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DISSERTATION

Clones and homogeneous structures

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Kurzfassung

Eine Struktur $\mathbb{A}$ heißt *homogen*, falls sich jeder Isomorphismus zwischen endlich erzeugten Unterstrukturen zu einem Automorphismus von $\mathbb{A}$ fortsetzen lässt. Eine Konstruktionsmethode solcher Strukturen wurde erstmals von Fraïssé in [Fra54] beschrieben; seitdem wurden homogene Strukturen zu einem Objekt von Interesse in zahlreichen Gebieten der Mathematik, insbesondere in der Modelltheorie, Ramsey Theorie, der Theorie unendlicher Permutationsgruppen und der theoretischen Informatik. In Kapitel 1 geben wir eine Einführung zu homogenen Strukturen und legen dabei den Schwerpunkt auf Resultate, die für den Rest der Arbeit von Bedeutung sind.

Ein wichtiges offenes Problem auf dem Gebiet ist die Vermutung von Thomas, die besagt, dass jede abzählbare homogene Struktur $\mathbb{A}$ in endlicher relationaler Sprache bis auf Interdefinierbarkeit nur endliche viele *Redukte* hat, sprich, dass in Logik erster Ordnung nur endlich viele Strukturen über $\mathbb{A}$ definiert werden können. In Kapitel 2 bestimmen wir alle Redukte jener homogenen gerichteten Graphen, die von Henson in [Hen72] beschrieben wurden. Wir zeigen, dass all diese kontinuum viele, homogene Digraphen nur endlich viele Redukte haben, was im Einklang mit Thomas' Vermutung steht. Als Korollar unserer Klassifizierung können wir zeigen, dass die symmetrische Gruppe auf einer abzählbaren Menge kontinuum viele nicht-isomorphe maximale abgeschlossene Untergruppen hat. Dies beantwortet einen offene Frage von Macpherson [BM16].

In Kapitel 3 bestimmen wir die Komplexität einer Familie von Problem aus der theoretischen Informatik, bei denen der Input aus quantorenfreien Formeln in der Sprache der Ordnungen besteht, und die Frage ist, ob es eine partielle Ordnung gibt, in der diese Formeln erfüllbar sind. Diese Probleme können als *constraint satisfaction problems* von Redukten der *random partial order* $\mathbb{P}$ modelliert werden, also jener homogenen Struktur, die als Fraïssé-Grenzwert der Klasse aller endlichen partiellen Ordnungen gebildet werden kann. Die Redukte von $\mathbb{P}$ wurden bereits in [PPP$^+$14] bis auf first-order Interdefinierbarkeit bestimmt. Wir verfeinern die dortigen Ergebnisse, und untersuchen Eigenschaften der primitiv positiven Theorie der Redukte. Dadurch können wir zeigen, dass alle obigen Erfüllbarkeitsprobleme entweder in P oder NP-vollständig sind. Dabei verwenden wir Methoden aus der universellen Algebra: Genauso wie die Automorphismengruppe $\mathrm{Aut}(\mathbb{A})$ einer ω-kategorischen Struktur $\mathbb{A}$, diese bis auf first-order Interdefinierbarkeit bestimmt, legt der sogenannten *Polymorphismenklon* $\mathrm{Pol}(\mathbb{A})$, also die Algebra aller Homomorphismen $\mathbb{A}^n \to \mathbb{A}$ die Struktur $\mathbb{A}$ bist auf primitiv-positive Definierbarkeit fest. Wir untersuchen folglich den Verband aller Polymorphismenklone, die $\mathrm{Aut}(\mathbb{P})$ enthalten.

Sowohl Kapitel 3 als auch Kapitel 2 bauen stark auf die Ramsey-theoretischen Methoden, die von Bodirsky und Pinsker in [BP15a] entwickelt wurden und sich

bereits vielfach als wichtiges Werkzeug im Klassifizieren von Redukten erwiesen haben.

In Kapitel 4 beschäftigen wir uns mit einer Fragestellung zur "Rekonstruierbarkeit" ω-kategorische Strukturen. Zwei ω-kategorische Strukturen sind first-order interdefinierbar (bzw. bi-interpretierbar), genau dann wenn ihre Automorphismengruppen übereinstimmen (bzw. topologisch isomorph sind). Dies motiviert die Fragestellung, inwieweit eine ω-kategorische Struktur bereits aus der Automorphismengruppe als *abstrakter* Gruppe rekonstruiert werden kann. Obwohl es zahlreichen positiven Resultate auf dem Gebiet gibt, wurde von Evans und Hewitt in [EH90] gezeigt, dass es zwei ω-kategorische Strukturen gibt, die zwar isomorphe, aber nicht topologisch isomorphe Automorphismengruppen haben. Wir zeigen, dass dieses Gegenbeispiel derart modifiziert werden kann, dass auch die Endomorphismen-Monoide und Polymorphismen-Klone der beiden Strukturen jeweils isomorph, aber nicht topologisch isomorph sind.

Die Kapitel 2, 3 und 4 entsprechen jeweils den Publikationen [AK15] (mit Lovkush Agarwal), [KP17] (mit Trung Van Pham) und [BEKP] (mit Manuel Bodirsky, David Evans und Michael Pinsker).

Abstract

A structure $\mathbb{A}$ is called *homogeneous*, if every isomorphism between its finitely generated substructures extends to an automorphism of $\mathbb{A}$. In this thesis we are studying homogeneous structures with countable domains. A method to construct such structures was introduced by Fraïssé in [Fra54]; since then homogeneous structures became subject of interest in various areas of mathematics, in particular model theory, infinite permutation groups, Ramsey theory and theoretical computer science. In Chapter 1 we give an introduction to homogeneous structures, focussing on theoretical background needed for the rest of the thesis.

An important open problem regarding homogeneous structures is Thomas' conjecture, which claims that every countable homogeneous structure in a finite relational language has finitely many reducts up to first-order interdefinability. In Chapter 2 of this thesis we classify the reducts of the homogeneous digraphs that were described by Henson in [Hen72]. For all of these continuum many *Henson digraphs* there are only finitely many first-order reducts, which is in accordance with Thomas' conjecture. Our proof uses the well-know fact that reducts of an ω-categorical structure $\mathbb{A}$ correspond to the closed supergroups of the automorphism group $\mathrm{Aut}(\mathbb{A})$. As a corollary of our result we can show that $\mathrm{Sym}(\omega)$, the symmetric group on a countable set, has continuum many non-isomorphic maximal closed subgroups, which answers an open question by Macpherson [BM16].

In Chapter 3 we discuss the complexity of a class of problems from theoretical computer science, where the input consists of quantifier-free formulas in the language of orders and the question is, whether there is a partial order that satisfies the formulas. These problems can be modeled as *constraint satisfaction problems* of reducts of the *random partial order* $\mathbb{P}$, which is a well-known homogeneous structure. The reducts of $\mathbb{P}$ were already classified in [PPP$^+$14] up to first-order definability. We refine this result and discuss properties of the primitive positive theory of those structures. This enables us to prove a dichotomy result: All the decision problems described above are either in P or NP-complete. As in the previous chapter we use algebraic methods to prove our result. The so called *polymorphism clone* $\mathrm{Pol}(\mathbb{A})$ of a structure $\mathbb{A}$ is the algebra consisting of all homomorphisms $\mathbb{A}^n \to \mathbb{A}$. For ω-categorical structures, the relations on A that are invariant under the action of $\mathrm{Pol}(\mathbb{A})$ are exactly the primitive positive definable relations in $\mathbb{A}$. Hence, studying reducts of $\mathbb{P}$ up to primitive positive definability is equivalent to study the lattice of closed clones containing $\mathrm{Aut}(\mathbb{P})$.

Both Chapter 2 and Chapter 3 rely on the Ramsey theoretical methods that were developed by Bodirsky and Pinsker in [BP15a] and proved to be a useful tool in studying reducts in many classifications.

In Chapter 4 of the thesis we discuss a questions about "reconstruction". It is a well known result that two ω-categorical structures are first-order interdefinable

(respectively bi-interpretable) if and only if their automorphism groups are equal (respectively topologically isomorphic). These facts motivate the question whether an ω-categorical structures can be already reconstructed from its automorphism group as an *abstract* group. In [EH90] Evans and Hewitt gave a counterexample of two ω-categorical structures with isomorphic, but not topologically isomorphic automorphism groups. We modify their result and construct two ω-categorical structures such that also the endomorphism monoids and polymorphism clones are isomorphic, but not topologically isomorphic.

The chapters 2, 3 and 4 correspond to the publications [AK15] (with Lovkush Agarwal), [KP17] (with Trung Van Pham) and [BEKP] (with Manuel Bodirsky, David Evans und Michael Pinsker).

Contents

Chapter 1

Introduction

In the course of this introduction we define homogeneous and ω-categorical structures and give a short overview how those concepts are related to the theory of infinite permutation groups and Ramsey theory. We also discuss the newer connection of homogeneous structures to constraint satisfaction problem and clones. We explain important definitions and theoretical background needed for the three main results of this thesis, which are presented in Chapter 2, 3 and 4.

This introduction is partially based on the survey article [Mac11], which offers a good overview over the field of homogeneous structures. For general notions from model theory we refer to [Hod97], for concepts regarding oligomorphic permutation groups we refer to [Cam90], for background on structural Ramsey theory we refer to [NR98] and the lecture notes [Paw16]. The topic of CPSs over homogeneous structures is treated in the survey [Bod12], a short summary of more up-to-date results can be found in the introduction section of [BKO$^+$17].

1.1 Basic notions from model theory

A first-order language L is a set of symbols, which can be divided into three types: constant symbols c_i, function symbols f_j, and relation symbols R_k, so $L = \{c_i : i \in I\} \cup \{f_j : j \in J\} \cup \{R_k : k \in K\}$ for some indexing sets I, J, K. Each function symbol f and each relation symbol R is associated with a nonzero natural number called the *arity* $\mathrm{ar}(f)$ respectively $\mathrm{ar}(R)$ of the symbol.

An *L-structure* $\mathbb{M}$ consists of a domain set M together with interpretations for the symbols of L. More precisely, every function symbol $f \in L$ with $\mathrm{ar}(f) = n$ is interpreted as a function $f^{\mathbb{M}} \colon M^n \to M$, and every relation symbol $R \in L$ with $\mathrm{ar}(R) = m$ is interpreted as an m-ary relation $R^{\mathbb{M}}$ on M, that is, a subset of M^n. Constant symbols are interpreted as unary, singleton relations. Note that this convention for constants might differ from the one used by other authors. We write $\mathbb{M} = (M; (c_i^{\mathbb{M}})_{i\in I}, (f_j^{\mathbb{M}})_{j\in J}, (R_k^{\mathbb{M}})_{k\in K})$ for an L-structure $\mathbb{M}$. If it is clear from the context, we do not distinguish between symbols R and their interpretations $R^{\mathbb{M}}$.

A structure in a language containing no functional symbols is called a *relational structure*. A structure in a language consisting only of function symbols is called an *algebra*.

For a given structure $\mathbb{M}$ in language L, the *(first-order) theory of* $\mathbb{M}$, short $\mathrm{Th}(\mathbb{M})$ is the set of all L-sentences in first-order logic that hold in $\mathbb{M}$. For a tuple $\bar{m} = (m_1, \ldots, m_n) \in M^n$, the *type of* $\bar{m}$, or short $\mathrm{tp}(\bar{m})$, is the set of all first-order formulas $\phi(\bar{x}) = \phi(x_1, x_2, \ldots, x_n)$, such that $\phi(\bar{m})$ holds in $\mathbb{M}$. The set of all types of $\mathbb{M}$ is denotes by $S(\mathbb{M})$.

Let $\mathbb{M}$ and $\mathbb{N}$ be two relational L-structures on domains M respectively N. Then we call a map $h\colon M \to N$ an *(L-)homomorphism*, if for every $R \in L$ and for every tuple $\bar{m} = (m_1, \ldots, m_n) \in R^{\mathbb{M}}$ implies $f(\bar{m}) := (f(m_1), \ldots, f(m_n)) \in R^{\mathbb{N}}$. If moreover $\bar{m} \in R^{\mathbb{M}} \leftrightarrow f(\bar{m}) \in R^{\mathbb{N}}$ holds, we call f a *strong homomorphism*. An injective, strong homomorphism is called an *embedding*. If $M \subseteq N$ and the identity function is an embedding of $\mathbb{M}$ into $\mathbb{N}$, we say that $\mathbb{M}$ is a *substructure* of $\mathbb{N}$ and write $\mathbb{M} \leq \mathbb{N}$. Bijective embeddings are called *isomorphisms*.

For tuples $\bar{a} \in M^n, \bar{b} \in N^n$, we say $\bar{a}$ and $\bar{b}$ are *isomorphic*, and write $\bar{a} \cong \bar{b}$, if the function $a_i \mapsto b_i$ for all i such that $1 \leq i \leq n$ is an isomorphism with respect to the relations inherited from $\mathbb{M}$ and $\mathbb{N}$.

A homomorphisms from a structure $\mathbb{M}$ to itself is called an *endomorphism* of $\mathbb{M}$. Under composition $\circ$, the set of endomorphisms of $\mathbb{M}$ forms the *endomorphism monoid* $\mathrm{End}(\mathbb{M})$. The embeddings of $\mathbb{M}$ into $\mathbb{M}$ also form a monoid, which will be referred to as the *monoid of self-embeddings* $\mathrm{Emb}(\mathbb{M})$. Finally, an isomorphism from a structure $\mathbb{M}$ to itself is called *automorphisms* of $\mathbb{M}$; with respect to composition the automorphisms of $\mathbb{M}$ form the *automorphism group* $\mathrm{Aut}(\mathbb{M})$.

1.2 Infinite permutation groups and transformation monoids

The theory of infinite permutation groups is one of the newer branches of group theory, and it has established connections with model theory as we will see in later sections of this introduction; at this point we just give some basic definitions and introduce the topology of pointwise convergence.

Let A and B be two sets. Then by A^B we denote the set of all functions from B to A. The set A^A together with the composition operation $\circ$ forms a monoid, which is called the *full transformation monoid of* A. By $\mathrm{Sym}(A)$ we denote the *symmetric group on* A, the group consisting of all permutations of A. Submonoids of A^A are called *transformation monoids* and subgroups of $\mathrm{Sym}(A)$ are called *permutation groups*.

There is a natural way to provide a topology on A^A by taking the product topology on A^A, where A bears the discrete topology. By this definition the sets of the form $\{f \in A^A : f(\bar{a}) = \bar{b}\}$, where $\bar{a}$ and $\bar{b}$ are finite tuples, form a basis of open subsets. It is not hard to check that the composition of functions $\circ : A^A \times A^A \to A^A$ is a continuous operation with respect to this topology, therefore the full transformation monoid A^A is a *topological monoid*. If A is countable, so without loss of generality $A = \omega$, the topology on A^A is induced by the complete metric

$$d\colon A^A \times A^A \to [0,1]$$
$$d(f,g) = 2^{-n} \text{ with } n = \min\{i : f(i) \neq g(i)\}.$$

So the closer f and g are to each other, the longer is the initial segment of A on which they agree. Hence a sequence of functions $(f_n)_{n\in\omega}$ converges, if for every $x \in A$ the sequence $f_n(x)$ is eventually constant. That is why this topology is called the *topology of pointwise convergence*.

All transformation monoids with the inherited subspace topology are also topological monoids. For a transformation monoid M a basis is given by all the cosets of the submonoids $M_{\bar{a}} := \{m \in M : m(\bar{a}) = \bar{a}\}$ for finite tuples $\bar{a}$. We call $M_{\bar{a}}$ the *stabilizer of* $\bar{a}$ *in* M. In particular also $\mathrm{Sym}(A)$ inherits the topology of pointwise convergence and is a *topological group* under it, i.e. a group in which both composition and inversion are continuous operations. If A is a countable set, $\mathrm{Sym}(A)$ with the subspace topology is in fact a *Polish group*: a complete metric is given by $\max(d(f,g), d(f^{-1}, g^{-1}))$, where d is defined as above. Note however that $\mathrm{Sym}(A)$ is not closed in A^A; its closure is the set of all injective function from A to A.

Clearly the endomorphism monoid of a given structure $\mathbb{A}$ is a transformation monoid on its domain A. One may ask, if also the opposite direction holds, so if also all transformation monoids are the endomorphism monoid for some structure $\mathbb{A}$. The answer is negative, it is not hard to see that all endomorphism monoids are *topologically closed* submonoids of A^A. But more holds:

We say that a relation $R \subseteq A^n$ is *invariant* under a function $f\colon A \to A$ (or *f preserves R*), if for all $\bar{a} \in R$ we have $f(\bar{a}) \in R$. This gives rise to a Galois connection between relational structures on one side and sets of functions on the other side. We write $\mathrm{Inv}(M)$ for the structure on A with all relations that are invariant under all functions of M. Clearly $\mathrm{End}(\mathrm{Inv}(M))$ is a monoid containing M and having the same invariant relations; it is not hard to see that $\mathrm{End}(\mathrm{Inv}(M))$ is the smallest closed transformation monoid containing M.

Analogously, for a given set of bijections $G \subseteq \mathrm{Sym}(A)$ we have that $\mathrm{Aut}(\mathrm{Inv}(G))$ is smallest closed group in $\mathrm{Sym}(A)$ containing G. We are going to use the following notational conventions:

Notation 1.2.1. For $F \subseteq A^A$ we write $\overline{F}$ for the topological closure of F in A^A. For a given set of permutations $F \subseteq \mathrm{Sym}(A)$ we write $\langle F \rangle$ for the smallest closed subgroup of $\mathrm{Sym}(A)$ containing F.

We would like to recap that, when talking about permutation groups and transformation monoids we work on three different structural levels: On the highest one we know the *action* on the domain set A. From that action we can derive the topology of pointwise convergence, which makes G to a topological group. If we forget also the topology, we end up with an abstract group. If it is needed to distinguish these structural levels we use the following notation, denoting actions by $\curvearrowright$ and topological objects by bold characters:

action on A	permutation group $G \leq \mathrm{Sym}(A)$	transformation monoid $M \leq A^A$
topology	topological group $\mathbf{G}$	topological monoid $\mathbf{M}$
abstract	group G	monoid M

This distinction will be essential in Chapter 4.

1.3 Homogeneous structures

In this section we define *homogeneity*, the first notion of central importance for this thesis. We discuss how homogeneous structures correspond to amalgamation classes and state Thomas' conjecture, which is a motivation for the results in Chapter 2 of this thesis.

Definition 1.3.1. A structure $\mathbb{M}$ is called *homogeneous* if every isomorphism between finitely generated substructures of $\mathbb{M}$ extends to an automorphism of $\mathbb{M}$.

In order to avoid conflicts with other similar notions, in the literature the above definition is sometimes also referred to as *ultra-homogeneity* (cf. [Hod97]). However, in this thesis there is not a need for such distinction, so we stick to the term *homogeneity.*

A basic example of a homogeneous structure is the rational order $(\mathbb{Q}; <)$; for this structure, homogeneity can be seen easily, since every finite order-preserving maps can be extended to an automorphisms by piecewise linear functions. The rational order can be constructed as a direct limit of finite total orders. This fact was generalized by Roland Fraïssé in [Fra54]; his technique is now known as Fraïssé's construction, and it constitutes the main method of constructing homogeneous structures. We outline it below.

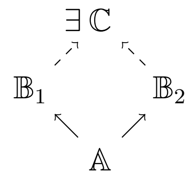

Figure 1.1: The amalgamation property

Definition 1.3.2. Let $\mathbb{M}$ an L-structure. The *age of* $\mathbb{M}$, denoted by $\mathrm{Age}(\mathbb{M})$, is the class of all finitely generated structures which can be embedded in $\mathbb{M}$.

Let $\mathcal{C}$ be a class of finitely generated L-structures. We then say that $\mathcal{C}$ has

- the *hereditary property (HP)*, if whenever $\mathbb{B} \in \mathcal{C}$ and $\mathbb{A}$ is a finitely generated substructure of $\mathbb{B}$, then $\mathbb{A} \in \mathcal{C}$,

- the *joint embedding property (JEP)*, if whenever $\mathbb{A}_1, \mathbb{A}_2 \in \mathcal{C}$, then there is $\mathbb{B} \in \mathcal{C}$ such that $\mathbb{A}_1$ and $\mathbb{A}_2$ both embed into $\mathbb{B}$.

- the *amalgamation property (AP)*, if for all $\mathbb{A}, \mathbb{B}_1, \mathbb{B}_2 \in \mathcal{C}$ and embeddings $e_1 \colon \mathbb{A} \to \mathbb{B}_1$ and $e_2 \colon \mathbb{A} \to \mathbb{B}_2$ there is a $\mathbb{C} \in \mathcal{C}$ and there are embeddings $f_1 \colon \mathbb{B}_1 \to \mathbb{C}$, $f_2 \colon \mathbb{B}_2 \to \mathbb{C}$ such that $f_1 \circ e_1 = f_2 \circ e_2$.

The age of any L-structure satisfies both HP and JEP. Conversely, a countable class of finitely generated L-structures with HP and JEP is always the age of some countable structure. So in a sense the amalgamation property is the only non-trivial property in the list above, which motivates the following definition:

Definition 1.3.3. We call a class $\mathcal{C}$ of finitely generated L-structures an *amalgamation class* if it contains countably many structures up to isomorphism, has the hereditary property, the joint embedding property, and the amalgamation property.

Fraïssé's Theorem now gives a one-to-one correspondence of homogeneous structures and amalgamation classes:

Theorem 1.3.4 (Fraïssé's Theorem [Fra54])**.** *Let L be a language and $\mathcal{C}$ an amalgamation class of finitely generated L-structures. Then there is an (up to isomorphism) unique countable homogeneous L-structure $\mathbb{M}$ such that $\mathrm{Age}(\mathbb{M}) = \mathcal{C}$. Conversely, if $\mathbb{M}$ is a countable homogeneous L-structure then $\mathrm{Age}(\mathbb{M})$ is an amalgamation class.* □

We remark that Fraïssé's Theorem can be stated in more abstract, category theoretical settings, see for instance [Kub14]. However in our context we only use the classical result by Fraïssé.

A useful consequence of Fraïssé's Theorem is that homogeneous structures have the *extension property*; in fact this property can also be used to characterize homogeneity:

Definition 1.3.5. We say that a structure $\mathbb{M}$ with $\mathrm{Age}(\mathbb{M}) = \mathcal{C}$ has the *extension property* (for $\mathcal{C}$) if and only if for every finitely generated substructure $\mathbb{A}$ of $\mathbb{M}$ and for every $\mathbb{B} \in \mathcal{C}$ such that $\mathbb{A}$ embeds into $\mathbb{B}$, there is a substructure $\mathbb{B}'$ of $\mathbb{M}$ that is isomorphic to $\mathbb{B}$, with $\mathbb{A} \leq \mathbb{B}'$.

Further note that, if $\mathbb{M}$ is a relational structure, it has the extension property if and only if it satisfies it for all extensions $\mathbb{B}$ of $\mathbb{A}$ by only one additional point. With that in mind it is for instance not hard to see that $(\mathbb{Q}; <)$ as a dense, unbounded linear order, has the extension property for linear orders.

A vast list of homogeneous structures can be constructed with the following stronger notion of amalgamation: For structures $\mathbb{A} = (A; (R^{\mathbb{A}})_{R \in L}), \mathbb{B}_1 = (B_1; (R^{\mathbb{B}_1})_{R \in L}), \mathbb{B}_2 = (B_2; (R^{\mathbb{B}_2})_{R \in L})$ such that $\mathbb{A}$ is a substructure of $\mathbb{B}_1$ and $\mathbb{B}_2$, we call the structure $\mathbb{C}$ the *free amalgam* of $\mathbb{B}_1$ and $\mathbb{B}_2$ over $\mathbb{A}$, if the domain of $\mathbb{C}$ is the union of $B_1 \cup B_2$ and the relations are also defined as the disjoint unions $R^{\mathbb{C}} = R^{\mathbb{B}_1} \cup R^{\mathbb{B}_2}$. We say a class of structures $\mathcal{C}$ has *free amalgamation* if for all triples $\mathbb{A}, \mathbb{B}_1, \mathbb{B}_2$ also their free amalgam is in $\mathcal{C}$. Then clearly $\mathcal{C}$ is an amalgamation class.

Whenever $\mathcal{F}$ is a set of finite L-structures, we let $\mathrm{Forb}(\mathcal{F})$ denote the class of all L-structures that do not embed any element of $\mathcal{F}$, and we refer to $\mathcal{F}$ as the set of *forbidden substructures*. It was observed by Henson in [Hen72] that, whenever $\mathcal{F}$ does only contain structures such that every pair of points is in some relation, then $\mathrm{Forb}(\mathcal{F})$ has free amalgamation. For instance, if L is the language of graphs and $\mathcal{F}$ consists of the 1-element loop, an undirected edge, and a set of tournaments, we obtain a so called Henson digraph as the Fraïssé limit of $\mathrm{Forb}(\mathcal{F})$. We are going to discuss these digraphs in Chapter 2.

1.3.1 Thomas' conjecture

Here, following [Tho91] and numerous subsequent authors, we define a *reduct* of a structure $\mathbb{A}$ to be a relational structure on the same domain all of whose relations have a first-order definition in $\mathbb{A}$ without parameters.

We can define a quasi-order on those reducts, by setting $\mathbb{B} \leq_{fo} \mathbb{C}$ if $\mathbb{B}$ is a reduct of $\mathbb{C}$. We are going to refer to the equivalence classes given by this quasi-order as the *first-order reducts* of $\mathbb{A}$. It is not hard to see that the partial order induced by $\leq_{fo}$ on the first-order reducts is actually a lattice.

Thomas' conjecture now states, that this lattice is finite, for certain homogeneous $\mathbb{A}$:

Conjecture 1.3.6 (Thomas [Tho91]). *Let $\mathbb{A}$ be a countable homogeneous structure in a finite relational language. Then $\mathbb{A}$ has only finitely many reducts up to first-order definability.*

Many well-known homogeneous structures have only finitely many first-order reducts, see [Cam76], [Tho91], [Tho96], [JZ08], [PPP$^+$14], [BPP15], [Aga16]. The methods used in these classifications will be outlined later in this introduction. So all known classifications of reducts are in accordance with Thomas' conjecture. Recently in [BCS16], it was shown that the countable atomless Boolean-algebra has infinitely many reducts, however this does not constitute a counterexample, since it is only a homogeneous structure in a finite *functional* language.

A major difficulty in proving that Thomas' conjecture is true in general, is that there is no direct connection between the concept of homogeneity and definability. In particular, homogeneity strongly depends on the language of the structure $\mathbb{A}$ and is not stable under forming reducts: it was already observed in [Tho91] that reducts of homogeneous structures in a finite language are not necessarily interdefinable with homogeneous structures in a finite language themselves.

In the next section we will define ω-categoricity, which is a weaker property than homogeneity, but is better suited to discuss questions about definability and relate them to infinite permutation groups.

1.4 Omega-categoricity

In this thesis we do not only consider homogeneous structures, but often also work in the broader framework of *ω-categorical structures*. In some sense ω-categorical structures are close to finite structures, since for every natural number n, they only have finitely many n-types. As we will see below, this finiteness condition gives rise to a one-to-one correspondence between ω-categorical structures and their automorphism groups.

Definition 1.4.1. A first-order theory T is called *ω-categorical*, if it has, up to isomorphisms, exactly one countable model. A countable structure $\mathbb{A}$ is called *ω-categorical*, if its theory $\mathrm{Th}(\mathbb{A})$ is *ω-categorical*.

Definition 1.4.2. A permutation group $G \leq \mathrm{Sym}(A)$ on a countable set A is called *oligomorphic* if for every $n \geq 1$ the (coordinatewise) action of G on A^n has only finitely many orbits.

Now the following theorem constitutes a bridge connecting model theory and permutation group theory, which is of central importance for us:

Theorem 1.4.3 (Engeler, Ryll-Nardzewski, Svenonius, 1959). *Let $\mathbb{A}$ be a countable structure. Then the following are equivalent:*

- $\mathbb{A}$ *is* ω*-categorical.*

- *For every* $n \in \omega$*, there are only finitely many* n*-types in* $\mathbb{A}$*.*

- *For every* $n \in \omega$ *there are only finitely many formulas* $\phi(x_1, \dots, x_n)$ *up to equivalent over* $\mathrm{Th}(\mathbb{A})$*.*

- *The automorphism group* $\mathrm{Aut}(\mathbb{A})$ *is oligomorphic.*

In this case the orbits of $\mathrm{Aut}(\mathbb{A})$ *on* A^n *correspond to the* n*-types of* $\mathbb{A}$ *and the invariant relations* $\mathrm{Inv}(\mathrm{Aut}(\mathbb{A}))$ *are exactly the relations that are first-order definable in* $\mathbb{A}$*.* □

Thus, from the point of view of automorphism groups, it is more natural to work with ω-categorical structures, rather than with homogeneous structures. However, every homogeneous structure $\mathbb{A}$ in finite relational language is ω-categorical: By definition of homogeneity, two n-tuples are in the same orbit of the action $\mathrm{Aut}(\mathbb{A}) \curvearrowright A^n$ if and only if they are isomorphic. So, since $\mathrm{Age}(\mathbb{A})$ contains for every n only finitely many n-substructures up to isomorphism, $\mathbb{A}$ is ω-categorical. Moreover we have the following characterization of homogeneous structures:

Theorem 1.4.4 ([Hod97])**.** *Let* L *be a finite relational language and let* $\mathbb{A}$ *be an* L*-structure. Then the following are equivalent*

- $\mathbb{A}$ *is homogeneous*

- *The theory of* $\mathbb{A}$ *is* ω*-categorical and has quantifier elimination.* □

By Theorem 1.4.3 also all of the reducts of a given ω-categorical structure are ω-categorical. Furthermore by Theorem 1.4.3, $\mathbb{B}$ is a reduct of $\mathbb{A}$, if and only if $\mathrm{Aut}(\mathbb{A}) \leq \mathrm{Aut}(\mathbb{B})$. Thus, determining the lattice of first-order reducts of a given ω-categorical structure $\mathbb{A}$ is equivalent to determining the lattice of closed subgroups of $\mathrm{Sym}(A)$ containing $\mathrm{Aut}(\mathbb{A})$.

1.4.1 Reconstruction

We saw that $\mathrm{Aut}(\mathbb{A})$ as permutation group allows us to "reconstruct" $\mathbb{A}$ up to first-order definability. In this section we will discuss how much information is captured in $\mathrm{Aut}(\mathbb{A})$, seen as topological or abstract group.

Definition 1.4.5. Let $\mathbb{A}, \mathbb{B}$ two structures on domains A and B and in not necessarily equal languages. Then we say that a partial function $I \colon A^n \to B$ is a *(first-order) interpretation* of $\mathbb{B}$ in $\mathbb{A}$, if

- I is surjective
- The domain of I is first-order definable in $\mathbb{A}$
- The pre-image of equality under I is first-order definable in $\mathbb{A}$
- The pre-image of every relation of $\mathbb{B}$ is first-order definable in $\mathbb{A}$

If such I exists, we say that $\mathbb{B}$ is *interpretable* in $\mathbb{A}$. If there is an interpretation I_1 of $\mathbb{B}$ in $\mathbb{A}$ and an interpretation I_2 of $\mathbb{A}$ in $\mathbb{B}$, such that also their compositions $I_1 \circ I_2$ and $I_2 \circ I_1$ are interpretations, then we say that $\mathbb{A}$ and $\mathbb{B}$ are *bi-interpretable*.

Interpretability is a key-definition in model theory, since most model-theoretical notions are preserved under it. If $\mathbb{B}$ is interpretable in $\mathbb{A}$, this implies that there is a continuous homomorphism $h\colon \mathrm{Aut}(\mathbb{A}) \to \mathrm{Aut}(\mathbb{B})$. For ω-categorical structures also the converse holds:

Theorem 1.4.6 ([AZ86]). *Let $\mathbb{A}$ be an ω-categorical structure. Then $\mathbb{B}$ is interpretable in $\mathbb{A}$ if and only if there is a continuous homomorphism $h\colon \mathrm{Aut}(\mathbb{A}) \to \mathrm{Aut}(\mathbb{B})$, such that $h(\mathrm{Aut}(\mathbb{A}))$ is oligomorphic.*
Furthermore $\mathbb{A}$ and $\mathbb{B}$ are bi-interpretable if and only if $\mathrm{Aut}(\mathbb{A})$ and $\mathrm{Aut}(\mathbb{B})$ are topologically isomorphic*, so if there is an isomorphism between them, which is also a homeomorphism.* □

Thus for ω-categorical $\mathbb{A}$, the automorphism group $\mathrm{Aut}(\mathbb{A})$ as a topological object allows us to "reconstruct" $\mathbb{A}$ up to first-order interpretations.

A natural thing to ask now is, how much information about an ω-categorical structure $\mathbb{A}$ is encoded in its automorphism group as *abstract algebraic object*. In particular we would like to know if it still determines $\mathbb{A}$ up to interpretability. There is a considerable literature about ω-categorical structures where the topology on the automorphisms is uniquely determined by the abstract automorphism group; this is for instance the case if $\mathrm{Aut}(\mathbb{A})$ has the so-called *small index property*, that is, all subgroups of countable index are open. (This is in fact stronger and equivalent to saying that all homomorphisms from $\mathrm{Aut}(\mathbb{A})$ to $\mathrm{Sym}(\omega)$ are continuous.) The small index property has for instance been shown

- for $\mathrm{Sym}(\omega)$ by Dixon, Neumann, and Thomas [DNT86];
- for $(\mathbb{Q}; <)$ and the atomless Boolean algebra by Truss [Tru89];
- for all ω-categorical ω-stable structures and the random graph in [HHLS93];
- for the Henson graphs by Herwig [Her98];

- more recently, for all homogeneous structures in finite relational language stemming from free amalgamation classes [SS17].

Other reconstructions results, using different methodology, were shown by Rubin in [Rub94]. However we will see in Chapter 4 that not all ω-categorical structures have this reconstruction property; a counterexample was given in [EH90].

1.5 CSPs over homogeneous structures

Constraint satisfaction problems or CSPs appear in almost every area of theoretical computer science, for instance in artificial intelligence, scheduling, computational linguistics, computational biology, verification, and algebraic computation. CSPs are a very general framework that allow us to phrase many different computational problems, depending on the set of constraint types, that are allowed as input. For a long time, the main focus of research was on CSPs of finite structures.

In this section we discuss how constraint satisfaction problems of homogeneous structures and their reducts can be used to encode naturally appearing satisfiability problems and we state a dichotomy conjecture for them.

Let $\mathcal{C}$ be an arbitrary class of finite structures in a finite relational language L and let Φ be a finite set of quantifier free L-formulas. Then we define $\mathcal{C}$-SAT(Φ) as the following decision problem:

$\mathcal{C}$-SAT(Φ):
INSTANCE: A finite set of variables W and a formula of the form $\phi_1 \wedge \phi_2 \wedge \cdots \wedge \phi_n$, where each ϕ_i is obtained by taking a formula from Φ and substituting with variables form W.
QUESTION: Is there a structure in $\mathcal{C}$, satisfying $\phi_1 \wedge \phi_2 \wedge \cdots \wedge \phi_n$?

If $\mathcal{C}$ is for instance the class of linear orders, then the above gives us a variety of scheduling problems; if $\mathcal{C}$ is the class of binary trees, respectively their leaf structures, we obtain phylogeny constraint satisfaction problems, if $\mathcal{C}$ is the class of graphs, we have satisfiability problems over graphs. The complexities of these problems were classified in [BK09], [BJP16] and [BP15a] respectively.

In the mentioned examples $\mathcal{C}$ is always an amalgamation class. As we will see this allows us to nicely rephrase the problems as *constraint satisfaction problem* of reducts of the corresponding homogeneous structure. This, in turn allows us to use tools from model theory and universal algebra in order to determine the complexity.

But what is a CSP? A first-order formula $\phi(x_1, \dots, x_n)$ in the language L is called *primitive positive* if it can be written using conjunction and existential quantification only, so it is of the form $\exists y_1, \dots, y_k\ (\psi_1 \wedge \dots \wedge \psi_m)$ where $\psi_1, \dots, \psi_m$ are all atomic L-formulas. The constraint satisfaction problem of a L-structure $\mathbb{A}$, short CSP($\mathbb{A}$), is then defined as the problem of deciding the primitive positive theory of $\mathbb{A}$:

> CSP($\mathbb{A}$):
> INSTANCE: A primitive positive sentence ϕ, i.e. a formula of the form
>
> $$\phi\colon \exists \bar{y}\ (\phi_1(\bar{y}) \wedge \dots \wedge \phi_m(\bar{y})) \text{ with } \phi_i \text{ atomic}$$
>
> QUESTION: Does ϕ hold in $\mathbb{A}$?

We call $\mathbb{A}$ the *template* of the constraint satisfaction problem CSP($\mathbb{A}$).

Now let $\mathcal{C}$ be an amalgamation class with Fraïssé limit $\mathbb{M}$ and let $\{\phi_1, \dots, \phi_n\}$ be a finite set of quantifier free L-formulas. We can associate this set with the reduct $(M; R_1, \dots, R_n)$ of $\mathbb{M}$ that we obtain by setting $(a_1, \dots, a_k) \in R_i$ if and only if $\phi_i(a_1, \dots, a_k)$ holds in $\mathbb{M}$. Then CSP$(M; R_1, \dots, R_n)$ and $\mathcal{C}$-SAT($\{\phi_1, \dots, \phi_n\}$) are essentially the same problem: Since the age of $\mathbb{M}$ is $\mathcal{C}$, a conjunction of formulas from $\{\phi_1, \dots, \phi_n\}$ can be satisfied in an element of $\mathcal{C}$ if and only if there is a substructure of $(M; R_1, \dots, R_n)$ satisfying it. On the other hand let $(M; R_1, \dots, R_n)$ be a reduct of $\mathbb{M}$. Since $\mathbb{M}$ is homogeneous it has quantifier elimination (see Theorem 1.4.4), so every relation R_i has a quantifier-free definition ϕ_i in $\mathbb{M}$. Then clearly $\mathcal{C}$-SAT($\{\phi_1, \dots, \phi_n\}$) = CSP$(M; R_1, \dots, R_n)$. Thus we would like to determine the complexity of the CSPs of the reducts of a given homogeneous structure $\mathbb{M}$.

The CSP of a *finite* structure is always in NP. Feder and Vardi conjectured in [FV99] that every CSP of a finite structure is either in P or NP-complete. Their dichotomy conjecture was recently proven to be true, independently in [Zhu17] and [Bul17], which makes finite CSPs to the largest known class of NP problems, where such a dichotomy holds.

The same conjecture is however false for infinite structures, in fact *every* computational problem is equivalent to a CSP, up to polynomial time. A simple counting argument shows that there are even homogeneous structure in finite relational language with undecidable CSPs (cf. [Bod12]). However if $\mathcal{C}$ is further *finitely bounded*, meaning that $\mathcal{C}$ can be described as a class of structures that do not embed a *finite* list of given forbidden structures, the corresponding CSPs are in NP. This lead to the following generalization of the Feder-Vardi conjecture:

Conjecture 1.5.1 (Bodirsky, Pinsker)**.** *Let $\mathbb{A}$ be a reduct of a finitely bounded homogeneous structure. Then* CSP($\mathbb{A}$) *is either solvable in polynomial time or NP-complete.*

In the following subsections we discuss the standard reductions that will allow us to compare the complexity of different constraint satisfaction problems. In Section 1.6 we see how we can link those reductions, in the ω-categorical case, to the concept to polymorphism clones.

1.5.1 Model-complete cores

Let $\mathbb{A}$ and $\mathbb{B}$ be two structures in the same language. We say that $\mathbb{A}$ and $\mathbb{B}$ are *homomorphically equivalent* if there is a homomorphisms from $\mathbb{A}$ to $\mathbb{B}$ and vice-versa. Since homomorphisms preserve primitive positive formulas, the constraint satisfaction problems CSP($\mathbb{A}$) and CSP($\mathbb{B}$) encode the same computational problem for homomorphic equivalent $\mathbb{A}$ and $\mathbb{B}$.

Definition 1.5.2. A structure $\mathbb{B}$ is called a *model-complete core*, if for every finite subset X of B and every endomorphism $f \in \mathrm{End}(\mathbb{B})$, there is an automorphism $g \in \mathrm{Aut}(\mathbb{B})$ such that $f|_X = g|_X$.

In other words, for a model-complete core the automorphism group Aut($\mathbb{B}$) is dense in the endomorphism monoid End($\mathbb{B}$) with respect to the topology of pointwise convergence. This definition generalizes the concept of *core* for finite structures introduced in [HN92]. By the following result model-complete cores give us a canonical and in a certain sense smallest representative of homomorphically equivalent structures:

Theorem 1.5.3. *Every ω-categorical structure $\mathbb{A}$ is homomorphically equivalent to a model-complete core $\mathbb{A}^c$ which is unique up to isomorphism. Furthermore the model-complete core $\mathbb{A}^c$ is ω-categorical or finite.* □

A model-theoretic proof of this fact was given by Bodirsky in [Bod07], a new proof in the language of transformation monoids be found in [BKO$^+$17]. By the following theorem of Bodirsky the complexity of the CSP of a model-complete core does not increase if we add finitely many constants.

Theorem 1.5.4 ([Bod07])**.** *Let $\mathbb{A}$ be a model-complete ω-categorical or finite core, and let a be an element of A. Then* CSP($\mathbb{A}$) *and* CSP($\mathbb{A}, c$) *have the same complexity, up to polynomial time.* □

1.5.2 Primitive positive definability and interpretability

Definition 1.5.5. Let $\mathbb{A}$ be a structure in language L. We then say a relation R is is *primitive positive definable* or short *pp-definable* in $\mathbb{A}$ if there is a primitive positive L-formula $\phi(x_1, \ldots, x_n)$ such that $(a_1, \ldots, a_n) \in R$ if and only if $\phi(a_1, \ldots, a_n)$ holds in $\mathbb{A}$.

If, for a given CSP, we substitute in an instance formula every relation by some pp-formula defining it, the size of the formula only grows linearly, which immediately gives us the following lemma:

Lemma 1.5.6 (Jeavons [Jea98])**.** *Let $\mathbb{A}$ be a relational structure in finite language, and let $\mathbb{B}$ be the structure on the same domain, such that every relation of $\mathbb{B}$ is primitive positive definable in $\mathbb{A}$. Then* CSP($\mathbb{B}$) *reduces to* CSP($\mathbb{A}$) *in polynomial time.* □

Lemma 1.5.6 implies that, when studying the CSPs of reducts of a homogeneous structures, we only have to study the reducts up to primitive positive definability. Let $\leq_{pp}$ denote the quasi-order on the reducts induced by primitive positive definability. Like in the first-order case, the partial order induced by $\leq_{pp}$ on its equivalence classes is in fact a complete lattice.

As for first-order definability, the notion of primitive positive definability can be generalized to primitive positive interpretations, which allows us to compare structures on different domains:

Definition 1.5.7. Let $\mathbb{A}, \mathbb{B}$ two structures on domains A and B. Then we say that a partial function $I\colon A^n \to B$ is a *primitive positive interpretation* (or short *pp-interpretation*) of $\mathbb{B}$ in $\mathbb{A}$, if

- I is surjective
- The domain of I is primitive positive definable in $\mathbb{A}$
- The pre-image of equality under I is primitive positive definable in $\mathbb{A}$
- The pre-image of every relation of $\mathbb{B}$ is primitive positive definable in $\mathbb{A}$

If such I exists, we say that $\mathbb{B}$ is *primitive positive interpretable* in $\mathbb{A}$. If there is an primitive positive interpretation I_1 of $\mathbb{B}$ in $\mathbb{A}$ and an primitive positive interpretation I_2 of $\mathbb{A}$ in $\mathbb{B}$, such that also their compositions $I_1 \circ I_2$ and $I_2 \circ I_1$ are also primitive positive interpretations, then we say that $\mathbb{A}$ and $\mathbb{B}$ are *primitive positive bi-interpretable.*

Then more general, the following lemma holds:

Lemma 1.5.8. *Let* $\mathbb{A}$ *and* $\mathbb{B}$ *be two structures in finite relational language. If* $\mathbb{A}$ *has a primitive positive interpretation, then* $\mathrm{CSP}(\mathbb{A})$ *reduces to* $\mathrm{CSP}(\mathbb{B})$ *in polynomial time.* □

We summarize that for finite or ω-categorical and finite structures $\mathbb{B}$ and $\mathbb{A}$ the complexity of $\mathrm{CSP}(\mathbb{B})$ reduces to $\mathrm{CSP}(\mathbb{A})$ in the following cases:

1. $\mathbb{B}$ is pp-interpretable in $\mathbb{A}$.
2. $\mathbb{B}$ is the model-complete core of $\mathbb{A}$.
3. $\mathbb{A}$ is a model-complete core and $\mathbb{B}$ is obtained by adding finitely many constants to the signature of $\mathbb{A}$.

It is a well-known fact that the finite structure

$$\mathbb{S} := (\{0,1\}; \{(1,0,0),(0,1,0),(0,0,1)\})$$

pp-interprets all finite structures, and that its CSP is NP-complete. In all know examples of structures $\mathbb{A}$ in the scope of the Conjecture 1.5.1 the only source of NP-hardness of $\mathrm{CSP}(\mathbb{A})$ is that $\mathbb{S}$ (and hence all finite structures) can be reduced to $\mathbb{A}$ by the reductions (1)-(3). This leads to the following stronger version of the dichotomy conjecture stated in Conjecture 1.5.1:

Conjecture 1.5.9. *Let* $\mathbb{A}$ *be a reduct of a finitely bounded homogeneous structure and let* $\mathbb{A}^c$ *be its core. Then*

1. *either* $\mathbb{A}^c$ *together with finitely many constants pp-interprets* $\mathbb{S}$ *and* $\mathrm{CSP}(\mathbb{A})$ *is NP-complete,*
2. *or* $\mathrm{CSP}(\mathbb{A})$ *is in P.*

We will recap this in the next section and give algebraic tools that help to distinguish the first from the second case.

1.6 Oligomorphic clones

We already saw in Section 1.4 that the automorphism group of an ω-categorical structure $\mathbb{A}$ captures its logical properties up to first-order definability, respectively interpretability. In this section we will see that there is a bigger algebraic object, called the *polymorphism clone* of $\mathbb{A}$ that captures all of the properties, up to *primitive positive* definability, respectively interpretability. This connection allows us to rephrase the reductions from Section 1.5 in the language of clones.

This so called *universal algebraic approach* to study CSPs was originally introduced by Bulatov, Jeavons and Krokhin in [BKJ05] for CSPs of finite structures and lead to big developments on the area of universal algebra. Their approach was generalized to the ω-categorical case by Bodirsky and Pinsker.

For a fixed finite base set A, we write A^{A^n} for the set of all n-ary operation on A, i.e. the functions $f\colon A^n \to A$. Then the set of all finitary operations $\bigcup_{n\geq 1} A^{A^n}$, together with composition of function is an algebraic object that is closed under composition.

Definition 1.6.1. A *function clone* $\mathscr{A}$ is a subset of $\bigcup_{n\geq 1} A^{A^n}$, such that

- $\mathscr{A}$ contains all projections $\pi_i^n(x_1,\ldots,x_n) = x_i$, for all indices $1 \leq i \leq n \in \omega$
- $\mathscr{A}$ is closed under composition. So if $f \in \mathscr{A} \cap A^{A^n}$ and $g_1,\ldots,g_n \in \mathscr{A} \cap A^{A^k}$, then also $f \circ (g_1,\ldots,g_n) \in \mathscr{A}$, where

  $$(f \circ (g_1,\ldots,g_n))(x_1,\ldots,x_{kn}) := f(g_1(x_1,\ldots,x_k),\ldots,g_n(x_{(k-1)n+1},\ldots,x_{kn})).$$

Function clones can be seen as a generalization of transformation monoids to higher arities. As transformation monoids, function clones bear the topology of pointwise convergence; a base of open neighborhoods for a function $f \in A^{A^n}$ is given by the sets $\{g \in A^{A^n} : g|_F = f|_F\}$ for all finite sets $F \subseteq A^n$. In this topology the composition of functions is continuous, hence it forms what we call a *topological clone.*

Let $\mathbb{A}$ be a relational structure with domain A. By $\mathbb{A}^n$ we denote the direct product of n-copies of $\mathbb{A}$. This is, a structure on A^n in the same language as $\mathbb{A}$, such that for $\bar{x}^{(1)},\ldots,\bar{x}^{(k)} \in A^n$ we set $(\bar{x}^{(1)},\ldots,\bar{x}^{(k)}) \in R$ if and only if $(x_i^{(1)},\ldots x_i^{(k)}) \in R$ in $\mathbb{A}$ for every coordinate $i \in [n]$.

Then an n-ary operation f is called a *polymorphism* of $\mathbb{A}$ if f is a homomorphism from $\mathbb{A}^n$ to $\mathbb{A}$. For every relation R on A we say that f *preserves* R if f is a polymorphism of $(A;R)$. Otherwise we say f *violates* R. Polymorphisms can be seen as a higher-arity generalization of endomorphisms; in particular endomorphisms are unary polymorphisms.

For a given structure $\mathbb{A}$ the set of all polymorphisms $\mathrm{Pol}(\mathbb{A})$ contains all the projections and is closed under composition, hence it is a function clone.

Definition 1.6.2. Let $\mathbb{A}$ be a relational structure. Then we call the set of all polymorphisms, the *polymorphism clone* of $\mathbb{A}$, or short $\mathrm{Pol}(\mathbb{A})$.

As for monoids, for a given function clone $\mathscr{A}$ the polymorphism clone $\mathrm{Pol}(\mathrm{Inv}\,(\mathscr{A}))$ of all the relations preserved by $\mathscr{A}$ is the closure of $\mathscr{A}$ with respect to the topology of pointwise convergence. Hence a function clone on A is closed if and only if it can be written as the polymorphism clone of some structure on A.

Definition 1.6.3. We call a function clone *oligomorphic* if and only if the group of its unary, invertible functions is an oligomorphic permutation group.

So a countable structure $\mathbb{A}$ is ω-categorical if and only if $\mathrm{Pol}(\mathbb{A})$ is oligomorphic. Now it is of central importance to us that primitive positive definability in ω-categorical (and finite) structures can be characterized by preservation under polymorphisms:

Theorem 1.6.4 (Bodirsky, Nešetřil [BN06])**.** *Let $\mathbb{A}$ be an ω-categorical structure. Then a relation is pp-definable in $\mathbb{A}$, if and only if it is preserved by the polymorphisms of $\mathbb{A}$.* □

Thus, by Lemma 1.5.6 the complexity of $\mathrm{CSP}(\mathbb{A})$ only depend on the polymorphism clone $\mathrm{Pol}(\mathbb{A})$ for ω-categorical $\mathbb{A}$.

Conversely, if a first-order definable relation is not pp-definable in $\mathbb{A}$, this can be witnessed by polymorphisms of bounded arity:

Theorem 1.6.5 (Bodirsky, Kara [BK09])**.** *Let $\mathbb{A}$ be a relational structure and let R be a k-ary relation that is a union of at most m orbits of* $\mathrm{Aut}(\mathbb{A})$ *on D^k. If $\mathbb{A}$ has a polymorphism f that violates R, then $\mathbb{A}$ also has an at most m-ary polymorphism that violates R.* □

1.6.1 Clone homomorphisms

Definition 1.6.6. A *clone homomorphism* from a function clone $\mathscr{A}$ to a function clone $\mathscr{B}$ is a mapping $\xi\colon \mathscr{A} \to \mathscr{B}$ which

- preserves arities: it sends every function in $\mathscr{A}$ to a function of the same arity in $\mathscr{B}$;
- preserves each projection: it sends the k-ary projection onto the i-th coordinate in $\mathscr{A}$ to the same projection in $\mathscr{B}$, for all $1 \leq i \leq k$;
- preserves composition: $\xi(f \circ (g_1, \dots, g_n)) = \xi(f) \circ (\xi(g_1), \dots, \xi(g_n))$ for all n-ary functions f and all m-ary functions $g_1, \dots, g_n$ in $\mathscr{A}$.

A mapping $\xi\colon \mathscr{A} \to \mathscr{B}$ with an inverse that is also a clone homomorphism is called a *clone isomorphism*.

As an analogue of the Theorem 1.4.6 we then obtain the following:

Theorem 1.6.7 ([BP15b])**.** *Let $\mathbb{B}$ be ω-categorical and $\mathbb{A}$ be an arbitrary structure. Then $\mathbb{A}$ has a primitive positive interpretation in $\mathbb{B}$ if and only if there is a continuous clone homomorphism $\xi\colon$ $\mathrm{Pol}(\mathbb{B}) \to \mathrm{Pol}(\mathbb{A})$, such that $\xi(\mathrm{Pol}(\mathbb{B}))$ is oligomorphic.*

Moreover, $\mathbb{A}$ and $\mathbb{B}$ are primitive positive bi-interpretable if and only if their polymorphism clones are topologically isomorphic. □

Note that by Lemma 1.5.8 this implies the following:

Observation 1.6.8. For every ω-categorical structure, the complexity of CSP($\mathbb{A}$) only depends on Pol($\mathbb{A}$) as topological clone.

This again leads to the question, as for automorphism groups, whether the topology is really essential for Theorem 1.6.7, or whether we can reconstruct the topology on Pol($\mathbb{A}$) from the pure algebraic structure of the polymorphism clone. Positive results in this respect are shown in [BPP17], [PP14], [PP15] and [TVG17], where the strategy is to lift the existing reconstruction results for the group to the monoid and clone level.

We are going to show in Chapter 4 that it is not possible in general to reconstruct the topology of an oligomorphic clone from its purely algebraic structure.

1.6.2 An algebraic dichotomy conjecture for infinite CSPs

We would like to recall Conjecture 1.5.9, which claims that the only source of NP-completeness of a given CSP of a reduct $\mathbb{A}$ of a finitely bounded homogeneous structure is, that the structure $\mathbb{S}$ has a pp-interpretation in the model complete core $\mathbb{A}^c$ extended by some constants. In this section we discuss the implications for the polymorphism clone of $\mathbb{A}$, respectively $\mathbb{A}^c$.

The polymorphism clone of $\mathbb{S}$ is the clone consisting of all projections on the set $\{0, 1\}$. We will denote this projection clone by $\mathbf{1}$. By Theorem 1.6.7, item (1) of Conjecture 1.5.9 is equivalent to the statement that there is a continuos clone homomorphism from some stabilizer of Pol($\mathbb{A}^c$) to $\mathbf{1}$. Barto and Pinsker could show that also item (2) has a nice, positive characterization in the language of clones, namely the existence of a Siggers operation modulo outer embeddings:

Theorem 1.6.9 ([BP16a])**.** *Let $\mathbb{A}$ be an ω-categorical model-complete core. The following are equivalent.*

1. $\mathbb{A}$ *does not pp-interpret* $\mathbb{S}$ *with parameters.*
2. *No stabilizer of* Pol($\mathbb{A}$) *maps homomorphically and continuously to* $\mathbf{1}$.
3. *No stabilizer of* Pol($\mathbb{A}$) *maps homomorphically to* $\mathbf{1}$.
4. Pol($\mathbb{A}$) *satisfies the* pseudo Siggers equation*, i.e., there exist* $e_1, e_2, f \in$ Pol($\mathbb{A}$) *such that the identity*

$$e_1 \circ f(x, y, x, z, y, z) = e_2 \circ f(y, x, z, x, z, y)$$

holds in Pol($\mathbb{A}$).

□

Hence Conjecture 1.5.9 has the following algebraic counterpart:

Conjecture 1.6.10. *Let* $\mathbb{A}$ *be a reduct of a finitely bounded homogeneous structure and let* $\mathbb{A}^c$ *be its core. Then*

1. *either there is a clone homomorphism* $\mathrm{Pol}(\mathbb{A}^c, c_1, \ldots, c_n) \to \mathbf{1}$ *and consequently* $\mathrm{CSP}(\mathbb{A})$ *is NP-complete,*
2. *or* $\mathrm{Pol}(\mathbb{A}^c)$ *contains a pseudo Siggers operation and* $\mathrm{CSP}(\mathbb{A})$ *is in P.*

Clearly the pseudo Siggers equation $e_1 \circ f(x, y, x, z, y, z) = e_2 \circ f(y, x, z, x, z, y)$ does not hold if f is any projection, which is why there cannot be a clone homomorphism to $\mathbf{1}$, not even a discontinuous one. In the analysis of actual CSPs over a given homogeneous structure, as in Chapter 3, we are going to identify operations f that satisfy the above, or other *non-trivial* equations, i.e. equations that cannot be satisfied by projections. Theses operations then give us structural information about the underlying reducts, which we can use to show tractability.

We remark that, the usage of homomorphic equivalence and pp-interpretations might not be optimal in the order in Conjecture 1.5.9. This lead to the statement of a second conjecture in [BOP] that avoids the concept of cores altogether and claims that NP-completeness occurs, if and only if $\mathbb{S}$ is homomorphically equivalent to some pp-interpretation of $\mathbb{A}$. Also this second conjecture can be stated in the language of clones, using the concept of h1-clone homomorphisms. We are however not going to delve into this, and remark that by a new result in [BKO$^+$17], the two resulting conjectures are equivalent for reducts of finitely bounded homogeneous structures.

1.7 The Ramsey property

For two structures $\mathbb{A}$ and $\mathbb{B}$ in the same language, let $\binom{\mathbb{B}}{\mathbb{A}}$ denote the set of all substructures of $\mathbb{B}$ that are isomorphic to $\mathbb{A}$. We then say that a class $\mathcal{C}$ of structures in the same language has the *Ramsey property*, if for all $\mathbb{A}, \mathbb{B} \in \mathcal{C}$, there is a $\mathbb{C} \in \mathcal{C}$ such that for all colorings $\chi \colon \binom{\mathbb{C}}{\mathbb{A}} \to \{0, 1\}$ there is a monochromatic copy of $\mathbb{B}$, i.e. there is a $\mathbb{B}' \in \binom{\mathbb{C}}{\mathbb{B}}$ such that χ is constant, when restricted to $\binom{\mathbb{B}'}{\mathbb{A}}$.

Under quite weak assumption (i.e. if $\mathcal{C}$ has HP and JEP), $\mathcal{C}$ having the Ramsey property implies that $\mathcal{C}$ is an amalgamation class [Neš05]. This motivates that we call a homogeneous structure a *Ramsey structure*, if its age has the Ramsey property. Not all homogeneous structures are Ramsey structures, but they often have "reasonable" expansions to Ramsey classes. This observation was also the start of Nešetřils classification programme, whose aim is to find precompact Ramsey expansions for amalgamation classes. For many well-known structures such expansions exist, but it was shown recently in [EHN16] that not every ω-categorical structure has such an expansion.

In the remarkable paper [KPT05], Kechris, Pestov and Todorcevic exhibited a connection between Ramsey notions and topological dynamics. They could prove that an ordered homogeneous structure is Ramsey, if and only if its automorphism group is extremely amenable.

1.7.1 Canonical functions

Canonical functions are a powerful concept with numerous applications in the study of automorphism groups, and polymorphism clones of countable structures with Ramsey properties.

Definition 1.7.1. Let $\mathbb{A}$ and $\mathbb{B}$ be two structures on domains A and B. Then a function $f\colon A \to B$ is called *canonical*, if for all tuples $\bar{a}, \bar{a}'$ in A with $\operatorname{tp}(\bar{a}) = \operatorname{tp}(\bar{a}')$, we have that $\operatorname{tp}(f(\bar{a})) = \operatorname{tp}(f(\bar{a}'))$ in $\mathbb{B}$.
Every canonical function f induces a function from the set of types of $\mathbb{A}$ to the set of types of $\mathbb{B}$. We refer to this function as the *behaviour* of f.

We will slightly abuse notation and use the same symbol for a canonical function and its behaviour function. For example, for an arbitrary structure $\mathbb{M}$, every automorphism $f \in \operatorname{Aut}(\mathbb{M})$ is a canonical function with the behaviour $f(p) = p$ for all types $p \in S(\mathbb{M})$.

The benefit of canonical functions is that they are particularly well-behaved and can be easily manipulated and analysed. Since ω-categorical structures have only finitely many types in every arity, this makes the analysis of function to a combinatorial problem. The following theorem grants us the existence of canonical functions generated by some given arbitrary function:

Theorem 1.7.2 (Bodirsky, Pinsker [BP16b]). *Let $\mathbb{A}$ be a Ramsey structure and $\mathbb{B}$ be ω-categorical. Then, for every $f\colon A \to B$ there is a $g \in \overline{\operatorname{Aut}(\mathbb{B}) \circ f \circ \operatorname{Aut}(\mathbb{A})}$ such that g is canonical from $\mathbb{A}$ to $\mathbb{B}$.* □

In particular, since expansions of Ramsey structures by finitely many constants are also Ramsey, this gives us the following corollary:

Corollary 1.7.3. *Let $\mathbb{A}$ be a Ramsey structure in finite relational language, let $f\colon A^l \to A$ be an arbitrary function and let $c_1, \ldots, c_n \in A$. Then there is a function $g \in \overline{\operatorname{Aut}(\mathbb{A}) \circ f \circ (\operatorname{Aut}(\mathbb{A}))^l}$, such that*

- *g agrees with f on $\{c_1, \ldots, c_n\}^l$ and*

- *g is canonical from $(\mathbb{A}, c_1, \ldots, c_n)^l$ to $\mathbb{A}$.*

□

This means that, whenever g witnesses a property on a finite set $\{c_1, \ldots, c_n\}$, we can find an f, generated by g and the automorphisms of $\mathbb{A}$, that has the same property and is furthermore canonical from $(\mathbb{A}^l, c_1, \ldots, c_n)$ to $\mathbb{A}$. This is an immense help in the study of reducts, and will become more evident in the proofs in Chapters 2 and Chapter 3.

Chapter 2

The reducts of Henson digraphs

In this chapter the reducts of the Henson digraphs are classified up to first-order definability. This result contributes to the large body of work concerning the two intimately related topics of reducts of countable structures and of closed subgroups of $\mathrm{Sym}(\omega)$. Motivation for this work comes from both areas.

In the topic of reducts, to our knowledge, this is the first time that the reducts of uncountably many homogeneous structures have been classified. In all cases only finitely many reducts appear. This supports Thomas' conjecture (Conjecture 1.3.6), which states that all countable homogeneous structures in a finite relational language have only finitely many reducts. Evidence for this conjecture has been building as there have been numerous classification results, e.g. [Cam76], [Tho91], [Tho96], [JZ08], [PPP+14], [BPP15], [Aga16].

The main tool used in this classification are the 'canonical functions' that we introduced in Section 1.7.1. The robustness and relative ease of the methodology is becoming more evident as several classifications have been achieved by their use, e.g. [PPP+14], [BPP15], [Bos15], [LP15], [Aga16], [BBWPP16].

In the topic of permutation groups, the main consequence of our result is a positive answer to a question of Macpherson, Question 5.10 in [BM16], which asked whether there are 2^ω pairwise non-conjugate maximal-closed subgroups of $\mathrm{Sym}(\omega)$ with $\mathrm{Sym}(\omega)$ bearing the pointwise convergence topology. Several related questions have recently been tackled. Independently, [BM16] and [BR13] showed that there exist non-oligomorphic maximal-closed subgroups of $\mathrm{Sym}(\omega)$, the existence of which was asked in [JZ08]. Also, independently, [KS16] and [BR13] positively answered Macpherson's question of whether there are maximal-closed subgroups of $\mathrm{Sym}(\omega)$ of countable cardinality. One question that remains open is whether every proper closed subgroup of $\mathrm{Sym}(\omega)$ is contained in a maximal-closed subgroup of $\mathrm{Sym}(\omega)$, (Question 7.7 in [MN96] and Question 5.9 in [BM16]).

The description of 2^ω maximal-closed subgroups follows from the classification of reducts by taking the automorphism groups of a suitably modified version of

Henson's construction of 2^ω pairwise non-isomorphic countable homogeneous digraphs ([Hen72]). A short argument shows that their automorphism groups are pairwise non-conjugate. However, we can say more: by Rubin's work on reconstruction ([Rub94]), the automorphism groups will be pairwise non-isomorphic as abstract groups.

We outline the structure of this chapter: In Section 2.1, we provide the necessary preliminary definitions and facts on the Henson digraphs. In Section 2.2, we prove the classification of the reducts of the Henson digraphs: In Section 2.2.1 we state the classification. In Section 2.2.2 we describe the reducts, establishing notation and important lemmas that are used in the rest of the paper. In Section 2.2.3 we carry out the combinatorial analysis of the possible behaviours of canonical functions, 2.2.4 contains the proof of the classification. Section 2.3 contains the denouement of the article: the classification is used to show that there exist 2^ω pairwise non-isomorphic maximal-closed subgroups of $\mathrm{Sym}(\omega)$.

2.1 Henson Digraphs

A *directed graph* $(V; E)$, or *digraph* for short, is in our context a relational structure on domain V with a binary irreflexive anti-symmetric relation $E \subseteq V^2$. We call V the set of vertices, and E is the set of edges. We say a digraph is *edgeless* if $E = \emptyset$. By L_n we denote the strict linear order on n-elements, regarded as a digraph.

A *tournament* is a digraph in which there is an edge between every pair of distinct vertices. Throughout this article, $\mathcal{T}$ will denote a set of finite tournaments. We will often refer to elements of $\mathcal{T}$ as forbidden tournaments.

Definition 2.1.1.

(i) Let $\mathcal{T}$ be a set of finite tournaments. We let $\mathrm{Forb}(\mathcal{T})$ be the class of finite digraphs D such that for all $T \in \mathcal{T}$, D does not embed T.

(ii) If $\mathcal{T}$ does not contain the 1-element tournament, we let $(D_{\mathcal{T}}; E_{\mathcal{T}})$ be the unique (up to isomorphism) countable homogeneous digraph whose age is $\mathrm{Forb}(\mathcal{T})$.

(iii) A *Henson digraph* is a digraph isomorphic to $(D_{\mathcal{T}}; E_{\mathcal{T}})$ where $\mathcal{T}$ is non-empty and does not contain the 1- or 2-element tournament.

The fact that $(D_{\mathcal{T}}; E_{\mathcal{T}})$ exists and is unique follows from the fact that $\mathrm{Forb}(\mathcal{T})$ is a (free) amalgamation class, see Theorem 1.3.4. This particular construction of digraphs was used by Henson in [Hen72] to show there exists uncountably many countable homogeneous digraphs.

If $\mathcal{T} = \emptyset$ then $(D_\mathcal{T}; E_\mathcal{T})$ is the generic digraph, the unique countable homogeneous digraph that embeds all finite digraphs. The reducts of the generic digraph are classified in [Aga16]. If $\mathcal{T}$ contains the 1-element tournament, then $\text{Forb}(\mathcal{T}) = \emptyset$. If $\mathcal{T}$ contains the 2-element tournament, then $(D_\mathcal{T}; E_\mathcal{T})$ is the countable edgeless digraph. These are degenerate cases which is why we defined the term Henson digraph to exclude these options.

Lemma 2.1.2. *Let $(D; E)$ be a Henson digraph.*

(i) $\text{Th}(D; E)$ *is ω-categorical and has quantifier elimination.*

(ii) Let $(D'; E')$ be a digraph such that $\text{Age}(D'; E') \subseteq \text{Age}(D; E)$*. Then $(D'; E')$ is embeddable in $(D; E)$.*

(iii) $(D; E)$ is connected: for every distinct $a, b \in D$, there is a path from a to b or from b to a. (In fact, an oriented path of length at most two.)

Proof. (i) Follows directly from Theorem 1.4.4.

(ii) This follows from the extension property (cf. Definition 1.3.5) and an induction argument.

(iii) Let $a, b \in D$ be distinct and without loss of generality suppose that there is no edge between a and b. Then, by the extension property, there is a $c \in D$ with $E(a, c)$ and $E(c, b)$.

□

In order to use the canonical functions machinery, we need to expand the Henson digraphs to ordered digraphs. This is described in the following definition.

Definition 2.1.3. (i) An *ordered digraph* is a digraph which is also linearly ordered. Formally, it is a structure $(V; E, <)$ where $(V; E)$ is a digraph and $(V; <)$ is a linear order.

(ii) We let $(D_\mathcal{T}; E_\mathcal{T}, <)$ be the unique (up to isomorphism) countable homogeneous ordered digraph such that a finite ordered digraph $(D; E, <)$ is embeddable in $(D_\mathcal{T}; E_\mathcal{T}, <)$ iff $(D; E) \in \text{Forb}(\mathcal{T})$.

(iii) We say $(D; E, <)$ is a *Henson ordered digraph* if $(D; E, <) \cong (D_\mathcal{T}; E_\mathcal{T}, <)$ for some $\mathcal{T}$.

Theorem 2.1.4. *All Henson ordered digraphs are Ramsey structures.*

Proof. This fact follows by applying the main theorem of [NR83], where it is shown that all free amalgamation classes, extended by linear orders, are Ramsey; additionally, the fact is stated in [LJTW14]. □

Hence we can apply the method of canonical functions. Corollary 1.7.3 implies the following for Henson ordered graphs:

Lemma 2.1.5. *Let $(D; E, <)$ be a Henson ordered digraph. Let $f \in \mathrm{Sym}(D)$ and $c_1, \ldots, c_n \in D$ be any vertices. Then there exists a function $g : D \to D$ such that*

(i) $g \in \overline{\langle \mathrm{Aut}(D; E) \cup \{f\} \rangle}$.

(ii) $g(c_i) = f(c_i)$ *for* $i = 1, \ldots, n$.

(iii) When regarded as a function from $(D; E, <, c_1, \ldots, c_n)$ to $(D; E)$, g is a canonical function.

2.2 Classification of the Reducts

For this section, we fix a Henson ordered digraph $(D; E, <)$ and let $\mathcal{T}$ be its set of forbidden tournaments.

2.2.1 Statement of the Classification

Definition 2.2.1.

(i) Recall that for $F \subseteq \mathrm{Sym}(D)$, $\langle F \rangle$ denotes the smallest closed subgroup of $\mathrm{Sym}(D)$ containing F. For brevity, when it is clear we are discussing supergroups of $\mathrm{Aut}(D; E)$, we may abuse notation and write $\langle F \rangle$ to mean $\langle F \cup \mathrm{Aut}(D; E) \rangle$.

(ii) We let $E^*(x, y)$ denote the relation defined by $E(y, x)$, i.e. the relation where the orientation all edges are flipped. We let $\overline{E}(x, y)$ denote the underlying (undirected) graph relation $E(x, y) \vee E(y, x)$ and $N(x, y)$ denote the non-edge relation $\neg\overline{E}(x, y)$.

(iii) A *Henson graph* is the Fraïssé limit of the class of all finite K_n-free undirected graphs, for some integer $n \geq 3$.

(iv) Assume $(D; E)$ is isomorphic to the digraph $(D; E^*)$ obtained by changing the direction of all its edges. In this case $-$ will denote a permutation of D such that for all $x, y \in D$, $E(-(x), -(y))$ iff $E(y, x)$.

(v) Assume $(D; E)$ is isomorphic to the digraph obtained by changing the direction of all the edges adjacent to one particular vertex of D. In this case sw will denote a permutation of D such that for some $a \in D$:

$$E(\mathrm{sw}(x), \mathrm{sw}(y)) \text{ if and only if } \begin{cases} E(x, y) \text{ and } x, y \neq a, \text{ OR,} \\ E(y, x) \text{ and } x = a \vee y = a \end{cases}$$

In words, $-$ is a function which changes the direction of all the edges of the digraph and sw is a function which changes the direction of those edges adjacent to one particularly vertex. The existence of $-$ or sw depends on which tournaments are forbidden; see Lemma 2.2.5. This explains the wording of Theorem 2.2.2(iii): if, for example, $-$ exists but sw does not, then $\max\{\mathrm{Aut}(D;E), \langle - \rangle, \langle \mathrm{sw} \rangle, \langle -, \mathrm{sw} \rangle\} = \langle - \rangle$. Also, note that the groups $\langle - \rangle$ and $\langle \mathrm{sw} \rangle$ are independent from the choice of the specific functions $-$ or sw; again see Lemma 2.2.5.

Theorem 2.2.2. *Let $(D;E)$ be a Henson digraph and let $G \leq \mathrm{Sym}(D)$ be a closed supergroup of $\mathrm{Aut}(D;E)$. Then:*

(i) $G \leq \mathrm{Aut}(D;\overline{E})$ *or* $G \geq \mathrm{Aut}(D;\overline{E})$

(ii) If $G < \mathrm{Aut}(D;\overline{E})$ *then* $G = \mathrm{Aut}(D;E), \langle - \rangle, \langle \mathrm{sw} \rangle$ *or* $\langle -, \mathrm{sw} \rangle$.

(iii) $(D;\overline{E})$ *is the random graph,* $(D;\overline{E})$ *is a Henson graph or* $(D;\overline{E})$ *is not homogeneous. In the last case* $\mathrm{Aut}(D;\overline{E})$ *is equal to* $\max\{\mathrm{Aut}(D;E), \langle - \rangle, \langle \mathrm{sw} \rangle, \langle -, \mathrm{sw} \rangle\}$ *and is a maximal-closed subgroup of* $\mathrm{Sym}(D)$.

The reducts of the random graph and the Henson graphs were classified by Thomas in [Tho91]: If $(D;\overline{E})$ is the random graph, the supergroups of $\mathrm{Aut}(D;\overline{E})$ are $\langle -_\Gamma \rangle$, $\langle \mathrm{sw}_\Gamma \rangle$, $\langle -_\Gamma, \mathrm{sw}_\Gamma \rangle$ and $\mathrm{Sym}(D)$, where $-_\Gamma \in \mathrm{Sym}(D)$ is a bijection which maps every edge to a non-edge and every non-edge to an edge and sw_Γ is a bijection which does the same but only for those edges adjacent to a particular vertex $a \in D$. A Henson graph has only two reducts, its automorphism group and the full symmetric group. As an immediate consequence we get the following corollary of Theorem 2.2.2:

Corollary 2.2.3. *Let $(D;E)$ be a Henson digraph. Then its lattice of reducts is a sublattice of the lattice in Figure 2.2.1. In particular, the lattice of reducts of $(D;E)$ is (isomorphic to) a sublattice of the lattice of reducts of the generic digraph ([Aga16]).*

2.2.2 Understanding the reducts

In this section, we establish several important lemmas that play prominent roles in the proof of Theorem 2.2.2. We omit most of the proofs of the lemmas. This is because they are relatively straightforward and are identical to the lemmas in [Aga16, Section 3]. Before we delve into the lemmas, we describe some terminology.

- Let $f, g : D \to D$ and $A \subseteq D$. We say f *behaves like* g *on* A if for all finite tuples $\bar{a} \in A$, $f(\bar{a})$ is isomorphic (as a finite digraph) to $g(\bar{a})$. If $A = D$, we simply say f *behaves like* g.

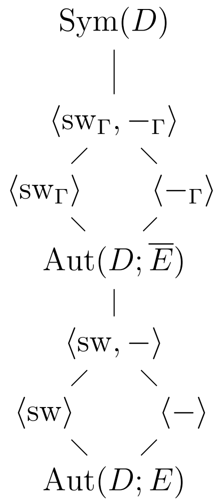

Figure 2.1: The (feasible) supergroups of $\mathrm{Aut}(D;E)$

- Let A, B be disjoint subsets of D. We say f *behaves like* sw *between* A *and* B if f switches the direction of all edges between A and B and preserves all non-edges between A and B.
- Let $A \subseteq D$. We let $\mathrm{sw}_A : D \to D$ denote a function that behaves like id on A and $D \setminus A$ and that behaves like sw between A and $D \setminus A$. Note that the existence of sw_A will depend on A and on $\mathcal{T}$.
- We overload the symbols $-$ and sw by letting them denote actions on finite tournaments. We say $\mathcal{T}$ *is closed under* $-$ if for every $T \in \mathcal{T}$, the tournament obtained from T by changing the direction of all its edges is in $\mathcal{T}$. We say $\mathcal{T}$ *is closed under* sw if for every $T \in \mathcal{T}$ and $t \in T$, the tournament obtained by changing the direction of those edges adjacent to t is in $\mathcal{T}$.

Remark 2.2.4. Note that if f and g are canonical and have the same behaviour, then f behaves like g. However the terminology does not match exactly as the notions of 'behaving like' is not only defined for canonical functions.

Lemma 2.2.5.

(i) $- : D \to D$ *exists if and only if* $\mathcal{T}$ *is closed under* $-$.

(ii) $\mathrm{sw} : D \to D$ *exists if and only if* $\mathcal{T}$ *is closed under* sw.

(iii) If sw *exists, then for all* $A \subseteq D$, sw_A *exists.*

(iv) $\langle - \rangle \supseteq \{f \in \mathrm{Sym}(D) : f$ *behaves like* $-\}$.

(v) $\langle \mathrm{sw} \rangle \supseteq \{f \in \mathrm{Sym}(D) :$ *there is* $A \subseteq D$ *such that* f *behaves like* $\mathrm{sw}_A\}$.

Proof.

(i) 'LHS ⇒ RHS': Suppose $-$ exists. To show $\mathcal{T}$ is closed under $-$, it suffices to show that if $T \notin \mathcal{T}$, then $-(T) \notin \mathcal{T}$. So suppose a finite tournament T is not in $\mathcal{T}$. Then T is embeddable in $(D; E)$. Then applying $-$ shows that $-(T)$ is embeddable in $(D; E)$, i.e. that $-(T) \notin \mathcal{T}$.

'RHS ⇒ LHS': To show $-$ exists, we need to show that $(D; E^*)$ is isomorphic to $(D; E)$. (Recall that $\phi^*(x, y) := \phi(y, x)$). By the uniqueness of Fraïssé limits, it suffices to show that $(D; E^*)$ is homogeneous and that $\mathrm{Age}(D; E^*) = \mathrm{Age}(D; E)$. That the ages are equal follows from the assumption that $\mathcal{T}$ is closed under $-$. That $(D; E^*)$ is homogeneous follows from the observation that for all $A, B \subseteq D$ and all $f : A \to B$, $f : (A; E|_A) \to (B; E|_B)$ is an isomorphism if and only if $f : (A; E^*|_A) \to (B; E^*|_B)$ is an isomorphism.

(ii) 'LHS ⇒ RHS': Apply the same argument as in (i) to prove this.

'RHS ⇒ LHS': Let $a \in D$, $X_{\mathrm{out}} = \{x \in D : E(a, x)\}$ and $X_{\mathrm{in}} = \{x \in D : E(x, a)\}$. Suppose we found an isomorphism $f : (X_{\mathrm{out}}; E) \to (X_{\mathrm{in}}; E)$. Then we can define sw as the function which maps a to a, maps elements of X_{out} using f and maps elements of X_{in} using f^{-1}. Thus to complete this proof, we need to prove that X_{out} and X_{in} are isomorphic digraphs. To do this, we will show that they are homogeneous and have the same age.

First we show that X_{out} is homogeneous. Note in advance that the same argument shows that X_{in} is homogeneous. Let $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in X_{\mathrm{out}}$ be isomorphic. Then $(a, a_1, \ldots, a_n)$ and $(a, b_1, \ldots, b_n)$ are isomorphic, so by homogeneity of $(D; E)$ there is $g \in \mathrm{Aut}(D; E)$ mapping $(a, a_1, \ldots, a_n)$ to $(a, b_1, \ldots, b_n)$. Since g fixes a, g fixes X_{out} setwise. Then the restriction of g to X_{out} is an automorphism of $(X_{\mathrm{out}}; E)$ mapping $(a_1, \ldots, a_n)$ to $(b_1, \ldots, b_n)$, as required.

Next we show that $\mathrm{Age}(X_{\mathrm{out}}) = \mathrm{Age}(X_{\mathrm{in}})$. Let A be a finite sub-digraph of X_{out}. Then let $A' = A \cup \{a\}$ and note that A' is an element of $\mathrm{Forb}(\mathcal{T})$. Now let A'' be the digraph obtained from A' by changing the direction of all the edges adjacent to a. Since $\mathcal{T}$ is closed under sw and $A' \in \mathrm{Forb}(\mathcal{T})$, A'' is also in $\mathrm{Forb}(\mathcal{T})$, so A'' is embeddable in $(D; E)$. By homogeneity, we may assume that the embedding maps $a \in A''$ to $a \in (D; E)$, so we have embedded A into X_{in}. Thus we have shown that $\mathrm{Age}(X_{\mathrm{out}}) \subseteq \mathrm{Age}(X_{\mathrm{in}})$. A symmetric argument shows that $\mathrm{Age}(X_{\mathrm{in}}) \subseteq \mathrm{Age}(X_{\mathrm{out}})$, so we are done.

(iii) Let $A \subseteq D$. Consider the digraph $(D; E')$ obtained from $(D; E)$ by changing the direction of the edges between A and $D \setminus A$ and leaving all other edges unchanged. If $(D; E')$ is embeddable in $(D; E)$, then sw_A exists as any embedding $(D; E') \to (D; E)$ has the desired property.

We will prove the contrapositive, so suppose sw_A does not exist. This implies that the digraph $(D; E')$ is not embeddable in $(D; E)$, which by Lemma 2.1.2 implies that $\mathrm{Age}(D; E') \nsubseteq \mathrm{Age}(D; E)$. This implies there exists $T \in \mathcal{T}$ which is embeddable in $(D; E')$; let g be such an embedding. Let $B = g^{-1}(g(T) \cap A)$, so B is a subset of T. Now consider the tournament T' obtained by applying the switch operation on T about every element of B. By choice of T and B, T' is isomorphic to $(g(T); E|_{g(T)})$. Hence T' is in the age of $(D; E)$ and so $T' \notin \mathcal{T}$. To summarise, we have $T \in \mathcal{T}, T' \notin \mathcal{T}$ and T' is obtained from T by switching. This means $\mathcal{T}$ is not closed under sw and so by (ii) sw does not exist, as required.

(iv)

(v) We omit these proofs for the reasons described at the start of this section.

□

Definition 2.2.6. Let G be a subgroup of $\mathrm{Sym}(D)$ and $n \in \mathbb{N}$. G is *n-transitive* if for all tuples $\bar{a}, \bar{b} \in D^n$ with distinct entries, there exists $g \in G$ such that $g(\bar{a}) = \bar{b}$. G is *n-homogeneous* if for all subsets $A, B \subseteq D$ of size n, there exists $g \in G$ such that $g(A) = B$.

Lemma 2.2.7. *Let $G \leq \mathrm{Sym}(D)$ be a closed supergroup of* $\mathrm{Aut}(D; E)$.

(i) If G is n-transitive for all $n \in \mathbb{N}$, then $G = \mathrm{Sym}(D)$.

(ii) If G is n-homogeneous for all $n \in \mathbb{N}$, then $G = \mathrm{Sym}(D)$.

(iii) Suppose that whenever $A \subseteq D$ is finite and has edges, there exists $g \in G$ such that $g(A)$ has less edges than in A. Then $G = \mathrm{Sym}(D)$.

(iv) Suppose that there exists a finite $A \subseteq D$ and $g \in G$ such that g behaves like id *on $D \backslash A$, g behaves like* id *between A and $D \backslash A$, and, g deletes at least one edge in A. Then, $G = \mathrm{Sym}(D)$.*

Notation 2.2.8. Let $a_1, \ldots, a_n, b_1, \ldots, b_n \in D$. We say $\bar{a}$ and $\bar{b}$ are isomorphic as graphs if $\overline{E}(a_i, a_j) \leftrightarrow \overline{E}(b_i, b_j)$ for all i, j.

Lemma 2.2.9. *Let $G \leq \mathrm{Sym}(D)$ be a closed supergroup of* $\mathrm{Aut}(D; E)$.

(i) *Suppose that whenever* $\bar{a}$ *and* $\bar{b}$ *are isomorphic as graphs, there exists* $g \in G$ *such that* $g(\bar{a}) = \bar{b}$. *Then* $G \geq \mathrm{Aut}(D; \overline{E})$.

(ii) *Suppose that for all* $A = \{a_1, \ldots, a_n\} \subseteq D$, *there exists* $g \in G$ *such that for all edges* (a_i, a_j) *in* A, $E(g(a_i), g(a_j))$ *if* $i < j$ *and* $E(g(a_j), g(a_i))$ *if* $i > j$. *(Intuitively, such a* g *is aligning the edges so they all point in the same direction.) Then,* $G \geq \mathrm{Aut}(D; \overline{E})$.

(iii) *Suppose that for all finite* $A \subseteq D$ *and all edges* $(a, a') \in A$ *there is* $g \in G$ *such that* g *changes the direction of* (a, a') *and behaves like* id *on all other edges and non-edges of* A. *Then* $G \geq \mathrm{Aut}(D; \overline{E})$.

(iv) *Suppose there is a finite* $A \subseteq D$ *and a* $g \in G$ *such that* g *behaves like* id *on* $D \backslash A$, g *behaves like* id *between* A *and* $D \backslash A$, *and* g *switches the direction of some edge in* A. *Then,* $G \geq \mathrm{Aut}(D; \overline{E})$.

Furthermore, in all of these cases we can also conclude that the underlying graph $(D; \overline{E})$ *is homogeneous.* □

2.2.3 Analysis of Canonical Functions

To help motivate the analysis we are about to undertake, we sketch a part of the proof of the main theorem. In the following, let G always denote a closed supergroup of $\mathrm{Aut}(D; E)$. One task will be to show that if $G > \mathrm{Aut}(D; E)$ then $G \geq \langle - \rangle$ or $G \geq \langle \mathrm{sw} \rangle$. Since $G > \mathrm{Aut}(D; E)$, G does not preserve the relation E, so there exist $g \in G$ and $c_1, c_2 \in D$ with $E(c_1, c_2)$ but $\neg E(g(c_1), g(c_2))$. Then by Lemma 2.1.5, we find a canonical function $f : (D; E, <, c_1, c_2) \to (D; E)$ that agrees with g on (c_1, c_2) and lies in the closure of G in the space of all unary functions. The behaviour of f will give us information about G. We only have to consider the behaviour of f on the 2-types, since $(D; E, <, c_1, c_2)$ has quantifier elimination and all relations are of arity ≤ 2. Therefore there are only finitely many possibilities for the behaviour of f, so we can check each case and show that G must contain $\langle - \rangle$ or $\langle \mathrm{sw} \rangle$.

Canonical functions from $(D; E, <)$ **to** $(D; E)$

We start our analysis of the behaviours with the simplest case, which is when no constants are added. This will simplify other proof steps later on.

- Let $\phi_1(x, y), \ldots, \phi_n(x, y)$ be formulas. We let $p_{\phi_1,\ldots,\phi_n}(x, y)$ denote the (partial) type determined by the formula $\phi_1(x, y) \wedge \ldots \wedge \phi_n(x, y)$.
- There are four 2-types in $(D; E)$: $p_=, p_E, p_{E^*}$ and p_N.

- There are seven 2-types in $(D; E, <)$: $p_{=}, p_{<,E}, p_{<,E^*}, p_{<,N}$, $p_{>,E}, p_{>,E^*}$ and $p_{>,N}$.

The following lemma contains a little 'trick' that proves useful during the analysis of the behaviours. Roughly, this lemma allows us to manipulate freely how finitely many elements are ordered, and its benefits will be seen shortly.

Lemma 2.2.10. *Let $\{a_1, \ldots, a_n\} \in (D; E, <)$ and let $\sigma \in S_n$. Then there exists $g \in \mathrm{Aut}(D; E)$ such that for all i, j, $E(a_i, a_j)$ if and only if $E(g(a_i), g(a_j))$, and for all i, j, $g(a_i) < g(a_j)$ if and only if $\sigma(i) < \sigma(j)$.*

Proof. Follows straightforwardly from the definition of the age of $(D; E, <)$ and the homogeneity of $(D; E)$. □

Lemma 2.2.11. *Let G be a closed supergroup of $\mathrm{Aut}(D; E)$, let $f \in \overline{G}$, and let f be canonical when considered as a function from $(D; E, <)$ to $(D; E)$.*

(i) If $f(p_{<,N}) = p_N, f(p_{<,E}) = p_{E^*}$ and $f(p_{<,E^*}) = p_E$, then $-$ exists and $- \in G$.

(ii) If $f(p_{<,N}) = p_N, f(p_{<,E}) = p_E$ and $f(p_{<,E^*}) = p_E$, then $(D; \overline{E})$ is a homogeneous graph and $G \geq \mathrm{Aut}(D; \overline{E})$.

(iii) If $f(p_{<,N}) = p_N, f(p_{<,E}) = p_{E^*}$ and $f(p_{<,E^*}) = p_{E^*}$, then $(D; \overline{E})$ is a homogeneous graph and $G \geq \mathrm{Aut}(D; \overline{E})$.

(iv) If $f(p_{<,N}) = p_E$ or $p_{E^*}, f(p_{<,E}) = p_N$ and $f(p_{<,E^*}) = p_N$, then $(D; \overline{E})$ is a homogeneous graph and $G \geq \mathrm{Aut}(D; \overline{E})$.

(v) If f has any other non-identity behaviour, then either we get a contradiction (i.e. that behaviour is not possible) or $G = \mathrm{Sym}(D)$.

Proof.

(i) By Lemma 2.2.5, to show $-$ exists, it suffices to show that if $T \notin \mathcal{T}$, then $-(T) \notin \mathcal{T}$. So let T be a finite tournament not in $\mathcal{T}$. This means T is embeddable in $(D; E)$; let $T' \subseteq (D; E)$ be isomorphic to T. Then the conditions in the lemma tell us that $f(T') \cong -(T)$, so $-(T)$ is embeddable in $(D; E)$, so $-(T) \notin \mathcal{T}$, as required.

Next we show $- \in G$. Since G is closed, it suffices to show that for all finite $\bar{a} \in D$ there exists $g \in G$ such that $g(\bar{a}) = -(\bar{a})$. So let $\bar{a} \in D$ be finite. By the conditions in the lemma, $f(\bar{a}) \cong -(\bar{a})$. By homogeneity, there exists $g_1 \in \mathrm{Aut}(D; E)$ mapping $f(\bar{a})$ to $-(\bar{a})$. Since $f \in \overline{G}$, there is $g_2 \in G$ such that $g_2(\bar{a}) = f(\bar{a})$. Letting $g = g_1 \circ g_2$ completes the argument.

(ii) We will use Lemma 2.2.9 (ii). Let $(a_1, \ldots, a_n) \in D$. By Lemma 2.2.10, there is $g_1 \in \mathrm{Aut}(D; E)$ such that $g(a_1) < g(a_2) < \ldots < g(a_n)$. Then, due to the conditions in the lemma, applying f aligns the edges of this tuple to point in the same direction. As $f \in \overline{G}$, there exists $g_2 \in G$ which agrees with f on $g_1(\bar{a})$. Letting $g = g_2 \circ g_1$ completes the argument.

Note: For the remaining arguments, we will no longer comment explicitly on the fact that $f \in \overline{G}$ implies that f can be imitated on a finite set by a function in G.

(iii) Use the same argument as (ii).

(iv) Let $\bar{a}$ be any tuple. Then apply f once to get $f(\bar{a})$. By Lemma 2.2.10, there is $g \in \mathrm{Aut}(D; E)$ such that $gf(\bar{a})$ is linearly ordered the same way as $\bar{a}$. Now apply f again. Observe that the behaviour of fgf on $\bar{a}$ matches the behaviour of the canonical function in (ii) or (iii). Thus, this case is reduced to one of those.

Terminology. In future, we use the phrase *applying f twice* to abbreviate the procedure of applying f, re-ordering the elements to match the ordering of the initial tuple, and applying f again.

(v) Case 1: $f(p_{<,N}) = p_N$. We are left with the behaviours where $f(p_{<,E}) = p_N$ or $f(p_{<,E^*}) = p_N$ (or both), as all the other possibilities have been dealt with above. Now for any finite $A \subseteq D$ that has edges, $f(A)$ has less edges than A does. So by Lemma 2.2.7 (iii), we conclude that $G = \mathrm{Sym}(D)$.

Case 2: $f(p_{<,N}) = p_E$
Case 2a: $f(p_{<,E}) = p_E$ and $f(p_{<,E^*}) = p_E$. For every all tuples $\bar{a}, \bar{b} \in D^n$ with pairwise distinct entries we have that $f(\bar{a}) \cong f(\bar{b}) \cong L_n$ (as digraphs), so G is n-transitive for all n and $G = \mathrm{Sym}(D)$ by Lemma 2.2.7 (i).

Case 2b: $f(p_{<,E}) = p_{E^*}$ and $f(p_{<,E^*}) = p_{E^*}$. Apply f twice and use the same argument as in Case 2a to show that $G = \mathrm{Sym}(D)$.

Case 2c: $f(p_{<,E}) = p_E$ and $f(p_{<,E^*}) = p_{E^*}$. We will show that this behaviour is not possible. Let $T \in \mathcal{T}$ be of minimal cardinality. Enumerate T as $T = (t_1, \ldots, t_n)$ so that we have an edge going from t_1 to t_2 (as opposed to t_2 to t_1). Now let $A = (a_1, \ldots, a_n)$ be the ordered digraph constructed as follows: Start with T, delete the edge (t_1, t_2), and add a linear order so that $a_1 < a_2$. As T was minimal, A can be embedded in $(D; E, <)$, so then $f(A) \subseteq (D; E)$. But by the construction of A, $f(A) \cong T$, so we have shown that T is embeddable in $(D; E)$. This contradicts that $T \in \mathcal{T}$.

Case 2d: $f(p_{<,E}) = p_{E^*}$ and $f(p_{<,E^*}) = p_E$. Applying f twice reduces to a case that is dual to Case 2c.

Case 2e: $f(p_{<,E}) = p_E$ and $f(p_{<,E^*}) = p_N$. Applying f twice reduces to Case 2a.

Case 2f: $f(p_{<,E}) = p_N$ and $f(p_{<,E^*}) = p_E$. Applying f twice reduces to Case 1.

Case 2g: $f(p_{<,E}) = p_{E^*}$ and $f(p_{<,E^*}) = p_N$. We will show that this behaviour is not possible. Let $T \in \mathcal{T}$ be of minimal cardinality. Observe that f^3 has the identity behaviour, so that $f^3(T) = T$. Now observe that $f^2(T)$ is a digraph that contains non-edges, so by the minimality of T, $f^2(T)$ can be embedded in $(D; E, <)$. But then applying f shows that $f(f^2(T))$ is embeddable in $(D; E)$, i.e. that $f^3(T) = T$ is embeddable in $(D; E)$. This contradicts that $T \in \mathcal{T}$.

Case 2h: $f(p_{<,E}) = p_N$ and $f(p_{<,E^*}) = p_{E^*}$. Using the same argument as in 2g shows that this case is not possible.

Case 3: $f(p_{<,N}) = p_{E^*}$. This case is is symmetric to Case 2.

□

Now we have seen an analysis, we provide more detailed intuition. Given some closed supergroup G of $\mathrm{Aut}(D; E)$, we want to know what functions it contains. Since G is closed, this amounts to knowing how G acts on finite tuples in D. But this is exactly the information a canonical function in $\overline{G}$ provides! For example, in (i) the canonical function tells us that G can behave like $-$ on any finite tuple, which implies that $G \geq \langle - \rangle$. The role of homogeneity is that it allows us to move between isomorphic tuples, so knowing how G acts on one tuple automatically tells us how G acts on all tuples isomorphic to that one tuple.

Canonical functions from $(D; E, <, \bar{c})$ to $(D; E)$

We now move on to the general situation where we have added constants $\bar{c} \in D$ to the structure. For convenience, we assume that $c_i < c_j$ for all $i < j$. Since $(D; E)$ is ω-categorical, $(D; E, \bar{c})$ is also ω-categorical, so the n-types of $(D; E, <, \bar{c})$ correspond to the orbits of $\mathrm{Aut}(D; E, <, \bar{c})$ acting on the set of n-tuples of D. For this reason, we often conflate the notion of types and orbits.

We need to describe the 2-types of $(D; E, <, \bar{c})$, and to do that we first need to describe the 1-types. There are two kinds of 1-types, i.e. two kinds of orbits. The first is a singleton, e.g. $\{c_1\}$. The other orbits are infinite and are determined by how their elements are related to the c_i. These infinite orbits are of the form $\{x \in D : \bigwedge_i (\phi_i(x, c_i) \wedge \psi_i(x, c_i))\}$, where $\phi_i \in \{<, >\}$ and $\psi_i \in \{E, E^*, N\}$.

Unlike in the case of the generic digraph, the substructures induced on these orbits will not necessarily be isomorphic to the original structure. For example, let

$\mathcal{T} = \{L_3\}$ and $\bar{c} = (c_1)$. Then consider the orbit $X = \{x \in D : x < c_1 \wedge E(x, c_1)\}$. If there was an edge, ab say, in X, then $\{c_1, a, b\}$ would be a copy of L_3. However, L_3 is forbidden. Thus, X contains no edges so in particular X is not isomorphic to $(D_{\mathcal{T}}; E_{\mathcal{T}}, <)$.

However, there are always orbits such that the substructures induced on them are isomorphic to the original structure. For example, regardless of $\mathcal{T}$, the orbit $X = \{x \in D : x < c_1 \wedge \bigwedge_i N(x, c_i)\}$ is isomorphic to $(D; E, <)$. These orbits form a central part of the argument so we give them a definition.

Definition 2.2.12. Let $\bar{c} \in D$ and $X \subseteq D$ be an orbit of $(D; E, <, \bar{c})$. We say X is *independent* if X is infinite and there are no edges between $\bar{c}$ and X.

The following lemma highlights the key feature of independent orbits that makes them useful.

Lemma 2.2.13. *Let X be an independent orbit of $(D; E, <, \bar{c})$.*

(i) Let $v \in D \backslash (X \cup \bar{c})$. Let $A = (a_0, \ldots, a_n)$ be a finite digraph in the age of $(D; E)$. Then there are $x_1, \ldots, x_n \in X$ such that $(a_0, a_1, \ldots, a_n) \cong (v, x_1, \ldots, x_n)$ as tuples in $(D; E, <, \bar{c})$.

(ii) The substructure induced on X is isomorphic to $(D; E)$.

Proof. Let k be the length of the tuple $\bar{c}$ and let x be any element of X. Consider the finite ordered digraph A' which is constructed as follows: start with A, add new vertices $c'_1, \ldots, c'_k$ and then add edges and an ordering so that we have $(a_0, c'_1, \ldots, c'_k) \cong (v, c_1, \ldots, c_k)$ and so that $(a_i, c'_1, \ldots, c'_k) \cong (x, c_1, \ldots, c_k)$ for all $i > 0$.

A' is embeddable in $(D; E, <)$ so let f be such an embedding. By composing with an automorphism of $(D; E, <)$ if necessary, we can assume that $f(c'_j) = c_j$ for $j = 1, \ldots, k$. Then letting $x_i = f(a_i)$ for $i = 1, \ldots, n$ completes the proof.

(ii) From (i), we know that the age of X equals the age of $(D; E)$, so it suffices to show that X is homogeneous. Let $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in X$ be isomorphic tuples, as ordered digraphs. Then $(c_1, \ldots, c_k, a_1, \ldots, a_n) \cong (c_1, \ldots, c_k, b_1, \ldots, b_n)$. By the homogeneity of $(D; E, <)$, there is $f \in \mathrm{Aut}(D; E, <)$ mapping $(c_1, \ldots, c_k, a_1, \ldots, a_n)$ to $(c_1, \ldots, c_k, b_1, \ldots, b_n)$. Since f fixes $\bar{c}$, f fixes X setwise, and so $f|_X$ is an automorphism of X mapping $\bar{a}$ to $\bar{b}$, as required. □

Notation 2.2.14. Let A, B be definable subsets of D and let $\phi_1(x, y), \ldots, \phi_n(x, y)$ be formulas. We let $p_{A,B,\phi_1,\ldots,\phi_n}(x, y)$ denote the (partial) type determined by the formula $x \in A \wedge y \in B \wedge \phi_1(x, y) \wedge \ldots \wedge \phi_n(x, y)$.

Using this notation, we can describe the 2-types of $(D; E, <, \bar{c})$. They are all of the form $p_{X,Y,\phi,\psi} = \{(a,b) \in D : a \in X, b \in Y, \phi(a,b) \text{ and } \psi(a,b)\}$, where X and Y are orbits, $\phi \in \{<, =, >\}$ and $\psi \in \{E, E^*, N\}$.

Our task now is to analyse the possibilities for $f(p_{X,Y,\phi,\psi})$, where f is a canonical function. It turns out that it is sufficient to study those cases where we assume X is an independent orbit. The first lemma deals with the situation when $X = Y$.

Lemma 2.2.15. *Let G be a closed supergroup of $\mathrm{Aut}(D; E)$, let $\bar{c} \in D$, let $f \in \overline{G}$, and let f be canonical when considered as a function from $(D; E, <, \bar{c})$ to $(D; E)$. Let $X \subseteq D$ be an independent orbit.*

(i) If $f(p_{X,X,<,N}) = p_N, f(p_{X,X,<,E}) = p_{E^*}$ and $f(p_{X,X,<,E^*}) = p_E$, then $-$ exists and $- \in G$.

(ii) If $f(p_{X,X,<,N}) = p_N, f(p_{X,X,<,E}) = p_E$ and $f(p_{X,X,<,E^*}) = p_E$, then $(D; \overline{E})$ is a homogeneous graph and $G \geq \mathrm{Aut}(D; \overline{E})$.

(iii) If $f(p_{X,X,<,N}) = p_N, f(p_{X,X,<,E}) = p_{E^*}$ and $f(p_{X,X,<,E^*}) = p_{E^*}$, then $(D; \overline{E})$ is a homogeneous graph and $G \geq \mathrm{Aut}(D; \overline{E})$.

(iv) If $f(p_{X,X,<,N}) = p_E$ or $p_{E^*}, f(p_{X,X,<,E}) = p_N$ and $f(p_{X,X,<,E^*}) = p_N$, then $(D; \overline{E})$ is a homogeneous graph and $G \geq \mathrm{Aut}(D; \overline{E})$.

(v) If f has any other non-identity behaviour, then either we get a contradiction or $G = \mathrm{Sym}(D)$.

Proof. Intuitively, since $X \cong (D; E)$, the canonical functions here provide us the same information as the canonical functions in Lemma 2.2.11, so we are done. More formally, one can copy the arguments from Lemma 2.2.11 and add minor adjustments as necessary. We do this for (i) as an example, and leave the rest to be checked by the reader.

First we show $-$ exists, so let T be a tournament not in $\mathcal{T}$. This means T is embeddable in $(D; E)$ and so, by Lemma 2.2.13, T is embeddable in X; let $T' \subseteq X$ be isomorphic to T. Then the conditions in the lemma tell us that $f(T') \cong -(T)$, so $-(T)$ is embeddable in $(D; E)$, so $-(T) \notin \mathcal{T}$, as required.

Next we show $- \in G$. Since G is closed, it suffices to show that for all finite $\bar{a} \in D$ there exists $g \in G$ such that $g(\bar{a}) = -(\bar{a})$. So let $\bar{a} \in D$ be finite. By Lemma 2.2.13, there is $\bar{a}' \in X$ isomorphic to $\bar{a}$. By the conditions in the lemma, $f(\bar{a}') \cong -(\bar{a})$. By homogeneity, there exist $g_1 \in \mathrm{Aut}(D; E)$ mapping $\bar{a}$ to $\bar{a}'$ and $g_2 \in \mathrm{Aut}(D; E)$ mapping $f(\bar{a}')$ to $-(\bar{a})$. Since $f \in \overline{G}$, there is $g_3 \in G$ such that $g_3(\bar{a}) = f(\bar{a})$. Letting $g = g_2 \circ g_3 \circ g_1$ completes the argument. $\square$

Next we look at the behaviour of f between an independent orbit X and any other orbit Y. This task is split depending on how X and Y relate with regard to the linear order.

Observation 2.2.16. There are two ways that two infinite orbits X and Y of $\mathrm{Aut}(D; E, <, \bar{c})$ can relate to each other with respect to the linear order $<$:

- All of the elements of one orbit, X say, are smaller than all of the elements of Y. This is abbreviated by '$X < Y$'.
- X and Y are interdense: $\forall x < x' \in X, \exists y \in Y$ such that $x < y < x'$ and vice versa.

The next lemma contains the analysis for the case where $X < Y$ or $X > Y$.

Lemma 2.2.17. *Let G be a closed supergroup of $\mathrm{Aut}(D; E)$, let $\bar{c} \in D$, let $f \in \overline{G}$, and let f be canonical when considered as a function from $(D; E, <, \bar{c})$ to $(D; E)$. Let $X \subseteq D$ be an independent orbit on which f behaves like id and let Y be an infinite orbit such that $X < Y$ or $X > Y$.*

(i) If $f(p_{X,Y,N}) = p_N, f(p_{X,Y,E}) = p_{E^*}$ and $f(p_{X,Y,E^*}) = p_E$, then sw exists and $\mathrm{sw} \in G$.

(ii) If $f(p_{X,Y,N}) = p_N, f(p_{X,Y,E}) = p_E$ and $f(p_{X,Y,E^*}) = p_E$, then $(D; \overline{E})$ is a homogeneous graph and $G \geq \mathrm{Aut}(D; \overline{E})$.

(iii) If $f(p_{X,Y,N}) = p_N, f(p_{X,Y,E}) = p_{E^*}$ and $f(p_{X,Y,E^*}) = p_{E^*}$, then $(D; \overline{E})$ is a homogeneous graph and $G \geq \mathrm{Aut}(D; \overline{E})$.

(iv) If $f(p_{X,Y,N}) = p_E$ or $p_{E^*}, f(p_{X,Y,E}) = p_N$ and $f(p_{X,Y,E^*}) = p_N$, then $(D; \overline{E})$ is a homogeneous graph and $G \geq \mathrm{Aut}(D; \overline{E})$.

(v) If f has any other non-identity behaviour, then either we get a contradiction or $G = \mathrm{Sym}(D)$.

Remark: We do not need to include $<$ or $>$ in the subscripts of the type because it is automatically determined by how X and Y are related to $\bar{c}$.

Proof. Assume that $X < Y$. The proof for the case $Y < X$ is symmetric. Let $y_0 \in Y$ be any element.

(i) The proof is analogous to that of Case (i) in Lemma 2.2.11 and is left as an exercise for the reader. Note that Lemma 2.2.13 is needed for this.

(ii) Using Lemma 2.2.9 (ii), it suffices to show that for any finite $A \subseteq D$ we can align all its edges by using functions in G. Let $A = \{a_1, \ldots, a_n\}$. First we map a_{n-1} to y_0 and the rest of A into X (possible by Lemma 2.2.13), and then apply f. Then we repeat but with a_{n-2} instead of a_{n-1}, then with a_{n-3}, and so on until a_1.

(iii) Same as (ii).

(iv) The same argument as in (ii) works but with a slight modification: the intuition is that whenever f was applied to some tuple $(a_0, \dots, a_n)$ in those proofs, here we apply f twice to get the same effect. To be more precise, the modification is as follows. Let $(a_0, \dots, a_n) \in D$. We first map this to an isomorphic copy $(y_0, x_1, \dots, x_n)$ for some $x_i \in X$. Then apply f. Then again we map this to an isomorphic tuple $(y_0, x'_1, \dots, x'_n)$ for some $x'_i \in X$. Then apply f a second time. The total effect of this procedure is the same as what the canonical function did in Case (ii) or (iii). Thus we have reduced this case to either (ii) or (iii).

Remark: For the rest of this proof, we will use the phrase "by applying f twice" to refer to the procedure described above.

(v) Case 1: $f(p_{<,N}) = p_N$. By a similar argument as in Case 1 of Lemma 2.2.11, $G = \mathrm{Sym}(D)$. Note that Lemma 2.2.13 is needed for this.

Case 2: $f(p_{X,Y,N}) = p_E$
Case 2a: $f(p_{X,Y,E}) = p_{E^*}$. We will show that this behaviour is not possible, in a similar fashion to Case 2c of Lemma 2.2.11. Let $T \in \mathcal{T}$ be of minimal size and enumerate T as $(t_0, t_1, \dots, t_n)$ so that t_0 has at least one edge going into it. Construct a digraph $A = (a_0, a_1, \dots, a_n)$ as follows: start with A being equal to T and then replace edges into a_0 with non-edges, replace edges out of a_0 with incoming edges, and leave all other edges of A the same.

Since T was minimal, $A \in \mathrm{Forb}(\mathcal{T})$ so A can be embedded in D. Furthermore, by Lemma 2.2.13 there are $x_i \in X$ such that $(a_0, a_1, \dots, a_n) \cong (y_0, x_1, \dots, x_n)$. Now apply f. By construction of A, $f(y_0, x_1, \dots, x_n) \cong (t_0, \dots, t_n)$. Thus, T is embeddable in D, contradicting $T \in \mathcal{T}$.

Case 2b: $f(p_{X,Y,E^*}) = p_{E^*}$. Use the same argument as Case 2a to show this is not possible.

Now there are only three behaviours left to analyse.

Case 2c: $f(p_{X,Y,E}) = p_E$ and $f(p_{X,Y,E^*}) = p_E$. We will show that $G = \mathrm{Sym}(D)$, by showing that every tuple $(a_0, \dots, a_{n-1}) \in D^n$ can be mapped to L_n using functions in G. We do this by induction on n. The base case $n = 1$ is trivial so let $n > 1$. By the inductive hypothesis we can assume that $(a_1, \dots, a_{n-1}) \cong L_{n-1}$. By Lemma 2.2.13 we map $\bar{a}$ to an isomorphic tuple $(y_0, x_1, \dots, x_{n-1})$ for some $x_i \in X$. Then applying f maps the tuple to a copy of L_n, as required.

Case 2d: $f(p_{X,Y,E}) = p_E$ and $f(p_{X,Y,E^*}) = p_N$. By applying f twice this case is reduced to Case 2c.

Case 2e: $f(p_{X,Y,E}) = p_N$ and $f(p_{X,Y,E^*}) = p_E$. By applying f twice this case is reduced to Case 1.

Case 3: $f(p_{X,Y,N}) = p_{E^*}$. This case is symmetric to Case 2.

□

In the proof above we only had to study the behaviour of f on $\{y_0\} \cup X$ for some element $y_0 \in Y$. The key property which allowed this is Lemma 2.2.13. This feature allows us to use these arguments with minimal modification to prove the subsequent lemmas.

The next lemma deals with the case where X and Y are interdense.

Lemma 2.2.18. *Let G be a closed supergroup of* $\mathrm{Aut}(D; E)$*, let $\bar{c} \in D$, let $f \in \overline{G}$, and let f be canonical when considered as a function from $(D; E, <, \bar{c})$ to $(D; E)$. Let $X \subseteq D$ be an independent orbit on which f behaves like* id *and let Y be an infinite orbit such that X and Y are interdense. Then at least one of the following holds.*

(i) f preserves all the edges and non-edges between X and Y.

(ii) f switches the direction of every edge between X and Y and preserves non-edges between X and Y. In this case sw *exists.*

(iii) $G \geq \mathrm{Aut}(D; \overline{E})$ and $(D; \overline{E})$ is a homogeneous graph.

(iv) $G = \mathrm{Sym}(D)$.

Proof. First just consider the increasing tuples from X to Y. With the same arguments as in Lemma 2.2.17 one can show that either

(a) $f(p_{X,Y,N,<}) = p_N, f(p_{X,Y,E,<}) = p_E$ and $f(p_{X,Y,E^*,<}) = p_{E*}$,

(b) $f(p_{X,Y,N,<}) = p_N, f(p_{X,Y,E,<}) = p_{E^*}$ and $f(p_{X,Y,E^*,<}) = p_E$,

(c) $G \geq \mathrm{Aut}(D; \overline{E})$ and $(D; \overline{E})$ is a homogeneous graph, or

(d) $G = \mathrm{Sym}(D)$.

If (c) or (d) is true we are done, so assume (a) or (b) is true. Similarly we can assume that f behaves like id or sw between decreasing tuples from X to Y. If the behaviours between increasing and decreasing tuples are the same, then (i) or (ii) will be true so we we would be done. Thus it remains to check what happens if f behaves like id on decreasing tuples and sw on increasing tuples. Explicitly we are assuming that:

$f(p_{X,Y,N,<}) = p_N, f(p_{X,Y,E,<}) = p_{E^*}, f(p_{X,Y,E^*,<}) = p_E$, and
$f(p_{X,Y,N,>}) = p_N, f(p_{X,Y,E,>}) = p_E, f(p_{X,Y,E^*,>}) = p_{E^*}$.

Let $\bar{a} = (a_0, a_1, \ldots, a_n) \in \mathrm{Forb}(\mathcal{T})$ be a digraph with at least one edge (a_0, a_1). We can consider $\bar{a}$ as an ordered digraph by setting $a_i < a_j \leftrightarrow i < j$. Then by Lemma 2.2.13 $\bar{a}$ has an isomorphic copy $\bar{b} = (b_0, b_1, \ldots, b_n)$ such that $b_1 \in Y$ and $b_i \in X$ for $i \neq 1$. All the edges of $\bar{b}$ are preserved under f, except for the edge (b_0, b_1) whose direction is switched. By Lemma 2.2.9, we conclude that $G \geq \mathrm{Aut}(D; \overline{E})$ and $(D; \overline{E})$ is a homogeneous graph. □

We end by looking at how f can behave between the constants $\bar{c}$ and the rest of the structure.

Lemma 2.2.19. *Let G be a closed supergroup of* $\mathrm{Aut}(D; E)$*, let $(c_1, \ldots, c_n) \in D$, let $f \in \overline{G}$, and let f be canonical when considered as a function from $(D; E, <, \bar{c})$ to $(D; E)$. Suppose that f behaves like* id *on $D^- := D \backslash \{c_1, \ldots, c_n\}$. Then at least one of the following holds.*

(i) For all $i, 1 \leq i \leq n$, f behaves like id *or like* sw *between c_i and D^-.*

(ii) $G \geq \mathrm{Aut}(D; \overline{E})$ and $(D; \overline{E})$ is a homogeneous graph.

(iii) $G = \mathrm{Sym}(D)$.

Proof. Fix some i, $1 \leq i \leq n$. Let $X_{\text{out}} = \{x \in D : x < c_1 \wedge E(c_i, x) \wedge \bigwedge_{j \neq i} N(c_j, x)\}$. Define X_{in} and X_N similarly, with $E(c_i, x)$ replaced with $E(x, c_i)$ and $N(x, c_i)$ respectively. Then for any finite digraph $(a_0, a_1, \ldots, a_n)$, there exist $x_1, \ldots, x_n \in X_{\text{out}} \cup X_{\text{in}} \cup X_N$ such that $(a_0, a_1, \ldots, a_n) \cong (c_i, x_1, \ldots, x_n)$. So by replicating the proof of Lemma 2.2.17 we can assume that f behaves like id or sw between c_i and $X_{\text{out}} \cup X_{\text{in}} \cup X_N$. Without loss of generality, we assume f behaves like id, because we can compose f with sw_{c_i} if necessary.

If f behaves like id between c_i and D^- we are done, so suppose there is an infinite orbit X such that f does not behave like id between c_i and X. Assume that there are edges from c_i into X - the arguments for the other two cases are similar.

Let A be a finite digraph in the age of D which contains an edge, ab say. Then observe that there is an embedding of A into D such that a is mapped to c_i, b is mapped into X, and the rest of A is mapped into $X_{\text{out}} \cup X_{\text{in}} \cup X_N$. Then applying f changes exactly the one edge (a, b) in A, so by Lemma 2.2.7 or Lemma 2.2.9 as appropriate, we are done. □

2.2.4 Proof of the classification

We would like to recall the statement of our main result that we are going to prove with the help of observations we made on canonical functions in the last section.

Theorem 2.2.2. *Let $(D; E)$ be a Henson digraph and let $G \leq \mathrm{Sym}(D)$ be a closed supergroup of* $\mathrm{Aut}(D; E)$. *Then:*

(i) $G \leq \mathrm{Aut}(D; \overline{E})$ *or* $G \geq \mathrm{Aut}(D; \overline{E})$

(ii) If $G < \mathrm{Aut}(D; \overline{E})$ *then* $G = \mathrm{Aut}(D; E), \langle - \rangle, \langle \mathrm{sw} \rangle$ *or* $\langle -, \mathrm{sw} \rangle$.

(iii) $(D; \overline{E})$ *is the random graph,* $(D; \overline{E})$ *is a Henson graph or* $(D; \overline{E})$ *is not homogeneous. In the last case* $\mathrm{Aut}(D; \overline{E})$ *is equal to* $\max\{\mathrm{Aut}(D; E), \langle - \rangle, \langle \mathrm{sw} \rangle, \langle -, \mathrm{sw} \rangle\}$ *and is a maximal-closed subgroup of* $\mathrm{Sym}(D)$.

Proof. (i) Suppose for contradiction that $G \ngeq \mathrm{Aut}(D; \overline{E})$ and $G \nleq \mathrm{Aut}(D; \overline{E})$. Because of the second assumption G violates the relation $\overline{E}$. By Lemma 2.1.5 this can be witnessed by a canonical function. Precisely, this means there are $c_1, c_2 \in D$ and $f \in \overline{G}$ such that $f : (D; E, <, c_1, c_2) \to (D; E)$ is a canonical function, $\overline{E}(c_1, c_2)$ and $N(f(c_1), f(c_2))$.

Now let X be an independent orbit of $(D; E, <, c_1, c_2)$.

Claim 1. We may assume that f behaves like id on X.

By Lemma 2.2.15 we know that f behaves like id or $-$ on X, otherwise G would contain $\mathrm{Aut}(D; \overline{E})$. If f behaves like $-$ on X, then we continue by replacing f by $- \circ f$.

Claim 2. We may assume that f behaves like id between X and every other infinite orbit Y.

Let Y be another infinite orbit. By the Lemmas 2.2.17 and 2.2.18, f behaves like id or sw between X and Y, as otherwise G would contain $\mathrm{Aut}(D; \overline{E})$. If f behaves like sw between them, then we simply replace f by $\mathrm{sw}_Y \circ f$. Note that one needs to check sw_Y is a legitimate function, but this has been done in Lemma 2.2.5 (iii).

Claim 3. We may assume that f behaves like id on every infinite orbit and between every pair of infinite orbits.

Suppose not, so there are infinite orbits Y_1 and Y_2 (possibly the same) and there are distinct $y_1, y_2 \in Y_1, Y_2$, respectively, such that $(y_1, y_2) \ncong f(y_1, y_2)$. Now for any finite digraph $(a_1, a_2, \ldots, a_n) \in \mathrm{Forb}(\mathcal{T})$ with $(y_1, y_2) \cong (a_1, a_2)$, we can find $x_3, \ldots, x_n \in X$ such that $(y_1, y_2, x_3, \ldots, x_n) \cong (a_1, \ldots, a_n)$ (This statement can be verified analogously to Lemma 2.2.13). Then f has the effect of only changing what happens between y_1 and y_2, since we know f behaves like id on X and between X and all other infinite orbits. In short, given any finite digraph, we can use f to change what happens between exactly two of the vertices of the digraph.

There are three options. If f creates an edge from a non-edge, then we we can use f to introduce a forbidden tournament, which gives a contradiction. If f deletes the edge or changes the direction of the edge, then by Lemma 2.2.7 or Lemma 2.2.9, as appropriate, we get that $G \geq \mathrm{Aut}(D; \overline{E})$.

Claim 4. We may assume that f behaves like id between $\{c_1, c_2\}$ and the union of all infinite orbits.

The follows immediately from Lemma 2.2.19, composing with sw_{c_i} if necessary.

Conclusion. We can assume that f behaves everywhere like the identity, except on (c_1, c_2), where it maps an edge to a non-edge. But then we get that $G = \mathrm{Sym}(D)$ by Lemma 2.2.7, completing the proof of (i).

The proof of (ii) follows exactly the same series of claims as in part (i) but with minor adjustments to how one starts and concludes. We go through one case as an example, leaving the rest to the reader. We will show that if $\mathrm{Aut}(D; E) < G \leq \mathrm{Aut}(D; \overline{E})$, then $G \geq \langle - \rangle$ or $G \geq \langle \mathrm{sw} \rangle$ (if they exist). So suppose $\mathrm{Aut}(D; E) < G \leq \mathrm{Aut}(D; \overline{E})$. Then G preserves non-edges but not the relation E. By Lemma 2.1.5, there is an edge $(c_1, c_2) \in E$ and a canonical function $f : (D; E, <, c_1, c_2) \to (D; E)$ which changes the direction of the edge (c_1, c_2). Suppose for contradiction that $G \not\geq \langle - \rangle$ and $G \not\geq \langle \mathrm{sw} \rangle$.

Let X be an independent orbit. By Lemma 2.2.15, f must behave like id on X and then by Lemma 2.2.17 and Lemma 2.2.18, f must behave like id between X and all other infinite orbits. By repeating the argument of Claim 3 above, f must behave like id on the union of infinite orbits and so by Lemma 2.2.19 f must behave like id between the constants and the union of infinite orbits. Now we are in the situation of Lemma 2.2.9 (iv), so we conclude that $G \geq \mathrm{Aut}(D; \overline{E})$, so $G \geq \langle - \rangle, \langle \mathrm{sw} \rangle$.

For (iii), note that $(D; \overline{E})$ embeds every finite edgeless graph and is connected (Lemma 2.1.2 (ii)). Hence, if $(D; \overline{E})$ is a homogeneous graph then $(D; \overline{E})$ has to be the random graph or a Henson graph, by the classification of countable homogeneous graphs ([LW80]).

Thus assume that $(D; \overline{E})$ is not a homogeneous graph. Let G' be equal to $\max\{\mathrm{Aut}(D; E), \langle - \rangle, \langle \mathrm{sw} \rangle, \langle -, \mathrm{sw} \rangle\}$. Now let G be a closed group such that $G' < G \leq \mathrm{Sym}(D)$. We want to show that $G = \mathrm{Sym}(D)$. By Lemma 2.1.5, there are $\bar{c} \in D$ and a canonical $f : (D; E, <, \bar{c}) \to (D; E)$ such that f cannot be imitated by any function of G' on $\bar{c}$. To be precise, we mean that for all $g \in G'$, $g(\bar{c}) \neq f(\bar{c})$.

Now we continue as in (i), proving that we may assume f behaves like id on the union of all infinite orbits and like id between $\bar{c}$ and the union of infinite orbits.

In doing so, we may have composed f with $-$ or sw_A for some A. Since $-$ and sw_A are elements of G', these compositions do not change the fact that f could not be imitated by G' on $\bar{c}$. In particular, $f(\bar{c}) \ncong \bar{c}$. Hence, we are in the situation of either Lemma 2.2.7 (iv) or Lemma 2.2.9 (iv). Thus, either $G = \mathrm{Sym}(D)$ and we are done, or $(D; \overline{E})$ is a homogeneous graph - contradiction.

We have shown that there are no closed groups in between G' and $\mathrm{Sym}(D)$. Since $\mathrm{Aut}(D; \overline{E})$ contains G' and is proper subgroup of $\mathrm{Sym}(D)$, we must conclude that $G' = \mathrm{Aut}(D; \overline{E})$, as required. □

2.3 2^ω pairwise non-isomorphic maximal-closed subgroups of $\mathrm{Sym}(\omega)$

Definition 2.3.1. Let G be a closed subgroup of $\mathrm{Sym}(\omega)$. We say that G is *maximal-closed* if $G \neq \mathrm{Sym}(\omega)$ and there are no closed groups G' such that $G < G' < \mathrm{Sym}(\omega)$.

We construct 2^ω pairwise non-isomorphic maximal-closed subgroups of $\mathrm{Sym}(\omega)$ by modifying Henson's construction of 2^ω pairwise non-isomorphic homogeneous countable digraphs and taking their automorphism groups. The modification is needed to ensure that the groups are maximal. A short argument will show that the automorphism groups are pairwise non-conjugate. The groups are even pairwise non-isomorphic, since by a result of Rubin [Rub94] automorphism groups of Henson digraphs are conjugate if and only if they are isomorphic as abstract groups.

Henson's construction in [Hen72] centres on finding an infinite anti-chain, with respect to embeddability, of finite tournaments.

Definition 2.3.2. Let $n \in \mathbb{N} \backslash \{0\}$. I_n denotes the n-element tournament obtained from the linear order L_n by changing the direction of the edges $(i, i+1)$ for $i = 1, \ldots, n-1$ and of the edge $(1, n)$.

By counting 3-cycles, Henson showed that $\{I_n : n \geq 6\}$ is an anti-chain. It is a short exercise to show that the 3-cycles in I_n are $(1,3,n)$,$(1,4,n)$,. . .,$(1,n-2,n)$,$(3,2,1)$,$(4,3,2)$,. . .,$(n,n-1,n-2)$. In particular, observe that I_n has at most two vertices through which there are more than four 3-cycles, namely the vertices 1 and n; this observation is useful in our modification.

The automorphism groups of the Henson digraphs constructed by forbidding any subset of these I_n's are not maximal: $\langle - \rangle$ and the automorphism group of the random graph are closed supergroups. By forbidding a few extra tournaments, however, we can ensure that the automorphism groups are maximal.

In a digraph, a *source*, respectively *sink*, is a vertex which only has outgoing, respectively incoming, edges adjacent to it. Then let T be a finite tournament that is not embeddable in I_n for any n and that contains a source but no sink. Such a T can be found, for example, by ensuring there are at least three vertices through which there are more than four 3-cycles.

Let $k = |T|$. Let $\mathcal{T} = \{T' : |T| = k+1, T \text{ is embeddable in } T'\}$. Then for $A \subseteq \mathbb{N}\backslash\{1,\ldots,k+1\}$, let $\mathcal{T}_A = \mathcal{T} \cup \{I_n : n \in A\}$. Then let D_A be the Henson digraph whose set of forbidden tournaments is $\mathcal{T}_A$. The automorphism groups of these D_A is the set of groups we want.

Theorem 2.3.3. $\{\mathrm{Aut}(D_A) : A \subseteq \mathbb{N}\backslash\{1,\ldots,k+1\}\}$ *is a set of* 2^ω *maximal-closed subgroups of* $\mathrm{Sym}(\omega)$ *which are pairwise non-isomorphic as abstract groups.*

Proof. We prove the Theorem in eight steps:

Claim 1. For all $A \subseteq \mathbb{N}\backslash\{1,\ldots,k+1\}, \mathcal{T}_A$ is not closed under $-$.

Let T' be obtained as follows: Change the direction of all the edges of T and then add a new vertex t which is a sink. Since T has no sinks, T can not be embedded into T', hence $T' \notin \mathcal{T}_A$. Now consider $-(T')$. By construction, T is embeddable in $-(T')$, so $-(T') \in \mathcal{T}_A$. Thus $\mathcal{T}_A$ is not preserved under $-$.

Claim 2. For all $A \subseteq \mathbb{N}\backslash\{1,\ldots,k+1\}, \mathcal{T}_A$ is not closed under sw.

Let T' be obtained as follows: Change the source s in T to a sink, and then add a new vertex which will be a sink of T'. Since T has no sinks, T can not be embedded into T', hence $T' \notin \mathcal{T}_A$. Now consider switching T' about s, to obtain T''. By construction, T is embeddable in T'', so $T'' \in \mathcal{T}_A$. Thus $\mathcal{T}_A$ is not preserved under sw.

Claim 3. For all $A \subseteq \mathbb{N}\backslash\{1,\ldots,k+1\}, (D_A, \overline{E})$ is not a Henson graph nor the random graph.

Finite linear orders do not embed any element of $\mathcal{T}_A$, thus are embeddable in D_A. Removing the direction of the edges in a finite linear order gives a complete graph, so $(D_A, \overline{E})$ is not K_n-free for any n, so $(D_A, \overline{E})$ is not a Henson graph.

Now let $U \subseteq D_A$ be isomorphic to T - this is possible as T has not been forbidden. Then there is no vertex $x \in D$ such that for all $u \in U$, $E(x,u) \vee E(u,x)$, because all tournaments containing T are forbidden. Hence $(D_A, \overline{E})$ does not satisfy the extension property of the random graph and so is not isomorphic to the random graph.

Claim 4. For all $A \subseteq \mathbb{N}\backslash\{1, \ldots, k+1\}$, $\mathrm{Aut}(D_A)$ is a maximal-closed subgroup of $\mathrm{Sym}(\mathbb{N})$.

This follows from the classification Theorem 2.2.2 and the previous three claims.

Claim 5. Let $A = \mathbb{N}\backslash\{1, \ldots, k+1\}$. Then $\mathcal{T}_A$ is an anti-chain with respect to embeddability.

Let $T_1, T_2 \in \mathcal{T}_A$ and suppose for contradiction that T_1 is embeddable in T_2. All elements of $\mathcal{T}_A$ have size at least $k+1$ and $|T_2|$ must be bigger than $|T_1|$, so $|T_2| \geq k+2$. Hence, $T_2 \notin \mathcal{T}$, so $T_2 = I_n$ for some $n \in A$. By Henson's arguments, T_1 cannot equal I_m for any $m \in A$. Thus $T_1 \in \mathcal{T}$, which implies that T is embeddable in I_n, contradicting our choice for T.

Claim 6. If $A, B \subseteq \mathbb{N}\backslash\{1, \ldots, k+1\}$ are not equal, then $D_A \not\cong D_B$.

Suppose, without loss of generality, that there is some n in A but not in B. Then I_n is not embeddable in D_A. To prove the claim, it suffices to show that I_n *is* embeddable in D_B. Suppose for contradiction that it is not. Hence, $I_n \notin \mathrm{Forb}(\mathcal{T}_B)$ which means that I_n embeds an element of $\mathcal{T}_B$. But this implies that $\mathcal{T}_{B\cup\{n\}}$ is not an anti-chain, contrary to Claim 5.

Claim 7. If $A, B \subseteq \mathbb{N}\backslash\{1, \ldots, k+1\}$ are not equal, then $\mathrm{Aut}(D_A)$ and $\mathrm{Aut}(D_B)$ are not conjugate.

We prove the contrapositive so suppose $\mathrm{Aut}(D_A)$ and $\mathrm{Aut}(D_B)$ are conjugate. Let $f : D_A \to D_B$ be a bijection witnessing this, so that $\mathrm{Aut}(D_A) = f^{-1}\mathrm{Aut}(D_B)f$. In particular this means that f maps orbits of $\mathrm{Aut}(D_A)$ to orbits of $\mathrm{Aut}(D_B)$, i.e., that f is canonical. f cannot map edges to non-edges or vice-versa, because non-edges are symmetric and edges are not. This leaves only two options: f behaves like id or f behaves like $-$. We can rule out the latter option because we know from (the proof of) Claim 1 that $\mathcal{T}$ is not closed under $-$. Hence, f behaves like id, which means f is an isomorphism, so by Claim 6 we conclude that $A = B$.

Claim 8. If $A, B \subseteq \mathbb{N}\backslash\{1, \ldots, k+1\}$ are not equal, then $\mathrm{Aut}(D_A)$ and $\mathrm{Aut}(D_B)$ are not isomorphic as pure groups.

This follows from Claim 7 and Rubin's reconstruction results [Rub94].

Together, Claim 4 and Claim 8 prove the theorem. □

Chapter 3

CSPs over the random partial order

Reasoning about temporal knowledge is a common task in various areas of computer science, for example Artificial Intelligence, Scheduling, Computational Linguistics and Operations Research. In many application temporal constraints are expressed as collections of relations between time points or time intervals. A typical computational problem is then to determine whether such a collection is satisfiable or not.

A lot of research in this area concerns only linear models of time. In particular there exists a complete classification of all satisfiability problems for linear temporal constraints in [BK09]. However, it has been observed many times that more complex time models are helpful, for instance in the analysis of concurrent and distributed systems or certain planning domains. A possible generalizations is to model time by partial orders (e.g. in [Lam86], [Ang89]).

Some cases of the arising satisfiability problems have already been studied in [BJ03]. We will give a complete classification in this chapter. Speaking more formally, let Φ be a set of quantifier-free formulas in the language consisting of a binary relation symbol $\leq$. Then Poset-SAT(Φ) is the following computational problem

Poset-SAT(Φ):
INSTANCE: A finite set of variables W and a formula of the form $\phi_1 \wedge \phi_2 \wedge \cdots \wedge \phi_n$, where each ϕ_i is obtained by taking a formula from Φ and substituting with variables form W.
QUESTION: Is there a partial order, satisfying $\phi_1 \wedge \phi_2 \wedge \cdots \wedge \phi_n$?

We are going to give a full complexity classification of problems of the form

Poset-SAT(Φ). In particular we are going to show that every such problem is NP-complete or solvable in polynomial time.

The class of partial orders is an amalgamation class, whose Fraïssé limit is called the *random partial order* $\mathbb{P} = (P; \leq)$. By the discussion in Section 1.5, the Poset-SAT(Φ) problems are equal to the constraint satisfaction problems of reducts of $\mathbb{P}$. We are going to study these reducts with the model theoretic and universal algebraic methods outlined in Sections 1.5 and 1.6 of the introduction.

A helpful result has already been established in the form of the classification of first-order reducts of the random partial order in [PPP+14]. In a fist step we extend this analysis to closed transformation monoids, identifying all possible model-complete cores. Informally this preclassification then implies that we can identify three types of Poset-SAT problems: (1) trivial ones (i.e., if there is a solution, there is a constant solution), (2) problems that can be reduced to the problems studied in [BK09] and (3) CSPs on templates that are already model-complete cores themselves.

So we only have to study problems in the third class. Here we will first identify a list NP-complete relations by reducing know finite NP-hard CSPs to them. By studying the polymorphism clones and their equational structure if those NP-hard relations are violated, we can show that the CSPs of all other reducts are in P.

This chapter has the following structure: Section 3.1 contains the preclassification, where we identify the model-complete cores of the reducts of $\mathbb{P}$. This is followed by the actual complexity analysis in Section 3.2 to Section 3.5 using the universal algebraic approach. In Section 3.6 we summarize our results. It turns out that a CSP of a model-complete reduct of $\mathbb{P}$ is NP-complete if one of four relations is pp-definable in it, and tractable otherwise (see Theorem 3.6.3). This complexity dichotomy corresponds to an algebraic dichotomy as in Conjecture 1.6.10; we even obtain slightly stronger equations in the tractable cases, see Corollary 3.6.5.

We fix some standard terminology and notation. By $\mathbb{P} = (P; \leq)$ we denote the *random partial order* (short: *random poset*), the Fraïssé limit of all finite partial order. In general we let $\leq$ always denote a partial order relation, i.e. a binary relation that is reflexive, antisymmetric and transitive. Let $<$ be the corresponding strict order defined by $x \leq y \wedge x \neq y$. And let $x \perp y$ denote the incomparability relation defined by $\neg(x \leq y) \wedge \neg(y \leq x)$. Sometimes we will write $x < y_1 \cdots y_n$ for the conjunction of the formulas $x < y_i$ for all $1 \leq i \leq n$. Similarly we will use $x \perp y_1 \cdots y_n$ if $x \perp y_i$ holds for all $1 \leq i \leq n$.

3.1 A pre-classification by model-complete cores

In this section we start our analysis of reducts of the random partial order $\mathbb{P} = (P; \leq)$. Our aim is to determine the model-complete core for every reduct $\mathbb{A}$ of $\mathbb{P}$. Therefore we want to discuss the endomorphism monoids $\mathrm{End}(\mathbb{A})$ containing $\mathrm{Aut}(\mathbb{P})$. Part of the work was already done in [PPP+14] where all the automorphism groups $\mathrm{Aut}(\mathbb{A}) \geq \mathrm{Aut}(\mathbb{P})$ were determined. Several parts of our proof are very similar to the group case; at that points we are going to directly refer to the corresponding proofs of [PPP+14].

As in Chapter 2, we are going to use canonical functions as a tool for the analysis, for which we need the following Ramsey result: If we look at the class of all structures $(A; \leq, \prec)$, where $\prec$ is a linear order that extends $\leq$, it is an amalgamation class and has the Ramsey property by [PTJW85]. Therefore its Fraïssé limit $(P; \leq, \prec)$ is an ordered homogeneous Ramsey structure.

We start by giving a description of the first-order reducts of $\mathbb{P}$. If we turn the partial order $\mathbb{P}$ upside-down, then the obtained partial order is again isomorphic to $\mathbb{P}$. Hence there exists a bijection $\updownarrow: P \to P$ such that for all $x, y \in P$ we have $x < y$ if and only if $\updownarrow(y) < \updownarrow(x)$. By the homogeneity of $\mathbb{P}$ it is easy to see that the closed monoid generated by $\updownarrow$ and $\mathrm{Aut}(\mathbb{P})$ does not depend on the choice of the function $\updownarrow$.

The class of all finite structures $(X; \leq, F)$, where $(X; \leq)$ is a partial order and F is upwards closed set is an amalgamation class. Its Fraïssé limit is isomorphic to $\mathbb{P}$ with an additional unary relation F. We say F is a *random filter* on $\mathbb{P}$. Note that F and $I = P \setminus F$ are both isomorphic to the random partial order. Furthermore for every pair $x \in I$ and $y \in F$ either $x < y$ or $x \perp y$ holds.

We define a new order relation $<_F$ on by setting $x <_F y$ if and only if

- $x, y \in F$ and $x < y$ or,
- $x, y \in I$ and $x < y$ or,
- $x \in I$, $y \in F$ and $x \perp y$.

It is shown in [PPP+14] that the resulting structure $(P; <_F)$ is isomorphic to $(P, <)$. We fix a map $\circlearrowright_F: P \to P$ that maps $(P; <)$ isomorphically to $(P, <_F)$. By the homogeneity of $\mathbb{P}$ one can see that the smallest closed monoid generated by $\circlearrowright$ and $\mathrm{Aut}(\mathbb{P})$ does not depend on the choice of the random filter F. We fix a random filter F and set $\circlearrowright := \circlearrowright_F$.

We recall that for $B \subseteq \mathrm{Sym}(P)$, $\langle B \rangle$ denotes the smallest closed subgroup of $\mathrm{Sym}(P)$ containing B. For brevity, when it is clear we are discussing supergroups of $\mathrm{Aut}(\mathbb{P})$, we may abuse notation and write $\langle B \rangle$ to mean $\langle B \cup \mathrm{Aut}(\mathbb{P}) \rangle$.

Theorem 3.1.1 (Theorem 1 from [PPP+14]). *Let* $\mathbb{A}$ *be a reduct of* $\mathbb{P}$. *Then* $\mathrm{Aut}(\mathbb{A})$ *is equal to one of the five groups* $\mathrm{Aut}(\mathbb{P})$, $\langle \updownarrow \rangle$, $\langle \circlearrowright \rangle$, $\langle \updownarrow, \circlearrowright \rangle$ *or* $\mathrm{Sym}(P)$. □

We are going to show the following extension of Theorem 3.1.1:

Proposition 3.1.2. *Let* $\mathbb{A}$ *be a reduct of* $\mathbb{P}$. *Then for* $\mathrm{End}(\mathbb{A})$ *at least one of the following cases applies:*

1. $\mathrm{End}(\mathbb{A})$ *contains a constant function,*
2. $\mathrm{End}(\mathbb{A})$ *contains a function* $g_<$ *that preserves* $<$ *and maps* P *onto a chain,*
3. $\mathrm{End}(\mathbb{A})$ *contains a function* $g_\perp$ *that preserves* $\perp$ *and maps* P *onto an antichain,*
4. *The automorphism group* $\mathrm{Aut}(\mathbb{A})$ *is dense in* $\mathrm{End}(\mathbb{A})$, *i.e.* $\mathbb{A}$ *is a model-complete core. So by the classification in Theorem 3.1.1,* $\mathrm{End}(\mathbb{A})$ *is the topological closure of* $\mathrm{Aut}(\mathbb{P})$, $\langle \updownarrow \rangle$, $\langle \circlearrowright \rangle$, $\langle \updownarrow, \circlearrowright \rangle$ *or* $\mathrm{Sym}(P)$ *in the space of all functions* P^P.

Before we start with the proof of Proposition 3.1.2 we want to point out its relevance for the complexity analysis of the CSPs on reducts of $\mathbb{P}$.

Constraint satisfaction problems on reducts of $(\mathbb{Q}; <)$ are called *temporal satisfaction problems*. The CSPs of reducts of a countable set with a predicate for equality $(\omega; =)$ are called *equality satisfaction problems*. For both classes a full complexity dichotomy is known, see [BK09] and [BK08]. As a corollary of Proposition 3.1.2 we get the following pre-classification of CSPs reducing all the cases where $\mathbb{A}$ is not a model-complete core to temporal or equality satisfaction problems:

Corollary 3.1.3. *Let* $\mathbb{A}$ *be a reduct of* $\mathbb{P}$. *Then one of the following holds*

1. $\mathrm{CSP}(\mathbb{A})$ *is trivial;*
2. *The model-complete core of* $\mathbb{A}$ *is a reduct of* $(\omega; =)$,
   so $\mathrm{CSP}(\mathbb{A})$ *is equal to an equality satisfaction problem;*
3. *The model-complete core of* $\mathbb{A}$ *is a reduct of* $(\mathbb{Q}; <)$,
   so $\mathrm{CSP}(\mathbb{A})$ *is equal to a temporal satisfaction problem;*
4. $\mathrm{End}(\mathbb{A})$ *is the topological closure of* $\mathrm{Aut}(\mathbb{P})$, $\langle \updownarrow \rangle$, $\langle \circlearrowright \rangle$ *or* $\langle \updownarrow, \circlearrowright \rangle$.

Proof. If there is a constant function in $\mathrm{End}(\mathbb{A})$, then $\mathrm{CSP}(\mathbb{A})$ accepts every instance, so we are in the first case. So let $\mathrm{End}(\mathbb{A})$ contain no constants.

Assume that $g_\perp \in \mathrm{End}(\mathbb{A})$. Since $g_\perp$ preserves $\perp$, the image of $(P; \perp)$ under $g_\perp$ is isomorphic to a countable antichain, or in other word, a countable set ω with a predicate for inequality $(\omega; \neq)$. Thus, for every reduct of $\mathbb{A}$ the image $g_\perp(\mathbb{A})$ can be seen as a reduct of $(\omega; \neq)$. Now clearly $\mathbb{A}$ and $g_\perp(\mathbb{A})$ are homomorphically equivalent. It is shown in [BK08] that every reduct of $(\omega; \neq)$ without constant endomorphisms is a model-complete core. So we are in the second case.

Now assume that $g_< \in \mathrm{End}(\mathbb{A})$ but $g_\perp \notin \mathrm{End}(\mathbb{A})$. Since $g_<$ preserves $<$ and is a chain, the image of $(P; <)$ under $g_<$ has to be isomorphic to the rational order $(\mathbb{Q}; <)$. Thus for every reduct of $\mathbb{A}$ the image $g_<(\mathbb{A})$ can be seen as a reduct of $\mathbb{Q}$. Now clearly $\mathbb{A}$ and $g_<(\mathbb{A})$ are homomorphically equivalent. It is shown in [BK09] that the model-complete core of every reduct of $(\mathbb{Q}, <)$ is either trivial, definable in $(\omega, \neq)$ or the reduct itself. So we are in the third case.

Note that also in the case where $\mathrm{End}(\mathbb{A}) = \overline{\mathrm{Sym}(P)}$ we have that $e_\perp \in \mathrm{End}(\mathbb{A})$. So by Proposition 3.1.2 we are only left with the cases where $\mathrm{End}(\mathbb{A})$ is the topological closure of $\mathrm{Aut}(\mathbb{P})$, $\langle \updownarrow \rangle$, $\langle \circlearrowright \rangle$ or $\langle \updownarrow, \circlearrowright \rangle$. $\square$

Let us define the following relations on P:

$$\begin{aligned}
\mathrm{Betw}(x,y,z) :=& (x < y \wedge y < z) \vee (z < y \wedge y < x). \\
\mathrm{Cycl}(x,y,z) :=& (x < y \wedge y < z) \vee (y < z \wedge z < x) \vee (z < x \wedge x < y) \vee \\
& (x < y \wedge z \perp xy) \vee (y < z \wedge x \perp yz) \vee (z < x \wedge y \perp zx). \\
\mathrm{Par}(x,y,z) :=& (x \perp yz \wedge y \perp z) \vee (x < yz \wedge y \perp z) \vee (x > yz \wedge y \perp z) \\
\mathrm{Sep}(x,y,z,t) :=& (\mathrm{Cycl}(x,y,z) \wedge \mathrm{Cycl}(y,z,t) \wedge \mathrm{Cycl}(x,y,t) \wedge \mathrm{Cycl}(x,z,t)) \vee \\
& (\mathrm{Cycl}(z,y,x) \wedge \mathrm{Cycl}(t,z,y) \wedge \mathrm{Cycl}(t,y,x) \wedge \mathrm{Cycl}(t,z,x)).
\end{aligned}$$

In Lemma 3.1.5 we are going to give a description of the monoids $\overline{\langle \updownarrow \rangle}$, $\overline{\langle \circlearrowright \rangle}$ and $\overline{\langle \updownarrow, \circlearrowright \rangle}$ as endomorphism monoids with the help of the above relations. We remark that Cycl and Par describes the orbits of triples under $\langle \circlearrowright \rangle$ and Sep describes the orbit of a linearly ordered 4-tuple under $\langle \updownarrow, \circlearrowright \rangle$.

Lemma 3.1.4. *The incomparability relation* $\perp$ *is pp-definable in* $(P; <, \mathrm{Cycl})$ *and* Par *is pp-definable in* $(P; \mathrm{Cycl})$.

Proof. To prove the first part of the lemma, let

$$\begin{aligned}
\psi(x,y,a,b,c,d) :=\ & x < a < c \wedge x < b < d \wedge y < c \wedge y < d \wedge \mathrm{Cycl}(x,a,y) \\
& \wedge \mathrm{Cycl}(x,b,y) \wedge \mathrm{Cycl}(y,c,b) \wedge \mathrm{Cycl}(y,d,a) \wedge \mathrm{Cycl}(b,d,c) \wedge \mathrm{Cycl}(a,c,d).
\end{aligned}$$

We claim that $x \perp y$ is equivalent to $\exists a,b,c,d\ \psi(x,y,a,b,c,d)$. It is not hard to verify that $x \perp y$ implies $\exists a,b,c,d\ \psi(x,y,a,b,c,d)$. For the other direction note that $\psi(x,y,a,b,c,d)$ implies that $x \neq y$ because $\mathrm{Cycl}(x,a,y)$ is part of the conjunction ψ.

Let us assume that $x < y$ and $\psi(x, y, a, b, c, d)$ holds for some elements $a, b, c, d \in P$. Then $\mathrm{Cycl}(x, a, y)$ implies that $a < y$, symmetrically we have $b < y$. Since $y < c, d$ we have that $a < d$ and $b < c$. Then $\mathrm{Cycl}(b, d, c)$ implies $d < c$ and $\mathrm{Cycl}(a, c, d)$ implies $c < d$, which is a contradiction.

Now assume that $y < x$ and $\psi(x, y, a, b, c, d)$ holds for some elements $a, b, c, d \in P$. Then we have $y < a, b$ by the transitivity of the order. Then $\mathrm{Cycl}(y, c, b)$ implies $c < b$ and $\mathrm{Cycl}(y, d, a)$ implies $d < a$. But this leads to the contradiction $a < c < b$ and $b < d < a$.

For the second part of the lemma let $s, t \in P$ be two elements with $s < t$. Then the set $X = \{x \in P : s < x < t\}$ is pp-definable in $(P; \mathrm{Cycl}, s, t)$ by the formula $\phi(x) := \mathrm{Cycl}(s, x, t)$. By a back-and-forth argument one can show the two structures $(X; \leq)$ and $(P; \leq)$ are isomorphic. The order relation, restricted to X is also pp-definable in $(P; \mathrm{Cycl}, s, t)$ by the equivalence

$$y <_{|X} z \leftrightarrow \phi(x) \wedge \phi(y) \wedge \mathrm{Cycl}(y, z, t).$$

Since $\perp$ is pp-definable in $(P; <, \mathrm{Cycl})$, we have that its restriction to X has a pp-definition in $(P; \mathrm{Cycl}, s, t)$. Therefore also the relation $R = \{(x, y, z) \in X^3 : x \perp y \wedge x \perp z \wedge z \perp y\}$ is pp-definable in $(P; \mathrm{Cycl}, s, t)$. Let $\phi(s, t, u, v, w)$ be a primitive positive formula defining R.

We claim that $\exists x, y\, \phi(x, y, u, v, w)$ is equivalent to $(u, v, w) \in \mathrm{Par}$. Let $(u, v, w) \in \mathrm{Par}$. The relation Par describes the orbit of a 3-element antichain under the action of $\langle \circlearrowright \rangle \subseteq \mathrm{End}(P; \mathrm{Cycl})$. So we can assume that (u, v, w) is a 3-antichain, otherwise we take an image under a suitable function form $\langle \circlearrowright \rangle$. Now let us take elements $s < t$ such that $s < uvw$ and $uvw < t$. Then clearly $\psi(s, t, u, v, w)$ has to hold.

Conversely let (s, t, u, v, w) be a tuple such that $\psi(s, t, u, v, w)$ holds. We can assume that $s < t$ (otherwise we take the image of (s, t, u, v, w) under a suitable function in $\langle \circlearrowright \rangle$). By what we proved above, (u, v, w) is antichain, hence it satisfies Par. □

Lemma 3.1.5.

1. $\mathrm{End}(P; <, \perp) = \overline{\mathrm{Aut}(\mathbb{P})}$
2. $\mathrm{End}(P; \mathrm{Betw}, \perp) = \overline{\langle \updownarrow \rangle}$
3. $\mathrm{End}(P; \mathrm{Cycl}) = \overline{\langle \circlearrowright \rangle}$
4. $\mathrm{End}(P; \mathrm{Sep}) = \overline{\langle \updownarrow, \circlearrowright \rangle}$

Proof.

1. Clearly $\mathrm{Aut}(\mathbb{P}) \subseteq \mathrm{End}(P; <, \perp)$. For the other inclusion let $f \in \mathrm{End}(P; <, \perp)$. Let $A \subseteq P$ be an arbitrary finite set. The restriction of f to a finite subset $A \subseteq P$ is an isomorphism between posets. By the homogeneity of $\mathbb{P}$ there is an automorphism $\alpha \in \mathrm{Aut}(\mathbb{P})$ such that $f \restriction A = \alpha \restriction A$.

2. Since $\updownarrow$ preserves Betw and $\perp$, we know that $\overline{\langle \updownarrow \rangle} \subseteq \mathrm{End}(P; \mathrm{Betw}, \perp)$ holds. For the opposite inclusion let $f \in \mathrm{End}(P; \mathrm{Betw}, \perp)$. If f preserves $<$, then $f \in \mathrm{End}(P; <, \perp)$ and we are done. Otherwise there is a pair of elements $c_1 < c_2$ with $f(c_1) > f(c_2)$. Let $d_1 < d_2$ be an other pair of points in P. Then there are $a_1, a_2 \in P$ such that $c_1 < c_2 < a_1 < a_2$ and $d_1 < d_2 < a_1 < a_2$. Since f preserves Betw, $f(a_1) > f(a_2)$ holds and hence also $f(d_1) > f(d_2)$. So f inverts the order, while preserving $\perp$. Therefore $\updownarrow \circ f \in \mathrm{End}(P; <, \perp)$. We conclude that $f \in \overline{\langle \updownarrow \rangle}$.

3. It is easy to see that $\overline{\langle \circlearrowright \rangle} \subseteq \mathrm{End}(P; \mathrm{Cycl})$. So let $f \in \mathrm{End}(P; \mathrm{Cycl})$. Clearly f is injective and preserves also the relation $\mathrm{Cycl}'(x, y, z) := \mathrm{Cycl}(y, x, z)$. By Lemma 3.1.4, f also preserves the relation Par. Furthermore $\overline{\langle \circlearrowright \rangle}$ is 2-transitive: This can be verified by the fact that for every two elements of P, we can find a $\alpha \in \mathrm{Aut}(\mathbb{P})$ that map one element to the random filter F and the other element to $P \setminus F$. So also $\mathrm{End}(P; \mathrm{Cycl})$ is 2-transitive. It follows that $\mathrm{End}(P; \mathrm{Cycl})$ also preserves the negation of Cycl. In other words, f is a self-embedding of $(P; \mathrm{Cycl})$. So, when restricted to a finite $A \subset P$, f is a partial isomorphism. By the results in [PPPS13] we know that $(P; \mathrm{Cycl})$ is a homogeneous structure. Hence for every finite $A \subset P$ we find an automorphism $\alpha \in \mathrm{Aut}(P; \mathrm{Cycl}) = \langle \circlearrowright \rangle$ such that $f \restriction A = \alpha \restriction A$.

4. Let $f \in \mathrm{End}(P; \mathrm{Sep})$. We claim that either f or $\updownarrow \circ f$ preserves Cycl. If we can prove our claim we are done by (3). First of all note that $\mathrm{Sep}(x, y, z, u)$ implies $\mathrm{Cycl}(x, y, z) \leftrightarrow \mathrm{Cycl}(y, z, u)$.

   Without loss of generality let there be a elements $x, y, z \in P$ with $\mathrm{Cycl}(x, y, z)$ and $\mathrm{Cycl}(f(x), f(y), f(z))$, otherwise we look at $\updownarrow \circ f$ instead of f. Let (r, s, t) be arbitrary tuple satisfying Cycl.

   We can always find elements $a < b < c$ in P that are incomparable with all entries of (x, y, z) and (r, s, t). Further we can choose elements $u, v \in P$ that are incomparable with (a, b, c) such that $z < u < v$ and $\mathrm{Sep}(x, y, z, u) \wedge \mathrm{Sep}(y, z, u, v)$ holds. This can be done by a case distinction and is left to the reader. By construction we have

   $$\mathrm{Sep}(x, y, z, u) \wedge \mathrm{Sep}(y, z, u, v) \wedge \mathrm{Sep}(z, u, v, a) \wedge \mathrm{Sep}(u, v, a, b) \wedge \mathrm{Sep}(v, a, b, c).$$

   So we have that $(f(x), f(y), f(z)) \in \mathrm{Cycl}$ if and only if $(f(a), f(b), f(c)) \in \mathrm{Cycl}$. Repeating the same argument for (r, s, t) gives us that $(f(r), f(s), f(t)) \in \mathrm{Cycl}$. So f preserves Cycl.

□

Recall that we obtain an ordered homogeneous Ramsey structure $(P; \leq, \prec)$ by taking the Fraïssé limit of the class of finite structures $(A; \leq, \prec)$, where $(A; \leq)$ is a partial order on A and $\prec$ an extension of $<$ to a total order. By Corollary 1.7.3 the following holds:

Lemma 3.1.6. *Let $f : P \to P$ and $c_1, \ldots, c_n \in P$ be any points. Then there exists a function $g : P \to P$ such that*

1. $g \in \overline{\langle \mathrm{Aut}(\mathbb{P}) \cup \{f\} \rangle}$.
2. $g(c_i) = f(c_i)$ *for* $i = 1, \ldots n$.
3. *Regarded as a function from $(P; \leq, \prec, \bar{c})$ to $(P; \leq)$, g is a canonical function.*

□

Let $\mathbb{A}$ be a reduct of $\mathbb{P}$. We are going to study all feasible behaviors of a canonical function $f : (P; \leq, \prec, \bar{c}) \to (P; \leq)$ when $f \in \mathrm{End}(\mathbb{A})$. Note that the behaviour of such f only depends on the behaviour on the 2-types because $(P; \leq, \prec, \bar{c})$ is homogeneous and its signature contains at most 2-ary relation symbols. Since there are only finitely many 2-types, the study of all possible behaviors of such canonical functions is a combinatorial problem. We introduce the following notation:

Notation 3.1.7. Let A, B be definable subsets of $\mathbb{P}$ and let $\phi_1(x, y), \ldots, \phi_n(x, y)$ be formulas. We let $p_{A,B,\phi_1,\ldots,\phi_n}(x, y)$ denote the (partial) type determined by the formula $x \in A \wedge y \in B \wedge \phi_1(x, y) \wedge \ldots \wedge \phi_n(x, y)$. Using this notation, we can describe the 2-types of $(P; \leq, \prec, \bar{c})$. They are all of the form $p_{X,Y,\phi,\psi} = \{(a, b) \in P^2 : a \in X, b \in Y, \phi(a, b)$ and $\psi(a, b)\}$, where X and Y are 1-types, $\phi \in \{=, <, >, \perp\}$ and $\psi \in \{=, \prec, \succ\}$.

Let X, Y be two distinct infinite 1-types of $(P; \leq, \prec, \bar{c})$. We write $X \overset{\perp}{\underset{<}{}} Y$ if there are pairs $(x, y), (x', y') \in X \times Y$ with $x < y$ and $x' \perp y'$.

When it is convenient for us we will abuse notation and write $\bar{c}$ to describe the set containing all entries of the tuple $\bar{c}$.

Observation 3.1.8. The structure $(P; \leq, \prec, \bar{c})$ is a homogeneous structure. If X is an 1-type of $(P; \leq, \prec, \bar{c})$ with infinite elements, then $(X; \leq, \prec)$ is isomorphic to $(P; \leq, \prec)$ itself. This can be seen by a back-and-forth argument. Similarly, if X and Y are 1-types of $(P; \leq, \prec, \bar{c})$ with infinite elements such that $X \overset{\perp}{\underset{<}{}} Y$ holds, then $X \cup Y$ is isomorphic to $(P; \leq)$ with X being a random filter. If we define $X \leq Y \leftrightarrow \exists (x, y) \in X \times Y\ (x \leq y)$ we get a partial order on the 1-types of $(P; \leq, \prec, \bar{c})$ (cf. Lemma 18 of [PPP+14]). But note that the 1-types of $(P; \leq, \prec, \bar{c})$ are not necessarily linearly ordered by $\prec$: There can be infinite 1-types X, Y and $(x, y), (x', y') \in X \times Y$ with $x \prec y$, $x \perp y$ and $y' \prec x'$, $x' \perp y'$.

In the following lemmas let $\mathbb{A}$ be always be a reduct of $\mathbb{P}$ and let $f \in \mathrm{End}(\mathbb{A})$ be a canonical function from $(P; \leq, \prec, \bar{c})$ to $(P; \leq)$.

Lemma 3.1.9. *Let X be a 1-type of $(P; \leq, \prec, \bar{c})$ with infinite elements. Then f behaves like id or $\updownarrow$ on X, otherwise* $\mathrm{End}(\mathbb{A})$ *contains a constant function, $g_<$ or $g_\perp$.*

Proof. Note that $(X; \leq, \prec)$ is isomorphic to $(P; \leq, \prec)$. Then we can prove the statement with the same arguments as in Lemma 8 of [PPP+14]. □

Lemma 3.1.10. *Let X, Y two infinite 1-types of $(P; \leq, \prec, \bar{c})$ with $X \underset{<}{\overset{\perp}{-}} Y$. Assume f behaves like id on X. Then f behaves like id or $\circlearrowright_X$ on $X \cup Y$, otherwise* $\mathrm{End}(\mathbb{A})$ *contains a constant function, $g_<$ or $g_\perp$.*

Proof. Assume that f does not contains a constant function, $g_<$ or $g_\perp$. Note that the union of X and Y is isomorphic to $\mathbb{P}$ and X is a random filter of $X \cup Y$. By following the arguments of Lemma 22 in [PPP+14] one can show that we only have the two possibilities that

1. $f(p_{X,Y,<}) = p_<$ and $f(p_{X,Y,\perp,\prec}) = p_\perp$ or
2. $f(p_{X,Y,<}) = p_\perp$ and $f(p_{X,Y,\perp,\prec}) = p_>$.

By Lemma 3.1.9 we may assume that f behaves like *id* or $\updownarrow$ on Y. But if f behaves like $\updownarrow$ on Y, the image of $y_1, y_2 \in Y$ and $x \in X$ with $x \prec y_1 < y_2$, $x \perp y_1$ and $x < y_2$ would be a non partially ordered set. So if the type $p_{X,Y,\perp,\succ}$ is empty, f behaves like *id* or $\circlearrowright_X$ on $X \cup Y$ and we are done.

If $p_{X,Y,\perp,\succ}$ is not empty, there are $x \in X$ and $y \in Y$ with $x \succ y$ and $x \perp y$. We claim that in this case $f(p_{X,Y,\perp,\succ}) = f(p_{X,Y,\perp,\prec})$. We only prove this claim for (1), the proof for (2) is the same.

Assume that $f(p_{X,Y,\perp,\succ}) = p_<$. Then let $x' \in X$ be an element such that $y \prec x'$ and $x < x'$ and $y \perp x'$. The fact that such an element exists can be verified by checking that the extension of $\{x, y\} \cup \bar{c}$ by such an element x' still lies in the age of $(P; \leq, \prec, \bar{c})$. By our assumption we then have $f(x) < f(x') < f(y)$, which contradicts to $f(x) \perp f(y)$.

Now assume that $f(p_{X,Y,\perp,\succ}) = p_>$. Then let $x' \in X$ be such that $x \prec y \prec x'$ and $x < y$ and $x' \perp xy$. Again the fact that x' exists can be verified by the homogeneity of $(P; \leq, \prec, \bar{c})$. Then $f(x) < f(y) < f(x')$, which contradicts to $f(x') \perp f(x')$. □

Lemma 3.1.11. *Either f behaves like id or $\updownarrow$ on every single 1-type or* $\mathrm{End}(\mathbb{A})$ *contains a constant function, $g_<$ or $g_\perp$.*

Proof. For every two infinite orbits $X < Y$ there is a infinite orbit Z with $X \overset{\perp}{<} Z$ and $Z \overset{\perp}{<} Y$. For every two infinite orbits $X \perp Y$ there is an infinite orbit Z with $X < Z$ and $Y < Z$. So this statement holds by Lemma 3.1.10. (cf Lemma 23 of[PPP$^+$14]) $\square$

Lemma 3.1.12. *Assume* End($\mathbb{A}$) *does not contains constant functions,* $g_<$ *or* $g_\perp$*. Then there is a* $g \in \overline{\langle \circlearrowright, \updownarrow \rangle} \cap \mathrm{End}(\mathbb{A})$ *such that* $g \circ f$ *is canonical from* $(P; \leq, \prec, \bar{c})$ *to* $(P; \leq)$ *and behaves like id on every set* $(P \setminus \bar{c}) \cup \{c\}$*, with* $c \in \bar{c}$*.*

Proof. By Lemma 3.1.11, f behaves like id or $\updownarrow$ on every infinite orbit. Without loss of generality we can assume that the first case holds, otherwise consider $\updownarrow \circ f$.

Let $X \overset{\perp}{<} Y$, $Y \overset{\perp}{<} Z$ and $X \overset{\perp}{<} Z$ or $X < Z$. If f behaves like id on $X \cup Y$ and $Y \cup Z$ it also has to behave like id on $X \cup Z$; otherwise the image of a triple $(x, y, z) \in X \times Y \times Z$ with $x < y < z$ would not be partially ordered. Let $X < Z$, $Y < Z$ and $X \perp Z$. Again, if f behaves like id on $X \cup Y$ and $Y \cup Z$ it also has to behave like id on $X \cup Z$, otherwise we get a contradiction.

By Lemma 3.1.10 f either behaves like id or like $\circlearrowright_X$ on the union two orbits $X \overset{\perp}{<} Y$. In the second case $\circlearrowright \in \mathrm{End}(\mathbb{A})$. The set $A = \{x \in P : y < x \vee y \perp x \text{ for all } y \in f(Y)\}$ is a union of orbits of $\mathrm{Aut}(P; \leq, \prec, \bar{c})$ and a random filter of P. So $\circlearrowright_A \circ f$ is canonical and behaves like id on $X \cup Y$. Repeating this step finitely many times gives us a function $g \in \langle \circlearrowright \rangle$ such that $g \circ f$ behaves like id on the union of infinite orbits, by the observations in the paragraph above.

It is only left to show that $g \circ f$ behaves like id between a given constant c in $\bar{c}$ and an infinite orbit X. Assume for example that $c < X$ and $g \circ f(p_{c,X,<}) = p_\perp$. Let $A \subseteq P$ with $a \in A$. By homogeneity of $\mathbb{P}$ we find an automorphism of $\mathbb{P}$ that maps a to c and all points that are greater than a to X. If we then apply $g \circ f$ and repeat this process at most $|A|$-times we can map A to an antichain. Thus $g_\perp \in \mathrm{End}(\mathbb{A})$ which contradicts to our assumption.

Similarly all other cases where $g \circ f$ does not behave like id between c and X contradict our assumptions. We leave the proof to the reader. Hence $g \circ f$ behaves like id everywhere except on $\bar{c}$. $\square$

Now we are ready to prove the main result of the section.

Proof of Proposition 3.1.2. Let $\mathbb{A}$ be a reduct of $\mathbb{P}$ such that End($\mathbb{A}$) does not contains constant functions, $g_<$ or $g_\perp$. We show that then End($\mathbb{A}$) is equal to $\overline{\mathrm{Aut}(\mathbb{P})}$, $\overline{\langle \updownarrow \rangle}$, $\overline{\langle \circlearrowright \rangle}$ or $\overline{\langle \updownarrow, \circlearrowright \rangle}$.

First assume that End($\mathbb{A}$) contains a non injective function. This can be witnessed by constants $c_1 \neq c_2$ and a function $f \in \mathrm{End}(\mathbb{A})$ with $f(c_1) = f(c_2)$ that is canonical as function $f : (P; \leq, \prec, c_1, c_2) \to (P; \leq)$. By Lemma 3.1.12 we can assume that f behaves like id everywhere except from c_1, c_2. But this is not possible, since there is a point in $a \in P$ with $a \perp c_1$ but $\neg(a \perp c_2)$. Since $f(c_1) = f(c_2)$ either

$<$ or $\perp$ is violated, which contradicts to f behaving like id everywhere except on $\{c_1, c_2\}$. So from now on let End($\mathbb{A}$) only contain injective functions.

Assume End($\mathbb{A}$) violates Sep. This can also be witnessed by a canonical function $f : (P; \leq, \prec, \bar{c}) \to (P; \leq)$ such that $\bar{c} \in$ Sep but $f(\bar{c}) \notin$ Sep. By Lemma 3.1.12 we can assume that f behaves like id on every set $(P \setminus \bar{c}) \cup \{c\}$, with $c \in \bar{c}$. If there are $c_i < c_j$ with $f(c_i) \perp f(c_j)$ it is easy to see that End($\mathbb{A}$) generates $g_\perp$ which contradicts to our assumptions. If there are $c_i < c_j$ or $c_i \perp c_j$ with $f(c_i) > f(c_j)$ let a be an element of $(P \setminus \bar{c})$ with $a < c_j$ and $a \perp c_i$. Then the image of a, c_i, c_j under f induces a non partially ordered structure - contradiction.

So End($\mathbb{A}$) preserves Sep. By Lemma 3.1.5 we know that End($\mathbb{A}$) $\subseteq \overline{\langle \updownarrow, \circlearrowright \rangle}$. If End($\mathbb{A}$) violates Cycl and Betw or Cycl and $\perp$ we can proof as in the paragraph above that End($\mathbb{A}$) $= \overline{\langle \updownarrow, \circlearrowright \rangle}$.

Similarly, if End($\mathbb{A}$) preserves Cycl but violates Betw or $\perp$ then End($\mathbb{A}$) $= \overline{\langle \circlearrowright \rangle}$.

If End($\mathbb{A}$) preserves Betw and $\perp$ but violates Cycl. Then End($\mathbb{A}$) $= \overline{\langle \updownarrow \rangle}$.

Finally, if End($\mathbb{A}$) preserves Betw, $\perp$ and Cycl we have End($\mathbb{A}$) $= \overline{\text{Aut}(\mathbb{P})}$. $\square$

3.2 The case where $<$ and $\perp$ are pp-definable

Throughout the remaining parts of this chapter we are going to study the complexity of CSP($\mathbb{A}$) for model-complete reducts $\mathbb{A}$ of $\mathbb{P}$. In this section, we start with the case where End($\mathbb{A}$) is the topological closure of the automorphism group of $\mathbb{P}$. In this case the two relations $<$ and $\perp$ are pp-definable by Theorem 1.6.5. So throughout this section let $\mathbb{A}$ be a reduct of $\mathbb{P}$ in which $<$ and $\perp$ are pp-definable. We are first going to discuss the binary part of the Pol($\mathbb{A}$). This will be essential for proving the dichotomy in this case.

Observation 3.2.1. The binary relation $x \overset{\perp}{\underset{<}{}} y$ defined by $x < y \vee x \perp y$ is equivalent to the primitive positive formula $\exists z\ (z < y) \wedge z \perp x$. Therefore $x \overset{\perp}{\underset{<}{}} y$ is pp-definable in $\mathbb{A}$.

By $e_<$ we denote an embedding of the structure $(P; <)^2$ into $(P; <)$. Clearly $e_<$ is canonical when regarded as map $e_< : (P; \leq, \prec)^2 \to (P; \leq)$. It has the following behaviour:

$e_<$	$=$	$<$	$>$	$\perp$
$=$	$=$	$\perp$	$\perp$	$\perp$
$<$	$\perp$	$<$	$\perp$	$\perp$
$>$	$\perp$	$\perp$	$>$	$\perp$
$\perp$	$\perp$	$\perp$	$\perp$	$\perp$

By $e_\leq$ we denote an embedding of $(P; \leq)^2$ into $(P; \leq)$ that is canonical function when regarded as map $e_\leq : (P; \leq, \prec)^2 \to (P; \leq)$. It has the following behaviour:

$e_\leq$	$=$	$<$	$>$	$\bot$
$=$	$=$	$<$	$>$	$\bot$
$<$	$<$	$<$	$\bot$	$\bot$
$>$	$>$	$\bot$	$>$	$\bot$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$

3.2.1 Horn tractable CSPs given by $e_<$ and $e_\leq$

The two functions $e_<$ or $e_\leq$ are of central interest to us. We will show in this section that if one of them is a polymorphisms of $\mathbb{A}$, then the problem CSP($\mathbb{A}$) is tractable.

Let $\mathbb{B}$ and $\mathbb{C}$ be relational structures of the same signature. We recall that map $h : \mathbb{B} \to \mathbb{C}$ is a *strong homomorphism* if $\bar{x} \in R \leftrightarrow h(\bar{x}) \in R$. By $\hat{\mathbb{B}}$ we denote the extension of $\mathbb{B}$ that contains the negation $\neg R$ for every R is in $\mathbb{B}$.

Theorem 3.2.2 (Proposition 14 from [BCKvO09]). *Let $\mathbb{B}$ be an ω-categorical structure and let $\mathbb{A}$ be a reduct of $\mathbb{B}$. Suppose* CSP($\hat{\mathbb{B}}$) *is tractable. If $\mathbb{A}$ has a polymorphism that is a strong homomorphism from $\mathbb{B}^2$ to $\mathbb{B}$, then also $\mathbb{A}$ is tractable.* □

By definition $e_<$ is a strong homomorphism from $(P;<)^2 \to (P;<)$ and $e_\leq$ s a strong homomorphism from $(P;\leq)^2 \to (P;\leq)$. Let $\not<$ respectively $\not\leq$ denote the negation of the order relation $<$ respectively $\leq$. One can see that every input to CSP($P;<,\not<$) and CSP($P;\leq,\not\leq$) is accepted as long as it does not contradict to the transitivity of $<$ respectively $\leq$. But this can be checked in polynomial time, thus the two problems are tractable. So by Theorem 3.2.2 every template $\mathbb{A}$ with polymorphism $e_<$ or $e_\leq$ gives us a tractable problem.

In the following theorem we additionally give a semantic characterization of these tractable problems via Horn formulas. This characterisation works also in the general setting, we refer to [BCKvO09] for the proof.

Lemma 3.2.3. *Let $\mathbb{A}$ be a reduct of $\mathbb{P}$. Suppose that $e_\leq \in$ Pol($\mathbb{A}$). Then* CSP($\mathbb{A}$) *is tractable and every relation in $\mathbb{A}$ is equivalent to Horn formula in $(P;\leq)$:*

$$x_{i_1} \leq x_{j_1} \wedge x_{i_2} \leq x_{j_2} \wedge \cdots \wedge x_{i_k} \leq x_{j_k} \to x_{i_{k+1}} \leq x_{j_{k+1}} \text{ or}$$
$$x_{i_1} \leq x_{j_1} \wedge x_{i_2} \leq x_{j_2} \wedge \cdots \wedge x_{i_k} \leq x_{j_k} \to \text{'false'}$$

Suppose that $e_< \in$ Pol($\mathbb{A}$). Then CSP($\mathbb{A}$) *is tractable and every relation in $\mathbb{A}$ is equivalent to a Horn formula in $(P;<)$, i.e. a formula of the form:*

$$x_{i_1} \lhd_1 x_{j_1} \wedge x_{i_2} \lhd_2 x_{j_2} \wedge \cdots \wedge x_{i_k} \lhd_k x_{j_k} \to x_{i_{k+1}} \lhd_{k+1} x_{j_{k+1}} \text{ or}$$
$$x_{i_1} \lhd_1 x_{j_1} \wedge x_{i_2} \lhd_2 x_{j_2} \wedge \cdots \wedge x_{i_k} \lhd_k x_{j_k} \to \text{'false'},$$

where $\lhd_i \in \{<,=\}$ for all $i = 1,\ldots,k+1$. □

3.2.2 Canonical binary functions on $(P; \leq, \prec)$

In the following text we are going to study the behaviour of binary functions $f \in \mathrm{Pol}(\mathbb{A})$ that are canonical seen as functions from $(P; \leq, \prec)^2$ to $(P; \leq)$, this will simplify later proof. We are in particular going to specify conditions for which $\mathrm{Pol}(\mathbb{A})$ contains $e_<$ or $e_\leq$.

Definition 3.2.4. Let $f : \mathbb{P}^2 \to \mathbb{P}$ be a function. Then f is called dominated on the first argument if

- $f(x, y) < f(x', y')$ for all $x < x'$ and
- $f(x, y) \perp f(x', y')$ for all $x \perp x'$.

We say f is dominated if f or $(x, y) \mapsto f(y, x)$ is dominated on the first argument.

We are going to prove the following lemma:

Proposition 3.2.5. *Let $\mathbb{A}$ be a reduct of $\mathbb{P}$ in which $<$ and $\perp$ are pp-definable. Let $f(x, y) \in \mathrm{Pol}(\mathbb{A})$ be canonical when seen as a function from $(P; \leq, \prec)^2$ to $(P; \leq)$. Then at least one of the following cases holds:*

- *f is dominated*
- $\mathrm{Pol}(\mathbb{A})$ *contains $e_<$*
- $\mathrm{Pol}(\mathbb{A})$ *contains $e_\leq$*

First of all we make some general observations for binary canonical functions preserving $<$ and $\perp$. We are again going to use the notation introduced in Notation 3.1.7. Let us fix a function $- : (P; \leq, \prec) \to (P; \leq, \prec)$ such that $x \prec y \leftrightarrow -y \prec -x$ holds. It is easy to see that such a function exists.

Lemma 3.2.6. *Let $f : (P; \leq, \prec)^2 \to (P; \leq)$ be canonical and $f \in \mathrm{Pol}(\mathbb{A})$. Then the following statements are true:*

1. $f(p_<, p_<) = p_<$, $f(p_\perp, p_\perp) = p_\perp$
2. $f(p, q) = -f(-p, -q)$, *for all types p, q.*
3. $f(p_<, p_{\perp,\prec})$, $f(p_<, p_{\perp,\succ})$, $f(p_{\perp,\succ}, p_<)$ *and* $f(p_{\perp,\prec}, p_<)$ *can only be equal to* $p_<$ *or* $p_\perp$.
4. *At least one of* $f(p_<, p_{\perp,\prec})$ *and* $f(p_{\perp,\prec}, p_<)$ *is equal to* $p_\perp$.
5. *At least one of* $f(p_<, p_{\perp,\succ})$ *and* $f(p_{\perp,\prec}, p_>)$ *is equal to* $p_\perp$.

6. *It is not possible that* $f(p_<, p_>) = p_=$ *holds.*

7. $f(p_{\perp,\prec}, p_<) = p_\perp \rightarrow f(p_\perp, p_=) = p_\perp$

Proof.

1. This is clear, since f is a polymorphism of $\mathbb{A}$ and hence preserves $<$ and $\perp$.

2. This is true by definition of $-$.

3. This is true since f preserves the relation $\overset{\perp}{<}$, see Observation 3.2.1.

4. Assume $f(p_<, p_{\perp,\prec}) = f(p_{\perp,\prec}, p_<) = p_<$. Let $a_1 \prec a_2 \prec a_3$ with $a_1 < a_2$, $a_3 \perp a_1 a_2$ and $b_1 \prec b_2 \prec b_3$ with $b_2 < b_3$, $b_1 \perp b_2 b_3$. By our assumption $f(a_1, b_1) < f(a_2, b_2) < f(a_3, b_3)$ holds, which is a contradiction to f preserving $\perp$.

5. This can be proven similarly to (4).

6. Assume that $f(p_<, p_>) = p_=$ holds. Let $a_1 \prec a_2 \prec a_3$ with $a_1 < a_3$, $a_2 < a_3$, $a_1 \perp a_2$ and $b_1 \succ b_2 \succ b_3$ with $b_1 > b_3$, $b_2 > b_3$, $b_1 \perp b_2$. Then $f(a_1, b_1) \perp f(a_2, b_2)$ but also $f(a_1, b_1) = f(a_3, b_3) = f(a_2, a_2)$ have to hold, which is a contradiction.

7. Assume that there are $a_1 \perp a_2$ and b such that $f(a_1, b) \leq f(a_2, b)$ holds. Then we take elements a_3 and b' with $a_2 < a_3$, $a_1 \perp a_3$, $a_1 \prec a_3$ and $b' > b$. Then $f(a_1, b) \leq f(a_2, b) < f(a_3, b')$ holds, which is a contradiction to $f(a_1, b) \perp f(a_3, b')$.

□

By Lemma 3.2.6 (2) we only have to consider pairs of types where the first entry is $p_=$, $p_<$ or $p_{\perp,\prec}$ when studying the behaviour of f. Further Lemma 3.2.6 implies that $f(x, y) \neq f(x', y')$ always holds for $x \neq x'$ and $y \neq y'$.

Lemma 3.2.7. *Let* $f \in \mathrm{Pol}(\mathbb{A})$. *Then the following are equivalent:*

1. $f(p_<, p_>) = p_<$

2. $f(p_<, p_{\perp,\succ}) = p_<$

3. $f(p_<, q) = p_<$ *for all 2-types* q

4. f *is dominated in the first argument*

Proof. It is clear that the implications (4) $\to$ (3) $\to$ (2) and (3) $\to$ (1) are true.

(1) $\to$ (3): Let $a_1 < a_2 < a_3$ and $b_1 b_3 < b_2$. Then $f(a_1, b_1) < f(a_2, b_2) < f(a_3, b_3)$ has to hold regardless if the type of (b_1, b_3) is $p_{\perp,\prec}$, $p_{\perp,\succ}$ or $p_=$. So $f(p_<, q) = p_<$ for all 2-types q.

(2) $\to$ (1): Let $a_1 < a_2 < a_3$ and $b_1 \succ b_2 \succ b_3$ with $b_1 > b_3$, $b_2 \perp b_1 b_3$. Then $f(a_1, b_1) < f(a_2, b_2) < f(a_3, b_3)$ implies $f(a_1, b_1) < f(a_3, b_3)$ and so $f(p_<, p_>) = p_<$.

(3) $\to$ (4): We have to consider all the pairs of 2-types where the first entry is $p_{\perp,\prec}$. By Lemma 3.2.6 (4) and (5) we know that $f(p_{\perp,\prec}, p_<) = f(p_{\perp,\prec}, p_>) = p_\perp$. From Lemma 3.2.6(7) follows that $f(p_\perp, p_=) = p_\perp$.

We want to point out that we did not require f to be canonical; it can be easily verified that all proof steps also work for general binary functions. □

Lemma 3.2.8. *Let $f : (P; \leq, \prec)^2 \to (P; \leq)$ be canonical and $f \in \mathrm{Pol}(\mathbb{A})$. If f is not dominated the following statements are true:*

1. $f(p_<, p_>) = f(p_<, p_{\perp,\succ}) = f(p_{\perp,\prec}, p_>) = p_\perp$.
2. $f(p_<, p_=) = p_<$ *or* $f(p_<, p_=) = p_\perp$.
3. $f(p_{\perp,\prec}, p_=) = p_\perp$ *or* $f(p_{\perp,\prec}, p_=) = p_<$.

Proof.

1. is a direct consequence of Lemma 3.2.7.
2. Suppose there are $a_1 < a_2$ and b such that $f(a_1, b) \geq f(a_2, b)$. Then we take elements $a_3, b' \in P$ with $a_2 \perp a_3$ $a_2 \succ a_3$, $a_1 < a_3$ and a $b' > b$. Then $f(a_2, b) \leq f(a_1, b) < f(a_3, b')$ holds, which is a contradiction to $f(a_2, b) \perp f(a_3, b')$.
3. Assume that there are $a_1 \perp a_2$, $a_1 \prec a_2$ and b such that $f(a_1, b) \geq f(a_2, b)$ holds. There are elements a_3 and b' with $a_2 > a_3$, $a_1 \perp a_3$, $a_1 \prec a_3$ and $b' < b$. Then $f(a_2, b) > f(a_3, b')$ and $f(a_1, b) \perp f(a_3, b')$. But this contradicts to our assumption.

□

Definition 3.2.9. Let us say a binary function is $\perp$-falling, if it has the same behaviour as $e_<$ respectively $e_\leq$ on pairs of partial type $(p_{\neq}, p_{\neq})$.

Lemma 3.2.10. *Let $f \in \mathrm{Pol}(\mathbb{A})$ be a canonical function $f : (P; \leq, \prec)^2 \to (P; \leq)$ of $\perp$-falling behaviour. Then $\mathrm{Pol}(\mathbb{A})$ contains $e_<$ or $e_\leq$.*

Proof. From Lemma 3.2.6 (7) follows that $f(p_\perp, p_=) = p_\perp$ and $f(p_=, p_\perp) = p_\perp$. By Lemma 3.2.8 we further know that $f(p_<, p_=), f(p_=, p_<) \in \{p_\perp, p_<\}$. So we have to do a simple case distinction:

- If $f(p_=, p_<) = f(p_<, p_=) = p_\perp$, then f behaves like $e_<$, hence $e_< \in \mathrm{Pol}(\mathbb{A})$.
- If $f(p_=, p_<) = p_<$ and $f(p_<, p_=) = p_\perp$, the function $(x, y) \to f(f(x, y), x)$ has the same behaviour as $e_<$, thus $e_< \in \mathrm{Pol}(\mathbb{A})$.
- Symmetrically if $f(p_=, p_<) = p_\perp$ and $f(p_<, p_=) = p_<$, the function $(x, y) \to f(f(y, x), y)$ has the same behaviour as $e_<$, thus $e_< \in \mathrm{Pol}(\mathbb{A})$.
- If $f(p_=, p_<) = p_< = f(p_<, p_=) = p_<$, then f has the same behaviour as $e_\leq$, thus $e_\leq \in \mathrm{Pol}(\mathbb{A})$.

□

We now give a simple criterium for the existence of a canonical $\perp$-falling function in $\mathrm{Pol}(\mathbb{A})$. This criterium will allows us to finish the proof of Proposition 3.2.5.

Lemma 3.2.11. *Assume that for every $k > 1$, every pair of tuples $\bar{a}, \bar{b} \in P^k$ and every indices $p, q \in [k]$ with $a_p < a_q$ and $\neg(b_p \leq b_q)$ there exists a binary function $g \in \mathrm{Pol}(\mathbb{A})$ such that $g(a_p, b_p) \perp g(a_q, b_q)$ and for all $i, j \in [k]$:*

1. *$a_i < a_j$ implies $g(a_i, b_i) < g(a_j, b_j)$ or $g(a_i, b_i) \perp g(a_j, b_j)$,*
2. *$a_i \perp a_j$ implies $g(a_i, b_i) \perp g(a_j, b_j)$.*

Then $\mathrm{Pol}(\mathbb{A})$ contains $e_<$ and $e_\leq$.

Proof. First we are going to show that for all $\bar{a}, \bar{b} \in P^k$ there is a binary function $f \in \mathrm{Pol}(\mathbb{A})$ that has $\perp$-falling on $(\bar{a}, \bar{b})$. To be more precise we want to construct an $f \in \mathrm{Pol}(\mathbb{A})$ such that:

- $f(a_i, b_i) < f(a_j, b_j)$ if $a_i < a_j$ and $b_i < b_j$,
- $f(a_i, b_i) \perp f(a_j, b_j)$ if $a_i < a_j$ and $\neg(b_p \leq b_q)$.
- $f(a_i, b_i) \perp f(a_j, b_j)$ if $a_i \perp a_j$ and $b_i \neq b_j$

We are going to construct f by a recursive argument.

Let $f^{(0)}(x, y) = g^{(0)}(x, y) = x$ and $\bar{a}^{(0)} = f^{(0)}(\bar{a}, \bar{b})$. If already $f^{(0)}$ has the desired properties we set $f(x, y) = f^{(0)}(x, y)$ and are done.

Otherwise, in the $(k+1)$-th recursion step, we are given a function $f^{(k)}(x, y)$ and a tuple $\bar{a}^{(k)} = f^{(k)}(\bar{a}, \bar{b})$. Let us assume that there are indices p, q with $a_p < a_q$, $\neg(b_p \leq b_q)$ and $a_p^{(k)} < a_q^{(k)}$. Then by our assumption there is a function $g^{(k+1)}(x, y) \in \mathrm{Pol}(\mathbb{A})$ such that $g^{(k+1)}(a_p^{(k)}, b_p) \perp g^{(k)}(a_p^{(k)}, b_p)$. We set $f^{(k+1)}(x, y) = g^{(k)}(f^{(k)}(x, y), y)$ and $\bar{a}^{(k)} = f^{(k)}(\bar{a}, \bar{b})$.

Note that by the properties (1) and (2) of the function g^k the only possible cases for f^k being not $\perp$-falling is the case above. It is clear that the recursion ends after finitely many steps.

So on every finite subset $X \times Y$ of P^2 we find a $\perp$-falling function. By a compactness argument there exists a $h \in \mathrm{Pol}(\mathbb{A})$ that is $\perp$-falling on P^2. It remains to show that there is also a canonical $\perp$-falling function in $\mathrm{Pol}(\mathbb{A})$.

By Theorem 1.7.2 we have that h is canonical on arbitrarily large substructures of P^2. Let $(F_n)_{n\in\omega}$ be an increasing sequence of finite substructures such that its union is equal to P. Then for every $n \in \omega$ there are $\alpha_1^{(n)}, \alpha_2^{(n)} \in \mathrm{Aut}(\mathbb{A})$ such that $f \circ (\alpha_1^{(n)}, \alpha_2^{(n)})$ is canonical on F_n. By thinning out the sequence we can assume that $f \circ (\alpha_1^{(n)}, \alpha_2^{(n)})$ has the same behaviour for every $n \in \omega$.

Since the behaviour $f \circ (\alpha_1^{(n)}, \alpha_2^{(n)})$ on all F_n is the same, we can inductively pick automorphisms $\beta_n \in \mathrm{Aut}(\mathbb{P})$ such that $\beta_n \circ f \circ (\alpha_1^{(n)}, \alpha_2^{(n)})$ agrees with $\beta_{n+1} \circ f \circ (\alpha_1^{(n+1)}, \alpha_2^{(n+1)})$ on F_n. The limit of this sequence is a canonical function in $\mathrm{Pol}(\mathbb{A})$ with $\perp$-falling behaviour.

By Lemma 3.2.10 we have that $e_<$ or $e_\leq$ is an element of $\mathrm{Pol}(\mathbb{A})$. This concludes the proof. □

Proof of Proposition 3.2.5. Let $f : (P; \leq, \prec)^2 \to (P; \leq)$ be canonical and $f \in \mathrm{Pol}(\mathbb{A})$. Let us assume that f is not dominated. By Lemma 3.2.8 we know $f(p_<, p_>) = f(p_<, p_{\perp,\succ}) = f(p_{\perp,\prec}, p_>) = p_\perp$.

By Lemma 3.2.6 (3) and (4) we have to look at the following cases:

1. $f(p_<, p_{\perp,\prec}) = f(p_{\perp,\prec}, p_<) = p_\perp$.
2. $f(p_<, p_{\perp,\prec}) = p_<$ and $f(p_{\perp,\prec}, p_<) = p_\perp$.
3. $f(p_<, p_{\perp,\prec}) = p_\perp$ and $f(p_{\perp,\prec}, p_<) = p_<$.

In the first case f has $\perp$-falling behaviour therefore we are done by Lemma 3.2.10.

For the remaining cases we can restrict ourselves to (2), otherwise we take $(x, y) \to f(y, x)$. From Lemma 3.2.6 (7) follows that $f(p_\perp, p_=) = p_\perp$. Thus $f(p_\perp, q) = p_\perp$ holds for every 2-type q.

We are going to show that then the conditions in Lemma 3.2.11 are satisfied. Let $\bar{a}, \bar{b} \in P^k$ be two tuples of arbitrary length k and let $p, q \in [k]$ such that $a_p < a_q$, $b_p \prec b_q$ and $b_p \perp b_q$ hold. Then let $\alpha \in \mathrm{Aut}(\mathbb{P})$ with $\alpha(b_p) \succ \alpha(b_q)$. Such an automorphism exists by the homogeneity of $\mathbb{P}$. Then we set $g(x, y) = f(x, \alpha(y))$.

Clearly $g(a_p, a_q) \perp g(b_p, b_q)$, since $\alpha(b_p) \succ \alpha(b_q)$. Also the other conditions in Lemma 3.2.11 are satisfied, by the properties of f. Therefore $\mathrm{Pol}(\mathbb{A})$ contains $e_<$ or $e_\leq$. □

3.3 The NP-hardness of Low

Let Low be the 3-ary relation defined by

$$\text{Low}(x, y, z) := (x < y \wedge z \bot xy) \vee (x < z \wedge y \bot xz).$$

Clearly $\bot$ and $<$ are pp-definable in Low. Note that Low is not preserved by $e_<$ or $e_\leq$, so CSP(P; Low) is not covered by the tractability result in Lemma 3.2.3. In this section we prove the NP-hardness of CSP(P; Low).

Lemma 3.3.1. *Let us define the relations*

$$\begin{aligned}\text{Abv}(x, y, z) :=& (y < x \wedge xy \bot z) \vee (z < x \wedge xz \bot y)\\ U(x, y, z) :=& (y < x \vee z < x) \wedge (y \bot z)\end{aligned}$$

Then Abv *and* U *are pp-definable in* Low*.*

Proof. Note that the formula

$$\phi(x, y, z, v) := \exists u \; u \bot v \wedge \text{Low}(u, y, z) \wedge \text{Low}(y, x, v) \wedge \text{Low}(z, x, v)$$

is equivalent to the statement that $v \bot x$ and $y \bot z$ and at least one element of $\{y, z\}$ is smaller that x and at most one element of $\{y, z\}$ is smaller than v
With that in mind one can see that

$$\exists v_1, v_2 \; \phi(x, y, z, v_1) \wedge \phi(v_2, y, z, x)$$

is equivalent to $\text{Abv}(x, y, z)$ and

$$\exists v \; \phi(x, y, z, v)$$

is equivalent to $U(x, y, z)$. □

Proposition 3.3.2. *Let $a, b \in P$ with $a \bot b$. There is a pp-interpretation of $\mathbb{S}$ in $(P; \text{Low}, a, b)$. Thus* CSP(P; Low) *is NP-hard.*

Proof. Let NAE be the Boolean relation $\{0, 1\}^3 \backslash \{(0, 0, 0), (1, 1, 1)\}$. It is easy to see that $\text{Pol}(\{0, 1\}, \text{NAE}, 0, 1)$ is the projection clone **1**. So by Theorem 1.6.9 it suffices to show that $(\{0, 1\}; \text{NAE}, 0, 1)$ has a pp-interpretation in $(P; \text{Low}, a, b)$ to prove the statement.

Let $D := \{x \in P : \text{Low}(x, a, b)\}, D_0 := \{x \in D : x < a\}, D_1 := \{x \in D : x < b\}$. Note that $D_0 \bot D_1$. Let $I : D \to \{0, 1\}$ be given by:

$$I(x) := \begin{cases} 0 & \text{if } x \in D_0 \\ 1 & \text{if } x \in D_1 \end{cases}.$$

Clearly the domain D of I is pp-definable in $(P; \mathrm{Low}, a, b)$. Since the order relation $<$ is pp-definable in Low also the sets D_0 and D_1 are pp-definable. Let $R = \{(x, y, z, t) \in P^4 : (x > y \vee x > z \vee x > t) \wedge \neg(x \leq yzt)\}$. We claim that the relation R is pp-definable in Low. Observe that $(x, y, z, t) \in R$ is equivalent to

$$\exists u, v\ (\mathrm{Abv}(x, u, v) \wedge U(x, y, u) \wedge U(x, z, u) \wedge U(x, t, v))$$

and therefore pp-definable in Low by Lemma 3.3.1. By the definition of R we have that $I(c_1, c_2, c_3) \in \mathrm{NAE}$ if and only if $(a, c_1, c_2, c_3) \in R$ and $(b, c_1, c_2, c_3) \in R$. Thus the preimage of NAE is pp-definable in $(P; \mathrm{Low}, a, b)$. □

The following lemma gives us an additional characterization of reducts, in which Low is pp-definable.

Lemma 3.3.3. *The relation* Low *is pp-definable in* $\mathbb{A}$ *if and only if every binary polymorphism of* $\mathbb{A}$ *is dominated.*

Proof. Every dominated function $f : P^2 \to P$ preserves Low. For the other direction observe that by Lemma 3.2.7 we have that f is dominated in the first argument if and only if $f(a_1, b_1) < f(a_2, b_2)$ for all $a_1 < a_2$ and $b_1 \perp b_2$. Note that Lemma 3.2.7 also works for non-canonical functions.
So if $f \in \mathrm{Pol}(\mathbb{A})$ is a binary, not dominated function, there are $a_1 < a_2, b_1 \perp b_2, a'_1 \perp a'_2$ and $b'_1 < b'_2$ such that $f(a_1, b_1) \perp f(a_2, b_2)$ and $f(a'_1, b'_1) \perp f(a'_2, b'_2)$. Hence f violates the relation

$$S(x_1, x_2, y_1, y_2) := (x_1 < x_2 \wedge y_1 \perp y_2) \vee (x_1 \perp x_2 \wedge y_1 < y_2).$$

But the relation S and Low are pp-interdefinable:

$$\begin{aligned} \mathrm{Low}(x, y, z) &\leftrightarrow S(x, y, x, z) \wedge y \perp z \\ S(x_1, x_2, y_1, y_2) &\leftrightarrow \exists u, v, w\ (\mathrm{Low}(x_1, x_2, u) \wedge \mathrm{Abv}(u, x_1, v), \\ &\qquad \wedge \mathrm{Low}(u, v, w) \wedge \mathrm{Abv}(w, y_1, v) \wedge \mathrm{Low}(y_1, y_2, w)). \end{aligned}$$

We conclude that f violates Low. □

3.4 Violating the Low relation

We saw in Lemma 3.2.3 that CSP($\mathbb{A}$) is tractable if $e_<$ or $e_\leq$ are polymorphisms of $\mathbb{A}$. By Proposition 3.3.2 we know that CSP($\mathbb{A}$) is NP-complete if Low is pp-definable in $\mathbb{A}$. In this section we are going to show that these results already cover all possible reducts where $<$ and $\perp$ are pp-definable.

Proposition 3.4.1. *Let $\mathbb{A}$ be a reduct of $\mathbb{P}$ such that $\perp$ and $<$ are pp-definable in $\mathbb{A}$. Then* Low *is not pp-definable in $\mathbb{A}$ if and only if* $\mathrm{Pol}(\mathbb{A})$ *contains one of the functions $e_<$ or $e_\leq$.*

Proof. Note that by Theorem 1.6.5 Low is not pp-definable in $\mathbb{A}$ if and only if there is a binary $f \in \mathrm{Pol}(\mathbb{A})$ violating Low. This means that there are $a, b, c \in P$ such that $a < b \wedge ab\perp c$ and $f(a,a) < f(b,c) \wedge f(a,a) < f(c,b)$, or $f(a,a)\perp f(b,c)$ and $f(a,a)\perp f(c,b)$.

We have only these two cases since f preserves $\overset{\perp}{<}$ and $\perp$. We can assume that $a \prec b \prec c$ since otherwise we can find an automorphism $\alpha \in \mathrm{Aut}(\mathbb{P})$ such that $\alpha(a) \prec \alpha(b) \prec \alpha(c)$. Then we consider the map $(x, y) \mapsto f(\alpha^{-1}(x), \alpha^{-1}(y))$ with three elements $\alpha(a)$, $\alpha(b)$ and $\alpha(c)$ instead.

By Lemma 3.1.6 we can assume that f is canonical as a function from $(P; <, \prec, a, b, c)^2$ to $(P; <)$. We deal with the two cases in Lemma 3.4.3 and Lemma 3.4.10 in the following subsections. □

Notation 3.4.2. For simplicities sake, a canonical binary function in this section means a function that is canonical as a function from $(P; \leq, \prec)^2 \to (P; <)$.
Let $f : P^2 \to P$ be a function and X, Y, X', Y' be subsets of P such that f is dominated on $X \times Y$ and $X' \times Y'$. We say that f has the *same domination* on $X \times Y$ and $X' \times Y'$ if f is dominated by the first argument on both $X \times Y$ and $X' \times Y'$, or dominated by the second argument on both $X \times Y$ and $X' \times Y'$. Otherwise, we say that f has the *different domination* on $X \times Y$ and $X' \times Y'$.

3.4.1 $f(a,a) < f(b,c) \wedge f(a,a) < f(c,b)$

The aim of this subsection is to prove the following lemma.

Lemma 3.4.3. *Let $f \in \mathrm{Pol}(\mathbb{A})$ be canonical as a function from $(P; <, \prec, a, b, c)^2$ to $(P; <)$. If $f(a,a) < f(b,c) \wedge f(a,a) < f(c,b)$ then* $\mathrm{Pol}(\mathbb{A})$ *contains $e_<$ or $e_\leq$.*

We define the following two sets:

- $B_1 := \{x \in P : x > c \wedge x\perp a \wedge x\perp b\}$,
- $B_2 := \{x \in P : x > b \wedge x > c\}$.

Let $x, y \in B_1 \cup B_2$. We say that x and y *are in the same orbit* if $x \in B_i$ and $y \in B_i$ for an $i \in [2]$.

Observation 3.4.4. B_1 and B_2 are orbits of $\mathrm{Aut}(P; <, \prec, a, b, c)$. By the homogeneity of $(P; \leq, \prec)$ we can show that $(B_1; \leq, \prec)$, $(B_2; \leq, \prec)$ are isomorphic to $(P; \leq, \prec)$. Further also the union of B_1 and B_2 is an isomorphic copy of $(P; \leq, \prec)$, in which B_1 forms a random filter.

If there is a canonical $g \in \mathrm{Pol}(\mathbb{A})$ that is not dominated, then Lemma 3.2.5 gives us that $e_<$ or $e_\leq$ is in $\mathrm{Pol}(\mathbb{A})$. So throughout the lemmata and corollaries below in this section, we assume that every binary canonical function in $\mathrm{Pol}(\mathbb{A})$ is dominated and $f(a,a) < f(b,c) \wedge f(a,a) < f(c,b)$.

Lemma 3.4.5. *f is dominated on $B_i \times B_j$ for every $i, j \in [2]$.*

Proof. For a contradiction, we assume that f is not dominated on $B_i \times B_j$. Since $(B_i; \leq, \prec)$ and $(B_j; \leq, \prec)$ are isomorphic to $(P; \leq, \prec)$, there are $\alpha : P \to B_i$ and $\beta : P \to B_j$ such that α is an isomorphism from $(P; \leq, \prec)$ to $(B_i; \leq, \prec)$ and β is an isomorphism from $(P; \leq, \prec)$ to $(B_j; \leq, \prec)$. Let $g : P^2 \to P$ be given by $g(x, y) := f(\alpha(x), \beta(y))$. It follows from Observation 3.4.4 that g is canonical and is not dominated, a contradiction. □

Lemma 3.4.6. *f has the same domination on all sets $B_i \times B_j$, $i, j \in [2]$.*

Proof. We claim that f has the same domination on $B_1 \times B_k$ and $B_2 \times B_k$ for any $k \in [2]$. For a contradiction, we assume that f does not have the same domination $B_1 \times B_k$ and $B_2 \times B_k$. Without loss of generality we can assume that f is dominated by the first argument on $B_1 \times B_k$ and dominated by the second argument on $B_2 \times B_k$. Let $x, y \in B_1, z, t \in B_2$ be such that $x < y \wedge y < z \wedge x \bot t$. Let $x', y', z', t' \in B_k$ be such that $x' \bot t' \wedge y' < z' \wedge z' < t'$. Since f is dominated by the first argument on $B_1 \times B_k$, we have $f(x, x') < f(y, y')$. Since f is dominated by the second argument on $B_2 \times B_k$, we have $f(z, z') < f(t, t')$. Since f preserves $<$, we have $f(y, y') < f(z, z')$. Thus $f(x, x') < f(t, t')$, a contradiction to the fact that f preserves $\bot$.

By considering the map $(x, y) \mapsto f(y, x)$ we have that f has the same domination on $B_k \times B_1$ and $B_k \times B_2$ for every $k \in [2]$. This implies that f has the same dominations on all products $B_i \times B_j, i, j \in [2]$. □

In the rest of this section, we assume that f is dominated by the first argument on $B_i \times B_j$ for every $i, j \in [2]$. The other case can be reduced to this case by considering the map $(x, y) \mapsto f(y, x)$.

Lemma 3.4.7. *Let $u, v \in B_1$ and $u' \in B_2, v' \in B_1$ be such that $u < v \vee u \bot v$. Then $f(u, u') \bot f(v, v')$.*

Proof. First, we claim that $f(u, u') > f(v, v') \vee f(u, u') \bot f(v, v')$. For a contradiction, we assume that $f(u, u') \leq f(v, v')$. Since f preserves $<$, we have $f(c, b) < f(u, u')$. Therefore $f(a, a) < f(c, b) < f(u, u') < f(v, v')$, a contradiction to the $\bot$-preservation of f. Thus the claim follows.

The proof is completed by showing that $f(u, u') > f(v, v')$ is impossible. For a contradiction, we assume that $f(u, u') > f(v, v')$. Let $s, t \in B_1$ be such that

$s \perp t \wedge s < v \wedge u < t$. Let $s' \in B_1, t' \in B_2$ be such that $s' \perp t'$. By the domination of f, we have $f(s,s') < f(v,v') \wedge f(u,u') < f(t,t')$. It follows from $f(u,u') > f(v,v')$, we have $f(s,s') < f(t,t')$, a contradiction to $\perp$-preservation of f. □

Lemma 3.4.8. *Let $u, v \in B_1$ be such that $u \perp v$. Then for every $u', v' \in B_1 \cup B_2$, we have $f(u,u') \perp f(v,v')$.*

Proof. For a contradiction, we assume that $\neg(f(u,u') \perp f(v,v'))$. Without loss of generality, we assume that $f(u,u') \leq f(v,v')$. Let $s, t \in B_1$ be such that $s < u \wedge v < t \wedge s \perp t$. Let $s', t' \in B_1 \cup B_2$ be such that $s' \perp t'$, s', u' are in the same orbit and t', v' are in the same orbit. By the domination of f, we have $f(s,s') < f(u,u') \wedge f(v,v') < f(t,t')$. Since $f(u,u') < f(v,v')$, we have $f(s,s') < f(t,t')$, a contradiction to the $\perp$-preservation of f. □

Lemma 3.4.9. *Let $u, v \in B_1$ and $u', v' \in B_1 \cup B_2$ be such that $u < v$. Then $f(u,u') < f(v,v') \vee f(u,u') \perp f(v,v')$.*

Proof. For a contradiction, we assume that $f(v,v') \leq f(u,u')$. Let $s, t \in B_1$ be such that $t < v \wedge u < s \wedge s \perp t$. Let $s', t' \in B_1 \cup B_2$ be such that $s' \perp t'$, s', u' are in the same orbit, and t', v' are in the same orbit. By the domination of f, we have $f(t,t') < f(v,v') \wedge f(u,u') < f(s,s')$. Since $f(v,v') < f(u,u')$, we have $f(t,t') < f(s,s')$, a contradiction to the $\perp$-preservation of f. □

Proof of Lemma 3.4.3. We are going to show that $\mathrm{Pol}(\mathbb{A})$ contains a function that behaves like $e_<$ or like $e_\leq$ by checking the conditions of Lemma 3.2.11.

So let $\bar{a}, \bar{b} \in P^k$ with $a_p < a_q$ and $\neg(b_p \leq b_q)$. We set $Y := \{b_i : b_i \geq b_p\}, Z := \{b_i : \neg(b_i \geq b_p)\}$. By definition we have $b_q \in Z$. By the homogeneity of $\mathbb{P}$, there is $\alpha \in \mathrm{Aut}(\mathbb{P})$ such that $\alpha(Y) \subseteq B_2$ and $\alpha(Z) \subseteq B_1$. Let $\beta \in \mathrm{Aut}(\mathbb{P})$ such that $\beta(\{a_i : i \in [k]\}) \subseteq B_1$. Let $g(x,y) := f(\beta(x), \alpha(y))$. Clearly, $g \in \mathrm{Pol}(\mathbb{A})$.
By Lemma 3.4.7 we have that $g(a_p, b_p) \perp g(a_q, b_q)$. Further we know by Lemma 3.4.9 that $g(a_i, b_i) < g(a_j, b_j)$ or $g(a_i, b_i) \perp g(a_j, b_j)$ holds for all $a_i < a_j$. By Lemma 3.4.8 we know that $g(a_i, b_i) \perp g(a_j, b_j)$ holds for all $a_i \perp a_j$. So the conditions of Lemma 3.2.11 are satisfied. Hence $e_<$ or $e_\leq$ is a polymorphism of $\mathbb{A}$.

□

3.4.2 $f(a,a) \perp f(b,c) \wedge f(a,a) \perp f(c,b)$

The aim of this section is to prove the following.

Lemma 3.4.10. *Let $f \in \mathrm{Pol}(\mathbb{A})$ be canonical as a function from $(P; <, \prec, a, b, c)^2$ to $(P; <)$. If $f(a,a) \perp f(b,c) \wedge f(a,a) \perp f(c,b)$, then $\mathrm{Pol}^{(2)}(\mathbb{A})$ contains $e_<$ or $e_\leq$.*

We define the following sets.

$$B_1 := \{x \in P : a < x < b \wedge x \perp c\}$$
$$B_2 := \{x \in P : x < b \wedge x < c \wedge x \perp a \wedge x \prec a\}.$$

Throughout the lemmata and corollaries below in this section, we assume that every binary canonical function in $\mathbb{A}$ is dominated and $f(a,a) \perp f(b,c) \wedge f(a,a) \perp f(c,b)$.

Observe that by the homogeneity of $(P; \leq; \prec)$ and the back-and-forth argument, we can show that $(B_1 \cup B_2; \leq, \prec)$ is isomorphic to $(P; \leq, \prec)$, with B_2 being a random filter. For every two k-tuples $\bar{x}$ and $\bar{y}$ in B_i^k, $\bar{x}$ and $\bar{y}$ are in the same orbit of $\mathrm{Aut}(\mathbb{P})$ if and only if $\bar{x}$ and $\bar{y}$ are in the same orbit of $\mathrm{Aut}(P; a, b, c)$.

Lemma 3.4.11. *f has the same domination on sets $B_i \times B_j, i, j \in [2]$.*

Proof. This lemma can be shown as in Lemma 3.4.5 and Lemma 3.4.6. □

In the rest of this section we assume that f is dominated by the first argument on $B_i \times B_j$ for every $i, j \in \{1, 2\}$. Similarly, to Lemma 3.4.7, we have the following.

Lemma 3.4.12. *Let $u, v \in B_1$ and $u' \in B_1, v' \in B_2$ be such that $u < v \vee u \perp v$. Then $f(u,u') \perp f(v,v')$.*

Proof. First we prove that $f(v,v') < f(u,u') \vee f(v,v') \perp f(u,u')$. For a contradiction we assume that $f(u,u') \leq f(v,v')$. Since $a < u \wedge a < u'$, we have $f(a,a) < f(u,u')$. Since $v < b \wedge v' < c$, we have $f(v,v') < f(b,c)$. Thus $f(a,a) < f(b,c)$, a contradiction to the fact that $f(a,a) \perp f(b,c)$. Thus $f(v,v') < f(u,u') \vee f(v,v') \perp f(u,u')$.

The proof is completed by showing that $f(u,u') > f(v,v')$ is impossible. For a contradiction, we assume that $f(u,u') > f(v,v')$. Let $s, t \in B_1$ be such that $s \perp t \wedge s < v \wedge u < t$. Let $s' \in B_2, t' \in B_1$ be such that $s' \perp t'$. By the domination of f, we have $f(s,s') < f(v,v') \wedge f(u,u') < f(t,t')$. It follows from $f(u,u') > f(v,v')$, we have $f(s,s') < f(t,t')$, a contradiction to $\perp$-preservation of f. □

Lemma 3.4.13. *Let $u, v \in B_1$ be such that $u \perp v$. Then for every $u', v' \in B_1 \cup B_2$, we have $f(u,u') \perp f(v,v')$.*

Proof. analogous to Lemma 3.4.8. □

Lemma 3.4.14. *Let $u, v \in B_1$ and $u', v' \in B_1 \cup B_2$ be such that $u < v$. Then $f(u,u') < f(v,v') \vee f(u,u') \perp f(v,v')$.*

Proof. analogous to Lemma 3.4.9. □

Proof of Lemma 3.4.10. We are again going to show that Pol($\mathbb{A}$) contains a function that behaves like $e_<$ or like $e_\leq$ by checking the conditions of Lemma 3.2.11.

So let $\bar{a}, \bar{b} \in P^k$ with $a_p < a_q$ and $\neg(b_p \leq b_q)$. We set $Y := \{b_i : b_i \geq b_p\}, Z := \{b_i : \neg(b_i \geq b_p)\}$. By definition we have $b_q \in Z$. By the homogeneity of $\mathbb{P}$, there is $\alpha \in \mathrm{Aut}(\mathbb{P})$ such that $\alpha(Y) \subseteq B_1$ and $\alpha(Z) \subseteq B_2$. Let $\beta \in \mathrm{Aut}(\mathbb{P})$ such that $\beta(\{a_i : i \in [k]\}) \subseteq B_1$. Let $g(x, y) := f(\beta(x), \alpha(y))$. Clearly, $g \in \mathrm{Pol}(\mathbb{A})$.
By Lemma 3.4.7 we have that $g(a_p, b_p) \perp g(a_q, b_q)$. Further we know by Lemma 3.4.9 that $g(a_i, b_i) < g(a_j, b_j)$ or $g(a_i, b_i) \perp g(a_j, b_j)$ holds for all $a_i < a_j$. By Lemma 3.4.8 we know that $g(a_i, b_i) \perp g(a_j, b_j)$ holds for all $a_i \perp a_j$. So the conditions of Lemma 3.2.11 are satisfied. Hence $e_<$ or $e_\leq$ is a polymorphism of $\mathbb{A}$. □

3.5 The NP-hardness of Betw, Sep and Cycl

By Corollary 3.1.3 we are now left with the cases where End($\mathbb{A}$) is equal to one of the monoids $\overline{\langle \updownarrow \rangle}$, $\overline{\langle \circlearrowright \rangle}$ or $\overline{\langle \updownarrow, \circlearrowright \rangle}$. We are going to deal with all these remaining cases in this section. Interestingly, we can treat them all similarly: By fixing finitely many constants $c_1, \ldots, c_n$ on $\mathbb{A}$ we obtain definable subsets of $(\mathbb{A}, c_1, \ldots, c_n)$ on which $<$ and Low are pp-definable. This enables us to reduce every every such case to the NP-completeness of Low.

Lemma 3.5.1. *Let $u, v \in P$ with $u < v$. Then the relations $<$ and* Low *are pp-definable in $(P, \mathrm{Betw}, \perp, u, v)$.*

Proof. It is easy to verify that there is a pp-definition of the order relation by the following equivalence:

$$x < y \leftrightarrow \exists a, b\ (\mathrm{Betw}(x, y, a) \wedge \mathrm{Betw}(y, a, b) \wedge \mathrm{Betw}(u, v, a) \wedge \mathrm{Betw}(v, a, b)).$$

The two maps $e_< : P^2 \to P$ and $e_\leq : P^2 \to P$ do not preserve Betw, since for every triple $\bar{a} = (a_1, a_2, a_3)$ with $a_1 < a_2 < a_3$ and $\bar{b} = (b_1, b_2, b_3)$ with $b_1 > b_2 > b_3$, the image of $(\bar{a}, \bar{b})$ forms an antichain.

By Proposition 3.4.1 we have that Low is pp-definable in $(P, \mathrm{Betw}, \perp, u, v)$. □

Proposition 3.5.2. *Let $\mathbb{A}$ be a reduct of $\mathbb{P}$ such that* End($\mathbb{A}$) $= \overline{\langle \updownarrow \rangle}$. *Then there are constants $u, v, w, t \in P$ such that $\mathbb{S}$ is pp-interpretable in $(\mathbb{A}, u, v, w, t)$. Hence* CSP($\mathbb{A}$) *is NP-complete.*

Proof. Note that the betweenness relation Betw is an orbit of End($\mathbb{A}$) $= \overline{\langle \updownarrow \rangle}$ on P^3. Now Theorem 1.6.5 implies that Betw is primitively positive definable in $\mathbb{A}$. For the same reason $\perp$ is pp-definable in $\mathbb{A}$. By Lemma 3.5.1 there is pp-definition of Low in $(\mathbb{A}, u, v)$. By Proposition 3.3.2 we can find a pp-interpretation of $\mathbb{S}$ in $(\mathbb{A}, u, v, w, t)$, where w, t are two additional constants. Hence CSP($\mathbb{A}$) is NP-complete. □

For the case where $\text{End}(\mathbb{A}) = \overline{\langle \circlearrowright \rangle}$, we first need the following lemma:

Lemma 3.5.3. *Let c, d be two constants in P such that $c < d$. Then there is a pp-interpretation of $(P; \text{Low})$ in $(P; \text{Cycl}, c, d)$*

Proof. Let $X := \{x \in P : c < x < d\}$. By using back-and-forth argument one can show easily that $(P; <)$ and $(X; <_{|X})$ are isomorphic. We first show that X (as a unary predicate) and $<_{|X}$ are pp-definable in $(P; \text{Cycl}, c, d)$. It is easy to verify that the set X can be defined in $(P; \text{Cycl}, c, d)$ by $\phi(x) := \text{Cycl}(c, x, d)$ and that $x <_{|X} y \leftrightarrow \phi(x) \wedge \phi(y) \wedge \text{Cycl}(c, x, y)$. Now a pp-interpretation of $(P; <, \text{Cycl})$ in $(P; \text{Cycl}, c, d)$ is simply given by the identity on X.

By Lemma 3.1.4 we have that $\perp$ is pp-definable in $(P; <, \text{Cycl})$. It is easy to verify that $e_<$ and $e_\leq$ do not preserve Cycl. Therefore, by Proposition 3.4.1, Low is pp-definable in $(P; <, \text{Cycl})$, which concludes the proof of the Lemma. ☐

Proposition 3.5.4. *Let $\mathbb{A}$ be a reduct of $\mathbb{P}$ such that $\text{End}(\mathbb{A}) = \overline{\langle \circlearrowright \rangle}$. Then there are constants $a, b, c, d \in P$ such that $\mathbb{S}$ is pp-interpretable in $(\mathbb{A}, a, b, c, d)$. Hence $\text{CSP}(\mathbb{A})$ is NP-complete.*

Proof. The cyclic order relation Cycl is an orbit of $\text{End}(\mathbb{A}) = \overline{\langle \circlearrowright \rangle}$ on P^3. So Theorem 1.6.5 implies that Cycl is primitively positive definable in $\mathbb{A}$. By Lemma 3.5.3 there is pp-definition of Low in $(\mathbb{A}, c, d)$ with $c < d$. By Proposition 3.3.2 we can find a pp-interpretation of $\mathbb{S}$ in $(\mathbb{A}, a, b, c, d)$, where a, b are two additional constants. Hence $\text{CSP}(\mathbb{A})$ is NP-complete. ☐

In the following, we prove the NP-hardness of $\text{CSP}(P; \text{Sep})$ by using the same proof idea as the proof of NP-hardness of $\text{CSP}(P; \text{Cycl})$ in Section 3.5.

Lemma 3.5.5. *Let c, d, u be constants in P such that $c < d < u$. Then $(P; \text{Low})$ has a pp-interpretation in $(P; \text{Sep}, c, d, u)$.*

Proof. Let $X := \{x \in P : d < x < u\}$. By using a back-and-forth argument, one can show easily that $(X; \leq)$ and $\mathbb{P}$ are isomorphic. Similarly as in the proof of Proposition 3.5.4, X and $<_{|X}$ are pp-definable in $(P; \text{Sep}, c, d, u)$ as follows.

The set X can be defined by the formula $\phi(x) := \text{Sep}(c, d, x, u)$, and $<_{|X}$ can be defined by $x <_{|X} y :\Leftrightarrow \phi(x) \wedge \phi(y) \wedge \text{Sep}(c, d, x, y)$. Also $\text{Cycl}(x, y, z)_{|X}$ can be defined by the primitive positive formula $\phi(x) \wedge \phi(y) \wedge \phi(z) \wedge \text{Sep}(c, x, y, z)$

So a pp-interpretation of $(P; <, \text{Cycl})$ in $(P; \text{Sep}, c, d, u)$ is simply given by the identity, restricted to X. By Lemma 3.5.3, Low is pp-definable in $(P; <, \text{Cycl})$, which concludes the proof of the Lemma. ☐

Proposition 3.5.6. *Let $\mathbb{A}$ be a reduct of $\mathbb{P}$ such that $\text{End}(\mathbb{A}) = \overline{\langle \updownarrow, \circlearrowright \rangle}$. Then there are constants $a, b, c, d, u \in P$ such that $\mathbb{S}$ is pp-interpretable in $(\mathbb{A}, a, b, c, d, u)$. Hence $\text{CSP}(\mathbb{A})$ is NP-complete.*

Proof. The relation Sep is an orbit of $\mathrm{End}(\mathbb{A}) = \overline{\langle \updownarrow, \circlearrowright \rangle}$ on P^3. So Theorem 1.6.5 implies that Sep is primitively positive definable in $\mathbb{A}$. By Lemma 3.5.5 there is pp-definition of Low in $(\mathbb{A}, c, d, u)$ with $c < d < u$. By Proposition 3.3.2 we can find a pp-interpretation of $\mathbb{S}$ in $(\mathbb{A}, a, b, c, d, u)$, where a, b are two additional constants. Hence CSP($\mathbb{A}$) is NP-complete. □

3.6 Main Results

In this section we complete the proof of our result that Poset-SAT(Φ) problems are either in P or NP-complete. This dichotomy corresponds to an model-theoretic dichotomy of the reducts of $\mathbb{P}$ and can nicely captured in the language of clones, see also the discussion in Section 1.6. We will show that Conjecture 1.6.10 holds for the reducts of $\mathbb{P}$, and that we obtain in fact some stronger equations in the tractable cases in Corollary 3.6.5. Furthermore we give a finite list of relations, that entirely describes the NP-complete cases and show that also the meta-problem of deciding whether Poset-SAT(Φ) is tractable or not for a given Φ is decidable.

Let $\mathbb{A}$ be a reduct of $\mathbb{P}$ and $\mathbb{A}^c$ be its model-complete core. Throughout this chapter we have studied the question whether there is a pp-interpretation of the structure $\mathbb{S}$ in $\mathbb{A}^c$, extended by finitely many constants or not.
First we recall the situation, where $<$ and $\perp$ are pp-definable.

Lemma 3.6.1. *Let $\mathbb{A}$ be a reduct of $\mathbb{P}$ in which $<$ and $\perp$ are pp-definable. Then the following are equivalent:*

1. *There is a binary $f \in \mathrm{Pol}(\mathbb{A})$ which is not dominated.*
2. *The relation* Low *is not pp-definable in $\mathbb{A}$.*
3. *$e_<$ or $e_\leq$ is a polymorphism of $\mathbb{A}$.*
4. *There is a binary $f \in \mathrm{Pol}(\mathbb{A})$ and endomorphisms $e_1, e_2 \in \mathrm{End}(\mathbb{A})$ such that*
$$e_1(f(x, y)) = e_2(f(y, x))$$
5. *For all $c_1, \ldots, c_n \in \mathbb{A}$ there is no clone homomorphism from $\mathrm{Pol}(\mathbb{A}, c_1, \ldots, c_n)$ onto* **1**.
6. *For all $c_1, \ldots, c_n \in \mathbb{A}$ there is no continuous clone homomorphism from $\mathrm{Pol}(\mathbb{A}, c_1, \ldots, c_n)$ onto* **1**.
7. *There is no pp-interpretation of $\mathbb{S}$ in any expansion of $\mathbb{A}$ by finitely many constants.*

Proof.
The equivalences of the points (5)-(7) hold for all ω-categorical structures $\mathbb{A}$ and were discussed in Theorem 1.6.9.
(1) $\leftrightarrow$ (2) This is the statement of Lemma 3.3.3.
(2) $\rightarrow$ (3): This is the statement of Proposition 3.4.1.
(3) $\rightarrow$ (4): Set $f = e_<$ respectively $f = e_\leq$.
(4) $\rightarrow$ (5): If there are $e_1, e_2, f \in \mathrm{Pol}(\mathbb{A})$ satisfying the equation $e_1(f(x,y)) = e_2(f(y,x))$ then there are also such polymorphisms fixing finitely many elements $c_1, \dots, c_n$. This is true for all ω-categorical cores, see Lemma 82 of [BJP16]. It follows that there is no clone homomorphism from $\mathrm{Pol}(\mathbb{A}, c_1, \dots, c_n)$ onto $\mathbf{1}$.
(7) $\rightarrow$ (2): This follows from the contraposition of Proposition 3.3.2. □

Thus, in this case, the non-existence of a pp-interpretation of $\mathbb{S}$ in $\mathbb{A}$ with finitely many parameters is equivalent to the existence of a pseudo-Siggers operation, but to even stronger equational condition $e_1(f(x,y)) = e_2(f(y,x))$. If we include the cases, in which $\mathbb{A}$ itself is not a model-complete core, we are able to show the following theorem:

Theorem 3.6.2. *Let $\mathbb{A}$ be a reduct of $\mathbb{P}$ and let $\mathbb{A}^c$ be the model-complete core of $\mathbb{A}$. Then the following are equivalent:*

1. *There is a binary $f \in \mathrm{Pol}(\mathbb{A}^c)$ and endomorphisms $e_1, e_2 \in \mathrm{End}(\mathbb{A}^c)$ such that*
$$e_1(f(x,y)) = e_2(f(y,x))$$
or there is a ternary $f \in \mathrm{Pol}(\mathbb{A}^c)$ and endomorphisms $e_1, e_2, e_3 \in \mathrm{End}(\mathbb{A}^c)$ such that
$$e_1(f(x,x,y)) = e_2(f(x,y,x)) = e_3(f(y,x,x)).$$

2. *There is a pseudo Siggers polymorphism, i.e. a function $f \in \mathrm{Pol}(\mathbb{A}^c)^{(6)}$ and endomorphism $e_1, e_2 \in \mathrm{End}(\mathbb{A}^c)$ such that*
$$e_1(f(x,y,x,z,y,z)) = e_2(f(y,x,z,x,z,y)).$$

3. *For all $c_1, \dots, c_n \in \mathbb{A}^c$ there is no clone homomorphism from $\mathrm{Pol}(\mathbb{A}^c)$ onto $\mathbf{1}$.*

4. *For all $c_1, \dots, c_n \in \mathbb{A}^c$ there is no continuous clone homomorphism from $\mathrm{Pol}(\mathbb{A}^c)$ onto $\mathbf{1}$.*

5. *There is no pp-interpretation of $\mathbb{S}$ in any expansion of $\mathbb{A}^c$ by finitely many constants.*

Proof. First of all we remark that the equivalence of the points (2)-(5) holds for all ω-categorical core structures and was discussed in Theorem 1.6.9.

In Proposition 3.1.2 we saw that the model-complete core $\mathbb{A}^c$ is either equal to $\mathbb{A}$ or a reduct of $(\mathbb{Q}, <)$ or $(\omega, =)$.

Suppose the core $\mathbb{A}^c$ is a reduct of $(\mathbb{Q}, <)$ or $(\omega, =)$. We know from the analysis of temporal constraint satisfaction problems that then the statement is true: By Theorem 10.1.1. in [Bod12] there is no pp-interpretation of $\mathbb{S}$ in $\mathbb{A}^c$, if and only if an equation $e_1(f(x, x, y)) = e_2(f(x, y, x)) = e_3(f(y, x, x))$ holds in $\mathrm{Pol}(\mathbb{A}^c)$.

So let $\mathbb{A} = \mathbb{A}^c$. By Lemma 3.6.1 the equivalence (1)$\leftrightarrow$(4) holds when $<$ and $\perp$ are pp-definable in $\mathbb{A}$. In the remaining cases $\mathrm{End}(\mathbb{A})$ is equal to $\overline{\langle \updownarrow \rangle}$, $\overline{\langle \circlearrowright \rangle}$ or $\overline{\langle \updownarrow, \circlearrowright \rangle}$ and we have a pp-interpretation of $\mathbb{S}$ in an extension of $\mathbb{A}$ with finitely many constants by Propositions 3.3.2, 3.5.2, 3.5.4 and 3.5.6. □

On the relational side, we can sum up our dichotomy result to the following complexity dichotomy:

Theorem 3.6.3. *Let $\mathbb{A}$ be a reduct of $\mathbb{P}$ in a finite relational language and a model-complete core. Under the assumption P≠NP either*

- *one of the relations* Low*,* Betw*,* Cycl*,* Sep *is pp-definable in $\mathbb{A}$ and* CSP($\mathbb{A}$) *is NP-complete or*
- CSP($\mathbb{A}$) *is in P.*

Proof. If Low, Betw, Cycl or Sep is pp-definable in $\mathbb{A}$, the CSP($\mathbb{A}$) is NP-complete by Propositions 3.3.2, 3.5.2, 3.5.4 and 3.5.6.

By Proposition 3.1.2 the only remaining case is the one, where $<$ and $\perp$ are pp-definable, but Low is not. In this case $e_<$ or $e_\leq$ is a polymorphism of $\mathbb{A}$ by Proposition 3.4.1. Lemma 3.2.3 then implies that the problem is tractable. □

Including also the non-model complete cores in the analysis we obtain the following result:

Corollary 3.6.4. *Let $\mathbb{A}$ be a reduct of $\mathbb{P}$ in a finite relational language. Under the assumption P≠NP the problem* CSP($\mathbb{A}$) *is either NP-complete or solvable in polynomial time. Further the "meta-problem" of deciding whether* CSP($\mathbb{A}$) *is tractable or NP-complete, is decidable.*

Proof. By Proposition 3.1.2 we know that either $\mathbb{A}$ is a model-complete core or $g_<$ or $g_\perp$ are endomorphisms of $\mathbb{A}$. In the first case the dichotomy holds by Theorem 3.6.3, in the second case $\mathbb{A}$ is homomorphically equivalent to a reduct of $(\mathbb{Q}, <)$ and the dichotomy holds by the result in [BK09] respectively [BK08].

The main result in [BPT13] implies that it is decidable if the relations $<$, $\perp$, Low, Betw, Cycl or Sep are pp-definable in $\mathbb{A}$. By Lemma 3.1.5 the question whether $\mathbb{A}$ is model-complete core or not is then also decidable. By Theorem 3.6.3 and Corollary 52 of [BK09] we have that the meta-problem is decidable. □

We finish with an algebraic version of our dichotomy that is a direct implication of Theorem 3.6.2:

Corollary 3.6.5. *Let* $\mathbb{A}$ *be a reduct of* $\mathbb{P}$ *in a finite relational language and let* $\mathbb{A}^c$ *be its model complete core. Under the assumption P≠NP either*

- CSP($\mathbb{A}$) *is NP-complete and all finite structures are pp-interpretable in* $\mathbb{A}^c$*, extended by finitely many constant, or*
- CSP($\mathbb{A}$) *is tractable and the conditions (1)-(5) in Theorem 3.6.2 hold.* □

Chapter 4

A counterexample to the reconstruction of ω-categorical structures from their endomorphism monoids

How much information about a structure $\mathbb{A}$ is coded into its automorphisms group $\mathrm{Aut}(\mathbb{A})$? We saw that classical model theory provides a strong form of reconstruction of $\mathbb{A}$ from $\mathrm{Aut}(\mathbb{A})$ when $\mathbb{A}$ is ω-categorical: $\mathrm{Aut}(\mathbb{A})$ is equal to $\mathrm{Aut}(\mathbb{B})$ as a permutation group for some structure $\mathbb{B}$ if and only if $\mathbb{B}$ has a first-order definition in $\mathbb{A}$, and vice versa. The assumption that $\mathbb{A}$ is ω-categorical is in some sense best possible for this type of reconstruction: it can be seen that when $\mathbb{A}$ has a countable signature, then the above reconstruction statement holds if and only if $\mathbb{A}$ is ω-categorical by Theorem 1.4.3.

The situation is more complicated when we only know that $\mathrm{Aut}(\mathbb{A})$ and $\mathrm{Aut}(\mathbb{B})$ are isomorphic as groups. To approach this question, it is essential to first examine $\mathrm{Aut}(\mathbb{A})$ and $\mathrm{Aut}(\mathbb{B})$ as topological groups, equipped with the topology of pointwise convergence. With this topology, automorphism groups of countable structures are precisely the closed subgroups of the full symmetric group $\mathrm{Sym}(\omega)$ on ω. By Theorem 1.4.6 $\mathrm{Aut}(\mathbb{A})$ and $\mathrm{Aut}(\mathbb{B})$ are isomorphic as topological groups (that is, via an isomorphism that is also a homeomorphism), then $\mathbb{A}$ and $\mathbb{B}$ are first-order bi-interpretable. Hence, we focus on the following subproblem: is it true that when $\mathrm{Aut}(\mathbb{A})$ and $\mathrm{Aut}(\mathbb{B})$ are isomorphic as groups, then they are also isomorphic as topological groups?

Rather surprisingly, isomorphisms between automorphism groups of countable structures are typically homeomorphisms, see also the discussion about the small index property in Section 1.4.1. And in fact, it is consistent with ZF + DC that *all* homomorphisms between closed subgroups of $\mathrm{Sym}(\omega)$ are continuous, and that all isomorphisms between closed subgroups of $\mathrm{Sym}(\omega)$ are homeomorphisms; see the end of Section 4.2.2 for more explanation. Using the existence of non-principal ultrafilters on ω, it is relatively easy to show that there are oligomorphic permutation groups with non-continuous homomorphisms to $\mathbb{Z}_2$. But it was open for a while whether for countable ω-categorical structures $\mathbb{A}$ and $\mathbb{B}$ the existence of an isomorphism between $\mathrm{Aut}(\mathbb{A})$ and $\mathrm{Aut}(\mathbb{B})$ implies the existence of an isomorphism which is additionally a homeomorphism. This problem was solved by Evans and Hewitt [EH90], by giving two structures $\mathbb{A}$ and $\mathbb{B}$ for which the answer was negative.

Natural objects that carry more information about a structure $\mathbb{A}$ than $\mathrm{Aut}(\mathbb{A})$ are its *endomorphism monoid* $\mathrm{End}(\mathbb{A})$ and more generally its *polymorphism clone* $\mathrm{Pol}(\mathbb{A})$, see Section 1.6. We are going to show the following theorem related to results of Lascar [Las89]; see also the discussion in Section 4.3.

Theorem 4.0.1. *There are countable ω-categorical structures $\mathbb{A}$, $\mathbb{B}$ such that $\mathrm{End}(\mathbb{A})$ and $\mathrm{End}(\mathbb{B})$ are isomorphic, but not topologically isomorphic.*

In fact, the two endomorphism monoids of the structures $\mathbb{A}$ and $\mathbb{B}$ will be the closures in ω^ω of the two automorphism groups which are isomorphic, but not topologically isomorphic, presented in [EH90]. Ironically, it is its non-continuity which makes the extension of the isomorphism between those groups to their closures non-trivial, giving rise to the present work.

It has been asked in [BPP17] whether there are ω-categorical structures whose polymorphism clones are isomorphic, but not topologically. Theorem 4.0.1 immediately implies a positive answer to this question: any two structures whose polymorphism clones consist essentially (that is, up to adding of dummy variables) of the functions in $\mathrm{End}(\mathbb{A})$ and $\mathrm{End}(\mathbb{B})$, respectively, are examples.

Corollary 4.0.2. *There are countable ω-categorical structures $\mathbb{A}$, $\mathbb{B}$ such that $\mathrm{Pol}(\mathbb{A})$ and $\mathrm{Pol}(\mathbb{B})$ are isomorphic, but not topologically isomorphic.*

The construction in [EH90] is based on a representation of profinite groups as quotients of oligomorphic groups, due to Hrushovski, and on a non-reconstruction result for profinite groups which uses the axiom of choice. The non-reconstruction lifts to the oligomorphic groups representing the profinite groups.

In the present paper we show that it lifts further to the closures of the oligomorphic groups. The method of embedding profinite groups into quotients of oligomorphic structures is quite powerful and might be useful in different contexts as well.

The structures constructed in our proof of Theorem 4.0.1 have an infinite relational language. We use a well-known construction due to Hrushovski to encode countable ω-categorical structures into structures with a finite relational language, and show that this encoding is compatible with our examples, roughly because the encoding preserves model-completeness. That way, we obtain the following main theorem of the present article.

Theorem 4.0.3. *There exists a countable ω-categorical structure $\mathbb{A}$ in a finite relational language such that none of* Aut($\mathbb{A}$), End($\mathbb{A}$), *and* Pol($\mathbb{A}$) *have* reconstruction *(cf. [BPP17]): that is, there exists a countable ω-categorical structure $\mathbb{B}$ such that* Aut($\mathbb{A}$) *and* Aut($\mathbb{B}$), End($\mathbb{A}$) *and* End($\mathbb{B}$), *and* Pol($\mathbb{A}$) *and* Pol($\mathbb{B}$) *are isomorphic, but not topologically isomorphic.*

4.1 Preliminaries

Most of the relevant notions needed for this chapter were introduced in Section 1.2 and Section 1.6. In order to distinguish better between (oligomorphic) permutation groups and (profinite) topological groups, we will denote the appearing oligomorphic permutation groups by capital greek letters (Φ, Σ, Λ) and the profinite topological groups by bold latin letters ($\mathbf{G}$, $\mathbf{G}'$).

For a subgroup $\mathbf{H}$ of $\mathbf{G}$ we write $\mathbf{H} \leq \mathbf{G}$, and we write $g\mathbf{H} := \{gh : h \in H\}$ for the (left-) coset of $\mathbf{H}$ in $\mathbf{G}$ containing g. We denote by $\mathbf{G}/\mathbf{H}$ the set of all cosets of $\mathbf{H}$ in $\mathbf{G}$. If $\mathbf{H}$ is a normal subgroup of $\mathbf{G}$ then $\mathbf{G}/\mathbf{H}$ carries a natural group structure which is a topological group with respect to the quotient topology. We write $\mathbf{G} \cong \mathbf{H}$ if $\mathbf{G}$ and $\mathbf{H}$ are isomorphic as groups, and $\mathbf{G} \cong_T \mathbf{H}$ if $\mathbf{G}$ and $\mathbf{H}$ are *topologically isomorphic*, that is, there exists an isomorphism which is also a homeomorphism. When forming direct products $\mathbf{G} \times \mathbf{H}$ of topological groups $\mathbf{G}$ and $\mathbf{H}$, then the group $\mathbf{G} \times \mathbf{H}$ is equipped with the product topology of $\mathbf{G}$ and $\mathbf{H}$.

For background on profinite groups, we refer to the text book of Ribes and Zalesskii [RZ00].

4.2 The Proof

4.2.1 Overview

The idea is to obtain the results in the following steps.

(1) There exist separable profinite groups $\mathbf{G}$ and $\mathbf{G}'$ which are abstractly but not topologically isomorphic: $\mathbf{G} \cong \mathbf{G}'$ but $\mathbf{G} \ncong_T \mathbf{G}'$.

(2) There is a oligomorphic permutation group Φ on a countable set such that for every separable profinite group $\mathbf{R}$ there exists a closed permutation group $\Sigma_{\mathbf{R}} \geq \Phi$ such that $\mathbf{R} \cong_T \Sigma_{\mathbf{R}}/\Phi$. Furthermore Φ can be characterized in the

topological group structure of $\Sigma_{\mathbf{R}}$ as the intersection of the open normal subgroups of finite index.

It would then be natural to continue by the following steps. However, we do not know whether (3) is true, so the argument will proceed in a less direct way, but still following the outline below.

(3) For the separable profinite groups $\mathbf{G}$ and $\mathbf{G}'$ from (1), the permutation groups $\Sigma_{\mathbf{G}}$ and $\Sigma_{\mathbf{G}'}$ are isomorphic.

(4) $\Sigma_{\mathbf{G}}$ and $\Sigma_{\mathbf{G}'}$ cannot be topologically isomorphic, since by (2) any topological isomorphism would have to send Φ onto itself, and so $\Sigma_{\mathbf{G}}/\Phi$ and $\Sigma_{\mathbf{G}'}/\Phi$ would be topologically isomorphic, contradicting (1).

(5) The isomorphism between the permutation groups $\Sigma_{\mathbf{G}}$ and $\Sigma_{\mathbf{G}'}$ extends to their topological closures $\overline{\Sigma_{\mathbf{G}}}$ and $\overline{\Sigma_{\mathbf{G}'}}$ in ω^ω. However, the closed monoids $\overline{\Sigma_{\mathbf{G}}}$ and $\overline{\Sigma_{\mathbf{G}'}}$ are not topologically isomorphic: otherwise we would obtain a topological isomorphism between $\Sigma_{\mathbf{G}}$ and $\Sigma_{\mathbf{G}'}$ by restricting any topological isomorphism between $\overline{\Sigma_{\mathbf{G}}}$ and $\overline{\Sigma_{\mathbf{G}'}}$, contradicting (4).

(6) The closed oligomorphic function clones containing precisely the essentially unary functions obtained from $\overline{\Sigma_{\mathbf{G}}}$ and $\overline{\Sigma_{\mathbf{G}'}}$ are isomorphic by extending the isomorphism between $\overline{\Sigma_{\mathbf{G}}}$ and $\overline{\Sigma_{\mathbf{G}'}}$ naturally. However, they are not topologically isomorphic as otherwise $\overline{\Sigma_{\mathbf{G}}}$ and $\overline{\Sigma_{\mathbf{G}'}}$ would be topologically isomorphic as well by restricting any topological isomorphism between the functions clones to their unary sort.

(7) $\Sigma_{\mathbf{G}}$ can be encoded in a structure in a finite language such that the above arguments still work.

We remark that the steps (1)-(3) have already been discussed in [EH90], but we are going to recapitulate them for the convenience of the reader and to build on the construction in the further steps. The profinite group $\mathbf{G}$ in (1) has been known for a long time [Wit54]. Its properties were used in [EH90] to construct the profinite group $\mathbf{G}'$ that is isomorphic, but not topologically isomorphic to it. The proof of step (2) is due to an idea of Cherlin and Hrushovski, and (7) to another idea of Hrushovski.

The biggest technical challenge is step (3), and similarly, step (5). It is worth noting that we do not know whether (3) and (5) are true in general; our proof depends on the particular structure of the group $\mathbf{G}$ from (1). In fact, our proof will deviate from the above presentation in that we will not directly work with $\mathbf{G}$ but with a factor thereof. We find it, however, useful to have the above schema in mind since it does reflect the general proof idea.

4.2.2 Profinite groups

In this section we are going to discuss the profinite group $\mathbf{G}$ that will be the basis of our counterexample. We say a subgroup $\mathbf{F}' \leq \mathbf{G}$ is a *complement* of a normal subgroup $\mathbf{F}$ of $\mathbf{G}$ iff $\mathbf{G} = \mathbf{F} \cdot \mathbf{F}'$ and $\mathbf{F} \cap \mathbf{F}'$ is the identity subgroup.

Proposition 4.2.1. *There exists a separable profinite group* $\mathbf{G}$ *with the following properties:*

- $\mathbf{G}$ *has a non-trivial, finite central subgroup* $\mathbf{F}$ *with a dense complement* $\mathbf{F}'$ *in* $\mathbf{G}$*;*
- *any complement of any finite central subgroup of* $\mathbf{G}$ *is dense in* $\mathbf{G}$*.*

The construction of this profinite group can be found in [EH90, Theorem 4.1], where it is also used to answer a question about relative categoricity. We remark that the same group had already been constructed in [Wit54] in a different context, namely to provide an example of a compact separable group with a non-compact commutator subgroup.

Lemma 4.2.2. *Let* $\mathbf{G}, \mathbf{F}$ *and* $\mathbf{F}'$ *be as in Proposition 4.2.1. Then:*

- $\mathbf{G}/\mathbf{F}$ *is a profinite group which is isomorphic, but not topologically isomorphic to* $\mathbf{F}'$*;*
- $\mathbf{G}$ *and* $\mathbf{G}/\mathbf{F} \times \mathbf{F}$ *are isomorphic as groups, but are not topologically isomorphic.*

Proof. Since $\mathbf{F}$ is central we have that $\mathbf{G} = \mathbf{F}' \cdot \mathbf{F}$. Since moreover $\mathbf{F}' \cap \mathbf{F}$ is the identity subgroup, every $g \in \mathbf{G}$ has a unique representation $g = f'f$, where $f' \in \mathbf{F}'$ and $f \in \mathbf{F}$. Hence every coset $g\mathbf{F}$ contains exactly one representative from $\mathbf{F}'$. So the restriction of the quotient homomorphism $\mathbf{G} \to \mathbf{G}/\mathbf{F}$ to $\mathbf{F}'$ is bijective and thus an isomorphism. Since $\mathbf{F}$ is closed, $\mathbf{G}/\mathbf{F}$ is a profinite group; in particular it is compact. By Proposition 4.2.1 $\mathbf{F}'$ is not closed in $\mathbf{G}$ and therefore not compact. So $\mathbf{G}/\mathbf{F}$ and $\mathbf{F}'$ cannot be topologically isomorphic.

Since $\mathbf{F}$ is central in $\mathbf{G}$, we have that $\mathbf{F}' \times \mathbf{F}$ is isomorphic to $\mathbf{G}$, and so is $\mathbf{G}/\mathbf{F} \times \mathbf{F}$ by the above. However, no isomorphism from $\mathbf{G}/\mathbf{F} \times \mathbf{F}$ to $\mathbf{G}$ can be a topological one. Otherwise, the image of $\mathbf{F}$ (viewed as a subgroup of $\mathbf{G}/\mathbf{F} \times \mathbf{F}$ in the natural embedding) would be central in $\mathbf{G}$ and so the image of $\mathbf{G}/\mathbf{F}$ would have to be a proper dense subgroup of $\mathbf{G}$, by Proposition 4.2.1. Therefore it would not be closed, contradicting compactness. □

Notation 4.2.3. From now on, we fix groups $\mathbf{G}$, $\mathbf{F}$, and $\mathbf{F}'$ as in Proposition 4.2.1. We moreover denote the isomorphism from $\mathbf{G}/\mathbf{F}$ onto $\mathbf{F}'$ which sends every class $g\mathbf{F}$ to the unique element in $g\mathbf{F} \cap \mathbf{F}'$ by κ.

We remark that the axiom of choice was used to show the existence of the pair of subgroups $\mathbf{F}, \mathbf{F}'$ in $\mathbf{G}$ in Proposition 4.2.1. This seems unavoidable: it is well-known that every Baire measurable homomorphism between Polish groups is continuous (see e.g. [Kec95]). Further the statement that every set is Baire measurable is consistent with ZF+DC ([She84]). Thus the existence of two separable profinite groups (respectively two closed oligomorphic groups) that are isomorphic, but not topologically isomorphic, cannot be proven in ZF+DC (see the discussion in [BP15b]). The insufficiency of ZF+DC to construct a non-continuous homomorphism between Polish groups was already observed in [Las91].

4.2.3 Encoding profinite groups as factors of oligomorphic groups

The next step is to describe a given separable profinite group as a factor of two oligomorphic permutation groups. Our argument is a generalization of an argument of Cherlin and Hrushovski, which can be used to show that there are oligomorphic groups without the small index property [Las82]. A similar construction is also used in [BPP] to show that there is an oligomorphic clone on a countable set with a discontinuous homomorphism onto the projection clone. The result also appears in [EH90].

Proposition 4.2.4. *There is a closed oligomorphic permutation group Φ on a countable set X such that for any separable profinite group $\mathbf{R}$ there exists a closed permutation group $\Sigma_{\mathbf{R}}$ such that $\Phi \leq \Sigma_{\mathbf{R}} \leq \mathrm{Sym}(X)$ and:*

- *Φ is a closed normal subgroup of $\Sigma_{\mathbf{R}}$,*
- *Φ is the intersection of the open subgroups of $\Sigma_{\mathbf{R}}$ of finite index,*
- *$\mathbf{R} \cong_T \Sigma_{\mathbf{R}}/\Phi$.*

Proof. We first prove the proposition for the special case $\mathbf{R} = \prod_{n\geq 1} \mathrm{Sym}(n)$. Let L be the language containing an n-ary relation symbol P_i^n for all integers $1 \leq i \leq n$. Then we consider the class of all finite L-structures such that

- for all $n \geq i \geq 1$: $P_i^n(\bar{x})$ implies that the entries of $\bar{x}$ are distinct;
- for all $n \geq 1$: $P_1^n, \ldots, P_n^n$ form a partition of the n-tuples with distinct entries.

It is easy to verify that this class is an amalgamation class. Thus there is a unique countable homogeneous structure $\mathbb{A}^* = (A, (P_i^n)_{n\geq i\geq 1})$ whose *age*, i.e., its set of finite induced substructures up to isomorphism, is equal to this class. Since the number of relations of any fixed arity in $\mathbb{A}^*$ is finite, $\mathbb{A}^*$ is ω-categorical. We set Φ to be the automorphism group of $\mathbb{A}^*$.

For every n, let $E^n(\bar{x}, \bar{y})$ be the $2n$-ary relation on A that holds if and only if $\bar{x}$ and $\bar{y}$ are members of the same partition class P_i^n. By definition, the relation E^n forms an equivalence relation on the n-tuples with distinct entries that has the sets P_i^n as equivalence classes. We set $\Sigma_{\mathbf{R}}$ to be the automorphism group of $(A, (E^n)_{n\geq 1})$. Clearly every E^n is definable in $\mathbb{A}^*$, so $\Phi \leq \Sigma_{\mathbf{R}}$. By verifying that $(A, (E^n)_{n\geq 1})$ has the extension property, one can easily see that it is a homogeneous structure.

Every function in $\Sigma_{\mathbf{R}}$ induces a permutation on the set $X_n := \{P_i^n : 1 \leq i \leq n\}$, for every $n \geq 1$. The action of $\Sigma_{\mathbf{R}}$ on the disjoint union of X_n gives us a homomorphism $\mu_{\mathbf{R}} \colon \Sigma_{\mathbf{R}} \to \mathbf{R}$. The homogeneity of $(A, (E^n)_{n\geq 1})$ guarantees that every permutation on a finite subset of $\bigcup_{n\geq 1} X_n$ (respecting the arities n) is induced by an element of $\Sigma_{\mathbf{R}}$. This fact, together with a standard back-and-forth-argument, implies that we can obtain every permutation on the full union $\bigcup_{n\geq 1} X_n$ as the action of an element of $\Sigma_{\mathbf{R}}$. In other words, $\mu_{\mathbf{R}}$ is surjective. Every stabilizer in $\Sigma_{\mathbf{R}}$ of a finite subset of $\bigcup_{n\geq 1} X_n$ is an open subgroup, hence $\mu_{\mathbf{R}}$ is continuous and open. The kernel of $\mu_{\mathbf{R}}$ is Φ, so we have $\Sigma_{\mathbf{R}}/\Phi \cong_T \mathbf{R}$.

Finally, we want to prove that Φ is the intersection of the open subgroups of $\Sigma_{\mathbf{R}}$ of finite index. It is clear that Φ contains this intersection, since Φ is the intersection of the preimages of all the stabilizers of X_n, $n \geq 1$, under the action of $\mu_{\mathbf{R}}$. It remains to show that Φ has no proper open subgroup of finite index.

Suppose that Φ has a proper open subgroup $\Lambda \leq \Phi$ of finite index. Since Λ is open, there is a finite tuple $\bar{y}$ of distinct elements in A such that its stabilizer $\Phi_{(\bar{y})}$ lies entirely in Λ. We will obtain a contradiction by studying the actions of Φ and Λ on $\bar{y}$. Let $O_\Phi(\bar{y}) := \{g(\bar{y}) : g \in \Phi\}$ and $O_\Lambda(\bar{y}) := \{g(\bar{y}) : g \in \Lambda\}$ be the orbits of $\bar{y}$ under these actions. Now $O_\Phi(\bar{y})$ can be partitioned into subsets of the form $gO_\Lambda(\bar{y})$, where $g \in \Phi$. This partition is clearly preserved under the action of Φ. For all $g \in \Phi$ the following holds:

$$g(\bar{y}) \in O_\Lambda(\bar{y}) \Leftrightarrow \exists h \in \Lambda(g(\bar{y}) = h(\bar{y})) \Leftrightarrow \exists h \in \Lambda(h^{-1} \circ g \in \Phi_{(\bar{y})}) \Leftrightarrow g \in \Lambda.$$

Thus the index $|\Lambda : \Phi|$ coincides with the number of partition classes $gO_\Lambda(\bar{y})$ in $O_\Phi(\bar{y})$. Since this index is greater than 1, there exists $\bar{b} \in O_\Phi(\bar{y})$ outside the class $O_\Lambda(\bar{y})$.

We next claim that there exists a tuple $\bar{a} \in O_\Lambda(\bar{y})$ such that all elements of the tuple $(\bar{a}, \bar{y})$ are distinct. Otherwise in every tuple $(\bar{a}, \bar{y})$ with $\bar{a} \in O_\Lambda(\bar{y})$ an equation $a_i = y_j$ holds; we will derive a contradiction. For all $i \in \omega$, pick $f_i \in \Phi$ such that for all $i \neq j$ the tuples $f_i(\bar{y})$ and $f_j(\bar{y})$ contain no common values. This is possible by the construction of $\mathbb{A}^*$. By our assumption, for every function g in the coset $f_i\Lambda$, $g(\bar{y})$ contains an element of the tuple $f_i(\bar{y})$. By the choice of the f_i, it follows that for $i \neq j$, the cosets $f_i\Lambda$ and $f_j\Lambda$ are disjoint. This is a contradiction to the finite index of Λ in Φ.

There exists $\bar{d} \in O_\Phi(\bar{y})$ such that the tuples $(\bar{y}, \bar{a}), (\bar{d}, \bar{a})$, and $(\bar{d}, \bar{b})$ lie in the same orbit with respect to the action of Φ. This follows from the extension property of $\mathbb{A}^*$. So there are functions $h_1, h_2 \in \Phi$ such that $h_1(\bar{y}, \bar{a}) = (\bar{d}, \bar{a})$ and $h_2(\bar{d}, \bar{a}) = (\bar{d}, \bar{b})$. Since Φ preserves our partition, $h_1(\bar{y}, \bar{a}) = (\bar{d}, \bar{a})$ implies that $\bar{d}$ lies in $O_\Lambda(\bar{y})$. But because of $h_2(\bar{d}, \bar{a}) = (\bar{d}, \bar{b})$ also $\bar{b}$ lies in the same class, which is a contradiction.

We have shown the proposition for $\mathbf{R} = \prod_{n\geq 1} \mathrm{Sym}(n)$. Let now $\mathbf{R}'$ be an arbitrary separable profinite group. As such, it is topologically isomorphic to a closed subgroup of $\mathbf{R}$, so without loss of generality let $\mathbf{R}' \leq \mathbf{R}$. We set $\Sigma_{\mathbf{R}'}$ to be the preimage of $\mathbf{R}'$ under $\mu_{\mathbf{R}}$. Clearly then $\mathbf{R}' \cong_T \Sigma_{\mathbf{R}'}/\Phi$. Again Φ is the intersection of all the stabilizers of X_n in $\Sigma_{\mathbf{R}'}$ for $n \geq 1$, implying that the intersection of all open subgroups of finite index in $\Sigma_{\mathbf{R}'}$ is contained in Φ. Since Φ has no proper open subgroup of finite index, they are equal. □

Notation 4.2.5. From now on let Φ be the oligomorphic permutation group defined in the proof of Proposition 4.2.4 and A be its domain. Also let $\mu_{\mathbf{R}} \colon \Sigma_{\mathbf{R}} \to \mathbf{R}$ be the quotient mapping described in the proof.

4.2.4 Lifting the isomorphism to the encoding groups

Let $\mathbf{G}$ be as in Proposition 4.2.1. The most natural next step in the proof might be to lift the non-topological isomorphism between $\mathbf{G}$ and $\mathbf{G}' := \mathbf{G}/\mathbf{F} \times \mathbf{F}$ to an isomorphism between $\Sigma_{\mathbf{G}}$ and $\Sigma_{\mathbf{G}'}$. However, we do not know if this is possible. Instead, we will work with $\Sigma_{\mathbf{G}/\mathbf{F}} \times \mathbf{F}$ and the closure of $\Sigma_{\mathbf{G}/\mathbf{F}}$ in a discontinuous action as a permutation group.

As technical preparation for this, we will now provide a particular representation of the topological group $\mathbf{G}$ as a permutation group (i.e., a topological isomorphism with a permutation group).

A representation of G as a permutation group

As a separable profinite group, $\mathbf{G}$ contains a countable sequence $(\mathbf{G}_i)_{i\in\omega}$ of open normal subgroups with trivial intersection. Since $\mathbf{G}$ is compact, the factor groups $\mathbf{G}/\mathbf{G}_i$ are finite. Letting $\mathbf{G}$ act on the disjoint union of the factor groups by translation, we obtain a topologically faithful action of $\mathbf{G}$, i.e., a representation of $\mathbf{G}$ as a closed permutation group on the countable set $\bigcup_{i\in\omega} \mathbf{G}/\mathbf{G}_i$. In particular, we then have a representation of the subgroup $\mathbf{F}'$ as a (non-closed) permutation group on $\bigcup_{i\in\omega} \mathbf{G}/\mathbf{G}_i$.

Recall that $\mathbf{F}'$ is naturally isomorphic to $\mathbf{G}/\mathbf{F}$, but not topologically isomorphic to it. In the following, we will pick the open normal subgroups $\mathbf{G}_i$ mentioned above in such a way that the restriction of the action of $\mathbf{F}'$ to $\bigcup_{i\geq 1} \mathbf{G}/\mathbf{G}_i$ (where $\mathbf{G}/\mathbf{G}_0$ is missing) will still be faithful and hence isomorphic to $\mathbf{F}'$; however, it will be topologically isomorphic to $\mathbf{G}/\mathbf{F}$, and in particular not topologically isomorphic to $\mathbf{F}'$. Note that the topology on $\mathbf{G}/\mathbf{F}$ is obtained from the topology of $\mathbf{F}'$ by factorizing modulo $\mathbf{F}$, and hence is coarser than the topology on $\mathbf{F}'$, making such an undertaking possible.

Our action of $\mathbf{G}$ on $\bigcup_{i\in\omega} \mathbf{G}/\mathbf{G}_i$ will moreover have the property that its restriction to $\mathbf{F}'$ will be isomorphic (as an action) to an action of $\mathbf{F}'$ on the disjoint union $\bigcup_{i\in\omega} \mathbf{F}'/\mathbf{F}'_i$ of certain coset spaces of $\mathbf{F}'$, rather than of $\mathbf{G}$. Hence, it can be defined from $\mathbf{F}'$ alone. In particular, since $\mathbf{F}'$ is dense in $\mathbf{G}$, the action of $\mathbf{G}$ can be reconstructed from $\mathbf{F}'$ and a particular sequence of normal subgroups $(\mathbf{F}'_i)_{i\in\omega}$ thereof. Note that not all of the $\mathbf{F}'_i$ will be open, since the action of $\mathbf{F}'$ on $\bigcup_{i\in\omega} \mathbf{F}'/\mathbf{F}'_i$ is not a closed permutation group. In fact, only $\mathbf{F}'_0$ will be non-open.

It is this particular representation of $\mathbf{G}$ as a permutation group which will allow us to lift isomorphisms to the oligomorphic permutation groups encoding our profinite groups. Note that we use the particular structure of $\mathbf{G}$, e.g., the density of $\mathbf{F}'$, to obtain the representation.

To obtain the desired open normal subgroups, we first pick a sequence $(\mathbf{H}_i)_{i\geq 1}$ of open normal subgroups of $\mathbf{G}/\mathbf{F}$ whose intersection is the identity. The sequence exists since $\mathbf{F}$ is closed and so $\mathbf{G}/\mathbf{F}$ is profinite. We now set $\mathbf{G}_i$ to be the preimage of $\mathbf{H}_i$ under the quotient mapping, i.e., $\mathbf{G}_i := \{hf \mid hF \in \mathbf{H}_i \text{ and } f \in \mathbf{F}\}$, for all $i \geq 1$. So each $\mathbf{G}_i$ is an open normal subgroup of $\mathbf{G}$, and $\bigcap_{i\geq 1} \mathbf{G}_i = \mathbf{F}$. To finish the construction, we pick an open normal subgroup $\mathbf{G}_0$ of $\mathbf{G}$ whose intersection with $\mathbf{F}$ is the identity; this is possible, because by profiniteness $\mathbf{G}$ contains a sequence of open normal subgroups with trivial intersection, and because $\mathbf{F}$ is finite. Finally, we set $\mathbf{F}'_i := \mathbf{F}' \cap \mathbf{G}_i$, for all $i \in \omega$.

Notation 4.2.6. We now fix $(\mathbf{G}_i)_{i\in\omega}$ and $(\mathbf{F}'_i)_{i\in\omega}$ as above, and let $\tau\colon \mathbf{G} \to \mathrm{Sym}(\bigcup_{i\in\omega} \mathbf{G}/\mathbf{G}_i)$ be the mapping which sends an element g of $\mathbf{G}$ to the permutation acting on $\bigcup_{i\in\omega} \mathbf{G}/\mathbf{G}_i$ by translation with g.

Lemma 4.2.7.

(1) *τ is faithful and continuous;*

(2) $\mathbf{F}$ *is the stabilizer of* $\bigcup_{i\geq 1} \mathbf{G}/\mathbf{G}_i$ *under the action* τ*;*

(3) *the restriction of* $\tau(\mathbf{F}')$ *to* $\bigcup_{i\geq 1} \mathbf{G}/\mathbf{G}_i$ *is a permutation group that is topologically isomorphic to* $\mathbf{G}/\mathbf{F}$*;*

(4) *the actions of* $\mathbf{F}'$ *on* $\bigcup_{i\in\omega} \mathbf{G}/\mathbf{G}_i$ *(via* τ*) and on* $\bigcup_{i\in\omega} \mathbf{F}'/\mathbf{F}'_i$ *(by translation) are isomorphic.*

(5) *the closure of* $\mathbf{F}'$ *in* $\mathrm{Sym}(\bigcup_{i\in\omega}\mathbf{F}'/\mathbf{F}'_i)$ *is isomorphic to* $\mathbf{G}$.

Proof.

(1) The elements of the family $(\mathbf{G}_i)_{i\in\omega}$ are open normal subgroups of $\mathbf{G}$ with trivial intersection. Thus τ is faithful and continuous.

(2) Since $\mathbf{F}$ is the intersection of all $(\mathbf{G}_i)_{i\geq 1}$, it is the stabilizer of $\bigcup_{i\geq 1}\mathbf{G}/\mathbf{G}_i$.

(3) For every $i \geq 1$ the quotient group $\mathbf{G}/\mathbf{G}_i$ is isomorphic to $(\mathbf{G}/\mathbf{F})/(\mathbf{G}_i/\mathbf{F})$. Thus the action of $\mathbf{F}'$ on $\bigcup_{i\geq 1}\mathbf{G}/\mathbf{G}_i$ is isomorphic to the action of $\mathbf{F}'$ on $\bigcup_{i\geq 1}(\mathbf{G}/\mathbf{F})/(\mathbf{G}_i/\mathbf{F})$, which is a representation of $\mathbf{G}/\mathbf{F}$ as permutation group since the intersection of the factors $(\mathbf{G}_i/\mathbf{F})$ is trivial by choice of the $\mathbf{G}_i$.

(4) Since $\mathbf{F}'$ is dense in $\mathbf{G}$ and all $\mathbf{G}_i$ are open, every coset in $\bigcup_{i\in\omega}\mathbf{G}/\mathbf{G}_i$ contains an element of $\mathbf{F}'$. Thus

$$\mathbf{G}/\mathbf{G}_i = \mathbf{F}'\mathbf{G}_i/\mathbf{G}_i \cong \mathbf{F}'/(\mathbf{F}'\cap\mathbf{G}_i) = \mathbf{F}'/\mathbf{F}'_i.$$

One can now easily verify that the actions of $\mathbf{F}'$ on $\bigcup_{i\in\omega}\mathbf{G}/\mathbf{G}_i$ and on $\bigcup_{i\in\omega}\mathbf{F}'/\mathbf{F}'_i$ are isomorphic.

(5) This follows from (4) as $\mathbf{F}'$ is dense in $\mathbf{G}$.

□

The lifting

We will now consider a discontinuous action of $\Sigma_{\mathbf{G}/\mathbf{F}}$, similarly to the action of $\mathbf{F}'$ on $\bigcup_{i\in\omega}\mathbf{G}/\mathbf{G}_i$ in Lemma 4.2.7, which is discontinuous if considered as an action of $\mathbf{G}/\mathbf{F}$ rather than of $\mathbf{F}'$: otherwise it would be closed as a permutation group, but its closure as a permutation group is topologically isomorphic to $\mathbf{G}$.

The quotient homomorphism $\mu_{\mathbf{G}/\mathbf{F}}\colon \Sigma_{\mathbf{G}/\mathbf{F}} \to \mathbf{G}/\mathbf{F}$ from Proposition 4.2.4 gives rise to an action of $\Sigma_{\mathbf{G}/\mathbf{F}}$ on the cosets $\bigcup_{i\in\omega}\mathbf{G}/\mathbf{G}_i$ by simply considering the composition $\tau\circ\kappa\circ\mu_{\mathbf{G}/\mathbf{F}}$. If we restrict this action to $\bigcup_{i\geq 1}\mathbf{G}/\mathbf{G}_i$ then it is continuous, as the composition of continuous functions. But if we regard the action on $\bigcup_{i\in\omega}\mathbf{G}/\mathbf{G}_i$, the action fails to be continuous, since the induced permutation group is topologically isomorphic to the non-closed $\mathbf{F}'$.

Recall that $\Sigma_{\mathbf{G}/\mathbf{F}}$ was defined as a closed, oligomorphic permutation group on a countable set A. Clearly, the combined action of $\Sigma_{\mathbf{G}/\mathbf{F}}$ on $A\cup\mathbf{G}/\mathbf{G}_0$ fails to be continuous. By χ we denote the embedding of $\Sigma_{\mathbf{G}/\mathbf{F}}$ into $\mathrm{Sym}(A\cup\mathbf{G}/\mathbf{G}_0)$. Then, analogously to $\mathbf{F}'$ in the profinite case, $\chi[\Sigma_{\mathbf{G}/\mathbf{F}}]$ is not closed in $\mathrm{Sym}(A\cup\mathbf{G}/\mathbf{G}_0)$.

Notation 4.2.8. Henceforth χ will denote the action of $\Sigma_{\mathbf{G}/\mathbf{F}}$ on $A\cup\mathbf{G}/\mathbf{G}_0$, and Γ the closure of $\chi[\Sigma_{\mathbf{G}/\mathbf{F}}]$ in $\mathrm{Sym}(A\cup\mathbf{G}/\mathbf{G}_0)$.

	Group	acting on	via	properties	image $\cong_T$	properties
(i)	$\mathbf{G}$	$\bigcup_{i\in\omega} \mathbf{G}/\mathbf{G}_i$	τ	faithful, cont.	$\mathbf{G}$	closed
(ii)	$\mathbf{G}/\mathbf{F}$	$\bigcup_{i\in\omega} \mathbf{G}/\mathbf{G}_i$	$\tau \circ \kappa$	faithful, discont.	$\mathbf{F}'$	non-closed
(iii)	$\Sigma_{\mathbf{G}/\mathbf{F}}$	$\bigcup_{i\in\omega} \mathbf{G}/\mathbf{G}_i$	$\tau \circ \kappa \circ \mu_{\mathbf{G}/\mathbf{F}}$	discontinuous	$\mathbf{F}'$	non-closed
(iv)	$\Sigma_{\mathbf{G}/\mathbf{F}}$	$\bigcup_{i\geq 1} \mathbf{G}/\mathbf{G}_i$	restr. of (iii)	continuous	$\mathbf{G}/\mathbf{F}$	closed
(v)	$\Sigma_{\mathbf{G}/\mathbf{F}}$	$A \cup \mathbf{G}/\mathbf{G}_0$	χ	faithful, discont.	$\chi[\Sigma_{\mathbf{G}/\mathbf{F}}]$	oligom., non-closed
(vi)	Γ	$A \cup \mathbf{G}/\mathbf{G}_0$	ext. of (v)	faithful, cont.	Γ	oligom., closed
(vii)	Γ	$\bigcup_{i\in\omega} \mathbf{G}/\mathbf{G}_i$	comb. of (iv), (vi)	continuous	$\mathbf{G}$	closed

Figure 4.1: Group actions, some of their properties, and the permutation groups they induce.

Figure 4.1 gives an overview to all the group actions we are considering.

Lemma 4.2.9.

(1) Γ *is a closed oligomorphic permutation group.*

(2) Γ *is the semidirect product* $\chi[\Sigma_{\mathbf{G}/\mathbf{F}}] \cdot \Gamma_{(A)}$.

(3) $\chi[\Phi]$ *is the intersection of the open subgroups of finite index in* Γ.

(4) $\Gamma/\chi[\Phi] \cong_T \mathbf{G}$.

(5) $\Gamma_{(A)}$ *is central in* Γ *and isomorphic to* $\mathbf{F}$.

(6) $\Gamma \cong \Sigma_{\mathbf{G}/\mathbf{F}} \times \mathbf{F}$.

Proof.

(1) Γ is closed by definition. As $\Sigma_{\mathbf{G}/\mathbf{F}}$ is oligomorphic on A and $\mathbf{G}/\mathbf{G}_0$ is finite, it follows that Γ is oligomorphic.

(2) The restriction function of Γ to A is a continuous homomorphism $|_A \colon \Gamma \to \Sigma_{\mathbf{G}/\mathbf{F}}$. Let $g \in \Gamma$ and let $(h_n)_{n\in\omega}$ be a sequence of permutations in $\Sigma_{\mathbf{G}/\mathbf{F}}$ such that $\chi(h_n)$ converges to g in Γ. Then $(h_n)_{n\in\omega}$ converges in $\Sigma_{\mathbf{G}/\mathbf{F}}$, since $h_n = |_A \circ \chi(h_n)$ for all $n \in \omega$. By h we denote its limit in $\Sigma_{\mathbf{G}/\mathbf{F}}$. The functions g and h are identical on A, thus $\chi(h)^{-1} \circ g \in \Gamma_{(A)}$. Moreover, $\chi[\Sigma_{\mathbf{G}/\mathbf{F}}]$ and $\Gamma_{(A)}$ have trivial intersection. Therefore Γ is the semidirect product of $\chi[\Sigma_{\mathbf{G}/\mathbf{F}}]$ and $\Gamma_{(A)}$.

(3) Note that χ is open and that it maps subgroups of finite index in $\Sigma_{\mathbf{G}/\mathbf{F}}$ to subgroups of finite index in Γ by (2). Since by Proposition 4.2.4 the permutation group Φ is the intersection of the open subgroups of finite index in $\Sigma_{\mathbf{G}/\mathbf{F}}$, we have that $\chi[\Phi]$ contains the intersection of open subgroups of finite index in Γ.

For the other inclusion we remark that $\chi[\Phi]$ fixes $\mathbf{G}/\mathbf{G}_0$. Therefore the restriction of χ to Φ is continuous. If now $\chi[\Phi]$ had a proper open subgroup of finite index, then its preimage under χ would be open and of finite index in Φ. Because of Proposition 4.2.4 it would be equal to Φ, a contradiction.

(4) By considering $\mu_{\mathbf{G}/\mathbf{F}} \circ |_A$ we get a continuous surjective homomorphism of Γ onto $\mathbf{G}/\mathbf{F}$. This gives us a continuous action of Γ on $\bigcup_{i\geq 1} \mathbf{G}/\mathbf{G}_i$, by further composing with the mapping $\tau \circ \kappa$. By additionally letting Γ act on $\mathbf{G}/\mathbf{G}_0$ by restriction of its domain we get a continuous action of Γ on $\bigcup_{i\in\omega} \mathbf{G}/\mathbf{G}_i$ (Item (vii) in Figure 4.1).

It is easily verified that $\chi[\Phi]$ is the kernel of the action of Γ on $\bigcup_{i\in\omega} \mathbf{G}/\mathbf{G}_i$. So $\Gamma/\chi[\Phi]$ is topologically isomorphic to the permutation group that Γ induces on $\bigcup_{i\in\omega} \mathbf{G}/\mathbf{G}_i$ via this action. By the definition of the action, if we consider its restriction to $\chi[\Sigma_{\mathbf{G}/\mathbf{F}}]$, then it induces the same permutation group on $\bigcup_{i\in\omega} \mathbf{G}/\mathbf{G}_i$ as the action of $\mathbf{G}/\mathbf{F}$ on $\bigcup_{i\in\omega} \mathbf{G}/\mathbf{G}_i$ – this permutation group is, by Lemma 4.2.7, topologically isomorphic to $\mathbf{F}'$. Since the action of Γ is continuous and Γ is the topological closure of $\chi[\Sigma_{\mathbf{G}/\mathbf{F}}]$ we get that the permutation group it induces is topologically isomorphic to the closure of the action of $\mathbf{F}'$ on $\bigcup_{i\in\omega} \mathbf{G}/\mathbf{G}_i$, which is in turn topologically isomorphic to $\mathbf{G}$. In conclusion we get that $\Gamma/\chi[\Phi] \cong_T \mathbf{G}$.

(5) In the action of Γ on $\bigcup_{i\in\omega} \mathbf{G}/\mathbf{G}_i$ from (4), the stabilizer of $\bigcup_{i\geq 1} \mathbf{G}/\mathbf{G}_i$ consists precisely of the elements of $\chi[\Phi] \cdot \Gamma_{(A)}$; this follows from (2) and the definition of the action. Since the permutation group induced by this action on $\bigcup_{i\in\omega} \mathbf{G}/\mathbf{G}_i$ coincides with the permutation group induced by the action τ of $\mathbf{G}$ on this set, and since the stabilizer of $\bigcup_{i\geq 1} \mathbf{G}/\mathbf{G}_i$ in the latter action is isomorphic to $\mathbf{F}$, we get that $\chi[\Phi] \cdot \Gamma_{(A)}$, factored by the kernel $\chi[\Phi]$, is isomorphic to $\mathbf{F}$. Hence, $\Gamma_{(A)}$ is isomorphic to $\mathbf{F}$. As $\mathbf{F}$ is a central subgroup of $\mathbf{G}$, $\Gamma_{(A)}$ is central in Γ.

(6) Since $\Gamma_{(A)}$ is a central normal subgroup, the semidirect product in (2) is a direct product. We conclude that, as groups:

$$\Gamma = \chi[\Sigma_{\mathbf{G}/\mathbf{F}}] \cdot \Gamma_{(A)} \cong \Sigma_{\mathbf{G}/\mathbf{F}} \times \mathbf{F}.$$

□

Notation 4.2.10. Let Δ be any closed oligomorphic permutation group on a countable set which is topologically isomorphic to $\Sigma_{\mathbf{G}/\mathbf{F}} \times \mathbf{F}$. The existence of Δ follows from the fact that $\Sigma_{\mathbf{G}/\mathbf{F}}$ is itself such a group and that $\mathbf{F}$ is finite.

Corollary 4.2.11. *The closed oligomorphic permutation groups Δ and Γ are isomorphic, but not topologically isomorphic.*

Proof. As we have seen in Lemma 4.2.9 (6), Δ and Γ are isomorphic as groups. Recall that $\chi[\Phi]$ is the intersection of the open subgroups of finite index in Γ, by Lemma 4.2.9 (3). By Proposition 4.2.4, Φ is the intersection of the open subgroups of finite index in $\Sigma_{\mathbf{G}/\mathbf{F}}$, and hence also in $\Sigma_{\mathbf{G}/\mathbf{F}} \times \mathbf{F}$. Thus any topological isomorphism from Γ to Δ sends $\chi[\Phi]$ onto Φ, and hence induces a topological isomorphism between the quotients $\Gamma/\chi[\Phi] \cong_T \mathbf{G}$ and $(\Sigma_{\mathbf{G}/\mathbf{F}}/\Phi) \times \mathbf{F} \cong_T \mathbf{G}/\mathbf{F} \times \mathbf{F}$, which is a contradiction to Lemma 4.2.2. □

4.2.5 Extending the isomorphism to the closures of the groups

Recall that, for a permutation group Θ, we denote by $\overline{\Theta}$ the topological closure of Θ in the space of all transformations on its domain, equipped with the topology of pointwise convergence.

Note that the elements of $\overline{\Theta}$ are precisely the elementary embeddings to itself of any structure whose automorphism group is Θ. Our aim in this section is to show that the monoids $\overline{\Delta}$ and $\overline{\Gamma}$ are isomorphic, but not topologically isomorphic. It is clear that $\overline{\Delta}$ and $\overline{\Gamma}$ are not topologically isomorphic, since the subgroups of invertible elements Δ and Γ are not. It is harder to show that they are isomorphic, since there seems to be no obvious way to carry it over from the permutation groups, the problem being the non-continuity of the isomorphism. We therefore need to further study the topological monoids $\overline{\Delta}$ and $\overline{\Gamma}$ and how they are related to the profinite group $\mathbf{G}$.

Lemma 4.2.12. *Let* $\mathbf{R}$ *be any separable profinite group. The continuous homomorphism* $\mu_{\mathbf{R}} \colon \Sigma_{\mathbf{R}} \to \Sigma_{\mathbf{R}}/\Phi \cong_T \mathbf{R}$ *extends to a continuous monoid homomorphism* $\overline{\mu_{\mathbf{R}}} \colon \overline{\Sigma_{\mathbf{R}}} \to \mathbf{R}$.

Proof. Recall that $\mu_{\mathbf{R}}$ was obtained via the action of $\Sigma_{\mathbf{R}}$ on $\bigcup_{n \geq 1} X_n$, where X_n consists of the equivalence classes of the relations E^n. Every element of $\overline{\Sigma_{\mathbf{R}}}$ agrees on every finite set with an element of $\Sigma_{\mathbf{R}}$. Therefore the functions in $\overline{\Sigma_{\mathbf{R}}}$ preserve the equivalence relations E^n and their negations for $n \geq 1$. Since every such relation has only finitely many equivalence classes, every element of $\overline{\Sigma_{\mathbf{R}}}$ induces a permutation on them. This action of $\overline{\Sigma_{\mathbf{R}}}$ on $\bigcup_{i \geq 1} X_i$ extends the action of $\Sigma_{\mathbf{R}}$ and gives us the continuous monoid homomorphism $\overline{\mu_{\mathbf{R}}}$. □

Recall the discontinuous action of $\Sigma_{\mathbf{G}/\mathbf{F}}$ on the cosets $\bigcup_{i \in \omega} \mathbf{G}/\mathbf{G}_i$ via the mapping $\tau \circ \kappa \circ \mu_{\mathbf{G}/\mathbf{F}}$ (Item (iii) in Figure 4.1). With the help of $\overline{\mu_{\mathbf{G}/\mathbf{F}}}$ we see that this action has a natural extension to $\overline{\Sigma_{\mathbf{G}/\mathbf{F}}}$. As before, the restriction of this action to $\bigcup_{i \geq 1} \mathbf{G}/\mathbf{G}_i$ is continuous, and the induced permutation group is isomorphic to $\mathbf{G}/\mathbf{F}$. It is with the action on $\mathbf{G}/\mathbf{G}_0$ that we lose the continuity.

By composing the continuous function $\overline{\mu_{\mathbf{G}/\mathbf{F}}} \circ |_A \colon \overline{\Gamma} \to \mathbf{G}/\mathbf{F}$ with the continuous action $\tau \circ \kappa$ of $\mathbf{G}/\mathbf{F}$ on $\bigcup_{i\geq 1} \mathbf{G}/\mathbf{G}_i$, we obtain a continuous action of $\overline{\Gamma}$ on $\bigcup_{i\geq 1} \mathbf{G}/\mathbf{G}_i$. By additionally letting $\overline{\Gamma}$ act on $\mathbf{G}/\mathbf{G}_0$ by restriction, we get a continuous action of $\overline{\Gamma}$ on $\bigcup_{i\in\omega} \mathbf{G}/\mathbf{G}_i$ which extends the action of Γ thereon.

Similarly to the situation with $\Sigma_{\mathbf{G}/\mathbf{F}}$, we can let $\overline{\Sigma_{\mathbf{G}/\mathbf{F}}}$ act on $A \cup \mathbf{G}/\mathbf{G}_0$, inducing an embedding $\overline{\chi}$ of $\overline{\Sigma_{\mathbf{G}/\mathbf{F}}}$ into the set of all transformations on $A \cup \mathbf{G}/\mathbf{G}_0$ which extends the group embedding χ from Lemma 4.2.9.

Lemma 4.2.13.

(1) $\overline{\Gamma} = \overline{\chi[\Sigma_{\mathbf{G}/\mathbf{F}}]}$.

(2) All elements of $\overline{\Gamma}$ *which stabilize (pointwise)* A *are invertible. Hence,* $\overline{\Gamma}_{(A)} = \Gamma_{(A)}$.

(3) The action of $\overline{\Gamma}$ *on* $\bigcup_{i\in\omega} \mathbf{G}/\mathbf{G}_i$ *induces a permutation group that is equal to* $\mathbf{G}$.

(4) $\overline{\Gamma}$ *is isomorphic to the monoid direct product* $\overline{\Sigma_{\mathbf{G}/\mathbf{F}}} \times \mathbf{F}$.

Proof.

(1) Γ was defined as the topological closure of $\chi[\Sigma_{\mathbf{G}/\mathbf{F}}]$ in $\mathrm{Sym}(A \cup \mathbf{G}/\mathbf{G}_0)$, so this is immediate.

(2) The functions in $\overline{\Gamma}$ are injective, so by finiteness of $\mathbf{G}/\mathbf{G}_0$ any element of $\overline{\Gamma}$ which fixes all points of A is bijective.

(3) The action of Γ on $\bigcup_{i\in\omega} \mathbf{G}/\mathbf{G}_i$ induces a permutation group that is topologically isomorphic to $\mathbf{G}$, by Lemma 4.2.9 (4). The action of $\overline{\Gamma}$ on $\bigcup_{i\in\omega} \mathbf{G}/\mathbf{G}_i$ extends this action. Since all permutations induced by the action of Γ have only finite orbits, and since the action of $\overline{\Gamma}$ is continuous, every element of $\overline{\Gamma}$ actually induces a permutation on $\bigcup_{i\in\omega} \mathbf{G}/\mathbf{G}_i$. Every such permutation is in turn already induced by the action of Γ, since the permutation group induced by this action is closed. Summarizing, the functions induced by the two actions coincide, and induce a permutation group which is topologically isomorphic to $\mathbf{G}$.

(4) Let $g \in \overline{\Gamma}$, and assume first that g fixes $\bigcup_{i\geq 1} \mathbf{G}/\mathbf{G}_i$ pointwise. So $g|_A \in \overline{\Sigma_{\mathbf{G}/\mathbf{F}}}$ fixes $\bigcup_{i\geq 1} \mathbf{G}/\mathbf{G}_i$ and $\overline{\chi}(g|_A)$ is the identity on $\mathbf{G}/\mathbf{G}_0$. Note that $\overline{\chi}(g|_A)$ agrees with g on A (but g may be non-identity on $\mathbf{G}/\mathbf{G}_0$).

By (3), there is $g' \in \Gamma$ which agrees with g on $\bigcup_{i\in\omega} \mathbf{G}/\mathbf{G}_i$. By 4.2.9 we can write $g' = e \cdot f$ where $e \in \chi(\Sigma_{\mathbf{G}/\mathbf{F}})$ and $f \in \Gamma_{(A)}$. As f, g' fix all of

$\bigcup_{i\geq 1}\mathbf{G}/\mathbf{G}_i$, the same is true of e. So $e \in \chi(\Phi)$ and therefore e fixes all of $\mathbf{G}/\mathbf{G}_0$. Thus $f \in \Gamma_{(A)}$ agrees with g' and therefore with g on $\bigcup_{i\in\omega}\mathbf{G}/\mathbf{G}_i$. So $g = \overline{\chi}(g|_A) \cdot f$ (as g agrees with $\overline{\chi}(g|_A)$ on A and f fixes all of A).

Now let g be arbitrary. There exists $h \in \Sigma_{\mathbf{G}/\mathbf{F}}$ such that g and h agree in their action on $\bigcup_{i\geq 1}\mathbf{G}/\mathbf{G}_i$, by (3). Then by the preceding case, $\chi(h)^{-1} \circ g$ is contained in $\overline{\chi}[\overline{\Sigma_{\mathbf{G}/\mathbf{F}}}] \cdot \Gamma_{(A)}$, and hence so is g.

Clearly $\overline{\chi}[\overline{\Sigma_{\mathbf{G}/\mathbf{F}}}] \cap \Gamma_{(A)}$ is the trivial group and $\overline{\chi}$ is a monoid isomorphism from $\overline{\Sigma_{\mathbf{G}/\mathbf{F}}}$ to its image. As $\Gamma_{(A)} \cong \mathbf{F}$, we have the result.

□

Let Δ be as in Notation 4.2.10.

Proposition 4.2.14. *The closed transformation monoids $\overline{\Delta}$ and $\overline{\Gamma}$ are isomorphic, but not topologically isomorphic.*

Proof. The group Δ is topologically isomorphic to $\Sigma_{\mathbf{G}/\mathbf{F}} \times \mathbf{F}$, thus $\overline{\Delta}$ is topologically isomorphic to $\overline{\Sigma_{\mathbf{G}/\mathbf{F}}} \times \mathbf{F}$. By Lemma 4.2.13 $\overline{\Gamma}$ is isomorphic to $\overline{\Sigma_{\mathbf{G}/\mathbf{F}}} \times \mathbf{F}$, so $\overline{\Delta}$ and $\overline{\Gamma}$ are isomorphic. If they were topologically isomorphic, then also the groups of invertible elements, equal to Δ and Γ respectively, would be topologically isomorphic. But this contradicts Corollary 4.2.11. □

4.2.6 Extending the isomorphism to the function clones

When Δ is any set of finitary functions on a given set, then there exists a smallest function clone containing it, the *function clone generated by* Δ. In the special case where Δ is a transformation monoid, this clone consists precisely of those functions which arise by adding dummy variables to the functions of the monoid. In this case, if Δ is topologically closed, then so is the function clone generated by Δ. Thus moving from a Δ to the clone it generates is an algebraic procedure, in contrast to the moving from a closed permutation group to its topological closure as a transformation monoid, which is topological. It is therefore much more straightforward to extend non-topological isomorphisms between closed transformation monoids to the clones they generate. The following proposition is easy, its proof can be found in [BPP17].

Proposition 4.2.15. *Let Σ, Λ be transformation monoids, and let $\xi\colon \Sigma \to \Lambda$ be a monoid isomorphism such that both ξ and its inverse function send constant functions to constant functions. Then ξ extends to an isomorphism between the function clones generated by Σ and Λ.*

Corollary 4.2.16. *The function clones generated by the transformation monoids $\overline{\Delta}$ and $\overline{\Gamma}$ are isomorphic, but not topologically isomorphic.*

Proof. By Propositions 4.2.14 and 4.2.15, the clones are isomorphic. Any topological isomorphism between them would yield a topological isomorphism between the monoids $\overline{\Delta}$ and $\overline{\Gamma}$ by restriction to the unary sort, and hence contradict Proposition 4.2.14. □

4.2.7 Encoding into a finite relational language

We have shown that there are ω-categorical structures $\mathbb{A}$ and $\mathbb{B}$ whose endomorphism monoids are isomorphic, but not topologically isomorphic. The structure $\mathbb{A}$ has an infinite signature, and it is easy to see from the theorem of Coquand, Ahlbrandt, and Ziegler [AZ86] that any structure $\mathbb{A}'$ whose automorphism group is topologically isomorphic to the one of $\mathbb{A}$ must have an infinite signature. In this section we are going to show that there is an ω-categorical structure in a finite language such that its automorphism group, its endomorphism monoid and its polymorphism clone do not have reconstruction.

The key ingredient for the counterexamples of the previous sections was Proposition 4.2.4. It gave us an encoding of the profinite group $\mathbf{G}/\mathbf{F}$ as the quotient of an oligomorphic group $\Sigma_{\mathbf{G}/\mathbf{F}}$ and the intersection of its open subgroups of finite index. Our primary goal in this section is to construct an oligomorphic permutation group $\tilde{\Sigma}$ that also encodes $\mathbf{G}/\mathbf{F}$ in the above sense and can be written as the automorphism group of a structure with finite signature. We will obtain $\tilde{\Sigma}$ with the help of a theorem due to Hrushovski, which states that every ω-categorical structure is definable on a definable subset of an ω-categorical structure with finite signature. In Proposition 4.2.17 we present Hrushovski's result and a proof sketch taken from [Hod97, Theorem 7.4.8] in order to refer to this construction later on.

Proposition 4.2.17. *Let $\mathbb{A}$ be a countable ω-categorical structure. Then there is a finite language L, containing a 1-ary predicate P, and an ω-categorical L-structure $\mathbb{B}$, such that the domain of $\mathbb{A}$ is equal to the elements of $\mathbb{B}$ satisfying P and the definable relations of $\mathbb{A}$ are exactly the definable relations of $\mathbb{B}$ restricted to P.*

Proof. We can assume that $\mathbb{A}$ is relational with atomic relations $R_1, R_2, \ldots$ where R_n has arity $l(n)$. We can also assume that every definable relation in $\mathbb{A}$ is equivalent to an atomic formula and that $l(n) \leq n$ for all $n \geq 1$. In particular, $\mathbb{A}$ has quantifier elimination and is homogeneous. Let L be the language consisting of the relation symbols P, Q, λ and ρ (all 1-ary), H (2-ary), and S (4-ary), and let L^+ be the union of L and the language of $\mathbb{A}$. Let T be the theory in L^+ which says:

- If $R_n(\bar{x})$ for some $n \geq 1$, then all entries of $\bar{x}$ satisfy P;
- $Q(x)$ if and only if $\neg P(x)$;

- if $\lambda(x)$ or $\rho(x)$, then $Q(x)$;
- if $H(x, y)$, then $Q(x)$ and $Q(y)$;
- if $S(x, y, a, b)$ then $Q(x)$, $Q(y)$, and $P(a)$, $P(b)$.

Let $\mathcal{M}$ be a model of T. Then we say a set of elements of $\mathcal{M}$ is an *n-pair* if it can be written as $\{a_1, \dots, a_{l(n)}, c_1, \dots, c_n\}$, where $n \geq 1$ and

- $P(a_i)$ holds for all $1 \leq i \leq l(n)$ and $Q(c_i)$ holds for all $1 \leq i \leq n$;
- the elements c_i are distinct and $H(c_i, c_j)$ holds iff $j \equiv i + 1 \mod n$;
- $\lambda(c_i)$ holds iff $i = 1$ and $\rho(c_i)$ holds iff $i = l(n)$;
- $S(c_h, c_i, a_k, a_m)$ holds iff $a_h = a_k$.

Note that if an n-pair $\{a_1, \dots, a_{l(n)}, c_1, \dots, c_n\}$ is given, we can uniquely recover the sequence $(c_1, \dots, c_n)$ and also the sequence $(a_1, \dots, a_{l(n)})$, which may contain repetitions. We say the n-pair *labels* the sequence $\bar{a} = (a_1, \dots, a_{l(n)})$.

Consider the class of finite models $\mathbb{B}'$ of T such that

- the restriction of $\mathbb{B}'$ to P and to the relations R_n is isomorphic to a finite substructure of $\mathbb{A}$;
- for every $n \geq 1$, if $\mathbb{B}'$ contains an n-pair which labels the sequence $\bar{a}$, then $B \models R_n(\bar{a})$.

By [Hod97, Theorem 7.4.8], this is an amalgamation class; let $\mathbb{B}^+$ be its Fraïssé limit.

Clearly, the restriction of $\mathbb{B}^+$ to the subset P and to the relations R_n is homogeneous and has the same age as $\mathbb{A}$. Therefore it is isomorphic to $\mathbb{A}$. Let $\mathbb{B}$ be the reduct of $\mathbb{B}^+$ in the language L. By construction $\mathbb{B}^+ \models R_n(\bar{a})$ holds if and only if some n-pair in $\mathbb{B}^+$ labels $\bar{a}$. Therefore every relation R_n is definable in $\mathbb{B}$. $\square$

In Proposition 4.2.17, the definable relations of $\mathbb{B}$ restricted to P are exactly the definable relations of $\mathbb{A}$. Hence the orbits of $\mathrm{Aut}(\mathbb{B})$ and the orbits of $\mathrm{Aut}(\mathbb{A})$ on tuples in P coincide. However we do not know if the restriction of $\mathrm{Aut}(\mathbb{B})$ to P is closed in the full group $\mathrm{Aut}(\mathbb{A})$, i.e. it might be a proper dense subgroup of $\mathrm{Aut}(\mathbb{A})$.

Lemma 4.2.18. *Let $\mathbb{A}$ be a countable ω-categorical homogeneous structure and $\mathbb{B}$ as constructed in Proposition 4.2.17. Then* $\mathrm{End}(\mathbb{B}) = \overline{\mathrm{Aut}(\mathbb{B})}$*, i.e.,* $\mathbb{B}$ *is a* model-complete core *(cf. [Bod07]).*

Proof. It is shown in [BP14] that for ω-categorical structure $\mathbb{B}$, $\mathrm{End}(\mathbb{B}) = \overline{\mathrm{Aut}(\mathbb{B})}$ holds if and only if every formula in $\mathbb{B}$ is equivalent to an existential positive formula. Let $\mathbb{B}^+$ be as in the proof of Proposition 4.2.17. Because of the homogeneity of $\mathbb{B}^+$, every L-formula in $\mathbb{B}$ is equivalent to a quantifier-free L^+-formula in $\mathbb{B}^+$. So it suffices to show that every quantifier free L^+-formula is equivalent to an existential positive L-formula in $\mathbb{B}^+$. We first prove the statement for an atomic formula $R_n(x_1, \ldots, x_{l(n)})$. By the construction of $\mathbb{B}^+$ we have

$$\mathbb{B}^+ \models R_n(x_1, \ldots, x_{l(n)}) \Leftrightarrow \mathbb{B}^+ \models \exists y_1, \ldots, y_n \left(\{x_1, \ldots, x_{l(n)}, y_1, \ldots, y_n\} \text{ is an } n\text{-pair}\right) .$$

The latter is an existential positive L-formula, since the definition of an n-pair did not require quantifiers or negations. For a general quantifier-free formula in $\mathbb{B}^+$ we can assume that the relations $(R_n)_{n\geq 1}$ only appear in positive form, since we introduced a relation symbol for every definable relation in $\mathbb{A}$. Applying the equivalence above for every such R_n then gives us an existential positive formula in L. □

From now on, let $\mathbb{A}$ be the canonical structure of the oligomorphic permutation group $\Sigma_{\mathbf{G/F}}$, i.e., the structure on the domain of $\Sigma_{\mathbf{G/F}}$ containing all relations which are invariant under $\Sigma_{\mathbf{G/F}}$. Let $\mathbb{B}$ and $\mathbb{B}^+$ be as in the proof of Proposition 4.2.17. Set $\tilde{\Sigma} := \mathrm{Aut}(\mathbb{B})$, and let $\tilde{\mu}\colon \tilde{\Sigma} \to \mathbf{G/F}$ be the composition of the restriction of $\tilde{\Sigma}$ to P and the homomorphism $\mu_{\mathbf{G/F}}$.

Recall the construction of $\Sigma_{\mathbf{G/F}}$ in Proposition 4.2.4. Let $\mathbb{A}^*$ be, as in the proof of that proposition, the structure $(A, (P_i^n)_{1\leq i\leq n})$. Recall that $\mathbb{A}^*$ is ω-categorical and homogeneous, and that all relations of $\mathbb{A}$ are definable in $\mathbb{A}^*$. By $\mathbb{B}^*$ we denote the expansion of $\mathbb{B}^+$ with the relations $(P_i^n)_{1\leq i\leq n}$ on its P-part. Let $\tilde{\Phi}$ be the automorphism group of $\mathbb{B}^*$.

Lemma 4.2.19. *The map $\tilde{\mu}\colon \tilde{\Sigma} \to \mathbf{G/F}$ is a continuous surjective homomorphism with kernel $\tilde{\Phi}$. Furthermore, $\tilde{\Phi}$ is the intersection of the open subgroups of finite index in $\tilde{\Sigma}$.*

Proof. As a composition of continuous homomorphisms, $\tilde{\mu}$ is a continuous homomorphism. As in Proposition 4.2.4 we can think about $\tilde{\mu}$ as an action of the elements of $\tilde{\Sigma}$ on the set $\bigcup_{n\geq 1} X_n$, where $X_n = \{P_1^n, \ldots, P_n^n\}$ for all $n \geq 1$. The functions in $\tilde{\Phi}$ are exactly those elements who stabilize all P_i^n pointwise, so $\tilde{\Phi}$ is indeed the kernel of $\tilde{\mu}$. Using the homogeneity of $\mathbb{B}^+$ and a back-and-forth argument as in Proposition 4.2.4 one can show that $\tilde{\mu}$ is surjective.

Note that the age of $\mathbb{B}^*$ consists exactly of those structures whose restriction to P lies in the age of $\mathbb{A}^*$ and whose reduct to the language L^+ lies in the age of $\mathbb{B}^+$. With this in mind it is easy to verify that $\mathbb{B}^*$ satisfies the extension property. Hence also $\mathbb{B}^*$ is homogeneous.

The subgroup of $\tilde{\Sigma}$ consisting of the elements that stabilize X_n pointwise for a fixed $n \geq 1$ is open and of finite index. The intersection of all such subgroups is equal to $\tilde{\Phi}$. Hence the intersection of all open subgroups of finite index in $\tilde{\Sigma}$ is contained in $\tilde{\Phi}$.

It remains to show that also the other inclusion holds; we follow the proof of Proposition 4.2.4. Assume that $\tilde{\Phi}$ has a proper open subgroup $\tilde{\Lambda}$ of finite index. Because of the openness of $\tilde{\Lambda}$, there is a tuple $\bar{y}$ such that the stabilizer $\tilde{\Phi}_{(\bar{y})}$ lies in $\tilde{\Lambda}$. Let $O_{\tilde{\Phi}}(\bar{y})$ and $O_{\tilde{\Lambda}}(\bar{y})$ denote the orbits of $\bar{y}$ under $\tilde{\Phi}$ and $\tilde{\Lambda}$, respectively. We will obtain a contradiction by studying the action of $\tilde{\Phi}$ on the partition of $O_{\tilde{\Phi}}(\bar{y})$ into blocks $gO_{\tilde{\Lambda}}(\bar{y})$ with $g \in \tilde{\Phi}$. The index $|\tilde{\Phi} : \tilde{\Lambda}|$ coincides with the number of partition classes $gO_{\tilde{\Lambda}}(\bar{y})$ in $O_{\tilde{\Phi}}(\bar{y})$.

Choose a tuple $\bar{a} \in O_{\tilde{\Lambda}}(\bar{y})$ and a tuple $\bar{b}$ from another partition class such that the entries of $(\bar{y}, \bar{a}, \bar{b})$ are pairwise disjoint. We claim that there is a $\bar{d} \in O_{\tilde{\Phi}}(\bar{y})$ such that $(\bar{y}, \bar{a})$, $(\bar{d}, \bar{a})$ and $(\bar{d}, \bar{b})$ lie in the same orbit of $\tilde{\Phi}$, which is a contradiction.

By the homogeneity of $\mathbb{B}^*$ two tuples lie in the same orbit of $\tilde{\Phi}$ if they satisfy the same relations in $\mathbb{B}^*$. We write $\bar{y} = (\bar{y}_P, \bar{y}_{\neg P})$, where the components of $\bar{y}_P$ satisfy P, and the components of $\bar{y}_{\neg P}$ do not satisfy P. Similarly, we write $\bar{a} = (\bar{a}_P, \bar{a}_{\neg P})$, $\bar{b} = (\bar{b}_P, \bar{b}_{\neg P})$. By the proof of Proposition 4.2.4, we can find a tuple $\bar{d}_P$ of elements of $\mathbb{A}^*$ such that $(\bar{y}_P, \bar{a}_P)$, $(\bar{d}_P, \bar{b}_P)$ and $(\bar{d}_P, \bar{a}_P)$ satisfy the same relations.

We wish to find a tuple $\bar{d}_{\neg P}$ of the same length as $\bar{y}_{\neg P}$ such that setting $\bar{d} := (\bar{d}_P, \bar{d}_{\neg P})$ we have that $(\bar{d}, \bar{a})$ and $(\bar{d}, \bar{b})$ lie in the same orbit as $(\bar{y}, \bar{a})$. To this end, let $\bar{d}_{\neg P}$ be a tuple of new variables of the right length. We endow the set of elements appearing in $\bar{y}, \bar{a}, \bar{b}$ and $\bar{d}$ with relations ρ, λ, H and S such that we obtain a structure in the age of $\mathbb{B}^*$, and such that $(\bar{d}, \bar{a})$ and $(\bar{d}, \bar{b})$ satisfy the same relations as $(\bar{y}, \bar{a})$; clearly, we can then realize these variables as elements of $\mathbb{B}^*$ and are done by homogeneity. When doing so we can also ensure that all quadruples of elements from $\bar{a}$, $\bar{b}$, and $\bar{d}$ for which S holds consist entirely of elements of $(\bar{d}, \bar{a})$ or of $(\bar{d}, \bar{b})$.

We claim that the resulting structure lies in the age of $\mathbb{B}^*$. Assume otherwise. Then the reduct of the structure in L^+ contains a n-pair that labels a tuple $\bar{x}$ with $\neg R_n(\bar{x})$. This n-pair has to contain elements of $\bar{d}$, otherwise this would be a contradiction to the fact that the union of the elements of $\bar{y}$, $\bar{a}$ and $\bar{b}$ induces a structure in the age of $\mathbb{B}^*$. Moreover, this n-pair lies entirely in $(\bar{d}, \bar{a})$ or $(\bar{d}, \bar{b})$, since S does not hold for any other tuples containing elements of $\bar{d}$. But then, by construction, also the union of $\bar{y}$ and $\bar{a}$ contains a n-pair that labels an $\bar{x}'$ with $\neg R_n(\bar{x}')$. This contradicts the fact that the union of the elements of $\bar{a}$ and $\bar{y}$ lies in the age of $\mathbb{B}^*$. This proves our claim.

Therefore there are functions $h_1, h_2 \in \tilde{\Phi}$ such that $h_1(\bar{y}, \bar{a}) = (\bar{d}, \bar{a})$ and $h_2(\bar{d}, \bar{a}) = (\bar{d}, \bar{b})$. Since Φ preserves our partition, $h_1(\bar{y}, \bar{a}) = (\bar{d}, \bar{a})$ implies that $\bar{d}$ lies in $O_\Lambda(\bar{y})$. But because of $h_2(\bar{d}, \bar{a}) = (\bar{d}, \bar{b})$ also $\bar{b}$ lies in the very same class, which is a contradiction. □

We are now ready to conclude this section with the proof of Theorem 4.0.3.

Proof of Theorem 4.0.3. In Lemma 4.2.19 we have shown that $\tilde{\mu} : \tilde{\Sigma} \to \mathbf{G}/\mathbf{F}$ is a surjective continuous homomorphism whose kernel $\tilde{\Phi}$ is the intersection of open subgroups with finite index in $\tilde{\Sigma} = \mathrm{Aut}(\mathbb{B})$. Let B be the domain of $\mathbb{B}$. We proceed as in Section 4.2.4: Via $\tilde{\mu}$ we can define an action of $\tilde{\Sigma}$ on $B \cup \mathbf{G}/\mathbf{G}_0$. This action is not continuous and has a non-open image, let $\tilde{\Gamma}$ be its closure in $\mathrm{Sym}(B \cup \mathbf{G}/\mathbf{G}_0)$. Then, following the exact same proof steps as in Lemma 4.2.9 and Corollary 4.2.11 we see that $\tilde{\Gamma}$ and $\tilde{\Sigma} \times \mathbf{F}$ are isomorphic, but not topologically isomorphic. By the same arguments as in Section 4.2.5 one can also prove that $\overline{\tilde{\Gamma}}$ and $\overline{\tilde{\Sigma}} \times \mathbf{F}$ are isomorphic as abstract monoids, but not topologically isomorphic.

Since $\mathbf{F}$ is finite and $\mathbb{B}$ has finite signature, there is a structure $\mathbb{C}$ with finite signature such that $\mathrm{End}(\mathbb{C}) \cong_T \mathrm{End}(\mathbb{B}) \times \mathbf{F}$. Then $\mathrm{Aut}(\mathbb{C})$ is topologically isomorphic to $\mathrm{Aut}(\mathbb{B}) \times \mathbf{F} = \tilde{\Sigma} \times \mathbf{F}$, which we know does not have reconstruction. By the model completeness of $\mathbb{B}$, we know that its automorphism group is dense in its endomorphism monoid. It follows that $\mathrm{End}(\mathbb{C}) \cong_T \mathrm{End}(\mathbb{B}) \times \mathbf{F} = \overline{\tilde{\Sigma}} \times \mathbf{F}$, proving that also the endomorphism monoid of $\mathbb{C}$ has no reconstruction. Finally, by including the relation $R(x, y, a, b) \leftrightarrow x = y \vee a = b$ in $\mathbb{C}$ one can ensure that the polymorphism clone of $\mathbb{C}$ consists of those functions arising from endomorphisms of $\mathbb{C}$ by adding dummy variables. By Proposition 4.2.15, $\mathrm{Pol}(\mathbb{C})$ and the function clone generated by $\tilde{\Gamma}$ are isomorphic, but not topologically isomorphic. □

We do not know whether $\tilde{\Gamma}$ can be represented as automorphism group of a structure with finite relational signature. Similarly, we do not know whether its closure $\overline{\tilde{\Gamma}}$ as a monoid is the endomorphism monoid of a structure with finite relational signature.

4.3 Open Problems

Because of the comments on the consistency of reconstruction for groups in Section 4.2.2, the following question is of central importance for the reconstruction of structures from their endomorphism monoid.

Question 4.3.1. *Let Σ be a closed oligomorphic subgroup of* $\mathrm{Sym}(\omega)$ *which has reconstruction. Does the monoid obtained as the closure of Σ in ω^ω have reconstruction?*

A positive answer would imply that it is consistent with ZF+DC that all monoids with a dense set of units have reconstruction. These monoids play a central role in the study of polymorphism clones of ω-categorical structures, in particular for the study of the computational complexity of constraint satisfaction problems (we refer to [BP15b, BP16a] for details).

In the course of the proof, we encountered natural questions that we had to leave open (for example at the beginning of Section 4.2.4). An answer to the following question will most probably shed some light on them.

Question 4.3.2. *Let* Γ *be a closed oligomorphic permutation group without reconstruction. Does the monoid closure of* Γ *also fail to have reconstruction?*

Lascar showed in [Las89] that if $\mathbb{A}$ and $\mathbb{B}$ are countable ω-categorical structures which are *G-finite*, then any isomorphism between their endomorphism monoids is a topological isomorphism when restricted to their automorphism groups. An early version of that article concluded with the question whether the assumption of G-finiteness could be dropped; the published version does not contain the question anymore. We remark that our example would be a counterexample to that question.

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Curriculum vitae

Personal data

Michael Kompatscher,
Born Dec. 3, 1989 in Bozen, Italy.
Son of Luis and Erika Kompatscher, 1 sister.
Italian citizenship.

School

1995-2000	Elementary school in Völs am Schlern.
2000-2003	Middle school “Leo Santifaller”, Kastelruth.
2003-2008	High school, Realgymnasium Bozen.

Studies

2008-2012	Bachelor “Mathematik in Technik und Naturwissenschaften”, TU Wien
2012-2014	Master “Mathematik in Technik und Naturwissenschaften”, TU Wien
2014-	PhD studies in Mathematics, TU Wien

Research visits

Sep-Dec 2016	Ramsey DocCourse 2016 at DIMATIA, Charles University in Prague

Awards

2015	Scholarship for outstanding studies awarded by the province of Bozen

Selected talks

Apr 2015	Endomorphism monoids of omega-categorical structures, International Seminar of the Institut für Algebra, TU Dresden
Jun 2015	Endomorphism monoids of omega-categorical structures, at TACL 2015, Ischia, Italy
Oct 2015	Algebraic methods in constraint satisfaction at Matej Bel University, Banská Bystrica
Nov 2015	A counterexample on the reconstruction of oligomorphic clones, at the Banff IRS workshop on Homogeneous structures
Dec 2015	Maximal subgroups of Sym(ω) via Henson digraphs, in the KAFKA seminar, Charles University Prague
Feb 2016	2^ω many maximal-closed subgroups of Sym(ω) via Henson digraphs, at New Pathways between Group Theory and Model Theory in Mülheim, Germany
May 2016	CSPs over the random partial order, AAA92, Prague
Aug 2016	Constraint satisfaction problems over the random poset, Logic Colloquium, Leeds
Nov 2016	An introduction to Ramsey theory, at the Fall school of the Algebra department of Charles University
Feb 2017	A new proof of the existence of cores of ω-categorical structures, AAA93, Bern
Mar 2017	A complexity dichotomy for poset constraint satisfaction, STACS2017, Hannover
May 2017	Completing edge-labelled graphs to metric spaces, Arbeitsgemeinschaft Diskrete Mathematik, Wien
Jun 2017	Linearization of certain non-trivial equations in oligomorphic clones, AAA94, Novi Sad
Jul 2017	Tutorial: Oligomorphic clones, Tianfu Universal Algebra Workshop, Chengdu, China

Publications

[ABWH+17] *Completing graphs to metric spaces*, with Andrés Aranda, David Bradley-Williams, Eng Keat Hng, Jan Hubička, Miltiadis Karamanlis, Matěj Koněcný, and Micheal Pawliuk. Electronic Notes in Discrete Mathematics, 61:53–60, 2017.
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[BKO+17] *The equivalence of two dichotomy conjectures for infinite domain constraint satisfaction problems* with Libor Barto, Michael Pinsker, Miroslav Olšák and Trung Van Pham.
Extended version available as arXiv preprint *Equations in oligomorphic clones and the Constraint Satisfaction Problem for ω-categorical structures*, arXiv:1612.07551.

[KP17] *A complexity dichotomy for poset constraint satisfaction*, with Trung Van Pham. In Proceedings of the 34th Symposium on Theoretical Aspects of Computer Science (STACS 2017), pages p47:1–47:12, 2017.
Extended version available as arXiv preprint, arXiv:1603.00082.

[AK15] $2^{\aleph_0}$ *pairwise non-isomorphic maximal-closed subgroups of* $\mathrm{Sym}(\omega)$ *via the classification of the reducts of the Henson digraphs*, 2015. Preprint, arXiv:1509.07674.

[BEKP] *A counterexample to the reconstruction of ω-categorical structures from their endomorphism monoids*, with Manuel Bodirsky, David Evans, and Michael Pinsker. To appear in the Israel Journal of Mathematics. arXiv preprint arXiv:1510.00356.

[Kom14] *Gegenbeispiele zu drei Vermutungen über Kategorizität.* Master thesis, supervised by Martin Goldstern, TU Wien, 2014.