrm{supp}(\Psi_{\bullet,T,j}))} &\le \rho^2_{\mathrm{unity}}\|v_j\|^2_{L^2(\mathrm{supp}(\Psi_{\bullet,T,j}))}.\end{aligned}$$

Inserting this in (5.3.10) gives

$$(1-\rho^2_{\mathrm{unity}})\|v_j\|^2_{L^2(\mathrm{supp}(\Psi_{\bullet,T,j}))} \le \frac{\mathrm{diam}(\mathrm{supp}(\Psi_{\bullet,T,j}))^{d-1+2\sigma}}{2|\mathrm{supp}(\Psi_{\bullet,T,j})|}|v_j|^2_{H^\sigma(\mathrm{supp}(\Psi_{\bullet,T,j}))}. \tag{5.3.11}$$

With (S4) and (M1)–(M3), we see that $\mathrm{diam}(\mathrm{supp}(\Psi_{\bullet,T,j})) \le \mathrm{diam}(\pi_\bullet^{q_{\mathrm{supp}}}(T)) \lesssim \mathrm{diam}(T) \simeq h_T$. Further, (S4) implies that $|\mathrm{supp}(\Psi_{\bullet,T,j})| \ge |T| = h_T^{d-1}$. Inserting this in (5.3.11) and using again (S4), we derive that

$$\|v_j\|^2_{L^2(T)} \le \|v_j\|^2_{L^2(\mathrm{supp}(\Psi_{\bullet,T,j}))} \lesssim h_T^{2\sigma}|v_j|^2_{H^\sigma(\mathrm{supp}(\Psi_{\bullet,T,j}))} \le h_T^{2\sigma}|v_j|^2_{H^\sigma(\pi_\bullet^{q_{\mathrm{supp}}}))}.$$

Altogether, this concludes the proof. □

The following lemma allows us to further estimate the term $|v|_{H^\sigma(\pi_\bullet^{q_{\mathrm{supp}}}(T))}$ of (5.3.9).

Lemma 5.3.5. *Let $q \in \mathbb{N}_0$ and $\mathcal{T}_\bullet \in \mathbb{T}$. Then, (M1)–(M4) imply the existence of a constant $C(q) > 0$ such that for all $v \in H^\sigma(\Gamma)^D$ and all $T \in \mathcal{T}_\bullet$ there holds that*

$$|v|^2_{H^\sigma(\pi_\bullet^q(T))} \le C(q) \sum_{\substack{T',T'' \in \Pi_\bullet^q(T) \\ T' \cap T'' \neq \emptyset}} |v|^2_{H^\sigma(T' \cup T'')}. \tag{5.3.12}$$

The constant depends only on the dimension d, σ, q, and the constants from (M1)–(M4).

Proof. Without loss of generality, we may assume that $D = 1$. We prove the assertion in two steps.
Step 1: Let $T_0, T_1, \dots, T_m$ be a *chain* of elements in $\Pi_\bullet^q(T)$ with $T_i \cap T_j = \emptyset$ for $|i-j| > 1$ and $T_i \cap T_j \neq \emptyset$ if $|i-j| = 1$, where $1 \le m \le q$. We set $T_i^j := \bigcup_{k=i}^j T_\ell$ for $i \le j$ and prove by induction on m that there exists a constant $C_1(m) > 0$ which depends only on d, σ, q, m, and (M2)–(M4), such that

$$|v|^2_{H^\sigma(T_0^m)} \le C_1(m) \sum_{i=0}^{m-1} |v|^2_{H^\sigma(T_i \cup T_{i+1})}. \tag{5.3.13}$$

For $m = 1$, (5.3.13) with $C_1(1) = 1$ even holds with equality. The induction hypothesis for $1 \le m-1 < q$ reads: For any chain $T_0, \dots, T_{m-1}$ of elements in $\Pi^q_\bullet(\mathcal{T})$, it holds that

$$|v|^2_{H^\sigma(T_0^{m-1})} \le C_1(m-1) \sum_{i=0}^{m-2} |v|^2_{H^\sigma(T_i \cup T_{i+1})}. \tag{5.3.14}$$

Let $T_m \in \Pi^q_\bullet(\mathcal{T})$ with $T_m \cap T_i = \emptyset$ for $i \le m-2$ and $T_m \cap T_i \neq \emptyset$ for $i = m-1$. Recall the abbreviation $V(x,y)$ from (5.3.5). The definition (5.1.1) of the Sobolev-Slobodeckij seminorm shows that

$$\begin{aligned}
|v|^2_{H^\sigma(T_0^m)} &= \int_{T_0^m}\int_{T_0^m} V(x,y)dxdy \\
&= \int_{T_0^{m-1}}\int_{T_0^{m-1}} V(x,y)dxdy + \int_{T_m}\int_{T_m} V(x,y)dxdy + 2\int_{T_m}\int_{T_0^{m-1}} V(x,y)dxdy \\
&= |v|^2_{H^\sigma(T_0^{m-1})} + |v|^2_{H^\sigma(T_m)} + 2\int_{T_m}\int_{T_0^{m-2}} V(x,y)dxdy + 2\int_{T_m}\int_{T_{m-1}} V(x,y)dxdy \\
&\le |v|^2_{H^\sigma(T_0^{m-1})} + |v|^2_{H^\sigma(T_{m-1}\cup T_m)} + 2\int_{T_m}\int_{T_0^{m-2}} V(x,y)dxdy.
\end{aligned}$$

With (5.3.14), we see that it remains to estimate $\int_{T_m}\int_{T_0^{m-2}} V(x,y)dxdy$. First, we note that for $x \in T_0^{m-2}, y \in T_m, z \in T_{m-1}$, it holds that

$$V(x,y) = \frac{|v(x)-v(y)|^2}{|x-y|^{d-1+2\sigma}} \le 2\frac{|v(x)-v(z)|^2}{|x-y|^{d-1+2\sigma}} + 2\frac{|v(z)-v(y)|^2}{|x-y|^{d-1+2\sigma}}. \tag{5.3.15}$$

Moreover, (M4) shows that $|x-y| \ge \operatorname{dist}(T_m, \Gamma \setminus \pi_\bullet(T_m)) \gtrsim \operatorname{diam}(T_m)$. Since $x, y, z \in T_0^m$, (M2) shows that $\max(|x-z|, |y-z|) \lesssim \operatorname{diam}(T_m)$. Hence we can proceed the estimate of (5.3.15)

$$V(x,y) \lesssim V(x,z) + V(z,y).$$

This implies that

$$\begin{aligned}
\int_{T_m}\int_{T_0^{m-2}} V(x,y)dxdy &= \frac{1}{|T_{m-1}|}\int_{T_{m-1}}\int_{T_m}\int_{T_0^{m-2}} V(x,y)dxdydz \\
&\lesssim \frac{1}{|T_{m-1}|}\int_{T_{m-1}}\int_{T_m}\int_{T_0^{m-2}} V(x,z) + V(y,z)dxdydz \\
&= \frac{1}{|T_{m-1}|}\left(\int_{T_{m-1}}\int_{T_0^{m-2}} |T_m| V(x,z)dxdz + \int_{T_{m-1}}\int_{T_{m-1}} |T_0^{m-2}| V(y,z)dydz\right) \\
&\le \frac{\max(|T_m|, |T_0^{m-2}|)}{|T_{m-1}|}\left(|v|^2_{H^\sigma(T_0^{m-1})} + |v|^2_{H^\sigma(T_{m-1}\cup T_m)}\right).
\end{aligned}$$

Note that $\max(|T_m|, |T_0^{m-2}|)/|T_{m-1}| \lesssim 1$ by (M2)–(M3). Together with the induction hypothesis (5.3.14), this concludes the induction step.

Step 2: We come to the assertion itself. By definition, we have that

$$|v|^2_{H^{1/2}(\pi^q_\bullet(T))} = \sum_{\widetilde{T}',\widetilde{T}''\in\Pi^q_\bullet(T)} \int_{\widetilde{T}'}\int_{\widetilde{T}''} V(x,y)dxdy.$$

Let $\widetilde{T}',\widetilde{T}'' \in \Pi^q_\bullet(T)$. First, we suppose that $\widetilde{T}' \neq \widetilde{T}'' = \emptyset$. Then, there exists a chain as in Step 1 with $\widetilde{T}' = T_0$ and $\widetilde{T}'' = T_m$. Step 1 proves that

$$\int_{\widetilde{T}'}\int_{\widetilde{T}''} V(x,y)dxdy \le |v|^2_{H^\sigma(T_0^m)} \lesssim \sum_{\substack{T',T''\in\Pi^q_\bullet(T)\\ T'\cap T''\neq\emptyset}} |v|^2_{H^\sigma(T'\cup T'')}.$$

If $\widetilde{T}' = \widetilde{T}''$, the same estimate holds true. Since the number of $\widetilde{T}',\widetilde{T}'' \in \Pi^q_\bullet(T)$ is uniformly bounded by a constant, which depends only on the constant of (M1) and q, this estimate concludes the proof. □

With the property (M5), one immediately derives the following Poincaré inequality.

Corollary 5.3.6. *Let $\mathcal{T}_\bullet \in \mathbb{T}$ and $T \in \mathcal{T}_\bullet$. Then, (M1)–(M5) and (S4) imply the existence of a constant $C_{\rm poinc} > 0$ such that for all $v \in H^1(\Gamma)^D$ which satisfy that $\langle v\,,\,\Psi_{\bullet,T,j}\rangle_{L^2(\Gamma)} = 0$ for all $j \in \{1,\dots,D\}$, where $\Psi_{\bullet,T,j}$ are the functions from (S4), there holds that*

$$\|h_\bullet^{-1}v\|_{L^2(T)} \le C_{\rm poinc}|v|_{H^1\left(\pi_\bullet^{q_{\rm supp}+1}(T)\right)}, \tag{5.3.16}$$

where $q_{\rm supp}$ is the constant from (S4). The constant $C_{\rm poinc}$ depends only on the dimension d, Γ, and the constants from (M1)–(M5) and (S4).

Proof. We apply Lemma 5.3.4 and Lemma 5.3.5 to see that

$$\|h_\bullet^{-1/2}v\|^2_{L^2(T)} \lesssim |v|^2_{H^{1/2}(\pi_\bullet^{q_{\rm supp}}(T))} \lesssim \sum_{\substack{T',T''\in\Pi_\bullet^{q_{\rm supp}}(T)\\ T'\cap T''\neq\emptyset}} |v|^2_{H^{1/2}(T'\cup T'')}.$$

For $T',T'' \in \mathcal{T}_\bullet$ with $T'\cap T'' \neq \emptyset$, we fix some point $z(T',T'') \in T'\cap T''$. With (M5), we continue our estimate

$$\begin{aligned}\|h_\bullet^{-1/2}v\|^2_{L^2(T)} \lesssim |v|^2_{H^{1/2}(\pi_\bullet^{q_{\rm supp}}(T))} &\lesssim \sum_{\substack{T',T''\in\Pi_\bullet^{q_{\rm supp}}(T)\\ T'\cap T''\neq\emptyset}} |v|^2_{H^{1/2}(\pi_\bullet(z(T',T'')))}\\ &\lesssim \sum_{\substack{T',T''\in\Pi_\bullet^{q_{\rm supp}}(T)\\ T'\cap T''\neq\emptyset}} {\rm diam}\big(\pi_\bullet(z(T',T''))\big)\|\nabla_\Gamma v\|^2_{L^2(\pi_\bullet(T'))}.\end{aligned}$$

(M1)–(M3) imply that $h_T \simeq h_\bullet$ on $\pi_\bullet^{q_{\rm supp}+1}(T)$, and that the last term of the latter estimate can be bounded from above (up to a multiplicative constant) by $\|h_\bullet^{1/2}\nabla_\Gamma v\|^2_{L^2\left(\pi_\bullet^{q_{\rm supp}+1}(T)\right)}$. This concludes the proof. □

With all the preparations, we can finally come to the main result of this section.

Proposition 5.3.7. *Let $\sigma \in (0,1)$ and $\mathcal{T}_\bullet \in \mathbb{T}$. Then, (M1)–(M4) and (S4) imply the existence of a constant $C_{\rm split} > 0$ such that for any $v \in H^\sigma(\Gamma)^D$ which satisfies that $\langle v\,,\,(\Psi_{\bullet,T,j})_j\rangle_{L^2(\Gamma)} = 0$ for all $T \in \mathcal{T}_\bullet$ and all $j \in \{1,\dots,D\}$, where $\Psi_{\bullet,T,j}$ are the functions from (S4), there holds that*

$$\|v\|^2_{H^\sigma(\Gamma)} \le C_{\rm split} \sum_{T\in\mathcal{T}_\bullet} \sum_{T'\in\Pi_\bullet(T)} |v|^2_{H^\sigma(T\cup T')}. \tag{5.3.17}$$

The constant $C_{\rm split}$ depends only on the dimension d, σ, Γ, and the constants from (M1)–(M4) and (S4).

Proof. Together with (M3), Lemma 5.3.2 proves that

$$\|v\|^2_{H^\sigma(\Gamma)} \lesssim \sum_{T\in\mathcal{T}_\bullet} \sum_{T'\in\Pi_\bullet(T)} |v|^2_{H^\sigma(T\cup T')} + \sum_{T\in\mathcal{T}_\bullet} h_T^{-2\sigma}\|v\|^2_{L^2(T)}.$$

It remains to estimate the second sum. With Lemma 5.3.4 and Lemma 5.3.5, we see that

$$\sum_{T\in\mathcal{T}_\bullet} h_T^{-2\sigma}\|v\|^2_{L^2(T)} \lesssim \sum_{T\in\mathcal{T}_\bullet} |v|^2_{H^\sigma(\pi_\bullet^{q_{\rm supp}}(T))} \lesssim \sum_{T\in\mathcal{T}_\bullet} \sum_{\substack{T',T''\in\Pi_\bullet^{q_{\rm supp}}(T)\\ T'\cap T''\neq\emptyset}} |v|^2_{H^\sigma(T'\cup T'')}. \tag{5.3.18}$$

If $T \in \mathcal{T}_\bullet$ and $T', T'' \in \Pi_\bullet^{q_{\rm supp}}(T)$ with $T'\cap T'' \neq \emptyset$, then $T \in \Pi_\bullet^{q_{\rm supp}}(T')$ and $T'' \in \Pi_\bullet(T')$. Plugging this into (5.3.18) shows that

$$\sum_{T\in\mathcal{T}_\bullet} h_T^{-2\sigma}\|v\|^2_{L^2(T)} \lesssim \sum_{T'\in\mathcal{T}_\bullet} \sum_{T\in\Pi_\bullet^{q_{\rm supp}}(T')} \sum_{T''\in\Pi_\bullet(T')} |v|^2_{H^\sigma(T'\cup T'')},$$

and $\#\Pi_\bullet^{q_{\rm supp}}(T') \lesssim 1$ (see (M1)) concludes the proof. □

As already mentioned, the converse inequality of (5.3.7) is trivially satisfied for any function $v \in H^\sigma(\Gamma)^D$.

Proposition 5.3.8. *Let $\sigma \in (0,1)$ and $\mathcal{T}_\bullet \in \mathbb{T}$. Then, (M1) implies the existence of a constant $C'_{\rm split} > 0$ such that for any $v \in H^\sigma(\Gamma)^D$, there holds that*

$$\sum_{T\in\mathcal{T}_\bullet} \sum_{T'\in\Pi_\bullet(T)} |v|^2_{H^\sigma(T\cup T')} \le C'_{\rm split}|v|^2_{H^\sigma(\Gamma)}.$$

The constant $C'_{\rm split}$ depends only on the constant from (M1).

Proof. With the abbreviation $V(x,y)$ of (5.3.5) and (M1), there holds that

$$\begin{aligned}\sum_{T\in\mathcal{T}_\bullet} \sum_{T'\in\Pi_\bullet(T)} |v|^2_{H^\sigma(T\cup T')} &= \sum_{T\in\mathcal{T}_\bullet} \sum_{T'\in\Pi_\bullet(T)} \Big(|v|^2_{H^\sigma(T)} + 2\int_T\int_{T'} V(x,y)\,dx\,dy + |v|^2_{H^\sigma(T')}\Big)\\ &= 2\sum_{T\in\mathcal{T}_\bullet} \sum_{T'\in\Pi_\bullet(T)} \Big(\int_T\int_T V(x,y)\,dx\,dy + \int_T\int_{T'} V(x,y)\,dx\,dy\Big)\\ &\le 2(C_{\rm patch}+1)|v|^2_{H^\sigma(\Gamma)}.\end{aligned}$$

This concludes the proof. □

The following easy corollary is the key ingredient for the proof of reliability (5.2.22).

Corollary 5.3.9. *Let $\mathcal{T}_\bullet \in \mathbb{T}$ and $\sigma \in \{0, 1/2, 1\}$. Then, (M1)–(M5) and (S4) imply the existence of a constant $C'_{\rm rel} > 0$ such that for any $v \in H^1(\Gamma)^D$ which satisfies that $\langle v\,,\,\Psi_{\bullet,T,j}\rangle_{L^2(\Gamma)} = 0$ for all $T \in \mathcal{T}_\bullet$ and all $j \in \{1, \dots, D\}$, where $\Psi_{\bullet,T,j}$ are the functions from (S4), there holds that*

$$\|v\|_{H^\sigma(\Gamma)} \le C'_{\rm rel}\|h_\bullet^{1-\sigma}\nabla_\Gamma v\|_{L^2(\Gamma)}. \tag{5.3.19}$$

The constant $C'_{\rm rel}$ depends only on the dimension d, Γ, as well as the constants from (M1)–(M5) and (S4).

Proof. First, let $\sigma = 1/2$. For all $T, T' \in \mathcal{T}_\bullet$ with $T \cap T'$, let $z(T, T') \in T \cap T'$. Proposition 5.3.7 together with (M5) proves that

$$\|v\|^2_{H^\sigma(\Gamma)} \lesssim \sum_{T\in\mathcal{T}_\bullet}\sum_{T'\in\Pi_\bullet(T)} |v|^2_{H^\sigma(\pi_\bullet(z(T,T')))} \lesssim \sum_{T\in\mathcal{T}_\bullet}\sum_{T'\in\Pi_\bullet(T)} \|h_\bullet^{1-\sigma}\nabla_\Gamma v\|^2_{L^2(\pi_\bullet(z(T,T')))}.$$

With (M1), we see that

$$\|v\|^2_{H^\sigma(\Gamma)} \lesssim \sum_{T\in\mathcal{T}_\bullet} \#\Pi_\bullet(T)\|h_\bullet^{1-\sigma}\nabla_\Gamma v\|^2_{L^2(\pi_\bullet(T))} \lesssim \|h_\bullet^{1-\sigma}\nabla_\Gamma v\|^2_{L^2(\Gamma)},$$

which concludes the proof. If $\sigma = 0$, the assertion follows easily from Corollary 5.3.6. If $\sigma = 1$, we use the assertion for $\sigma = 0$ to see that $\|v\|_{H^1(\Gamma)} \lesssim \||\Gamma|^{1/(d-1)}\nabla_\Gamma v\|_{L^2(\Gamma)} + \|\nabla_\Gamma v\|_{L^2(\Gamma)}$. ☐

5.3.2 Reliability (5.2.22)

Let $\mathcal{T}_\bullet \in \mathbb{T}$. Since $\mathfrak{V} : H^{-1/2}(\Gamma)^D \to H^{1/2}(\Gamma)^D$ is an isomorphism, we have that

$$\|\phi - \Phi_\bullet\|_{H^{-1/2}(\Gamma)} \simeq \|\mathfrak{V}(\phi - \Phi_\bullet)\|_{H^{1/2}(\Gamma)}. \tag{5.3.20}$$

Due to Galerkin orthogonality (5.2.15), we can apply Corollary 5.3.9 to obtain that

$$\|\phi - \Phi_\bullet\|_{H^{-1/2}(\Gamma)} \simeq \|f - \mathfrak{V}\Phi_\bullet\|_{H^{1/2}(\Gamma)} \lesssim \|h_\bullet^{1/2}\nabla_\Gamma(f - \mathfrak{V}\Phi_\bullet)\|_{L^2(\Gamma)} = \eta_\bullet.$$

Remark 5.3.10. *The equivalence (5.3.20), Proposition 5.3.7, and Proposition 5.3.8 show that*

$$\|f - \mathfrak{V}\Phi_\bullet\|^2_{H^{1/2}(\Gamma)} \simeq \sum_{T\in\mathcal{T}_\bullet}\sum_{T'\in\Pi_\bullet(T)} |f - \mathfrak{V}\Phi_\bullet|^2_{H^{1/2}(T\cup T')}. \tag{5.3.21}$$

This is even true for arbitrary $f \in H^{1/2}(\Gamma)^D$ without the additional restriction $f \in H^1(\Gamma)^D$. In particular,

$$F_\bullet(T)^2 := \sum_{T'\in\Pi_\bullet(T)} |f - \mathfrak{V}\Phi_\bullet|^2_{H^{1/2}(T\cup T')} \quad \text{for all } T \in \mathcal{T}_\bullet \tag{5.3.22}$$

provides a local error indicator. The corresponding error estimator $\mathcal{F}_\bullet$ is often referred to as Faermann estimator*. In BEM, it is the only known estimator which is reliable and efficient (without further assumptions as, e.g., the saturation assumption [FLP08, Section 1]). Obviously, one could replace the residual estimator η_ℓ in Algorithm 5.2.4 by $\mathcal{F}_\ell$. However, due to the lack of an h-weighting factor, it is unclear whether the reduction property (E2) of Section 5.3.2 is satisfied. [FFME+14, Theorem 7] proves at least plain convergence of $\mathcal{F}_\ell$ even for $f \in H^{1/2}(\Gamma)^D$ if one uses piecewise constants on affine triangulations of Γ as ansatz space. The proof immediately extends to our current abstract situation, where the assumptions (M1)–(M5), (R2)–(R3), and (S1)–(S2) are employed. The key ingredient is the construction of an equivalent mesh-size function $\widetilde{h}_\bullet \in L^\infty(\Gamma)$ which is contractive on each element which touches a refined element, i.e., there exists a uniform constant $0 < \rho_{\rm ctr} < 1$ such that*

$$\widetilde{h}_\circ|_T \le \rho_{\rm ctr}\widetilde{h}_\bullet|_T \quad \text{for all } \mathcal{T}_\circ \in \texttt{refine}(\mathcal{T}_\bullet) \text{ and all } T \in \Pi_\bullet(\mathcal{T}_\bullet \setminus \mathcal{T}_\circ). \tag{5.3.23}$$

The existence of such a function is proved in [CFPP14, Section 8.7] for shape-regular triangular meshes, where the proof works verbatim for our situation.

5.3.3 Convergence of perturbations

Nestedness (S2) ensures that $\mathcal{X}_\infty := \overline{\bigcup_{\ell\in\mathbb{N}_0}\mathcal{X}_\ell}$ is a closed subspace of $H^{-1/2}(\Gamma)^D$ and hence admits a unique Galerkin solution $\Phi_\infty \in \mathcal{X}_\infty$. Note that Φ_ℓ is also a Galerkin approximation of Φ_∞. Hence, the Céa lemma (5.2.16) with ϕ replaced by Φ_∞ proves that $\|\Phi_\infty - \Phi_\ell\|_{H^{-1/2}(\Gamma)} \to 0$ as $\ell \to \infty$. In particular, we obtain that $\lim_{\ell\to\infty}\|\Phi_{\ell+1} - \Phi_\ell\|_{H^{-1/2}(\Gamma)} = 0$.

5.3.4 An inverse inequality for $\mathfrak{V}$

In Proposition 5.3.15, we establish an inverse inequality for the single-layer operator $\mathfrak{V}$. Throughout this section, the ellipticity of $\mathfrak{V}$ is not needed, wherefore we can drop this assumption here. For the Laplace operator $\mathfrak{P} = -\Delta$, such an estimate was already proved in [FKMP13, Theorem 3.1] for shape-regular triangulations of a polyhedral boundary Γ. Independently, [Gan13] derived a similar result for globally smooth Γ and arbitrary symmetric and elliptic boundary integral operators. In [AFF+17, Theorem 3.1], [FKMP13, Theorem 3.1] is generalized to piecewise polynomial ansatz functions on shape-regular curvilinear triangulations. In particular, our Proposition 5.3.15 does not only extend this result to arbitrary general meshes as in Section 5.2.1, but is also novel for the Lamé and the Helmholtz equation. The proof works very similar as in [AFF+17, Section 4]. We start with the following lemma, which was proved in [CP06, Theorem 4.1] on shape-regular triangulations.

Lemma 5.3.11. *For $\mathcal{T}_\bullet \in \mathbb{T}$, let $\mathcal{P}^0(\mathcal{T}_\bullet)^D \subset L^2(\Gamma)^D$ be the set of all functions whose D components are $\mathcal{T}_\bullet$-piecewise constant functions on Γ. Let $P_\bullet : L^2(\Gamma)^D \to \mathcal{P}^0(\mathcal{T}_\bullet)^D$ be the corresponding L^2-projection. Then, (M1) and (M3) imply for arbitrary $\sigma \in (0,1)$ the existence of a constant $C > 0$ such that*

$$\|(1 - P_\bullet)\psi\|_{H^{-\sigma}(\Gamma)} \le C\|h_\bullet^\sigma\psi\|_{L^2(\Gamma)} \quad \text{for all } \psi \in L^2(\Gamma). \tag{5.3.24}$$

The constant C depends only on the dimension D, the boundary Γ, σ, and the constants from (M3).

Proof. Let $v \in H^\sigma(\Gamma)^D$. Then, there holds that

$$\langle v\,,\,(1-P_\bullet)\psi\rangle = \langle v\,,\,(1-P_\bullet)\psi\rangle_{L^2(\Gamma)} = \langle (1-P_\bullet)v\,,\,\psi\rangle_{L^2(\Gamma)}.$$

Further, the Cauchy–Schwarz inequality proves that

$$\langle v\,,\,(1-P_\bullet)\psi\rangle \le \sum_{T\in\mathcal{T}_\bullet} \|(1-P_\bullet)v\|_{L^2(T)}\|\psi\|_{L^2(T)}.$$

Since $P_\bullet v$ is nothing but the integral mean on each element $T\in\mathcal{T}_\bullet$, we see with Lemma 5.3.3 and (M3) that $\|(1-P_\bullet)v\|_{L^2(T)} \lesssim h_T^\sigma |v|_{H^\sigma(T)}$. Together with the Cauchy–Schwarz inequality and the definition of $\|\cdot\|_{H^\sigma(\Gamma)}$, this shows that

$$\langle v\,,\,(1-P_\bullet)\psi\rangle \lesssim \sum_{T\in\mathcal{T}_\bullet} |v|_{H^\sigma(T)}\|h_\bullet^\sigma\psi\|_{L^2(T)} \le \|v\|_{H^\sigma(\Gamma)}\|h_\bullet^\sigma\psi\|_{L^2(\Gamma)}.$$

Since $H^{-\sigma}(\Gamma)^D$ is the dual space of $H^\sigma(\Gamma)^D$, and $\|\cdot\|_{H^{-\sigma}(\Gamma)}$ is equivalent to the dual norm, where the constants depend only on D and Γ, the latter inequality concludes the proof. ☐

Remark 5.3.12. *Obviously, (5.3.24) holds as well for $\sigma = 0$. If one additionally assumes (M2) and (M4)–(M5), it is also satisfied for $\sigma = 1$. This follows from Corollary 5.3.6 and similar arguments as above. However, we will only apply (5.3.24) for $\sigma = 1/2$.*

In contrast to [AFF+17], we cannot use the Caccioppoli type inequality from [Mor08, Lemma 5.7.1] which is only shown for the Poisson problem there. Therefore, we prove the following generalization. For an open set $O\subset\mathbb{R}^d$ and an arbitrary $u\in H^2(O)$, we abbreviate $|u|_{H^1(O)} := \|\nabla u\|_{L^2(O)}$ and $|u|_{H^2(O)} := (\sum_{i=1}^d |\partial_i u|^2_{H^1(O)})^{1/2}$.

Lemma 5.3.13. *Let $r>0$, $x\in\mathbb{R}^d$ and $u\in H^1(B_{2r}(x))^D$ be a weak solution of $\mathfrak{P}u = 0$. Then, $u|_{B_r(x)} \in C^\infty(B_r(x))^D$ and there exists a constant $C>0$ such that*

$$|u|_{H^2(B_r(x))} \le C\big(\|u\|_{L^2(B_{2r}(x))} + \frac{1+r+r^2}{r}|u|_{H^1(B_{2r}(x))}\big). \tag{5.3.25}$$

The constant C depends only on the dimensions d, D, and the coefficients of the partial differential operator $\mathfrak{P}$.

Proof. By [McL00, Theorem 4.16], there holds that $u|_{B_{3r/2}(x)} \in H^k(B_{3r/2}(x))^D$ for all $k\in\mathbb{N}_0$, and the Sobolev embedding theorem proves that $u|_{B_{3r/2}(x)} \in C^\infty(B_{3r/2}(x))^D$. In particular, u is a strong solution of $\mathfrak{P}u = 0$ on $B_{3r/2}(x)$. To prove (5.3.25), let $\lambda\in\mathbb{R}^D$ be an arbitrary constant vector, and define $\widetilde{u} := u\circ\varphi$ with the affine bijection $\varphi: B_{3/2}(0)\to B_{3r/2}(x)$, $\varphi(\widetilde{y}) = r\widetilde{y}+x$ for $\widetilde{y}\in B_{3/2}(0)$. Since the coefficients of $\mathfrak{P}$ are constant and u is a strong solution, there holds for all $\widetilde{y}\in B_{3/2}(0)$ with $y := \varphi(\widetilde{y})$ that

$$\begin{aligned} -\sum_{i=1}^d\sum_{i'=1}^d \partial_i(A_{ii'}\partial_{i'}(\widetilde{u}-\lambda))(\widetilde{y}) &= -\sum_{i=1}^d\sum_{i'=1}^d \partial_i(A_{ii'}\partial_{i'}(u-\lambda))(y)\,r^2 \\ &= -r^2\Big(\sum_{i=1}^d b_i\partial_i(u-\lambda)(y) + c(u-\lambda)(y) + c\,\lambda\Big). \end{aligned} \tag{5.3.26}$$

We define the right-hand side as $\widetilde{f} \in C^\infty(B_{3/2}(0))$, i.e.,

$$\widetilde{f}(\widetilde{y}) := -r^2\Big(\sum_{i=1}^{d} b_i\partial_i(u-\lambda)(\varphi(\widetilde{y})) + c(u-\lambda)(\varphi(\widetilde{y})) + c\,\lambda\Big)$$

This shows that $\widetilde{u} - \lambda$ is a strong (and thus weak) solution of a *strongly elliptic* (see Section 5.1.3) system of second-order partial differential equations with smooth coefficients and smooth right-hand side. The application of [McL00, Theorem 4.16] yields the existence of a constant $C_1 > 0$, which depends only on d, D, and the coefficients of the matrices $A_{ii'}$, such that

$$|\widetilde{u}-\lambda|_{H^2(B_1(0))} \le C_1\big(\|\widetilde{u}-\lambda\|_{H^1(B_{3/2}(0))} + \|\widetilde{f}\|_{L^2(B_{3/2}(0))}\big). \tag{5.3.27}$$

Standard scaling arguments prove that

$$\begin{aligned}
|\widetilde{u}-\lambda|_{H^2(B_1(0))} &\simeq \frac{r^2}{r^{d/2}}\,|u|_{H^2(B_r(x))},\\
\|\widetilde{u}-\lambda\|_{L^2(B_{3/2}(0))} &\simeq \frac{1}{r^{d/2}}\|u-\lambda\|_{L^2(B_{3r/2}(x))},\\
|\widetilde{u}-\lambda|_{H^1(B_{3/2}(0))} &\simeq \frac{r}{r^{d/2}}|u|_{H^1(B_{3r/2}(x))},\\
\|\widetilde{f}\|_{L^2(B_{3/2}(0))} &\lesssim \frac{r^2}{r^{d/2}}|u|_{H^1(B_{3r/2}(x))} + \frac{r^2}{r^{d/2}}\|u-\lambda\|_{L^2(B_{3r/2}(x))} + r^2|\lambda|.
\end{aligned}$$

Plugging this into (5.3.27), we obtain that

$$|u|_{H^2(B_r(x))} \lesssim \Big(\frac{1+r^2}{r^2}\|u-\lambda\|_{L^2(B_{3r/2}(x))} + \frac{1+r}{r}|u|_{H^1(B_{3r/2}(x))} + r^{d/2}|\lambda|\Big). \tag{5.3.28}$$

We choose λ as the integral mean $\lambda := \int_{B_{3r/2}(x)} u(y)dy/|B_{3r/2}(x)|$. The Cauchy–Schwarz inequality implies that

$$|\lambda| \lesssim \|u\|_{L^1(B_{3r/2}(x))}/|B_{3r/2}(x)| \le \|u\|_{L^2(B_{3r/2}(x))}/|B_{3r/2}(x)|^{1/2} \simeq r^{-d/2}\|u\|_{L^2(B_{3r/2}(x))}.$$

Using this and the Poincaré inequality in (5.3.28) shows that

$$|u|_{H^2(B_r(x))} \lesssim \Big(\|u\|_{L^2(B_{3r/2}(x))} + \frac{1+r+r^2}{r}|u|_{H^1(B_{3r/2}(x))}\Big).$$

Finally, the assumption $r < R$ comes into play and guarantees that $(1+r+r^2)/r \le (1+R+R^2)/r$. Together with the fact that $B_{3r/2}(x) \subset B_{2r}(x)$, this concludes the proof. □

Remark 5.3.14. *Throughout the whole thesis, Lemma 5.3.13 is the only place where we need the assumption that the coefficients of the differential operator $\mathfrak{P}$ are constant instead of bounded C^∞ functions as in [McL00]. Indeed, with the definition $\widetilde{A}_{ii'} := A_{ii'} \circ \varphi$, one could try to modify (5.3.26) in the proof of Lemma 5.3.13 as follows*

$$\begin{aligned}
-\sum_{i=1}^{d}\sum_{i'=1}^{d}\partial_i(\widetilde{A}_{ii'}\partial_{i'}(\widetilde{u}-\lambda))(\widetilde{y}) &= -\sum_{i=1}^{d}\sum_{i'=1}^{d}\partial_i(A_{ii'}\partial_{i'}(u-\lambda))(y)\,r^2\\
&= -r^2\Big(\sum_{i=1}^{d} b_i(y)\partial_i(u-\lambda)(y) + c(y)(u-\lambda)(y) + c(y)\,\lambda\Big).
\end{aligned}$$

The resulting system is still strongly elliptic. However, the constant C_1 depends now on the coefficients of this new system, i.e., on the matrices $\widetilde{A}_{ii'}$ instead of $A_{ii'}$, and thus possibly on r.

For the proof of the next proposition, we need the linear and continuous *single-layer potential* from [McL00, Theorem 6.11]

$$\widetilde{\mathfrak{V}} : H^{-1/2}(\Gamma)^D \to H^1(O)^D, \tag{5.3.29}$$

where O is an arbitrary bounded domain with $\Gamma \subset O$. The single-layer operator $\mathfrak{V}$ is just the trace of $\widetilde{\mathfrak{V}}$, i.e.,

$$\mathfrak{V} = \widetilde{\mathfrak{V}}(\cdot)|_\Gamma : H^{-1/2}(\Gamma)^D \to H^{1/2}(\Gamma)^D; \tag{5.3.30}$$

see [McL00, pages 219–220]. Indeed, for $\psi \in L^\infty(\Gamma)$, [McL00, pages 201–202] states the following integral representation

$$(\widetilde{\mathfrak{V}}\psi)(x) = \int_\Gamma G(x-y)\psi(y)\,dy \quad \text{for all } x \in O. \tag{5.3.31}$$

Proposition 5.3.15. *Suppose (M1)–(M5). For $\mathcal{T}_\bullet \in \mathbb{T}$, let $w_\bullet \in L^\infty(\Gamma)$ be a function which satisfies for some $\alpha > 0$ and all $T \in \mathcal{T}_\bullet$ that*

$$\|w_\bullet\|_{L^\infty(T)} \le \alpha w_\bullet(x) \quad \textit{for almost all } x \in \pi_\bullet(T). \tag{5.3.32}$$

Then, there exists a constant $C_{\mathrm{inv},\mathfrak{V}} > 0$ such that for all $\psi \in L^2(\Gamma)^D$, there holds that

$$\|w_\bullet \nabla_\Gamma \mathfrak{V}\psi\|_{L^2(\Gamma)} \le C_{\mathrm{inv},\mathfrak{V}}\big(\|w_\bullet/h_\bullet^{1/2}\|_{L^\infty(\Gamma)}\|\psi\|_{H^{-1/2}(\Gamma)} + \|w_\bullet\psi\|_{L^2(\Gamma)}\big). \tag{5.3.33}$$

The constant $C_{\mathrm{inv},\mathfrak{V}}$ depends only on (M1)–(M5), Γ, the coefficients of $\mathfrak{P}$, and the admissibility constant α. The particular choice $w_\bullet = h_\bullet^{1/2}$ shows that

$$\|h_\bullet^{1/2}\nabla_\Gamma \mathfrak{V}\psi\|_{L^2(\Gamma)} \le C_{\mathrm{inv},\mathfrak{V}}\big(\|\psi\|_{H^{-1/2}(\Gamma)} + \|h_\bullet^{1/2}\psi\|_{L^2(\Gamma)}\big). \tag{5.3.34}$$

Proof. By (M4) and with the abbreviation

$$\delta_1(T) := \frac{\mathrm{diam}(T)}{2C_{\mathrm{cent}}}$$

and $U_T := B_{\delta_1(T)}(T)$, there holds for all $T \in \mathcal{T}_\bullet$ that

$$U_T \cap \Gamma \subset B_{2\delta_1(T)}(T) \cap \Gamma \subset \pi_\bullet(T). \tag{5.3.35}$$

This provides us with an open covering of $\Gamma \subset \bigcup_{T\in\mathcal{T}_\bullet} U_T$. We show that it is even locally finite in the sense that there exists a constant $C > 0$ with $\#\{T \in \mathcal{T}_\bullet : x \in U_T\} \le C$ for all $x \in \mathbb{R}^d$: Let $T_0 \in \mathcal{T}_\bullet$ with $x \in U_{T_0}$ such that $\delta_1(T_0)$ is minimal, and let $x_0 \in T_0$ with $|x - x_0| < \delta_1(T_0)$. Let $x \in \bigcup_{T\in\mathcal{T}_\bullet} U_T$. If $T \in \mathcal{T}_\bullet$ with $x \in U_T$, the triangle inequality yields that $\mathrm{dist}(x_0, T) < 2\delta_1(T)$. By choice of $\delta_1(T)$, (M4) and (M1) imply that

$$\#\{T \in \mathcal{T}_\bullet : x \in U_T\} \le \#\{T \in \mathcal{T}_\bullet : x_0 \in \pi_\bullet(T)\} \le C_{\mathrm{patch}}^2. \tag{5.3.36}$$

We fix (independently of $\mathcal{T}_\bullet$) a bounded domain $U \subset \mathbb{R}^d$ with $U_T \subset U$ for all $T \in \mathcal{T}_\bullet$. We define for $T \in \mathcal{T}_\bullet$ the near-field and the far-field of $u_{\mathfrak{V}} := \widetilde{\mathfrak{V}}\psi$ by

$$u_{\mathfrak{V},T}^{\text{near}} := \widetilde{\mathfrak{V}}(\psi\chi_{\Gamma\cap U_T}) \quad \text{and} \quad u_{\mathfrak{V},T}^{\text{far}} := \widetilde{\mathfrak{V}}(\psi\chi_{\Gamma\setminus U_T}). \tag{5.3.37}$$

In the following five steps, the near-field and the far-field are estimated separately. The first two steps are devoted to the near-field, whereas the last three steps deal with the far-field.

Step 1: We consider the near-field. We show that for all $T \in \mathcal{T}_\bullet$, all $\mathcal{T}_\bullet$-piecewise (componentwise) constant functions $\Psi_\bullet^T \in \mathcal{P}^0(\mathcal{T}_\bullet)^D$ with $\mathrm{supp}(\Psi_\bullet^T) \subseteq \pi_\bullet(T)$ satisfy that

$$\|\widetilde{\mathfrak{V}}\Psi_\bullet^T\|_{H^1(U_T)} \lesssim \|h_\bullet^{1/2}\Psi_\bullet^T\|_{L^2(\pi_\bullet(T))}. \tag{5.3.38}$$

For $x \in U_T \setminus \Gamma$, (5.3.31) implies that

$$(\nabla\widetilde{\mathfrak{V}}\Psi_\bullet^T)(x) = \sum_{T'\in\Pi_\bullet(T)} \int_{T'} \nabla_x G(x-y)\, dy\, \Psi_\bullet^T|_{T'}.$$

With (M1), we derive that

$$|(\nabla\widetilde{\mathfrak{V}}\Psi_\bullet^T)(x)|^2 \lesssim \sum_{T'\in\Pi_\bullet(T)} \Big(\int_{T'} |\nabla_x G(x-y)|\, dy\Big)^2 |\Psi_\bullet^T|_{T'}|^2. \tag{5.3.39}$$

Similarly, one sees that

$$|(\widetilde{\mathfrak{V}}\Psi_\bullet^T)(x)|^2 \lesssim \sum_{T'\in\Pi_\bullet(T)} \Big(\int_{T'} |G(x-y)|\, dy\Big)^2 |\Psi_\bullet^T|_{T'}|^2. \tag{5.3.40}$$

According to [McL00, Theorem 6.3 and Corollary 6.5], the fundamental solution G satisfies for all $x, y \in U$ that

$$|\nabla_x G(x-y)| \lesssim |x-y|^{-d+1} \quad \text{and} \quad |G(x-y)| \lesssim \max(|\log|x-y||, |x-y|^{-d+2}).$$

Together with (5.3.39)–(5.3.40), this implies that

$$\|\widetilde{\mathfrak{V}}\Psi_\bullet^T\|_{H^1(U_T)}^2 \lesssim \sum_{T'\in\Pi_\bullet(T)} |\Psi_\bullet^T|_{T'}|^2 \int_{U_T} \Big(\int_{T'} |x-y|^{-d+1}\, dy\Big)^2 dx. \tag{5.3.41}$$

To proceed, we prove the following estimate for arbitrary $T' \in \Pi_\bullet(T)$

$$\int_{U_T} \Big(\int_{T'} |x-y|^{-d+1}\, dy\Big)^2 dx \lesssim h_T^d. \tag{5.3.42}$$

Since Ω is a Lipschitz domain, $\pi_\bullet(T)$ is (up to a rigid motion) a subset of a Lipschitz hypograph $\big\{x \in \mathbb{R}^d : x_d = \zeta_j(x_1, \dots, x_{d-1})\big\}$ for some Lipschitz mapping $\zeta : \mathbb{R}^{d-1} \to \mathbb{R}$ provided that its diameter $\mathrm{diam}(\pi_\bullet(T))$ is sufficiently small. (Otherwise we must have $\mathrm{diam}(T) \simeq 1$ and (5.3.42) is trivially satisfied.) Note that the mapping $Z : \mathbb{R}^d \to \mathbb{R}^d$ with $Z(\widetilde{x}) := \zeta(\widetilde{x}_1, \dots, \widetilde{x}_{d-1}) + \widetilde{x}_d$ for $\widetilde{x} \in \mathbb{R}^d$ is bi-Lipschitz and onto. Due to (M2)–(M3), there exists a point $\widetilde{x}_0 \in \mathbb{R}^{d-1} \times \{0\}$ and a generic constant $C > 0$ such that $\widetilde{U}_T := Z^{-1}(U_T) \subseteq B_{Ch_T}(\widetilde{x}_0)$ and $\widetilde{T}' := Z^{-1}(T') \subseteq B_{Ch_T}(\widetilde{x}_0) \cap (\mathbb{R}^{d-1} \times \{0\})$. The transformation formula [EG92, Section 3.3.3] yield that

$$\int_{U_T} \Big(\int_{T'} |x-y|^{-d+1}\, dy \Big)^2 dx = \int_{\widetilde{U}_T} \Big(\int_{\widetilde{T}'} |Z(\widetilde{x}) - Z(\widetilde{y})|^{-d+1} \sqrt{1 + |\nabla\zeta(\widetilde{y})|^2}\, d\widetilde{y} \Big)^2 |\det DZ(\widetilde{x})|\, d\widetilde{x}$$
$$\simeq \int_{\widetilde{U}_T} \Big(\int_{\widetilde{T}'} |\widetilde{x} - \widetilde{y}|^{-d+1}\, d\widetilde{y} \Big)^2 d\widetilde{x} \le h_T^d \int_{B_C(0)} \Big(\int_{B_C(0)\cap(\mathbb{R}^{d-1}\times\{0\})} |s-t|^{-d+1}\, dt \Big)^2 ds.$$

The last integral in the latter estimate is finite, wherefore we conclude (5.3.42). Plugging (5.3.42) into (5.3.41) and using (M2)–(M3), we conclude Step 1.

Step 2: We show that $u^{\rm near}_{\mathfrak{V},T} \in H^1(U)$ and $u^{\rm near}_{\mathfrak{V},T}|_\Gamma \in H^1(\Gamma)$ with the near-field bound

$$\sum_{T\in\mathcal{T}_\bullet} \|w_\bullet \nabla_\Gamma u^{\rm near}_{\mathfrak{V},T}\|^2_{L^2(T)} + \sum_{T\in\mathcal{T}_\bullet} \|w_\bullet/h_\bullet^{1/2}\|^2_{L^\infty(T)} \|u^{\rm near}_{\mathfrak{V},T}\|^2_{H^1(U_T)} \lesssim \|w_\bullet \psi\|^2_{L^2(\Gamma)}. \tag{5.3.43}$$

First, we apply the stability of $\mathfrak{V} : L^2(\Gamma)^D \to H^1(\Gamma)^D$ of (5.1.11) to see for each $T \in \mathcal{T}_\bullet$ that

$$\|\nabla_\Gamma u^{\rm near}_{\mathfrak{V},T}\|_{L^2(T)} \le \|\mathfrak{V}(\psi\chi_{U_T\cap\Gamma})\|_{H^1(\Gamma)} \lesssim \|\psi\chi_{U_T\cap\Gamma}\|_{L^2(\Gamma)} = \|\psi\|_{L^2(U_T\cap\Gamma)}.$$

Summing the last estimate over all elements $T \in \mathcal{T}_\bullet$ and using (5.3.35)–(5.3.36) as well as the admissibility (5.3.32), we derive that

$$\sum_{T\in\mathcal{T}_\bullet} \|w_\bullet \nabla_\Gamma u^{\rm near}_{\mathfrak{V},T}\|^2_{L^2(T)} \lesssim \sum_{T\in\mathcal{T}_\bullet} \|w_\bullet\|^2_{L^\infty(T)} \|\psi\|^2_{L^2(U_T\cap\Gamma)} \simeq \|w_\bullet\psi\|^2_{L^2(\Gamma)}.$$

It remains to bound the second term in (5.3.43). With the notation of Lemma 5.3.11, we decompose the near field for $T \in \mathcal{T}_\bullet$ as $u^{\rm near}_{\mathfrak{V},T} = \widetilde{\mathfrak{V}} P_\bullet(\psi\chi_{U_T\cap\Gamma}) + \widetilde{\mathfrak{V}}(1-P_\bullet)(\psi\chi_{U_T\cap\Gamma})$. The property (5.3.35) shows that $\operatorname{supp}(P_\bullet(\psi\chi_{U_T\cap\Gamma})) \subseteq \pi_\bullet(T)$. In particular, we can apply Step 1 with $\Psi^T_\bullet := P_\bullet(\psi\chi_{U_T\cap\Gamma})$, which yields together with the local L^2-stability of $P_\bullet$ and (M1)–(M3) that

$$\begin{aligned} &\sum_{T\in\mathcal{T}_\bullet} \|w_\bullet/h_\bullet^{1/2}\|^2_{L^\infty(T)} \|\widetilde{\mathfrak{V}} P_\bullet(\psi\chi_{U_T\cap\Gamma})\|^2_{H^1(U_T)} \\ &\quad\lesssim \sum_{T\in\mathcal{T}_\bullet} \|w_\bullet/h_\bullet^{1/2}\|^2_{L^\infty(T)} \|h_\bullet^{1/2} P_\bullet(\psi\chi_{U_T\cap\Gamma})\|^2_{L^2(\pi_\bullet(T))} \lesssim \|w_\bullet\psi\|^2_{L^2(\Gamma)}. \end{aligned} \tag{5.3.44}$$

Next, we exploit the stability $\widetilde{\mathfrak{V}} : H^{-1/2}(\Gamma)^D \to H^1(U)^D$ of (5.3.29) as well as the approximation property of Lemma (5.3.11). Together with (5.3.35) and (M1)–(M3), we obtain that

$$\begin{aligned} &\sum_{T\in\mathcal{T}_\bullet} \|w_\bullet/h_\bullet^{1/2}\|^2_{L^\infty(T)} \|\widetilde{\mathfrak{V}}(1-P_\bullet)(\psi\chi_{U_T\cap\Gamma})\|_{H^1(U_T)} \\ &\quad\lesssim \sum_{T\in\mathbb{T}_\bullet} \|w_\bullet/h_\bullet^{1/2}\|^2_{L^\infty(T)} \|(1-P_\bullet)(\psi\chi_{U_T\cap\Gamma})\|_{H^{-1/2}(\Gamma)} \\ &\quad\lesssim \sum_{T\in\mathcal{T}_\bullet} \|w_\bullet/h_\bullet^{1/2}\|^2_{L^\infty(T)} \|h_\bullet^{1/2}\psi\chi_{U_T\cap\Gamma}\|^2_{L^2(\Gamma)} \simeq \|w_\bullet\psi\|^2_{L^2(\Gamma)}. \end{aligned} \tag{5.3.45}$$

Finally, we combine (5.3.44)–(5.3.45) to bound the second term in (5.3.43).
Step 3: We consider the far-field. We set $\Omega^{\text{ext}} := \mathbb{R}^d \setminus \overline{\Omega}$. According to [McL00, Theorem 6.11], for all $T \in \mathcal{T}_\bullet$, $u^{\text{far}}_{\mathfrak{V},T}$ is a solution of the transmission problem

$$
\begin{aligned}
\mathfrak{P} u^{\text{far}}_{\mathfrak{V},T} &= 0 && \text{on } \Omega \cup \Omega^{\text{ext}},\\
[u^{\text{far}}_{\mathfrak{V},T}] &= 0 && \text{in } H^{1/2}(\Gamma)^D,\\
[\mathfrak{D}_\nu u^{\text{far}}_{\mathfrak{V},T}] &= -\psi \chi_{\Gamma\setminus U_T} && \text{in } H^{-1/2}(\Gamma)^D,
\end{aligned}
$$

where $[\cdot]$ resp. $[\mathfrak{D}_\nu(\cdot)]$ denotes the jump of the traces resp. the conormal derivatives (see (4.2.11) resp. [McL00, page 117] for a precise definition) across the boundary Γ. Twofold integration by parts [McL00, Theorem 4.4] on Ω resp. Ω^{ext} that uses these equalities shows that $\langle u^{\text{far}}_{\mathfrak{V},T}\,, \mathfrak{P}^* v\rangle = 0$ for all $x \in T$ and all test functions $v \in C^\infty(B_{\delta_1(T)/2}(x))$ with compact support. Here, $\mathfrak{P}^*$ denotes the adjoint partial differential operator to $\mathfrak{P}$. This shows that $u^{\text{far}}_{\mathfrak{V},T}$ satisfies the equation $\mathfrak{P} u^{\text{far}}_{\mathfrak{V},T} = 0$ weakly on $B_{\delta_1(T)/2}(x) \subset U_T$ for all $x \in T$. Hence, we can apply Lemma 5.3.13 to see that $u^{\text{far}}_{\mathfrak{V},T} \in C^\infty(B_{\delta_1(T)/4}(x))$ with

$$
|u^{\text{far}}_{\mathfrak{V},T}|_{H^2(B_{\delta_1(T)/4}(x))} \lesssim \|u^{\text{far}}_{\mathfrak{V},T}\|_{L^2(B_{\delta_1(T)/2}(x))} + \operatorname{diam}(T)^{-1} |u^{\text{far}}_{\mathfrak{V},T}|_{H^1(B_{\delta_1(T)/2}(x))}. \tag{5.3.46}
$$

Step 4: With the latter inequality at hand, we prove the following local far-field bound for the single-layer potential $\widetilde{\mathfrak{V}}$

$$
\|h_\bullet^{1/2} \nabla_\Gamma u^{\text{far}}_{\mathfrak{V},T}\|_{L^2(T)} \le \|h_\bullet^{1/2} \nabla u^{\text{far}}_{\mathfrak{V},T}\|_{L^2(T)} \lesssim \|u^{\text{far}}_{\mathfrak{V},T}\|_{H^1(U_T)}. \tag{5.3.47}
$$

The first estimate of (5.3.47) follows from the fact that, for C^1 functions v, the surface gradient $\nabla_\Gamma v$ is the orthogonal projection of the gradient ∇u onto the tangent plane, i.e., $\nabla_\Gamma v = \nabla v - (\nabla v \cdot \nu)\nu$, where ν denotes the outer normal vector; see, e.g., [ME14, Lemma 2.22].

To derive the second one, we first show an auxiliary trace inequality. Let $B = B_r(x)$ be an arbitrary open ball in $\mathbb{R}^d$ and $v \in H^1(B')$ with $B' := B_{2r}(x)$. According to [McL00, Theorem 3.6], there exists a smooth indicator function $\widetilde{\chi}_B \in C^\infty(\mathbb{R}^d)$ which only takes values in $[0,1]$ such that $\widetilde{\chi}_B = 1$ on $\overline{B}$, $\widetilde{\chi}_B = 0$ on $\mathbb{R}^d \setminus B'$, and $\|\nabla \widetilde{\chi}_B\|_{L^\infty(\mathbb{R}^d)} \le C r^{-1}$ for some generic constant $C > 0$. Together with the trace inequality of Proposition 4.2.2, we see that

$$
\begin{aligned}
\|v\|^2_{L^2(\Gamma\cap\overline{B})} \le \|\widetilde{\chi}_B v\|^2_{L^2(\Gamma)} &\lesssim \|\widetilde{\chi}_B v\|^2_{L^2(\Omega)} + \|\widetilde{\chi}_B v\|_{L^2(\Omega)} \|\nabla(\widetilde{\chi}_B v)\|_{L^2(\Omega)}\\
&\lesssim r^{-1}\|v\|^2_{L^2(B')} + \|v\|_{L^2(B')} \|\nabla v\|_{L^2(B')}.
\end{aligned} \tag{5.3.48}
$$

By Step 3, we can apply (5.3.48) to $v := \partial_j u^{\text{far}}_{\mathfrak{V},T}$ on $B := B_{\delta_1(T)/4}(x)$ with $x \in T \in \mathcal{T}_\bullet$. Together with (5.3.46) and the abbreviation $B'' := B_{\delta_1(T)}(x)$, this yields that

$$
|u^{\text{far}}_{\mathfrak{V},T}|^2_{H^1(\Gamma\cap\overline{B})} \lesssim h_T^{-1} |u^{\text{far}}_{\mathfrak{V},T}|^2_{H^1(B')} + |u^{\text{far}}_{\mathfrak{V},T}|_{H^1(B')} |u^{\text{far}}_{\mathfrak{V},T}|_{H^2(B')} \lesssim h_T^{-1} \|u^{\text{far}}_{\mathfrak{V},T}\|^2_{H^1(B'')}. \tag{5.3.49}
$$

To exploit the latter estimate, we use a covering argument for $T \in \mathcal{T}_\bullet$. The set $\mathcal{F} := \{\overline{B_{\delta_1(T)/4}(x)} \,:\, x \in T\}$, is a cover of T consisting of closed balls with $\sup_{B\in\mathcal{F}} \operatorname{diam}(B) < \infty$, where T is the set of their midpoints. Besicovitch's covering theorem [EG92, Section 1.5.2] implies the existence of a constant $N_d \in \mathbb{N}$, which depends only on the dimension d, and countable subsets $\mathcal{G}_j \subseteq \mathcal{F}$, $j = 1, \dots, N_d$, where the elements of each $\mathcal{G}_j$ are pairwise disjoint, such that $T \subseteq \bigcup_{j=1}^{N_d} \bigcup_{B \in \mathcal{G}_j} B$. We define $\mathcal{G}_j'' := \{B_{\delta_1(T)}(x) : \overline{B_{\delta_1(T)/4}(x)} \in \mathcal{G}_j\}$. Since the elements of $\mathcal{G}_j$ are pairwise disjoint and all balls have the same radius $\delta_1(T)/4$, there is a constant $N_d'' \in \mathbb{N}$, which depends only on the dimension d, such that at most N_d'' elements of $\mathcal{G}_j''$ overlap. Therefore, (5.3.49) leads to

$$\|\nabla u^{\text{far}}_{\mathfrak{V},T}\|^2_{L^2(T)} \le \sum_{j=1}^{N_d} \sum_{\overline{B} \in \mathcal{G}_j} \|\nabla u^{\text{far}}_{\mathfrak{V},T}\|^2_{L^2(T \cap \overline{B})} \lesssim h_T^{-1} \sum_{j=1}^{N_d} \sum_{B'' \in \mathcal{G}_j''} \|u^{\text{far}}_{\mathfrak{V},T}\|^2_{H^1(B'')} \le h_T^{-1} N_d N_d'' \|u^{\text{far}}_{\mathfrak{V},T}\|^2_{H^1(U_T)},$$

and thus to (5.3.47).

Step 5: Finally, we prove the following far-field bound for $\widetilde{\mathfrak{V}}$

$$\begin{aligned}\sum_{T \in \mathcal{T}_\bullet} \|w_\bullet \nabla_\Gamma u^{\text{far}}_{\mathfrak{V},T}\|^2_{L^2(T)} &\le \sum_{T \in \mathcal{T}_\bullet} \|w_\bullet \nabla u^{\text{far}}_{\mathfrak{V},T}\|^2_{L^2(T)} \\ &\lesssim \|w_\bullet / h_\bullet^{1/2}\|^2_{L^\infty(\Gamma)} \|\psi\|^2_{H^{-1/2}(\Gamma)} + \|w_\bullet \psi\|^2_{L^2(\Gamma)}.\end{aligned} \tag{5.3.50}$$

By definition (5.3.37), (5.3.50) together with (5.3.43) of Step 2 will conclude the proof of the proposition. The estimate (5.3.47) and the definition (5.3.37) show that

$$\begin{aligned}\sum_{T \in \mathcal{T}_\bullet} \|w_\bullet \nabla_\Gamma u^{\text{far}}_{\mathfrak{V},T}\|^2_{L^2(T)} &\lesssim \sum_{T \in \mathcal{T}_\bullet} \|w_\bullet / h_\bullet^{1/2}\|^2_{L^\infty(\Gamma)} \|u^{\text{far}}_{\mathfrak{V},T}\|^2_{H^1(U_T)} \\ &\lesssim \sum_{T \in \mathcal{T}_\bullet} \|w_\bullet / h_\bullet^{1/2}\|^2_{L^\infty(T)} \|u^{\text{near}}_{\mathfrak{V},T}\|^2_{H^1(U_T)} + \sum_{T \in \mathcal{T}_\bullet} \|w_\bullet / h_\bullet^{1/2}\|^2_{L^\infty(\Gamma)} \|\widetilde{\mathfrak{V}}\psi\|^2_{H^1(U_T)}.\end{aligned} \tag{5.3.51}$$

The first term in (5.3.51) can be bounded with the near-field bound (5.3.43). For the second one, we apply (5.3.36) and the stability $\widetilde{\mathfrak{V}} : H^{-1/2}(\Gamma)^D \to H^1(U)^D$ of (5.3.29) to see that

$$\sum_{T \in \mathcal{T}_\bullet} \|w_\bullet / h_\bullet^{1/2}\|^2_{L^\infty(T)} \|\widetilde{\mathfrak{V}}\psi\|^2_{H^1(U_T)} \lesssim \|w_\bullet / h_\bullet^{1/2}\|^2_{L^\infty(\Gamma)} \|\widetilde{\mathfrak{V}}\psi\|^2_{H^1(U)} \lesssim \|w_\bullet / h_\bullet^{1/2}\|^2_{L^\infty(\Gamma)} \|\psi\|^2_{H^{-1/2}(\Gamma)}.$$

This concludes the proof. □

Remark 5.3.16. *[AFF+17] does not only treat the single-layer operator $\mathfrak{V} : H^{-1/2}(\Gamma)^D \to H^{1/2}(\Gamma)^D$, but also derives similar inverse estimates as in (5.3.33) for the* double-layer operator $\mathfrak{K} : H^{1/2}(\Gamma)^D \to H^{1/2}(\Gamma)^D$, *the* adjoint double-layer operator $\mathfrak{K}' : H^{-1/2}(\Gamma)^D \to H^{-1/2}(\Gamma)^D$, *and the* hyper-singular operator $\mathfrak{W} : H^{1/2}(\Gamma)^D \to H^{-1/2}(\Gamma)^D$*; see, e.g., [McL00, page 218] for a precise definition (where these operators are denoted by $T, \widetilde{T}^*$, and R). Although they only considered the Laplace problem, with the techniques of the proof of Proposition 5.3.15, their result can be extended to arbitrary partial differential operators $\mathfrak{P}$ with constant coefficients as in Section 5.1.3. Indeed, one can show for all $\psi \in L^2(\Gamma)^D$ and $v \in H^1(\Gamma)^D$ that*

$$\begin{aligned}\|w_\bullet \nabla_\Gamma \mathfrak{V}\psi\|_{L^2(\Gamma)} + \|w_\bullet \mathfrak{K}'\psi\|_{L^2(\Gamma)} &\lesssim \big(\|w_\bullet / h_\bullet^{1/2}\|_{L^\infty(\Gamma)} \|\psi\|_{H^{-1/2}(\Gamma)} + \|w_\bullet \psi\|_{L^2(\Gamma)}\big), \\ \|w_\bullet \nabla_\Gamma \mathfrak{K} v\|_{L^2(\Gamma)} + \|w_\bullet \mathfrak{W} v\|_{L^2(\Gamma)} &\lesssim \big(\|w_\bullet / h_\bullet^{1/2}\|_{L^\infty(\Gamma)} \|v\|_{H^{1/2}(\Gamma)} + \|w_\bullet \nabla_\Gamma v\|_{L^2(\Gamma)}\big).\end{aligned}$$

For the second inequality, one additionally needs the assumption that $\mathfrak{P}$ has no lowest-order terms, i.e., $c = 0$. Although it is likely that also the general case $c \neq 0$ is valid, the analysis of [AFF+17] exploits the fact that the double-layer potential $\widetilde{\mathfrak{K}} : H^{1/2}(\Gamma)^D \to H^1(\Omega)^D$ *(see, e.g, [McL00] for a definition) of the Laplacian satisfies that $\widetilde{\mathfrak{K}}x$ is constant for arbitrary constant functions $x \in \mathbb{R}^D$. In general, this is only satisfied if the considered partial differential operator has no lowest-order terms.*

5.3.5 Stability on non-refined elements (E1)

We show that the assumptions (M1)–(M5) and (S1)–(S2) imply stability (E1), i.e., the existence of $C_{\rm stab} \geq 1$ such that for all $\mathcal{T}_\bullet \in \mathbb{T}$, and all $\mathcal{T}_\circ \in \texttt{refine}(\mathcal{T}_\bullet)$, it holds that

$$|\eta_\circ(\mathcal{T}_\bullet \cap \mathcal{T}_\circ) - \eta_\bullet(\mathcal{T}_\bullet \cap \mathcal{T}_\circ)| \leq C_{\rm stab}\|\Phi_\circ - \Phi_\bullet\|_{H^{-1/2}(\Gamma)}. \tag{5.3.52}$$

In Section 5.3.6, we will fix the constant C_ϱ for the perturbations (5.3.1) such that $C_{\rm stab} \leq C_\varrho$. The reverse triangle inequality and the fact that $h_\circ = h_\bullet$ on $\bigcup(\mathcal{T}_\bullet \cap \mathcal{T}_\circ)$ prove that

$$\begin{aligned}|\eta_\circ(\mathcal{T}_\bullet \cap \mathcal{T}_\circ) - \eta_\bullet(\mathcal{T}_\bullet \cap \mathcal{T}_\circ)| &= \big|\|h_\circ^{1/2}\nabla_\Gamma\mathfrak{V}(\phi - \Phi_\circ)\|_{L^2(\bigcup(\mathcal{T}_\bullet\cap\mathcal{T}_\circ))} - \|h_\bullet^{1/2}\nabla_\Gamma\mathfrak{V}(\phi - \Phi_\bullet)\|_{L^2(\bigcup(\mathcal{T}_\bullet\cap\mathcal{T}_\circ))}\big|\\ &\leq \|h_\circ^{1/2}\nabla_\Gamma\mathfrak{V}(\Phi_\circ - \Phi_\bullet)\|_{L^2(\bigcup(\mathcal{T}_\bullet\cap\mathcal{T}_\circ))}\\ &\leq \|h_\circ^{1/2}\nabla_\Gamma\mathfrak{V}(\Phi_\circ - \Phi_\bullet)\|_{L^2(\Gamma)}.\end{aligned}$$

(S2) shows that $\Phi_\circ - \Phi_\bullet \in \mathcal{X}_\circ$. Therefore, the inverse inequalities from (S1) and (5.3.34) are applicable, which implies (5.3.52). The constant $C_{\rm stab}$ depends only on d, D, Γ, the coefficients of $\mathfrak{P}$, and the constants from (M1)–(M5) and (S1).

5.3.6 Reduction on refined elements (E2)

We show that the assumptions (M1)–(M5), (R2)–(R3), and (S1)–(S2) imply reduction on refined elements (E2), i.e., the existence of $C_{\rm red} \geq 1$ and $0 < \rho_{\rm red} < 1$ such that for all $\mathcal{T}_\bullet \in \mathbb{T}$ and all $\mathcal{T}_\circ \in \texttt{refine}(\mathcal{T}_\bullet)$, there holds that

$$\eta_\circ(\mathcal{T}_\circ \setminus \mathcal{T}_\bullet)^2 \leq \rho_{\rm red}\,\eta_\bullet(\mathcal{T}_\bullet\setminus\mathcal{T}_\circ)^2 + C_{\rm red}\|\Phi_\circ - \Phi_\bullet\|^2_{H^{-1/2}(\Gamma)}. \tag{5.3.53}$$

With this, we can fix the constant for the perturbations (5.3.1) as

$$C_\varrho := \max(C_{\rm stab}, C_{\rm red}^{1/2}). \tag{5.3.54}$$

First, we apply the triangle inequality

$$\begin{aligned}\eta_\circ(\mathcal{T}_\circ \setminus \mathcal{T}_\bullet) &= \|h_\circ^{1/2}\nabla_\Gamma\mathfrak{V}(\phi - \Phi_\circ)\|_{L^2(\bigcup\mathcal{T}_\circ\setminus\mathcal{T}_\bullet)}\\ &\leq \|h_\circ^{1/2}\nabla_\Gamma\mathfrak{V}(\phi - \Phi_\bullet)\|_{L^2(\bigcup\mathcal{T}_\circ\setminus\mathcal{T}_\bullet)} + \|h_\circ^{1/2}\nabla_\Gamma\mathfrak{V}(\Phi_\circ - \Phi_\bullet)\|_{L^2(\bigcup\mathcal{T}_\circ\setminus\mathcal{T}_\bullet)}.\end{aligned}$$

(R2)–(R3) show that $\bigcup(\mathcal{T}_\circ \setminus \mathcal{T}_\bullet) = \bigcup(\mathcal{T}_\bullet \setminus \mathcal{T}_\circ)$ and $h_\circ \leq \rho_{\rm son}^{1/(d-1)} h_\bullet$ on $\bigcup(\mathcal{T}_\bullet \setminus \mathcal{T}_\circ)$. Thus, we can proceed the estimate as follows

$$\begin{aligned}\eta_\circ(\mathcal{T}_\circ \setminus \mathcal{T}_\bullet) &\leq \rho_{\rm son}^{1/(2d-2)}\|h_\bullet^{1/2}\nabla_\Gamma\mathfrak{V}(\phi - \Phi_\bullet)\|_{L^2(\bigcup\mathcal{T}_\bullet\setminus\mathcal{T}_\circ)} + \|h_\circ^{1/2}\nabla_\Gamma\mathfrak{V}(\Phi_\circ - \Phi_\bullet)\|_{L^2(\bigcup\mathcal{T}_\circ\setminus\mathcal{T}_\bullet)}\\ &= \rho_{\rm son}^{1/(2d-2)}\eta_\bullet(\mathcal{T}_\bullet \setminus \mathcal{T}_\circ) + \|h_\circ^{1/2}\nabla_\Gamma\mathfrak{V}(\Phi_\circ - \Phi_\bullet)\|_{L^2(\bigcup\mathcal{T}_\circ\setminus\mathcal{T}_\bullet)}.\end{aligned}$$

Since $\Phi_\bullet \in \mathcal{X}_\bullet \subseteq \mathcal{X}_\circ$ according to (S2), we can apply the inverse estimates (S1) and (5.3.34). Together with the Young's inequality, we derive for arbitrary $\delta > 0$ that

$$\eta_\circ(\mathcal{T}_\circ \setminus \mathcal{T}_\bullet)^2 \leq (1+\delta)\rho_{\rm son}^{1/(d-1)}\eta_\bullet(\mathcal{T}_\bullet \setminus \mathcal{T}_\circ)^2 + (1+\delta^{-1})C_{{\rm inv},\mathfrak{V}}^2(1+C_{\rm inv})^2\|\Phi_\circ - \Phi_\bullet\|_{H^{-1/2}(\Gamma)}^2.$$

Choosing $\delta > 0$ sufficiently small, we obtain (5.3.53). The constant $C_{\rm red}$ depends only on d, D, Γ, the coefficients of $\mathfrak{P}$, and the constants from (M1)–(M5), (R2)–(R3), and (S1).

5.3.7 General quasi-orthogonality (E3)

According to Theorem 2.3.1 (i), Section 5.3.3, Section 5.3.5, and Section 5.3.6 already imply estimator convergence $\lim_{\ell\to\infty}\eta_\ell = 0$. Therefore, reliability (5.2.22) implies error convergence $\lim_{\ell\to\infty}\|\phi - \Phi_\ell\|_{H^{-1/2}(\Gamma)} = 0$. In particular, we obtain that $\phi \in \mathcal{X}_\infty = \overline{\bigcup_{\ell\in\mathbb{N}_0}\mathcal{X}_\ell}$. As in Section 4.3.4, we show that the latter inclusion $\phi \in \mathcal{X}_\infty$, reliability (5.2.22), and (S2) imply general quasi-orthogonality (E3), i.e., the existence of

$$0 \leq \varepsilon_{\rm qo} < \sup_{\delta>0}\frac{1-(1+\delta)(1-(1-\rho_{\rm red})\theta)}{2+\delta^{-1}}, \tag{5.3.55}$$

and $C_{\rm qo} \geq 1$ such that

$$\sum_{j=\ell}^{\ell+N}(C_\varrho\|\Phi_{j+1} - \Phi_j\|_{H^{-1/2}(\Gamma)}^2 - \varepsilon_{\rm qo}\eta_j^2) \leq C_{\rm qo}\eta_\ell^2 \quad \text{for all } \ell, N \in \mathbb{N}_0. \tag{5.3.56}$$

Recall that we already fixed the constant C_ϱ in (5.3.54). Again, the key ingredient is provided by the abstract Lemma 4.3.2. We choose $\mathcal{H} := H^{-1/2}(\Gamma)^D$ with $\mathcal{H}^* = (H^{-1/2}(\Gamma)^D)^*$ and $\mathcal{H}_\ell := \mathcal{X}_\ell$ for all $\ell \in \mathbb{N}_0$. Note that, $H^{1/2}(\Gamma)^D$ is a realization of $(H^{-1/2}(\Gamma)^D)^*$ with equivalent norms. To define the involved operators, we first introduce the *principal part* of $\mathfrak{P}$ as the corresponding partial differential operator without lower-order terms

$$\mathfrak{P}_0 v := -\sum_{i=1}^{d}\sum_{i'=1}^{d}\partial_i(A_{ii'}\partial_{i'}v). \tag{5.3.57}$$

According to [McL00, Lemma 4.5], the principal part is also coercive on $H_0^1(\Omega)^D$. We denote its corresponding single-layer operator which can be defined as in Section 5.1.3 by

$$\mathfrak{V}_0 : H^{-1/2}(\Gamma)^D \to H^{1/2}(\Gamma)^D. \tag{5.3.58}$$

With this, we set $\mathfrak{A} := \mathfrak{V}_0$ as well as $\mathfrak{C} := \mathfrak{V} - \mathfrak{V}_0$, which gives $\mathfrak{B} = \mathfrak{V}$ in Lemma 4.3.2. Our assumption $A_{ii'}^\top = A_{i'i}$ easily implies that $\mathfrak{V}_0$ is symmetric; see, e.g., [McL00, page 218]. To see compactness of $\mathfrak{C}$, one can proceed as in [SS11, Lemma 3.9.8], where the assertion is proved for the Helmholtz operator: First, one shows as in [McL00, Theorem 6.1] that the corresponding *Newton potentials* satisfy the mapping property $\widetilde{\mathfrak{N}} - \widetilde{\mathfrak{N}}_0 : H^\sigma(\mathbb{R}^d)^D \to H^{\sigma+3}(\mathbb{R}^d)^D$ for all $\sigma \in \mathbb{R}$. In combination with Rellich's compactness theorem [McL00, Theorem 3.27], one can now adapt the proof of [McL00, Theorem 6.11] which yields compactness of $\mathfrak{C}$. Recall that we already observed at the beginning of the current subsection that $\phi \in \mathcal{X}_\infty$. Altogether, we see that Lemma 4.3.2 is applicable. The rest follows along the lines of Step 2 from Section 4.3.4, where $C_{\rm qo}$ depends only on the dimension D, the boundary Γ, the chosen $\varepsilon_{\rm qo}$, the perturbation constant C_ϱ, the reliability constant $C_{\rm rel}$, the coefficients of $\mathfrak{P}$, and the sequence $(\Phi_\ell)_{\ell\in\mathbb{N}_0}$.

Remark 5.3.17. *If the bilinear form $\langle \mathfrak{V}\,\cdot\,,\,\cdot\rangle$ is symmetric, (5.3.56) follows from the Pythagoras theorem $\|\phi-\Phi_j\|_{\mathfrak{V}}^2 + \|\Phi_{j+1}-\Phi_j\|_{\mathfrak{V}}^2 = \|\phi-\Phi_j\|_{\mathfrak{V}}^2$ in the $\mathfrak{V}$-induced energy norm $\|\psi\|_{\mathfrak{V}}^2 := \langle \mathfrak{V}\psi\,,\,\psi\rangle$ and norm equivalence*

$$\sum_{j=\ell}^{\ell+N} \|\Phi_{j+1}-\Phi_j\|_{H^{-1/2}(\Gamma)}^2 \simeq \sum_{j=\ell}^{\ell+N} \|\Phi_{j+1}-\Phi_j\|_{\mathfrak{V}}^2 = \|\phi-\Phi_\ell\|_{\mathfrak{V}}^2 - \|\phi-\Phi_{\ell+N}\|_{\mathfrak{V}}^2$$
$$\lesssim \|\phi-\Phi_\ell\|_{H^{-1/2}(\Gamma)}^2.$$

Together with reliability (5.2.22), this proves (5.3.56) even for $\varepsilon_{\rm qo}=0$, and $C_{\rm qo}$ is independent of the sequence $(\Phi_\ell)_{\ell\in\mathbb{N}_0}$.

5.3.8 Discrete reliability (E4)

The proof of (E4) is inspired by [FKMP13, Proposition 5.3] which considers piecewise constants on shape-regular triangulations as ansatz space. Under the assumptions (M1)–(M5), (5.2.12), and (S1)–(S6), we show that there exist $C_{\rm drel}, C_{\rm ref} \ge 1$ such that for all $\mathcal{T}_\bullet \in \mathbb{T}$ and all $\mathcal{T}_\circ \in \texttt{refine}(\mathcal{T}_\bullet)$, the subset

$$\mathcal{R}_{\bullet,\circ} := \Pi_\bullet^{q_{\rm supp}+q_{\rm loc}+2}(\mathcal{T}_\bullet \setminus \mathcal{T}_\circ) \tag{5.3.59}$$

satisfies that

$$C_\varrho\|\Phi_\circ-\Phi_\bullet\|_{H^{-1/2}(\Gamma)} \le C_{\rm drel}\,\eta_\bullet(\mathcal{R}_{\bullet,\circ}), \quad \mathcal{T}_\bullet\setminus\mathcal{T}_\circ \subseteq \mathcal{R}_{\bullet,\circ}, \text{ and } \#\mathcal{R}_{\bullet,\circ} \le C_{\rm ref}(\#\mathcal{T}_\circ - \#\mathcal{T}_\bullet).$$

The last two properties are obvious with $C_{\rm ref} = C_{\rm patch}^{q_{\rm supp}+q_{\rm loc}+2}$ by validity of (M1) and (5.2.12). Before we start the proof itself, we provide the following lemma about certain smooth characteristic functions.

Lemma 5.3.18. *Let $\mathcal{T}_\bullet \in \mathbb{T}$ and $\mathcal{S} \subseteq \mathcal{T}_\bullet$. Suppose (M1)–(M4). Then there exists a function $\widetilde{\chi}_{\mathcal{S}} \in H^1(\Gamma)$ such that for almost all $x \in \Gamma$*

$$\begin{aligned} &\widetilde{\chi}_{\mathcal{S}}(x) = 1 && \text{if } x \in \textstyle\bigcup \mathcal{S}, \\ &0 \le \widetilde{\chi}_{\mathcal{S}}(x) \le 1 && \text{if } x \in \pi_\bullet(\mathcal{S}), \\ &\widetilde{\chi}_{\mathcal{S}}(x) = 0 && \text{if } x \notin \pi_\bullet(\mathcal{S}). \end{aligned} \tag{5.3.60}$$

Further, there exists a constant $C>0$ such that for almost all $x\in\Gamma$, there holds that

$$|\nabla_\Gamma \widetilde{\chi}_{\mathcal{S}}(x)| \le C h_\bullet(x)^{-1} \tag{5.3.61}$$

The constant C depends only on the dimension d and the constants from (M1)–(M4).

Proof. In the following three steps, we will even prove the existence of a function $\widetilde{\chi}_{\mathcal{S}} \in C^\infty(O)$ with an open superset $O \supset \Gamma$ such that the restriction to Γ has the desired properties. With the constants from (M1)–(M2) and (M4), we introduce the following abbreviations for $T \in \mathcal{T}_\bullet$

$$\delta_1(T) := \frac{\operatorname{diam}(T)}{2C_{\mathrm{patch}}C_{\mathrm{locuni}}C_{\mathrm{cent}}}, \quad \delta_2(T) := \frac{\operatorname{diam}(T)}{C_{\mathrm{cent}}}, \quad \delta_3(T) := \frac{\operatorname{diam}(T)}{2C_{\mathrm{cent}}}. \tag{5.3.62}$$

Step 1: First, we construct an equivalent smooth mesh-size function $\widetilde{\delta}_\bullet \in C^\infty(\mathbb{R}^d)$. Let $K_1 \in C^\infty(\mathbb{R}^d)$ be a standard mollifier with $0 \le K_1 \le 1$ on $B_1(0)$, $K_1 = 0$ on $\mathbb{R}^d \setminus B_1(0)$, and $\int_{\mathbb{R}^d} K_1 \, dx = 1$. For $s > 0$, we set $K_s(\cdot) := K_1(\cdot/s)s^{-d}$. With the convolution operator, we define

$$\widetilde{\delta}_\bullet := \sum_{T\in\mathcal{T}_\bullet} \delta_1(T)\, \chi_{B_{\delta_2(T)}(T)} * K_{\delta_2(T)}. \tag{5.3.63}$$

Note that $\operatorname{supp}(\chi_{B_{\delta_2(T)}(T)} * K_{\delta_2(T)}) \subseteq B_{2\delta_2(T)}(T)$ for $T \in \mathcal{T}_\bullet$. Thus, (M4) and the choice (5.3.62) of $\delta_2(T)$ yields that $\operatorname{supp}(\chi_{B_{\delta_2(T)}(T)} * K_{\delta_2(T)}) \cap \Gamma \subseteq \pi_\bullet(T)$. Together with (M1)–(M2) and $0 \le \chi_{B_{\delta_2(T)}(T)} * K_{\delta_2(T)} \le 1$, this implies for the interior T'° of any $T' \in \mathcal{T}_\bullet$ that

$$\widetilde{\delta}_\bullet|_{T'^\circ} \le \sum_{T\in\mathcal{T}_\bullet} \delta_1(T)\chi_{\pi_\bullet(T)}|_{T'^\circ} = \sum_{T\in\Pi_\bullet(T')} \delta_1(T) \le C_{\mathrm{patch}}C_{\mathrm{locuni}}\, \delta_1(T').$$

By continuity of $\widetilde{\delta}_\bullet$, this estimate is also satisfied if T'° is replaced by T', i.e., $\widetilde{\delta}_\bullet|_{T'} \le C_{\mathrm{patch}}C_{\mathrm{locuni}}\delta_1(T')$. The fact that $\chi_{B_{\delta_2(T')}(T')} * K_{\delta_2(T')} = 1$ on T' shows that also the converse estimate is valid. This leads to

$$\delta_1(T') \le \widetilde{\delta}_\bullet|_{T'} \le C_{\mathrm{patch}}C_{\mathrm{locuni}}\, \delta_1(T') \quad \text{for all } T' \in \mathcal{T}_\bullet. \tag{5.3.64}$$

In particular, this proves the existence of an open set $O \supset \Gamma$ such that $\widetilde{\delta}_\bullet > 0$ on O. Finally, we consider the gradient of $\widetilde{\delta}_\bullet$ for $x \in \Gamma$. Recall that $\operatorname{supp}(\chi_{B_{\delta_2(T)}(T)} * K_{\delta_2(T)}) \subseteq \pi_\bullet(T)$. Together with the Hölder inequality, $\|\nabla K_s\|_{L^1(\mathbb{R}^d)} \lesssim s^{-1}$, and (M1)–(M2), this proves that

$$\begin{aligned} |\nabla\widetilde{\delta}_\bullet(x)| &= \sum_{T\in\mathcal{T}_\bullet} \delta_1(T)\, \chi_{\pi_\bullet(T)}(x)\, |\chi_{B_{\delta_2(T)}(T)} * \nabla K_{\delta_2(T)}(x)| \\ &\lesssim \sum_{T\in\mathcal{T}_\bullet} \delta_1(T)\, \chi_{\pi_\bullet(T)}(x)\delta_2(T)^{-1} \lesssim 1. \end{aligned} \tag{5.3.65}$$

Step 2: We set $\widetilde{\mathcal{S}} := \bigcup \{B_{\delta_3(T)}(T) \,:\, T \in \mathcal{S}\}$. For $x \in O$, we define the quasi-convolution

$$\widetilde{\chi}_{\mathcal{S}}(x) := \int_{\mathbb{R}^d} \chi_{\widetilde{\mathcal{S}}}(y)\, K_{\widetilde{\delta}_\bullet(x)}(x-y)\, dy.$$

Since $\widetilde{\delta}_\bullet > 0$ on $O \supset \Gamma$, there holds that $\widetilde{\chi}_{\mathcal{S}} \in C^\infty(O)$, and thus $\widetilde{\chi}_{\mathcal{S}}|_\Gamma \in H^1(\Gamma)$; see, e.g., [McL00, pages 98–99]. In this step, we verify (5.3.60). Since $\operatorname{supp}(K_s) = B_s(0)$, there holds that

$$\widetilde{\chi}_{\mathcal{S}}(x) = \int_{B_{\widetilde{\delta}_\bullet(x)}(x)} \chi_{\widetilde{\mathcal{S}}}(y)\, K_{\widetilde{\delta}_\bullet(x)}(x-y)\, dy. \tag{5.3.66}$$

We observe that $\widetilde{\chi}_{\mathcal{S}}(x) = 1$ if $B_{\widetilde{\delta}_\bullet(x)}(x) \subseteq \widetilde{\mathcal{S}}$. Due to $\widetilde{\delta}_\bullet|_{T'} \le \delta_3(T')$ for all $T' \in \mathcal{T}_\bullet$ (which follows from (5.3.62) and (5.3.64)), this is particularly satisfied if $x \in \bigcup \mathcal{S}$. Moreover, (5.3.66) shows that $0 \le \widetilde{\chi}_{\mathcal{S}}(x) \le 1$ for all $x \in \mathbb{R}^d$, and $\widetilde{\chi}_{\mathcal{S}}(x) = 0$ if $B_{\widetilde{\delta}_\bullet(x)}(x) \cap \widetilde{\mathcal{S}} = \emptyset$. It remains to prove that $x \in \Gamma \setminus \pi_\bullet(\mathcal{S})$ implies that $B_{\widetilde{\delta}_\bullet(x)}(x) \cap \widetilde{\mathcal{S}} = \emptyset$. We prove the contraposition. Let $x \in \Gamma$ and suppose that $B_{\widetilde{\delta}_\bullet(x)}(x) \cap \widetilde{\mathcal{S}} \neq \emptyset$. Then, there exists $T \in \mathcal{S}$ and $y \in \mathbb{R}^d$ such that $|x-y| < \widetilde{\delta}_\bullet(x)$ and $\mathrm{dist}(y,T) < \delta_3(T)$. The triangle inequality yields that

$$\mathrm{dist}(x,T) \le |x-y| + \mathrm{dist}(y,T) < \widetilde{\delta}_\bullet(x) + \delta_3(T) \le 2\max(\widetilde{\delta}_\bullet(x), \delta_3(T)). \tag{5.3.67}$$

Now, we differ two different cases: If $\widetilde{\delta}_\bullet(x) \le \delta_3(T)$, then we have that $\mathrm{dist}(x,T) < 2\delta_3(T)$. The choice (5.3.62) of $\delta_3(T)$ together with (M4) shows that $x \in \pi_\bullet(T) \subseteq \pi_\bullet(\mathcal{S})$. If $\widetilde{\delta}_\bullet(x) > \delta_3(T)$, then we have that $\mathrm{dist}(x,T) \le 2\widetilde{\delta}_\bullet(x)$. Let $T' \in \mathcal{T}_\bullet$ with $x \in T'$ and $z \in T$ with $|x-z| = \mathrm{dist}(x,T)$. Together with (5.3.64) and (5.3.67), this yields that

$$\mathrm{dist}(z,T') \le |x-z| = \mathrm{dist}(x,T) < 2\widetilde{\delta}_\bullet(x) \le 2C_{\mathrm{patch}}C_{\mathrm{locuni}}\,\delta_1(T').$$

The choice (5.3.62) of $\delta_1(T')$ together with (M4) implies that even an open neighborhood of z is contained in $\pi_\bullet(T')$. We conclude that $T' \in \Pi_\bullet(T)$, and thus $x \in T' \subseteq \pi_\bullet(T)$.
Step 3: Finally, we prove (5.3.61). We recall that $\widetilde{\delta}_\bullet > 0$ on O; see Step 1. With the identity matrix $I \in \mathbb{R}^{d\times d}$ and the matrix $(x-y)(\nabla\widetilde{\delta}_\bullet(x))^\top \in \mathbb{R}^{d\times d}$, elementary calculations prove for all $x \in O \supset \Gamma$ and all $y \in \mathbb{R}^d$ that

$$\begin{aligned}\big(\nabla_x\big(K_{\widetilde{\delta}_\bullet(x)}(x-y)\big)\big)^\top = \Big(\nabla K_1\Big(\frac{x-y}{\widetilde{\delta}_\bullet(x)}\Big)\Big)^\top \frac{\widetilde{\delta}_\bullet(x)I - (x-y)(\nabla\widetilde{\delta}_\bullet(x))^\top}{\widetilde{\delta}_\bullet(x)^2}\,\widetilde{\delta}_\bullet(x)^{-d} \\ + K_1\Big(\frac{x-y}{\widetilde{\delta}_\bullet(x)}\Big)\widetilde{\delta}_\bullet(x)^{-d-1}(-d)(\nabla\widetilde{\delta}_\bullet(x))^\top.\end{aligned}$$

Considering the norm yields that

$$\big|\nabla_x\big(K_{\widetilde{\delta}_\bullet(x)}(x-y)\big)\big| \lesssim \widetilde{\delta}_\bullet(x)^{-d-1} + |x-y||\nabla\widetilde{\delta}_\bullet(x)|\,\widetilde{\delta}_\bullet(x)^{-d-2} + \widetilde{\delta}_\bullet(x)^{-d-1}|\nabla\widetilde{\delta}_\bullet(x)|.$$

Together with $\mathrm{supp}(K_s) = B_s(0)$, this shows for all $x \in \Gamma$ that

$$\begin{aligned}|\nabla\widetilde{\chi}_{\mathcal{S}}(x)| &= \Big|\int_{\mathbb{R}^d} \chi_{\widetilde{\mathcal{S}}}(y)\nabla_x\big(K_{\widetilde{\delta}_\bullet(x)}(x-y)\big)\,dy\Big| \\ &\lesssim \int_{B_{\widetilde{\delta}_\bullet(x)}(x)} \widetilde{\delta}_\bullet(x)^{-d-1} + |x-y||\nabla\widetilde{\delta}_\bullet(x)|\,\widetilde{\delta}_\bullet(x)^{-d-2} + \widetilde{\delta}_\bullet(x)^{-d-1}|\nabla\widetilde{\delta}_\bullet(x)|\,dy \\ &\lesssim \widetilde{\delta}_\bullet(x)^{-1}(1 + \|\nabla\widetilde{\delta}_\bullet\|_{L^\infty(\Gamma)}).\end{aligned}$$

Thus, (5.3.64)–(5.3.65), and (M2) prove that $|\nabla\widetilde{\chi}_{\mathcal{S}}(x)| \lesssim h_\bullet(x)^{-1}$ for almost all $x \in \Gamma$. Moreover, for smooth functions, the surface gradient ∇_Γ is the orthogonal projection of the gradient ∇ onto the tangent plane; see, e.g., [ME14, Lemma 2.22]). With the outer normal vector ν, this implies that $\nabla_\Gamma\widetilde{\chi}_{\mathcal{S}} = \nabla\widetilde{\chi}_{\mathcal{S}} - (\nabla\widetilde{\chi}_{\mathcal{S}}\cdot\nu)\nu$ almost everywhere on Γ, and concludes the proof with the previous estimate. $\square$

Remark 5.3.19. *For shape-regular triangular meshes as in [FKMP13, FFK+14], the proof of Lemma 5.3.18 simplifies a lot. Indeed, one can define $\widetilde{\chi}_{\mathcal{S}}$ with the help of standard hat functions on Γ; see [FKMP13, Section 5.3].*

Now, we prove discrete reliability (E4) in three steps.
Step 1: For $\mathcal{S}_1 := \mathcal{T}_\bullet \cap \mathcal{T}_\circ$, let $J_{\bullet,\mathcal{S}_1}$ be the corresponding projection operator from (S5)–(S6). Ellipticity (5.1.12), nestedness (S2) of the ansatz spaces, and the definition (5.2.14) of the Galerkin approximations yield that

$$\begin{aligned}\|\Phi_\circ - \Phi_\bullet\|_{H^{-1/2}(\Gamma)}^2 &\lesssim \langle \mathfrak{V}(\Phi_\circ - \Phi_\bullet)\,,\,\Phi_\circ - \Phi_\bullet\rangle_{L^2(\Gamma)}\\ &= \langle \mathfrak{V}(\phi - \Phi_\bullet)\,,\,(1 - J_{\bullet,\mathcal{S}_1})(\Phi_\circ - \Phi_\bullet)\rangle_{L^2(\Gamma)}.\end{aligned}$$

(S3) shows that $(\Phi_\circ - \Phi_\bullet)|_{\pi_\bullet^{\rm proj}(T)} \in \big\{\Psi_\bullet|_{\pi_\bullet^{\rm proj}(T)}\,:\,\Psi_\bullet \in \mathcal{X}_\bullet\big\}$ for any $T \in \mathcal{T}_\bullet \setminus \Pi_\bullet^{q_{\rm loc}}(\mathcal{T}_\bullet \setminus \mathcal{T}_\circ)$. Moreover, one easily sees (as in (4.5.13)) that

$$\Pi_\bullet^{q_{\rm loc}}(T) \subseteq \mathcal{T}_\bullet \cap \mathcal{T}_\circ = \mathcal{S}_1 \quad \text{for all } T \in \mathcal{T}_\bullet \setminus \Pi_\bullet^{q_{\rm loc}}(\mathcal{T}_\bullet \setminus \mathcal{T}_\circ). \tag{5.3.68}$$

Hence, the local projection property (S5) of $J_{\bullet,\mathcal{S}_1}$ is applicable and proves that $J_{\bullet,\mathcal{S}_1}(\Phi_\circ - \Phi_\bullet) = \Phi_\circ - \Phi_\bullet$ on $\Gamma \setminus \pi_\bullet^{q_{\rm loc}}(\mathcal{T}_\bullet \setminus \mathcal{T}_\circ)$. Altogether, we conclude with Lemma 5.3.18 and $\mathcal{S}_2 := \Pi_\bullet^{q_{\rm loc}}(\mathcal{T}_\bullet \setminus \mathcal{T}_\circ)$ that

$$\|\Phi_\circ - \Phi_\bullet\|_{H^{-1/2}(\Gamma)}^2 \lesssim \langle \widetilde{\chi}_{\mathcal{S}_2}\,\mathfrak{V}(\phi - \Phi_\bullet)\,,\,(1 - J_{\bullet,\mathcal{S}_1})(\Phi_\circ - \Phi_\bullet)\rangle_{L^2(\Gamma)}. \tag{5.3.69}$$

We bound the two terms $\langle \widetilde{\chi}_{\mathcal{S}_2}\,\mathfrak{V}(\phi - \Phi_\bullet)\,,\,\Phi_\circ - \Phi_\bullet\rangle_{L^2(\Gamma)}$ and $\langle \widetilde{\chi}_{\mathcal{S}_2}\,\mathfrak{V}(\phi - \Phi_\bullet)\,,\,J_{\bullet,\mathcal{S}_1}(\Phi_\circ - \Phi_\bullet)\rangle_{L^2(\Gamma)}$ separately. Since $H^{-1/2}(\Gamma)^D$ is the dual space of $H^{1/2}(\Gamma)^D$, there holds that

$$\langle \widetilde{\chi}_{\mathcal{S}_2}\,\mathfrak{V}(\phi - \Phi_\bullet)\,,\,\Phi_\circ - \Phi_\bullet\rangle_{L^2(\Gamma)} \le \|\widetilde{\chi}_{\mathcal{S}_2}\,\mathfrak{V}(\phi - \Phi_\bullet)\|_{H^{1/2}(\Gamma)}\|\Phi_\circ - \Phi_\bullet\|_{H^{-1/2}(\Gamma)}. \tag{5.3.70}$$

The Cauchy–Schwarz inequality shows that

$$\begin{aligned}&\langle \widetilde{\chi}_{\mathcal{S}_2}\,\mathfrak{V}(\phi - \Phi_\bullet)\,,\,J_{\bullet,\mathcal{S}_1}(\Phi_\circ - \Phi_\bullet)\rangle_{L^2(\Gamma)}\\ &\quad\le \|h_\bullet^{-1/2}\widetilde{\chi}_{\mathcal{S}_2}\,\mathfrak{V}(\phi - \Phi_\bullet)\|_{L^2(\Gamma)}\|h_\bullet^{1/2}J_{\bullet,\mathcal{S}_1}(\Phi_\circ - \Phi_\bullet)\|_{L^2(\Gamma)}.\end{aligned}$$

Since $J_{\bullet,\mathcal{S}_1} : L^2(\Gamma)^D \to \big\{\Psi_\bullet \in \mathcal{X}_\bullet\,:\,\Psi_\bullet|_{\bigcup(\mathcal{T}_\bullet\setminus\mathcal{S}_1)} = 0\big\}$, it holds that $\operatorname{supp}(J_{\bullet,\mathcal{S}_1}(\Phi_\circ - \Phi_\bullet)) \subseteq \bigcup(\mathcal{T}_\bullet \cap \mathcal{T}_\circ)$. This together with the fact that $h_\bullet = h_\circ$ on $\bigcup(\mathcal{T}_\bullet \cap \mathcal{T}_\circ)$, the local L^2-stability (S6) and (M1)–(M3) implies that

$$\begin{aligned}&= \|h_\bullet^{-1/2}\widetilde{\chi}_{\mathcal{S}_2}\,\mathfrak{V}(\phi - \Phi_\bullet)\|_{L^2(\Gamma)}\|h_\circ^{1/2}J_{\bullet,\mathcal{S}_1}(\Phi_\circ - \Phi_\bullet)\|_{L^2(\bigcup(\mathcal{T}_\bullet\cap\mathcal{T}_\circ))}\\ &\lesssim \|h_\bullet^{-1/2}\widetilde{\chi}_{\mathcal{S}_2}\,\mathfrak{V}(\phi - \Phi_\bullet)\|_{L^2(\Gamma)}\|h_\circ^{1/2}(\Phi_\circ - \Phi_\bullet)\|_{L^2(\Gamma)}.\end{aligned}$$

With the inverse inequality (S1) applied to $\Phi_\circ - \Phi_\bullet \in \mathcal{X}_\circ$ (see (S2)), the latter estimate implies that

$$\langle \widetilde{\chi}_{\mathcal{S}_2}\,\mathfrak{V}(\phi - \Phi_\bullet)\,,\,J_{\bullet,\mathcal{S}_1}(\Phi_\circ - \Phi_\bullet)\rangle_{L^2(\Gamma)} \lesssim \|h_\bullet^{-1/2}\widetilde{\chi}_{\mathcal{S}_2}\mathfrak{V}(\phi - \Phi_\bullet)\|_{L^2(\Gamma)}\|\Phi_\circ - \Phi_\bullet\|_{H^{-1/2}(\Gamma)}. \tag{5.3.71}$$

Plugging (5.3.70) and (5.3.71) into (5.3.69) shows that

$$\|\Phi_\circ - \Phi_\bullet\|_{H^{-1/2}(\Gamma)} \lesssim \|h_\bullet^{-1/2}\widetilde{\chi}_{\mathcal{S}_2}\mathfrak{V}(\phi - \Phi_\bullet)\|_{L^2(\Gamma)} + \|\widetilde{\chi}_{\mathcal{S}_2}\mathfrak{V}(\phi - \Phi_\bullet)\|_{H^{1/2}(\Gamma)}. \tag{5.3.72}$$

Step 2: Next, we deal with the first summand of (5.3.72). First, we note that $\operatorname{supp}(\widetilde{\chi}_{\mathcal{S}_2}) \subseteq \pi_\bullet^{q_{\rm loc}+1}(\mathcal{T}_\bullet \setminus \mathcal{T}_\circ)$ and $0 \le \widetilde{\chi}_{\mathcal{S}_2} \le 1$ (see (5.3.60)) imply that

$$\|h_\bullet^{-1/2}\widetilde{\chi}_{\mathcal{S}_2}\mathfrak{V}(\phi - \Phi_\bullet)\|_{L^2(\Gamma)} \le \|h_\bullet^{-1/2}\mathfrak{V}(\phi - \Phi_\bullet)\|_{L^2\left(\pi_\bullet^{q_{\rm loc}+1}(\mathcal{T}_\bullet\setminus\mathcal{T}_\circ)\right)}. \tag{5.3.73}$$

By Galerkin orthogonality (5.2.15), we see that $\mathfrak{V}(\phi - \Phi_\bullet)$ is L^2-orthogonal to all functions of $\mathcal{X}_\bullet$ which includes in particular the functions $\Psi_{\bullet,T,j}$ from (S4). Hence, we can apply Corollary 5.3.6. Together with (M1)–(M3) and recalling (5.3.59), we prove that

$$\|h_\bullet^{-1/2}\widetilde{\chi}_{\mathcal{S}_2}\mathfrak{V}(\phi - \Phi_\bullet)\|_{L^2(\Gamma)} \lesssim \|h_\bullet^{1/2}\nabla_\Gamma\mathfrak{V}(\phi - \Phi_\bullet)\|_{L^2\left(\pi_\bullet^{q_{\rm supp}+q_{\rm loc}+2}(\mathcal{T}_\bullet\setminus\mathcal{T}_\circ)\right)} = \eta_\bullet(\mathcal{R}_{\bullet,\circ}).$$

Step 3: It remains to consider the second summand of (5.3.72). Lemma 5.3.2 in conjunction with shape-regularity (M3) implies that

$$\|\widetilde{\chi}_{\mathcal{S}_2}\mathfrak{V}(\phi - \Phi_\bullet)\|_{H^{1/2}(\Gamma)}^2 \lesssim \sum_{T\in\mathcal{T}_\bullet}\sum_{T'\in\Pi_\bullet(T)} |\widetilde{\chi}_{\mathcal{S}_2}\mathfrak{V}(\phi - \Phi_\bullet)|_{H^{1/2}(T\cup T')}^2 + \|h_\bullet^{-1/2}\widetilde{\chi}_{\mathcal{S}_2}\mathfrak{V}(\phi - \Phi_\bullet)\|_{L^2(\Gamma)}.$$

We have already dealt with the second summand in Step 2. For the first one, we fix again some $z(T,T') \in T \cap T'$ for any $T \in \mathcal{T}_\bullet, T' \in \Pi_\bullet(T)$. (M1)–(M3) and (M5) show that

$$\begin{aligned}\sum_{T\in\mathcal{T}_\bullet}\sum_{T'\in\Pi_\bullet(T)} |\widetilde{\chi}_{\mathcal{S}_2}\mathfrak{V}(\phi - \Phi_\bullet)|_{H^{1/2}(T\cup T')}^2 &\le \sum_{T\in\mathcal{T}_\bullet}\sum_{T'\in\Pi_\bullet(T)} |\widetilde{\chi}_{\mathcal{S}_2}\mathfrak{V}(\phi - \Phi_\bullet)|_{H^{1/2}(\pi_\bullet(z(T,T')))}^2 \\ &\le \sum_{T\in\mathcal{T}_\bullet}\sum_{T'\in\Pi_\bullet(T)} \|h_\bullet^{1/2}\nabla_\Gamma(\widetilde{\chi}_{\mathcal{S}_2}\mathfrak{V}(\phi - \Phi_\bullet))\|_{L^2(\pi_\bullet(z(T,T'))}^2 \lesssim \|h_\bullet^{1/2}\nabla_\Gamma(\widetilde{\chi}_{\mathcal{S}_2}\mathfrak{V}(\phi - \Phi_\bullet))\|_{L^2(\Gamma)}^2.\end{aligned}$$

With the product rule and (5.3.61), we continue our estimate

$$\|\widetilde{\chi}_{\mathcal{S}_2}\mathfrak{V}(\phi - \Phi_\bullet)\|_{H^{1/2}(\Gamma)}^2 \lesssim \|h_\bullet^{-1/2}\mathfrak{V}(\phi - \Phi_\bullet))\|_{L^2(\operatorname{supp}(\widetilde{\chi}_{\mathcal{S}_2}))}^2 + \|h_\bullet^{1/2}\nabla_\Gamma\mathfrak{V}(\phi - \Phi_\bullet))\|_{L^2(\operatorname{supp}(\widetilde{\chi}_{\mathcal{S}_2}))}^2.$$

Note that we have already dealt with the first summand in Step 2 (see (5.3.73)). Finally, $\operatorname{supp}(\widetilde{\chi}_{\mathcal{S}_2}) \subseteq \pi_\bullet^{q_{\rm loc}+1}(\mathcal{T}_\bullet \setminus \mathcal{T}_\circ)$ (see (5.3.60)) and $\Pi_\bullet^{q_{\rm loc}+1}(\mathcal{T}_\bullet \setminus \mathcal{T}_\circ) \subseteq \mathcal{R}_{\bullet,\circ}$ (see (5.3.59)) prove for the second one that

$$\|h_\bullet^{1/2}\nabla_\Gamma\mathfrak{V}(\phi - \Phi_\bullet))\|_{L^2(\operatorname{supp}(\widetilde{\chi}_{\mathcal{S}_2}))}^2 \le \eta_\bullet(\mathcal{R}_{\bullet,\circ})^2.$$

With this, we conclude the proof of discrete reliability (E4). The constant $C_{\rm drel}$ depends only on C_ϱ, d, D, Γ, and the constants from (M1)–(M5) and (S1)–(S6).

5.3.9 Refinement axioms (T1)–(T3)

Clearly, the properties (R1), (R4), and (R5) are even slightly stronger versions of the axioms (T1)–(T3).

5.4 Boundary element method with hierarchical splines

In this section, we consider $\Omega \subset \mathbb{R}^d$ with $d \geq 3$. We introduce hierarchical splines on the boundary Γ and propose a local mesh-refinement strategy. To this end, we assume the existence of a mesh $\{\Gamma_m : m \in \{1, \dots, M\}\}$ of Γ in the sense of Section 5.2.1 such that each surface Γ_m can be parametrized over $\widehat{\Gamma}_m := [0,1]^{d-1}$. We use the notation from Section 3.4 (with an additional index m for the surface Γ_m), where we have already introduced hierarchical splines in the *parameter domain* $\widehat{\Gamma}_m$. The main result of this section is Theorem 5.4.5 which states that hierarchical splines together with the proposed mesh-refinement strategy fit into the abstract setting of Section 5.2 and are hence covered by Theorem 5.2.5. The proof of Theorem 5.4.5 is given in Section 5.5.

5.4.1 Parametrization of the boundary

We assume that for all $m \in \{1, \dots, M\}$, the surface Γ_m can be parametrized via a bi-Lipschitz mapping

$$\gamma_m : \widehat{\Gamma}_m \to \Gamma_m, \tag{5.4.1}$$

where $\widehat{\Gamma}_m = [0,1]^{d-1}$. In particular, Lemma 5.2.1 (applied on the interior of $\widehat{\Gamma}_m$) shows that γ_m is almost everywhere differentiable, and there exists a constant $C_\gamma > 0$ such that

$$C_\gamma^{-1} |s - t| \leq |\gamma_m(s) - \gamma_m(t)| \leq C_\gamma |s - t| \quad \text{for all } s, t \in \widehat{\Gamma}_m, \tag{5.4.2a}$$

and the Gram determinant satisfies that

$$C_\gamma^{-(d-1)} \leq \sqrt{\det(D\gamma_m^\top(t) D\gamma_m(t))} \leq C_\gamma^{d-1} \quad \text{for almost all } t \in \widehat{\Gamma}_m. \tag{5.4.2b}$$

We define the set of nodes $\mathcal{N}_\gamma := \bigcup_{m=1}^M \{\gamma_m(\widehat{z}) : \widehat{z} \in \{0,1\}^{d-1}\}$. We suppose that there are no (initial) hanging nodes, i.e., the intersection $\Gamma_m \cap \Gamma_{m'}$ with $m \neq m'$ is either empty or a common (transformed) lower-dimensional hyperrectangle $\gamma_m(\widehat{E}_m) = \gamma_{m'}(\widehat{E}_{m'})$, where $\widehat{E}_m$ and $\widehat{E}_{m'}$ are the convex hulls of at most $d-2$ points in $\{0,1\}^{d-1}$. Moreover, with the node patch $\pi_\gamma(z) := \bigcup \{\Gamma_m : m \in \{1, \dots, M\} \wedge z \in \Gamma_m\}$ for $z \in \mathcal{N}_\gamma$, we suppose the following compatibility assumption for the different parametrizations: For all nodes $z \in \mathcal{N}_\gamma$, there exists a polytope $\overline{\pi}_\gamma(z) \subset \mathbb{R}^{d-1}$, i.e., a polygon for $d = 3$ and a polyhedron for $d = 4$, and a bi-Lipschitz mapping

$$\gamma_z : \overline{\pi}_\gamma(z) \to \pi_\gamma(z) \tag{5.4.3}$$

such that $\gamma_z^{-1} \circ \gamma_m$ is an affine bijection for all $m \in \{1, \dots, M\}$ with $\Gamma_m \subseteq \pi_\gamma(z)$. Put into words, this means that each node patch $\pi_\gamma(z)$ can be flattened, where the corresponding bi-Lipschitz mapping restricted to any contained surface Γ_m essentially coincides with the parametrization γ_m. In particular, this prohibits the case $\pi_\gamma(z) = \Gamma$. The same assumption is also made in [SS11, Assumption 4.3.25] for curvilinear triangulations. It particularly implies that the parametrizations essentially coincide at the boundary of the surfaces, i.e., for all $m \neq m'$ with non-empty intersection $E := \Gamma_m \cap \Gamma_{m'} \neq \emptyset$, there holds with $\widehat{E}_m := \gamma_m^{-1}(E)$ and $\widehat{E}_{m'} := \gamma_{m'}^{-1}(E)$

$$\gamma_m|_{\widehat{E}_m} = \gamma_{m'} \circ \big(\gamma_{m'}^{-1} \circ \gamma_z \circ \gamma_z^{-1} \circ \gamma_m|_{\widehat{E}_m}\big), \tag{5.4.4}$$

where $\gamma_{m'}^{-1} \circ \gamma_z \circ \gamma_z^{-1} \circ \gamma_m|_{\widehat{E}_m} : \widehat{E}_m \to \widehat{E}_{m'}$ is an affine bijection. By possibly enlarging C_γ from (5.4.2), we can assume that

$$C_\gamma^{-1}|s-t| \leq |\gamma_z(s) - \gamma_z(t)| \leq C_\gamma |s-t| \quad \text{for all } s,t \in \overline{\pi}_\gamma(z). \tag{5.4.5}$$

5.4.2 Hierarchical meshes and splines on the boundary

For $m \in \{1,\dots,M\}$, let $(p_{1,m},\dots,p_{d-1,m})$ be a vector of fixed polynomial degrees in $\mathbb{N}$, and set

$$p_{\max} := \max\big\{p_{i,m} \,:\, i \in \{1,\dots,d\} \wedge m \in \{1,\dots,M\}\big\}. \tag{5.4.6}$$

Let

$$\widehat{\mathcal{K}}_{0,m} = (\widehat{\mathcal{K}}_{1(0,m)},\dots,\widehat{\mathcal{K}}_{(d-1)(0,m)}) \tag{5.4.7}$$

be a fixed initial $(d-1)$-dimensional vector of $p_{i,m}$-open knot vectors as in Section 3.3.2, where we additionally suppose that all interior knots $t_{i(0,m),j} \in (0,1)$ even satisfy that

$$\#_{i(0,m)} t_{i(0,m),j} \leq p_{i,m} \quad \text{for all } i \in \{1,\dots,d-1\}, j \in \{2+p_{i,m},\dots,N_{i(0,m)}-1\}. \tag{5.4.8}$$

For any corresponding hierarchical mesh $\widehat{\mathcal{T}}_{\bullet,m}$, we define $\widehat{\mathcal{X}}_{\bullet,m}$ as the space of all hierarchical splines on $\widehat{\Gamma}_m = [0,1]^{d-1}$ as

$$\begin{aligned}\widehat{\mathcal{X}}_{\bullet,m} &:= \widehat{\mathcal{S}}^{(p_{1,m},\dots,p_{d-1,m})}(\widehat{\mathcal{K}}_{0,m}, \widehat{\mathcal{T}}_{\bullet,m})^D \\ &\subset \big\{\widehat{\psi} \in C^0(\widehat{\Gamma}_m)^D \,:\, \widehat{\psi}|_T \in C^\infty(\widehat{T})^D \text{ for all } \widehat{T} \in \widehat{\mathcal{T}}_{\bullet,m}\big\}.\end{aligned} \tag{5.4.9}$$

In order to transform the definitions from the parameter domain $\widehat{\Gamma}_m$ to the boundary part Γ_m, we use the parametrizations from Section 5.4.1. All previous definitions can now also be made on each part Γ_m, just by pulling them from the parameter domain via the bi-Lipschitz mapping γ_m. For these definitions, we drop the symbol $\widehat{\cdot}$. If $\widehat{\mathcal{T}}_{\bullet,m}$ is a hierarchical mesh in the parameter domain $\widehat{\Gamma}_m$, we define the corresponding mesh on Γ_m as $\mathcal{T}_{\bullet,m} := \big\{\gamma_m(\widehat{T}) \,:\, \widehat{T} \in \widehat{\mathcal{T}}_{\bullet,m}\big\}$. In particular, we have that $\mathcal{T}_{0,m} = \big\{\gamma_m(\widehat{T}) \,:\, \widehat{T} \in \widehat{\mathcal{T}}_{0,m}\big\}$. Moreover, let $\mathbb{T}_m := \big\{\mathcal{T}_{\bullet,m} \,:\, \widehat{\mathcal{T}}_{\bullet,m} \in \widehat{\mathbb{T}}_m\big\}$ denote the set of all admissible meshes on Γ_m, where $\widehat{\mathbb{T}}_m$ is the set of all admissible hierarchical meshes on $\widehat{\Gamma}_m$ in the sense of Section 3.4.4. For an arbitrary hierarchical mesh $\mathcal{T}_{\bullet,m}$ on Γ_m, we introduce the corresponding hierarchical spline space

$$\mathcal{X}_{\bullet,m} := \big\{\widehat{\Psi}_{\bullet,m} \circ \gamma_m^{-1} \,:\, \widehat{\Psi}_{\bullet,m} \in \widehat{\mathcal{X}}_{\bullet,m}\big\} \subset C^0(\Gamma_m)^D \subset L^2(\Gamma_m)^D \subset H^{-1/2}(\Gamma_m)^D. \tag{5.4.10}$$

Finally, it remains to define hierarchical meshes and splines on Γ itself. This can be done by gluing the previous definitions for the single surfaces Γ_m together. For all $m \in$

$\{1,\dots,M\}$, let $\mathcal{T}_{\bullet,m}$ be a hierarchical mesh on Γ_m. We define the corresponding hierarchical mesh on Γ as $\mathcal{T}_\bullet := \bigcup_{m=1}^M \mathcal{T}_{\bullet,m}$. Clearly, $\mathcal{T}_\bullet$ is a mesh in the sense of Section 5.2.1, where

$$\widehat{T} := \gamma_m^{-1}(T) \quad \text{and} \quad \gamma_T := \gamma_m|_{\widehat{T}} \quad \text{for } T \in \mathcal{T}_{\bullet,m} \text{ with } m \in \{1,\dots,M\}, \tag{5.4.11}$$

and we can use the notation from there. For $T \in \mathcal{T}_\bullet$

$$\mathrm{level}(T) := \mathrm{level}(\widehat{T}). \tag{5.4.12}$$

We call $\mathcal{T}_\bullet$ *admissible* if the mesh satisfies the following two properties:

- All partial meshes are admissible, i.e., $\mathcal{T}_{\bullet,m} \in \mathbb{T}_m$ for all $m \in \{1,\dots,M\}$.
- There are no hanging nodes on the boundary of the surfaces Γ_m, i.e., the intersection $T \cap T'$ for $T \in \mathcal{T}_{\bullet,m}, T' \in \mathcal{T}_{\bullet,m'}$ with $m \neq m'$ is either empty or a common (transformed) lower-dimensional hyperrectangle.

We define the set of all admissible hierarchical meshes on Γ as $\mathbb{T}$, and suppose that the initial mesh on Γ is admissible, i.e.,

$$\mathcal{T}_0 = \bigcup_{m=1}^M \mathcal{T}_{0,m} \in \mathbb{T}. \tag{5.4.13}$$

For an arbitrary hierarchical mesh $\mathcal{T}_\bullet$ on Γ, the corresponding hierarchical splines read

$$\mathcal{X}_\bullet := \big\{\Psi_\bullet : \Gamma \to \mathbb{R}^D \,:\, \Psi_\bullet|_{\Gamma_m} \in \mathcal{X}_{\bullet,m} \text{ for all } m \in \{1,\dots,M\}\big\} \subset L^2(\Gamma)^D \subset H^{-1/2}(\Gamma)^D. \tag{5.4.14}$$

Remark 5.4.1. (a) *The property that there are no hanging nodes implies local quasi-uniformity at the boundaries $\partial\Gamma_m$, i.e., if $T \in \mathcal{T}_{\bullet,m}, T' \in \mathcal{T}_{\bullet,m'}$ with $m \neq m'$ have non-empty intersection $T \cap T' \neq \emptyset$, then* $|\mathrm{level}(T) - \mathrm{level}(T')| \le 1$. *Indeed, the intersection $T \cap T'$ of $T \in \mathcal{T}_{\bullet,m}, T' \in \mathcal{T}_{\bullet,m'}$ is either empty or a common (transformed) lower-dimensional hyperrectangle. Thus, if* $\dim(T \cap T') \ge 1$, *then* $\mathrm{level}(\widehat{T}) = \mathrm{level}(\widehat{T'})$. *If* $\dim(T \cap T') = 0$, *i.e., if $T \cap T'$ is only a point, there exists a sequence of elements $T_1 \in \mathcal{T}_{\bullet,m_1}, \dots, T_J \in \mathcal{T}_{\bullet,m_J}$ with $T_1 = T$ and $T_J = T'$ such that $m_j \neq m_{j+1}$ and* $\dim(T_j \cap T_{j+1}) \ge 1$ *for all $j \in \{1,\dots,J-1\}$. The previous argumentation yields that* $\mathrm{level}(\widehat{T}) = \mathrm{level}(\widehat{T'})$.

(b) *Since, the ansatz space only needs to be a subset of $H^{-1/2}(\Gamma)^D$, the property that there are no hanging nodes can actually be replaced by local quasi-uniformity at the boundaries $\partial\Gamma_m$ as in* (a)*, which is sufficient for the following analysis. However, for the* hyper-singular integral equation *which appears for the Neumann problem $\mathfrak{P}u = 0$ in Ω with $\mathfrak{D}_\nu u = \phi$ on Γ for some given $\phi \in H^{-1/2}(\Gamma)^D$; see, e.g., [McL00, pages 229–231], the ansatz functions must be in $H^{1/2}(\Gamma)^D$. In this case, the natural choice for the ansatz space is $\mathcal{X}_\bullet \cap C^0(\Gamma)^D$, which is even a subset of $H^1(\Gamma)^D$. If one supposes that there are no hanging nodes, one can define a local basis with the help of Proposition 4.5.3. The knowledge of such a basis is not only essential for an efficient implementation, but is also needed, e.g., for the definition of a quasi-interpolation operator.*

5.4.3 Refinement of hierarchical meshes

In this section, we present a concrete refinement algorithm to specify the setting of Section 5.2.2. We start in the parameter domain. Recall that we call a hierarchical mesh $\widehat{\mathcal{T}}_{\circ,m}$ finer than another hierarchical mesh $\widehat{\mathcal{T}}_{\bullet,m}$ if $\widehat{\Omega}^k_{\bullet,m} \subseteq \widehat{\Omega}^k_{\circ,m}$ for all $k \in \mathbb{N}_0$. This just means that $\widehat{\mathcal{T}}_{\circ,m}$ is obtained from $\widehat{\mathcal{T}}_{\bullet,m}$ by iterative dyadic bisections of the elements in $\widehat{\mathcal{T}}_{\bullet,m}$. To bisect an element $\widehat{T} \in \widehat{\mathcal{T}}_{\bullet,m}$, one just has to add it to the set $\widehat{\Omega}^{\mathrm{level}(\widehat{T})+1}_{\bullet,m}$; see (5.4.22) below. In this case, the corresponding spaces are nested according to (3.4.18), i.e.,

$$\widehat{\mathcal{X}}_{\bullet,m} \subseteq \widehat{\mathcal{X}}_{\circ,m}. \tag{5.4.15}$$

To transfer this definition onto the surface Γ_m for $m \in \{1, \dots, M\}$, we essentially just drop the symbol $\widehat{\cdot}$. We say that a hierarchical mesh $\mathcal{T}_{\circ,m}$ on Γ_m is *finer* than another hierarchical mesh $\mathcal{T}_{\bullet,m}$ on Γ_m, if the corresponding meshes in the parameter domain satisfy this relation, i.e., if $\widehat{\mathcal{T}}_{\circ,m}$ is finer than $\widehat{\mathcal{T}}_{\bullet,m}$. In this case, there holds that

$$\mathcal{X}_{\bullet,m} \subseteq \mathcal{X}_{\circ,m}. \tag{5.4.16}$$

Finally, we call a hierarchical mesh $\mathcal{T}_\circ$ on Γ *finer* than another hierarchical mesh $\mathcal{T}_\bullet$ on Γ, if the corresponding partial meshes satisfy this relation, i.e., if $\mathcal{T}_{\circ,m}$ is finer than $\mathcal{T}_{\bullet,m}$ for all $m \in \{1, \dots, M\}$. In this case, there holds that

$$\mathcal{X}_\bullet \subseteq \mathcal{X}_\circ. \tag{5.4.17}$$

Let $\mathcal{T}_\bullet$ be a hierarchical mesh and $T \in \mathcal{T}_\bullet$, and hence $T \in \mathcal{T}_{\bullet,m}$ for some $m \in \{1, \dots, M\}$. Moreover, let $\widehat{T} = \gamma_m^{-1}(T)$ be the corresponding element in the parameter domain. We define the sets of *neighbors*

$$\mathbf{N}_{\bullet,m}(T) := \big\{\gamma_m(\widehat{T}') \,:\, \widehat{T}' \in \mathbf{N}_{\bullet,m}(\widehat{T})\big\}, \tag{5.4.18}$$

where $\mathbf{N}_{\bullet,m}(\widehat{T}) = \big\{\widehat{T}' \in \widehat{\mathcal{T}}_{\bullet,m} \,:\, \exists \widehat{\beta} \in \widehat{\mathcal{B}}_{\bullet,m} \quad \widehat{T}, \widehat{T}' \subseteq \mathrm{supp}(\widehat{\beta})\big\}$ is the set from (3.4.24), and

$$\mathbf{N}_\bullet(T) := \mathbf{N}_{\bullet,m}(T) \cup \bigcup_{m' \neq m} \big\{T' \in \mathcal{T}_{\bullet,m} \,:\, T \cap T' \neq \emptyset\big\}. \tag{5.4.19}$$

Further, we define the sets of *bad neighbors*

$$\mathbf{N}^{\mathrm{bad}}_{\bullet,m}(T) := \big\{\gamma_m(\widehat{T}') \,:\, \widehat{T}' \in \mathbf{N}^{\mathrm{bad}}_{\bullet,m}(\widehat{T})\big\}, \tag{5.4.20}$$

where $\mathbf{N}^{\mathrm{bad}}_{\bullet,m}(\widehat{T}) = \big\{\widehat{T}' \in \mathbf{N}_{\bullet,m}(\widehat{T}) \,:\, \mathrm{level}(\widehat{T}') = \mathrm{level}(\widehat{T}) - 1\big\}$ is the set from (4.4.9), and

$$\mathbf{N}^{\mathrm{bad}}_\bullet(T) := \mathbf{N}^{\mathrm{bad}}_{\bullet,m}(T) \cup \bigcup_{m' \neq m} \big\{T' \in \mathcal{T}_{\bullet,m} \,:\, \dim(T \cap T') > 0\big\} \tag{5.4.21}$$

Algorithm 5.4.2. ***Input:*** *Hierarchical mesh* $\mathcal{T}_\bullet$ *, marked elements* $\mathcal{M}_\bullet =: \mathcal{M}^{(0)}_\bullet \subseteq \mathcal{T}_\bullet$.

(i) *Iterate the following steps* (a)–(b) *for* $i = 0, 1, 2, \dots$ *until* $\mathcal{U}^{(i)}_\bullet = \emptyset$*:*

(a) *Define* $\mathcal{U}_\bullet^{(i)} := \bigcup_{T\in\mathcal{M}_\bullet^{(i)}} \{T' \in \mathcal{T}_\bullet \setminus \mathcal{M}_\bullet^{(i)} : T' \in \mathbf{N}_\bullet^{\mathrm{bad}}(T)\}$.

(b) *Define* $\mathcal{M}_\bullet^{(i+1)} := \mathcal{M}_\bullet^{(i)} \cup \mathcal{U}_\bullet^{(i)}$.

(ii) *Dyadically bisect all* $T \in \mathcal{M}_\bullet^{(i)}$ *in the parameter domain by adding the corresponding* $\widehat{T} \in \widehat{\mathcal{T}}_{\bullet,m}$ *to the set* $\widehat{\Omega}_{\bullet,m}^{\mathrm{level}(\widehat{T})+1}$ *and obtain a finer hierarchical mesh* $\mathcal{T}_\circ$, *where for all* $m \in \{1,\dots,M\}$ *and* $k \in \mathbb{N}$

$$\widehat{\Omega}_{\circ,m}^k = \widehat{\Omega}_{\bullet,m}^k \cup \bigcup \{\widehat{T} \in \widehat{\mathcal{T}}_{\bullet,m} : \gamma_m(\widehat{T}) \in \mathcal{M}_\bullet^{(i)} \wedge \mathrm{level}(\widehat{T}) = k-1\}. \tag{5.4.22}$$

Output: *Refined mesh* $\mathcal{T}_\circ = \mathtt{refine}(\mathcal{T}_\bullet, \mathcal{M}_\bullet)$.

Clearly, $\mathtt{refine}(\mathcal{T}_\bullet, \mathcal{M}_\bullet)$ is finer than $\mathcal{T}_\bullet$. For any hierarchical mesh $\mathcal{T}_\bullet$ on Γ, we define $\mathtt{refine}(\mathcal{T}_\bullet)$ as the set of all hierarchical meshes $\mathcal{T}_\circ$ on Γ such that there exist hierarchical meshes $\mathcal{T}_{(0)},\dots,\mathcal{T}_{(J)}$ and marked elements $\mathcal{M}_{(0)},\dots,\mathcal{M}_{(J-1)}$ with $\mathcal{T}_\circ = \mathcal{T}_{(J)} = \mathtt{refine}(\mathcal{T}_{(J-1)}, \mathcal{M}_{(J-1)}),\dots,\mathcal{T}_{(1)} = \mathtt{refine}(\mathcal{T}_{(0)}, \mathcal{M}_{(0)})$, and $\mathcal{T}_{(0)} = \mathcal{T}_\bullet$. Note that $\mathtt{refine}(\mathcal{T}_\bullet, \emptyset) = \mathcal{T}_\bullet$, wherefore $\mathcal{T}_\bullet \in \mathtt{refine}(\mathcal{T}_\bullet)$. The following proposition characterizes the set $\mathtt{refine}(\mathcal{T}_\bullet)$. In particular, it shows that $\mathtt{refine}(\mathcal{T}_0) = \mathbb{T}$.

Proposition 5.4.3. *If* $\mathcal{T}_\bullet \in \mathbb{T}$, *then* $\mathtt{refine}(\mathcal{T}_\bullet)$ *coincides with the set of all admissible hierarchical meshes* $\mathcal{T}_\circ$ *that are finer than* $\mathcal{T}_\bullet$.

Proof. The proof is achieved similarly as that of Proposition 4.4.2. Therefore, we mainly focus on the differences.

Step 1: We show that $\mathcal{T}_\circ := \mathtt{refine}(\mathcal{T}_\bullet, \mathcal{M}_\bullet) \in \mathbb{T}$ for arbitrary marked elements $\mathcal{M}_\bullet \subseteq \mathcal{T}_\bullet$. As in Step 1 of the proof of Proposition 4.4.2, one derives that $\mathcal{T}_{\circ,m} \in \mathbb{T}_m$ for all $m \in \{1,\dots,M\}$. Thus, it remains to verify the compatibility conditions across the boundary of the surfaces Γ_m. Let $T, T' \in \mathcal{T}_\circ$ with non-empty intersection and $T \subseteq \Gamma_m, T' \subseteq \Gamma_{m'}$, where $m, m' \in \{1,\dots,M\}$ with $m \neq m'$. We consider four different cases.

(i) Let $T, T' \in \mathcal{T}_\bullet \cap \mathcal{T}_\circ$. Since $\mathcal{T}_\bullet$ is admissible, one immediately sees that the intersection of T and T' is a common (transformed) lower-dimensional hyperrectangle.

(ii) Let $T, T' \in \mathcal{T}_\circ \setminus \mathcal{T}_\bullet$. Then, there are $T_\bullet, T'_\bullet \in \mathcal{T}_\bullet$ with $T \subsetneqq T_\bullet \subseteq \Gamma_m$ and $T' \subsetneqq T'_\bullet \subseteq \Gamma_{m'}$. Admissibility of $\mathcal{T}_\bullet$ shows that $T_\bullet$ and $T'_\bullet$ share a common (transformed) lower-dimensional hyperrectangle. Due to (5.4.4), and since T resp. T' results from one single bisection of $T_\bullet$ resp. $T'_\bullet$, this property holds as well for T and T'.

(iii) Let $T \in \mathcal{T}_\circ \setminus \mathcal{T}_\bullet$ and $T' \in \mathcal{T}_\bullet \cap \mathcal{T}_\circ$. Then, there is $T_\bullet \in \mathcal{T}_\bullet$ with $T \subsetneqq T_\bullet \subseteq \Gamma_m$. By admissibility of $\mathcal{T}_\bullet$, $T_\bullet$ and T' share a common (transformed) lower-dimensional hyperrectangle. It is not possible that they share more than one single point, since otherwise the refinement of $T_\bullet$ would also lead to the refinement of $T' \in \mathcal{T}_\bullet$.

(iv) Let $T \in \mathcal{T}_\bullet \cap \mathcal{T}_\circ$ and $T' \in \mathcal{T}_\circ \setminus \mathcal{T}_\bullet$. Clearly, this case can be treated as case (iii).

Step 2: It is clear that an arbitrary $\mathcal{T}_\circ \in \mathtt{refine}(\mathcal{T}_\bullet)$ is finer than $\mathcal{T}_\bullet$. By induction, Step 1 concludes the inclusion $\mathtt{refine}(\mathcal{T}_\bullet) \subseteq \mathbb{T}$.

Step 3: To show the converse inclusion, let $\mathcal{T}_\circ$ be an admissible mesh which is finer than $\mathcal{T}_\bullet$. We prove that $\mathcal{T}_\circ$ is finer than $\mathcal{T}_\star := \texttt{refine}(\mathcal{T}_\bullet, \{T\})$, where $T \in \mathcal{T}_\bullet \setminus \mathcal{T}_\circ$ is arbitrary. We suppose that the assertion is false and lead this to contradiction. As in Step 3 from the proof of Proposition 4.4.2, one derives the existence of some $T' \in (\mathcal{T}_\bullet \setminus \mathcal{T}_\star) \cap \mathcal{T}_\circ$ such that $T' \in \mathbf{N}_\bullet^{\rm bad}(T'')$ for some $T'' \in \mathcal{T}_\bullet \setminus (\mathcal{T}_\star \cup \mathcal{T}_\circ)$.

The definition of $\mathbf{N}_\bullet^{\rm bad}(T'')$ implies that either $T' \in \mathbf{N}_{\bullet,m}^{\rm bad}(T'')$ with $T', T'' \subseteq \Gamma_{m'}$, or T' and T'' are contained in different boundary parts $\Gamma_{m'}$ and $\Gamma_{m''}$ and they share a common lower-dimensional hyperrectangle. The first case can be treated exactly as in Step 3 of the corresponding proof for FEM. For the second case, let $T''_\circ \in \mathcal{T}_\circ$ with $T''_\circ \subsetneqq T''$ such that the intersection of T' and $T''_\circ$ is non-empty as well. We see that $T' \cap T''_\circ$ is not a common (transformed) lower-dimensional hyperrectangle. Since $T', T''_\circ \in \mathcal{T}_\circ$, this finally contradicts the admissibility of $\mathcal{T}_\circ$.

Step 4: With Step 2–3, one concludes the remaining inclusion as in Step 4 from the proof of Proposition 4.4.2. ☐

Remark 5.4.4. *In Section 5.5.11, we will give a basis of (transformed) hierarchical B-splines and truncated hierarchical B-splines for $\mathcal{X}_\bullet$ on $\mathcal{T}_\bullet \in \mathbb{T}$. Actually, the proposed refinement strategy of Algorithm 5.4.2 was designed for hierarchical B-splines; see also Proposition 3.4.3. However,* (3.4.22) *implies that Proposition 3.4.3 holds accordingly for truncated hierarchical B-splines. Moreover, if one applies the refinement strategy of Algorithm 5.4.2,* (3.4.28) *shows that the computation of the truncated hierarchical B-splines simplifies significantly.*

5.4.4 Optimal convergence for hierarchical splines

Altogether, we have specified the abstract framework of Section 5.2 to hierarchical meshes and splines. The following theorem is the second main result of this chapter. It shows that all assumptions of Theorem 5.2.5 are satisfied for the present hierarchical approach. The proof is given in Section 5.5.

Theorem 5.4.5. *Hierarchical splines on admissible meshes satisfy the abstract assumptions* (M1)–(M5), (R1)–(R5), *and* (S1)–(S6) *from Section 5.2, where the constants depend only on the dimensions d, D, the (fixed) number M of boundary parts Γ_m, the parametrizations γ_m and γ_z, the initial meshes $\widehat{\mathcal{T}}_{0,m}$, and the polynomial orders $(p_{1,m}, \dots, p_{d-1,m})$ for $m \in \{1, \dots, M\}$ and $z \in \mathcal{N}_\gamma$. By Theorem 5.2.5, this implies reliability* (5.2.22) *of the error estimator, and linear convergence* (5.2.23) *at optimal rate* (5.2.24) *for the adaptive strategy from Algorithm 5.2.4.*

Remark 5.4.6. (a) *Theorem 5.4.5 is still valid if one replaces the ansatz space $\mathcal{X}_\bullet$ by* rational hierarchical splines, *i.e., by the set*

$$\mathcal{X}_\bullet^{W_0} := \Big\{ W_0^{-1} \Psi_\bullet : \Psi_\bullet \in \mathcal{X}_\bullet \Big\}, \tag{5.4.23}$$

where $\widehat{W}_{0,m} := W_0 \circ \gamma_m$ is a fixed positive weight function in the initial space of hierarchical splines $\widehat{\mathcal{S}}^{(p_{1,m},\dots,p_{d-1,m})}(\widehat{\mathcal{K}}_{0,m}, \widehat{\mathcal{T}}_{0,m})$ for all $m \in \{1, \dots, M\}$. With the B-spline basis $\widehat{\mathcal{B}}_{0,m}$

on $\widehat{\mathcal{T}}_{0,m}$, we additionally suppose that $\widehat{W}_{0,m}$ can be written as

$$\widehat{W}_{0,m} = \sum_{\widehat{\beta}\in\widehat{\mathcal{B}}_{0,m}} w_{0,m,\widehat{\beta}}\,\widehat{\beta} \quad \text{with non-negative coefficients } w_{0,m,\widehat{\beta}} \geq 0. \tag{5.4.24}$$

We will prove this generalization in Section 5.5.15. In this case, the constants depend additionally on W_0.

(b) *Moreover, Theorem 5.4.5 still holds true if newly inserted knots have a multiplicity higher than one, i.e., if one uses, as in Remark 3.4.1, the uniformly refined knots $\widehat{\mathcal{K}}_{\mathrm{uni}(k,(q_{1,m},\dots,q_{d-1,m})),m}$ with $1 \leq q_{i,m} \leq p_{i,m}$ instead of $\widehat{\mathcal{K}}_{\mathrm{uni}(k),m}$ to define (rational) hierarchical splines. The corresponding proof works verbatim.*

(c) *Finally, if one defines for an element $\widehat{T}$ of a hierarchical mesh $\widehat{\mathcal{T}}_{\bullet,m}$ its neighbours $\mathbf{N}_{\bullet,m}(\widehat{T})$ as in Remark 3.4.6, and adapts the definition of admissibility and* `refine`$(\cdot,\cdot)$ *accordingly, one can also allow for lowest-order polynomial degrees $p_{i,m} \in \mathbb{N}_0$ as well as full knot multiplicities $q_{i,m} = p_{i,m} + 1$.*

5.5 Proof of Theorem 5.4.5

This section is devoted to the proof of Theorem 5.4.5, i.e., to the verification of the mesh properties (M1)–(M5), the refinement properties (R1)–(R5), and the boundary element space properties (S1)–(S6).

5.5.1 Verification of (M1)

Let $\mathcal{T}_\bullet \in \mathbb{T}$ and $T \in \mathcal{T}_\bullet$. We split the patch as follows

$$\Pi_\bullet(T) = \bigcup_{m=1}^{M} (\Pi_\bullet(T) \cap \mathcal{T}_{\bullet,m}) \subseteq \bigcup_{m=1}^{M} \big\{\Pi_\bullet(E) \cap \mathcal{T}_{\bullet,m} \,:\, E \text{ lower-dim. hyperrect. of } T\big\}.$$

If $\Pi_\bullet(E) \cap \mathcal{T}_{\bullet,m} \neq \emptyset$, there exists $T' \in \mathcal{T}_{\bullet,m}$ with $\emptyset \neq E \cap T' \subseteq T \cap T'$. By admissibility, this implies that E is even a common (transformed) lower-dimensional hyperrectangle of T and T'. In particular, this leads to $\Pi_\bullet(E) \cap \mathcal{T}_{\bullet,m} \subseteq \Pi_\bullet(T') \cap \mathcal{T}_{\bullet,m} = \Pi_{\bullet,m}(T')$, where $\Pi_{\bullet,m}(T') := \big\{\gamma_m(\widehat{T}) \,:\, \widehat{T} \in \Pi_{\bullet,m}(\widehat{T}')\big\}$. Since $\mathcal{T}_{\bullet,m} \in \mathbb{T}_m$, one sees as in Section 4.5.1 that $\#\Pi_{\bullet,m}(T') \lesssim 1$. Altogether, we derive that $\#\Pi_\bullet(T) \leq C_{\mathrm{patch}}$ with a constant C_{patch} which depends only on d and M.

5.5.2 Verification of (M2)

Let $\mathcal{T}_\bullet \in \mathbb{T}$ and $T, T' \in \mathcal{T}_\bullet$ with $T \cap T' \neq \emptyset$. If both T and T' are in the same boundary part, i.e., $T, T' \subseteq \Gamma_m$ for some $m \in \{1,\dots,M\}$, one sees as in Section 4.5.1 that $\mathrm{diam}(\widehat{T}) \simeq \mathrm{diam}(\widehat{T}')$ for $\widehat{T} = \gamma_m^{-1}(T)$ and $\widehat{T}' = \gamma_m^{-1}(T')$. Bi-Lipschitz continuity of γ_m yields that $\mathrm{diam}(T) \simeq \mathrm{diam}(T')$. Otherwise, let $m, m' \in \{1,\dots,M\}$ with $m \neq m'$, $T \subseteq \Gamma_m$ and $T' \subseteq \Gamma_{m'}$. By admissibility of $\mathcal{T}_\bullet$ and Remark 5.4.1, we see that $\mathrm{level}(T) = \mathrm{level}(T')$. This implies that $\mathrm{diam}(\widehat{T}) \simeq \mathrm{diam}(\widehat{T}')$ for $\widehat{T} = \gamma_{m'}^{-1}(T)$ and $\widehat{T}' = \gamma_m^{-1}(T')$, and thus $\mathrm{diam}(T) \simeq \mathrm{diam}(T')$ due to bi-Lipschitz continuity of γ_m and $\gamma_{m'}$. This concludes local quasi-uniformity (M2), where the constant C_{locuni} depends only the dimension d, the constant C_γ, and the initial meshes $\widehat{\mathcal{T}}_{0,m}$ for $m \in \{1,\dots,M\}$.

5.5.3 Verification of (M3)

Let $\mathcal{T}_\bullet \in \mathbb{T}$, $T \in \mathcal{T}_\bullet$, and $m \in \{1, \dots, M\}$ with $T \subseteq \Gamma_m$. We abbreviate $\widehat{T} := \gamma_m^{-1}(T)$. As the refinement procedure `refine` only uses uniform bisection of an element in the parameter domain, we see that $\operatorname{diam}(\widehat{T})^{d-1} \simeq |\widehat{T}|$, where the hidden constants depend only on the dimension d and the initial mesh $\widehat{\mathcal{T}}_{0,m}$. Since γ_m is bi-Lipschitz, we see that $\operatorname{diam}(\widehat{T}) \simeq \operatorname{diam}(T)$. Moreover, (5.4.2) shows that $|\widehat{T}| \simeq |T|$. Altogether, we conclude that $\operatorname{diam}(T)^{d-1} \simeq |T|$, where the hidden constants depend only on d, C_γ, and $\widehat{\mathcal{T}}_{0,m}$.

5.5.4 Verification of (M4)

Let $\mathcal{T}_\bullet \in \mathbb{T}$, $T \in \mathcal{T}_\bullet$, and $m \in \{1, \dots, M\}$ with $T \subseteq \Gamma_m$. We show that there exists $r \simeq \operatorname{diam}(T)$ with $B_r(T) \cap \Gamma \subset \pi_\bullet(T)$, which concludes (M4).

Step 1: According to Lemma 3.4.5 and Remark 5.4.1, admissibility $\mathcal{T}_\bullet \in \mathbb{T}$ shows that $|\operatorname{level}(T) - \operatorname{level}(T')| \le 1$ for all $T' \in \Pi_\bullet(T)$. Since we only use dyadic bisection, there exists an upper bound for the number of possible configurations of T and $\pi_\bullet(T)$ depending only on the initial meshes $\widehat{\mathcal{T}}_{0,m'}$ and (an upper bound for) $\operatorname{level}(T)$. In particular, this implies that $\operatorname{diam}(T) \lesssim \operatorname{dist}(T, \Gamma \setminus \pi_\bullet(T))$, but the hidden constant still depends on (an upper bound for) $\operatorname{level}(T)$. We see that it only remains to consider small elements T with high level.

Step 2: In this step, we show that there exists $z \in \mathcal{N}_\gamma$ and a generic constant $C > 0$ such that $\pi_\bullet(T) \subseteq \pi_\gamma(z)$ and $B_C(T) \cap \Gamma \subseteq \pi_\gamma(z)$ if $\operatorname{level}(T)$ is sufficiently high. Without loss of generality, we assume that $T \cap \gamma_m([0,1/2]^{d-1}) \neq \emptyset$ and set $z := \gamma_m(0)$. Note that $\gamma_m([0,1/2]^{d-1}) \cap \mathcal{N}_\gamma = \{z\}$. Since we assumed that the surfaces have no hanging nodes, $z \notin \Gamma_{m'}$ implies that $\Gamma_{m'} \cap \gamma_m([0,1/2]^{d-1}) = \emptyset$ for all $m' \in \{1, \dots, M\}$. We abbreviate

$$C_m := \min_{\substack{m' \in \{1,\dots,M\} \\ z \notin \Gamma_{m'}}} \operatorname{dist}\big(\Gamma_{m'}, \gamma_m([0,1/2]^{d-1})\big) = \operatorname{dist}(\Gamma \setminus \pi_\gamma(z), \gamma_m([0,1/2]^{d-1}) > 0.$$

Let $\operatorname{level}(T)$ be sufficiently high such that $\operatorname{diam}(\pi_\bullet(T)) < C_m$, which is possible due to (M1)–(M2). Note that this choice depends only on the dimension d, the constant C_γ, and the initial meshes. With the assumption that $T \cap \gamma_m([0,1/2]^{d-1}) \neq \emptyset$, we derive that $\pi_\bullet(T) \subseteq \pi_\gamma(z)$. The same argument proves that $B_{C_m/2}(T) \cap \Gamma \subseteq \pi_\gamma(z)$ if $\operatorname{diam}(T) < C_m/2$.

Step 3: Due to Step 2, we may consider the set $\gamma_z^{-1}(\pi_\bullet(T)) \subseteq \overline{\pi}_\gamma(z)$ provided that $\operatorname{level}(T)$ is sufficiently high. Recall that $|\operatorname{level}(T) - \operatorname{level}(T')| \le 1$ for all $T' \in \Pi_\bullet(T)$; see Step 1. With the assumptions for the mapping γ_z of Section 5.4.1, and since we only use dyadic bisection, we see that the number of possible shapes of $\gamma_z^{-1}(T)$ and $\gamma_z^{-1}(\pi_\bullet(T))$ is uniformly bounded. In particular, there exists $\overline{r} \simeq \operatorname{diam}(\gamma_z^{-1}(T))$ with $B_{\overline{r}}(\gamma_z^{-1}(T)) \subseteq \gamma_z^{-1}(\pi_\bullet(T))$. Bi-Lipschitz continuity of γ_z and Step 2 yield the existence of $r \simeq \operatorname{diam}(T)$ with $r < C_m/2$ such that

$$B_r(T) \cap \Gamma = B_r(T) \cap \pi_\gamma(z) \subset \pi_\bullet(T).$$

Together with Step 1, this concludes (M4), where the constant C_{cent} depends only on the dimension d, the parametrizations γ_m and γ_z, and the initial meshes $\widehat{\mathcal{T}}_{0,m}$ for $m \in \{1, \dots, M\}$ and $z \in \mathcal{N}_\gamma$.

5.5.5 Verification of (M5)

We show that there are only finitely many reference point patches. Then, Proposition 5.2.2 will conclude (M5). Let $\mathcal{T}_\bullet \in \mathbb{T}$ and $z \in \Gamma$. According to Lemma 3.4.5 and Remark 5.4.1, admissibility $\mathcal{T}_\bullet \in \mathbb{T}$ shows that $|\mathrm{level}(T) - \mathrm{level}(T')| \le 1$ for all $T' \in \Pi_\bullet(z)$. Since there are no hanging nodes in $\mathcal{N}_\gamma$, there exists $z' \in \mathcal{N}_\gamma$ such that $\pi_\bullet(z) \subseteq \pi_\gamma(z')$. With the assumptions for the mapping $\gamma_{z'}$ of Section 5.4.1, and since we only use dyadic bisection, we see that the number of possible shapes of $\overline{\pi}_\bullet(z) := \gamma_{z'}^{-1}(\pi_\bullet(z)) \subseteq \overline{\pi}_\gamma(z')$ is uniformly bounded. More precisely, there exists a finite set $\{\widehat{\omega}_j : j \in \{1,\dots,J\}\}$ of connected subsets $\widehat{\omega}_j \subset \mathbb{R}^{d-1}$ such that for arbitrary $z \in \Gamma$ and corresponding $z' \in \mathcal{N}_\gamma$ there exist $j \in \{1,\dots,J\}$ and an affine bijection $\gamma_{\overline{\pi}_\bullet(z)} : \widehat{\omega}_j \to \overline{\pi}_\bullet(z)$ with

$$\frac{|\gamma_{\overline{\pi}_\bullet(z)}(s) - \gamma_{\overline{\pi}_\bullet(z)}(t)|}{\mathrm{diam}(\overline{\pi}_\bullet(z))} \simeq |s-t| \quad \text{for all } s,t \in \widehat{\omega}_j. \tag{5.5.1}$$

Since $\gamma_{z'}$ is bi-Lipschitz, there holds that $\mathrm{diam}(\overline{\pi}_\bullet(z)) \simeq \mathrm{diam}(\pi_\bullet(z))$, and we see for the mapping $\gamma_{\pi_\bullet(z)} := \gamma_{z'} \circ \gamma_{\overline{\pi}_\bullet(z)}$ that

$$\frac{|\gamma_{\pi_\bullet(z)}(s) - \gamma_{\pi_\bullet(z)}(t)|}{\mathrm{diam}(\pi_\bullet(z))} \simeq |s-t| \quad \text{for all } s,t \in \widehat{\omega}_j. \tag{5.5.2}$$

Thus, the application of Proposition 5.2.2 (on the interior of $\widehat{\omega}_j$) concludes (M5).

The constant C_{semi} depends only on the dimension d, the parametrizations γ_m and $\gamma_{z'}$, and the initial meshes $\widehat{\mathcal{T}}_{0,m}$ for $m \in \{1,\dots,M\}$ and $z' \in \mathcal{N}_\gamma$.

5.5.6 Verification of (R1)–(R3)

The son estimate (R1) is trivially satisfied with $C_{\mathrm{son}} = 2^{d-1}$ since each refined element is split into exactly 2^{d-1} sons. The union of sons property (R2) holds by definition. Finally, the proof of (R3) works just as in Section 4.5.3, where one now has to use (5.4.2) instead of (4.4.2). The constant ρ_{son} depends only on d and C_γ.

5.5.7 Verification of (R4)

We imitate the proof of Section 4.5.4. For a mesh $\mathcal{T}_\bullet \in \mathbb{T}$ and an arbitrary set $\mathcal{M}_\bullet$, we define $\texttt{refine}(\mathcal{T}_\bullet, \mathcal{M}_\bullet) := \texttt{refine}(\mathcal{T}_\bullet, \mathcal{M}_\bullet \cap \mathcal{T}_\bullet)$ and note that $\texttt{refine}(\mathcal{T}_\bullet, \emptyset) = \mathcal{T}_\bullet$. With this notation, Lemma 4.5.1 is also valid in the current situation. Recalling the definitions (5.4.19) resp. (5.4.21) of $\mathbf{N}_\bullet(\cdot)$ resp. $\mathbf{N}_\bullet^{\mathrm{bad}}(\cdot)$, the proof can be essentially copied.

Lemma 5.5.1. *Let $\mathcal{T}_\bullet \in \mathbb{T}$. Then, there holds that*

$$\texttt{refine}(\mathcal{T}_\bullet, \mathcal{M}_\bullet) = \texttt{refine}(\texttt{refine}(\dots(\texttt{refine}(\mathcal{T}_\bullet, \{T_1\})\dots, \{T_{n-1}\}), \{T_n\}) \tag{5.5.3}$$

for arbitrary $\mathcal{M}_\bullet = \{T_1,\dots,T_n\} \subseteq \mathcal{T}_\bullet$ with $n \in \mathbb{N}$. □

Let $\mathcal{T}_\bullet \in \mathbb{T}$. For $T \in \mathcal{T}_\bullet$ with $T \subseteq \Gamma_m$ for some $m \in \{1,\dots,M\}$, we define $\mathrm{mid}(T)$ as $\gamma_m(\mathrm{mid}(\widehat{T}))$, where $\mathrm{mid}(\widehat{T})$ denotes the midpoint of the corresponding element $\widehat{T}$ in the parameter domain $\widehat{\Gamma}_m$. Now, let $T, T' \in \mathcal{T}_\bullet$ with $T' \in \mathbf{N}_\bullet(T)$, and let $m, m' \in \{1,\dots,M\}$

with $T \subseteq \Gamma_m$ and $T' \subseteq \Gamma_{m'}$. Therefore, either $T' \in \mathbf{N}_{\bullet,m}(T)$, or T and T' are in different boundary parts, i.e., $m \neq m'$, and the intersection $T \cap T'$ is non-empty. In both cases admissibility of $\mathcal{T}_\bullet$ provides $|\mathrm{level}(T) - \mathrm{level}(T')| \leq 1$; see Lemma 3.4.5 and Remark 5.4.1. For the first case, (4.5.4) and bi-Lipschitz continuity of γ_m show that

$$|\mathrm{mid}(T) - \mathrm{mid}(T')| \lesssim 2^{-\mathrm{level}(T)}. \tag{5.5.4}$$

Clearly, the same holds true in the second case. This particularly implies that Lemma 4.5.2 holds accordingly. Indeed, the proof can be copied verbatim (up to the symbol $\widehat{\cdot}$).

Lemma 5.5.2. *There exists a constant $C > 0$ such that for all $\mathcal{T}_\bullet \in \mathbb{T}$, $T' \in \mathcal{T}_\bullet$, and $\mathcal{T}_\circ = \mathtt{refine}(\mathcal{T}_\bullet, \{T'\})$, it holds that*

$$|\mathrm{mid}(T) - \mathrm{mid}(T')| \leq C\, 2^{-\mathrm{level}(T)} \quad \text{for all } T \in \mathcal{T}_\circ \setminus \mathcal{T}_\bullet, \tag{5.5.5}$$

where $C > 0$ depends only on the dimension d, the constant C_γ, the initial meshes $\widehat{\mathcal{T}}_{0,m}$, the polynomial degrees and $(p_{1,m}, \dots, p_{d-1,m})$ for $m \in \{1, \dots, M\}$. □

Also the property (4.5.7) is still valid: For $T \in \mathcal{T}_\bullet$ and $\mathcal{T}_\circ := \mathtt{refine}(\mathcal{T}_\bullet, \{T\})$ there holds that

$$\mathrm{level}(T') \leq \mathrm{level}(T) + 1 \quad \text{for all refined } T' \in \mathcal{T}_\circ \setminus \mathcal{T}_\bullet. \tag{5.5.6}$$

This follows from the fact that all elements $T'' \in \mathcal{T}_\bullet \setminus \mathcal{T}_\circ$ that are refined satisfy that $\mathrm{level}(T'') \leq \mathrm{level}(T)$, wherefore their children T' satisfy (5.5.6). With this last observation, we can argue as in the proof of [BGMP16, Theorem 12] to show the closure estimate (R4). The constant C_{clos} depends only on d, C_γ, $\widehat{\mathcal{T}}_{0,m}$, and $(p_{1,m}, \dots, p_{d-1,m})$ for $m \in \{1, \dots, M\}$.

5.5.8 Verification of (R5)

Let $\mathcal{T}_\bullet, \mathcal{T}_\star \in \mathbb{T}$. For each $m \in \{1, \dots, M\}$, let $\widehat{\mathcal{T}}_{\circ,m}$ be the common refinement of $\widehat{\mathcal{T}}_{\bullet,m}$ and $\widehat{\mathcal{T}}_{\star,m}$ of Section 4.5.5. We already saw there that $\mathcal{T}_{\bullet,m} \in \mathbb{T}_m$, and that

$$\#\mathcal{T}_{\circ,m} \leq \#\mathcal{T}_{\bullet,m} + \#\mathcal{T}_{\star,m} - \#\mathcal{T}_{0,m}. \tag{5.5.7}$$

Summing all components gives the overlay estimate

$$\#\mathcal{T}_\circ \leq \#\mathcal{T}_\bullet + \#\mathcal{T}_\star - \#\mathcal{T}_0. \tag{5.5.8}$$

We still have to show that $\mathcal{T}_\circ$ is a refinement of $\mathcal{T}_\bullet$ and $\mathcal{T}_\star$. Clearly, $\mathcal{T}_\circ$ is finer than these meshes. By Proposition 5.4.3, we just have to verify admissibility of $\mathcal{T}_\circ$. Let $T, T' \in \mathcal{T}_\circ$ with non-empty intersection, and suppose that $T \subseteq \Gamma_m$ and $T' \subseteq \Gamma_{m'}$ for some $m, m' \in \{1, \dots, M\}$ with $m \neq m'$. We have to show that $T \cap T'$ is a common (transformed) lower-dimensional hyperrectangle. Without loss of generality, we assume that $T \in \mathcal{T}_\bullet$ and $T' \in \mathcal{T}_\star$. Further, we may assume that $\dim(T \cap T') > 0$. Then, by definition of $\mathcal{T}_\circ$, there exist $T_\star \in \mathcal{T}_\star$ and $T'_\bullet \in \mathcal{T}_\bullet$ with $T \subseteq T_\star \subseteq \Gamma_m$ and $T' \subseteq T'_\bullet \subseteq \Gamma_{m'}$. Obviously, T and $T'_\bullet$ have non-empty intersection as well. Hence, admissibility of $\mathcal{T}_\bullet$ shows that $T \cap T'_\bullet$ is a common (transformed) hyperrectangle. We suppose that T' is obtained from $T'_\bullet$ via iterative bisections, i.e., $T' \subsetneqq T'_\bullet$, and lead this to a contradiction. The intersection $T \cap T'$ is only a proper subset of a (transformed) hyperrectangle of T. Since $T_\star \supseteq T$, the same holds for $T_\star$ instead of T, i.e., $T_\star \cap T'$ is only a proper subset of a (transformed) hyperrectangle of $T_\star$. Thus, admissibility of $\mathcal{T}_\star$ leads to a contradiction, and we see that $T' = T'_\bullet$ sharing a common (transformed) hyperrecangle with T.

5.5.9 Verification of (S1)

For piecewise constants and piecewise affine functions on a triangulation of the boundary of a polyhedral domain Ω, the inverse estimate (S1) is already found in [DFG+04, Theorem 4.7]. [GHS05, Theorem 3.6] and [Geo08, Theorem 3.9] generalized the result to arbitrary piecewise polynomials on curvilinear triangulations. In the recent own work [FGHP17, Proposition 4.1] and based on the ideas of [DFG+04], we proved (S1) for non-rational splines on a one-dimensional piecewise smooth boundary Γ. In the proof, we derived the following abstract criterion for the ansatz functions which is sufficient for the inverse inequality (S1). Although, there, we only considered $d = 2$, the proof works for arbitrary dimension $d \geq 2$.

Proposition 5.5.3. *Let $\mathcal{T}_\bullet \in \mathbb{T}$ be a general mesh as in Section 5.2.1 which satisfies (M1)–(M5). We assume that the Lipschitz constants of the mappings $\gamma_T : \widehat{T} \to T$ are uniformly bounded, i.e., there exists a constant $C_{\rm lip} > 0$ such that*

$$C_{\rm lip}^{-1} \leq \frac{|\gamma_T(s) - \gamma_T(t)|}{|s-t|} \leq C_{\rm lip} \quad \text{for all } s, t \in \widehat{T} \text{ with } T \in \mathcal{T}_\bullet. \tag{5.5.9}$$

Moreover, let $\psi \in L^2(\Gamma)$ satisfy the following assumption: There exists a constant $\rho_{\rm inf} \in (0,1)$, such that for all $T \in \mathcal{T}_\bullet$ there exists a hyperrectangular subset R_T of the interior T° (i.e., R_T has the form $R_T = \gamma_T(\widehat{R}_T)$ with $\widehat{R}_T = \prod_{i=1}^{d-1}[a_{T,i}, b_{T,i}] \subset \widehat{T}^\circ$ for some real numbers $a_{T,i} < b_{T,i}$) such that $|R_T| \geq \rho_{\rm inf}|T|$, ψ does not change its sign on R_T, and

$$\inf_{x \in R_T} |\psi(x)| \geq \rho_{\rm inf} \|\psi\|_{L^\infty(T)}, \tag{5.5.10}$$

where inf *denotes here the essential infimum. We further assume that the shape-regularity constants of the sets $\widehat{R}_T$ are uniformly bounded, i.e., there exists a constant $C_{\rm rec} > 0$ such that*

$$\max\left\{\frac{b_{T,i} - a_{T,i}}{b_{T,i'} - a_{T,i'}} : T \in \mathcal{T}_\bullet \wedge i, i' \in \{1, \dots, d-1\}\right\} \leq C_{\rm rec} \quad \text{for all } T \in \mathcal{T}_\bullet. \tag{5.5.11}$$

Then, there exists a constant $C_{\rm inv} > 0$ such that

$$\|h_\bullet^{1/2}\psi\|_{L^2(\Gamma)} \leq C_{\rm inv}\|\psi\|_{H^{-1/2}(\Gamma)}. \tag{5.5.12}$$

The constant $C_{\rm inv}$ depends only on d, $C_{\rm lip}$, $\rho_{\rm inf}$, $C_{\rm rec}$, and (M1)–(M5).

Proof. We split the proof into three steps.
Step 1: We construct a suitable test function $v \in H^{1/2}(\Gamma)$. For $T \in \mathcal{T}_\bullet$, we define a bubble function B_T on Γ via

$$\widehat{B}_T(t) := \prod_{i=1}^{d-1}\left(\frac{t_i - a_{T,i}}{b_{T,i} - a_{T,i}} \cdot \frac{b_{T,i} - t_i}{b_{T,i} - a_{T,i}}\right), \quad B_T(x) := \begin{cases} \widehat{B}_T \circ \gamma_T^{-1}(x) & \text{if } x \in R_T, \\ 0 & \text{else.} \end{cases}$$

It satisfies that $0 \le B_T \le 1$ and $\operatorname{supp}(B_T) = R_T$. A standard scaling argument together with Lemma 5.2.1 proves that

$$|R_T| \le \|B_T\|_{L^2(R_T)}^2 \lesssim \|B_T\|_{L^1(R_T)} \lesssim |R_T|, \tag{5.5.13}$$

where the hidden constants depend only on $d, C_{\rm rec}$, and $C_{\rm lip}$. Moreover, B_T is Lipschitz continuous, which implies that $B_T \in H^1(\Gamma)$; see, e.g., [ME14, Theorem 2.28]. Again, a standard scaling argument together with the chain rule (5.1.3) and Lemma 5.2.1 proves that

$$|R_T|^{2/(d-1)} \|\nabla_\Gamma B_T\|_{L^2(R_T)}^2 \lesssim \|B_T\|_{L^2(R_T)}^2, \tag{5.5.14}$$

where the hidden constant depends only on $d, C_{\rm rec}$, and $C_{\rm lip}$. We define the coefficients

$$c_T := \operatorname{sgn}(\psi|_{R_T}) h_T \inf_{x \in R_T} |\psi(x)|. \tag{5.5.15}$$

By definition of the dual norm, it holds that

$$\|\psi\|_{H^{-1/2}(\Gamma)} \ge \frac{|\langle v \,,\, \psi\rangle|}{\|v\|_{H^{1/2}(\Gamma)}} \quad \text{with, e.g., } v := \sum_{T \in \mathcal{T}_\bullet} c_T B_T \in H^1(\Gamma) \subset H^{1/2}(\Gamma). \tag{5.5.16}$$

Step 2: We estimate the numerator in (5.5.16). The definition (5.5.15) shows that

$$|\langle v \,,\, \psi\rangle| = \Big| \sum_{T \in \mathcal{T}_\bullet} \int_T \psi(x) c_T B_T(x)\, dx \Big| \ge \sum_{T \in \mathcal{T}_\bullet} h_T \inf_{x \in R_T} |\psi(x)|^2 \, \|B_T\|_{L^1(R_T)}.$$

The application of (5.5.10) and (5.5.13), together with the fact that $|T| \simeq |R_T|$ proves that

$$|\langle v \,,\, \psi\rangle| \gtrsim \sum_{T \in \mathcal{T}_\bullet} h_T \|\psi\|_{L^\infty(T)}^2 \|B_T\|_{L^1(R_T)} \gtrsim \sum_{T \in \mathcal{T}_\bullet} h_T \, \|\psi\|_{L^2(T)}^2 = \|h_\bullet^{1/2}\psi\|_{L^2(\Gamma)}^2. \tag{5.5.17}$$

Step 3: It remains to estimate the denominator $\|v\|_{H^{1/2}(\Gamma)}$ in (5.5.16) from above by $\|h_\bullet^{1/2}\psi\|_{L^2(\Gamma)}$. Similarly as in the proof of Corollary 5.3.9, one easily derives from Lemma 5.3.2 with (M1)–(M3) and (M5) that

$$\|v\|_{H^{1/2}(\Gamma)}^2 \lesssim \|h_\bullet^{-1/2} v\|_{L^2(\Gamma)}^2 + \|h_\bullet^{1/2} \nabla_\Gamma v\|_{L^2(\Gamma)}^2.$$

Note that $\|h_\bullet^{1/2}\nabla_\Gamma v\|_{L^2(T)}^2 = h_T c_T^2 \|\nabla_\Gamma B_T\|_{L^2(R_T)}^2$ as well as $\|h_\bullet^{-1/2} v\|_{L^2(T)}^2 = h_T^{-1} c_T^2 \|B_T\|_{L^2(R_T)}^2$ for all $T \in \mathcal{T}_\bullet$. Thus, we see with (5.5.14) and $h_T = |T|^{1/(d-1)} \simeq |R_T|^{1/(d-1)}$ that

$$\begin{aligned}
\|v\|_{H^{1/2}(\Gamma)}^2 \quad &\lesssim \quad \|h_\bullet^{-1/2} v\|_{L^2(\Gamma)}^2 + \sum_{T \in \mathcal{T}_\bullet} h_T c_T^2 \|\nabla_\Gamma B_T\|_{L^2(R_T)}^2 \\
&\overset{(5.5.14)}{\lesssim} \quad \|h_\bullet^{-1/2} v\|_{L^2(\Gamma)}^2 + \sum_{T \in \mathcal{T}_\bullet} h_T c_T^2 |R_T|^{-2/(d-1)} \|B_T\|_{L^2(R_T)}^2 \\
&\simeq \quad \|h_\bullet^{-1/2} v\|_{L^2(\Gamma)}^2.
\end{aligned}$$

With (5.5.13) and (5.5.15), we proceed

$$\begin{aligned}
\|h_\bullet^{-1/2} v\|_{L^2(\Gamma)}^2 &= \sum_{T\in\mathcal{T}_\bullet} h_T^{-1} c_T^2 \|B_T\|_{L^2(R_T)}^2 \\
&\overset{(5.5.13)}{\simeq} \sum_{T\in\mathcal{T}_\bullet} h_T^{-1} c_T^2 |R_T| \\
&\overset{(5.5.15)}{=} \sum_{T\in\mathcal{T}_\bullet} h_T \inf_{x\in R_T} |\psi(x)|^2 |R_T| \\
&\le \sum_{T\in\mathcal{T}_\bullet} h_T \|\psi\|_{L^2(R_T)}^2 \le \|h_\bullet^{1/2}\psi\|_{L^2(\Gamma)}^2.
\end{aligned}$$

This concludes the proof. ☐

To apply Proposition 5.5.3 to hierarchical splines, we need the next elementary lemma which was already proved in the recent own work [FGHP17, Proposition 4.1].

Lemma 5.5.4. *Let $p \in \mathbb{N}_0$ be a fixed polynomial degree, and let I be a compact interval with $|I| > 0$. Then, there exists a constant $\rho \in (0,1)$ such that for all polynomials P of degree p on I, there exists some interval $[a,b] \subset I^\circ$ of length $(b-a) \ge \rho|I|$ such that P does not change its sign on $[a,b]$ and*

$$\min_{t\in[a,b]} |P(t)| \ge \rho\, \|P\|_{L^\infty(I)}. \tag{5.5.18}$$

The constant ρ depends only on p.

Proof. We only prove the assertion for $I = [0,1]$. The general case follows immediately by a scaling argument. Instead of considering general polynomials $\mathcal{P}^p(0,1)$ of degree p, it is sufficient to consider the following subset

$$\mathcal{M} := \big\{P \in \mathcal{P}^p(0,1) \,:\, \|P\|_{L^\infty(0,1)} = 1\big\}.$$

Note that $\mathcal{M}$ is a compact subset of $L^\infty(0,1)$ and that differentiation $(\cdot)' : \mathcal{P}^p(0,1) \to \mathcal{P}^{p-1}(0,1)$ with $\mathcal{P}^{-1}(0,1) := \{0\}$ is continuous due to finite dimension. In particular, this implies boundedness $\sup_{P\in\mathcal{M}} \|P'\|_{L^\infty(0,1)} \le C < \infty$. We may assume that $C > 2$. For given $P \in \mathcal{M}$, we define an interval $[\widetilde{a}, \widetilde{b}] \subseteq I$ having all the desired properties but $[\widetilde{a}, \widetilde{b}] \subset I^\circ$: Without loss of generality, we assume that the maximum of $|P|$ is attained at some $\widetilde{a} \in [0,1/2]$ and that $P(\widetilde{a}) = 1$. We set $t_0 := \widetilde{a} + 1/C \in (\widetilde{a}, 1]$ and $\widetilde{b} := \widetilde{a} + 1/(2C) \in (\widetilde{a}, 3/4]$. Then, $(\widetilde{b} - \widetilde{a}) = 1/(2C)$ and for all $t \in [\widetilde{a}, \widetilde{b}]$ it holds that

$$1/2 \le C(t_0 - t) = P(\widetilde{a}) + C(\widetilde{a} - t) \le P(\widetilde{a}) + \|P'\|_{L^\infty(0,1)}(\widetilde{a} - t) \le P(t) = |P(t)|.$$

Altogether, we have that

$$\widetilde{\rho} := 1/(2C) \le 1/2 \le \min_{t\in[\widetilde{a},\widetilde{b}]} |P(t)| \quad \text{and} \quad \widetilde{b} - \widetilde{a} = \widetilde{\rho}.$$

Now, we shrink the interval $[\widetilde{a}, \widetilde{b}]$ around its midpoint, i.e., we choose $a := (\widetilde{a}+\widetilde{b})/2 - (\widetilde{b}-\widetilde{a})/4$ and $b := (\widetilde{a}+\widetilde{b})/2 + (\widetilde{b}-\widetilde{a})/4$. Clearly, $[a,b] \subset I^\circ$ has the desired properties with $\rho := \widetilde{\rho}/2$. ☐

Finally, we come to the proof of the inverse inequality (S1). Let $\mathcal{T}_\bullet \in \mathbb{T}$ be an admissible hierarchical mesh on Γ. We recall that $\mathcal{X}_\bullet$ is a product space of transformed hierarchical splines. Thus, we can assume without loss of generality that we are in the scalar case, i.e., $D = 1$. We show that all $\Psi_\bullet \in \mathcal{X}_\bullet \subset L^2(\Gamma)$ satisfy the assumptions of Proposition 5.5.3 and hence conclude that $\|h_\bullet^{1/2}\Psi_\bullet\|_{L^2(\Gamma)} \lesssim \|\Psi_\bullet\|_{H^{-1/2}(\Gamma)}$. We have already seen that (M1)–(M5) are satisfied. Moreover, (5.5.9) is trivially satisfied since each γ_T is just the restriction of some γ_m to $\widehat{T} = \gamma_m^{-1}(T)$, where $m \in \{1, \ldots, M\}$. For $T \in \mathcal{T}_\bullet$, we abbreviate $\widehat{\Psi}_\bullet := \Psi_\bullet \circ \gamma_T$. Due to the regularity (5.4.2) of the parametrizations γ_m, it is sufficient to find a uniform constant $\widehat{\rho}_{\inf} \in (0,1)$ and a shape-regular hyperrectangle $\widehat{R}_T \subset \widehat{T}^\circ$ such that $|\widehat{R}_T| \geq \widehat{\rho}_{\inf}|\widehat{T}|$, $\widehat{\Psi}_\bullet$ does not change sign on $\widehat{R}_T$, and

$$\inf_{t \in \widehat{R}_T} |\widehat{\Psi}_\bullet(t)| \geq \widehat{\rho}_{\inf} \|\widehat{\Psi}_\bullet\|_{L^\infty(\widehat{T})}. \tag{5.5.19}$$

Indeed, one sees as in Section 4.5.3 that $|\widehat{R}_T| \geq \widehat{\rho}_{\inf}|\widehat{T}|$ implies that $|R_T| \geq \rho_{\inf}|T|$ for some uniform constant $\rho_{\inf} \in (0,1)$. Recall that $\widehat{\Psi}_\bullet$ coincides with a tensor-product polynomial P. Hence, there exist polynomials P_i of degree $p_{\max}$ such that $P(t) = \prod_{i=1}^{d-1} P_i(t_i)$. With the notation $\widehat{T} = \prod_{i=1}^{d-1} \widehat{T}_i$ and $\widehat{R}_T = \prod_{i=1}^{d-1} (\widehat{R}_T)_i$, we see that the latter inequality is satisfied if for all $i \in \{1, \ldots, d-1\}$ it holds that

$$\inf_{t_i \in (\widehat{R}_T)_i} |P_i(t_i)| \geq \widehat{\rho}_{\inf}^{\,1/(d-1)} \|P_i\|_{L^\infty(\widehat{T}_i)}. \tag{5.5.20}$$

We define $(\widehat{R}_T)_i$ as the interval of Lemma 5.5.4 corresponding to the polynomial P_i on the interval $I = \widehat{T}_i$. With the constant ρ of Lemma 5.5.4, we set $\widehat{\rho}_{\inf} := \rho^{d-1}$. Then, (5.5.20), and therefore (5.5.19) is satisfied. Moreover, one sees that $|\widehat{R}_T| \geq \widehat{\rho}_{\inf}|\widehat{T}|$, and that $\widehat{\Psi}_\bullet$ does not change its sign on $\widehat{R}_T \subset \widehat{T}^\circ$. It remains to prove shape-regularity (5.5.11). Since, the refinement procedure `refine` only uses uniform bisection of elements, the element $\widehat{T}$ is shape-regular in the sense that $|\widehat{T}_i| \simeq |\widehat{T}_{i'}|$ for all $i, i' \in \{1, \ldots, d-1\}$. This, together with $|(\widehat{R}_T)_i| \geq \rho|\widehat{T}_i|$, proves (5.5.11). Altogether, we conclude (S1), where the constant C_{inv} depends only on the dimensions d, D, the (fixed) number M of boundary parts Γ_m, the parametrizations γ_m and γ_z, the initial meshes $\widehat{\mathcal{T}}_{0,m}$, and the polynomial orders $(p_{1,m}, \ldots, p_{d-1,m})$ for $m \in \{1, \ldots, M\}$ and $z \in \mathcal{N}_\gamma$.

5.5.10 Verification of (S2)

Let $\mathcal{T}_\bullet \in \mathbb{T}$ and $\mathcal{T}_\circ \in \texttt{refine}(\mathcal{T}_\bullet)$. The nestedness $\mathcal{X}_\bullet \subseteq \mathcal{X}_\circ$ was already stated in (5.4.17).

5.5.11 Basis of hierarchical splines on the boundary

In this section, we give a basis for $\mathcal{X}_\bullet$. For $m \in \{1, \ldots, M\}$, recall the definition of hierarchical B-splines $\widehat{\mathcal{B}}_{\bullet,m}$ from Section 3.4.2. Then,

$$\mathcal{X}_{\bullet,m} = \mathrm{span}(\mathcal{B}_{\bullet,m}^D) \quad \text{with} \quad \mathcal{B}_{\bullet,m} := \big\{\widehat{\beta} \circ \gamma_m^{-1} \,:\, \widehat{\beta} \in \widehat{\mathcal{B}}_{\bullet,m}\big\}, \tag{5.5.21}$$

where $\mathcal{B}_{\bullet,m}^D$ forms even a basis. For $\beta \in \mathcal{B}_{\bullet,m}^D$, let $\mathrm{Trunc}_\bullet(\beta) := \mathrm{Trunc}_{\bullet,m}(\beta) := \mathrm{Trunc}_{\bullet,m}(\widehat{\beta} \circ \gamma_m) \circ \gamma_m^{-1}$ denote the componentwise truncation of Section 3.4.3 transformed onto Γ_m. Then, another basis is given by

$$\mathcal{X}_{\bullet,m} = \operatorname{span}\big\{\operatorname{Trunc}_{\bullet,m}(\beta) \,:\, \beta \in \mathcal{B}_{\bullet,m}^D\big\}. \tag{5.5.22}$$

If we identify functions in $L^2(\Gamma_m)^D$ with their extension (by zero) in $L^2(\Gamma)^D$, there holds that

$$\mathcal{X}_\bullet = \operatorname{span}(\mathcal{B}_\bullet^D) \quad \text{with} \quad \mathcal{B}_\bullet := \bigcup_{m=1}^M \mathcal{B}_{\bullet,m}, \tag{5.5.23}$$

where $\mathcal{B}_\bullet^D$ forms even a basis. Another basis is given by

$$\mathcal{X}_\bullet = \operatorname{span}\big\{\operatorname{Trunc}_\bullet(\beta) \,:\, \beta \in \mathcal{B}_\bullet^D\big\}. \tag{5.5.24}$$

5.5.12 Verification of (S3)

Let $\mathcal{T}_\bullet \in \mathbb{T}$, $\mathcal{T}_\circ \in \texttt{refine}(\mathcal{T}_\bullet)$, and $T \in \mathcal{T}_\bullet \setminus \Pi_\bullet^{q_{\rm loc}}(\mathcal{T}_\bullet \setminus \mathcal{T}_\circ)$, where we set $q_{\rm loc} := q_{\rm proj} + 2(p_{\max}+1)$. Later, $q_{\rm proj}$ will be fixed as $q_{\rm proj} := 2(p_{\max}+1)$. Since $\mathcal{X}_\bullet$ is a product space of transformed hierarchical splines, we can assume without loss of generality that $D=1$. The proof now works essentially as in Section 4.5.9. There holds that

$$\big\{\Psi_\bullet|_{\pi_\bullet^{q_{\rm proj}}(T)} \,:\, \Psi_\bullet \in \mathcal{X}_\bullet\big\} = \operatorname{span}\big\{\beta|_{\pi_\bullet^{q_{\rm proj}}(T)} \,:\, \beta \in \mathcal{B}_\bullet \wedge |\operatorname{supp}(\beta) \cap \pi_\bullet^{q_{\rm proj}}(T)| > 0\big\}$$

as well as

$$\big\{\Psi_\circ|_{\pi_\bullet^{q_{\rm proj}}(T)} \,:\, \Psi_\circ \in \mathcal{X}_\circ\big\} = \operatorname{span}\big\{\beta|_{\pi_\bullet^{q_{\rm proj}}(T)} \,:\, \beta \in \mathcal{B}_\circ \wedge |\operatorname{supp}(\beta) \cap \pi_\bullet^{q_{\rm proj}}(T)| > 0\big\}.$$

We show that

$$\big\{\beta \in \mathcal{B}_\bullet \,:\, |\operatorname{supp}(\beta) \cap \pi_\bullet^{q_{\rm proj}}(T)| > 0\big\} = \big\{\beta \in \mathcal{B}_\circ \,:\, |\operatorname{supp}(\beta) \cap \pi_\bullet^{q_{\rm proj}}(T)| > 0\big\}. \tag{5.5.25}$$

First, let β be an element of the left-hand side. Then, Remark 3.4.4 implies $\operatorname{supp}(\beta) \subseteq \pi_\bullet^{q_{\rm loc}}(T)$. As in (4.5.13), one easily verifies that $\Pi_\bullet^{q_{\rm loc}}(T) \subseteq \mathcal{T}_\bullet \cap \mathcal{T}_\circ$. In particular, we see that $\operatorname{supp}(\widehat{\beta}) \subseteq \bigcup(\widehat{\mathcal{T}}_{\bullet,m} \cap \widehat{\mathcal{T}}_{\circ,m})$ for the corresponding $\widehat{\beta} := \beta \circ \gamma_m$ in the parameter domain $\widehat{\Gamma}_m$. This proves that no element within $\operatorname{supp}(\widehat{\beta})$ is changed during refinement, i.e., $\widehat{\Omega}_{\bullet,m}^k \cap \operatorname{supp}(\widehat{\beta}) = \widehat{\Omega}_{\circ,m}^k \cap \operatorname{supp}(\widehat{\beta})$ for all $k \in \mathbb{N}_0$. Thus, (3.4.13) proves that $\widehat{\beta} \in \widehat{\mathcal{B}}_{\circ,m}$, and hence $\beta \in \mathcal{B}_\circ$. The proof works the same if we start with some β in the right-hand side of (5.5.25). This proves (5.5.25) and therefore (S3).

5.5.13 Verification of (S4)

Let $\mathcal{T}_\bullet \in \mathbb{T}$. First, we recall that $\mathcal{X}_\bullet$ is a product space of transformed hierarchical splines. With Remark 5.2.3, we can thus assume without loss of generality that $D=1$. [Fae00, Lemma 2.6] resp. [Fae02, Lemma 3.5] prove a similar version of (S4) for splines on a one-dimensional boundary Γ resp. for certain piecewise polynomials of degree $0,1,5$, and 6 on curvilinear triangulations of a two-dimensional boundary Γ. There, the proof follows from direct calculations, where [Fae00, Lemma 2.6] actually only proves (S4) for splines of degree 2. In contrast, we will make use of the following abstract result.

Proposition 5.5.5. *Let $D = 1$. Let $\mathcal{T}_\bullet \in \mathbb{T}$ be a general mesh as in Section 5.2.1 which satisfies (M1)–(M3). Assume that there exists a finite subset $\overline{\mathcal{B}}_\bullet \subset \mathcal{X}_\bullet$, which satisfies the following three properties:*

(i) *Non-negativity: Each $\overline{\beta} \in \overline{\mathcal{B}}_\bullet$ is non-negative.*

(ii) *Locality: There is some $q'_{\rm supp} \in \mathbb{N}_0$ such that for all $\overline{\beta} \in \overline{\mathcal{B}}_\bullet$ there exists an element $T_{\overline{\beta}} \in \mathcal{T}_\bullet$ with $\mathrm{supp}(\overline{\beta}) \subseteq \pi_\bullet^{q'_{\rm supp}}(T_{\overline{\beta}})$.*

(iii) *Partition of unity: It holds that $\sum_{\overline{\beta}\in\overline{\mathcal{B}}_\bullet} \overline{\beta} = 1$.*

Then, (S4) is satisfied with $q_{\rm supp} = 2q'_{\rm supp}$, and the constant $\rho_{\rm unity}$ depends only on (M1)–(M3) and $q'_{\rm supp}$.

Proof. Let $T \in \mathcal{T}_\bullet$. We set

$$\Psi_{\bullet,T,1} := \Psi_{\bullet,T} := \sum_{\substack{\overline{\beta}\in\overline{\mathcal{B}}_\bullet \\ T\subseteq \mathrm{supp}(\overline{\beta})}} \overline{\beta}.$$

This implies that $0 \le \Psi_{\bullet,T} \le 1$ and $\Psi_{\bullet,T}|_T = 1$, wherefore we have that $T \subseteq \mathrm{supp}(\Psi_{\bullet,T})$. Note that $T \subseteq \mathrm{supp}(\overline{\beta})$ implies that $T \subseteq \mathrm{supp}(\overline{\beta}) \subseteq \pi_\bullet^{q'_{\rm supp}}(T_{\overline{\beta}})$. In particular, we obtain that $T \in \Pi_\bullet^{q'_{\rm supp}}(T_{\overline{\beta}})$, and hence $\pi_\bullet^{q'_{\rm supp}}(T_{\overline{\beta}}) \subseteq \pi_\bullet^{q_{\rm supp}}(T)$ with $q_{\rm supp} := 2q'_{\rm supp}$. We conclude that $\mathrm{supp}(\Psi_{\bullet,T}) \subseteq \pi_\bullet^{q_{\rm supp}}(T)$. Finally, there holds that

$$\begin{aligned}\int_{\mathrm{supp}(\Psi_{\bullet,T})} (1-\Psi_{\bullet,T})^2\,dx \le \int_{\mathrm{supp}(\Psi_{\bullet,T})\setminus T} (1-0)^2\,dx &= |\mathrm{supp}(\Psi_{\bullet,T})| - |T| \\ &= \Big(1 - \frac{|T|}{|\mathrm{supp}(\Psi_{\bullet,T})|}\Big)|\mathrm{supp}(\Psi_{\bullet,T})| \\ &\le \Big(1 - \frac{|T|}{|\pi_\bullet^{q_{\rm supp}}(T)|}\Big)|\mathrm{supp}(\Psi_{\bullet,T})| \le \rho_{\rm unity}^2 |\mathrm{supp}(\Psi_{\bullet,T})|,\end{aligned}$$

where $0 < \rho_{\rm unity} < 1$ depends only on (M1)–(M3) and $q'_{\rm supp}$. ∎

We choose $\overline{\mathcal{B}}_\bullet = \{\mathrm{Trunc}_\bullet(\beta) : \beta \in \mathcal{B}_\bullet\}$ in Proposition 5.5.5. Then, (i) follows from (3.4.22), (ii) with $q'_{\rm supp} = 2(p_{\max}+1)$ from Remark 3.4.4, and (iii) from (3.4.23). This concludes the proof of (S4), where $q_{\rm supp} = 4(p_{\max}+1)$, and $\rho_{\rm unity}$ depends only on the dimension d, the number M of boundary parts Γ_m, the constant C_γ, the initial meshes $\widehat{\mathcal{T}}_{0,m}$, and $(p_{1,m},\dots,p_{d-1,m})$ for $m \in \{1,\dots,M\}$.

5.5.14 Verification of (S5)–(S6)

Let $\mathcal{T}_\bullet$ and $\mathcal{S} \subseteq \mathcal{T}_\bullet$. Since $\mathcal{X}_\bullet$ is a product space of transformed hierarchical splines, we may assume without loss of generality that $D = 1$. For $m \in \{1,\dots,M\}$, we set $\widehat{\mathcal{S}}_m := \{\gamma_m^{-1}(T) : T \in \mathcal{S} \cap \mathcal{T}_{\bullet,m}\}$. Let $\widehat{\beta}^*$ be the dual basis functions and $\widehat{T}_{\widehat{\beta}}$ their corresponding support (which depends on $\widehat{\mathcal{T}}_{\bullet,m}$) of Section 3.4.5. We define the operator $J_{\bullet,\mathcal{S}} : L^2(\Gamma) \to \{\Psi_\bullet \in \mathcal{X}_\bullet : \Psi_\bullet|_{\bigcup(\mathcal{T}_\bullet\setminus\mathcal{S})} = 0\}$ via

$$(J_{\bullet,\mathcal{S}}\psi) \circ \gamma_m := \widehat{J}_{\bullet,m,\widehat{\mathcal{S}}_m}(\psi \circ \gamma_m) \quad \text{for all } m \in \{1,\dots,M\}, \tag{5.5.26}$$

where

$$\widehat{J}_{\bullet,m,\widehat{\mathcal{S}}_m} : L^2(\widehat{\Gamma}_m) \to \widehat{\mathcal{X}}_{\bullet,m}, \quad \widehat{\psi} \mapsto \sum_{\substack{\widehat{\beta}\in\widehat{\mathcal{B}}_{\bullet,m} \\ \mathrm{supp}(\widehat{\beta})\subseteq\bigcup\widehat{\mathcal{S}}_m}} \int_{\widehat{T}_{\widehat{\beta}}} \widehat{\beta}^*\widehat{\psi}\,dx\,\mathrm{Trunc}_{\bullet,m}(\widehat{\beta}). \tag{5.5.27}$$

Note that the sum is always finite since the maximal level is bounded. Recall that $0 \le \mathrm{Trunc}_{\bullet,m}(\widehat{\beta}) \le \widehat{\beta}$ (see (3.4.22)), wherefore $J_{\bullet,\mathcal{S}}$ clearly maps into the desired space $\{\Psi_\bullet \in \mathcal{X}_\bullet : \Psi_\bullet|_{\bigcup(\mathcal{T}_\bullet\setminus\mathcal{S})} = 0\}$.

We come to the verification of the properties (S5)–(S6). Let $q_{\mathrm{proj}} := 2(p_{\max}+1)$ and $q_{\mathrm{loc}} = q_{\mathrm{proj}} + 2(p_{\max}+1)$ of Section 5.5.12. Moreover, let $T \in \mathcal{T}_\bullet$ with $\Pi_\bullet^{q_{\mathrm{loc}}}(T) \subseteq \mathcal{S}$ and $m \in \{1,\dots,M\}$ with $T \subseteq \Gamma_m$. Recall the notation $\widehat{T} = \gamma_m^{-1}(T)$. For all $\psi \in L^2(\Gamma)$, there holds with the abbreviation $\widehat{\psi} := \psi \circ \gamma_m$ that

$$(J_{\bullet,\mathcal{S}}\psi) \circ \gamma_m|_{\widehat{T}} = (\widehat{J}_{\bullet,m,\widehat{\mathcal{S}}_m}\widehat{\psi})|_{\widehat{T}} = \sum_{\substack{\widehat{\beta}\in\widehat{\mathcal{B}}_{\bullet,m} \\ \mathrm{supp}(\widehat{\beta})\subseteq\bigcup\widehat{\mathcal{S}}_m}} \int_{\widehat{T}_{\widehat{\beta}}} \widehat{\beta}^*\widehat{\psi}\,dx\,\mathrm{Trunc}_{\bullet,m}(\widehat{\beta})|_{\widehat{T}}.$$

Note that $\mathrm{Trunc}_{\bullet,m}(\widehat{\beta})|_{\widehat{T}}$ vanishes if $|\mathrm{supp}(\widehat{\beta})\cap\widehat{T}| = 0$. Due to Remark 3.4.4, $|\mathrm{supp}(\widehat{\beta})\cap\widehat{T}| > 0$ implies that $\mathrm{supp}(\widehat{\beta}) \subseteq \Pi_{\bullet,m}^{2(p_{\max}+1)}(\widehat{T}) \subseteq \Pi_{\bullet,m}^{q_{\mathrm{loc}}}(\widehat{T})$. We abbreviate $\Pi_{\bullet,m}^{q_{\mathrm{loc}}}(T) := \{\gamma(\widehat{T}') : \widehat{T}' \in \Pi_{\bullet,m}^{q_{\mathrm{loc}}}(\widehat{T})\}$ and note that $\Pi_{\bullet,m}^{q_{\mathrm{loc}}}(T) \subseteq \Pi_\bullet^{q_{\mathrm{loc}}}(T) \cap \mathcal{T}_{\bullet,m} \subseteq \mathcal{S} \cap \mathcal{T}_{\bullet,m}$, which yields that $\Pi_{\bullet,m}^{q_{\mathrm{loc}}}(\widehat{T}) \subseteq \widehat{\mathcal{S}}_m$. Hence, we see that

$$(J_{\bullet,\mathcal{S}}\psi) \circ \gamma_m|_{\widehat{T}} = \sum_{\widehat{\beta}\in\widehat{\mathcal{B}}_{\bullet,m}} \int_{\widehat{T}_{\widehat{\beta}}} \widehat{\beta}^*\widehat{\psi}\,dx\,\mathrm{Trunc}_{\bullet,m}(\widehat{\beta})|_{\widehat{T}}.$$

The right-hand side just coincides with the quasi-interpolation $(\widehat{I}_{\bullet,m}\widehat{\psi})|_{\widehat{T}}$ corresponding to the mesh $\widehat{\mathcal{T}}_{\bullet,m}$ of Section 3.4.5. If ψ satisfies that $\psi|_{\pi_\bullet^{q_{\mathrm{proj}}}(T)} \in \{\Psi_\bullet|_{\pi_\bullet^{q_{\mathrm{proj}}}(T)} : \Psi_\bullet \in \mathcal{X}_\bullet\}$, and hence $\widehat{\psi}|_{\pi_{\bullet,m}^{q_{\mathrm{proj}}}(\widehat{T})} \in \{\widehat{\Psi}_{\bullet,m}|_{\pi_{\bullet,m}^{q_{\mathrm{proj}}}(\widehat{T})} : \widehat{\Psi}_{\bullet,m} \in \widehat{\mathcal{X}}_{\bullet,m}\}$, Proposition 3.4.9 proves that

$$(J_{\bullet,\mathcal{S}}\psi) \circ \gamma_m|_{\widehat{T}} = (\widehat{I}_{\bullet,m}\widehat{\psi})|_{\widehat{T}} = \widehat{\psi}|_{\widehat{T}}.$$

This proves the local projection property (S5).

Finally, we prove local L^2-stability (S6). Let again $T \in \mathcal{T}_\bullet$ and $m \in \{1,\dots,M\}$ with $T \subseteq \Gamma_m$. With the notation from before, the boundedness of the Gram determinant (5.4.2) shows that

$$\|J_{\bullet,\mathcal{S}}\psi\|_{L^2(T)} \simeq \|\widehat{J}_{\bullet,m,\widehat{\mathcal{S}}_m}\widehat{\psi}\|_{L^2(\widehat{T})}.$$

Exactly as in the proof of Proposition 3.4.9, one estimates

$$\|\widehat{J}_{\bullet,m,\widehat{\mathcal{S}}_m}\widehat{\psi}\|_{L^2(\widehat{T})} \lesssim \|\widehat{\psi}\|_{L^2(\pi^{q_{\rm loc}}_{\bullet,m}(\widehat{T}))}.$$

Thus, from the boundedness of the Gram determinant (5.4.2), we derive with $\pi^{q_{\rm loc}}_{\bullet,m}(T) := \gamma_m(\pi^{q_{\rm loc}}_{\bullet,m}(\widehat{T}))$ that

$$\|\widehat{\psi}\|_{L^2(\pi^{q_{\rm loc}}_{\bullet,m}(\widehat{T}))} \simeq \|\psi\|_{L^2(\pi^{q_{\rm loc}}_{\bullet,m}(T))} \leq \|\psi\|_{L^2(\pi^{q_{\rm loc}}_{\bullet}(T))},$$

which concludes (S6). The constant $C_{\rm sz}$ depends only on the dimension d, the constant C_γ, the initial meshes $\widehat{\mathcal{T}}_{\bullet,m}$, and the polynomial orders $(p_{1,m},\dots,p_{d-1,m})$ for $m \in \{1,\dots,M\}$.

5.5.15 Proof of Theorem 5.4.5 for rational hierarchical splines

As mentioned in Remark 5.4.6, Theorem 5.4.5 is still valid if one replaces the ansatz space $\mathcal{X}_\bullet$ for $\mathcal{T}_\bullet \in \mathbb{T}$ by rational hierarchical splines, i.e., by the set

$$\mathcal{X}^{W_0}_\bullet = \Big\{W_0^{-1}\Psi_\bullet : \Psi_\bullet \in \mathcal{X}_\bullet\Big\}, \tag{5.5.28}$$

where $\widehat{W}_{0,m} = W_0 \circ \gamma_m \in \widehat{\mathcal{S}}^{(p_{1,m},\dots,p_{d-1,m})}(\widehat{\mathcal{K}}_{0,m},\widehat{\mathcal{T}}_{0,m})$ is a fixed positive weight function in the initial space of hierarchical splines for all $m \in \{1,\dots,M\}$, where we additionally assume the representation (5.4.24). Indeed, the mesh properties (M1)–(M5) as well as the refinement properties (R1)–(R5) of Section 5.2 are independent of the discrete spaces. To verify the validity of Theorem 5.4.5 in the rational setting, it thus only remains to verify the properties (S1)–(S6) for the rational boundary element spaces.

To see the inverse estimate (S1), it is again sufficient to consider $D = 1$. In Section 5.5.9, we proved (S1) for $\mathcal{X}_\bullet$ by applying Proposition 5.5.3 for all $\Psi_\bullet \in \mathcal{X}_\bullet$. With the notation from Section 5.5.9, we showed that

$$\inf_{x\in R_T}|\Psi_\bullet(x)| \geq \rho_{\rm inf}\|\Psi_\bullet\|_{L^\infty(T)} \quad \text{for all } T \in \mathcal{T}_\bullet, \Psi_\bullet \in \mathcal{X}_\bullet,$$

where $\Psi_\bullet$ does not change its sign on R_T. With $0 < w_{\rm min} := \inf_{x\in\Gamma} W_0(x)$, $w_{\rm max} := \sup_{x\in\Gamma} W_0(x)$, and $\widetilde{\rho}_{\rm inf} := \rho_{\rm inf}w_{\rm min}/w_{\rm max}$, this yields for all $\Psi_\bullet \in \mathcal{X}_\bullet$ that

$$\widetilde{\rho}_{\rm inf}\|W_0^{-1}\Psi_\bullet\|_{L^\infty(T)} \leq \frac{\rho_{\rm inf}}{w_{\rm max}}\|\Psi_\bullet\|_{L^\infty(T)} \leq \frac{1}{w_{\rm max}}\inf_{x\in R_T}|\Psi_\bullet(x)| \leq \inf_{x\in R_T}|W_0^{-1}\Psi_\bullet(x)|.$$

In particular, the conditions for Proposition 5.5.3 are also satisfied for the functions in $\mathcal{X}^{W_0}_\bullet$, which concludes (S1).

The properties (S2)–(S3) depend only on the numerator of the rational hierarchical splines and thus transfer.

For the proof of (S4), we exploit the representation (5.4.24) to verify the conditions of the abstract Proposition 5.5.5. Again, we assume without loss of generality that $D = 1$. Let $\mathcal{T}_\bullet \in \mathbb{T}$. Note that $\widehat{W}_{0,m}$ is also an element of the standard tensor-product spline space $\widehat{\mathcal{S}}^{(p_{1,m},\dots,p_{d-1,m})}(\widehat{\mathcal{K}}_{{\rm uni}(k),m})$ for all $m \in \{1,\dots,M\}$ and $k \in \mathbb{N}_0$. In particular, it can be written as linear combination of B-splines in $\widehat{\mathcal{B}}_{{\rm uni}(k),m}$. The representation (5.4.24) and the two-scale relation with only non-negative coefficients between bases of consecutive levels of Section 3.4 yields that the corresponding coefficients are non-negative. Therefore, [SM16, Theorem 1] or [GJS14, Theorem 12] imply that also the coefficients of the linear combination of $\widehat{W}_{0,m}$ in $\{\mathrm{Trunc}_{\bullet,m}(\widehat{\beta}) : \widehat{\beta} \in \widehat{\mathcal{B}}_{\bullet,m}\}$ are non-negative, i.e.,

$$\widehat{W}_{0,m} = \sum_{\widehat{\beta}\in\widehat{\mathcal{B}}_{\bullet,m}} \widetilde{w}_{\bullet,m,\widehat{\beta}}\,\mathrm{Trunc}_{\bullet,m}(\widehat{\beta}) \quad \text{with non-negative coefficients } \widetilde{w}_{\bullet,m,\widehat{\beta}} \geq 0. \tag{5.5.29}$$

If we identify functions in $L^2(\Gamma_m)$ with their extension (by zero) in $L^2(\Gamma)$, we can choose

$$\overline{\mathcal{B}}_\bullet := \bigcup_{m=1}^{M} \Big\{ \Big(\frac{\widetilde{w}_{\bullet,m,\widehat{\beta}}}{\widehat{W}_{0,m}} \mathrm{Trunc}_{\bullet,m}(\widehat{\beta}) \Big) \circ \gamma_m^{-1} : \widehat{\beta} \in \widehat{\mathcal{B}}_{\bullet,m} \Big\} \subset \mathcal{X}_\bullet^{W_0}.$$

As in Section 5.5.13, one sees that this choice satisfies the assumptions of Proposition 5.5.5.
To see (S5) and (S6), we define the corresponding projection operator

$$J_{\bullet,\mathcal{S}}^{W_0} : L^2(\Gamma)^D \to \{\Psi_\bullet \in \mathcal{X}_\bullet : \Psi_\bullet|_{\bigcup(\mathcal{T}_\bullet\setminus\mathcal{S})} = 0\}, \quad \psi \mapsto W_0^{-1} J_{\bullet,\mathcal{S}}(W_0\psi). \tag{5.5.30}$$

The desired properties transfer immediately from the non-rational case.

5.6 Numerical experiments with hierarchical splines

In this section, we empirically investigate the performance of Algorithm 5.2.4 in two typical situations: In Section 5.6.1, the solution is generically singular at the edges of $\Gamma = \partial\Omega$. In Section 5.6.2, the solution is nearly singular at one point.

We consider the 3D Laplace operator $\mathfrak{P} := -\Delta$ as partial differential operator. The corresponding fundamental solution reads

$$G(z) := \frac{1}{4\pi}\frac{1}{|z|} \quad \text{for all } z \in \mathbb{R}^3 \setminus \{0\}. \tag{5.6.1}$$

As already mentioned in Section 5.1.3, the corresponding single-layer operator $\mathfrak{V} : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is elliptic.

In the first example (Section 5.6.1), we consider the exterior Laplace–Dirichlet problem

$$\begin{aligned} -\Delta u &= 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega}, \\ u &= g \quad \text{on } \Gamma, \end{aligned} \tag{5.6.2a}$$

for given Dirichlet data $g \in H^{1/2}(\Gamma)$, together with the far field boundary condition

$$u(x) = \mathcal{O}\Big(\frac{1}{|x|}\Big) \quad \text{as } |x| \to \infty. \tag{5.6.2b}$$

Then, (5.6.2) can be equivalently rewritten as integral equation (5.1.14); see, e.g., [McL00, Theorem 7.15 and Theorem 8.9], [Ste08a, Section 7.5], or [SS11, Section 3.4.2.2]. Indeed, the (exterior) normal derivative $\phi := \partial_\nu u$ of the weak solution u of (5.6.2) satisfies the integral equation (5.1.14) with $f := (\mathfrak{K} - 1/2)g$, i.e.,

$$\mathfrak{V}\phi = (\mathfrak{K} - 1/2)g, \tag{5.6.3}$$

where

$$\mathfrak{K} : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma) \tag{5.6.4}$$

denotes the *double-layer operator*. According to [SS11, Corollary 3.3.12 and Theorem 3.3.13], if Γ is piecewise smooth and if $g \in L^\infty(\Gamma)$, there holds for all $x \in \Gamma$ the representation

$$\mathfrak{K}g(x) = \int_\Gamma g(y)\partial_{\nu(y)}G(x,y)\,dy \quad \text{if } \Gamma \text{ is smooth in } x \text{ and } g \text{ is continuous at } x. \tag{5.6.5}$$

In the second example (Section 5.6.2), we consider the interior Laplace–Dirichlet problem

$$\begin{aligned} -\Delta u &= 0 \quad \text{in } \Omega, \\ u &= g \quad \text{on } \Gamma, \end{aligned} \tag{5.6.6}$$

for given Dirichlet data $g \in H^{1/2}(\Gamma)$. Then, (5.6.6) can be equivalently rewritten as integral equation (5.1.14); see, e.g., [McL00, Theorem 7.6], [Ste08a, Section 7.1], or [SS11, Section 3.4.2.1]. Indeed, the normal derivative $\phi := \partial_\nu u$ of the weak solution u of (5.6.6) satisfies the integral equation (5.1.14) with $f := (\mathfrak{K} + 1/2)g$, i.e.,

$$\mathfrak{V}\phi = (\mathfrak{K} + 1/2)g, \tag{5.6.7}$$

where $\mathfrak{K}$ denotes again the double-layer operator (5.6.4).

The integral representation (5.6.4) is satisfied for both considered examples. Indeed, the surfaces Γ_m of the boundary $\Gamma = \bigcup_{m=1}^M \Gamma_m$ are parametrized via rational splines, i.e., for each $m \in \{1, \dots, M\}$ there exist polynomial orders $p_{1(\gamma,m)}, p_{2(\gamma,m)} \in \mathbb{N}$, a two-dimensional vector $\widehat{\mathcal{K}}_{\gamma,m} = (\widehat{\mathcal{K}}_{1(\gamma,m)}, \widehat{\mathcal{K}}_{2(\gamma,m)})$ of $p_{i(\gamma,m)}$-open knot vectors with multiplicity smaller or equal to $p_{i(\gamma,m)}$ for the interior knots, and a positive spline weight function $\widehat{W}_{\gamma,m} \in \widehat{\mathcal{S}}^{(p_{1(\gamma,m)},p_{2(\gamma,m)})}(\widehat{\mathcal{K}}_{\gamma,m})$ such that the parametrization $\gamma_m : \widehat{\Gamma}_m \to \Gamma_m$ satisfies that

$$\gamma_m \in \big\{\widehat{W}_{\gamma,m}^{-1}\,\widehat{S} \,:\, \widehat{S} \in \widehat{\mathcal{S}}^{(p_{1(\gamma,m)},p_{2(\gamma,m)})}(\widehat{\mathcal{K}}_{\gamma,m})^3\big\}. \tag{5.6.8}$$

Based on the knots $\widehat{\mathcal{K}}_{\gamma,m}$ for the geometry, we choose the initial knots $\widehat{\mathcal{K}}_{0,m}$ for the discretization. As basis for the considered ansatz spaces of (non-rational) hierarchical splines, we use the basis given in (5.5.23). To (approximately) calculate the Galerkin matrix and the right-hand side vector, we proceed as in [SS11, Chapter 5] where all singular integrals are transformed via Duffy transformations and then computed with tensor Gauss quadrature. For the (dense) Galerkin matrix, we do not apply any matrix compression techniques such as wavelet methods [BCR91, DHS06, HR10], fast multipole methods [GR87, TM12, DHK$^+$17], or $\mathcal{H}$-matrix methods [Hac99, MZBF15]. To calculate the weighted-residual error estimator[5] (5.2.17), we employ formula (5.1.2) for the surface gradient and

[5] To ease computation, we replace $h_T = |T|^{1/2}$ in (5.2.17) by the equivalent term $\mathrm{diam}(\Gamma)\,|\widehat{T}|^{1/2}$. Here, $\widehat{T}$ denotes the corresponding element of $T \in \mathcal{T}_{\ell,m}$ in the parameter domain $\widehat{\Gamma}_m$.

use again tensor Gauss quadrature. To this end, we approximate $\nabla((f - \mathfrak{V}\Phi_\ell) \circ \gamma_m)$ on an element $\widehat{T} \in \widehat{\mathcal{T}}_{\ell,m}$ by the gradient of the polynomial interpolation of the residual $f - \mathfrak{V}\Phi_\ell$ as in [Kar12, Section 7.1.5]. In particular, we have to evaluate the residual at some quadrature points which can be done (approximately) using appropriate Duffy transformations and tensor Gauss quadrature as in [Gan14, Sections 5.1–5.2].

To (approximately) calculate the energy error, we proceed as follows: Let $\Phi_\ell \in \mathcal{X}_\ell$ be the Galerkin approximation of the ℓ-th step with the corresponding coefficient vector $\boldsymbol{c_\ell}$. Further, let $\boldsymbol{V_\ell}$ be the Galerkin matrix. With Galerkin orthogonality (5.2.15) and the energy norm $\|\phi\|_{\mathfrak{V}}^2 = \langle \mathfrak{V}\phi \,,\, \phi \rangle$ obtained by Aitken's Δ^2-extrapolation, we can compute the energy error as

$$\|\phi - \Phi_\ell\|_{\mathfrak{V}}^2 = \|\phi\|_{\mathfrak{V}}^2 - \|\Phi_\ell\|_{\mathfrak{V}}^2 = \|\phi\|_{\mathfrak{V}}^2 - \boldsymbol{V_\ell c_\ell} \cdot \boldsymbol{c_\ell}. \tag{5.6.9}$$

5.6.1 Solution with edge singularities on cube

In the first experiment, we consider the cube

$$\Omega := (0, 1/10)^3. \tag{5.6.10}$$

Each of the six faces Γ_m of Ω can be parametrized by non-rational splines of degree $p_{1(\gamma,m)} := p_{2(\gamma,m)} := 1$ corresponding to the knot vectors $\widehat{\mathcal{K}}_{1(\gamma,m)} := \widehat{\mathcal{K}}_{2(\gamma,m)} := (0,0,1,1)$; see [GHP17, Section 6.1]. We choose the right-hand side $f := 1$ in (5.1.14). Note that the constant function 1 satisfies the Laplace problem, wherefore (5.6.7) implies that $\mathfrak{K}1 = -1/2$. We conclude that

$$f = (\mathfrak{K} - 1/2)g \quad \text{with } g := -1. \tag{5.6.11}$$

This means that the considered integral equation stems from an exterior Laplace–Dirichlet problem (5.6.2). In particular, we expect singularities at the non-convex edges of $\mathbb{R}^3 \setminus \overline{\Omega}$, i.e., at all edges of the cube Ω.

We consider polynomial degrees $p \in \{0, 1, 2\}$. For the initial ansatz space with spline degree $p_{1,m} := p_{2,m} := p$ for all $m \in \{1, \dots 6\}$, we choose the initial knot vectors $\widehat{\mathcal{K}}_{1(0,m)} := \widehat{\mathcal{K}}_{2(0,m)} := (0, \dots 0, 1, \dots, 1)$ for all $m \in \{1, \dots, 6\}$, where the multiplicity of 0 and 1 is $p+1$. We choose the parameters of Algorithm 5.2.4 as $\theta = 0.5$ and $C_{\min} = 1$, where we use the refinement strategy of Remark 5.4.6 (c) in the lowest-order case $p = 0$. For comparison, we also consider uniform refinement, where we mark all elements in each step, i.e., $\mathcal{M}_\ell = \mathcal{T}_\ell$ for all $\ell \in \mathbb{N}_0$. This leads to uniform bisection of all elements. In Figure 5.1, one can see some adaptively generated hierarchical meshes. In Figure 5.2 and Figure 5.3, we plot the energy error $\|\phi - \Phi_\ell\|_{\mathfrak{V}}$ and the error estimator η_ℓ against the number of elements $\#\mathcal{T}_\ell$. All values are plotted in a double logarithmic scale such that the experimental convergence rates are visible as the slope of the corresponding curves. Although we only proved reliability (5.2.22) of the employed estimator, the curves for the error and the estimator are parallel in each case, which numerically indicates reliability and efficiency. The uniform approach always leads to the suboptimal convergence rate $\mathcal{O}((\#\mathcal{T}_\ell)^{-1/3})$ due to the edge singularities. Independently on the chosen polynomial degree p, the adaptive approach leads approximately to the rate $\mathcal{O}((\#\mathcal{T}_\ell)^{-1/2})$. For smooth solutions ϕ, one would expect the rate $\mathcal{O}((\#\mathcal{T}_\ell)^{-3/4-p/2})$; see [SS11, Corollary 4.1.34]. However, according to Theorem 5.4.5, the achieved rate is optimal if one uses the proposed refinement strategy and the resulting hierarchical splines. The reduced optimal convergence rate is probably due to the edge singularites. A similar convergence behavior is also witnessed in [FL07, Section 5.2] for the lowest-order case $p = 0$. [FL07] additionally considers anisotropic refinement which recovers the optimal convergence rate $\mathcal{O}((\#\mathcal{T}_\ell)^{-3/4})$.

5.6.2 Nearly singular solution on quarter pipe

We consider the quarter pipe

$$\Omega := \big\{10^{-1}(r\cos(\beta), r\sin(\beta), z) : \ r \in (1/2,1) \wedge \beta \in (0,\pi/2) \wedge z \in (0,1)\big\}; \tag{5.6.12}$$

see Figure 5.4. We split the boundary Γ into the six surfaces

$$\begin{aligned}
\Gamma_1 &:= \big\{10^{-1}(\cos(\beta)/2, \sin(\beta)/2, z) \, : \, \beta \in (0,\pi/2) \wedge z \in (0,1)\big\}\\
\Gamma_2 &:= \big\{10^{-1}(r, 0, z) \, : \, r \in (1/2,1) \wedge z \in (0,1)\big\}\\
\Gamma_3 &:= \big\{10^{-1}(\cos(\beta), \sin(\beta), z) \, : \, \beta \in (0,\pi/2) \wedge z \in (0,1)\big\}\\
\Gamma_4 &:= \big\{10^{-1}(0, r, z) \, : \, r \in (1/2,1) \wedge z \in (0,1)\big\}\\
\Gamma_5 &:= \big\{10^{-1}(r\cos(\beta), r\sin(\beta), 0) \, : \, r \in (1/2,1) \wedge \beta \in (0,\pi/2)\big\}\\
\Gamma_6 &:= \big\{10^{-1}(r\cos(\beta), r\sin(\beta), 1) \, : \, r \in (1/2,1) \wedge \beta \in (0,\pi/2)\big\}
\end{aligned}$$

$\Gamma_1, \Gamma_3, \Gamma_5$, and Γ_6 can be parametrized by rational splines of degree $p_{1(\gamma,m)} := 2, p_{2(\gamma,m)} := 1$ corresponding to the knot vectors $\widehat{\mathcal{K}}_{1(\gamma,m)} := (0,0,0,1,1,1), \widehat{\mathcal{K}}_{2(\gamma,m)} := (0,0,1,1)$; see [PT97, Chapter 8]. The affine surfaces Γ_2 and Γ_4 can be parametrized by non-rational splines of degree $p_{1(\gamma,m)} := p_{2(\gamma,m)} := 1$ corresponding to the knot vectors $\widehat{\mathcal{K}}_{1(\gamma,m)} := \widehat{\mathcal{K}}_{2(\gamma,m)} := (0,0,1,1)$; see [GHP17, Section 6.1].

We prescribe the exact solution of the interior Laplace–Dirichlet problem (5.6.6) as the shifted fundamental solution

$$u(x) := G(x - y_0) = \frac{1}{4\pi}\frac{1}{|x-y_0|}, \tag{5.6.13}$$

with $y_0 := 10^{-1}(0.95 \cdot 2^{-3/2}, 0.95 \cdot 2^{-3/2}, 1/2) \in \mathbb{R}^3 \setminus \overline{\Omega}$. Although u is smooth on $\overline{\Omega}$, it is nearly singular at the midpoint $\widetilde{y}_0 := 10^{-1}(2^{-3/2}, 2^{-3/2}, 1/2)$ of Γ_1. We consider the corresponding integral equation (5.6.7). The normal derivative $\phi = \partial_\nu u$ of u reads

$$\phi(x) = -\frac{1}{4\pi}\frac{x-y_0}{|x-y_0|^3} \cdot \nu(x). \tag{5.6.14}$$

We consider polynomial degrees $p \in \{0,1,2\}$. For the initial ansatz space with spline degree $p_{1,m} := p_{2,m} := p$ for all $m \in \{1,\dots 6\}$, we choose the initial knot vectors $\widehat{\mathcal{K}}_{1(0,m)} := \widehat{\mathcal{K}}_{2(0,m)} := (0,\dots 0,1,\dots,1)$ for all $m \in \{1,\dots,6\}$, where the multiplicity of 0 and 1 is $p+1$. We choose the parameters of Algorithm 5.2.4 as $\theta = 0.5$ and $C_{\min} = 1$, where we use the refinement strategy of Remark 5.4.6 (c) in the lowest-order case $p = 0$. For comparison, we also consider uniform refinement, where we mark all elements in each step, i.e., $\mathcal{M}_\ell = \mathcal{T}_\ell$ for all $\ell \in \mathbb{N}_0$. This leads to uniform bisection of all elements. In Figure 5.4, one can see some adaptively generated hierarchical meshes. In the Figure 5.5 and Figure 5.6, we plot the energy error $\|\phi - \Phi_\ell\|_{\mathfrak{V}}$ and the error estimator η_ℓ against the number of elements $\#\mathcal{T}_\ell$. All values are plotted in a double logarithmic scale such that the experimental convergence rates are visible as the slope of the corresponding curves. In all cases, the lines of the error and the error estimator are parallel, which numerically indicates reliability and efficiency. Since the solution ϕ is smooth, the uniform and the adaptive approach both lead to the optimal asymptotic convergence rate $\mathcal{O}((\#\mathcal{T}_\ell)^{-3/4-p/2})$. However, ϕ is nearly singular at $\widetilde{y}_0$, wherefore adaptivity yields a much better multiplicative constant.

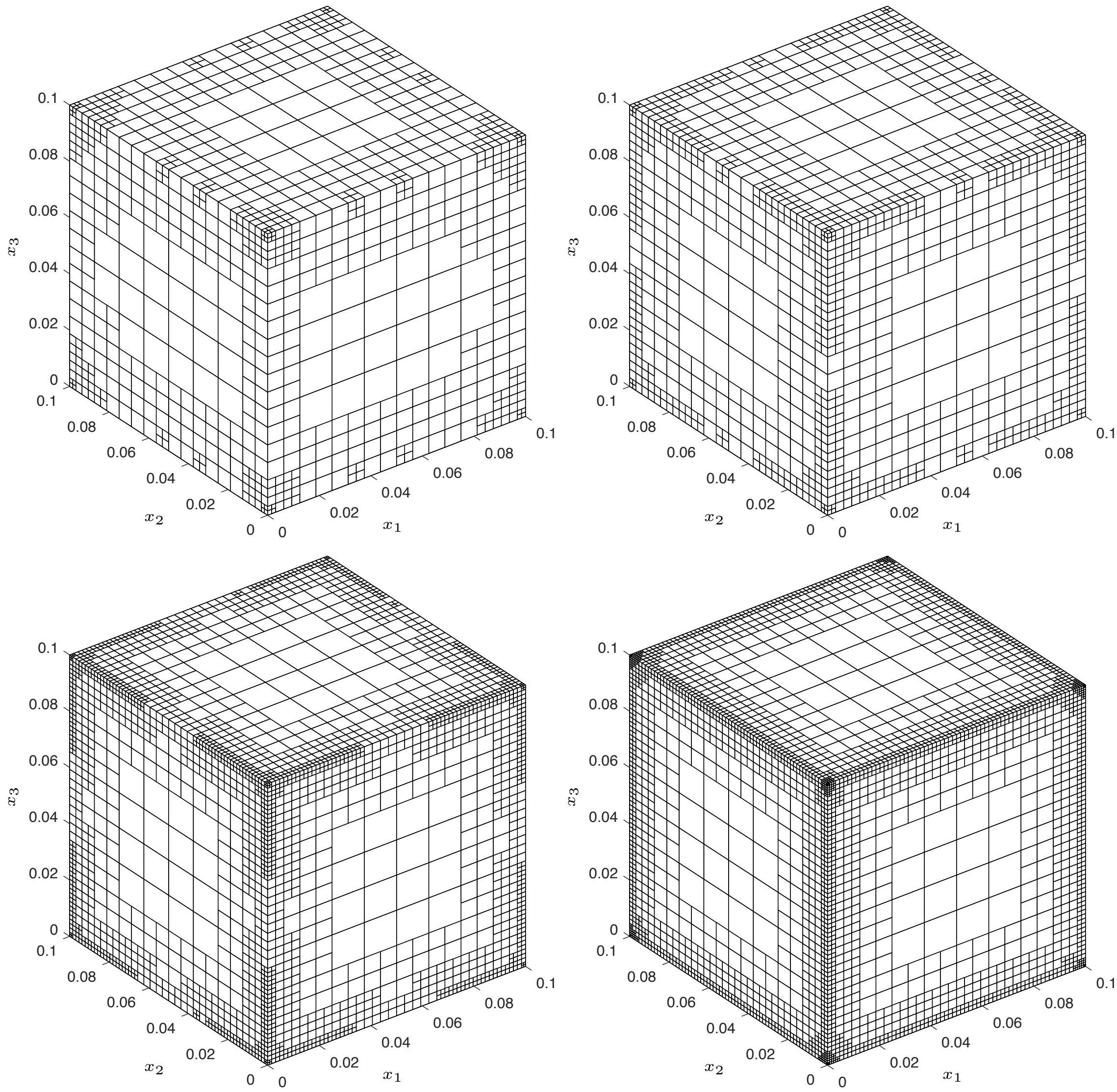


Figure 5.1: Experiment with solution with edge singularities on cube of Section 5.6.1. Hierarchical meshes $\mathcal{T}_8, \mathcal{T}_{10}, \mathcal{T}_{11}, \mathcal{T}_{13}$ generated by Algorithm 5.2.4 (with $\theta = 0.5$) for hierarchical splines of degree $p = 1$.

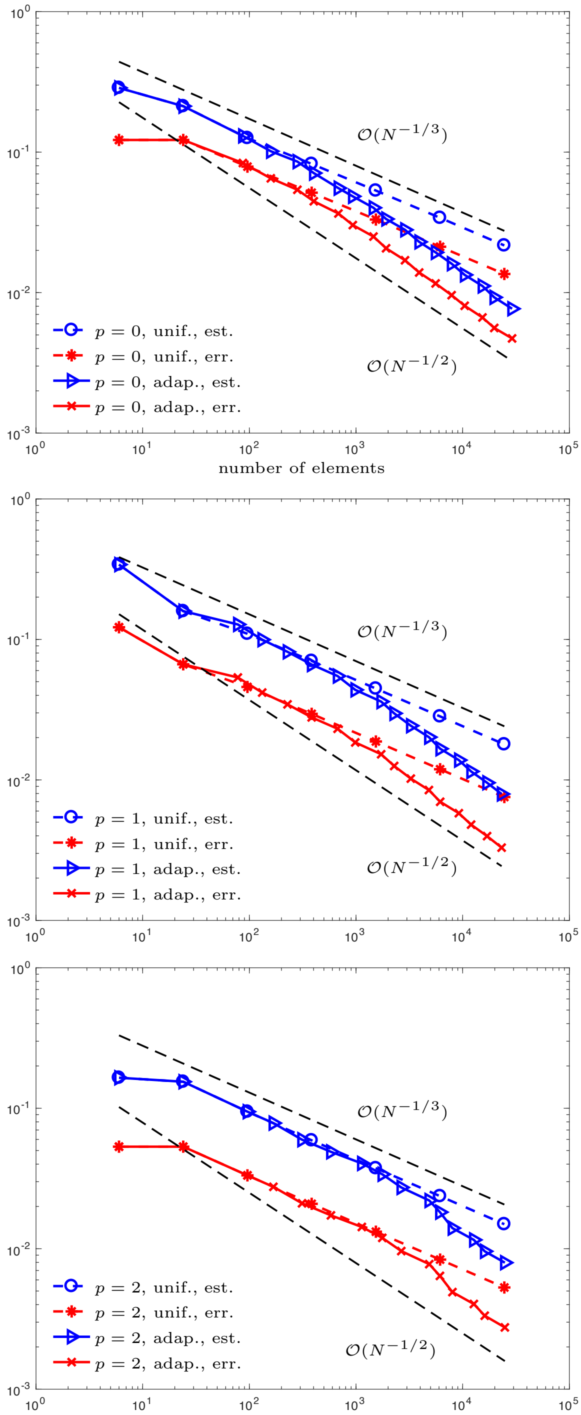


Figure 5.2: Experiment with solution with edge singularities on cube of Section 5.6.1. Energy error $\|\phi - \Phi_\ell\|_{\mathfrak{V}}$ and estimator η_ℓ of Algorithm 5.2.4 for hierarchical splines of degree $p \in \{0,1,2\}$ are plotted versus the number of elements $\#\mathcal{T}_\ell$. Uniform and adaptive ($\theta = 0.5$) refinement is considered.

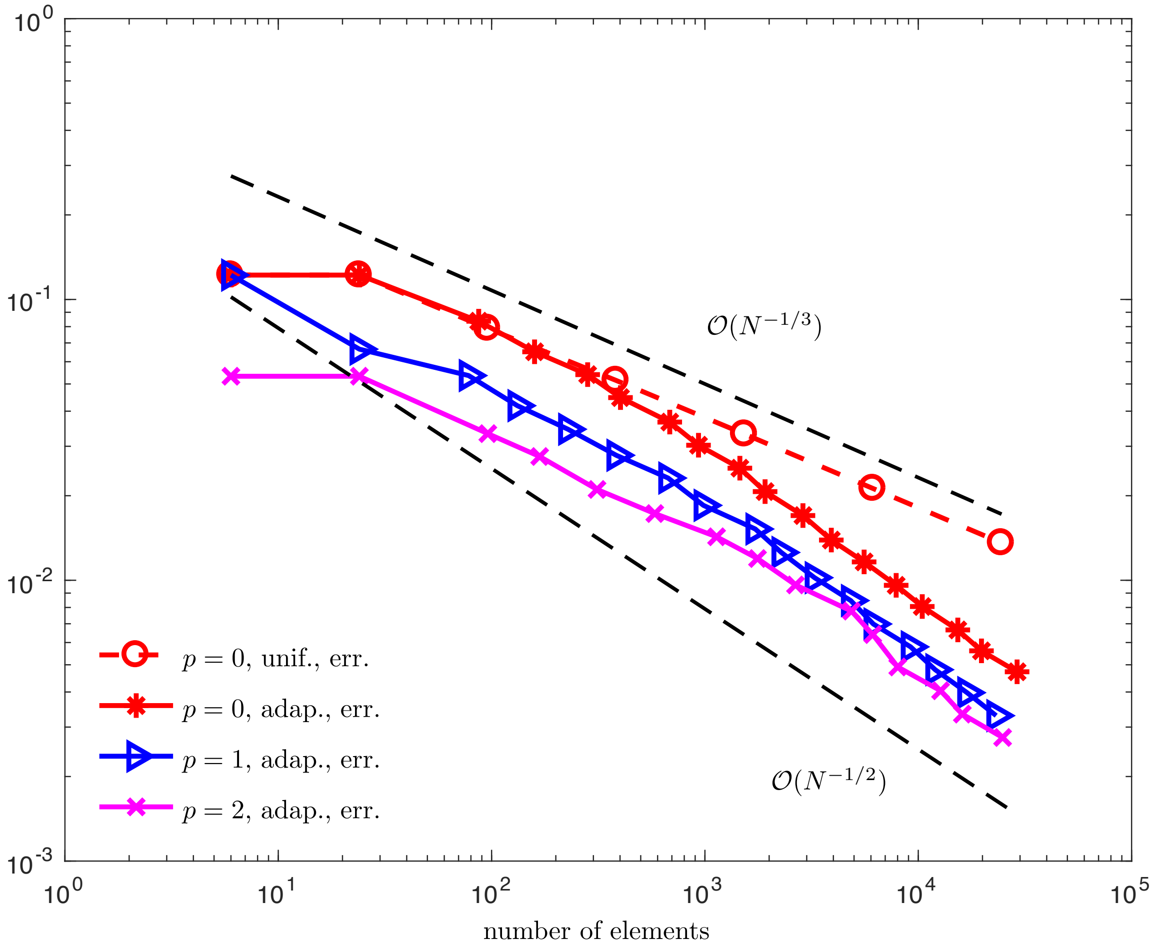


Figure 5.3: Experiment with solution with edge singularities on cube of Section 5.6.1. The energy errors $\|\phi - \Phi_\ell\|_{\mathfrak{V}}$ of Algorithm 5.2.4 for hierarchical splines of degree $p \in \{0, 1, 2\}$ are plotted versus the number of elements $\#\mathcal{T}_\ell$. Uniform (for $p = 0$) and adaptive ($\theta = 0.5$ for $p \in \{0, 1, 2\}$) refinement is considered.

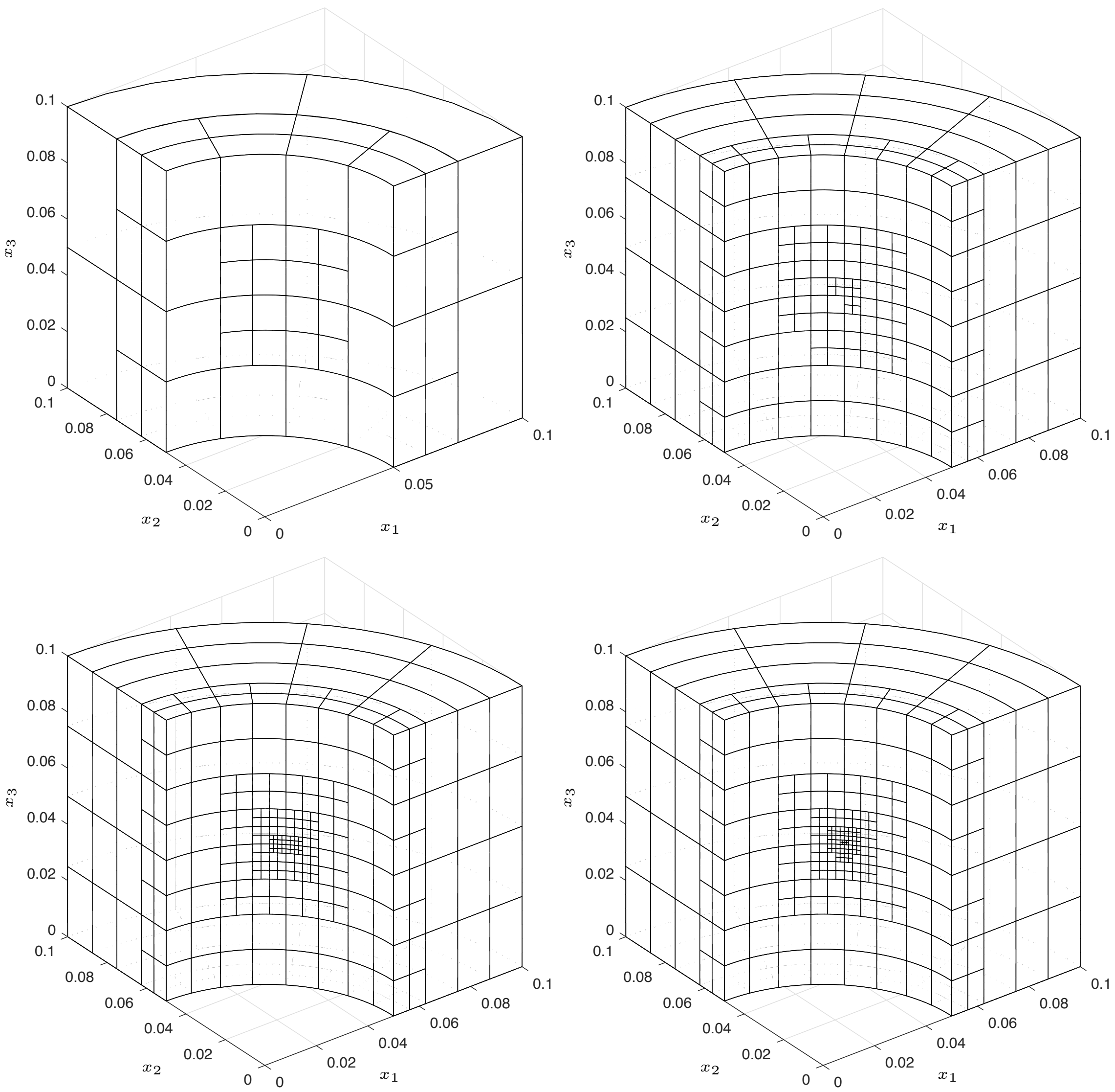


Figure 5.4: Experiment with nearly singular solution on quarter pipe of Section 5.6.2. Hierarchical meshes $\mathcal{T}_4, \mathcal{T}_7, \mathcal{T}_9, \mathcal{T}_{10}$ generated by Algorithm 5.2.4 (with $\theta = 0.5$) for hierarchical splines of degree $p = 1$.

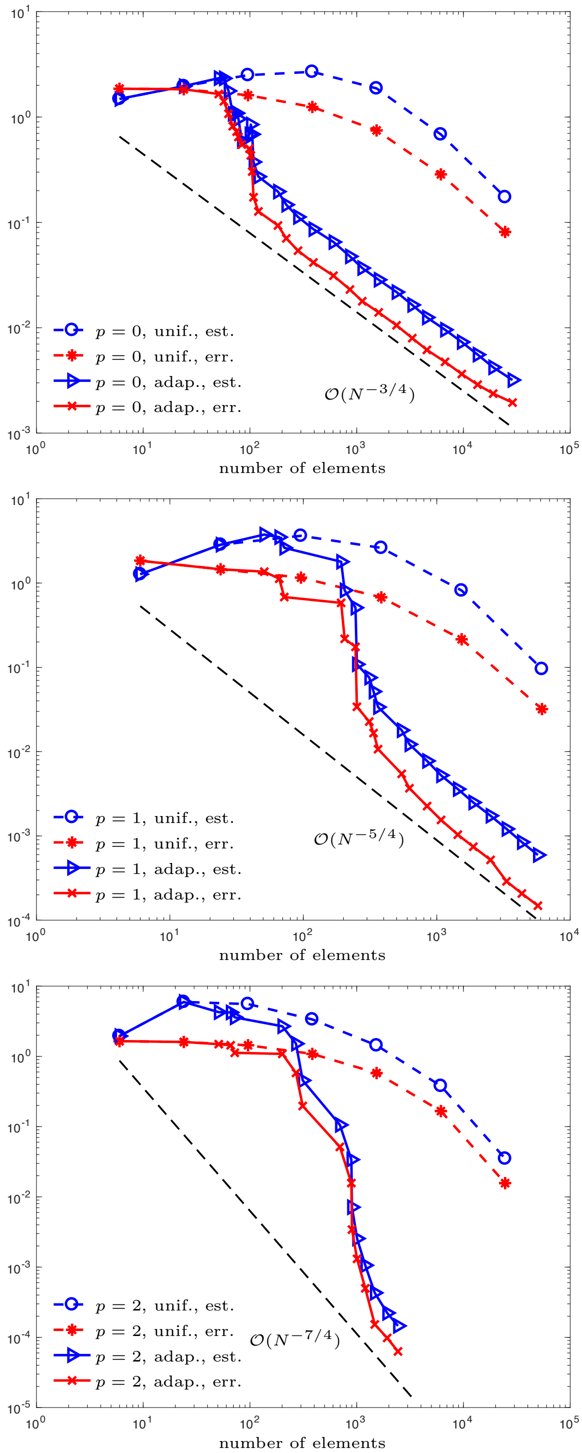


Figure 5.5: Experiment with nearly singular solution on quarter pipe of Section 5.6.2. Energy error $\|\phi - \Phi_\ell\|_{\mathfrak{Y}}$ and estimator η_ℓ of Algorithm 5.2.4 for hierarchical splines of degree $p \in \{0, 1, 2\}$ are plotted versus the number of elements $\#\mathcal{T}_\ell$. Uniform and adaptive ($\theta = 0.5$) refinement is considered.

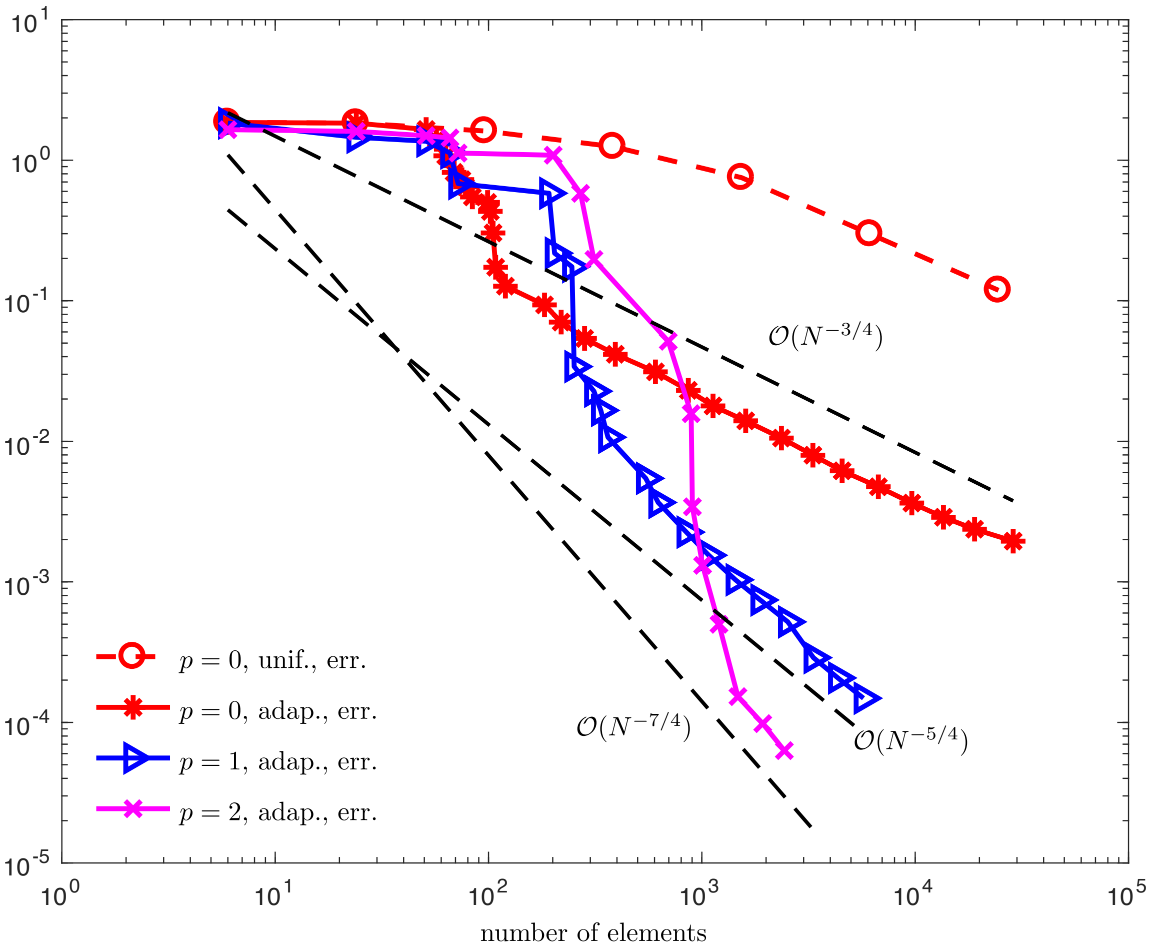


Figure 5.6: Experiment with nearly singular solution on quarter pipe of Section 5.6.2. The energy errors $\|\phi - \Phi_\ell\|_{\mathfrak{V}}$ of Algorithm 5.2.4 for hierarchical splines of degree $p \in \{0, 1, 2\}$ are plotted versus the number of elements $\#\mathcal{T}_\ell$. Uniform (for $p = 2$) and adaptive ($\theta = 0.5$ for $p \in \{0, 1, 2\}$) refinement is considered.

5.7 Boundary element method with one-dimensional splines

In this section, we consider the model problem (4.1.1) for $d = 2$. To ease presentation and without loss of generality, we assume that the one-dimensional boundary $\Gamma \subset \mathbb{R}^2$ is connected. If Γ consists of finitely many connected components, the following results hold accordingly. We introduce univariate splines on Γ and propose a node-based local mesh-refinement. In contrast to the previous refinement strategies that we have considered, this strategy does not only use element bisections but also increases certain knot multiplicities leading to local regularity reduction of the ansatz functions. We use the notation of Section 3.3, where we already introduced standard splines in the *parameter domain*. We define a node-based version of the residual error indicators of the previous section; see (5.7.17) below. These are used to steer an adaptive algorithm (Algorithm 5.7.3), which was developed and numerically investigated in the recent own work [FGP15].

The main result of this section is Theorem 5.7.4 which states that the estimator is reliable and the adaptive algorithm leads to linear convergence at optimal algebraic rate. This theorem is already found in the recent own works [FGHP16, FGHP17], where [FGHP16] proves reliability, and [FGHP17] proves linear convergence at optimal rate. It is covered by the abstract framework of Chapter 2. The verification of the corresponding axioms is done in Section 5.8.

5.7.1 Parametrization of the boundary

We set $\widehat{\Gamma} := [0, 1]$, and assume that the boundary Γ can be parametrized by a closed continuous curve

$$\gamma : \widehat{\Gamma} \to \Gamma \quad \text{with } \gamma(0) = \gamma(1) \tag{5.7.1}$$

such that the restriction $\gamma|_{[0,1)}$ is bijective. Throughout and by abuse of notation, we write γ^{-1} for the inverse of $\gamma|_{[0,1)}$ resp. $\gamma|_{(0,1]}$. The meaning will be clear from the context. Moreover, we suppose the existence of finitely many points $\widehat{\mathcal{N}}_\gamma = \big\{\widehat{z}_{\gamma,m} : m \in \{0, \dots, M\}\big\} \subset [0, 1]$ with $0 = \widehat{z}_{\gamma,0} < \widehat{z}_{\gamma,1} < \cdots < \widehat{z}_{\gamma,M} \le 1$ such that, with $\widehat{\Gamma}_m := [\widehat{z}_{\gamma,m-1}, \widehat{z}_{\gamma,m}]$ and $\Gamma_m := \gamma(\widehat{\Gamma}_m)$,

$$\gamma_m := \gamma|_{\widehat{\Gamma}_m} : \widehat{\Gamma}_m \to \Gamma_m \tag{5.7.2}$$

is bi-Lipschitz for all $m \in \{1, \dots, M\}$. In particular, Lemma 5.2.1 (applied on the interior of $\widehat{\Gamma}_m$) shows that γ_m is almost everywhere differentiable, and there exists a constant $C_\gamma > 0$ such that

$$C_\gamma^{-1}|s - t| \le |\gamma_m(s) - \gamma_m(t)| \le C_\gamma|s - t| \quad \text{for all } s, t \in \widehat{\Gamma}_m \tag{5.7.3a}$$

and

$$C_\gamma^{-1} \le |\gamma_m'(t)| \le C_\gamma \quad \text{for almost all } t \in \widehat{\Gamma}_m. \tag{5.7.3b}$$

We additionally suppose that $M \ge 3$ so that each node patch $\pi_\gamma(z) := \bigcup\big\{\Gamma_m : m \in \{1, \dots, M\} \wedge z \in \Gamma_m\big\}$ for $z \in \mathcal{N}_\gamma := \big\{\gamma(\widehat{z}_{\gamma,m}) : m \in \{1, \dots, M\}\big\}$ can be transformed to some interval: For all nodes $z = \widehat{z}_{\gamma,m}$ for $m \in \{1,\dots,M\}$, we define an interval $\overline{\pi}_\gamma(z) \subset \mathbb{R}$ and a mapping

$$\gamma_z : \overline{\pi}_\gamma(z) \to \pi_\gamma(z), \tag{5.7.4}$$

via $\gamma_z := \gamma|_{[\widehat{z}_{\gamma,m-1},\widehat{z}_{\gamma,m+1}]}$ for $m < M$ resp. $\gamma_z|_{[\widehat{z}_{\gamma,M-1},1]} := \gamma|_{[\widehat{z}_{\gamma,M-1},1]}$ and $\gamma_z|_{[1,\widehat{z}_{\gamma,1}+1]} := \gamma((\cdot)-1)|_{[1,\widehat{z}_{\gamma,1}+1]}$ for $m = M$. Then, γ_z is automatically Lipschitz continuous. We additionally assume that also the inverse γ_z^{-1} is Lipschitz continuous, wherefore γ_z is a bi-Lipschitz mapping.

5.7.2 One-dimensional splines on the boundary

Let $p \in \mathbb{N}_0$ be a fixed polynomial degree. For any p-open knot vector $\widehat{\mathcal{K}}_\bullet$ on $[0,1]$, we define the space of all splines on $\widehat{\Gamma} = [0,1]$ as

$$\widehat{\mathcal{X}}_\bullet := \widehat{\mathcal{S}}^p(\widehat{\mathcal{K}}_\bullet)^D \subset \big\{\widehat{\psi} : \widehat{\Gamma} \to \mathbb{R}^D \,:\, \widehat{\psi}|_T \in C^\infty(\widehat{T})^D \text{ for all } \widehat{T} \in \widehat{\mathcal{T}}_\bullet\big\}. \tag{5.7.5}$$

In order to transform the definitions from the parameter domain $\widehat{\Gamma}$ to the boundary Γ, we use the parametrization γ of Section 5.7.1. All previous definitions can now also be made on Γ, just by pulling them from the parameter domain via the mapping γ. For these definitions, we drop the symbol $\widehat{\cdot}$. If $\widehat{\mathcal{K}}_\bullet = (t_{\bullet,0},\dots,t_{\bullet,N_\bullet})$ is a p-open knot vector in the parameter domain $\widehat{\Gamma}$, we define the corresponding knot vector $\mathcal{K}_\bullet$ on Γ as the vector $(\gamma(t_{\bullet,0}),\dots,\gamma(t_{\bullet,N_\bullet}))$. Moreover, we define the nodes $\mathcal{N}_\bullet := \big\{\gamma(\widehat{z}) \,:\, \widehat{z} \in \widehat{\mathcal{N}}_\bullet\big\}$ on Γ as the set of all knots. If $\widehat{\mathcal{T}}_\bullet$ is the corresponding mesh in the parameter domain, we set $\mathcal{T}_\bullet := \big\{\gamma(\widehat{T}) \,:\, \widehat{T} \in \widehat{\mathcal{T}}_\bullet\big\}$. Clearly, $\mathcal{T}_\bullet$ is a mesh in the sense of Section 5.2.1, where

$$\widehat{T} := \gamma^{-1}(T) \quad \text{and} \quad \gamma_T := \gamma|_{\widehat{T}} \quad \text{for } T \in \mathcal{T}_\bullet, \tag{5.7.6}$$

and we can use the notation from there. We introduce the corresponding spline space on Γ

$$\mathcal{X}_\bullet := \big\{\Psi_\bullet \circ \gamma^{-1} \,:\, \Psi_\bullet \in \mathcal{X}_\bullet\big\} \subset L^2(\Gamma)^D \subset H^{-1/2}(\Gamma)^D. \tag{5.7.7}$$

According to (3.3.8), a basis of $\mathcal{X}_\bullet$ is given by the (transformed) B-splines

$$\mathcal{X}_\bullet = \operatorname{span}(\mathcal{B}_\bullet^D) \quad \text{with} \quad \mathcal{B}_\bullet := \big\{\widehat{B}_{\bullet,j,p} \circ \gamma^{-1} \,:\, j \in \{1,\dots,N_\bullet\}\big\}. \tag{5.7.8}$$

5.7.3 Refinement of knot vectors

In this section, we present a concrete refinement algorithm which uses both bisection and knot multiplicity increase. To this end, we also introduce an auxiliary refinement algorithm which fits into the setting of Section 5.2.2. We start in the parameter domain. Recall that we call a p-open knot vector $\widehat{\mathcal{K}}_\circ$ finer than another p-open knot vector $\widehat{\mathcal{K}}_\bullet$ if $\widehat{\mathcal{K}}_\bullet$ is a subsequence of $\widehat{\mathcal{K}}_\circ$. In this case, (3.3.9) implies that the corresponding spaces are nested, i.e.,

$$\widehat{\mathcal{X}}_\bullet \subseteq \widehat{\mathcal{X}}_\circ. \tag{5.7.9}$$

To transfer this definition onto the boundary Γ, we essentially just drop the symbol $\widehat{\cdot}$. We say that a p-open knot vector $\mathcal{K}_\circ$ on Γ is *finer* than another p-open knot vector $\mathcal{K}_\bullet$ on Γ, if the corresponding knots in the parameter domain satisfy this relation, i.e., if $\widehat{\mathcal{K}}_\circ$ is finer than $\widehat{\mathcal{K}}_\bullet$. In this case, there holds that

$$\mathcal{X}_\bullet \subseteq \mathcal{X}_\circ. \tag{5.7.10}$$

Let $\mathcal{K}_0$ be a fixed initial p-open knot vector on Γ such that

$$\mathcal{N}_\gamma \subseteq \mathcal{N}_0. \tag{5.7.11}$$

We set

$$\widehat{\kappa}_0 := \max\Big\{\frac{|\widehat{T}|}{|\widehat{T}'|} : \widehat{T}, \widehat{T}' \in \widehat{\mathcal{T}}_0 \text{ with } T \cap T' \neq \emptyset\Big\}. \tag{5.7.12}$$

For a p-open knot vector $\mathcal{K}_\bullet$ on Γ and $T \in \mathcal{T}_\bullet$, we define the set

$$\Pi_\bullet^{\mathrm{bad}}(T) := \big\{T' \in \Pi_\bullet(T) \,:\, |\widehat{T}'| > \widehat{\kappa}_0 |\widehat{T}|\big\}. \tag{5.7.13}$$

With this, we can formulate the first auxiliary refinement procedure of [AFF+13, Algorithm 2].

Algorithm 5.7.1. ***Input:*** *p-open knot vector $\mathcal{K}_\bullet$, marked elements $\mathcal{M}_\bullet =: \mathcal{M}_\bullet^{(0)} \subseteq \mathcal{T}_\bullet$.*

(i) *Iterate the following steps* (a)–(b) *for $i = 0, 1, 2, \ldots$ until $\mathcal{U}_\bullet^{(i)} = \emptyset$:*
   (a) *Define $\mathcal{U}_\bullet^{(i)} := \bigcup_{T \in \mathcal{M}_\bullet^{(i)}} \big\{T' \in \mathcal{T}_\bullet \setminus \mathcal{M}_\bullet^{(i)} \,:\, T' \in \Pi_\bullet^{\mathrm{bad}}(T)\big\}$.*
   (b) *Define $\mathcal{M}_\bullet^{(i+1)} := \mathcal{M}_\bullet^{(i)} \cup \mathcal{U}_\bullet^{(i)}$.*

(ii) *Bisect all $T \in \mathcal{M}_\bullet^{(i)}$ in the parameter domain by inserting the midpoint of the corresponding $\widehat{T} \in \widehat{\mathcal{T}}_\bullet$ with multiplicity one in the knot vector $\widehat{\mathcal{K}}_\bullet$ and obtain a finer knot vector $\widehat{\mathcal{K}}_\circ$.*

Output: *Refined p-open knot vector $\mathcal{K}_\circ = \mathtt{refine}(\mathcal{K}_\bullet, \mathcal{M}_\bullet)$.*

The next algorithm is the main refinement strategy which we will use in the adaptive Algorithm 5.7.3. In contrast to Algorithm 5.7.1, it receives marked nodes instead of marked elements as input and also uses knot multiplicity increase for refinement.

Algorithm 5.7.2. ***Input:*** *p-open knot vector $\mathcal{K}_\bullet$, marked nodes $\mathcal{M}_\bullet \subseteq \mathcal{N}_\bullet$.*

(i) *Define the set of marked elements $\mathcal{M}'_\bullet := \emptyset$.*

(ii) *If both nodes of an element $T \in \mathcal{T}_\bullet$ belong to $\mathcal{M}_\bullet$, mark the element T by adding it to $\mathcal{M}'_\bullet$.*

(iii) *For all other nodes in $\mathcal{M}_\bullet$, increase the multiplicity if it is less or equal to $p+1$. Otherwise mark the elements which contain one of these nodes, by adding them to $\mathcal{M}'_\bullet$.*

(iv) *With obtained knot vector* $\mathcal{K}_\star$, *define* $\mathcal{K}_\circ := \texttt{refine}(\mathcal{K}_\star, \mathcal{M}'_\bullet)$.

Output: *Refined p-open knot vector* $\mathcal{K}_\circ = \texttt{refine}(\mathcal{K}_\bullet, \mathcal{M}_\bullet)$.

Clearly, $\texttt{refine}(\mathcal{K}_\bullet, \mathcal{M}_\bullet)$ is finer than $\mathcal{K}_\bullet$. For any p-open knot vector $\mathcal{K}_\bullet$ on Γ, we define $\texttt{refine}(\mathcal{K}_\bullet)$ as the set of all p-open knot vectors $\mathcal{K}_\circ$ on Γ such that there exist p-open knot vectors $\mathcal{K}_{(0)}, \dots, \mathcal{K}_{(J)}$ and marked nodes $\mathcal{M}_{(0)}, \dots, \mathcal{M}_{(J-1)}$ with $\mathcal{K}_\circ = \mathcal{K}_{(J)} = \texttt{refine}(\mathcal{K}_{(J-1)}, \mathcal{M}_{(J-1)}), \dots, \mathcal{K}_{(1)} = \texttt{refine}(\mathcal{K}_{(0)}, \mathcal{M}_{(0)})$, and $\mathcal{K}_{(0)} = \mathcal{K}_\bullet$. Note that $\texttt{refine}(\mathcal{K}_\bullet, \emptyset) = \mathcal{K}_\bullet$, wherefore $\mathcal{K}_\bullet \in \texttt{refine}(\mathcal{K}_\bullet)$. We define the set of all *admissible* p-open knot vectors on Γ as

$$\mathbb{K} := \texttt{refine}(\mathcal{K}_0). \tag{5.7.14}$$

Similarly as in Proposition 5.4.3, one shows that $\mathcal{K}_\bullet \in \mathbb{K}$ implies that

$$|\widehat{T}|/|\widehat{T}'| \leq 2\widehat{\kappa}_0 \quad \text{for all } T, T' \in \mathcal{T}_\bullet \text{ with } T \cap T' \neq \emptyset. \tag{5.7.15}$$

This is also proved in [AFF+13, Theorem 3]. Indeed, one can show (as in Proposition 5.4.3) that $\mathbb{K}$ coincides with the set of all p-open knot vectors $\mathcal{K}_\bullet$ which are obtained via iterative bisections in the parameter domain and arbitrary knot multiplicity increases which satisfy (5.7.15). Further, we define the corresponding *admissible* meshes on Γ

$$\mathbb{T} := \big\{\mathcal{T}_\bullet \,:\, \mathcal{K}_\bullet \in \mathbb{K}\big\}, \tag{5.7.16}$$

which coincides with the set of all meshes which result from iterative bisections in the parameter domain, and which satisfy (5.7.15).

5.7.4 Error estimator

Let $\mathcal{K}_\bullet \in \mathbb{K}$. Due to the mapping property (5.1.11) and $\mathcal{X}_\bullet \subset L^2(\Gamma)^D$, there holds that $\mathfrak{V}\Psi_\bullet \in H^1(\Gamma)^D$ for all $\Psi_\bullet \in \mathcal{X}_\bullet$. This allows to employ a node-based version of the weighted-residual *a posteriori* error estimator of Section 5.2.4

$$\eta_\bullet := \eta_\bullet(\mathcal{N}_\bullet) \quad \text{with} \quad \eta_\bullet(\mathcal{S})^2 := \sum_{z\in\mathcal{S}} \eta_\bullet(z)^2 \text{ for all } \mathcal{S} \subseteq \mathcal{N}_\bullet, \tag{5.7.17a}$$

where, for all $z \in \mathcal{N}_\bullet$, the local refinement indicators read

$$\eta_\bullet(z)^2 := |\pi_\bullet(z)| |f - \mathfrak{V}\Phi_\bullet|^2_{H^1(\pi_\bullet(z))}. \tag{5.7.17b}$$

5.7.5 Adaptive algorithm

We consider the following adaptive algorithm.

Algorithm 5.7.3. ***Input:*** *Dörfler parameter* $\theta \in (0, 1]$ *and marking constant* $C_{\min} \in [1, \infty]$.
Loop: *For each* $\ell = 0, 1, 2, \dots,$ *iterate the following steps:*

(i) *Compute Galerkin approximation* $\Phi_\ell \in \mathcal{X}_\ell$.

(ii) *Compute refinement indicators* $\eta_\ell(z)$ *for all nodes* $z \in \mathcal{N}_\ell$.

(iii) *Determine a set of marked nodes $\mathcal{M}_\ell \subseteq \mathcal{N}_\ell$ which has up to the multiplicative constant $C_{\min}$ minimal cardinality, such that the following Dörfler marking is satisfied*

$$\theta\,\eta_\ell^2 \le \eta_\ell(\mathcal{M}_\ell)^2. \tag{5.7.18}$$

(iv) *Generate refined knot vector $\mathcal{K}_{\ell+1} := \texttt{refine}(\mathcal{K}_\ell, \mathcal{M}_\ell)$.*

Output: *Refined knot vectors $\mathcal{K}_\ell$ and corresponding Galerkin approximations Φ_ℓ with error estimators η_ℓ for all $\ell \in \mathbb{N}_0$.*

5.7.6 Optimal convergence for one-dimensional splines

Recall that, for $\mathcal{K}_\bullet \in \mathbb{K}$, $N_\bullet + 1$ denotes the number of all knots in the parameter domain $[0,1]$. We define

$$\mathbb{K}(N) := \big\{\mathcal{K}_\bullet \in \mathbb{K} \,:\, N_\bullet - N_0 \le N\big\} \quad \text{for all } N \in \mathbb{N}_0 \tag{5.7.19}$$

and for all $s > 0$

$$C_{\rm approx}(s) := \sup_{N\in\mathbb{N}_0} \min_{\mathcal{K}_\bullet \in \mathbb{K}(N)} (N+1)^s\,\eta_\bullet \in [0,\infty]. \tag{5.7.20}$$

We say that the solution $\phi \in H^{-1/2}(\Gamma)^D$ lies in the *approximation class s with respect to the estimator* if

$$\|\phi\|_{\mathbb{A}_s^{\rm est}} := C_{\rm approx}(s) < \infty. \tag{5.7.21}$$

By definition, $\|\phi\|_{\mathbb{A}_s^{\rm est}} < \infty$ implies that the error estimator $\eta_\bullet$ on the optimal knots vectors $\mathcal{K}_\bullet$ decays at least with rate $\mathcal{O}\big(N_\bullet^{-s}\big)$. The following main theorem states that each possible rate $s > 0$ is in fact realized by Algorithm 5.7.3. The proof is given in Section 5.8 and is also found in the recent own work [FGHP17, Theorem 3.2]. It essentially follows from its abstract counterpart Theorem 2.3.1 by verifying the axioms of Section 2.3. In particular, Theorem 5.7.4 (i) states reliability which was verified for the current setting in the recent own works [FGHP16, Theorem 4.4] and [Gan14, Theorem 3.8].

Theorem 5.7.4. *Let $(\mathcal{K}_\ell)_{\ell\in\mathbb{N}_0}$ be the sequence of knots generated by Algorithm 5.7.3. Then, there hold:*

(i) *The residual error estimator satisfies reliability, i.e., there exists a constant $C_{\rm rel} > 0$ such that*

$$\|\phi - \Phi_\bullet\|_{H^{-1/2}(\Gamma)} \le C_{\rm rel}\eta_\bullet \quad \textit{for all } \mathcal{K}_\bullet \in \mathbb{K}. \tag{5.7.22}$$

(ii) *For arbitrary $0 < \theta \le 1$ and $C_{\min} \in [1,\infty]$, the residual error estimator converges linearly, i.e., there exist constants $0 < \rho_{\rm lin} < 1$ and $C_{\rm lin} \ge 1$ such that*

$$\eta_{\ell+j}^2 \le C_{\rm lin}\rho_{\rm lin}^j \eta_\ell^2 \quad \textit{for all } j,\ell \in \mathbb{N}_0. \tag{5.7.23}$$

(iii) *There exists a constant* $0 < \theta_{\rm opt} \le 1$ *such that for all* $0 < \theta < \theta_{\rm opt}$ *and* $C_{\rm min} \in [1,\infty)$, *the estimator converges at optimal rate, i.e., for all* $s > 0$ *there exist constants* $c_{\rm opt}, C_{\rm opt} > 0$ *such that*

$$c_{\rm opt}\|\phi\|_{\mathbb{A}_s^{\rm est}} \le \sup_{\ell\in\mathbb{N}_0} (N_\ell - N_0 + 1)^s\,\eta_\ell \le C_{\rm opt}\|\phi\|_{\mathbb{A}_s^{\rm est}}. \tag{5.7.24}$$

All involved constants $C_{\rm rel}, C_{\rm lin}, \rho_{\rm lin}, \theta_{\rm opt}$, *and* $C_{\rm opt}$ *depend only on the dimension* D, *the coefficients of the differential operator* $\mathfrak{P}$, *the parametrization* γ, *the polynomial order* p, *and the initial mesh* $\widehat{\mathcal{T}}_0$, *while* $C_{\rm lin}, \rho_{\rm lin}$ *depend additionally on* θ *and the sequence* $(\Phi_\ell)_{\ell\in\mathbb{N}_0}$, *and* $C_{\rm opt}$ *depends furthermore on* $C_{\rm min}$ *and* $s > 0$. *The constant* $c_{\rm opt}$ *depends only on* p, N_0, s, *and if there exists* ℓ_0 *with* $\eta_{\ell_0} = 0$, *also on* ℓ_0 *and* η_0.

Remark 5.7.5. *If the bilinear form* $\langle \mathfrak{V}(\cdot)\,,\,\cdot\rangle$ *is symmetric, then* $C_{\rm lin}$, $\rho_{\rm lin}$, *and* $C_{\rm opt}$ *are independent of* $(\Phi_\ell)_{\ell\in\mathbb{N}_0}$; *see Remark 5.3.17.*

Remark 5.7.6. *Theorem 5.7.4 is still valid if one replaces the ansatz space* $\mathcal{X}_\bullet$ *by* rational one-dimensional splines, *i.e., by the set*

$$\mathcal{X}_\bullet^{W_0} := \Big\{W_0^{-1}\Psi_\bullet : \Psi_\bullet \in \mathcal{X}_\bullet\Big\}, \tag{5.7.25}$$

where $\widehat{W}_0 := W_0\circ\gamma$ *is a fixed positive weight function in the initial space of splines* $\widehat{\mathcal{S}}^p(\widehat{\mathcal{K}}_0)$. *With the B-spline basis* $\widehat{\mathcal{B}}_0 = \big\{\widehat{B}_{0,j,p}|_{[0,1)} : j \in \{1,\dots,N_\bullet\}\big\}$, *we even suppose that* $\widehat{W}_0$ *can be written as*

$$\widehat{W}_0 = \sum_{j=1}^{N_0} w_{0,j}\,\widehat{B}_{0,j,p}|_{[0,1)} \quad \textit{with non-negative coefficients } w_{0,j} \ge 0. \tag{5.7.26}$$

We will prove this version in Section 5.8.11. Then, the constants depend additionally on W_0.

Remark 5.7.7. *If one modifies the adaptive Algorithm 5.7.3 such that* η_ℓ *denotes again the element-based residual error estimator of Section 5.2.4,* $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ *is a set of marked elements, and* `refine`$(\mathcal{K}_\ell, \mathcal{M}_\ell)$ *results from Algorithm 5.7.1 (which does not use knot multiplicity increase) instead of Algorithm 5.7.2, it fits into the abstract framework of Section 5.2. Indeed, all the assumptions from there are satisfied, which can be proved similarly as in Section 5.5; see also Section 5.8. In particular, Theorem 5.2.5 is applicable and guarantees linear convergence of the estimator at optimal algebraic rate. Again, one can also use rational splines as in Remark 5.7.6.*

Remark 5.7.8. *If* $\mathfrak{P} := -\Delta$ *is chosen as the Laplace operator, [Sch16] proves that Theorem 5.7.4 resp. the generalization of Remark 5.7.6 holds accordingly for integral equations of the form*

$$\mathfrak{W}u = f, \tag{5.7.27}$$

where $\mathfrak{W} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ *denotes the* hyper-singular operator *and* $f \in H^1(\Gamma)$ *is a given right-hand side. Such equations arise from the solution of Neumann problems of* *the form $\mathfrak{P}u = 0$ in Ω with $\mathfrak{D}_\nu u = \phi$ on Γ for some $\phi \in H^{-1/2}(\Gamma)$; see, e.g., [McL00, pages 229–231] for more details. In this case, one has to choose continuous ansatz spaces $\mathcal{Y}_\bullet := \mathcal{X}_\bullet \cap C^0(\Gamma) \subset H^1(\Gamma) \subset H^{1/2}(\Gamma)$. Note that all functions $\mathcal{X}_\bullet$ are continuous on $\Gamma \setminus \{\gamma(0)\}$ provided that the corresponding knot multiplicities are bounded by p instead of $p+1$. Hence, the adapted algorithm of [Sch16] only increases knot multiplicities up to the value p. Since, for the Laplace operator, $\mathfrak{W}$ is only elliptic up to constant functions, the analysis of [Sch16] requires an additional stabilization term for the induced bilinear form.*

5.8 Proof of Theorem 5.7.4

In Section 5.8.2, we show reliability (5.2.22). The proof works as in Section 5.3.2. To prove Theorem 5.7.4 (ii)–(iii), we want to apply the abstract results of Chapter 2. However, at first glance, the adaptive Algorithm 5.7.3 does not fit exactly in the framework of Chapter 2. Indeed, in each refinement step, one refines the current knot vector based on some marked nodes instead of purely refining the mesh based on some marked mesh elements. Nevertheless, we can equivalently reformulate Algorithm 5.7.3 such that it is a special realization of its abstract counterpart Algorithm 2.2.1. To this end, we introduce for an admissible knot vector $\mathcal{K}_\bullet \in \mathbb{K}$, the corresponding set of *extended node patches*

$$\widetilde{\mathcal{N}}_\bullet := \big\{\widetilde{\pi}_\bullet(z) \,:\, z \in \mathcal{N}_\bullet\big\} \quad \text{with} \quad \widetilde{\pi}_\bullet(z) := \big(\pi_\bullet(z), \#_\bullet z_{\bullet,\mathrm{left}}, \#_\bullet z, \#_\bullet z_{\bullet,\mathrm{right}}\big) \tag{5.8.1}$$

where $z_{\bullet,\mathrm{left}} \in \mathcal{N}_\bullet \cap \pi_\bullet(z)$ is (with respect to γ) the left node and $z_{\bullet,\mathrm{right}} \in \mathcal{N}_\bullet \cap \pi_\bullet(z)$ is the right node in the patch $\pi_\bullet(z)$. Hence, $\widetilde{\pi}_\bullet(z)$ is just the patch $\pi_\bullet(z)$ with the multiplicities of the nodes that it contains. We define the set of all *admissible* sets of extended node patches

$$\widetilde{\mathbb{N}} := \big\{\widetilde{\mathcal{N}}_\bullet \,:\, \mathcal{K}_\bullet \in \mathbb{K}\big\}. \tag{5.8.2}$$

Note that $\mathbb{K}$ and $\widetilde{\mathbb{N}}$ are in a bijective relation, and the knowledge of the knot vector $\mathcal{K}_\bullet \in \mathbb{K}$ implies the knowledge of the corresponding set of extended node patches $\widetilde{\mathcal{N}}_\bullet \in \widetilde{\mathbb{N}}$ and vice versa. To concretize the setting of Chapter 2, we choose the set of general meshes $\mathbb{T}$ of Section 2.2.1 (which should not be mistaken for the set of admissible meshes $\mathbb{T}$ of (5.7.16)) as $\widetilde{\mathbb{N}}$. With the refinement strategy of Algorithm 5.7.1, we define for $\widetilde{\mathcal{N}}_\bullet \in \widetilde{\mathbb{N}}$, $\widetilde{\mathcal{M}}_\bullet \subseteq \widetilde{\mathcal{N}}_\bullet$, and corresponding nodes $\mathcal{M}_\bullet := \big\{z \in \mathcal{N}_\bullet \,:\, \widetilde{\pi}_\bullet(z) \in \widetilde{\mathcal{M}}_\bullet\big\}$,

$$\mathtt{refine}(\widetilde{\mathcal{N}}_\bullet, \widetilde{\mathcal{M}}_\bullet) := \widetilde{\mathcal{N}}_\circ \quad \text{with} \quad \mathcal{K}_\circ = \mathtt{refine}(\mathcal{K}_\bullet, \mathcal{M}_\bullet). \tag{5.8.3}$$

Note that $\widetilde{\mathcal{M}}_\bullet \subseteq \widetilde{\mathcal{N}}_\bullet \setminus \widetilde{\mathcal{N}}_\circ$, i.e., each marked extended node patch is changed (by inserting a knot in it) during refinement. Similarly as in Section 2.2.1, we define $\mathtt{refine}(\widetilde{\mathcal{N}}_\bullet)$ as the set of all refinements. Moreover, we define for $\widetilde{\mathcal{S}} \subseteq \widetilde{\mathcal{N}}_\bullet$ the number of all corresponding knots

$$\mu(\widetilde{\mathcal{S}}) := \sum_{\widetilde{\pi} \in \widetilde{\mathcal{S}}} \mu(\widetilde{\pi}) \quad \text{with} \quad \mu(\widetilde{\pi}_\bullet(z)) := \#_\bullet z \text{ for all } z \in \mathcal{N}_\bullet. \tag{5.8.4}$$

Recall the abbreviation $N_\bullet + 1$ for the number of all knots in the parameter domain $[0,1]$. Since $\#_\bullet 0 = p+1$, we see that

$$\mu(\widetilde{\mathcal{N}}_\bullet) = N_\bullet - p \quad \text{and} \quad \mu(\widetilde{\mathcal{N}}_\bullet) - \mu(\widetilde{\mathcal{N}}_0) = N_\bullet - N_0. \tag{5.8.5}$$

We define

$$\widetilde{\mathbb{N}}(N) := \{\widetilde{\mathcal{N}}_\bullet \in \widetilde{\mathbb{N}} : \mu(\widetilde{\mathcal{N}}_\bullet) - \mu(\widetilde{\mathcal{N}}_0) \le N\} \quad \text{for all } N \in \mathbb{N}_0. \tag{5.8.6}$$

Altogether, we have a particular realization of Section 2.2.1. Now, we concretize Section 2.2.2. For $\widetilde{\mathcal{N}}_\bullet \in \widetilde{\mathbb{N}}$, we set

$$\eta_\bullet := \eta_\bullet(\widetilde{\mathcal{N}}_\bullet) \quad \text{with} \quad \eta_\bullet(\widetilde{\mathcal{S}})^2 := \sum_{\widetilde{\pi} \in \widetilde{\mathcal{S}}} \eta_\bullet(\widetilde{\pi})^2 \text{ for all } \widetilde{\mathcal{S}} \subseteq \widetilde{\mathcal{N}}_\bullet, \tag{5.8.7a}$$

where, for all $z \in \mathcal{N}_\bullet$, the local refinement indicators read

$$\eta_\bullet(\widetilde{\pi}_\bullet(z))^2 := |\pi_\bullet(z)| |f - \mathfrak{V}\Phi_\bullet|^2_{H^1(\pi_\bullet(z))}. \tag{5.8.7b}$$

We consider the following adaptive algorithm.

Algorithm 5.8.1. ***Input:*** *Dörfler parameter $\theta \in (0,1]$ and marking constant $C^\mu_{\min} \in [1,\infty]$.*
Loop: *For each $\ell = 0,1,2,\dots,$ iterate the following steps:*

(i) *Compute Galerkin approximation $\Phi_\ell \in \mathcal{X}_\ell$.*

(ii) *Compute refinement indicators $\eta_\ell(\widetilde{\pi})$ for all extended node patches $\widetilde{\pi} \in \widetilde{\mathcal{N}}_\ell$.*

(iii) *Determine a set of marked extended node patches $\widetilde{\mathcal{M}}_\ell \subseteq \widetilde{\mathcal{N}}_\ell$ which is up to the multiplicative constant $C^\mu_{\min}$ minimal with respect to μ, such that the following Dörfler marking is satisfied*

$$\theta\, \eta_\ell^2 \le \eta_\ell(\widetilde{\mathcal{M}}_\ell)^2. \tag{5.8.8}$$

(iv) *Generate refined $\widetilde{\mathcal{N}}_{\ell+1} := \mathtt{refine}(\widetilde{\mathcal{N}}_\ell, \widetilde{\mathcal{M}}_\ell)$.*

Output: *Refined sets $\widetilde{\mathcal{N}}_\ell$ and corresponding Galerkin approximations Φ_ℓ with error estimators η_ℓ for all $\ell \in \mathbb{N}_0$.*

Let $\widetilde{\mathcal{N}}_\bullet \in \widetilde{\mathbb{N}}$. Since the maximal knot multiplicity is $p+1$, we see that $\#\mathcal{S} \le \mu(\widetilde{\mathcal{S}}) \le (p+1)\#\mathcal{S}$ for arbitrary $\widetilde{\mathcal{S}} \subseteq \widetilde{\mathcal{N}}_\bullet$ with corresponding nodes $\mathcal{S} \subseteq \mathcal{N}_\bullet$. Further, with the node-based estimator from (5.7.17), there holds by definition $\eta_\bullet(\widetilde{\pi}_\bullet(z)) = \eta_\bullet(z)$ for all $z \in \mathcal{N}_\bullet$. This shows that the output of Algorithm 5.7.3 can be seen as output of Algorithm 5.8.1, if one chooses for both algorithms the same θ and $C^\mu_{\min} = (p+1)C_{\min}$ with $C_{\min} \in [1,\infty]$ of Algorithm 5.7.3. Further, (5.8.5) implies that the approximation constants coincide for all $s > 0$, i.e.,

$$\|\phi\|_{\mathbb{A}^{\rm est}_s} = \sup_{N \in \mathbb{N}_0} \min_{\mathcal{K}_\bullet \in \mathbb{K}(N)} (N+1)^s\, \eta_\bullet = \sup_{N \in \mathbb{N}_0} \min_{\widetilde{\mathcal{N}}_\bullet \in \widetilde{\mathbb{N}}(N)} (N+1)^s\, \eta_\bullet. \tag{5.8.9}$$

Altogether, we see that Theorem 5.7.4 (ii)–(iii) follows from Corollary 2.3.4 if $\eta_\bullet$ is locally equivalent to an estimator $\widetilde{\eta}_\bullet$ which satisfies the axioms (E1)–(E4) and the refinement axioms (T1)–(T3) are satisfied. To define a suitable equivalent estimator, we replace the weight $|\pi_\bullet(z)|$ in (5.8.7) by an equivalent weight $\widetilde{h}_{\widetilde{\pi}_\bullet(z)}$ which uniformly contracts if a knot is inserted in the patch $\pi_\bullet(z)$. Similarly as in the recent own work [FGHP17, Proposition 4.2], we construct such a weight in the following proposition.

Proposition 5.8.2. *For $\widetilde{\mathcal{N}}_\bullet \in \widetilde{\mathbb{N}}$ and $z \in \mathcal{N}_\bullet$, we define with a constant $0 < \rho_{\rm eq} < 1$ which depends only on $\widehat{\kappa}_0$ and p (and which is fixed in the proof)*

$$\widetilde{h}_{\widetilde{\pi}_\bullet(z)} := |\gamma^{-1}(\pi_\bullet(z))|\, \rho_{\rm eq}^{\#_\bullet z_{\bullet,\rm left} + \#_\bullet z + \#_\bullet z_{\bullet,\rm right}}. \tag{5.8.10}$$

Then, there exists a constant $C_{\rm eq} > 0$ such that

$$C_{\rm eq}^{-1}|\pi_\bullet(z)| \le \widetilde{h}_{\widetilde{\pi}_\bullet(z)} \le C_{\rm eq}|\pi_\bullet(z)|, \tag{5.8.11}$$

where $C_{\rm eq}$ depends only on $C_\gamma, \widehat{\kappa}_0$, and p. If additionally $\widetilde{\mathcal{N}}_\circ \in$ `refine`$(\widetilde{\mathcal{N}}_\bullet)$, then there exists a constant $0 < \rho_{\rm ctr} < 1$ such that for all $z \in \mathcal{N}_\bullet$ with $\widetilde{\pi}_\bullet(z) \in \widetilde{\mathcal{N}}_\bullet \setminus \widetilde{\mathcal{N}}_\circ$ and all $z' \in \mathcal{N}_\circ$ with $z' = z$ or $z' \in \pi_\bullet(z) \setminus \mathcal{N}_\bullet$, there holds that

$$\widetilde{h}_{\widetilde{\pi}_\circ(z')} \le \rho_{\rm ctr}\, \widetilde{h}_{\widetilde{\pi}_\bullet(z)}. \tag{5.8.12}$$

where $\rho_{\rm ctr}$ depends only on $\widehat{\kappa}_0$ and p.

Proof. (5.8.11) follows immediately from the regularity (5.7.3) of γ. To see (5.8.12), note that $\widetilde{\pi}_\bullet(z) \in \widetilde{\mathcal{N}}_\bullet \setminus \widetilde{\mathcal{N}}_\circ$ implies that at least one new knot is inserted in the patch $\pi_\bullet(z)$.

First, we suppose that thereby no bisection is used, wherefore only knot multiplicities are increased within $\pi_\bullet(z)$. Due to our assumption for z', this implies that $z' = z$. There holds that

$$\begin{aligned}\widetilde{h}_{\widetilde{\pi}_\circ(z')} &= |\gamma^{-1}(\pi_\circ(z))|\, \rho_{\rm eq}^{\#_\circ z_{\circ,\rm left} + \#_\circ z + \#_\circ z_{\circ,\rm right}} \\ &\le |\gamma^{-1}(\pi_\bullet(z))|\, \rho_{\rm eq}^{\#_\bullet z_{\bullet,\rm left} + \#_\bullet z + \#_\bullet z_{\bullet,\rm right}+1} = \rho_{\rm eq}\, \widetilde{h}_{\widetilde{\pi}_\bullet(z)}.\end{aligned}$$

Now, we suppose that at least one bisection takes place within $\pi_\bullet(z)$. By local quasi-uniformity (5.7.15), there holds that $|\gamma^{-1}(\pi_\circ(z'))| \le \rho_1|\gamma^{-1}(\pi_\bullet(z))|$ with a constant $0 < \rho_1 < 1$ that depends only on $\widehat{\kappa}_0$. We choose $0 < \rho_{\rm eq} < 1$ sufficiently large such that $\rho_1\, \rho_{\rm eq}^{-3p} < 1$. Since the maximal knot multiplicity is $p+1$, this yields that

$$\begin{aligned}\widetilde{h}_{\widetilde{\pi}_\circ(z')} &= |\gamma^{-1}(\pi_\circ(z'))|\, \rho_{\rm eq}^{\#_\circ z_{\circ,\rm left} + \#_\circ z + \#_\circ z_{\circ,\rm right}} \\ &\le \rho_1|\gamma^{-1}(\pi_\bullet(z))|\, \rho_{\rm eq}^{\#_\bullet z_{\bullet,\rm left} + \#_\bullet z + \#_\bullet z_{\bullet,\rm right}-3p} = \rho_1\, \rho_{\rm eq}^{-3p}\, \widetilde{h}_{\widetilde{\pi}_\bullet(z)}.\end{aligned}$$

The choice $\rho_{\rm ctr} := \max(\rho_{\rm eq}, \rho_1\, \rho_{\rm eq}^{-3p})$ concludes the proof. □

For $\widetilde{\mathcal{N}}_\bullet \in \widetilde{\mathbb{N}}$, we define the locally equivalent estimator

$$\widetilde{\eta}_\bullet := \widetilde{\eta}_\bullet(\widetilde{\mathcal{N}}_\bullet) \quad \text{with} \quad \widetilde{\eta}_\bullet(\widetilde{\mathcal{S}})^2 := \sum_{\widetilde{\pi} \in \widetilde{\mathcal{S}}} \widetilde{\eta}_\bullet(\widetilde{\pi})^2 \text{ for all } \widetilde{\mathcal{S}} \subseteq \widetilde{\mathcal{N}}_\bullet, \tag{5.8.13a}$$

where, for all $z \in \mathcal{N}_\bullet$, the local refinement indicators read

$$\widetilde{\eta}_\bullet(\widetilde{\pi}_\bullet(z))^2 := \widetilde{h}_{\widetilde{\pi}_\bullet(z)} \, |f - \mathfrak{V}\Phi_\bullet|^2_{H^1(\pi_\bullet(z))}. \tag{5.8.13b}$$

To apply Corollary 2.3.4, we prove in the following subsections the estimator axioms (E1)–(E4) for the equivalent estimator $\widetilde{\eta}_\bullet$ as well as the refinement axioms (T1)–(T3). The perturbation $\varrho_{\bullet,\circ}$ is chosen as $C_\varrho \|\Phi_\circ - \Phi_\bullet\|_{H^{-1/2}(\Gamma)}$ with some constant $C_\varrho > 0$ which is fixed later in Section 5.8.5. For the proof, we verify adapted versions of the assumed properties of the abstract Section 5.2. This can essentially be done as in Section 5.5. With these, we can derive the axioms similarly as in Section 5.3.

5.8.1 Mesh properties (M1)–(M5)

In this section, we verify the mesh properties of Section 5.2.1 such that we can use some of the auxiliary results of Section 5.5. Clearly, (M1) is trivially satisfied with $C_{\text{patch}} = 2$. (M2) follows from the regularity (5.7.3) of γ as well as (5.7.15). Further, (5.7.3) yields (M3). (M4) is proved similarly as in Section 5.5.4. Finally, (M5) follows from Proposition 5.2.2 since there exists only one reference point patch: Indeed, due to the assumptions on γ, for all $z \in \Gamma$, there exists a bi-Lipschitz mapping $\gamma_{\pi_\bullet(z)} : [0,1] \to \pi_\bullet(z)$ such that

$$\frac{|\gamma_{\pi_\bullet(z)}(s) - \gamma_{\pi_\bullet(z)}(t)|}{\operatorname{diam}(\pi_\bullet(z))} \simeq |s-t| \quad \text{for all } s,t \in [0,1]. \tag{5.8.14}$$

The constants of (M2)–(M5) depend only the parametrization γ, the polynomial order p, and the initial mesh $\widehat{\mathcal{T}}_0$.

5.8.2 Reliability (5.7.22)

Let $\mathcal{K}_\bullet \in \mathbb{K}$. There holds Galerkin orthogonality

$$\langle f - \mathfrak{V}\Phi_\bullet \,,\, \Psi_\bullet \rangle_{L^2(\Gamma)} = 0 \quad \text{for all } \Psi_\bullet \in \mathcal{X}_\bullet. \tag{5.8.15}$$

In Section 5.8.7, we will prove (S4) for $\mathcal{X}_\bullet$ associated to arbitrary $\mathcal{K}_\bullet \in \mathbb{K}$. In particular, we can apply Corollary 5.3.9. Together with the fact that $\mathfrak{V} : H^{-1/2}(\Gamma)^D \to H^{1/2}(\Gamma)^D$ is an isomorphism, we obtain that

$$\begin{aligned}\|\phi - \Phi_\bullet\|_{H^{-1/2}(\Gamma)} \lesssim \|\mathfrak{V}(\phi - \Phi_\bullet)\|_{H^{1/2}(\Gamma)} &= \|f - \mathfrak{V}\Phi_\bullet\|_{H^{1/2}(\Gamma)} \\ &\lesssim \|h_\bullet^{1/2} \nabla_\Gamma (f - \mathfrak{V}\Phi_\bullet)\|_{L^2(\Gamma)} = \eta_\bullet.\end{aligned}$$

Remark 5.8.3. *As in Remark 5.3.10, one sees that*

$$\begin{aligned}\|f - \mathfrak{V}\Phi_\bullet\|^2_{H^{1/2}(\Gamma)} &\simeq \sum_{T \in \mathcal{T}_\bullet} \sum_{T' \in \Pi_\bullet(T)} |f - \mathfrak{V}\Phi_\bullet|^2_{H^{1/2}(T \cup T')} \\ &= 2 \sum_{z \in \mathcal{N}_\bullet} |f - \mathfrak{V}\Phi_\bullet|^2_{H^{1/2}(\pi_\bullet(z))}.\end{aligned} \tag{5.8.16}$$

Again, this is even true for arbitrary $f \in H^{1/2}(\Gamma)^D$ without the additional regularity $f \in H^1(\Gamma)^D$. In particular,

$$\digamma_\bullet(z)^2 := |f - \mathfrak{V}\Phi_\bullet|^2_{H^{1/2}(\pi_\bullet(z))} \quad \text{for all } z \in \mathcal{N}_\bullet \tag{5.8.17}$$

provides a local error indicator. The corresponding error estimator $\digamma_\bullet$ is often referred to as Faermann estimator; see also Remark 5.3.10. Obviously, one could replace the residual estimator η_ℓ in Algorithm 5.7.3 by the Faermann estimator $\digamma_\ell$. However, due to the lack of an h-weighting factor, it is unclear whether the reduction property (E2) of Section 5.3.2 is satisfied. Based on the ideas of [FFME+14, Theorem 7], we proved in the recent own work [FGHP17, Theorem 3.4] that one obtains at least estimator convergence $\lim_{\ell\to\infty} \digamma_\ell = 0$ and due to reliability also error convergence $\lim_{\ell\to\infty} \|\phi - \Phi_\ell\|_{H^{-1/2}(\Gamma)} = 0$. We even proved this assertion for rational splines as in Remark 5.7.6. Although [FGHP17, Theorem 3.4] only treats the Laplace problem, the proof immediately extends to the current situation if one exploits the generalized inverse inequality (5.3.34).

5.8.3 An inverse inequality for splines

We prove the following analogous version of (S1).

Proposition 5.8.4. *There exists a constant $C_{\rm inv} > 0$ such that for all $\mathcal{K}_\bullet \in \mathbb{K}$ with corresponding ansatz space $\mathcal{X}_\bullet$, there holds for all $\Psi_\bullet \in \mathcal{X}_\bullet$ that*

$$\|h_\bullet^{1/2}\Psi_\bullet\|_{L^2(\Gamma)} \le C_{\rm inv}\, \|\Psi_\bullet\|_{H^{-1/2}(\Gamma)}. \tag{5.8.18}$$

The constant $C_{\rm inv}$ depends only on the parametrization γ, the polynomial order p, and the initial mesh $\widehat{\mathcal{T}}_0$.

Proof. Since $\mathcal{X}_\bullet$ is a product space of transformed one-dimensional splines, we can assume without loss of generality that we are in the scalar case, i.e., $D = 1$. Similarly as in Section 5.5.9, we show that all $\Psi_\bullet \in \mathcal{X}_\bullet$ satisfy the assumptions of Proposition 5.5.3, which concludes the proof. The condition (5.5.9) is trivially satisfied since each γ_T is just the restriction of some γ to $\widehat{T} = \gamma^{-1}(T)$. Indeed, (5.7.3) yields that $C_{\rm lip} \le C_\gamma$. For $T \in \mathcal{T}_\bullet$, we abbreviate $\widehat{\Psi}_\bullet := \Psi_\bullet \circ \gamma_T$. Due to the regularity (5.7.3) of the parametrization γ, it is sufficient to find a uniform constant $\widehat{\rho}_{\rm inf} \in (0,1)$ and some interval $\widehat{R}_T \subset \widehat{T}^\circ$ such that $|\widehat{R}_T| \ge \widehat{\rho}_{\rm inf}|\widehat{T}|$, $\widehat{\Psi}_\bullet$ does not change sign on $\widehat{R}_T$, and

$$\inf_{t\in\widehat{R}_T} |\widehat{\Psi}_\bullet(t)| \ge \widehat{\rho}_{\rm inf}\|\widehat{\Psi}_\bullet\|_{L^\infty(\widehat{T})}. \tag{5.8.19}$$

Indeed, one sees as in Section 4.5.3 that $|\widehat{R}_T| \ge \widehat{\rho}_{\rm inf}|\widehat{T}|$ implies that $|R_T| \ge \rho_{\rm inf}|T|$ for some uniform constant $\rho_{\rm inf} \in (0,1)$. Recall that $\widehat{\Psi}_\bullet$ is just a polynomial of degree p. We define $\widehat{R}_T$ as the interval from Lemma 5.5.4 corresponding to the polynomial $\widehat{\Psi}_\bullet$ on the interval $I := \widehat{T}$. With the constant ρ from Lemma 5.5.4, we set $\widehat{\rho}_{\rm inf} := \rho$. Then, (5.8.19) is satisfied. Moreover, one sees that $|\widehat{R}_T| \ge \widehat{\rho}_{\rm inf}|\widehat{T}|$, and that $\widehat{\Psi}_\bullet$ does not change its sign on $\widehat{R}_T \subset \widehat{T}^\circ$. □

5.8.4 Stability on non-refined elements (E1)

We show the existence of $C_{\rm stab} \geq 1$ such that for all $\widetilde{\mathcal{N}}_\bullet \in \widetilde{\mathbb{N}}$, and all $\widetilde{\mathcal{N}}_\circ \in \texttt{refine}(\widetilde{\mathcal{N}}_\bullet)$, it holds that

$$|\widetilde{\eta}_\circ(\widetilde{\mathcal{N}}_\bullet \cap \widetilde{\mathcal{N}}_\circ) - \widetilde{\eta}_\bullet(\widetilde{\mathcal{N}}_\bullet \cap \widetilde{\mathcal{N}}_\circ)| \leq C_{\rm stab} \|\Phi_\circ - \Phi_\bullet\|_{H^{-1/2}(\Gamma)}.$$

In Section 5.8.5, we will fix the constant C_ϱ for the perturbations such that $C_{\rm stab} \leq C_\varrho$. The reverse triangle inequality and the fact that $\pi_\bullet(z) = \pi_\circ(z)$ if $\widetilde{\pi}_\bullet(z) \in \widetilde{\mathcal{N}}_\bullet \cap \widetilde{\mathcal{N}}_\circ$ prove that

$$\begin{aligned}
&|\widetilde{\eta}_\circ(\widetilde{\mathcal{N}}_\bullet \cap \widetilde{\mathcal{N}}_\circ) - \widetilde{\eta}_\bullet(\widetilde{\mathcal{N}}_\bullet \cap \widetilde{\mathcal{N}}_\circ)| \\
&\leq \Big| \sum_{\substack{z \in \mathcal{N}_\circ \\ \widetilde{\pi}_\circ(z) \in \widetilde{\mathcal{N}}_\bullet \cap \widetilde{\mathcal{N}}_\circ}} \big(\|\widetilde{h}^{1/2}_{\widetilde{\pi}_\circ(z)} \nabla_\Gamma (f - \mathfrak{V}\Phi_\circ)\|_{L^2(\pi_\circ(z))} - \|\widetilde{h}^{1/2}_{\widetilde{\pi}_\circ(z)} \nabla_\Gamma (f - \mathfrak{V}\Phi_\bullet)\|_{L^2(\pi_\circ(z))}\big)^2 \Big|^{1/2} \\
&\leq \Big| \sum_{\substack{z \in \mathcal{N}_\circ \\ \widetilde{\pi}_\circ(z) \in \widetilde{\mathcal{N}}_\bullet \cap \widetilde{\mathcal{N}}_\circ}} \|\widetilde{h}^{1/2}_{\widetilde{\pi}_\circ(z)} \nabla_\Gamma \mathfrak{V}(\Phi_\circ - \Phi_\bullet)\|^2_{L^2(\pi_\circ(z))} \Big|^{1/2}.
\end{aligned}$$

With the regularity (5.7.3) of γ, local quasi-uniformity (5.7.15), and the equivalence (5.8.11), we proceed

$$|\widetilde{\eta}_\circ(\widetilde{\mathcal{N}}_\bullet \cap \widetilde{\mathcal{N}}_\circ) - \widetilde{\eta}_\bullet(\widetilde{\mathcal{N}}_\bullet \cap \widetilde{\mathcal{N}}_\circ)| \lesssim \|h_\circ^{1/2} \nabla_\Gamma \mathfrak{V}(\Phi_\circ - \Phi_\bullet)\|_{L^2(\Gamma)}. \tag{5.8.20}$$

(5.7.10) shows that $\Phi_\circ - \Phi_\bullet \in \mathcal{X}_\circ$. Therefore, the inverse inequalities (5.3.34) and (5.8.18) are applicable, which concludes the proof of (E1). The constant $C_{\rm stab}$ depends only on the dimension D, the coefficients of the differential operator $\mathfrak{P}$, the parametrization γ, the polynomial order p, and the initial mesh $\widehat{\mathcal{T}}_0$.

5.8.5 Reduction on refined elements (E2)

We show the existence of $C_{\rm red} \geq 1$ and $0 < \rho_{\rm red} < 1$ such that for all $\widetilde{\mathcal{N}}_\bullet \in \widetilde{\mathbb{N}}$ and all $\widetilde{\mathcal{N}}_\circ \in \texttt{refine}(\widetilde{\mathcal{N}}_\bullet)$, there holds that

$$\widetilde{\eta}_\circ(\widetilde{\mathcal{N}}_\circ \setminus \widetilde{\mathcal{N}}_\bullet)^2 \leq \rho_{\rm red}\, \widetilde{\eta}_\bullet(\widetilde{\mathcal{N}}_\bullet \setminus \widetilde{\mathcal{N}}_\circ)^2 + C_{\rm red} \|\Phi_\circ - \Phi_\bullet\|^2_{H^{-1/2}(\Gamma)}.$$

With this, we can fix the constant for the perturbations as

$$C_\varrho := \max(C_{\rm stab}, C_{\rm red}^{1/2}). \tag{5.8.21}$$

First, we apply the triangle inequality and the Young inequality to see for arbitrary $\delta > 0$ that

$$\begin{aligned}
\widetilde{\eta}_\circ(\widetilde{\mathcal{N}}_\circ \setminus \widetilde{\mathcal{N}}_\bullet) &= \sum_{\substack{z\in\mathcal{N}_\circ\\ \widetilde{\pi}_\circ(z)\in\widetilde{\mathcal{N}}_\circ\setminus\widetilde{\mathcal{N}}_\bullet}} \|\widetilde{h}^{1/2}_{\widetilde{\pi}_\circ(z)}\nabla_\Gamma(f-\mathfrak{V}\Phi_\circ)\|^2_{L^2(\pi_\circ(z))}\\
&\le \sum_{\substack{z\in\mathcal{N}_\circ\\ \widetilde{\pi}_\circ(z)\in\widetilde{\mathcal{N}}_\circ\setminus\widetilde{\mathcal{N}}_\bullet}} \Big(\|\widetilde{h}^{1/2}_{\widetilde{\pi}_\circ(z)}\nabla_\Gamma(f-\mathfrak{V}\Phi_\bullet)\|_{L^2(\pi_\circ(z))} + \|\widetilde{h}^{1/2}_{\widetilde{\pi}_\circ(z)}\nabla_\Gamma\mathfrak{V}(\Phi_\circ-\Phi_\bullet)\|_{L^2(\pi_\circ(z))}\Big)^2\\
&\le (1+\delta)\sum_{\substack{z\in\mathcal{N}_\circ\\ \widetilde{\pi}_\circ(z)\in\widetilde{\mathcal{N}}_\circ\setminus\widetilde{\mathcal{N}}_\bullet}} \|\widetilde{h}^{1/2}_{\widetilde{\pi}_\circ(z)}\nabla_\Gamma(f-\mathfrak{V}\Phi_\bullet)\|^2_{L^2(\pi_\circ(z))}\\
&\quad + (1+\delta^{-1})\sum_{\substack{z\in\mathcal{N}_\circ\\ \widetilde{\pi}_\circ(z)\in\widetilde{\mathcal{N}}_\circ\setminus\widetilde{\mathcal{N}}_\bullet}} \|\widetilde{h}^{1/2}_{\widetilde{\pi}_\circ(z)}\nabla_\Gamma\mathfrak{V}(\Phi_\circ-\Phi_\bullet)\|^2_{L^2(\pi_\circ(z))}.
\end{aligned}$$

The second term can be estimated as in Section 5.8.4. To bound the first one, we split each patch $\pi_\circ(z) = T_{\circ,\mathrm{left}}(z)\cup T_{\circ,\mathrm{right}}(z)$ into a (with respect to the parametrization γ) left and a right element in $\mathcal{T}_\circ$. We obtain that

$$\begin{aligned}
&\sum_{\substack{z\in\mathcal{N}_\circ\\ \widetilde{\pi}_\circ(z)\in\widetilde{\mathcal{N}}_\circ\setminus\widetilde{\mathcal{N}}_\bullet}} \|\widetilde{h}^{1/2}_{\widetilde{\pi}_\circ(z)}\nabla_\Gamma(f-\mathfrak{V}\Phi_\bullet)\|^2_{L^2(\pi_\circ(z))}\\
&= \sum_{\substack{z\in\mathcal{N}_\circ\\ \widetilde{\pi}_\circ(z)\in\widetilde{\mathcal{N}}_\circ\setminus\widetilde{\mathcal{N}}_\bullet}} \|\widetilde{h}^{1/2}_{\widetilde{\pi}_\circ(z)}\nabla_\Gamma(f-\mathfrak{V}\Phi_\bullet)\|^2_{L^2(T_{\circ,\mathrm{left}}(z))} + \sum_{\substack{z\in\mathcal{N}_\circ\\ \widetilde{\pi}_\circ(z)\in\widetilde{\mathcal{N}}_\circ\setminus\widetilde{\mathcal{N}}_\bullet}} \|\widetilde{h}^{1/2}_{\widetilde{\pi}_\circ(z)}\nabla_\Gamma(f-\mathfrak{V}\Phi_\bullet)\|^2_{L^2(T_{\circ,\mathrm{right}}(z))}.
\end{aligned}$$

Note that the domains in the first resp. second sum do not overlap. Let $z\in\mathcal{N}_\circ$ with $\widetilde{\pi}_\circ(z)\in\widetilde{\mathcal{N}}_\circ\setminus\widetilde{\mathcal{N}}_\bullet$. If $z\in\mathcal{N}_\bullet$, we define $z' := z$, where $\widetilde{\pi}_\bullet(z')\in\widetilde{\mathcal{N}}_\bullet\setminus\widetilde{\mathcal{N}}_\circ$. Otherwise, there exists a unique $z'\in\mathcal{N}_\bullet$ with $z\in T_{\bullet,\mathrm{left}}(z')$, where $T_{\bullet,\mathrm{left}}(z')$ is defined analogously as above. Again, this implies that $\widetilde{\pi}_\bullet(z')\in\widetilde{\mathcal{N}}_\bullet\setminus\widetilde{\mathcal{N}}_\circ$. Altogether, we see with the contraction property (5.8.12) that

$$\begin{aligned}
&\sum_{\substack{z\in\mathcal{N}_\circ\\ \widetilde{\pi}_\circ(z)\in\widetilde{\mathcal{N}}_\circ\setminus\widetilde{\mathcal{N}}_\bullet}} \|\widetilde{h}^{1/2}_{\widetilde{\pi}_\circ(z)}\nabla_\Gamma(f-\mathfrak{V}\Phi_\bullet)\|^2_{L^2(T_{\circ,\mathrm{left}}(z))}\\
&\quad\le \sum_{\substack{z'\in\mathcal{N}_\bullet\\ \widetilde{\pi}_\bullet(z')\in\widetilde{\mathcal{N}}_\bullet\setminus\widetilde{\mathcal{N}}_\circ}} \sum_{\substack{z\in\mathcal{N}_\circ\\ z=z'\vee z\in T_{\bullet,\mathrm{left}}(z')\setminus\mathcal{N}_\bullet}} \|\widetilde{h}^{1/2}_{\widetilde{\pi}_\circ(z)}\nabla_\Gamma(f-\mathfrak{V}\Phi_\bullet)\|^2_{L^2(T_{\circ,\mathrm{left}}(z))}\\
&\quad\le \sum_{\substack{z'\in\mathcal{N}_\bullet\\ \widetilde{\pi}_\bullet(z')\in\widetilde{\mathcal{N}}_\bullet\setminus\widetilde{\mathcal{N}}_\circ}} \rho_{\mathrm{ctr}}\,\|\widetilde{h}^{1/2}_{\widetilde{\pi}_\bullet(z')}\nabla_\Gamma(f-\mathfrak{V}\Phi_\bullet)\|^2_{L^2(T_{\bullet,\mathrm{left}}(z'))}.
\end{aligned}$$

The same holds for the right elements. Hence, we end up with

$$\begin{aligned}
\sum_{\substack{z\in\mathcal{N}_\circ\\ \widetilde{\pi}_\circ(z)\in\widetilde{\mathcal{N}}_\circ\setminus\widetilde{\mathcal{N}}_\bullet}} \|\widetilde{h}^{1/2}_{\widetilde{\pi}_\circ(z)}\nabla_\Gamma(f-\mathfrak{V}\Phi_\bullet)\|^2_{L^2(\pi_\circ(z))} &\le \rho_{\mathrm{ctr}} \sum_{\substack{z\in\mathcal{N}_\bullet\\ \widetilde{\pi}_\bullet(z')\in\widetilde{\mathcal{N}}_\bullet\setminus\widetilde{\mathcal{N}}_\circ}} \|\widetilde{h}^{1/2}_{\widetilde{\pi}_\bullet(z')}\nabla_\Gamma(f-\mathfrak{V}\Phi_\bullet)\|^2_{L^2(\pi_\bullet(z'))}\\
&= \rho_{\mathrm{ctr}}\,\widetilde{\eta}_\bullet(\widetilde{\mathcal{N}}_\bullet\setminus\widetilde{\mathcal{N}}_\circ)^2.
\end{aligned}$$

Choosing δ sufficiently small such that $\rho_{\rm red} := (1+\delta)\rho_{\rm ctr} < 1$ concludes the proof. The constant $C_{\rm red}$ depends only on the dimension D, the coefficients of the differential operator $\mathfrak{P}$, the parametrization γ, the polynomial order p, and the initial mesh $\widehat{\mathcal{T}}_0$.

5.8.6 General quasi-orthogonality (E3)

Exactly as in Section 5.3.3, one shows convergence of the perturbations $\lim_{\ell\to\infty} \|\Phi_{\ell+1} - \Phi_\ell\|_{H^{-1/2}(\Gamma)} = 0$. Therefore, the proof of general quasi-orthogonality (E3) can be copied verbatim from Section 5.3.7.

5.8.7 Discrete reliability (E4)

We show that there exist $q_{\rm drel} \in \mathbb{N}_0$ and $C_{\rm drel}, C_{\rm ref} \geq 1$ such that for all $\mathcal{K}_\bullet \in \mathbb{K}$ and all $\mathcal{K}_\circ \in \mathtt{refine}(\mathcal{K}_\bullet)$, the subset

$$\widetilde{\mathcal{R}}_{\bullet,\circ} := \widetilde{\Pi}_\bullet^{q_{\rm drel}}(\widetilde{\mathcal{N}}_\bullet \setminus \widetilde{\mathcal{N}}_\circ) := \big\{\widetilde{\pi}_\bullet(z) \,:\, \exists \widetilde{\pi}_\bullet(z') \in \widetilde{\mathcal{N}}_\bullet \setminus \widetilde{\mathcal{N}}_\circ \quad z \in \mathcal{N}_\bullet \cap \Pi_\bullet^{q_{\rm drel}}(z')\big\} \tag{5.8.22}$$

satisfies that

$$C_\varrho \|\Phi_\circ - \Phi_\bullet\|_{H^{-1/2}(\Gamma)} \leq C_{\rm drel}\, \widetilde{\eta}_\bullet(\widetilde{\mathcal{R}}_{\bullet,\circ}), \quad \widetilde{\mathcal{N}}_\bullet \setminus \widetilde{\mathcal{N}}_\circ \subseteq \widetilde{\mathcal{R}}_{\bullet,\circ}, \text{ and } \mu(\widetilde{\mathcal{R}}_{\bullet,\circ}) \leq C_{\rm ref}(\mu(\widetilde{\mathcal{N}}_\circ) - \mu(\widetilde{\mathcal{N}}_\bullet)).$$

The second property $\widetilde{\mathcal{N}}_\bullet \setminus \widetilde{\mathcal{N}}_\circ \subseteq \widetilde{\mathcal{R}}_{\bullet,\circ}$ is obvious. Since the maximal knot multiplicity is bounded by $p+1$, we have that

$$\mu(\widetilde{\mathcal{R}}_{\bullet,\circ}) \lesssim \mu(\widetilde{\mathcal{N}}_\bullet \setminus \widetilde{\mathcal{N}}_\circ),$$

where the hidden constant depends only on p and $q_{\rm drel}$. Note that $\widetilde{\pi}_\bullet(z) \in \widetilde{\mathcal{N}}_\bullet \setminus \widetilde{\mathcal{N}}_\circ$ holds only if a knot is inserted in the corresponding patch $\pi_\bullet(z)$, where a new knot can be inserted in at most three old patches. Since $\mu(\widetilde{\mathcal{N}}_\circ) - \mu(\widetilde{\mathcal{N}}_\bullet)$ is the number of all new knots, we see that

$$\mu(\widetilde{\mathcal{N}}_\bullet \setminus \widetilde{\mathcal{N}}_\circ) \leq 3\big(\mu(\widetilde{\mathcal{N}}_\circ) - \mu(\widetilde{\mathcal{N}}_\bullet)\big).$$

Now, we devote ourselves to the first property $\|\Phi_\circ - \Phi_\bullet\|_{H^{-1/2}(\Gamma)} \lesssim \widetilde{\eta}_\bullet(\widetilde{\mathcal{R}}_{\bullet,\circ})$. We prove the assertion in five steps. First, we derive similar versions of (S1)–(S6).

Step 1: In Proposition 5.8.4, we already proved that (S1) holds accordingly. By (5.7.10), nestedness (S2) with $\mathcal{K}_\bullet, \mathcal{K}_\circ \in \mathbb{K}$ instead of $\mathcal{T}_\bullet, \mathcal{T}_\circ \in \mathbb{T}$ is also satisfied.

Step 2: Next, we prove an adapted version of (S3): For $q_{\rm proj} \in \mathbb{N}_0$ which will be fixed later as $q_{\rm proj} := p$, and $q_{\rm loc} := q_{\rm proj}+p$, there holds for all $\mathcal{K}_\circ \in \mathtt{refine}(\mathcal{K}_\bullet)$, $T \in \mathcal{T}_\bullet \setminus \Pi_\bullet^{q_{\rm loc}+1}\big(\{z \in \mathcal{N}_\bullet \,:\, \widetilde{\pi}_\bullet(z) \in \widetilde{\mathcal{N}}_\bullet \setminus \widetilde{\mathcal{N}}_\circ\}\big)$, and $\Psi_\circ \in \mathcal{X}_\circ$ that

$$\Psi_\circ|_{\pi_\bullet^{q_{\rm proj}}(T)} \in \big\{\Psi_\bullet|_{\pi_\bullet^{q_{\rm proj}}(T)} \,:\, \Psi_\bullet \in \mathcal{X}_\bullet\big\}. \tag{5.8.23}$$

To see this, we argue as in Section 5.5.12.

First, we verify by contradiction that[6]

$$\Pi_\bullet^{q_{\rm loc}}(T) \subseteq \Pi_\bullet\big(\{z \in \mathcal{N}_\bullet \,:\, \widetilde{\pi}_\bullet(z) \in \widetilde{\mathcal{N}}_\bullet \cap \widetilde{\mathcal{N}}_\circ\}\big). \tag{5.8.24}$$

[6]The proof reveals that this is even true for arbitrary $q \in \mathbb{N}_0$ instead of $q_{\rm loc} = q_{\rm proj} + p$.

Suppose there exists $T' \in \Pi_\bullet^{q_{\rm loc}}(T)$ with $T' \notin \Pi_\bullet\big(\{z \in \mathcal{N}_\bullet : \widetilde{z} \in \widetilde{\mathcal{N}}_\bullet \cap \widetilde{\mathcal{N}}_\circ\}\big)$. This is equivalent to $T \in \Pi_\bullet^{q_{\rm loc}}(T')$ and $T' \in \mathcal{T}_\bullet \setminus \Pi_\bullet\big(\{z \in \mathcal{N}_\bullet : \widetilde{\pi}_\bullet(z) \in \widetilde{\mathcal{N}}_\bullet \cap \widetilde{\mathcal{N}}_\circ\}\big)$, which yields that $T \in \Pi_\bullet^{q_{\rm loc}}\big(\mathcal{T}_\bullet \setminus \Pi_\bullet\big(\{z \in \mathcal{N}_\bullet : \widetilde{\pi}_\bullet(z) \in \widetilde{\mathcal{N}}_\bullet \cap \widetilde{\mathcal{N}}_\circ\}\big)\big)$. Note that

$$\mathcal{T}_\bullet \setminus \Pi_\bullet\big(\{z \in \mathcal{N}_\bullet : \widetilde{\pi}_\bullet(z) \in \widetilde{\mathcal{N}}_\bullet \cap \widetilde{\mathcal{N}}_\circ\} \subseteq \Pi_\bullet\big(\{z \in \mathcal{N}_\bullet : \widetilde{\pi}_\bullet(z) \in \widetilde{\mathcal{N}}_\bullet \setminus \widetilde{\mathcal{N}}_\circ\},$$

since T'' in the left-hand side implies that $z \notin T''$ for all $\widetilde{\pi}_\bullet(z) \in \widetilde{\mathcal{N}}_\bullet \cap \widetilde{\mathcal{N}}_\circ$, but $T'' \cap \mathcal{N}_\bullet \neq \emptyset$, which implies the existence of $\widetilde{\pi}_\bullet(z) \in \widetilde{\mathcal{N}}_\bullet \setminus \widetilde{\mathcal{N}}_\circ$ with $z \in T''$. Altogether, we see that

$$T \in \Pi_\bullet^{q_{\rm loc}+1}\big(\{z \in \mathcal{N}_\bullet : \widetilde{\pi}_\bullet(z) \in \widetilde{\mathcal{N}}_\bullet \setminus \widetilde{\mathcal{N}}_\circ\}\big),$$

which contradicts our assumption for T and thus proves (5.8.24).

Next, we prove (5.8.23). Since $\mathcal{X}_\bullet$ is a product space of (transformed) splines, we can assume without loss of generality that $D = 1$. There holds that

$$\big\{\Psi_\bullet|_{\pi_\bullet^{q_{\rm proj}}(T)} : \Psi_\bullet \in \mathcal{X}_\bullet\big\} = \mathrm{span}\big\{\beta|_{\pi_\bullet^{q_{\rm proj}}(T)} : \beta \in \mathcal{B}_\bullet \wedge |\mathrm{supp}(\beta) \cap \pi_\bullet^{q_{\rm proj}}(T)| > 0\big\}$$

as well as

$$\big\{\Psi_\circ|_{\pi_\bullet^{q_{\rm proj}}(T)} : \Psi_\circ \in \mathcal{X}_\circ\big\} = \mathrm{span}\big\{\beta|_{\pi_\bullet^{q_{\rm proj}}(T)} : \beta \in \mathcal{B}_\circ \wedge |\mathrm{supp}(\beta) \cap \pi_\bullet^{q_{\rm proj}}(T)| > 0\big\}.$$

We show that

$$\big\{\beta \in \mathcal{B}_\bullet : |\mathrm{supp}(\beta) \cap \pi_\bullet^{q_{\rm proj}}(T)| > 0\big\} = \big\{\beta \in \mathcal{B}_\circ : |\mathrm{supp}(\beta) \cap \pi_\bullet^{q_{\rm proj}}(T)| > 0\big\}. \tag{5.8.25}$$

First, let β be an element of the left-hand side. By Lemma 3.2.1 (ii), $\mathrm{supp}(\beta)$ is connected and consists of at most $p+1$ elements, which implies $\mathrm{supp}(\beta) \subseteq \pi_\bullet^{q_{\rm loc}}(T)$. We show by contradiction that no knots are inserted in $\pi_\bullet^{q_{\rm loc}}(T)$ and thus in $\mathrm{supp}(\beta)$ during the refinement from $\mathcal{K}_\bullet$ to $\mathcal{K}_\circ$. Due to (5.8.24), a corresponding node $z' \in \mathcal{N}_\circ$ would satisfy $z' \in \pi_\bullet(z)$ for some $z \in \mathcal{N}_\bullet$ with $\widetilde{\pi}_\bullet(z) \in \widetilde{\mathcal{N}}_\bullet \cap \widetilde{\mathcal{N}}_\circ$. Since $\widetilde{\mathcal{N}}_\bullet \cap \widetilde{\mathcal{N}}_\circ$ is just the set of all (extended) node patches where no new knot is inserted, this leads to a contradiction. Hence, Lemma 3.2.1 (iii) proves that $\beta \in \mathcal{B}_\circ$. The proof works the same if we start with some β in the right-hand side of (5.8.25). This proves (5.8.25) and hence (S3).

Step 3: (S4) still holds true if $\mathcal{X}_\bullet$ is associated to an arbitrary knot vector $\mathcal{K}_\bullet \in \mathbb{K}$: According to Remark 5.2.3, we can assume without loss of generality that $D = 1$. Then, (S4) follows from Proposition 5.5.5, where the (transformed) B-splines $\overline{\mathcal{B}}_\bullet := \mathcal{B}_\bullet$ satisfy the required assumptions due to Lemma 3.2.1.

Step 4: For $\mathcal{K}_\bullet \in \mathbb{K}$ and a subset of the corresponding mesh $\mathcal{S} \subseteq \mathcal{T}_\bullet$, we construct a quasi-interpolatorion projection $J_{\bullet,\mathcal{S}} : L^2(\Gamma)^D \to \big\{\Psi_\bullet \in \mathcal{X}_\bullet : \Psi_\bullet|_{\bigcup(\mathcal{T}_\bullet \setminus \mathcal{S})} = 0\big\}$ which satisfies (S5)–(S6) with $q_{\rm proj} := p$ and $q_{\rm sz} := p$. Since $\mathcal{X}_\bullet$ is a product space of (transformed) splines, we may assume without loss of generality that $D = 1$. With the definition $\widehat{\mathcal{S}} := \big\{\gamma^{-1}(T') : T' \in \mathcal{S}\big\}$ and the dual functions $\widehat{B}^*_{\bullet,j,p}$ of Section 3.3.3, we start in the parameter domain

$$\widehat{J}_\bullet : L^2(0,1) \to \widehat{\mathcal{X}}_\bullet, \quad \widehat{\psi} \mapsto \sum_{\substack{j=1 \\ \mathrm{supp}(\widehat{B}_{\bullet,j,p}) \subseteq \bigcup \widehat{\mathcal{S}}}}^{N_\bullet} \int_0^1 \widehat{B}^*_{\bullet,j,p}(t)\, \widehat{\psi}(t)\, dt\, \widehat{B}_{\bullet,j,p}|_{[0,1)}. \tag{5.8.26}$$

By definition, $\widehat{J}_\bullet$ even maps into $\{\widehat{\Psi}_\bullet \in \widehat{\mathcal{X}}_\bullet : \widehat{\Psi}_\bullet|_{\bigcup(\widehat{\mathcal{T}}_\bullet \setminus \widehat{\mathcal{S}})} = 0\}$. With this, we can define

$$J_{\bullet,\mathcal{S}} : L^2(\Gamma) \to \{\Psi_\bullet \in \mathcal{X}_\bullet : \Psi_\bullet|_{\bigcup(\mathcal{T}_\bullet \setminus \mathcal{S})} = 0\}, \quad \psi \mapsto \widehat{J}_\bullet(\psi \circ \gamma) \circ \gamma^{-1}. \tag{5.8.27}$$

To show (S5), let $T \in \mathcal{T}_\bullet$ with $\Pi_\bullet^{q_{\rm sz}}(T) \subseteq \mathcal{S}$ and let $\widehat{T}$ be the corresponding element in the parameter domain. Since $\{\gamma(\widehat{T}') : \widehat{T}' \in \Pi_\bullet(\widehat{T})\} \subseteq \Pi_\bullet(T)$, this particularly implies that $\Pi_\bullet^{q_{\rm sz}}(\widehat{T}) \subseteq \widehat{\mathcal{S}}$. Further, let $\psi \in L^2(\Gamma)$ and $\widehat{\psi} := \psi \circ \gamma$. There holds that

$$(J_\bullet \psi) \circ \gamma|_{\widehat{T}} = (\widehat{J}_\bullet \widehat{\psi})|_{\widehat{T}} = \sum_{\substack{j=1 \\ \operatorname{supp}(\widehat{B}_{\bullet,j,p}) \subseteq \bigcup \widehat{\mathcal{S}}}}^{N_\bullet} \int_0^1 \widehat{B}^*_{\bullet,j,p}(t)\, \widehat{\psi}(t)\, dt\, \widehat{B}_{\bullet,j,p}|_{\widehat{T}}.$$

The term $\widehat{B}_{\bullet,j,p}|_{\widehat{T}}$ does not vanish only if $|\operatorname{supp}(\widehat{B}_{\bullet,j,p}) \cap \widehat{T}| > 0$. Due to Lemma 3.2.1 (ii), this requires $\operatorname{supp}(\widehat{B}_{\bullet,j,p}) \subseteq \pi_\bullet^{q_{\rm sz}}(\widehat{T})$. Hence, $\Pi_\bullet^{q_{\rm sz}}(\widehat{T}) \subseteq \widehat{\mathcal{S}}$ implies that

$$(J_\bullet \psi) \circ \gamma|_{\widehat{T}} = \sum_{j=1}^{N_\bullet} \int_0^1 \widehat{B}^*_{\bullet,j,p}(t)\, \widehat{\psi}(t)\, dt\, \widehat{B}_{\bullet,j,p}|_{\widehat{T}}.$$

The right-hand side just coincides with the quasi-interpolation $(\widehat{I}_\bullet \widehat{\psi})|_{\widehat{T}}$ of Section 3.3.3. If ψ satisfies that $\psi|_{\pi_\bullet^{q_{\rm proj}}(T)} \in \{\Psi_\bullet|_{\pi_\bullet^{q_{\rm proj}}(T)} : \Psi_\bullet \in \mathcal{X}_\bullet\}$, and hence $\widehat{\psi}|_{\pi_\bullet^{q_{\rm proj}}(\widehat{T})} \in \{\widehat{\Psi}_\bullet|_{\pi_\bullet^{q_{\rm proj}}(\widehat{T})} : \widehat{\Psi}_\bullet \in \widehat{\mathcal{X}}_\bullet\}$, Proposition 3.3.1 proves that

$$(J_{\bullet,\mathcal{S}} \psi) \circ \gamma|_{\widehat{T}} = (\widehat{I}_\bullet \widehat{\psi})|_{\widehat{T}} = \widehat{\psi}|_{\widehat{T}}.$$

This concludes the local projection property (S5).

Finally, we prove local L^2-stability (S6). Let again $T \in \mathcal{T}_\bullet$. With the abbreviations from before, the regularity (5.7.3) of γ shows that

$$\|J_{\bullet,\mathcal{S}} \psi\|_{L^2(T)} \simeq \|\widehat{J}_{\bullet,\widehat{\mathcal{S}}} \widehat{\psi}\|_{L^2(\widehat{T})}.$$

As in the proof of Proposition 3.3.1, the local quasi-uniformity (5.7.15) yields that

$$\|\widehat{J}_{\bullet,\widehat{\mathcal{S}}} \widehat{\psi}\|_{L^2(\widehat{T})} \lesssim \|\widehat{\psi}\|_{L^2(\pi_\bullet^{q_{\rm sz}}(\widehat{T}))}.$$

Thus, the regularity (5.7.3) of γ implies that

$$\|\widehat{\psi}\|_{L^2(\pi_\bullet^{q_{\rm sz}}(\widehat{T}))} \lesssim \|\psi\|_{L^2(\pi_\bullet^{q_{\rm sz}}(T))},$$

which concludes (S6). The constant $C_{\rm sz}$ depends only the constant C_γ, the polynomial order p, and the initial mesh $\widehat{\mathcal{T}}_0$.

Step 5: We set

$$q_{\rm drel} := q_{\rm supp} + \max(q_{\rm loc}, q_{\rm sz}) + 2.$$

Replacing $\mathcal{T}_\bullet \cap \mathcal{T}_\circ$ by $\Pi_\bullet\big(\{z \in \mathcal{N}_\bullet \,:\, \widetilde{\pi}_\bullet(z) \in \widetilde{\mathcal{N}}_\bullet \cap \widetilde{\mathcal{N}}_\circ\}\big)$ as well as $\mathcal{T}_\bullet \setminus \mathcal{T}_\circ$ by $\Pi_\bullet\big(\{z \in \mathcal{N}_\bullet \,:\, \widetilde{\pi}_\bullet(z) \in \widetilde{\mathcal{N}}_\bullet \setminus \widetilde{\mathcal{N}}_\circ\}\big)$ and using the properties from the Steps 1–4, one can show exactly as in the Steps 1–3 of Section 5.3.8 that

$$\|\Phi_\circ - \Phi_\bullet\|_{H^{-1/2}(\Gamma)} \lesssim \|h_\bullet^{1/2}\nabla_\Gamma \mathfrak{V}(\phi - \Phi_\bullet)\|_{L^2\left(\pi_\bullet^{q_{\rm drel}+1}(\{z\in\mathcal{N}_\bullet : \widetilde{\pi}_\bullet(z)\in\widetilde{\mathcal{N}}_\bullet\setminus\widetilde{\mathcal{N}}_\circ\})\right)},$$

where the adapted (5.3.68) follows from (5.8.24). Recalling the definition (5.8.22) of $\widetilde{\mathcal{R}}_{\bullet,\circ}$, we see that

$$\pi_\bullet^{q_{\rm drel}+1}(\{z \in \mathcal{N}_\bullet : \widetilde{\pi}_\bullet(z) \in \widetilde{\mathcal{N}}_\bullet \setminus \widetilde{\mathcal{N}}_\circ\}) = \bigcup \big\{\pi_\bullet(z) \,:\, z \in \mathcal{N}_\bullet \wedge \widetilde{\pi}_\bullet(z) \in \widetilde{\mathcal{R}}_{\bullet,\circ}\big\}.$$

Therefore, we obtain with the regularity (5.7.3) of γ, local quasi-uniformity (5.7.15), and the equivalence (5.8.11) that

$$\begin{aligned} C_\varrho\|\Phi_\circ - \Phi_\bullet\|_{H^{-1/2}(\Gamma)} \leq\ & C_{\rm drel}\Big(\sum_{\substack{z\in\mathcal{N}_\bullet\\ \widetilde{\pi}_\bullet(z)\in\widetilde{\mathcal{R}}_{\bullet,\circ}}} \|\widetilde{h}_{\widetilde{\pi}_\bullet(z)}\nabla_\Gamma\mathfrak{V}(\Phi_\circ - \Phi_\bullet)\|^2_{L^2(\pi_\bullet(z))}\Big)^{1/2} \\ =\ & C_{\rm drel}\widetilde{\eta}_\bullet(\widetilde{\mathcal{R}}_{\bullet,\circ}), \end{aligned}$$

where the constant $C_{\rm drel}$ depends only on the perturbation constant C_ϱ, the dimension D, the coefficients of the differential operator $\mathfrak{P}$, the parametrization γ, the polynomial order p, and the initial mesh $\widehat{\mathcal{T}}_0$.

5.8.8 Son estimate (T1)

Let $\ell \in \mathbb{N}_0$. During the refinement from the extended node patches $\widetilde{\mathcal{N}}_\ell$ to $\widetilde{\mathcal{N}}_{\ell+1}$ of Algorithm 5.8.1, one can increase the multiplicity of at most $\#\mathcal{N}_\ell$ nodes, and only $\#\mathcal{T}_\ell = \#\mathcal{N}_\ell$ bisections can take place. Therefore, the number of newly inserted knots is bounded by $2\#\mathcal{N}_\ell \leq 2\mu(\widetilde{\mathcal{N}}_\ell)$. With $C_{\rm son} := 3$, we see that

$$\mu(\widetilde{\mathcal{N}}_{\ell+1}) \leq C_{\rm son}\mu(\widetilde{\mathcal{N}}_\ell).$$

5.8.9 Closure estimate (T2)

Let $\ell \in \mathbb{N}_0$. For all $j \in \mathbb{N}_0$, let $\widetilde{\mathcal{N}}_j$ be the extended node patches, $\widetilde{\mathcal{M}}_j \subseteq \widetilde{\mathcal{N}}_j$ the marked extended node patches with corresponding nodes $\mathcal{M}_j \subseteq \mathcal{N}_j$, and $\mathcal{M}'_j \subseteq \mathcal{T}_j$ the marked elements of Algorithm 5.7.2 and Algorithm 5.8.1. With the auxiliary refinement Algorithm 5.7.1, we recursively define $\mathcal{K}_{(0)} := \mathcal{K}_0$ and $\mathcal{K}_{(j)} := \texttt{refine}(\mathcal{K}_{(j-1)}, \mathcal{M}'_{j-1})$ for $j \in \mathbb{N}$. Note that $\mathcal{T}_{(j)} = \mathcal{T}_j$ for all $j \in \mathbb{N}_0$. [AFF+13, Theorem 3] shows that these meshes satisfy the closure estimate, i.e.,

$$\#\mathcal{T}_\ell - \#\mathcal{T}_0 \leq C'_{\rm clos}\sum_{j=0}^{\ell-1}\#\mathcal{M}'_j, \tag{5.8.28}$$

where the constant $C'_{\rm clos} > 0$ depends only on the initial mesh $\widehat{\mathcal{T}}_0$. The proof works as in Section 5.5.7. Since $\mu(\widetilde{\mathcal{N}}_\ell) - \mu(\widetilde{\mathcal{N}}_0)$ is just the number of newly inserted knots, this term can be written as the number of all bisections $\#\mathcal{T}_\ell - \#\mathcal{T}_0$ plus the number of all multiplicity increases. Since, only the multiplicity of marked nodes can be increased, we derive with (5.8.28) that

$$\mu(\widetilde{\mathcal{N}}_\ell) - \mu(\widetilde{\mathcal{N}}_0) \le \#\mathcal{T}_\ell - \#\mathcal{T}_0 + \sum_{j=0}^{\ell-1} \#\mathcal{M}_j \le C'_{\rm clos} \sum_{j=0}^{\ell-1} \#\mathcal{M}'_j + \sum_{j=0}^{\ell-1} \#\mathcal{M}_j.$$

Note that the refinement strategy of Algorithm 5.7.2 yields that $\#\mathcal{M}'_j \le 2\#\mathcal{M}_j$. Together with $\#\mathcal{M}_j \le \mu(\widetilde{\mathcal{M}}_j)$, this concludes the proof of (T2).

5.8.10 Overlay property (T3)

Let $\widetilde{\mathcal{N}}_\bullet, \widetilde{\mathcal{N}}_\star \in \widetilde{\mathbb{N}}$. We define $\widetilde{\mathcal{N}}_\circ$ via $\mathcal{N}_\circ := \mathcal{N}_\bullet \cup \mathcal{N}_\star$ with $\#_\circ z := \max(\#_\bullet z, \#_\star z)$ for all $z \in \mathcal{N}_\circ$. [AFF$^+$13, Theorem 3] shows that the corresponding mesh $\mathcal{T}_\circ$ satisfies local quasi-uniformity (5.7.15). The simple proof works as in Section 5.5.8. Thus, we see that the corresponding knots $\mathcal{K}_\circ$ are admissible, wherefore $\widetilde{\mathcal{N}}_\circ \in \widetilde{\mathbb{N}}$. Clearly, we have that $\widetilde{\mathcal{N}}_\circ \in \texttt{refine}(\widetilde{\mathcal{N}}_\bullet) \cap \texttt{refine}(\widetilde{\mathcal{N}}_\star)$. Further, by definition, there holds that

$$\mu(\widetilde{\mathcal{N}}_\circ) \le \mu(\widetilde{\mathcal{N}}_\bullet) + \mu(\widetilde{\mathcal{N}}_\star) - \mu(\widetilde{\mathcal{N}}_0).$$

5.8.11 Proof of Theorem 5.7.4 for rational hierarchical splines

As mentioned in Remark 5.7.6, Theorem 5.7.4 is still valid if one replaces the ansatz space $\mathcal{X}_\bullet$ for $\mathcal{K}_\bullet \in \mathbb{K}$ by rational hierarchical splines, i.e., by the set

$$\mathcal{X}_\bullet^{W_0} = \Big\{ W_0^{-1}\Psi_\bullet : \Psi_\bullet \in \mathcal{X}_\bullet \Big\}, \tag{5.8.29}$$

where $\widehat{W}_0 = W_0 \circ \gamma^{-1}$ is a fixed positive weight function in the initial space of splines $\widehat{\mathcal{S}}^p(\widehat{\mathcal{K}}_0)$, where we additionally assume the representation (5.7.26). The mesh properties (M1)–(M5) as well as the refinement axioms (T1)–(T3) are independent of the discrete spaces. To verify the validity of Theorem 5.7.4 in the rational setting, it thus only remains to verify the axioms (E1)–(E4) for the rational boundary element spaces. Note that these axioms hinge only on the similar versions of (S1)–(S6) from Section 5.8.7.

To see the inverse estimate (S1), i.e., Proposition 5.8.4, in the rational setting, it is again sufficient to consider $D = 1$. We proved Proposition 5.8.4 for $\mathcal{X}_\bullet$ by applying Proposition 5.5.3 for all $\Psi_\bullet \in \mathcal{X}_\bullet$. With the notation from the proof of Proposition 5.8.4, we showed for all $T \in \mathcal{T}_\bullet$ that

$$\inf_{x \in R_T} |\Psi_\bullet(x)| \ge \rho_{\rm inf} \|\Psi_\bullet\|_{L^\infty(T)} \quad \text{for all } \Psi_\bullet \in \mathcal{X}_\bullet,$$

where $\Psi_\bullet$ does not change sign on R_T. With $0 < w_{\rm min} := \inf_{x\in\Gamma} W_0(x)$, $w_{\rm max} := \sup_{x\in\Gamma} W_0(x)$, and $\widetilde{\rho}_{\rm inf} := \rho_{\rm inf} w_{\rm min}/w_{\rm max}$, this yields for all $\Psi_\bullet \in \mathcal{X}_\bullet$ that

$$\widetilde{\rho}_{\rm inf} \|W_0^{-1}\Psi_\bullet\|_{L^\infty(T)} \le \frac{\rho_{\rm inf}}{w_{\rm max}} \|\Psi_\bullet\|_{L^\infty(T)} \le \frac{1}{w_{\rm max}} \inf_{x\in R_T} |\Psi_\bullet(x)| \le \inf_{x \in R_T} |W_0^{-1}\Psi_\bullet(x)|.$$

In particular, the conditions for Proposition 5.5.3 are also satisfied for the functions in $\mathcal{X}_\bullet^{W_0}$, which concludes (S1).

The (adapted) properties (S2)–(S3) depend only on the numerator of the rational splines and thus transfer.

For the proof of (S4), we exploit the representation (5.7.26) to verify the conditions of the abstract Proposition 5.5.5. Again, we assume without loss of generality that $D = 1$. Let $\widehat{\mathcal{K}}_\bullet \in \widehat{\mathbb{K}}$. Note that $\widehat{W}_0$ is also an element of the spline space $\widehat{\mathcal{S}}^p(\widehat{\mathcal{K}}_\bullet)$. In particular, it can be written as linear combination of B-splines in $\widehat{\mathcal{B}}_\bullet$. The representation (5.7.26) and the two-scale relation with only non-negative coefficients between bases of consecutive levels of Section 3.4 yields that the corresponding coefficients are non-negative. This implies that

$$\widehat{W}_0 = \sum_{j=1}^{N_\bullet} w_{\bullet,j}\, \widehat{B}_{\bullet,j,p}|_{[0,1)} \quad \text{with non-negative coefficients } w_{\bullet,j} \geq 0.$$

With the choice

$$\overline{\mathcal{B}}_\bullet := \Big\{ \frac{w_{\bullet,j}\widehat{B}_{\bullet,j,p}}{\widehat{W}_0} \circ \gamma^{-1} : j \in \{1, \dots, N_\bullet\} \Big\},$$

Lemma 3.2.1 shows that the assumptions of Proposition 5.5.5 are satisfied.

To see the (adapted) properties (S5)–(S6), we define the corresponding projection operator

$$J_{\bullet,\mathcal{S}}^{W_0} : L^2(\Gamma)^D \to \big\{ \Psi_\bullet \in \mathcal{X}_\bullet \,:\, \Psi_\bullet|_{\bigcup(\mathcal{T}_\bullet \setminus \mathcal{S})} = 0 \big\}, \quad \psi \mapsto W_0^{-1} J_{\bullet,\mathcal{S}}(W_0\psi). \tag{5.8.30}$$

The desired properties transfer immediately from the non-rational case.

5.9 Numerical experiments with one-dimensional splines

In this section, we empirically investigate the performance of Algorithm 5.7.3 in three typical situations: In Section 5.9.1, the solution is piecewise smooth on $\Gamma = \partial\Omega$ with certain jumps which locally require discontinuous ansatz functions. In Section 5.9.2, the solution exhibits a generic (i.e., geometry induced) singularity.

We consider the Laplace–Dirichlet problem

$$\begin{aligned} -\Delta u &= 0 \quad \text{in } \Omega, \\ u &= g \quad \text{on } \Gamma, \end{aligned} \tag{5.9.1}$$

for given Dirichlet data $g \in H^{1/2}(\Gamma)$. The fundamental solution of $-\Delta$ is given by

$$G(z) := -\frac{1}{2\pi} \log |z| \quad \text{for all } z \in \mathbb{R}^2 \setminus \{0\}. \tag{5.9.2}$$

To guarantee ellipticity of the corresponding single-layer operator $\mathfrak{V}$, we additionally suppose that $\operatorname{diam}(\Omega) < 1$; see Section 5.1.3. Then, (5.9.1) can be equivalently rewritten as integral equation (5.1.14); see, e.g., [McL00, Theorem 7.6], [Ste08a, Section 7.1], or [SS11, Section 3.4.2.1]. Indeed, the normal derivative $\phi := \partial_\nu u$ of the weak solution u of (5.9.1) satisfies the integral equation (5.1.14) with $f := (\mathfrak{K} + 1/2)g$, i.e.,

$$\mathfrak{V}\phi = (\mathfrak{K} + 1/2)g, \tag{5.9.3}$$

where

$$\mathfrak{K} : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma) \tag{5.9.4}$$

denotes the *double-layer operator*. According to [SS11, Corollary 3.3.12 and Theorem 3.3.13], if Γ is piecewise smooth and if $g \in L^\infty(\Gamma)$, there holds for all $x \in \Gamma$ the representation

$$\mathfrak{K}g(x) = \int_\Gamma g(y)\partial_{\nu(y)}G(x,y)\,dy \quad \text{if } \Gamma \text{ is smooth in } x \text{ and } g \text{ is continuous at } x. \tag{5.9.5}$$

These conditions are satisfied for all considered examples. Indeed, the boundary Γ is parametrized via rational splines, i.e., there exists a polynomial order $p_\gamma \in \mathbb{N}$, a p_γ-open knot vector $\widehat{\mathcal{K}}_\gamma$ on $[0,1]$, and a positive spline weight function

$$\widehat{W}_\gamma = \sum_{j=1}^{N_\gamma} w_{\gamma,j}\,\widehat{B}_{\gamma,j,p}|_{[0,1)} \quad \text{with non-negative coefficients } w_{\gamma,j} \geq 0. \tag{5.9.6}$$

such that

$$\gamma \in \big\{\widehat{W}_\gamma^{-1}\,\widehat{S} \,:\, \widehat{S} \in \widehat{\mathcal{S}}^{p_\gamma}(\widehat{\mathcal{K}}_\gamma)^2\big\} \tag{5.9.7}$$

Based on the knots $\widehat{\mathcal{K}}_\gamma$ for the geometry, we choose the initial knots $\widehat{\mathcal{K}}_0$ for the discretisation such that (at least) the corresponding nodes coincide, i.e., $\widehat{\mathcal{N}}_0 = \widehat{\mathcal{N}}_\gamma$. As basis for the considered ansatz spaces, we use (5.7.8). To (approximately) calculate the Galerkin matrix, the right-hand side vector, and the weighted-residual error estimator[7] (5.7.17), we transform the singular integrands into a sum of a smooth part and a logarithmically singular part. Then, we use adapted Gauss quadrature to compute the resulting integrals with appropriate accuracy; see [Gan14, Section 5] for details. For the (dense) Galerkin matrix, we do not apply any matrix compression techniques such as wavelet methods [BCR91, DHS06, HR10], fast multipole methods [GR87, TM12, DHK+17], or $\mathcal{H}$-matrix methods [Hac99, MZBF15].

To (approximately) calculate the energy error, we proceed as follows: Let $\Phi_\ell \in \mathcal{X}_\ell$ be the Galerkin approximation of the ℓ-th step with the corresponding coefficient vector $\boldsymbol{c_\ell}$. Further, let $\boldsymbol{V_\ell}$ be the Galerkin matrix. With Galerkin orthogonality (5.8.15) and the energy norm $\|\phi\|_{\mathfrak{V}}^2 = \langle \mathfrak{V}\phi\,,\,\phi\rangle$ obtained by Aitken's Δ^2-extrapolation, we can compute the energy error as

$$\|\phi - \Phi_\ell\|_{\mathfrak{V}}^2 = \|\phi\|_{\mathfrak{V}}^2 - \|\Phi_\ell\|_{\mathfrak{V}}^2 = \|\phi\|_{\mathfrak{V}}^2 - \boldsymbol{V_\ell c_\ell} \cdot \boldsymbol{c_\ell}. \tag{5.9.8}$$

[7] To ease computation, we replace the term $|\pi_\ell(z)|$ in the error indicators $\eta_\ell(z) = \||\pi_\ell(z)|^{1/2}\nabla_\Gamma(f - \mathfrak{V}\Phi_\ell)\|_{\pi_\ell(z)}$ by the equivalent term $\mathrm{diam}(\Gamma)\,\widehat{h}_\ell$. Here, $\widehat{h}_\ell \in L^\infty(\Gamma)$ denotes the mesh-width function with $\widehat{h}_\ell|_T = |\gamma^{-1}(T)|$ for all $T \in \mathcal{T}_\ell$.

5.9.1 Jump solution on square

We consider the Laplace–Dirichlet problem (5.9.1) on the square

$$\Omega := (0, 1/4)^2; \tag{5.9.9}$$

see Figure 5.7. The boundary Γ is parametrized on $[0, 1]$ by a (non-rational) spline curve of degree $p_\gamma := 1$, where $\widehat{\mathcal{K}}_\gamma := (0, 0, 1/4, 1/2, 3/4, 1, 1)$. We prescribe the exact solution of (5.9.1) as

$$u(x_1, x_2) := \sinh(2\pi x_1)\cos(2\pi x_2), \tag{5.9.10}$$

and consider the corresponding integral equation (5.9.3). The normal derivative $\phi = \partial_\nu u$ of u reads

$$\phi(x_1, x_2) = 2\pi \begin{pmatrix} \cosh(2\pi x_1)\cos(2\pi x_2) \\ \sinh(2\pi x_1)\cos(2\pi x_2) \end{pmatrix} \cdot \nu(x_1, x_2). \tag{5.9.11}$$

It is smooth up to four points as can be seen in Figure 5.8.

We employ splines of degree $p := p_\gamma$ with initial knots $\widehat{\mathcal{K}}_0 := \widehat{\mathcal{K}}_\gamma$. The parameters of Algorithm 5.7.3 are chosen as $\theta = 0.75$ and $C_{\min} = 1$. For comparison, we also consider uniform refinement, where we mark all nodes in each step, i.e., $\mathcal{M}_\ell = \mathcal{N}_\ell$ for all $\ell \in \mathbb{N}_0$. Note that this leads to uniform bisection (without knot multiplicity increase) of all elements. In Figure 5.9, the corresponding errors and error estimators are illustrated. All values are plotted in a double logarithmic scale such that the experimental convergence rates are visible as the slope of the corresponding curves. Although we only proved reliability (5.7.22) of the employed estimator, the curves for the error and the estimator are parallel in each case, which numerically indicates reliability and efficiency. The solution $\phi \circ \gamma$ has jumps at the points $t = 1/4$, $t = 1/2$, and $t = 1$ resp. $t = 0$. As the knots $\widehat{\mathcal{K}}_\gamma$ used for the parametrization of Γ all have multiplicity one, the functions of the isogeometric initial ansatz space are continuous at the points $t = 1/4$, $t = 1/2$. Uniform refinement leads to the suboptimal rate $\mathcal{O}(N^{-1})$ for the energy error, whereas adaptive refinement increases the knot multiplicity at these problematic points and leads again to the optimal rate $\mathcal{O}(N^{-3/2-p}) = \mathcal{O}(N^{-5/2})$; see [SS11, Corollary 4.1.34].

5.9.2 Singular solution on pacman geometry

We consider the Laplace–Dirichlet problem (5.9.1) on the pacman geometry

$$\Omega := \left\{ r \begin{pmatrix} \cos(\beta) \\ \sin(\beta) \end{pmatrix} : r \in (0, 1/4) \wedge \beta \in (-\pi/2\tau, \pi/2\tau) \right\}, \tag{5.9.12}$$

with $\tau := 4/7$; see Figure 5.10. The geometry is parametrized on $[0, 1]$ by a rational spline curve of degree $p_\gamma := 2$, where

$$\widehat{\mathcal{K}}_\gamma := (0, 0, 0, 1/6, 1/6, 1/3, 1/3, 1/2, 1/2, 2/3, 2/3, 5/6, 5/6, 1, 1, 1), \tag{5.9.13}$$

and with the abbreviation $w := \cos(\pi/\tau)$,

$$(w_{\gamma,1}, \dots, w_{\gamma,13}) := (1, w, 1, w, 1, 1, 1, 1, 1, w, 1, w, 1); \tag{5.9.14}$$

see [FGP15, Section 5.3]. We prescribe the exact solution of (5.9.1) as

$$u(x_1, x_2) := r^\tau \cos(\tau\beta) \tag{5.9.15}$$

in polar coordinates $(x_1, x_2) = r(\cos\beta, \sin\beta)$ with $\beta \in (-\pi, \pi)$. We consider the corresponding integral equation (5.9.3). The normal derivative $\phi = \partial_\nu u$ of u reads

$$\phi(x_1, x_2) = \begin{pmatrix} \cos(\beta)\cos(\tau\beta) + \sin(\beta)\sin(\tau\beta) \\ \sin(\beta)\cos(\tau\beta) - \cos(\beta)\sin(\tau\beta) \end{pmatrix} \cdot \nu(x_1, x_2) \cdot \tau \cdot r^{\tau-1} \tag{5.9.16}$$

and has a generic singularity at the origin. In Figure 5.11, the solution ϕ is plotted over the parameter domain. The singularity is located at $t = 1/2$ and two jumps are located at $t = 1/3$ resp. $t = 2/3$.

First, we make a pure isogeometric approach and choose the polynomial degree $p := p_\gamma$, the initial knots $\widehat{\mathcal{K}}_0 := \widehat{\mathcal{K}}_\gamma$, and $\widehat{W}_0 := \widehat{W}_\gamma$; see Remark 5.7.6. We choose the parameters of Algorithm 5.7.3 as $\theta = 0.75$ and $C_{\min} = 1$. For comparison, we also consider uniform refinement, where we mark all nodes in each step, i.e., $\mathcal{M}_\ell = \mathcal{N}_\ell$ for all $\ell \in \mathbb{N}_0$. Note that this leads to uniform bisection (without knot multiplicity increase) of all elements. In Figure 5.12, the corresponding errors and error estimators are plotted. All values are plotted in a double logarithmic scale such that the experimental convergence rates are visible as the slope of the corresponding curves. Although we only proved reliability (5.7.22) of the employed estimator, the curves for the error and the estimator are parallel in each case, which numerically indicates reliability and efficiency. Since the solution lacks regularity, uniform refinement leads to the suboptimal rate $\mathcal{O}(N^{-4/7})$ for the energy error, whereas adaptive refinement leads to the optimal rate $\mathcal{O}(N^{-3/2-p}) = \mathcal{O}(N^{-7/2})$. For adaptive refinement, Figure 5.13 provides a histogram of the knots in $[a, b]$ of the last refinement step. We observe that at $1/2$, where the singularity occurs, mainly h-refinement is used. Instead, at the two jump points $1/3$ and $2/3$, the adaptive algorithm just increases the multiplicity of the corresponding knots to its maximum allowing for discontinuous ansatz functions.

Next, we consider non-rational splines with

$$\begin{aligned}
p &:= 0 \quad \text{and} \quad \widehat{\mathcal{K}}_0 := \big(0, \tfrac{1}{6}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{5}{6}, 1\big), \\
p &:= 1 \quad \text{and} \quad \widehat{\mathcal{K}}_0 := \big(0, 0, \tfrac{1}{6}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{5}{6}, 1, 1\big), \\
p &:= 2 \quad \text{and} \quad \widehat{\mathcal{K}}_0 := \big(0, 0, 0, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{2}{3}, \tfrac{5}{6}, \tfrac{5}{6}, 1, 1, 1\big), \\
p &:= 3 \quad \text{and} \quad \widehat{\mathcal{K}}_0 := \big(0, 0, 0, 0, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{2}{3}, \tfrac{2}{3}, \tfrac{5}{6}, \tfrac{5}{6}, \tfrac{5}{6}, 1, 1, 1, 1\big),
\end{aligned}$$

i.e., the initial ansatz space mimics (if possible) the smoothness of the geometry representation γ at the nodes $\widehat{\mathcal{N}}_0 = \widehat{\mathcal{N}}_\gamma$. Note that γ is only continuous and not necessarily differentiable at the nodes. Again, we consider adaptive refinement with $\theta = 0.75$ and $C_{\min} = 1$ and uniform refinement. For $p = 2$, we compare in Figure 5.14 the energy errors with the isogeometric approach from before. In Figure 5.15 and 5.9.3, we plot the errors and the estimators for $p \in \{0, 1, 2, 3\}$. Again, adaptive refinement leads to the optimal convergence rate $\mathcal{O}(N^{-3/2-p})$.

5.9.3 Singular solution on heart geometry

We consider the Laplace–Dirichlet problem (5.9.1) on the heart geometry (consisting of two semicircles and a square)

$$\begin{aligned}\Omega := &\left\{ r \begin{pmatrix}\cos(\beta)\\ \sin(\beta)\end{pmatrix} - \begin{pmatrix}-1/8\\ -1/8\end{pmatrix} : r \in [0, \sqrt{2}/8) \wedge \beta \in [\pi/4, 5\pi/4] \right\}\\ &\cup \left\{ r \begin{pmatrix}\cos(\beta)\\ \sin(\beta)\end{pmatrix} - \begin{pmatrix}1/8\\ -1/8\end{pmatrix} : r \in [0, \sqrt{2}/8) \wedge \beta \in [-\pi/4, 3\pi/4] \right\}\\ &\cup \operatorname{co}\left(\left\{ \begin{pmatrix}-1/8\\ -1/8\end{pmatrix}, \begin{pmatrix}0\\ -1/2\end{pmatrix}, \begin{pmatrix}1/8\\ -1/8\end{pmatrix}, \begin{pmatrix}0\\ 0\end{pmatrix} \right\} \right)^\circ\end{aligned} \tag{5.9.17}$$

where $\operatorname{co}(\cdot)^\circ$ denotes the interior of the convex hull of a set; see Figure 5.17. The geometry is parametrized on $[0, 1]$ by a rational spline curve of degree $p_\gamma := 2$, where

$$\widehat{\mathcal{K}}_\gamma := (0, 0, 0, 1/6, 1/6, 1/3, 1/3, 1/2, 1/2, 2/3, 2/3, 5/6, 5/6, 1, 1, 1), \tag{5.9.18}$$

and with the abbreviation $w := 1/\sqrt{2}$,

$$(w_{\gamma,1}, \dots, w_{\gamma,13}) := (1, w, 1, w, 1, 1, 1, w, 1, 1, 1, 1, 1); \tag{5.9.19}$$

see [CHB09, Section 2.4.1.1]. We set $\tau := 2/3$ and prescribe the exact solution of (5.9.1) as

$$u(x_1, x_2) := r^\tau \cos\left(\tau(\beta + \pi/2)\right) \tag{5.9.20}$$

in polar coordinates $(x_1, x_2) = r(\cos\beta, \sin\beta)$ with $\beta \in (-3\pi/2, \pi/2)$. We consider the corresponding integral equation (5.9.3). With the abbreviation $\widetilde{\beta} := \beta + \pi/2$, the normal derivative $\phi = \partial_\nu u$ of u reads

$$\phi(x_1, x_2) = \begin{pmatrix}\cos(\widetilde{\beta})\cos(\tau\widetilde{\beta}) + \sin(\widetilde{\beta})\sin(\tau\widetilde{\beta})\\ \sin(\widetilde{\beta})\cos(\tau\widetilde{\beta}) - \cos(\widetilde{\beta})\sin(\tau\widetilde{\beta})\end{pmatrix} \cdot \nu(x_1, x_2) \cdot \tau \cdot r^{\tau-1} \tag{5.9.21}$$

and has a generic singularity at the origin. In Figure 5.18, the solution ϕ is plotted over the parameter domain. The singularity is located at $t = 1/2$.

First, we make a pure isogeometric approach and choose the polynomial degree $p := p_\gamma$, the initial knots $\widehat{\mathcal{K}}_0 := \widehat{\mathcal{K}}_\gamma$, and $\widehat{W}_0 := \widehat{W}_\gamma$; see Remark 5.7.6. We choose the parameters of Algorithm 5.7.3 as $\theta = 0.75$ and $C_{\min} = 1$. For comparison, we also consider uniform refinement, where we mark all nodes in each step, i.e., $\mathcal{M}_\ell = \mathcal{N}_\ell$ for all $\ell \in \mathbb{N}_0$, which leads to uniform bisection (without knot multiplicity increase) of all elements. In Figure 5.19, the corresponding errors and error estimators are plotted. All values are plotted in a double logarithmic scale such that the experimental convergence rates are visible as the slope of the corresponding curves. Again, the curves for the error and the estimator are parallel in each case. Since the solution lacks regularity, uniform refinement leads to the suboptimal rate $\mathcal{O}(N^{-2/3})$ for the energy error, whereas adaptive refinement leads to the optimal rate $\mathcal{O}(N^{-3/2-p}) = \mathcal{O}(N^{-7/2})$.

Next, we consider non-rational splines with

$$
\begin{aligned}
p &:= 0 \quad \text{and} \quad \widehat{\mathcal{K}}_0 := \big(0, \tfrac{1}{6}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{5}{6}, 1\big),\\
p &:= 1 \quad \text{and} \quad \widehat{\mathcal{K}}_0 := \big(0, 0, \tfrac{1}{6}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{5}{6}, 1, 1\big),\\
p &:= 2 \quad \text{and} \quad \widehat{\mathcal{K}}_0 := \big(0, 0, 0, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{2}{3}, \tfrac{5}{6}, \tfrac{5}{6}, 1, 1, 1\big),\\
p &:= 3 \quad \text{and} \quad \widehat{\mathcal{K}}_0 := \big(0, 0, 0, 0, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{2}{3}, \tfrac{2}{3}, \tfrac{5}{6}, \tfrac{5}{6}, \tfrac{5}{6}, 1, 1, 1, 1\big),
\end{aligned}
$$

i.e., the initial ansatz space mimics (if possible) the smoothness of the geometry representation γ at the nodes $\widehat{\mathcal{N}}_0 = \widehat{\mathcal{N}}_\gamma$. Note that γ is only continuous and not necessarily differentiable at the nodes. As before, we consider adaptive refinement with $\theta = 0.75$ and $C_{\min} = 1$ and uniform refinement. For $p = 2$, we compare in Figure 5.20 the energy errors with the isogeometric approach from before. In Figure 5.21 and 5.9.3, we plot the errors and the estimators for $p \in \{0, 1, 2, 3\}$. Again, adaptive refinement leads to the optimal convergence rate $\mathcal{O}(N^{-3/2-p})$.

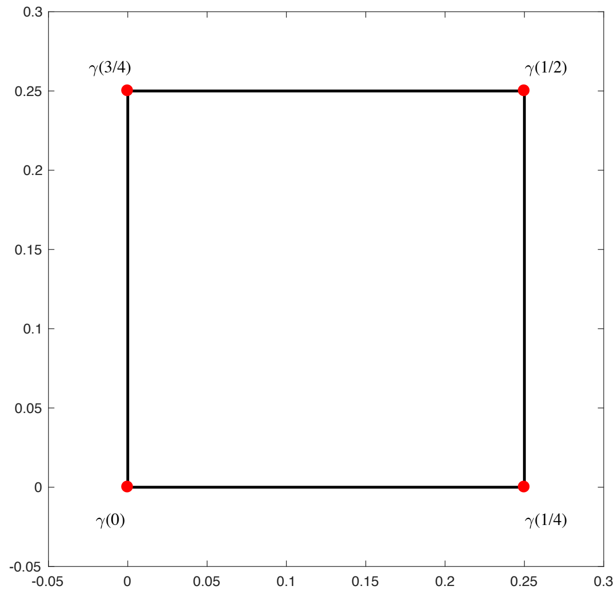


Figure 5.7: Geometry and initial nodes for the experiment of Section 5.9.1.

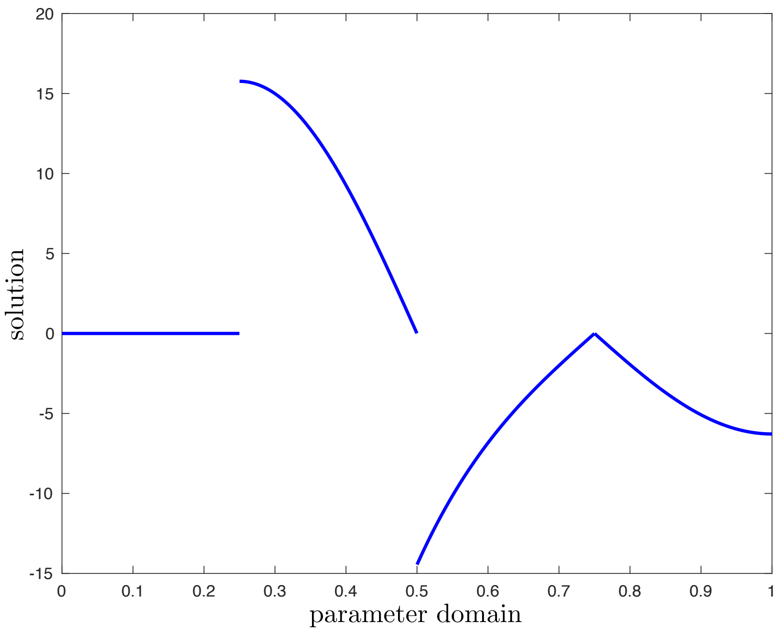


Figure 5.8: Experiment with jump solution on square of Section 5.9.1. The solution $\phi \circ \gamma$ is plotted on the parameter parameter domain.

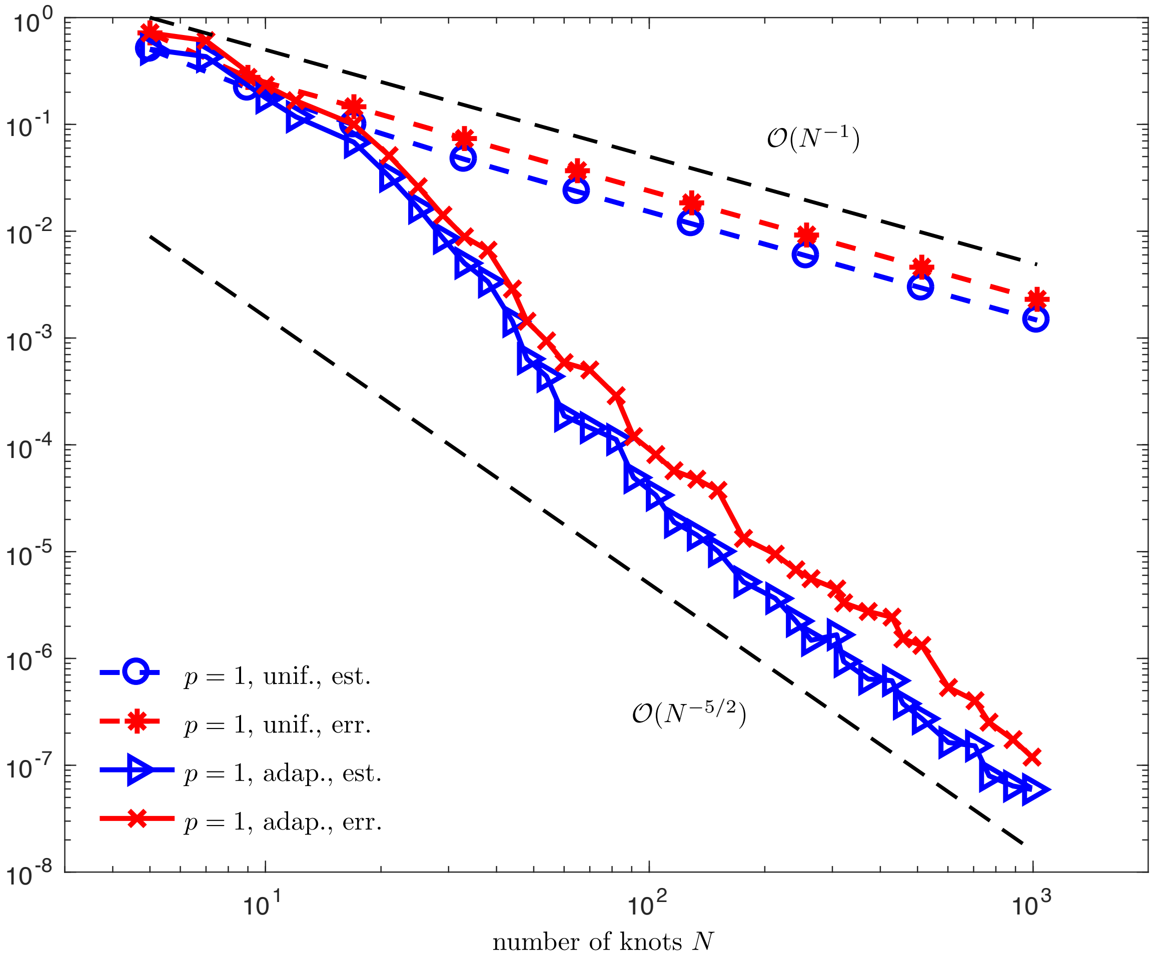


Figure 5.9: Experiment with jump solution on square of Section 5.9.1. Energy error $\|\phi - \Phi_\ell\|_{\mathfrak{V}}$ and estimator η_ℓ of Algorithm 5.7.3 for splines of degree $p = 1$ are plotted versus the number of knots N. Uniform and adaptive ($\theta = 0.75$) refinement is considered.

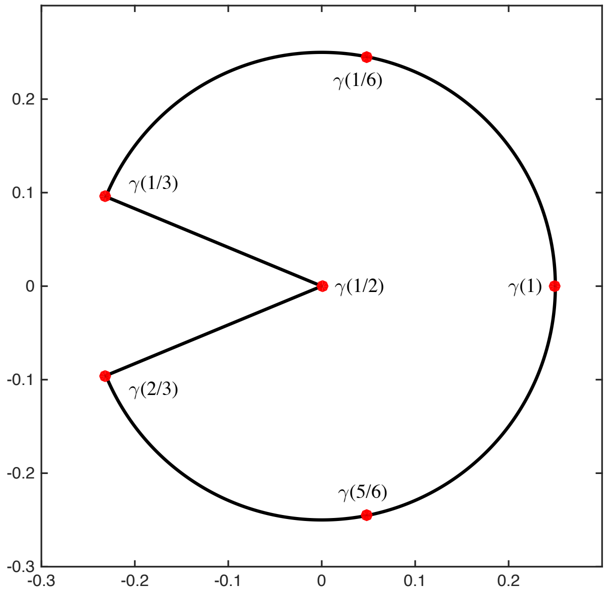


Figure 5.10: Geometry and initial nodes for the experiment of Section 5.9.2.

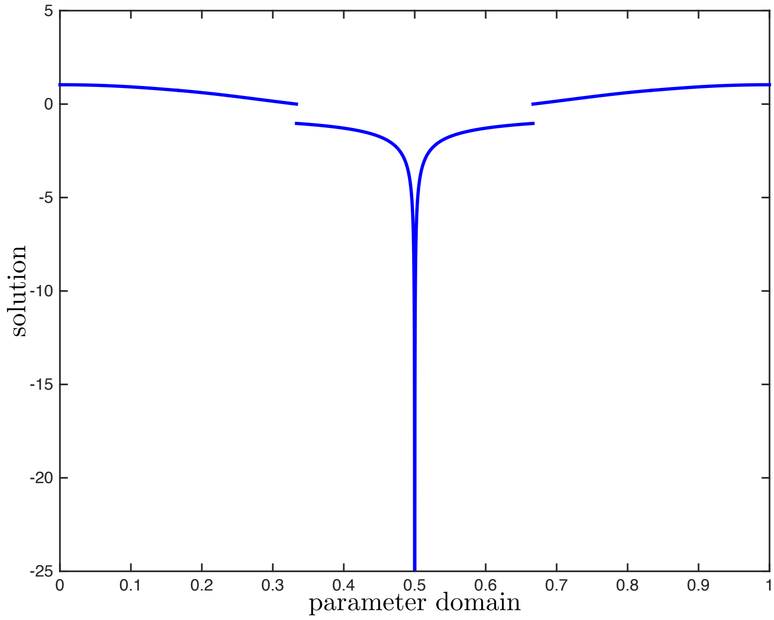


Figure 5.11: Experiment with singular solution on pacman geometry of Section 5.9.2. The singular solution $\phi \circ \gamma$ is plotted on the parameter domain, where 0.5 corresponds to the origin, where ϕ is singular.

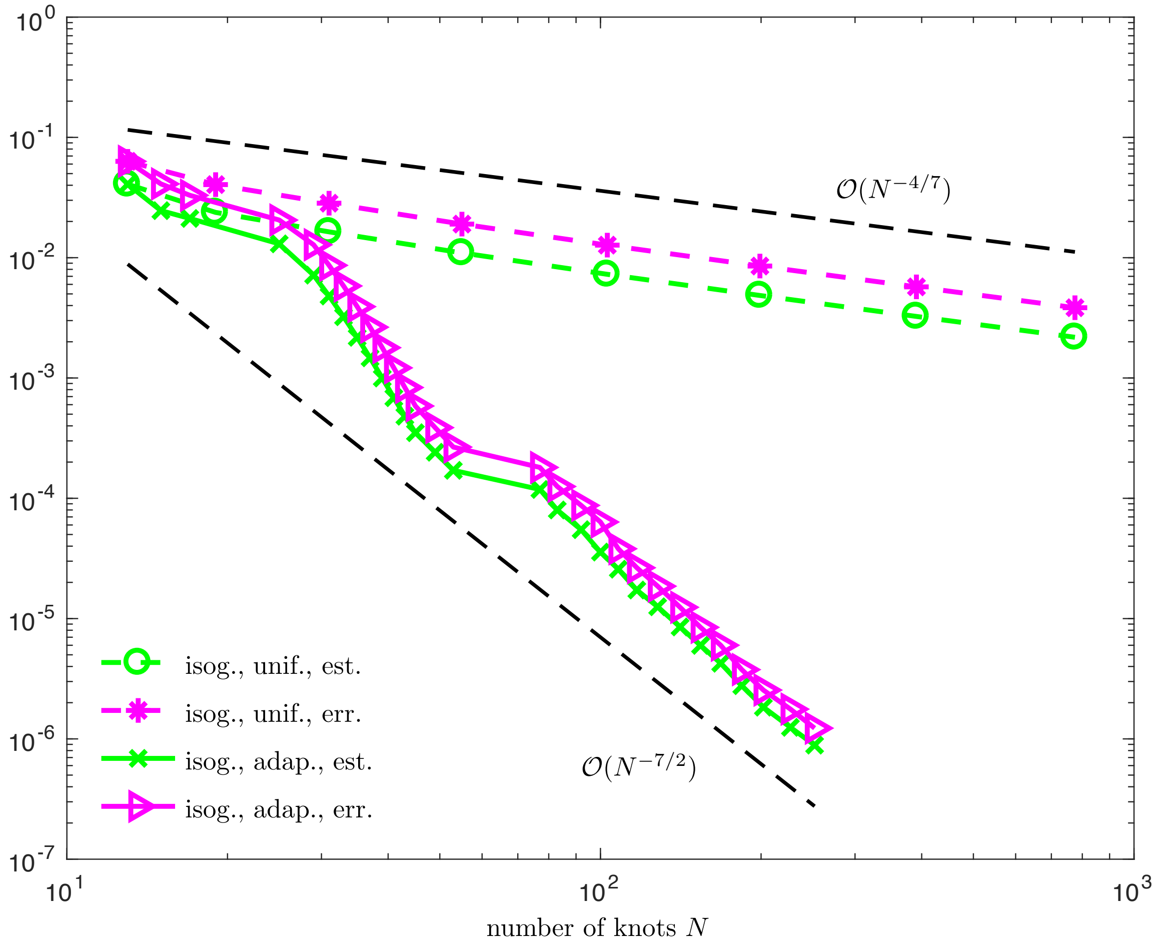


Figure 5.12: Experiment with singular solution on pacman geometry of Section 5.9.2. Energy error $\|\phi - \Phi_\ell\|_{\mathfrak{V}}$ and estimator η_ℓ of Algorithm 5.7.3 for rational splines of degree $p = 2$ are plotted versus the number of knots N. Uniform and adaptive ($\theta = 0.75$) refinement is considered.

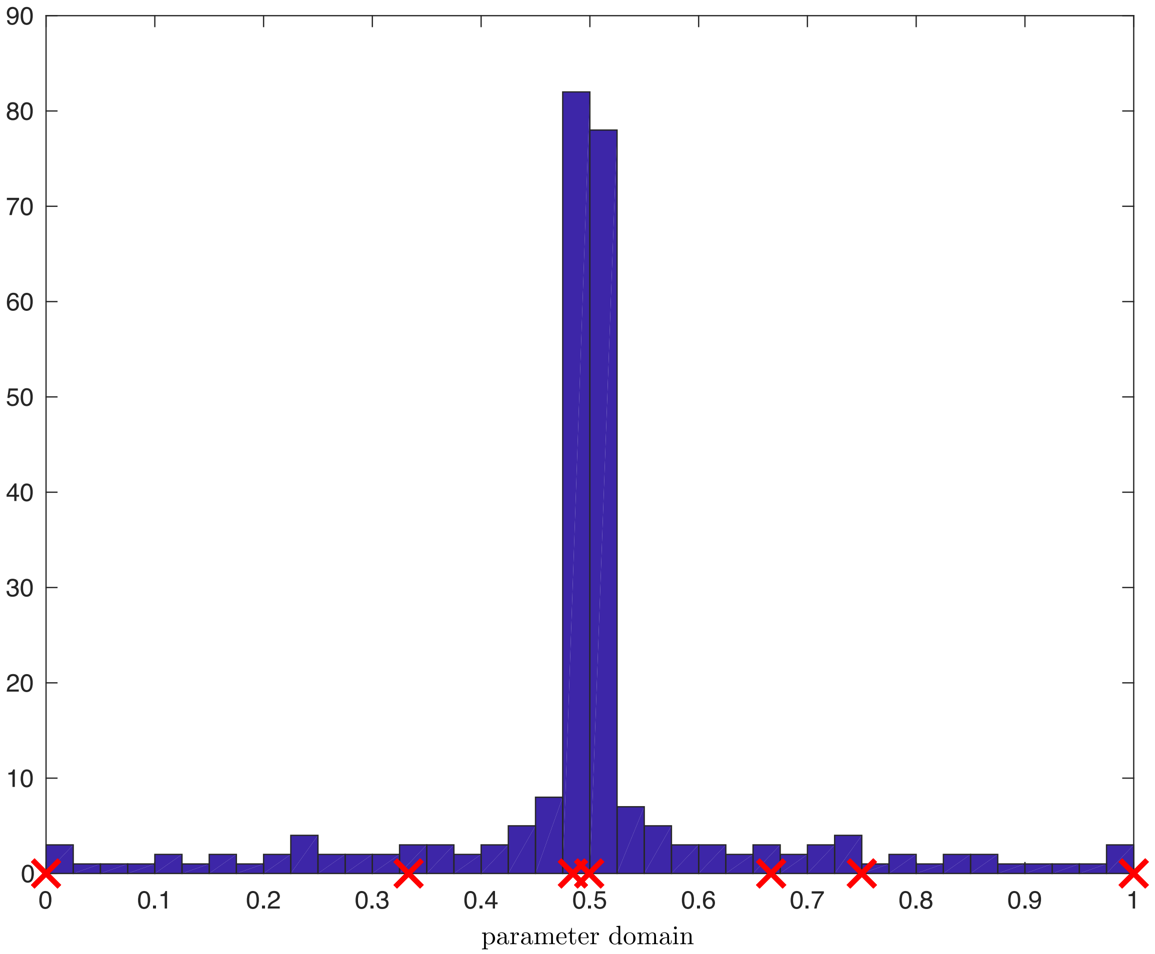


Figure 5.13: Experiment with singular solution on pacman geometry of Section 5.9.2. Histogram of number of knots over the parameter domain for the knot vector $\mathcal{K}_{28}$ generated by Algorithm 5.7.3 (with $\theta = 0.75$) for rational splines of degree $p = 2$. Knots with maximal multiplicity $p + 1 = 3$ are marked.

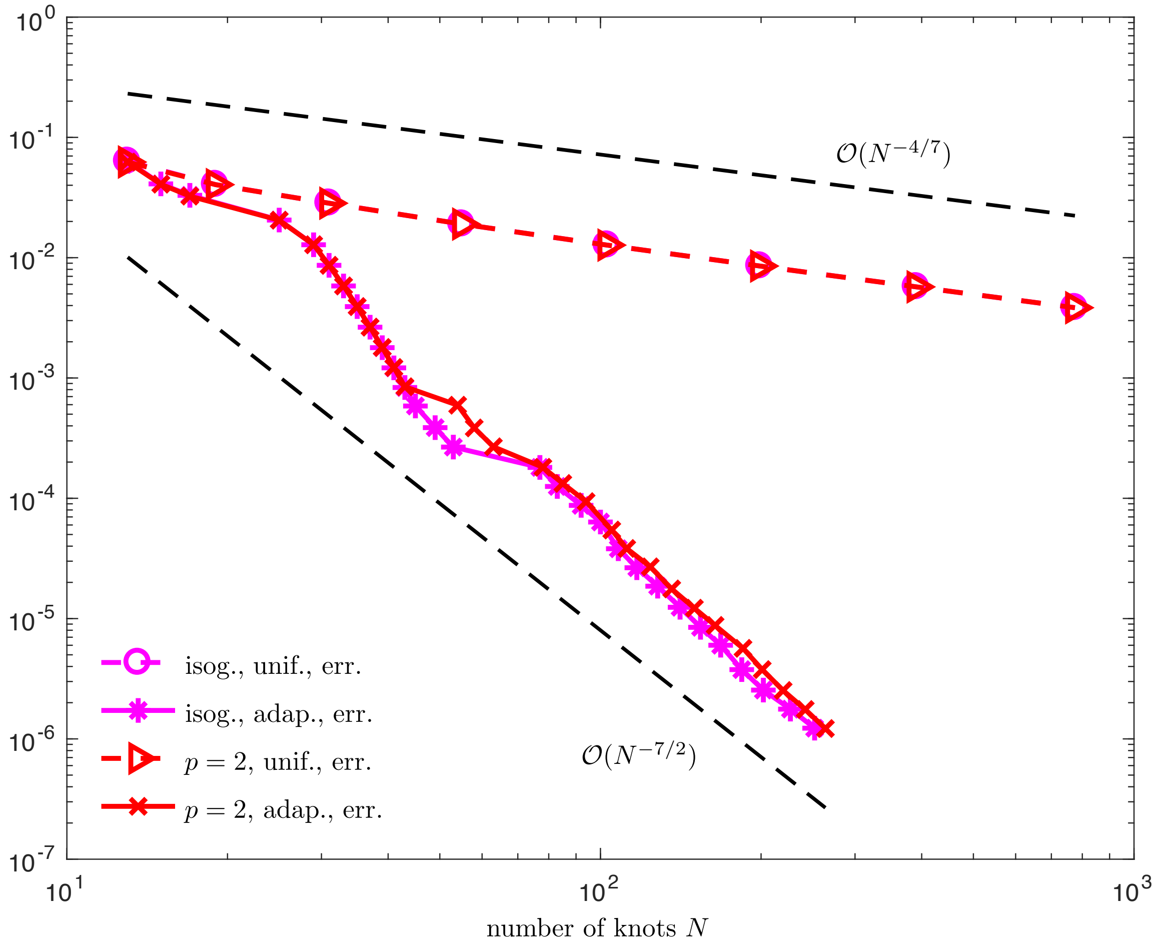


Figure 5.14: Experiment with singular solution on pacman geometry of Section 5.9.2. Energy error $\|\phi - \Phi_\ell\|_{\mathfrak{V}}$ and estimator η_ℓ of Algorithm 5.7.3 for (rational) splines of degree $p = 2$ are plotted versus the number of knots N. Uniform and adaptive ($\theta = 0.75$) refinement is considered.

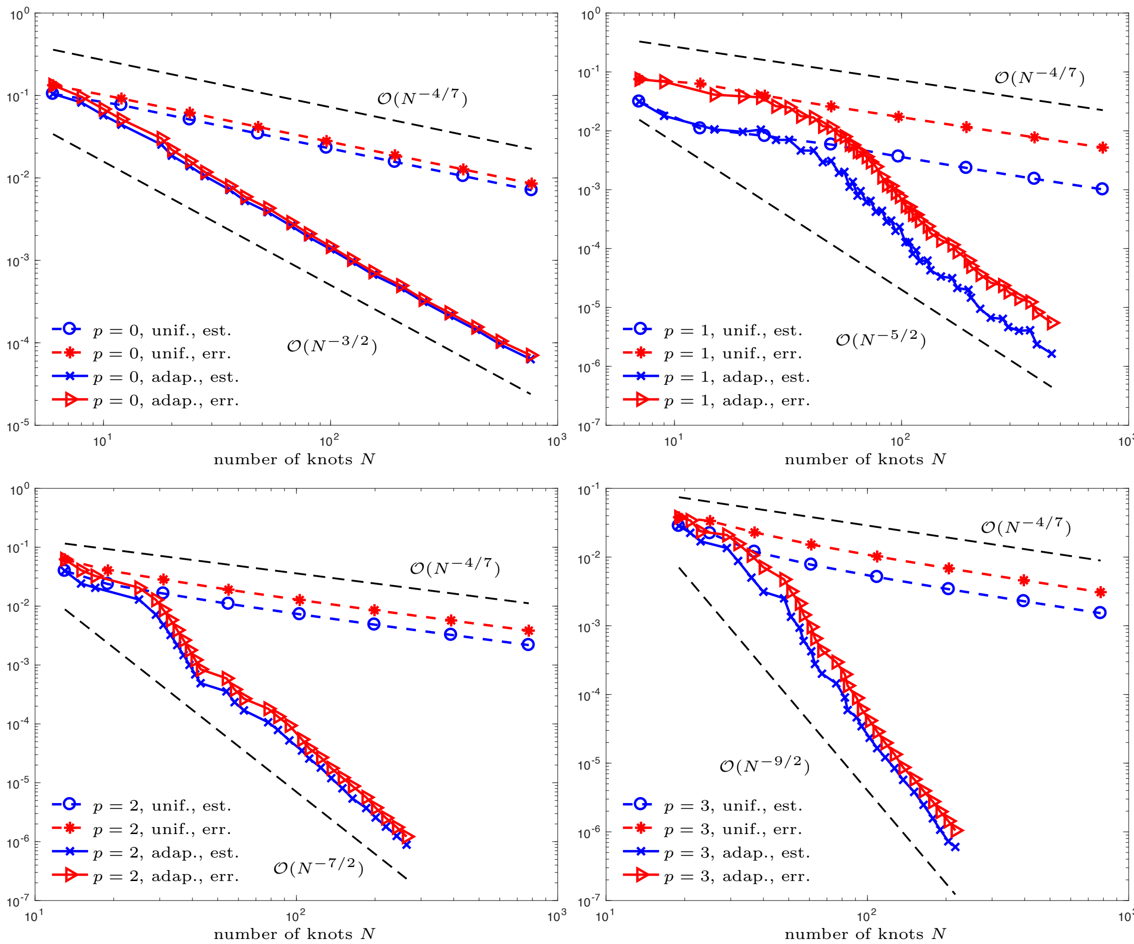


Figure 5.15: Experiment with singular solution on pacman geometry of Section 5.9.2. Energy error $\|\phi - \Phi_\ell\|_{\mathfrak{V}}$ and estimator η_ℓ of Algorithm 5.7.3 for splines of degree $p \in \{0, 1, 2, 3\}$ are plotted versus the number of knots N. Uniform and adaptive ($\theta = 0.75$) refinement is considered.

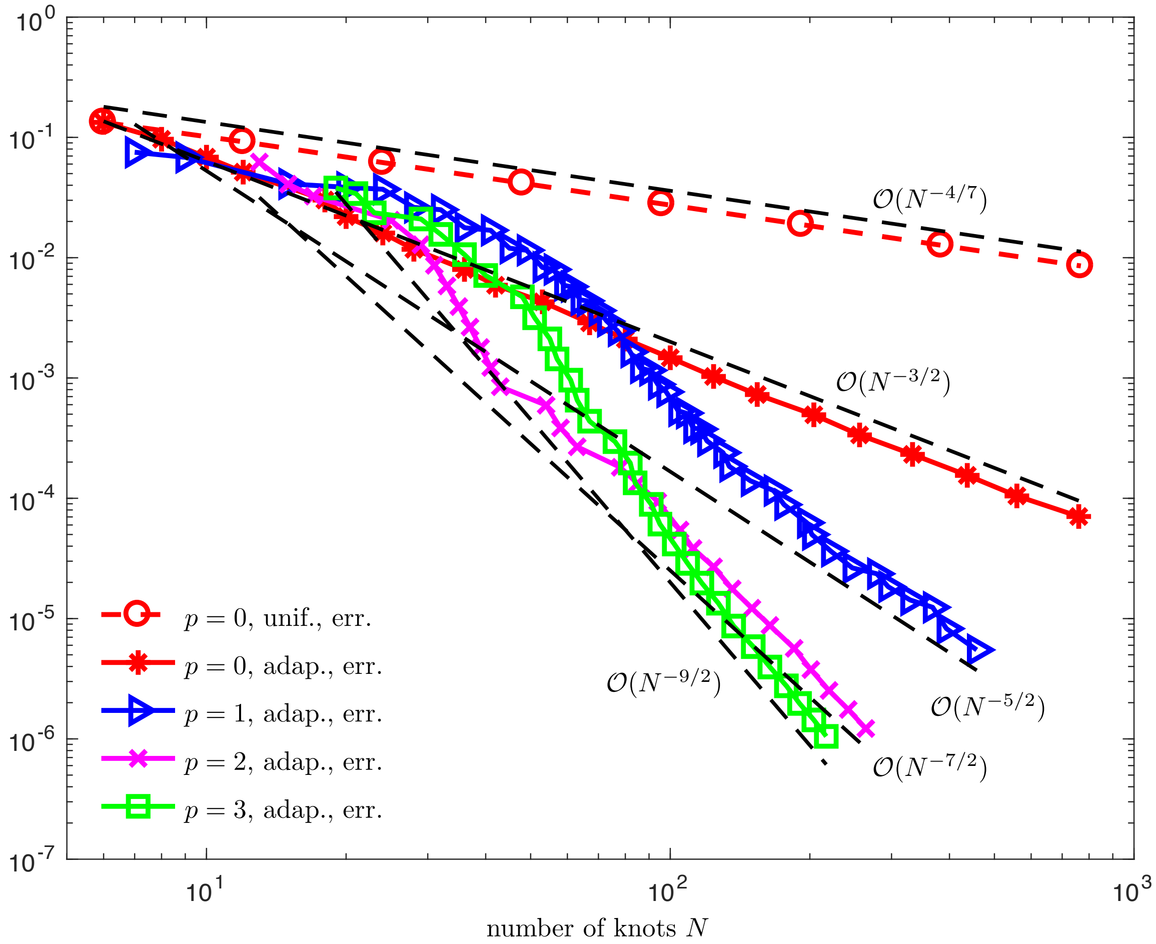


Figure 5.16: Experiment with singular solution on pacman geometry of Section 5.9.2. The energy errors $\|\phi - \Phi_\ell\|_{\mathfrak{V}}$ of Algorithm 5.7.3 for splines of degree $p \in \{0, 1, 2, 3\}$ are plotted versus the number of knots N. Uniform (for $p = 0$) and adaptive ($\theta = 0.75$ for $p \in \{0, 1, 2, 3\}$) refinement is considered.

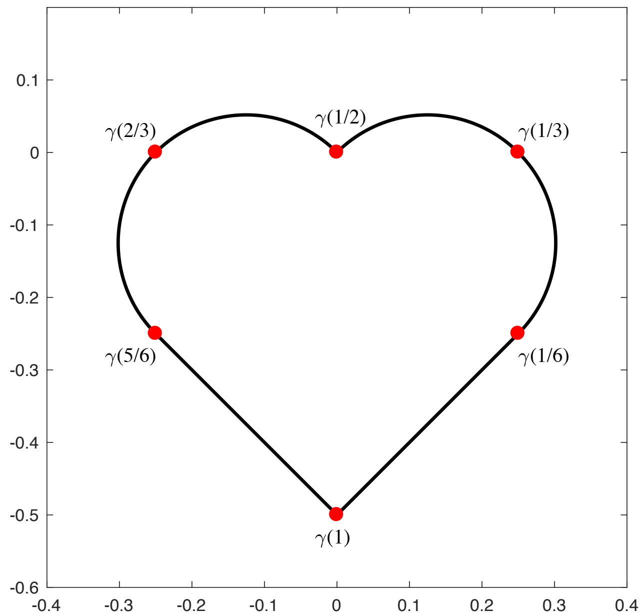


Figure 5.17: Geometry and initial nodes for the experiment of Section 5.9.3.

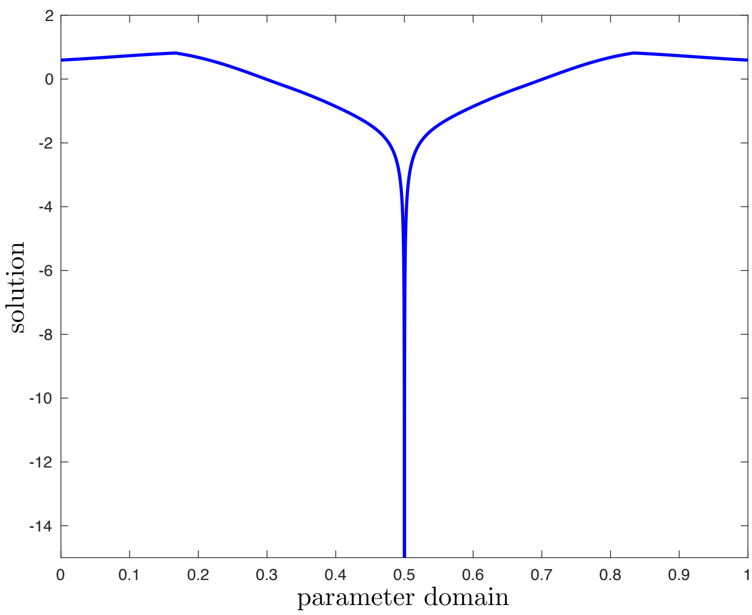


Figure 5.18: Experiment with singular solution on heart geometry of Section 5.9.3. The solution $\phi \circ \gamma$ is plotted on the parameter domain.

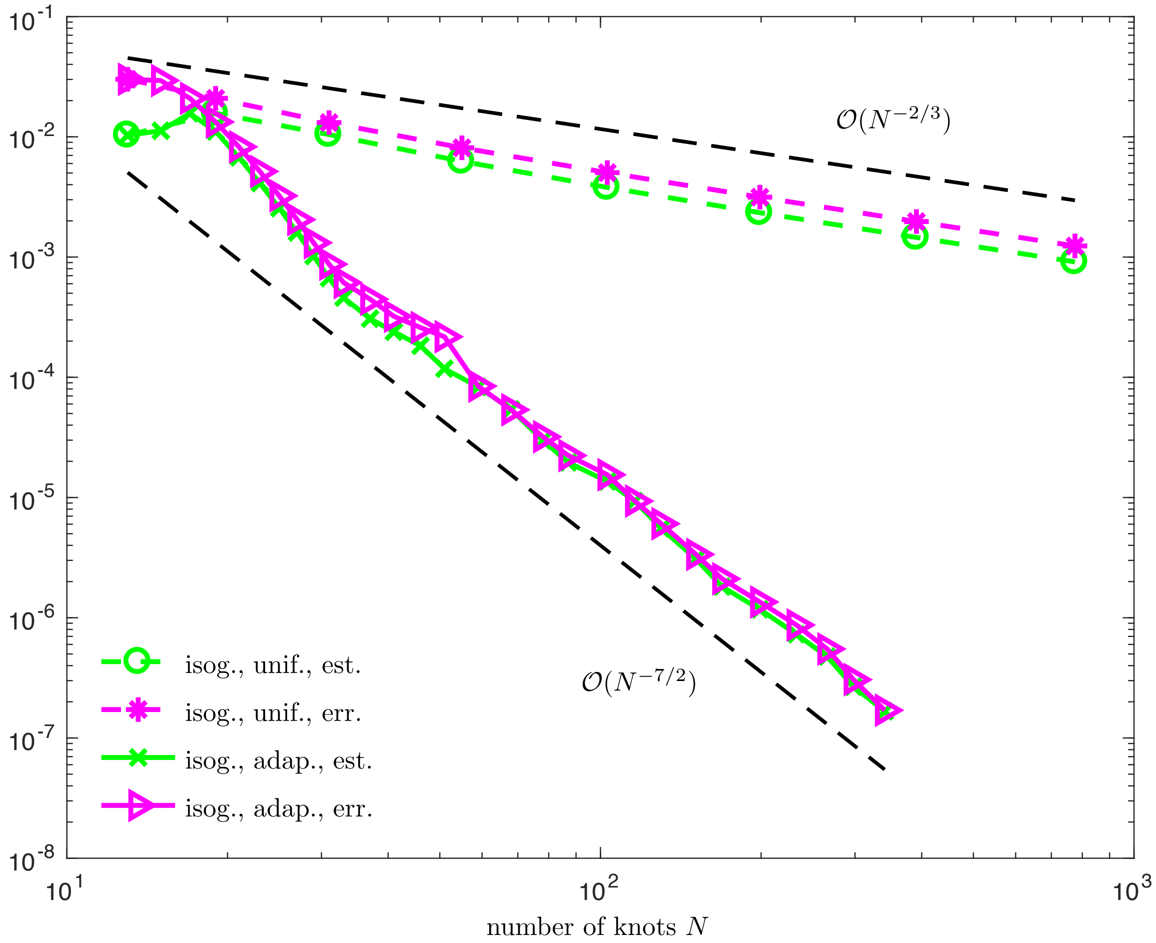


Figure 5.19: Experiment with singular solution on heart geometry of Section 5.9.3. Energy error $\|\phi - \Phi_\ell\|_{\mathfrak{Y}}$ and estimator η_ℓ of Algorithm 5.7.3 for rational splines of degree $p = 2$ are plotted versus the number of knots N. Uniform and adaptive ($\theta = 0.75$) refinement is considered.

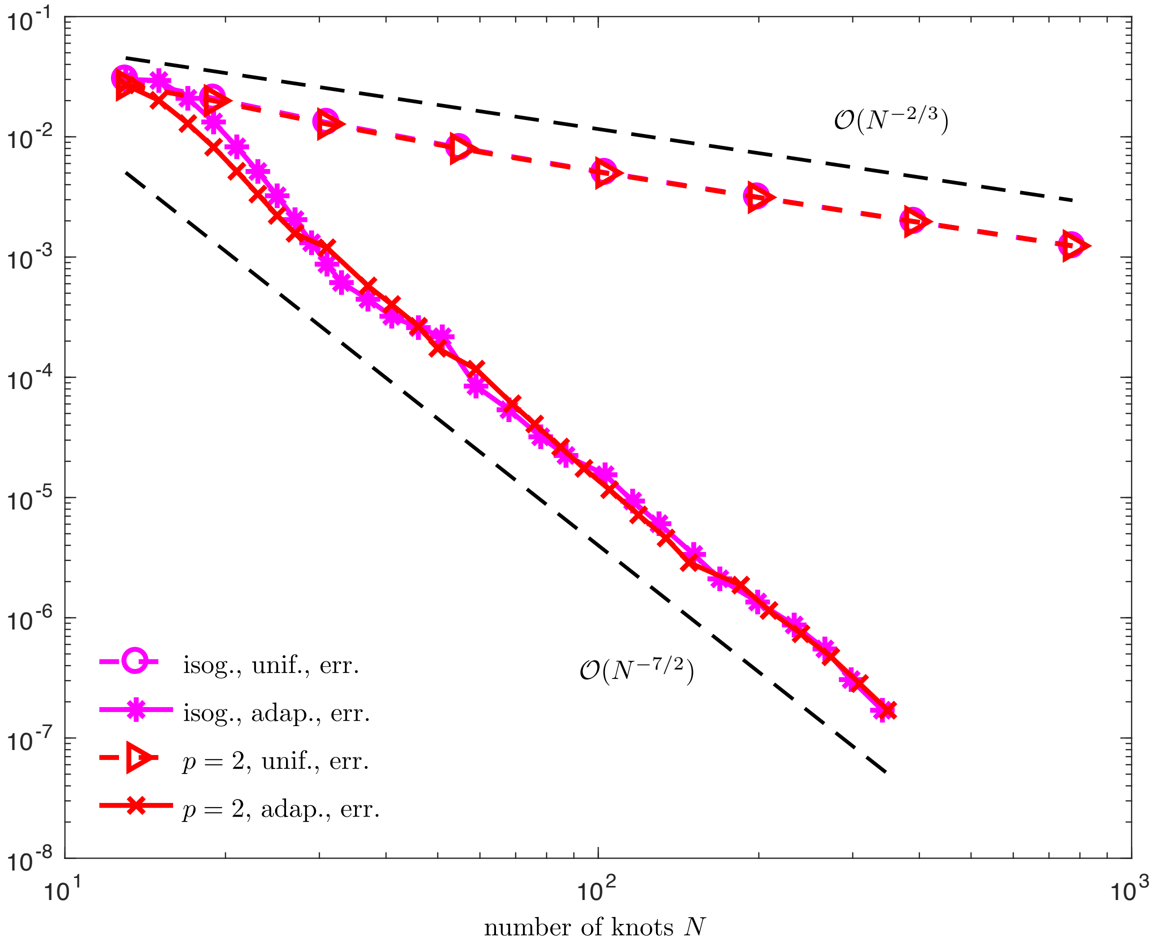


Figure 5.20: Experiment with singular solution on heart geometry of Section 5.9.3. Energy error $\|\phi - \Phi_\ell\|_{\mathfrak{V}}$ and estimator η_ℓ of Algorithm 5.7.3 for (rational) splines of degree $p = 2$ are plotted versus the number of knots N. Uniform and adaptive ($\theta = 0.75$) refinement is considered.

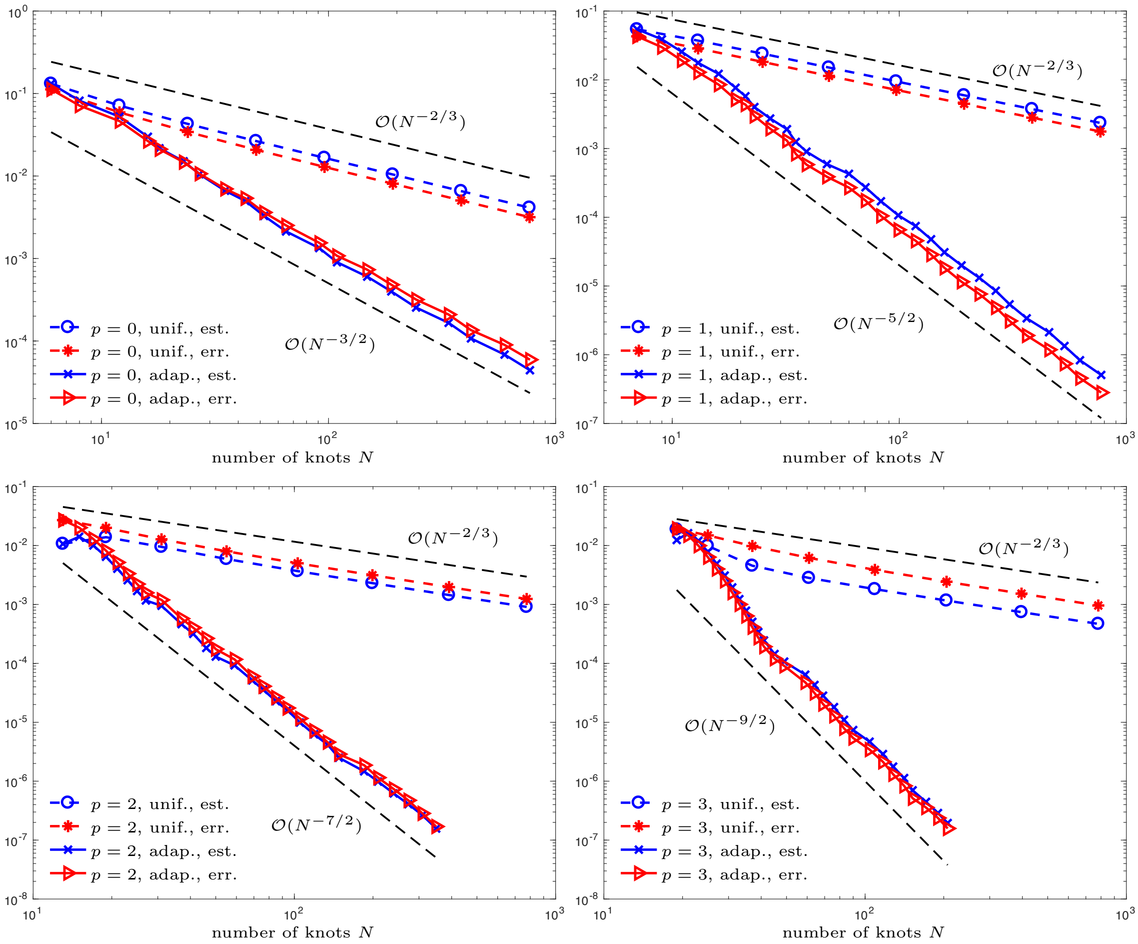


Figure 5.21: Experiment with singular solution on heart geometry of Section 5.9.3. Energy error $\|\phi - \Phi_\ell\|_{\mathfrak{V}}$ and estimator η_ℓ of Algorithm 5.7.3 for splines of degree $p \in \{0, 1, 2, 3\}$ are plotted versus the number of knots N. Uniform and adaptive ($\theta = 0.75$) refinement is considered.

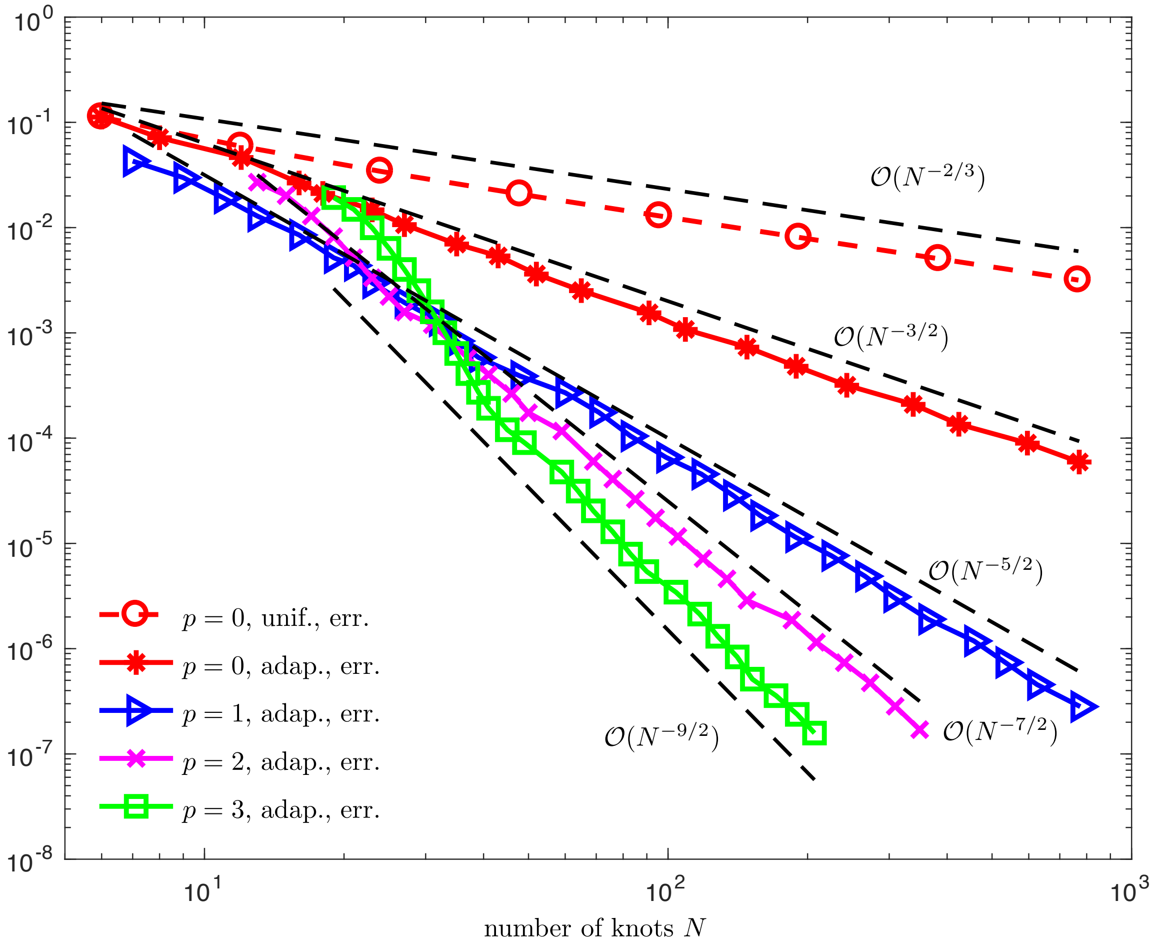


Figure 5.22: Experiment with singular solution on heart geometry of Section 5.9.3. The energy errors $\|\phi - \Phi_\ell\|_{\mathfrak{V}}$ of Algorithm 5.7.3 for splines of degree $p \in \{0, 1, 2, 3\}$ are plotted versus the number of knots N. Uniform (for $p = 0$) and adaptive ($\theta = 0.75$ for $p \in \{0, 1, 2, 3\}$) refinement is considered.

Bibliography

[ACD$^+$17] Alessandra Aimi, Francesco Calabrò, Mauro Diligenti, Maria L. Sampoli, Giancarlo Sangalli, and Alessandra Sestini. New efficient assembly in isogeometric analysis for symmetric galerkin boundary element method. *arXiv preprint*, 1703.10016, 2017.

[AFF$^+$13] Markus Aurada, Michael Feischl, Thomas Führer, Michael Karkulik, and Dirk Praetorius. Efficiency and optimality of some weighted-residual error estimator for adaptive 2D boundary element methods. *Comput. Methods Appl. Math.*, 13(3):305–332, 2013.

[AFF$^+$17] Markus Aurada, Michael Feischl, Thomas Führer, Michael Karkulik, J. Melenk, and Dirk Praetorius. Local inverse estimates for non-local boundary integral operators. *Math. Comp.*, 86(308):2651–2686, 2017.

[AHJ$^+$09] Martin Aigner, Christoph Heinrich, Bert Jüttler, Elisabeth Pilgerstorfer, Bernd Simeon, and Anh-Vu Vuong. Swept volume parameterization for isogeometric analysis. In *IMA Conference on the Mathematics of Surfaces*, pages 19–44, 2009.

[AO00] Mark Ainsworth and J. Tinsley Oden. *A posteriori error estimation in finite element analysis*. John Wiley & Sons, New York, 2000.

[Ape99] Thomas Apel. *Anisotropic finite elements: local estimates and applications.* Teubner, Stuttgart, 1999.

[BBdVC$^+$06] Yuri Bazilevs, Lourenco Beirão da Veiga, J. Austin Cottrell, Thomas J. R. Hughes, and Giancarlo Sangalli. Isogeometric analysis: approximation, stability and error estimates for h-refined meshes. *Math. Models Methods Appl. Sci.*, 16(07):1031–1090, 2006.

[BCR91] Gregory Beylkin, Ronald Coifman, and Vladimir Rokhlin. Fast wavelet transforms and numerical algorithms I. *Comm. Pure Appl. Math.*, 44(2):141–183, 1991.

[BDD04] Peter Binev, Wolfgang Dahmen, and Ron DeVore. Adaptive finite element methods with convergence rates. *Numer. Math.*, 97(2):219–268, 2004.

[BdVBC$^+$13] Lourenco Beirão da Veiga, Franco Brezzi, Andrea Cangiani, Gianmarco Manzini, L. Donatella Marini, and Alessandro Russo. Basic principles of virtual element methods. *Math. Models Methods Appl. Sci.*, 23(01):199–214, 2013.

[BdVBSV13] Lourenco Beirão da Veiga, Annalisa Buffa, Giancarlo Sangalli, and Rafael Vázquez. Analysis-suitable T-splines of arbitrary degree: definition, linear independence and approximation properties. *Math. Models Methods Appl. Sci.*, 23(11):1979–2003, 2013.

[BdVBSV14] Lourenco Beirão da Veiga, Annalisa Buffa, Giancarlo Sangalli, and Rafael Vázquez. Mathematical analysis of variational isogeometric methods. *Acta numerica*, 23:157–287, 2014.

[BG16a] Annalisa Buffa and Eduardo M. Garau. A posteriori error estimators for hierarchical B-spline discretizations. *arXiv preprint*, 1611.07816, 2016.

[BG16b] Annalisa Buffa and Eduardo M. Garau. Refinable spaces and local approximation estimates for hierarchical splines. *IMA J. Numer. Anal.*, 37(3):1125–1149, 2016.

[BG16c] Annalisa Buffa and Carlotta Giannelli. Adaptive isogeometric methods with hierarchical splines: error estimator and convergence. *Math. Models Methods Appl. Sci.*, 26(01):1–25, 2016.

[BG17] Annalisa Buffa and Carlotta Giannelli. Adaptive isogeometric methods with hierarchical splines: optimality and convergence rates. Technical report, École Polytechnique Fédérale de Lausanne, March 2017.

[BGMP16] Annalisa Buffa, Carlotta Giannelli, Philipp Morgenstern, and Daniel Peterseim. Complexity of hierarchical refinement for a class of admissible mesh configurations. *Comput. Aided Geom. Design*, 47:83–92, 2016.

[BHP17] Alex Bespalov, Alexander Haberl, and Dirk Praetorius. Adaptive FEM with coarse initial mesh guarantees optimal convergence rates for compactly perturbed elliptic problems. *Comput. Methods Appl. Mech. Engrg.*, 317:318–340, 2017.

[Car97] Carsten Carstensen. An a posteriori error estimate for a first-kind integral equation. *Math. Comp.*, 66(217):139–155, 1997.

[CF01] Carsten Carstensen and Birgit Faermann. Mathematical foundation of a posteriori error estimates and adaptive mesh-refining algorithms for boundary integral equations of the first kind. *Eng. Anal. Bound. Elem.*, 25(7):497–509, 2001.

[CFPP14] Carsten Carstensen, Michael Feischl, Marcus Page, and Dirk Praetorius. Axioms of adaptivity. *Comput. Math. Appl.*, 67(6):1195–1253, 2014.

[CHB09] J. Austin Cottrell, Thomas J. R. Hughes, and Yuri Bazilevs. *Isogeometric analysis: toward integration of CAD and FEA*. John Wiley & Sons, New York, 2009.

[CKNS08] J. Manuel Cascon, Christian Kreuzer, Ricardo H. Nochetto, and Kunibert G. Siebert. Quasi-optimal convergence rate for an adaptive finite element method. *SIAM J. Numer. Anal.*, 46(5):2524–2550, 2008.

[CMPS04] Carsten Carstensen, Matthias Maischak, Dirk Praetorius, and Ernst P. Stephan. Residual-based a posteriori error estimate for hypersingular equation on surfaces. *Numer. Math.*, 97(3):397–425, 2004.

[CMS01] Carsten Carstensen, Matthias Maischak, and Ernst P. Stephan. A posteriori error estimate and h-adaptive algorithm on surfaces for Symm's integral equation. *Numer. Math.*, 90(2):197–213, 2001.

[CP06] Carsten Carstensen and Dirk Praetorius. Averaging techniques for the effective numerical solution of symm's integral equation of the first kind. *SIAM J. Sci. Comput.*, 27(4):1226–1260, 2006.

[CS96] Carsten Carstensen and Ernst P. Stephan. Adaptive boundary element mevthods for some first kind integral equations. *SIAM J. Numer. Anal.*, 33(6):2166–2183, 1996.

[dB86] Carl de Boor. B (asic)-spline basics. Technical report, Mathematics Research Center, University of Wisconsin-Madison, 1986.

[dB01] Carl de Boor. *A practical guide to splines.* Springer, New York, 2001.

[DFG+04] Wolfgang Dahmen, Birgit Faermann, Ivan G. Graham, Wolfgang Hackbusch, and Stefan A. Sauter. Inverse inequalities on non-quasi-uniform meshes and application to the mortar element method. *Math. Comp.*, 73(247):1107–1138, 2004.

[DHK+17] Jürgen Dölz, Helmut Harbrecht, Stefan Kurz, Sebastian Schöps, and Felix Wolf. A fast isogeometric BEM for the three dimensional Laplace- and Helmholtz problems. *arXiv preprint*, 1708.09162, 2017.

[DHP16] Jürgen Dölz, Helmut Harbrecht, and Michael Peters. An interpolation-based fast multipole method for higher-order boundary elements on parametric surfaces. *Internat. J. Numer. Methods Engrg.*, 108(13):1705–1728, 2016.

[DHS06] Wolfgang Dahmen, Helmut Harbrecht, and Reinhold Schneider. Compression techniques for boundary integral equations—asymptotically optimal complexity estimates. *SIAM J. Numer. Anal.*, 43(6):2251–2271, 2006.

[DLP13] Tor Dokken, Tom Lyche, and Kjell F. Pettersen. Polynomial splines over locally refined box-partitions. *Comput. Aided Geom. Design*, 30(3):331–356, 2013.

[DNPV12] Eleonora Di Nezza, Giampiero Palatucci, and Enrico Valdinoci. Hitchhiker' s guide to the fractional Sobolev spaces. *Bull. Sci. Math.*, 136(5):521–573, 2012.

[Dör96] Willy Dörfler. A convergent adaptive algorithm for Poisson's equation. *SIAM J. Numer. Anal.*, 33(3):1106–1124, 1996.

[EG92] Lawrence C. Evans and Ronald F. Gariepy. *Measure theory and fine properties of functions*. CRC Press, Boca Raton, 1992.

[Fae00] Birgit Faermann. Localization of the Aronszajn-Slobodeckij norm and application to adaptive boundary elements methods. Part I. The two-dimensional case. *IMA J. Numer. Anal.*, 20(2):203–234, 2000.

[Fae02] Birgit Faermann. Localization of the Aronszajn-Slobodeckij norm and application to adaptive boundary element methods Part II. the three-dimensional case. *Numer. Math.*, 92(3):467–499, 2002.

[Fei15] Michael Feischl. *Rate optimality of adaptive algorithms*. PhD thesis, Institute for Analysis and Scientific Computing, TU Wien, 2015.

[FFH+15] Michael Feischl, Thomas Führer, Norbert Heuer, Michael Karkulik, and Dirk Praetorius. Adaptive boundary element methods: A posteriori error estimators, adaptivity, convergence, and implementation. *Arch. Comput. Methods Engrg.*, 22:309–389, 2015.

[FFK+14] Michael Feischl, Thomas Führer, Michael Karkulik, J. Markus Melenk, and Dirk Praetorius. Quasi-optimal convergence rates for adaptive boundary element methods with data approximation, Part I: Weakly-singular integral equation. *Calcolo*, 51(4):531–562, 2014.

[FFK+15] Michael Feischl, Thomas Führer, Michael Karkulik, J. Markus Melenk, and Dirk Praetorius. Quasi-optimal convergence rates for adaptive boundary element methods with data approximation, Part II: Hyper-singular integral equation. *Electron. Trans. Numer. Anal.*, 44:153–176, 2015.

[FFKP14] Michael Feischl, Thomas Führer, Michael Karkulik, and Dirk Praetorius. ZZ-type a posteriori error estimators for adaptive boundary element methods on a curve. *Eng. Anal. Bound. Elem.*, 38:49–60, 2014.

[FFME+14] Michael Feischl, Thomas Führer, Gregor Mitscha-Eibl, Dirk Praetorius, and Ernst P. Stephan. Convergence of adaptive bem and adaptive fem-bem coupling for estimators without h-weighting factor. *Comput. Meth. in Appl. Math.*, 14(4):485–508, 2014.

[FFP14] Michael Feischl, Thomas Führer, and Dirk Praetorius. Adaptive FEM with optimal convergence rates for a certain class of nonsymmetric and possibly nonlinear problems. *SIAM J. Numer. Anal.*, 52(2):601–625, 2014.

[FGH+16] Michael Feischl, Gregor Gantner, Alexander Haberl, Dirk Praetorius, and Thomas Führer. Adaptive boundary element methods for optimal convergence of point errors. *Numer. Math.*, 132(3):541–567, 2016.

[FGHP16] Michael Feischl, Gregor Gantner, Alexander Haberl, and Dirk Praetorius. Adaptive 2D IGA boundary element methods. *Eng. Anal. Bound. Elem.*, 62:141–153, 2016.

[FGHP17] Michael Feischl, Gregor Gantner, Alexander Haberl, and Dirk Praetorius. Optimal convergence for adaptive iga boundary element methods for weakly-singular integral equations. *Numer. Math.*, 136(1):147–182, 2017.

[FGP15] Michael Feischl, Gregor Gantner, and Dirk Praetorius. Reliable and efficient a posteriori error estimation for adaptive IGA boundary element methods for weakly-singular integral equations. *Comput. Methods Appl. Mech. Engrg.*, 290:362–386, 2015.

[FKMP13] Michael Feischl, Michael Karkulik, J. Markus Melenk, and Dirk Praetorius. Quasi-optimal convergence rate for an adaptive boundary element method. *SIAM J. Numer. Anal.*, 51(2):1327–1348, 2013.

[FL07] Samuel Ferraz-Leite. A Posteriori Fehlerschätzer für die Symmsche Integralgleichung in 3D. Master's thesis, Institute for Analysis and Scientific Computing, TU Wien, 2007.

[FLP08] Samuel Ferraz-Leite and Dirk Praetorius. Simple a posteriori error estimators for the h-version of the boundary element method. *Computing*, 83(4):135–162, 2008.

[Gan13] Tsogtgerel Gantumur. Adaptive boundary element methods with convergence rates. *Numer. Math.*, 124(3):471–516, 2013.

[Gan14] Gregor Gantner. Adaptive isogeometric BEM. Master's thesis, Institute for Analysis and Scientific Computing, TU Wien, 2014.

[GD17] Y. P. Gong and C. Y. Dong. An isogeometric boundary element method using adaptive integral method for 3D potential problems. *J. Comput. Appl. Math.*, 319:141–158, 2017.

[Geo08] Emmanuil Georgoulis. Inverse-type estimates on hp-finite element spaces and applications. *Math. Comp.*, 77(261):201–219, 2008.

[GHP17] Gregor Gantner, Daniel Haberlik, and Dirk Praetorius. Adaptive IGAFEM with optimal convergence rates: Hierarchical B-splines. *Math. Models Methods Appl. Sci.*, https://doi.org/10.1142/S0218202517500543, 2017.

[GHPS17] Gregor Gantner, Alexander Haberl, Dirk Praetorius, and Bernhard Stiftner. Rate optimal adaptive FEM with inexact solver for nonlinear operators. *IMA J. Numer. Anal.*, https://doi.org/10.1093/imanum/drx050, 2017.

[GHS05] Ivan G. Graham, Wolfgang Hackbusch, and Stefan A. Sauter. Finite elements on degenerate meshes: inverse-type inequalities and applications. *IMA J. Numer. Anal.*, 25(2):379–407, 2005.

[GJS12] Carlotta Giannelli, Bert Jüttler, and Hendrik Speleers. THB-splines: The truncated basis for hierarchical splines. *Comput. Aided Geom. Design*, 29(7):485–498, 2012.

[GJS14] Carlotta Giannelli, Bert Jüttler, and Hendrik Speleers. Strongly stable bases for adaptively refined multilevel spline spaces. *Adv. Comput. Math.*, 40(2), 2014.

[GR87] Leslie Greengard and Vladimir Rokhlin. A fast algorithm for particle simulations. *J. Comput. Phys.*, 73(2):325–348, 1987.

[Hab] Daniel Haberlik. Adaptive isogeometrische Finite Elemente Methode mit hierarchischen Splines. Bachelor's thesis, Institute for Analysis and Scientific Computing, TU Wien, 2016.

[Hac95] Wolfgang Hackbusch. *Integral equations: Theory and numerical treatment.* Birkhäuser, Basel, 1995.

[Hac99] Wolfgang Hackbusch. A sparse matrix arithmetic based on $\mathcal{H}$-matrices. I: Introduction to $\mathcal{H}$-matrices. *Computing*, 62(2):89–108, 1999.

[HAD14] Luca Heltai, Marino Arroyo, and Antonio DeSimone. Nonsingular isogeometric boundary element method for Stokes flows in 3D. *Comput. Methods Appl. Mech. Engrg.*, 268:514–539, 2014.

[HCB05] Thomas J. R. Hughes, J. Austin Cottrell, and Yuri Bazilevs. Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement. *Comput. Methods Appl. Mech. Engrg.*, 194(39):4135–4195, 2005.

[HKMP17] Paul Hennig, Markus Kästner, Philipp Morgenstern, and Daniel Peterseim. Adaptive mesh refinement strategies in isogeometric analysis—A computational comparison. *Comput. Methods Appl. Mech. Engrg.*, 316:424–448, 2017.

[HMT09] Steve Hofmann, Marius Mitrea, and Michael Taylor. Singular integrals and elliptic boundary problems on regular Semmes–Kenig–Toro domains. *Int. Math. Res. Not.*, 2010(14):2567–2865, 2009.

[HR10] Helmut Harbrecht and Maharavo Randrianarivony. From computer aided design to wavelet BEM. *Comput. Vis. Sci.*, 13(2):69–82, 2010.

[JKD14] Kjetil A. Johannessen, Trond Kvamsdal, and Tor Dokken. Isogeometric analysis using LR B-splines. *Comput. Methods Appl. Mech. Engrg.*, 269:471–514, 2014.

[JRK15] Kjetil A. Johannessen, Filippo Remonato, and Trond Kvamsdal. On the similarities and differences between classical hierarchical, truncated hierarchical and LR B-splines. *Comput. Methods Appl. Mech. Engrg.*, 291:64–101, 2015.

[Kar12] Michael Karkulik. *Zur Konvergenz und Quasioptimalität adaptiver Randelementmethoden*. PhD thesis, Institute for Analysis and Scientific Computing, TU Wien, 2012.

[KHZvE17] Sören Keuchel, Nils Christian Hagelstein, Olgierd Zaleski, and Otto von Estorff. Evaluation of hypersingular and nearly singular integrals in the isogeometric boundary element method for acoustics. *Comput. Methods Appl. Mech. Engrg.*, 325:488–504, 2017.

[Kra98] Rainer Kraft. *Adaptive und linear unabhängige Multilevel B-Splines und ihre Anwendungen*. PhD thesis, Universität Stuttgart, 1998.

[KVVdZvB14] Gokturk Kuru, Clemens V. Verhoosel, Kristoffer G. Van der Zee, and E. Harald van Brummelen. Goal-adaptive isogeometric analysis with hierarchical splines. *Comput. Methods Appl. Mech. Engrg.*, 270:270–292, 2014.

[MCK08] Tobias Martin, Elaine Cohen, and Mike Kirby. Volumetric parameterization and trivariate B-spline fitting using harmonic functions. In *Proceedings of the 2008 ACM symposium on Solid and physical modeling*, pages 269–280, 2008.

[McL00] William McLean. *Strongly elliptic systems and boundary integral equations*. Cambridge University Press, Cambridge, 2000.

[ME14] Gregor Mitscha-Eibl. Adaptive BEM und FEM-BEM-Kopplung für die Lamé-Gleichung. Master's thesis, Institute for Analysis and Scientific Computing, TU Wien, 2014.

[MNS00] Pedro Morin, Ricardo H. Nochetto, and Kunibert G. Siebert. Data oscillation and convergence of adaptive FEM. *SIAM J. Numer. Anal.*, 38(2):466–488, 2000.

[Mor08] Charles B. Jr. Morrey. *Multiple integrals in the calculus of variations*. Springer, Berlin, reprint of the 1966 original edition, 2008.

[Mor16] Philipp Morgenstern. Globally structured three-dimensional analysis-suitable T-splines: Definition, linear independence and m-graded local refinement. *SIAM J. Numer. Anal.*, 54(4):2163–2186, 2016.

[MP15] Philipp Morgenstern and Daniel Peterseim. Analysis-suitable adaptive T-mesh refinement with linear complexity. *Comput. Aided Geom. Design*, 34:50–66, 2015.

[MZBF15] Benjamin Marussig, Jürgen Zechner, Gernot Beer, and Thomas-Peter Fries. Fast isogeometric boundary element method based on independent field approximation. *Comput. Methods Appl. Mech. Engrg.*, 284:458–488, 2015.

[NV11] Ricardo H. Nochetto and Andreas Veeser. Primer of adaptive finite element methods. In *Multiscale and adaptivity: Modeling, numerics and applications*, pages 125–225. Springer, 2011.

[NZW+17] B. H. Nguyen, Xiaoying Zhuang, Peter Wriggers, Timon Rabczuk, M. E. Mear, and Han D. Tran. Isogeometric symmetric Galerkin boundary element method for three-dimensional elasticity problems. *Comput. Methods Appl. Mech. Engrg.*, 323:132–150, 2017.

[PGK+09] Costas Politis, Alexandros I. Ginnis, Panagiotis D. Kaklis, Kostas Belibassakis, and Christian Feurer. An isogeometric BEM for exterior potential-flow problems in the plane. In *2009 SIAM/ACM Joint Conference on Geometric and Physical Modeling*, pages 349–354, 2009.

[PT97] Les Piegl and Wayne Tiller. *The NURBS book*. Springer, New York, second edition, 1997.

[PTC13] Michael J. Peake, Jon Trevelyan, and Graham Coates. Extended isogeometric boundary element method (XIBEM) for two-dimensional Helmholtz problems. *Comput. Methods Appl. Mech. Engrg.*, 259:93–102, 2013.

[Rud91] Walter Rudin. *Functional analysis*. McGraw-Hill, New York, second edition, 1991.

[SBLT13] Robert N. Simpson, Stéphane P. A. Bordas, Haojie Lian, and Jon Trevelyan. An isogeometric boundary element method for elastostatic analysis: 2D implementation aspects. *Comput. & Structures*, 118:2–12, 2013.

[SBTR12] Robert N. Simpson, Stéphane P. A. Bordas, Jon Trevelyan, and Timon Rabczuk. A two-dimensional isogeometric boundary element method for elastostatic analysis. *Comput. Methods Appl. Mech. Engrg.*, 209:87–100, 2012.

[Sch07] Larry Schumaker. *Spline functions: Basic theory*. Cambridge University Press, Cambridge, 2007.

[Sch16] Stefan Schimanko. Adaptive isogeometric boundary element method for the hyper-singular integral equation. Master's thesis, Institute for Analysis and Scientific Computing, TU Wien, 2016.

[SLSH12] Michael A. Scott, Xin Li, Thomas W. Sederberg, and Thomas J. R. Hughes. Local refinement of analysis-suitable T-splines. *Comput. Methods Appl. Mech. Engrg.*, 213:206–222, 2012.

[SM16] Hendrik Speleers and Carla Manni. Effortless quasi-interpolation in hierarchical spaces. *Numer. Math.*, 132(1):155–184, 2016.

[SS11] Stefan A. Sauter and Christoph Schwab. *Boundary element methods*. Springer, Berlin, 2011.

[SSE+13] Michael A. Scott, Robert N. Simpson, John A. Evans, Scott Lipton, Stéphane P. A. Bordas, Thomas J. R. Hughes, and Thomas W. Sederberg. Isogeometric boundary element analysis using unstructured T-splines. *Comput. Methods Appl. Mech. Engrg.*, 254:197–221, 2013.

[Ste07] Rob Stevenson. Optimality of a standard adaptive finite element method. *Found. Comput. Math.*, 7(2):245–269, 2007.

[Ste08a] Olaf Steinbach. *Numerical approximation methods for elliptic boundary value problems*. Springer, New York, 2008.

[Ste08b] Rob Stevenson. The completion of locally refined simplicial partitions created by bisection. *Math. Comp.*, 77(261):227–241, 2008.

[TM12] Toru Takahashi and Toshiro Matsumoto. An application of fast multipole method to isogeometric boundary element method for Laplace equation in two dimensions. *Eng. Anal. Bound. Elem.*, 36(12):1766–1775, 2012.

[Ver13] Rüdiger Verfürth. *A posteriori error estimation techniques for finite element methods*. Oxford University Press, Oxford, 2013.

[VGJS11] Anh-Vu Vuong, Carlotta Giannelli, Bert Jüttler, and Bernd Simeon. A hierarchical approach to adaptive local refinement in isogeometric analysis. *Comput. Methods Appl. Mech. Engrg.*, 200(49):3554–3567, 2011.

[XKW17] Gang Xu, Tsz-Ho Kwok, and Charlie C. L. Wang. Isogeometric computation reuse method for complex objects with topology-consistent volumetric parameterization. *Comput.-Aided Des.*, 91:1–13, 2017.

[XMDG13] Gang Xu, Bernard Mourrain, Régis Duvigneau, and André Galligo. Analysis-suitable volume parameterization of multi-block computational domain in isogeometric applications. *Comput.-Aided Des.*, 45(2):395–404, 2013.

Curriculum Vitae

Personal Data

Name	Gregor Gantner
Academic degree	Master of Science
Date of birth	May 25th, 1990
Place of birth	Hollabrunn, Austria
Email	gregor.gantner@asc.tuwien.ac.at

Education

since 08/2014	PhD student, supervised by Dirk Praetorius, Member of Doctoral program Dissipation and Dispersion in Nonlinear PDEs
10/2012 – 06/2014	Master studies in technical mathematics at TU Wien
10/2008 – 10/2012	Bachelor studies in mathematics in science and technology at TU Wien with interruption 03/2009 – 09/2009 due to alternative service (Zivildienst)
2000 – 2008	BG und BRG Hollabrunn

Publications in Scientific Journals

Gregor Gantner, Daniel Haberlik, Dirk Praetorius: Adaptive IGAFEM with optimal convergence rates: Hierarchical B-splines. *Mathematical Models and Methods in Applied Sciences*, accepted for publication, 2017.

Gregor Gantner, Alexander Haberl, Dirk Praetorius, Bernhard Stiftner: Rate optimal adaptive FEM with inexact solver for nonlinear operators, *IMA Journal of Numerical Analysis*, accepted for publication, 2017.

Michael Feischl, Gregor Gantner, Alexander Haberl, Dirk Praetorius: Optimal convergence for adaptive IGA boundary element methods for weakly-singular integral equations. *Numerische Mathematik*, 2017.

Michael Feischl, Gregor Gantner, Alexander Haberl, Dirk Praetorius: Adaptive 2D IGA boundary element methods. *Engineering Analysis with Boundary Elements*, 2016.

Michael Feischl, Thomas Führer, Gregor Gantner, Alexander Haberl, Dirk Praetorius: Adaptive boundary element methods for optimal convergence of point errors. *Numerische Mathematik*, 2016.

Michael Feischl, Gregor Gantner, Dirk Praetorius: Reliable and efficient a posteriori error estimation for adaptive IGA boundary element methods for weakly-singular integral equations. *Computer Methods in Applied Mechanics and Engineering*, 2015.

Proceedings

Gregor Gantner, Alexander Haberl, Dirk Praetorius, Bernhard Stiftner: Rate optimal adaptive FEM with inexact solver for strongly monotone operators. *Oberwolfach Workshop on Adaptive Algorithms*, Oberwolfach, 2016.

Michael Feischl, Gregor Gantner, Dirk Praetorius: A posteriori error estimation for adaptive IGA boundary element methods, *11th World Congress on Computational Mechanics (WCCM XI)*, Barcelona, 2014.

Florian Judex, Markus Brychta, Gregor Gantner, Reiner Braun: Method to assess the load shifting potential by using buildings as a thermal storage. *2nd Central European Symposium on Building Physics*, Vienna, 2013.

Supervised Theses

Stefan Schimanko (Supervisor: Gregor Gantner, Dirk Praetorius): Adaptive isogeometric boundary element method for the hyper-singular integral equation, Master's thesis, Institute for Analysis and Scientific Computing, TU Wien, 2016.

Daniel Haberlik (Supervisor: Gregor Gantner, Dirk Praetorius): Adaptive isogeometrische Finite Elemente Methode mit hierarchischen Splines. Bachelor's thesis, Institute for Analysis and Scientific Computing, TU Wien, 2016.

Juliana Kainz (Supervisor: Thomas Führer, Gregor Gantner, Dirk Praetorius): Stabile Implementierung von HILBERT S2P1, Bachelor's thesis, Institute for Analysis and Scientific Computing, TU Wien, 2015.

Academic Theses

Gregor Gantner (Supervisor: Michael Feischl, Dirk Praetorius): Adaptive isogeometric BEM, Master's thesis, Institute for Analysis and Scientific Computing, TU Wien, 2014.

Gregor Gantner (Supervisor: Michael Kaltenbäck): Positiv definite Funktionen in harmonischer Analysis, Bachelor's thesis, Institute for Analysis and Scientific Computing, TU Wien, 2012.