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The twin hypothesis of education and retirement

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Abstract

Common models examining the effects of demographic change and the efficiency of policy reforms often suffer from two important shortcomings: (i) interaction between an individual's schooling and retirement decision is disregarded and (ii) a realistic demographic population structure is absent. In order to overcome these limitations, we combine two papers of Ben J. Heijdra and Ward E. Romp who developed a continuous time overlapping generations model in the manner of Blanchard (1985) allowing for age-dependent mortality rates. In our modification of their work, agents choose both length of schooling and retirement age endogenously. Pension benefits are calculated from lifetime wage earnings which positively depend on the level of education. On the aggregate, dynamics are rather complicated due to generational turnover effects and a human capital externality. Nevertheless, we present a comparative static analysis of long-run effects as well as numerical simulations for the transition paths following demographic shocks and fiscal stimuli. We find that not controlling for individual adjustments in education and retirement at the same time will overestimate the negative impacts of aging on the macroeconomy. Similarly, the economic impact of education reforms is found to be much stronger if not only schooling but also retirement is treated as endogenous.

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1 Introduction

Pension policy design and pension reform have been fields of intensive discussion in the last decades. In developed countries, rising longevity combined with persistently low fertility rates puts increasing financial pressure on existing Pay-as-you-go (PAYG) pension schemes. The share of retirees among the population increases at the expense of the population in working ages, who are the main financiers of a PAYG pension scheme. As an example, the OECD-wide old-age dependency ratio, which relates the population over age 65 to the working-age population, is projected to double from 25% in 2010 to almost 50% until 2050, see OECD (2011, pp.166–167). Moreover, a shrinking labor force implies falling per capita output levels unless there is sufficiently high productivity growth at the same time. This reduces domestic consumption possibilities and increases the need for imports. In light of this development, drastic reforms seem necessary in order to maintain fiscal sustainability of social security systems and to retain current levels of per capita output. Popular reform steps include increasing contribution rates, lowering pension benefits, raising the mandatory retirement age or increasing incentives of postponing retirement.

There exists a vast literature relating to the question whether PAYG schemes are useful at all or if a shift to a capital funded system should be considered. Indeed, this discussion has always been prominent in the literature, see for instance Sinn (2000), Knell (2011) or Barr and Diamond (2006), who provide a comprehensive overview. In this light, most authors compare the performance of a PAYG system to a fully funded system in isolation, abstracting from the interaction of the pension system with other publicly financed services such as the education system. Notable exceptions include Casarico (1998), Kaganovich and Zilcha (1999), Kaganovich and Meier (2012) and Iturbe-Ormaetxe and Valera (2012).

Kaganovich and Zilcha (1999) find that the presence of a PAYG scheme increases long-run economic growth if individuals exhibit either a sufficiently high concern for retirement income or a low concern for their children's education. A similar result was obtained by Kaganovich and Meier (2012), who assume that individual human capital–and therefore individual wage–is stochastic and the education tax rate is subject to public voting. "The scenario where [...] education tax rate and growth rate of per capita GDP will be higher under a PAYG scheme than under the respective funded system is likely in an environment with [...] moderate degree of inequality, strong public education system and a culture where parents give no more than a moderate priority to the human capital attainment of their children. We notice that all the conditions listed above are characteristic of developed economies." Iturbe-Ormaetxe and Valera (2012) show that even if parents do not directly benefit from their children's human capital, increasing the funded pillar of the pension system decreases public support for the education system. A stronger intergenerational link of the pension system encourages the working generation to spend more money on the education of the young generation since they finance part of their future pension benefit.

Although we do not want to contribute to the "PAYG vs. Fully Funded" discussion, the papers mentioned above provide an important understanding why joint modeling of the pension and the education system is necessary. Apart from the economic consequences of a social security scheme and its design, some studies investigate whether such a transfer scheme is able to improve social welfare:

Docquier and Michel (1999) and Docquier et al. (2007) argue that individual investments into education are less than socially optimal because they disregard the positive externality in human capital accumulation. Their result is that on pure efficiency grounds pensions should be negative: "In a world without myopia, inequality and/or credit market imperfections, taxing the retirees is a relevant option to decentralize the optimal path of human and physical capitals." This finding indeed disregards the main reasons why intergenerational transfer schemes exist. Giving up these assumptions may imply very different results:

If individuals act myopic (shortsightedly) and save too little for their elder days, Andersen and Bhattacharya (2011) show that a sufficiently high degree of myopia can raise a welfare case for a pension system. However, their model does not consider education. Regarding incomplete financial markets, Boldrin and Montes (2005) assume that young individuals are not allowed to finance education by borrowing against their future income. In absence of public transfers, young people would remain uneducated, which is clearly not socially optimal. Taxing the working generation alone is not feasible to improve utility levels of all generations–a future transfer is required for compensation. This results in an "intergenerational contract", where the young generation receives the means necessary for education from the working generation and finances its future pension in return.

Although not explicitly focusing on welfare, Glomm and Kaganovich (2003) consider heterogeneous agents and study the effect of policy reforms on the inequality of human capital distribution in the economy. They allow for a potential interaction between the education and the social security tax rate through the government budget. If private and public investments in education are substitutes, increasing public education funding has an equalizing effect, while the induced decline in pension spending could eventually counteract this improvement.

The interrelation between public education and the pension system was also illustrated by Rojas (2004). Motivated by the Spanish case, he studies the effect of higher public education spending using a rather sophisticated CGE (computable general equilibrium) model. He finds that the reform brings about changes in the university enrollment rate and the educational composition of the population. The fertility decision is endogenous and the average fertility rate falls since better educated wish to have less children. This, together with the assumption that better education induces a higher life expectancy, changes the age distribution and raises the old-age dependency ratio. In order to keep the government budget balanced, an increase in the social security tax rate is necessary, which reduces the welfare benefits of the policy reform. Though he controls for endogenous fertility and education, the retirement age is exogenous.

All of the above papers capture to some extent an interaction between education and retirement, either in individual decision making or in public funding. However, the models are static in the sense that they assume a constant population composition. None of them explicitly accounts for the changing age distribution which we have already observed for several decades and is going to intensify in the future.[1] By contrast, Vogel et al. (2012) use a large-scale overlapping generations (OLG) model with 74 age groups to study the macroeconomic effect of the projected demographic change in selected developed countries. Their results predict that endogenous human capital accumulation, accompanied by an increase in the (exogenous) retirement age, is able to significantly dampen welfare losses. The growing individual human capital levels can even make up for the rising dependency-ratio and generate an increasing time path of per capita output (instead of a falling profile under exogenous schooling).

The strong impact of aging on the schooling process, as highlighted by Vogel et al. (2012), was indeed our driving motivation to consider education in addition to retirement as two potentially interrelated adjustment channels of individuals in an aging economy. Nevertheless, their model is extremely detailed as it allows for two interacting open economies, endogenous factor prices, endogenous labor supply and depreciation of human capital. Therefore, replicating this framework would have been very demanding and well beyond the scope of a master's thesis.

An earlier contribution broaching the issue of education and retirement in an aging society was that of Boucekkine et al. (2002). They use a continuous time OLG model à la Blanchard (1985) with age-dependent mortality rates together with a maximum attainable age. As common, there is an externality in human capital accumulation.[2] Completely abstaining from physical assets, output is generated only by human capital. The authors find that (in case of an interior solution) the retirement age is proportional to length of schooling. Furthermore, a rise in life expectancy increases optimal length of schooling and the optimal retirement age. On the aggregate, the authors focus on the economy's balanced growth path and study the impact of aging on the equilibrium growth rate. They find that the effect of aging on growth crucially depends on the initial level of life expectancy. "The effect [...] is positive for countries with a relatively low life expectancy, but could be negative in more advanced countries."

The model of Boucekkine et al. (2002) was later extended by Echevarría (2004) to include physical capital accumulation, a production technology using both physical and human capital as input factors, and depreciation of both types of capital. Nevertheless, the impacts of population aging remain very similar to the original model. Another extension by Echevarría and Iza (2006) again starts with the original framework of Boucekkine et al. (2002) but additionally considers a pension scheme. However, their analysis is limited to the comparison of balanced growth paths without paying attention to the complex transitional dynamics of the model.

[1] Rojas (2004) actually reverses the argument and argues that population aging may be partly caused by changes in public education spending. However, he does not clarify how exogenous changes in the mortality profile (e.g. by improvements in health care) affect university enrollment.

[2] Individual human capital is directly proportional to per capita human capital at the agent's date of birth. Hence the model gives rise to endogenous growth.

A second series of papers taking up the ideas of Boucekkine et al. (2002) was that of Heijdra and Romp. They also use a continuous time OLG setting but allow for a more general mortality function. Different to Boucekkine et al., a small open economy with both physical and human capital accumulation is considered. The model is able to capture exogenous or endogenous growth. Besides analytically determining the long-run effects of various shocks, the authors also present a detailed analysis of transitional dynamics. The base model was presented in Heijdra and Romp (2008) and only considered life-cycle saving and consumption. It was then extended to feature education and retirement, but not both at the same time.

> *We have chosen to study start-up education and retirement in separate papers in order to obtain simple and intuitive results. It is, of course, quite feasible to combine the two decisions in a single computable general equilibrium (CGE).* Heijdra and Romp (2009b, p.603)

In particular, Heijdra and Romp (2009a) allows for endogenous human capital accumulation but excludes the possibility to retire. Whereas Heijdra and Romp (2009b) considers an endogenous retirement age but takes human capital as exogenously given.[3]

The idea of this thesis is to combine this two papers into a joint model where individuals choose both length of schooling and retirement age endogenously–just as proposed by the authors. The goal is (i) to determine how individual decision making differs from the original papers and (ii) to investigate how the dynamics of aggregate variables are affected. In our view, having both choice variables in the same model does not add much complexity but proves to be very insightful. To keep the model simple, however, we abstain from an early eligibility age as in Heijdra and Romp (2009b). Instead, we include a more realistic wage-dependent pension benefit formula with a constant accrual rate as incorporated into a great number of pension schemes (e.g. in the Austrian scheme).

Our results show that reduced adult mortality encourages people to increase length of schooling and to postpone retirement. We find that these adjustments are higher when we control for both variables at the same time. This is due to a positive interaction between education and retirement. Longer education time spans induce higher retirement ages and vice versa.

On the aggregate, we find that not controlling for individual adjustments in education and retirement at the same time will overestimate the negative impact of aging on the macroeconomy. We show this by comparing the model response of a mortality shock between three different specifications: One where individuals adjust length of schooling but continue to retire at the initial retirement age, one where retirement can be revised but education time spans remain fixed, and one where the individual has both variables under control. An increase in remaining life expectancy at age 60 from 21.2 to 26 years is found to decrease the long-run level of per capita output by 4.7% (3.3%) when only education (retirement) is allowed to vary. If agents can alter both decisions, the output drops only by 2.5%. Although the quantitative dimension of this result seems negligible, it gives a hint that the negative economic effect of longevity may be overestimated by more than 30% in common models where only retirement is endogenous.

Interestingly, we find even more pronounced differences in the model outcomes for reforms of the public education system. For a 20% increase of the education subsidy, the gain in long-run output is twice as high in the full model specification as in the setting where only adjustments in schooling are considered.

We proceed as follows: Chapters 2 and 3 provide a comprehensive overview of the model setup and restate the most important findings of Heijdra and Romp (2009a) and Heijdra and Romp (2009b), respectively. In Chapter 4, the two models are combined into a joint model with endogenous human capital formation and retirement. Chapter 5 conducts a comparative static analysis of individual optimal behavior and long-run economic performance with respect to fiscal and demographic shocks. Whereas Chapter 6 presents the short-run transitional dynamics using a particular parametrization of the model. Chapter 7 concludes. Mathematical proofs and other technical details can be found in the Appendix.

For a quick read-through we advise the reader to first become familiar with the demographic setup in Section 2.1.1 as well as with the notation of Heijdra and Romp (2009a) and Heijdra and Romp (2009b) by reading the first parts of Sections 2.1.2 and 3.1.1 (you can stop where the model solution procedure is presented). Thereafter, jumping directly to Chapter 4 should be feasible without causing too much confusion.

[3] In fact, both papers are originally chapters from Romp (2007).

2 Endogenous human capital formation without retirement

The second chapter of this thesis provides a comprehensive overview of the model setup and important results of Heijdra and Romp (2009a). The model builds on an earlier work of Heijdra and Romp (2008), where the authors analyzed life-cycle consumption and saving patterns under a realistic demographic process. The most significant difference between this paper and Heijdra and Romp (2009a) is the endogeneity of human capital. Efficient units of labor supply are the outcome of a schooling process at the beginning of an agent's life. Years at school are endogenously chosen and determine the level of individual wage and lifetime income. Moreover, an intergenerational spillover effect in education is considered. A cohort's schooling productivity, i.e. how fast *individual* human capital is accumulated, depends on *per capita* human capital at the cohort's date of birth. This assumption implies that an agent's schooling decision does not only influence his own lifetime utility, but also the welfare of all future generations.

The following section presents the model setup on the individual and on the aggregate level. We obtain the optimal consumption path and the first order condition for schooling. The remaining two sections are devoted to the effects of demographic shocks and fiscal stimuli on individual optimal behavior and economic aggregates.

2.1 Model setup

The model framework builds on the continuous time overlapping generations model developed by Blanchard (1985) drawing from the previous work of Yaari (1965). Compared to the more commonly used discrete time setting, a continuous time framework allows for a much greater flexibility in modeling mortality risk, lifetime uncertainty and demographic changes.

As the reader will face many different variables in the following, we want to prevent possible troubles with some remarks concerning notational conventions.

> *Individual variables* in general depend on the cohort-index, v, and on the current date, t. Individual variables which have counterparts on the aggregate level, such as consumption or human capital, are denoted by a bar, i.e. $\bar{c}(v,t)$ or $\bar{h}(v,t)$. Where no confusion arises, plain lower case letters are used, e.g. $e(v)$ for schooling length of a cohort v individual. Individual *optimal* levels are additionally denoted by an asterisk. For instance, $\bar{c}^*(v,t)$ refers to the utility maximizing consumption level of a cohort v individual at time t.
>
> For the *per capita* equivalents of these variables, lower case letters are used, e.g. $c(t)$ or $h(t)$. The corresponding economic *aggregates* (hardly used) are written in capitals, e.g. $C(t)$ or $H(t)$. The steady state level of a variable is indicated by a hat (and the time index is omitted), e.g. $\hat{c}$ or $\hat{h}$.
>
> Derivatives denoted by a dot refer to the derivative with respect to time, e.g. $\dot{\bar{c}}(v,t) := \partial\bar{c}(v,t)/\partial t$.

Before presenting the individual lifetime optimization problem and the resulting first order conditions, it is advisable to first introduce the demographic setup of the economy. This will be used throughout the whole thesis and–although quite straightforward–is crucial to understand the model results.

2.1.1 Demography

At each point in time, an infinite number of cohorts is alive. Each cohort consists of perfectly homogeneous agents. While identical within the cohort, they are allowed to be heterogeneous across the cohort dimension.

All individuals are subject to mortality risk. For an individual born at time v (equally considered as the cohort-index) the instantaneous mortality rate $m(u,\psi_m(v))$ naturally depends on his current age u but may also depend on a parameter $\psi_m(v)$, which links to the cohort's time of birth. Therefore, it is possible to analyze shocks in cohort-specific longevity. Heijdra and Romp (2009a, Assumption 1) make the following assumptions concerning the mortality process:

Assumption 2.1 *The mortality function has the following properties:*

(i) $m(u, \psi_m)$ *is non-negative, continuous and non-decreasing in age,* $\frac{\partial m(u,\psi_m)}{\partial u} \geq 0$.

(ii) $m(u, \psi_m)$ *is convex in age,* $\frac{\partial^2 m(u,\psi_m)}{\partial u^2} \geq 0$.

(iii) $m(u, \psi_m)$ *is non-increasing in* ψ_m *for all ages,* $\frac{\partial m(u,\psi_m)}{\partial \psi_m} \leq 0$, *and strictly decreasing at some* u.[4]

(iv) The effect of ψ_m *on the mortality function is non-decreasing in age,* $\frac{\partial^2 m(u,\psi_m)}{\partial \psi_m \partial u} \leq 0$.

The corresponding cumulative mortality rate is

$$M(u, \psi_m(v)) := \int_0^u m(\alpha, \psi_m(v))\, d\alpha.$$

At each point in time, a new cohort with size $L(v, v) = b(v)L(v)$ enters the economy where $L(v)$ is population size at time v and the crude birth rate, $b(v)$, is exogenous but may vary with time. According this mortality law, at any date $t > v$, the number of surviving cohort v individuals is

$$L(v, t) = L(v, v)e^{-M(t-v,\psi_m(v))} = b(v)L(v)e^{-M(t-v,\psi_m(v))}. \tag{2.1}$$

Total population size at time t is naturally defined as

$$L(t) := \int_{-\infty}^{t} L(v, t)\, dv.$$

The population growth rate is then $n(t) := \dot{L}(t)/L(t)$ and, for any $t \geq t_0$, total population size can equally be expressed as $L(t) = L(t_0)e^{N(t_0,t)}$ where $N(t_0, t) := \int_{t_0}^{t} n(\tau)\, d\tau$ is the cumulative population growth rate between t_0 and t. Using this expression for $t_0 = v$ and substituting (2.1) gives the share of cohort v in the population at time $t \geq v$,

$$l(v, t) := \frac{L(v, t)}{L(t)} = b(v)e^{-[N(v,t)+M(t-v,\psi_m(v))]}. \tag{2.2}$$

By definition, the cohort weights sum up to one, $\int_{-\infty}^{t} l(v, t)\, dv = 1$, and therefore the following identity holds for any t:

$$1 = \int_{-\infty}^{t} b(v)e^{-[N(v,t)+M(t-v,\psi_m(v))]}\, dv. \tag{2.3}$$

Section A.4.2 demonstrates how this equation can be used to calculate the time path of the population growth rate, $n(t)$.

2.1.2 Individual plans

At time t, an individual born in $v \leq t$ maximizes remaining expected lifetime utility,[5]

$$\Lambda(v, t) := e^{M(t-v)} \int_t^{\infty} U(\bar{c}(v, \tau))e^{-[\theta(\tau-t)+M(\tau-v)]}\, d\tau, \tag{2.4}$$

with respect to the future consumption stream and the number of years in education, $e(v)$. U is the felicity function and $\bar{c}(v, \tau)$ is individual consumption in τ. The agent discounts future utility not only with the rate of time preference, $\theta > 0$, but also takes his conditional survival probabilities into account.[6]

[4] Although the latter is not part of their original assumption, it seems reasonable, and necessary to establish some further results of their paper.

[5] For simplicity, we drop dependency of m and M on $\psi_m(v)$, but keep in mind that mortality may be cohort-specific.

[6] According to (2.1), the unconditional probability of surviving until t is $P[T \geq t] = L(v, t)/L(v, v) = e^{-M(t-v)}$. Then, by definition, the conditional probability for a cohort v individual alive in t to survive until $\tau \geq t$ is

$$P[T \geq \tau \,|\, T \geq t] = \frac{P[T \geq \tau \cap T \geq t]}{P[T \geq t]} = \frac{P[T \geq \tau]}{P[T \geq t]} = e^{M(t-v)-M(\tau-v)}.$$

The felicity function is assumed to be of CES type,

$$U(\bar{c}) := \begin{cases} \frac{\bar{c}^{1-1/\sigma}-1}{1-1/\sigma} & \text{for } \sigma \neq 1, \\ \ln \bar{c} & \text{for } \sigma = 1, \end{cases} \tag{2.5}$$

where $\sigma > 0$ is the elasticity of intertemporal substitution.

Individual assets, $\bar{a}(v,t)$, accumulate according to

$$\dot{\bar{a}}(v,t) = [r + m(t-v)]\bar{a}(v,t) + \bar{w}(v,t) - \bar{g}(v,t) - \bar{c}(v,t) \tag{2.6}$$

where $\bar{w}(v,t)$ is the individual wage rate and $\bar{g}(v,t)$ are net taxes. Assets can be invested in domestic firms, $\bar{k}(v,t)$, domestic governmental bonds, $\bar{d}(v,t)$, and foreign bonds, $\bar{f}(v,t)$, such that

$$\bar{a}(v,t) = \bar{k}(v,t) + \bar{d}(v,t) + \bar{f}(v,t). \tag{2.7}$$

The agent is perfectly indifferent between these investment options because all gain the same rate of return. In line with Yaari (1965), existence of actuarially fair annuity contracts is assumed. As a result, the rate of return is age-dependent and equal to the sum of the market interest rate, r, and the instantaneous mortality risk, $m(t-v)$.[7]

Individual wage income, $\bar{w}(v,t)$, is determined by

$$\bar{w}(v,t) := w(t)\bar{h}(v,t) \tag{2.8}$$

where $w(t)$ is the economy's rental rate of human capital and $\bar{h}(v,t)$ denotes individual human capital stock, which can be equally considered as effective units of labor supply. While the first variable is determined by the market equilibrium, the latter is under the agent's control. We assume that the agent participates in full time schooling until the age of $e(v)$ and inelastically supplies his time resources to the labor market thereafter.

Individual human capital is given by

$$\bar{h}(v,t) := \begin{cases} 0 & \text{for } 0 \leq u \leq e(v), \\ A_H h(v)^{\phi} e(v) & \text{for } u > e(v), \end{cases} \qquad 0 \leq \phi \leq 1, \tag{2.9}$$

where we write $u := t - v$ for the individual's age at time t. Human capital remains zero while the agent is being educated. Afterwards, the level of $\bar{h}(v,t)$ positively depends on schooling time, $e(v)$, and on economy-wide average human capital at the cohort's date of birth, $h(v)$. A_H is an exogenously set productivity index. The parameter $\phi \in [0,1]$ captures the extent of intergenerational knowledge spillover. Unless $\phi = 0$, there is a positive externality in human capital formation. The schooling decision of a cohort v individual does not only affect his own utility, but also influences the behavior of future generations via per capita human capital, which is defined later on.

Notice that individual human capital remains constant once schooling is completed and does not depreciate. This assumption was later relaxed by Heijdra and Reijnders (2012). In a very similar setting, they consider that after schooling, $\bar{h}(v,t)$ increases due to working experience and decreases due to deterioration of knowledge and skills through the aging process.[8]

[7] Actuarial fairness implies that the insurance companies make no profit. Assume that $L(v,t)$ people make an investment of one euro at time t. The insurance company re-invests the money at the (goods or bonds) market and earns the rental rate of capital, r. However, life is risky and some people who invested their money in t die before $t + \Delta t$ and lose their annuity claims. Hence the insurance company only needs to reward $L(v, t + \Delta t)$ people. Under actuarial fair conditions, the interest rate offered by the insurance company, $r^A(v,t)$, fulfills

$$(1 + r\Delta t)L(v,t) = \left[1 + r^A(v,t)\Delta t\right] L(v, t+\Delta t).$$

Collecting terms yields

$$r^A(v,t) = r\frac{L(v,t)}{L(v,t+\Delta t)} - \frac{[L(v,t+\Delta t) - L(v,t)]/\Delta t}{L(v,t+\Delta t)}.$$

Taking the limit for $\Delta t \to 0$ and noting $[\partial L(v,t)/\partial t]/L(v,t) = -m(t-v)$ gives $r^A(v,t) = r + m(t-v)$.

[8] However, their analysis is restricted to the optimal individual behavior and the comparison of steady states. The complex dynamics of the aggregate economy are not considered.

Last but not least, the tax system is given by

$$\bar{g}(v,t) := \begin{cases} A_H h(v)^\phi w(t)[z(t) - s_E] & \text{for } 0 \leq u < e(v), \\ A_H h(v)^\phi w(t)[z(t) + t_L e(v)] & \text{for } u \geq e(v), \end{cases} \tag{2.10}$$

where s_E is the educational subsidy rate, t_L is the labor income tax rate and $z(t)$ is a lump sum tax. All tax instruments are indexed to the value of marginal schooling productivity, $A_H h(v)^\phi$. It turns out that this indexing assumption is crucial for the model solution because it ensures that the optimal education decision is cohort-independent. Individuals born at a date with higher $h(v)$ have higher returns to schooling on wage income but also pay higher taxes. Therefore, on the margin, schooling productivity does not influence an individual's education decision.

We can now formulate the individual dynamic optimization problem: A young individual chooses years in schooling, $e(v)$, and a time path for consumption $\bar{c}(v,\tau)$ (for $\tau \in [t,\infty)$) in order to maximize $\Lambda(v,t)$ given in (2.4) subject to (2.6)–(2.10), taking the level of financial assets in the planning period, $\bar{a}(v,t)$, as given. Additionally, a non-negativity constraint for remaining schooling time, $v - t + e(v) \geq 0$,[9] and a No-Ponzi-Game (NPG) condition[10] are imposed.

By integrating the asset accumulation equation, taking account of the NPG condition and $\bar{a}(v,v) = 0$, i.e. agents are born without financial assets, the lifetime budget constraint can be written as

$$e^{M(t-v)} \int_t^\infty \bar{c}(v,\tau) e^{-[r(\tau-t)+M(\tau-t)]}\, d\tau = \bar{a}(v,t) + \overline{li}(v,t,e(v)). \tag{2.11}$$

The present value of (remaining) lifetime consumption equals initial financial wealth, $\bar{a}(v,t)$, plus the discounted stream of (remaining) lifetime after-tax wage income, $\overline{li}(v,t)$, which is given by

$$\begin{aligned} \overline{li}(v,t) := A_H h(v)^\phi e^{M(t-v)} \Bigg[& s_E \int_t^{\max\{t,v+e(v)\}} w(\tau) e^{-[r(\tau-t)+M(\tau-v)]}\, d\tau \, + \\ (1-t_L)e(v) \int_{\max\{t,v+e(v)\}}^\infty & w(\tau) e^{-[r(\tau-t)+M(\tau-v)]}\, d\tau - \int_t^\infty z(\tau) w(\tau) e^{-[r(\tau-t)+M(\tau-v)]}\, d\tau \Bigg]. \end{aligned} \tag{2.12}$$

The individual problem can be solved in two steps:[11]

1. Obtain the optimal consumption path for a given education time span.
2. Determine the optimal length of schooling.

Step 1: Consumption. The time path for consumption is chosen such as to maximize (2.4) subject to (2.6), taking all the rest as given. Solving the according Hamiltonian system gives the familiar Euler equation for consumption growth,[12]

$$\frac{\dot{\bar{c}}^*(v,\tau,e(v))}{\bar{c}^*(v,\tau,e(v))} = \sigma(r-\theta), \tag{2.13}$$

which yields the optimal consumption function

$$\bar{c}^*(v,\tau,e(v)) = e^{\sigma(r-\theta)(\tau-t)}\, \bar{c}^*(v,t,e(v)), \quad \tau \geq t. \tag{2.14}$$

[9] This inequality holds as long as individual age in t is below the optimal value of $e(v)$. Older agents, who have already completed their education ($t - v > e(v)$), only choose consumption, but cannot re-enter education. Therefore, labor market entry is assumed to be an absorbing state.

[10] The NPG condition reads $\lim_{\tau\to\infty} \bar{a}(v,\tau)e^{-r(\tau-t)-M(\tau-v)+M(t-v)} = 0$. It ensures that an individual cannot finance its debt by making further debt.

[11] In the original paper (Heijdra and Romp, 2009a) the order is reversed. First, optimal schooling time is chosen to maximize lifetime income, $\overline{li}(v,t)$. Then, the consumption path is calculated. However, this procedure is only feasible when the effect of $e(v)$ on lifetime utility comes exclusively via lifetime income. We already want to familiarize the reader with the general approach as used in Chapter 3 and 4.

[12] Please refer to Section A.2.1 of the Appendix for an explicit derivation.

By substituting (2.14) into (2.11), the lifetime budget constraint can be rewritten

$$\Delta(u, r')\ \bar{c}^*(v,t,e(v)) = \bar{a}(v,t) + \overline{li}(v,t,e(v)) \tag{2.15}$$

where Δ is the "demographic discount function" introduced in Heijdra and Romp (2008) and defined as[13]

$$\Delta(x,\lambda) := e^{\lambda x + M(x)} \int_x^\infty e^{-[\lambda\alpha + M(\alpha)]}\, d\alpha, \quad x \geq 0. \tag{2.16}$$

The auxiliary variable $u = t - v$ is again the agent's age in t. The interest rate used for discounting the future income stream is $r' := r - \sigma(r - \theta)$, i.e. the market interest rate corrected for the optimal growth rate of consumption.

Now, as $\bar{a}(v,t)$ is predetermined in t and lifetime income does not depend on consumption, Eq. (2.15) pins down (conditional) optimal consumption in t. Thus, the marginal (and average) propensity to consume out of wealth is $1/\Delta(u, r')$. It can be shown (see Proposition A.1 in the Appendix) that this factor rises with age. Intuitively, as death becomes more likely, individuals are encouraged to consume a larger part of their financial wealth since no bequests are allowed.

Step 2: Education. Education affects lifetime utility only through consumption, which is determined by lifetime income according to (2.15). Therefore, the optimal length of schooling can be found by maximizing the expression for lifetime income (2.12) with respect to $e(v)$.

Taking account of (2.8)–(2.10), the first order condition reads

$$(1 - t_L) \int_{v+e^*(v)}^\infty w(\tau) e^{-[r(\tau - t) + M(\tau - v)]}\, d\tau = \big[(1 - t_L)e^*(v) - s_E\big] w(v + e^*(v)) e^{-[re^*(v) + M(e^*(v))]}. \tag{2.17}$$

The left hand side represents the marginal benefit from increasing schooling time due to a higher wage rate. Whereas the right hand side is the marginal cost of postponing labor market entry. By staying in school, the agent loses wage income during this period but avoids to pay labor income tax and continuously receives the education subsidy. Therefore, marginal cost is falling both in s_E and t_L.

Within a small open economy, the market interest rate, r, is constant and exogenously given, implying that the wage rate is constant as well, $w(t) \equiv w$. Therefore, (2.17) reduces to

$$\Delta(e^*(v), r) = e^*(v) - \frac{s_E}{1 - t_L}. \tag{2.18}$$

The left-hand side of this condition is again the demographic discount function as defined in (2.16). By Proposition A.1(ii)–(iii) in the Appendix, this expression is positive and non-increasing in e, while the right-hand side is negative at $e = 0$ and linearly increasing. Therefore, optimal schooling time is unique and positive. Furthermore, for a constant mortality process, the schooling decision is independent from the agent's date of birth, $e^*(v) \equiv e^*$.

Notice that this result crucially depends on the definition of the tax system in (2.10). If the education subsidy was not proportional to $h(v)^\phi$, this term would not cancel in Eq. (2.17) and the optimal schooling decision would explicitly depend on the cohort index.

2.1.3 Per capita plans

Per capita variables at time t are calculated as the integral of individual variables multiplied by the cohort weights $l(v,t)$. For instance, per capita consumption is defined as

$$c(t) := \int_{-\infty}^t \bar{c}(v,t) l(v,t)\, dv.$$

[13] In particular, $\Delta(x,\lambda)$ is the mortality corrected annuity factor of an age x individual who earns an interest rate of λ. This means that if an agent of age x is promised to receive p euros at each point in time from today until his death, the (expected) present value of this payment is $\Delta(x,\lambda) \cdot p$. See Proposition A.1 in the Appendix for some important properties of the Δ-function.

By differentiating $c(t)$ with respect to time and noting (2.2) as well as (2.13), we obtain per capita consumption growth,

$$\dot{c}(t) = b(t)\bar{c}(t,t) + \sigma(r-\theta)c(t) - n(t)c(t) - \int_{-\infty}^{t} \bar{c}(v,t)l(v,t)m(t-v,\psi_m(v))\,dv. \tag{2.19}$$

The first term represents consumption of newborns in t, the second captures consumption growth of all surviving individuals. The third term corrects for population growth and the last term for consumption of those individuals passing away in t. Hence, the evolution of the per capita variable at time t is determined by individual variables of all cohorts born in $(-\infty, t]$ and their share in the population. Such generational turnover effects are characteristic for overlapping generations models which feature an age distribution, see for instance Heijdra and Ligthart (2000).

Analogously, per capita human capital is defined as

$$h(t) := \int_{-\infty}^{t} \bar{h}(v,t)l(v,t)\,dv. \tag{2.20}$$

Per capita financial assets, $a(t) := \int_{-\infty}^{t} \bar{a}(v,t)l(v,t)\,dv$, accumulate according to

$$\dot{a}(t) = [r - n(t)]a(t) + \omega(t) - g(t) - c(t)$$

where $r - n(t)$ is the net interest rate, $\omega(t) := \int_{-\infty}^{t} \bar{w}(v,t)l(v,t)\,dv = wh(t)$ is per capita wage income before taxes and $g(t) := \int_{-\infty}^{t} \bar{g}(v,t)l(v,t)\,dv$ is per capita net tax payment.

2.1.4 Firms

Firms are perfectly competitive and use both physical capital, $K(t)$, and human capital, $H(t) := h(t)L(t)$, to produce a homogeneous good $Y(t)$ with Cobb-Douglas production technology. Profit maximization implies that marginal productivities of the inputs equal their factor prices. The market interest rate, r, is given internationally, thereby pinning down the capital labor ratio $K(t)/H(t)$ used in production. As this ratio is time-invariant, the same holds for the wage rate.

In an economic equilibrium, aggregate investment in domestic firms equals physical capital used in production. Any difference is either held in governmental bonds or invested abroad, see Eq. (2.7).

2.1.5 Government

The government uses the lump sum tax, z, to balance its earnings and expenditures. Instead of facing a period-by-period budget constraint, policy makers are allowed to run up debts and participate in perfect tax smoothing. However, a solvency condition has to be met. The explicit formulas are stated in Section 4.4 but not needed for the moment.

2.2 Demographic shocks

In general, the model allows for any demographic process consistent with Assumption 2.1. However, for the numerical analysis, Heijdra and Romp use a Gompertz-Makeham (G-M) mortality law as it fits the Dutch demographic data quite well. The instantaneous mortality rate for a G-M process is given by[14]

$$m(u) = \mu_0 + \mu_1 e^{\mu_2 u}.$$

The first term, μ_0, represents the baseline mortality, which is independent from u. The second term is age-dependent. μ_2 controls for the slope of the mortality rate, while μ_1 is a multiplicative scaling factor. In the special case $\mu_0 = 0$, the logarithm of the death rate is linear in age, i.e. $\ln m(u) = \ln \mu_1 + \mu_2 u$. $\mu_0 > 0$ additionally accounts for accidents or infections which occur regardless of age, see Preston et al. (2001, p.192).

[14] The estimated parameters for the cohort born in 1930 are $\mu_0 = 0.573 \times 10^{-3}$, $\mu_1 = 0.312 \times 10^{-4}$ and $\mu_2 = 0.095$.

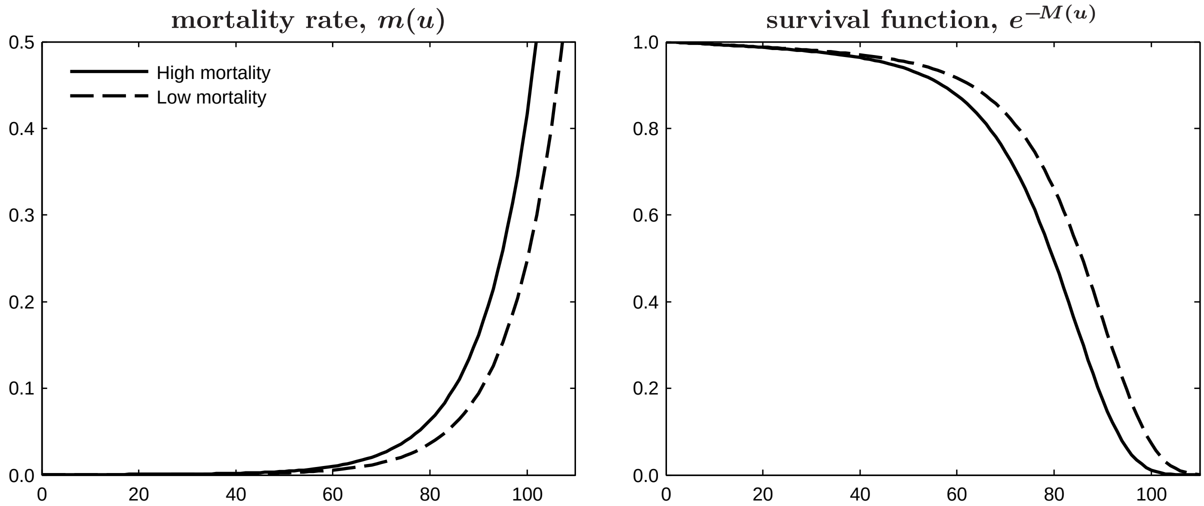

Figure 1: Mortality rate and survival function of the G-M mortality process

The cumulative mortality rate of the G-M mortality law corresponds to

$$M(u) = \mu_0 u + \frac{\mu_1}{\mu_2}\left[e^{\mu_2 u} - 1\right].$$

For the analysis of demographic shocks, let us write the mortality rate again in the form $m(u, \psi_m)$. Notice that ψ_m is a scalar, while the G-M process depends on three parameters. Hence we assume that the G-M parameter vector is a function of ψ_m, i.e. $(\mu_0, \mu_1, \mu_2)' = f(\psi_m)$. One can easily verify that, for example, $f(\psi_m) := (\mu_0, -\psi_m, \mu_2)'$ fulfills all properties imposed in Ass. 2.1. The solid line in Figure 1 depicts the mortality rate and the survival function for the parameters given in Table 1 (p. 42). The dashed line refers to a shock in adult mortality caused by a 50% decline in μ_1 and a 1.8% increase in μ_2.[15]

In the following, we examine the impact of mortality changes and fertility shocks on population growth, optimal schooling time and per capita human capital.

2.2.1 Effects on population growth

The growth rate of the population depends both on the birth rate and on the survival probabilities. In a demographic steady state, we have $b(v) \equiv b$, $\psi_m(v) \equiv \psi_m$ and $n(t) \equiv \hat{n}$. Hence (2.3) simplifies to

$$1 = b \int_{-\infty}^{t} e^{-[\hat{n}(t-v)+M(t-v,\psi_m)]}\, dv = b\, \Delta(0, \hat{n}; \psi_m). \tag{2.21}$$

Notice that the demographic discount function, Δ, also depends on ψ_m via the mortality process. This equation implicitly defines the long-run population growth rate, $\hat{n}$, as a function of b and ψ_m. Implicit differentiation of (2.21) gives the partial derivatives

$$\frac{\partial \hat{n}}{\partial b} = -\frac{\Delta(0, \hat{n}; \psi_m)}{b\, \partial\Delta(0, \hat{n}; \psi_m)/\partial \hat{n}} > 0 \tag{2.22}$$

and

$$\frac{\partial \hat{n}}{\partial \psi_m} = -\frac{\partial\Delta(0, \hat{n}; \psi_m)/\partial\psi_m}{\partial\Delta(0, \hat{n}; \psi_m)/\partial \hat{n}} > 0 \tag{2.23}$$

where the signs follow from Proposition A.3 in the Appendix. Both a higher birth rate and an increase in longevity lead to a higher steady state population growth rate. Nevertheless, there are significant differences in the transition path, see Figure 2. Assume that the economy resides in a demographic

[15]The underlying analytical dependence could be expressed as $f(\psi_m) = \big(\mu_0, (1-0.5\psi_m)\mu_1, (1+0.018\psi_m)\mu_2\big)'$

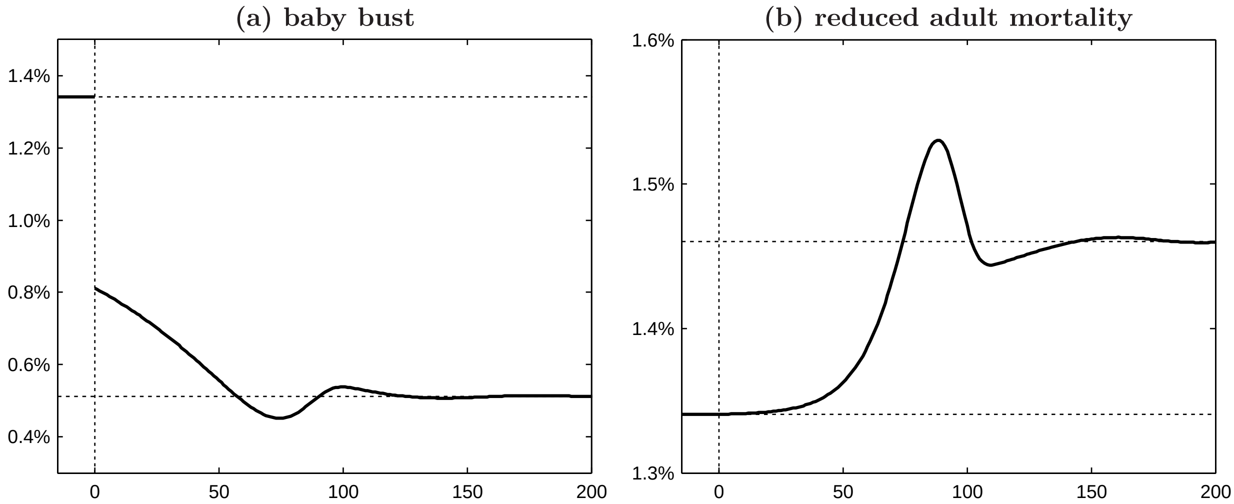

Figure 2: Dynamics of the population growth rate, $n(t)$

steady state until $t = 0$. A baby bust immediately brings a downwards jump in the population growth rate, while a decrease in adult mortality results in a gradual change. This is due to the assumption that longevity shocks are embodied (i.e. only cohorts born after the shock are affected), while fertility shocks affect pre- and post-shock generations in the same way.

In both cases, cyclical adjustments to the new long-run population growth rate can be observed. In general, transition to the new equilibrium takes much longer when the driving factor is longevity, because cohorts born before the shock still face the old mortality process. However, this inherence is somewhat unrealistic. In developed countries mortality gains are mainly achieved by improvements in medication and health care. It is likely that living cohorts benefit from these enhancements as well–at least to some degree. Nevertheless, we adopt this convention regarding mortality shocks since it simplifies the numerical solution of the model significantly.

2.2.2 Effects on schooling time

To analyze the impact of demographics shocks on schooling time, first notice that the mortality parameter ψ_m enters the first order condition for education through the demographic discount function,

$$\Delta(e^*, r; \psi_m) = e^* - \frac{s_E}{1 - t_L}. \tag{2.24}$$

To simplify notation, we omit dependence of ψ_m and e^* on the cohort index v.

As the birth rate, b, does not enter the optimality condition, a fertility shock does not influence individual behavior (neither education nor consumption). A longevity shock, however, encourages individuals to spend more years on schooling,

$$\frac{\partial e^*}{\partial \psi_m} = \frac{\partial \Delta(e^*, r; \psi_m)/\partial \psi_m}{1 - \partial \Delta(e^*, r; \psi_m)/\partial e^*} > 0 \tag{2.25}$$

where the sign follows from Prop. A.1(ii) and Prop. A.3(iii).

Decreasing adult mortality increases life expectancy at birth and–even more distinctly–the expected working period. This increases the marginal benefit from schooling because the future wage rate is determined by human capital investment at the beginning of life. As higher lifetime earnings translate into a higher consumption flow and thereby into a higher lifetime utility, the education time span rises.

Indeed, Heijdra and Romp emphasize that their model "clarifies that the crucial determinant of the schooling decision is *adult life expectancy*, not the expected planning horizon at birth" and, moreover, that "a decrease in child mortality increases expected remaining life time at birth but leaves the optimal schooling period unchanged". See Heijdra and Romp (2009a, Ch. 4) for a detailed discussion.

2.2.3 Effects on per capita human capital

Transitional dynamics of per capita variables are rather complex due to the generational turnovers as indicated in (2.19). Therefore, we only present the long-run effects of demographic shocks at this point and refer the interested reader to Heijdra and Romp (2009a, Ch. 5.2).

In the long-run demographic and economic equilibrium, the per capita stock of human capital defined in Eq. (2.20) fulfills[16]

$$\hat{h} = A_H \hat{h}^{\phi} e^* \int_{-\infty}^{t-e^*} l(v,t)\, dv,$$

which can be rewritten

$$\hat{h}^{1-\phi} = A_H \cdot e^* \cdot \hat{x} \tag{2.26}$$

where

$$\hat{x} := \hat{x}(e^*, b, \psi_m) := b \int_{e^*}^{\infty} e^{-[\hat{n}(b,\psi_m)u + M(u,\psi_m)]}\, du \tag{2.27}$$

is the steady state labor market participation rate, i.e. the proportion of workers in the population, and $\hat{n}(b, \psi_m)$ is implicitly defined by (2.21).

If the intergenerational human capital spillover parameter, ϕ, is less than unity, i.e. $0 \leq \phi < 1$, Eq. (2.26) implies a unique steady state level of human capital, $\hat{h}$. As noted in Section 2.1.4, both capital and output are proportional to human capital. It can be shown that the steady state levels of the remaining per capita variables are fully determined by $\hat{h}$ and $\hat{n}$. Furthermore, the steady state growth rates of all aggregate variables are identical and equal to the long-run population growth rate, $\hat{n}$.

The knife-edge case, $\phi = 1$, gives rise to endogenous growth and shall not be considered here, see Heijdra and Romp (2009a, Appendix B).

Fertility shock. First we study the impact of a fertility shock. As e^* is independent from b, a change in the birth rate affects long-run human capital only via the participation rate,[17]

$$\frac{\partial \hat{h}^{1-\phi}}{\partial b} = A_H \cdot e^* \cdot \frac{\partial \hat{x}}{\partial b} < 0.$$

The effect is negative because a higher fertility increases the mass of young individuals in the population, which in turn decreases the labor market participation rate.

Pure schooling shock. By contrast, a change in mortality affects $\hat{h}$ via two channels, the education decision and the age distribution. Therefore, it seems reasonable to study the partial effect of a pure education shock first. Differentiating (2.26)–(2.27) and substituting (2.18) yields

$$\frac{\partial \hat{h}^{1-\phi}}{\partial e^*} = A_H \left[\hat{x} + e^* \frac{\partial \hat{x}}{\partial e^*}\right] = b A_H e^{-[\hat{n}e^* + M(e^*,\psi_m)]} \left[\underbrace{\Delta(e^*, \hat{n}) - \Delta(e^*, r)}_{>0} - \frac{s_E}{1 - t_L}\right]. \tag{2.28}$$

In absence of an education subsidy, $s_E = 0$, the effect of a schooling shock is unambiguously positive when dynamic efficiency is assumed in the long run. In this case, $r > \hat{n}$, and the term in square brackets is positive by Proposition A.1(i).

For a non-zero subsidy, however, the aggregate effect may be negative even if $r > \hat{n}$. In this case, the positive effect of higher individual education is more than offset by the simultaneous decline in labor force participation. The economy is then "over-educated" in the sense that the same level of per capita

[16] Notice that, although variables denoted by a hat often refer to growth rates in the literature, we use this sign to indicate steady state levels.

[17] A proof can be found in Heijdra and Romp (2009a, Appendix A). However, in Proposition A.5 stated in the Appendix, we provide a different derivation which we consider more intuitive.

human capital could be achieved with shorter schooling periods. This scenario is more likely when the effective education subsidy, $s_E/(1-t_L)$, is high because this implies that the level of schooling was already comparatively high before the shock. Heijdra and Romp proceed under the following reasonable assumption which renders the effect of a schooling shock positive:

Assumption 2.2 *The steady state net interest rate, $r-\hat{n}$, is sufficiently high to ensure*

$$\Delta(e^*, \hat{n}) - \Delta(e^*, r) > \frac{s_E}{(1-t_L)}.$$

Mortality shock. It can be shown that the total effect of a mortality shock amounts to

$$\frac{\partial \hat{h}^{1-\phi}}{\partial \psi_m} = \frac{\partial \hat{h}^{1-\phi}}{\partial e^*} \cdot \frac{\partial e^*}{\partial \psi_m} + A_H \cdot e^* \cdot \frac{\partial \hat{x}}{\partial \psi_m} > 0.$$

The first term captures the *indirect* effect of aging operating via the schooling decision. It is positive according to (2.25) and Assumption 2.2. Clearly, higher longevity increases optimal length of schooling, which in turn increases per capita human capital.

The second term represents the *direct* effect of aging coming via the change in age composition. As there is no possibility to retire in this model, lowering adult mortality directly translates into a larger mass of workers in the population.[18]

2.3 Fiscal shocks

The fiscal system is determined by the income tax rate, t_L, and the education subsidy, s_E. If either of them increases, the marginal cost of schooling declines and optimal schooling time rises accordingly,

$$\frac{\partial e^*}{\partial [s_E/(1-t_L)]} = \frac{1}{1 - \partial \Delta(e^*, r; \psi_m)/\partial e^*} > 0.$$

Under Assumption 2.2, this directly translates into the steady state per capita human capital stock,

$$\frac{\partial \hat{h}^{1-\phi}}{\partial [s_E/(1-t_L)]} = \frac{\partial \hat{h}^{1-\phi}}{\partial e^*} \cdot \frac{\partial e^*}{\partial [s_E/(1-t_L)]} > 0.$$

2.4 Conclusion

In this section, we summarized the main results of Heijdra and Romp (2009a). Not surprisingly, higher longevity and more generous education subsidies increase the optimal education time span. We have also seen that demographic shocks influence the aggregate economy via two channels. Potential changes in individual optimal behavior are accompanied by an altered age distribution. We find that mortality improvements yield a positive effect on per capita human capital, $\partial \hat{h}/\partial \psi_m > 0$.

However, it should be highlighted that this result crucially depends on the model assumptions. In particular, individuals are not allowed to retire. Therefore, lower adult mortality increases the share of working individuals, $\partial \hat{x}/\partial \psi_m > 0$. Indeed, we show in Chapter 3 that in the case of a finite retirement age it is likely to observe $\partial \hat{x}/\partial \psi_m < 0$, which renders the effect of population aging on human capital unambiguous. The numerical example which we will present in Chapter 6 even reveals a negative relationship between longevity and per capita human capital, $\partial \hat{h}/\partial \psi_m < 0$, because the increase in working periods is far too weak to compensate for the falling participation rate (which is caused by a higher old-age dependency ratio).

[18] An analytical proof for $\frac{\partial \hat{x}}{\partial \psi_m} \geq 0$ can be found in Proposition A.6. Unfortunately, its counterpart in Heijdra and Romp (2009a, Appendix A) contains some minor errors.

3 Endogenous retirement with exogenous human capital

The second work on which the model in Chapter 4 is based is Heijdra and Romp (2009b). The model setup of this paper shares many similarities with Heijdra and Romp (2009a), though it differs in some important aspects. Most notably, the model features a retirement possibility and allows for a (very general) pension scheme. However, individual human capital (efficiency units of labor) is now exogenously given and the individual's wage rate, $\bar{w}(v,t)$, is exogenous as well. Moreover, there is no human capital externality on the aggregate level, and the population is only considered from biological age 20 onwards.

In the following section, we summarize the model setup on the individual and on the aggregate level. The remaining two sections again study the effect of demographic and fiscal shocks on individual behavior and on the aggregate economy. Actually, the macroeconomic impact is not discussed in Heijdra and Romp (2009b) but enables us to identify important differences to the results of the previous chapter.

3.1 Model setup

The demographic setup is identical to Section 2.1.1. The definition of per capita variables (except for human capital) as well as the setup of the private and the public sector also parallels Chapter 2. Consequently, the main part of this section will present the individual problem and discuss the first order condition for retirement.

3.1.1 Individual plans

Individuals are assumed to start working full time at biological age 20 (corresponding to $u = 0$) but are permitted to retire at some time, $R(v)$. Allowing for age dependent disutility of work, $D(u)$, remaining lifetime utility of a cohort v individual reads

$$\Lambda(v,t) = e^{M(t-v)} \int_t^{\infty} [U(\bar{c}(v,\tau)) - I(\tau - v, R(v))D(\tau - v)]e^{-[\theta(\tau-t)+M(\tau-v)]}\, d\tau \tag{3.1}$$

where I is the indicator function in general terms defined as

$$I(u,u_0) := \begin{cases} 1 & \text{for} \quad 0 \leq u < u_0, \\ 0 & \text{for} \quad u_0 \leq u. \end{cases} \tag{3.2}$$

Accordingly, $I(\tau - v, R(v))$ captures the individual's labor market status at time τ. The disutility function is assumed to be non-decreasing in age, $D' \geq 0$, and thereby stipulates that working becomes more and more burdensome as one gets older.

The asset accumulation equation reads

$$\dot{\bar{a}}(v,t) = [r + m(u)]\bar{a}(v,t) + I(u, R(v))(1 - t_L)\bar{w}(u) + [1 - I(u, R(v))]\bar{p}(u, R(v)) - \bar{c}(v,t) - z(t) \tag{3.3}$$

where here and in the following we again write $u = t - v$ shorthand for the agent's age in t. Agents earn wages, $\hat{w}$, while at work and a pension, $\hat{p}$, afterwards. The government taxes wage income at a constant rate, t_L, and collects a lump sum tax, z, from all inhabitants.[19] Unlike in the previous chapter, individual human capital is exogenous and represented by an age-dependent productivity profile, $E(u)$. The individual wage rate is proportional to the worker's productivity and therefore also age-dependent,

$$\bar{w}(u) := wE(u) \tag{3.4}$$

where w is the market rental rate of labor (which is again constant due to the assumption of a small open economy).

[19] In the original paper, Heijdra and Romp allow the income tax to be time-dependent in order to study the effect of a tax reform where the government adjusts the lump-sum tax z and uses the income tax to balance its budget. Nevertheless, we will treat t_L as a time-invariant parameter like in Chapter 2.

Although there is no policy restriction in choosing the retirement date (i.e. no minimum or maximum retirement age), the authors consider an early eligibility age, R_E. Retired agents receive their pension benefit only from R_E onward. If $R(v) < R_E$, individual expenditures during the age interval $[R(v), R_E)$ must be fully covered by income from financial assets and/or reducing savings (or increasing borrowing). For simplicity, the benefit formula depends solely on the retirement age,

$$\bar{p}(u, R(v)) := \begin{cases} 0 & \text{for } u < R_E, \\ B(R(v)) & \text{for } u \geq R_E, \end{cases} \tag{3.5}$$

where B is non-decreasing in its argument and might be discontinuous at some levels of R.

An agent of cohort v chooses a time path for consumption $\bar{c}(v, \tau)$ (for $\tau \in [t, \infty)$) and a retirement age $R(v)$ in order to maximize $\Lambda(v, t)$ given in (3.1) subject to (3.3)–(3.5), taking the level of financial assets in the planning period, $\bar{a}(v, t)$, as given. Additionally, an NPG condition is imposed (see footnote 10) and the retirement decision is irreversible.

The individual problem is again solved in two steps:

1. Obtain the optimal consumption path for a given retirement age.
2. Determine the optimal retirement age.

Step 1: Consumption In analogy to Section 2.1.2, the optimal consumption decision reads

$$\bar{c}^*(v, \tau, R(v)) = e^{\sigma(r-\theta)(\tau-t)}\, \bar{c}^*(v, t, R(v)), \quad \tau \geq t, \tag{3.6}$$

$$\bar{c}^*(v, t, R(v)) = \frac{\bar{a}(v, t) + \overline{li}(v, t, R(v))}{\Delta(u, r')}, \tag{3.7}$$

where lifetime income is now given by

$$\overline{li}(v, t, R(v)) := e^{ru+M(u)} \Bigg[(1 - t_L) \int_{\min\{u, R(v)\}}^{R(v)} \bar{w}(s) e^{-[rs+M(s)]}\, ds + B(R(v)) \int_{\max\{R_E, R(v)\}}^{\infty} e^{-[rs+M(s)]}\, ds \\ - \int_t^{\infty} z(\tau) e^{-[r(\tau-v)+M(\tau-v)]}\, d\tau \Bigg]$$

and we again use $u = t - v$ to prevent notational overload. The first line represents the present value of wage and pension income, while the second line comprises future tax payments.

Step 2: Retirement To obtain the optimal retirement decision, the above consumption rule is substituted into the lifetime utility function (3.1). For a working individual, the resulting *concentrated* utility function reads[20]

$$\bar{\Lambda}(v, t) := e^{\theta u + M(u)} \left[\int_t^{\infty} U(\bar{c}^*(v, \tau, R(v))) e^{-[\theta(\tau-v)+M(\tau-v)]}\, d\tau - \int_u^{R(v)} D(s) e^{-[\theta s + M(s)]}\, ds \right]. \tag{3.8}$$

The retirement age is now found by maximizing $\bar{\Lambda}$ with respect to $R(v)$. We show in Section A.2.2 of the Appendix that, by using (3.6)–(3.7) and taking account of the CES utility, the resulting first order condition (FOC) can be rewritten

$$U'(\bar{c}^*(v, t, R(v))) \frac{\partial \overline{li}(v, t, R(v))}{\partial R(v)} = D(R(v)) e^{-\theta[R(v)-u]-[M(R(v))-M(u)]} \tag{3.9}$$

where consumption in t is given in (3.7). The left-hand side represents the marginal utility that can be financed by the additional wage income, while the right-hand side captures the marginal disutility of postponing retirement.

[20] For retired individuals, $R(v)$ is no longer a choice variable as retirement is an absorbing state.

However, Eq. (3.9) is rather inconvenient to analyze the properties of the optimal retirement age (particularly its response to parameter variations). The reason is that the right-hand side can either be increasing or decreasing in $R(v)$, depending on the parametrization. D is increasing with age, while the exponential term is decreasing. Therefore, we follow Heijdra and Romp and define the *transformed retirement age*,[21]

$$S(u,R) := e^{ru+M(u)} \int_0^R e^{-[rs+M(s)]}\, ds, \quad 0 \le u \le R,$$

which is a continuous, monotonically increasing and strictly concave transformation of R. Hence S is invertible with derivative

$$\frac{\partial R}{\partial S} = e^{r(R-u)+M(R)-M(u)} > 0.$$

By multiplying both sides of the FOC (3.9) with $\partial R/\partial S$, we obtain[22]

$$U'(\bar{c}^*(v,t,R(v)))\frac{d\overline{li}(v,t,R(v))}{dS(v)} = D(R(v))e^{(r-\theta)[R(v)-u]} \tag{3.10}$$

where the right-hand side is now indeed unambiguously increasing in $R(v)$. Since the right-hand side is positive and $U' > 0$, any solution of (3.10) has to lie on the upward sloping part of the lifetime income function. A sufficient condition for

$$\frac{d\overline{li}}{dS} = \frac{\partial \overline{li}}{\partial R}\frac{\partial R}{\partial S} = (1-t_L)\bar{w}(R) - B(R) + B'(R)\Delta(R,r) \tag{3.11}$$

to be positive–and hence that an optimal retirement age can exist–is that the replacement rate,

$$NRR(R) := \frac{B(R)}{(1-t_L)\bar{w}(R)}, \tag{3.12}$$

is less or equal than one around $R = R^*$ (or at least does not exceed unity by much).

Notice that Eq. (3.11) only applies for $R \ge R_E$. Otherwise,

$$\frac{d\overline{li}}{dS} = (1-t_L)\bar{w}(R) + B'(R)e^{rR+M(R)} \int_{R_E}^{\infty} e^{-[rs+M(s)]}\, ds.$$

However, we restrict our attention to the case where the optimal retirement age is equal to or greater than the early eligibility age, $R^* \ge R_E$.

3.1.2 Macroeconomic aggregation

The model demography is the same as presented in Section 2.1.1. The behavior of private firms and the public sector follows the discussion in Sections 2.1.4 and 2.1.5, respectively. Per capita variables are defined as in Section 2.1.3, the only exception being per capita human capital stock, which now reads

$$h(t) := \int_{-\infty}^{t} E(t-v)I(t-v,R(v))l(v,t)\, dv.$$

3.2 Demographic shocks

In order to analyze the effects of mortality shocks, the mortality function is rewritten in the form $m(u,\psi_m)$. Dependence of m on the parameter ψ_m is as stipulated in Assumption 2.1(iii)–(iv). Besides this *biological aging effect*, the model allows to control for *productive aging*. This comprises changes in the life-cycle profile of disutility of work, D, or efficiency of labor, E, and is captured by the parameters ψ_D and ψ_E, respectively.

[21] Intuitively, $S(u,R)$ gives the value at age u of an annuity which is earned from date of birth until retirement, R.

[22] This multiplication actually transforms the FOC which the agent faces at age u into the equivalent condition at age R. Changing from a present value to a current value formulation eliminates the mortality risk from the right-hand side of (3.9).

A rise in ψ_D is assumed to lower working disutility, $\partial D(u, \psi_D)/\partial\psi_D \leq 0$ with strict inequality around the optimal retirement age $u = R^*$. Such a decrease in disutility could be driven by improvements in working conditions or due to less physically demanding production technologies.

The shape of the efficiency profile is determined by ψ_E. Increasing this parameter has a non-negative impact on productivity at all ages, $\partial E(u, \psi_E)/\partial\psi_E \geq 0$ with strict inequality around $u = R^*$. As individual human capital coincides with E, increasing old workers' productivity could equally be regarded as increasing their innate "value" on the labor market. In other words, ψ_E measures the importance of experience in the production process. As the parameter rises, experienced workers get more valuable and more effective compared to their younger colleagues.

3.2.1 Effects on population growth

Since the demographic setup is identical to the model presented in Chapter 2, the comparative static effects of the long-run population growth rate can be seen from Section 2.2.1. Higher fertility and lower adult-mortality both increase the population growth rate. Clearly, neither a change in working disutility nor productivity affects population growth.

3.2.2 Effects on retirement

In order to determine the response of the optimal retirement age to parameter variations, we stipulate that the marginal effect from postponing retirement on lifetime income, $d\overline{li}/dS$, is decreasing in R around R^*. This guarantees unique solvability of Eq. (3.10) with respect to R near the optimum which we require in the following analysis.[23]

Like the model presented in Chapter 2, a fertility shock does not influence individual decision making at all. Nevertheless, there are effects on the *per capita* level of human capital which will be presented in the next section.

In analyzing the effects of aging, we distinguish between biological and productive aging as motivated above. To simplify notation, we omit dependence of R on the cohort index v in the following.

Productive aging. A change in disutility of work decreases the marginal cost of prolonged working (right-hand side of (3.10)), while the marginal gain (left-hand side) is unaffected. This encourages individuals to postpone retirement. Analytically, implicit differentiation of the FOC (3.10) gives

$$\frac{\partial R^*}{\partial \psi_D} = -\zeta_0 \frac{\partial D(R^*, \psi_D)}{\partial \psi_D} e^{(r-\theta)(R^*-u)} > 0 \tag{3.13}$$

where ζ_0 is a positive constant.[24]

Conversely, a change in the age profile of labor efficiency leaves the marginal cost unchanged but affects the marginal gain of longer labor market participation. Keeping R fixed, lifetime income rises due to higher wages, thereby increasing consumption at all ages and decreasing marginal utility. At the same time, the marginal effect of postponing retirement on lifetime income increases because wage rates around the former retirement age are higher. Therefore, we observe a negative wealth and a positive substitution effect, which renders the total effect ambiguous,

$$\frac{\partial R^*}{\partial \psi_E} = -\zeta_0 \left\{ U''(\bar{c}^*) \underbrace{\frac{\partial \bar{c}^*}{\partial \psi_E}}_{>0} \frac{d\overline{li}}{dS} + U'(\bar{c}^*) \underbrace{\frac{\partial}{\partial \psi_E}\left[\frac{d\overline{li}}{dS}\right]}_{>0} \right\} \gtreqless 0. \tag{3.14}$$

[23]According to (3.11), a sufficient condition for $\frac{\partial}{\partial R}\left[\frac{d\overline{li}}{dS}\right] < 0$ around $R = R^*$ is that the pension benefit B is concave, i.e. $B''(R) > 0$, and the wage profile is non-increasing above the optimal retirement age, $\bar{w}'(R) \leq 0$ for $R > R^*$ (see Assumption 1 in Heijdra and Romp (2009b)).

[24]In particular, $\zeta_0 := -1/\Gamma'(R^*)$ where $\Gamma(R) := U'(\bar{c}^*)\frac{d\overline{li}}{dS} - D(R)e^{(r-\theta)(R-u)}$ resembles the first order condition (3.10). $\Gamma'(R^*)$ is positive due to the assumption that $\frac{\partial}{\partial R}\left[\frac{d\overline{li}}{dS}\right](R^*) < 0$.

Biological aging. The sensitivity analysis is more complicated when we consider a shock in old-age mortality, $\partial M(u, \psi_m)/\partial \psi_m \leq 0$. Altogether, there are three components determining the total effect:

First of all, there is a potentially ambiguous effect on lifetime income. The discount rate decreases and puts more weight on future income streams. In an economy without pensions, $B = 0$, the agents are likely to suffer income losses. For illustration, the marginal effect on lifetime income of a newborn is

$$\frac{\partial \overline{li}(v, v, R)}{\partial \psi_m} = -\int_0^R \bar{w}(s)\frac{\partial M(s, \psi_m)}{\partial \psi_m} e^{-[rs+M(s)]}\, ds + \int_0^\infty z(v+s)\frac{\partial M(s, \psi_m)}{\partial \psi_m} e^{-[rs+M(s)]}\, ds.$$

As a parameter change in ψ_m primarily affects mortality at older ages, the first term can be relatively small, leading to a decrease in lifetime income. In presence of a social security scheme, however, the agents receive pensions when old. Due to the lower mortality rate, the expected length of retirement rises and more weight is given to future pension payments. Therefore, agents perceive an additional positive effect on lifetime income.

Secondly, not only the effect on lifetime income but also the reaction of individual consumption is unclear. As life expectancy rises, the individual has to finance his consumption out of lifetime income for a longer period of time. Even if lifetime income increases due to the shock, the financial gain may still be too small to afford higher consumption levels throughout one's whole life. In this case, the agent is encouraged to postpone retirement.

Thirdly, there is yet another channel influencing the retirement decision if $B'(R^*) > 0$. In this case, the marginal effect of postponing retirement on lifetime income increases due to the lower discount rate,

$$\frac{\partial}{\partial \psi_m}\left[\frac{d\overline{li}(v, t, R)}{dS}\right] = B'(R)\frac{\partial \Delta(R, r; \psi_m)}{\partial \psi_m} > 0,$$

and the individual has an additional incentive to postpone retirement.

To sum up, $\partial R^*/\partial \psi_m > 0$ seems to be a likely outcome of sensible calibrations. A possible increase in lifetime income is most likely too low to prevent individual consumption from falling. This, together with the increasing marginal benefit from postponing retirement, encourages the agent to retire later.

3.2.3 Effects on per capita human capital

Having identified the individual response to parameter shocks, we now focus on per capita human capital. Although this section is not covered in Heijdra and Romp (2009b), we regard the aggregate economic effect as worth considering. For we expect the dynamics of the combined model in Chapter 4 to be a combination of this model and the model covered in the previous chapter. However, our attention is again limited to the steady state case.

The equilibrium stock of per capita human capital is given by

$$\hat{h} = \int_0^{R^*} E(u)l(u)\, du = b\int_0^{R^*} E(u)e^{-[\hat{n}(b,\psi_m)u+M(u,\psi_m)]}\, du. \tag{3.15}$$

We again consider fertility shocks, shocks in productivity and disutility of work as well as shocks in adult mortality.

Fertility shock. It can be shown that[25]

$$\frac{\partial \hat{h}}{\partial b}\frac{b}{\hat{h}} = \frac{1}{\bar{u}}\left[\bar{u} - \int_0^{R^*} u\frac{E(u)l(u)}{\int_0^{R^*} E(u)l(u)\, du}\, du\right]$$

where $\bar{u} := \int_0^\infty u\, l(u)\, du$ is the mean age of the population. The second term corresponds to the mean age of the labor force measured in efficiency units.

[25] This is a direct application of Lemma A.4(i) with $\varphi = E$, $e = 0$ and $R = R^*$.

For a constant productivity profile, $E(u) \equiv E$, the term in square brackets is simply the difference in mean age between the total population and the workforce. In this model framework, the latter figure is necessarily lower. To see this, remember that $u = 0$ refers to biological age 20, which is assumed to be the age of labor market entry. Hence the model only considers cohorts from their "economic birth" at age 20 onwards. As a result, the mean age of the working population (cohorts with $u \in [0, R^*)$), is below the mean age of the total model population (all cohorts with $u \geq 0$).[26]

Intuitively, a birth rate shock increases the share of young individuals. As these immediately start working, labor force participation increases. In case of a flat efficiency profile, this translates directly into per capita human capital. However, if young agents were significantly less productive than their older colleagues, the effect on per capita human capital might be negative.

Productive aging. A change in disutility of older workers affects human capital exclusively via the change in optimal retirement age. As $\partial \hat{h}/\partial R^* = E(R^*)l(R^*) > 0$, Eq. (3.13) implies

$$\frac{\partial \hat{h}}{\partial \psi_D} = \frac{\partial \hat{h}}{\partial R^*}\frac{\partial R^*}{\partial \psi_D} > 0.$$

The effect of an increase in the productivity of older workers is again unclear,

$$\frac{\partial \hat{h}}{\partial \psi_E} = \frac{\partial \hat{h}}{\partial R^*}\frac{\partial R^*}{\partial \psi_E} + \int_0^{R^*} \frac{\partial E(u, \psi_E)}{\partial \psi_E} l(u)\, du \gtreqless 0.$$

However, an increase in ψ_E not only affects retirement ages, but also the whole efficiency profile of the labor force. As a result, the ambiguous effect on retirement (first term) is accompanied by a higher efficiency of the remaining workers (second term). This means that the aggregate effect of productivity gains could still be positive, even if $\partial R^*/\partial \psi_E$ is negative.

Mortality shock. A increase in old-age mortality affects both retirement ages and the age distribution,

$$\frac{d\hat{h}}{d\psi_m} = \frac{\partial \hat{h}}{\partial R^*}\frac{\partial R^*}{\partial \psi_m} + \frac{\partial \hat{h}}{\partial \psi_m}.$$

Without making further assumptions, the sign of both terms is unclear. However, as argued in Section 3.2, the effect of a reduction in mortality on retirement age is likely to be positive, $\partial R^*/\partial \psi_m > 0$, which renders the first term positive.

The second term, $\partial \hat{h}/\partial \psi_m$, accounts for the demographic change induced by rising longevity. Although the analytical expression looks rather complicated, it simplifies a bit for the case of a flat productivity profile, $E(u) \equiv E$,[27]

$$\frac{\partial \hat{h}}{\partial \psi_m} = E \cdot \frac{\partial \hat{x}}{\partial \psi_m} = \mathbb{C}\left[\frac{\int_0^{R^*} M_\psi(u, \psi_m)\lambda(u)\, du}{\int_0^\infty M_\psi(u, \psi_m) l(u)\, du} - \frac{\bar{u}_x}{\bar{u}}\right] \leq 0$$

where $\mathbb{C}$ is a positive constant, $\bar{u}_x := \int_0^{R^*} u\lambda(u)\, du$ is the mean age of the workforce and $\lambda(u) := l(u)/\hat{x}$ is the weight of the cohort aged u in the working population. The sign follows from Proposition A.6(ii).

The expression in square brackets essentially measures at which ages the mortality gains occur. The first term compares the average marginal effect of ψ_m on mortality rates between working population and total population. The second one compares the respective mean ages. As the distribution of marginal

[26] An alternative way to formulate the model would be to consider all cohorts from their biological birth onwards and impose a fixed education time span of $e = 20$ years. Although individual behavior is very similar, this setup significantly changes the dynamics of per capita human capital. A birth rate shock then not only increases the mass of younger workers (which is the case here) but also the mass of non-working cohorts aged below 20. See also Sections 5.2 and 5.4.

[27] The expression for an arbitrary efficiency profile $E(.)$ can easily be inferred from Lemma A.4(ii) in setting $e = 0$, $R = R^*$ and $\varphi = E$.

longevity gains in the population is extremely right skewed (significant gains only happen at old ages), the first term will be comparatively small.

To sum up, a mortality shock affects steady state per capita human capital in two ways which render the total effect ambiguous: On the one hand, the optimal retirement age is likely to increase, thereby raising human capital. On the other hand, the weight of working ages in the population decreases, which–in case of a flat efficiency profile–lowers human capital. However, this negative effect will be weaker when the efficiency profile is strictly decreasing at higher ages.

3.3 Fiscal shocks

The pension system is determined by the income tax rate, t_L, and the pension benefit formula, $B(R)$. The following relations can be obtained by implicit differentiation of the FOC (3.10). A tax increase has an ambiguous effect on retirement ages,

$$\frac{\partial R^*}{\partial t_L} = \zeta_1 e^{ru+M(u)} \int_u^{R^*} \bar{w}(s) e^{-[rs+M(s)]}\, ds - \zeta_2 \bar{w}(R^*) \gtreqless 0$$

where ζ_1 and ζ_2 are positive constants.[28] When the net wage falls, individuals are encouraged to retire later to compensate for the loss in lifetime income. On the other hand, there is an incentive to retire earlier since the loss in (net) wage income involved with retiring is lower for a higher labor tax rate.

Raising the pension benefit *level*, $B(R)$, unambiguously decreases the optimal retirement age,

$$\frac{\partial R^*}{\partial B(R^*)} = -\zeta_1 e^{ru+M(u)} \int_{R^*}^{\infty} e^{-[rs+M(s)]}\, ds - \zeta_2 < 0,$$

as it increases lifetime income and the pension replacement rate at the same time. Conversely, increasing the *slope* of the pension formula encourages individuals to postpone retirement,

$$\frac{\partial R^*}{\partial B'(R^*)} = \zeta_2 \Delta(R^*, r) > 0,$$

because working an additional period then has a stronger effect on lifetime income.

As all fiscal shocks affect steady state per capita human capital only via the retirement decision, $\hat{h}$ responds in the same way as R^*. For instance, human capital falls if the pension system becomes more generous,

$$\frac{\partial \hat{h}}{\partial B(R^*)} = \frac{\partial \hat{h}}{\partial R^*} \frac{\partial R^*}{\partial B(R^*)} < 0.$$

3.4 Conclusion

This chapter gave a comprehensive overview of Heijdra and Romp (2009b) supplemented by a comparative static analysis of aggregate economic effects. Most importantly, we found that lower adult mortality encourages individuals to retire later. However, the effect on per capita human capital is ambiguous since the weight of retirees in the population increases at the same time, $\partial \hat{x}/\partial \psi_m \leq 0$. By contrast, lower working disutility of older agents implies both later retirement and higher human capital in the long run (since the demography is unaffected).

Regarding pension reforms, we found that increasing the benefit level promotes early retirement due to the income effect. If instead the slope of the benefit formula is raised, agents tend to postpone retirement. The same results can be established for the macroeconomic impact of pension reforms on per capita human capital.

[28] In particular, $\zeta_1 := -\zeta_0\, U''(\bar{c}^*)/\Delta(u, r')$ and $\zeta_2 := \zeta_0\, U'(\bar{c}^*)$.

4 Endogenous human capital formation and retirement

We now combine the models presented in Chapters 2 and 3 in a straightforward manner. Agents can choose both schooling length *and* retirement age endogenously. Unlike in Chapter 2, we assume that they suffer disutility from both education and working. Although potentially different, we keep analysis as simple as possible and consider the same age-dependent disutility function $D(u)$ in both the schooling and the working period. This is for instance in line with Boucekkine et al. (2002).

4.1 Individual plans

Remaining lifetime utility of a cohort v individual at time $t \geq v$ is identical to Chapter 3,

$$\Lambda(v,t) = e^{\theta u + M(u)} \int_t^{\infty} \big[U(\bar{c}(v,\tau)) - I(\tau - v, R(v))D(\tau - v)\big] e^{-[\theta(\tau - v) + M(\tau - v)]}\, d\tau, \tag{4.1}$$

where here and in the following we again write $u = t - v$ for the agent's age in t to simplify notation. I is the indicator function defined in (3.2). Since $I(\tau - v, R(v)) = 1$ for $\tau < R(v)$, the individual suffers age-dependent disutility until retirement.

Combining the asset accumulation equations of the two models yields

$$\begin{aligned} \dot{\bar{a}}(v,t) = [r + m(u)]\bar{a}(v,t) + I(u, e(v))\, \bar{s}_E(v,t) + I(u, R(v))(1 - t_L)\bar{w}(v,t) \\ + [1 - I(u, R(v))]\, \bar{p}(v, e(v), R(v)) - \bar{c}(v,t) - \bar{z}(v,t). \end{aligned} \tag{4.2}$$

Assets can be invested in domestic firms, $\bar{k}(v,t)$, domestic governmental bonds, $\bar{d}(v,t)$, and foreign bonds, $\bar{f}(v,t)$, such that

$$\bar{a}(v,t) = \bar{k}(v,t) + \bar{d}(v,t) + \bar{f}(v,t). \tag{4.3}$$

Education subsidies and lump sum taxes are indexed to marginal schooling productivity as in Chapter 2,

$$\bar{s}_E(v,t) := s_E\, w(t) A_H h(v)^{\phi} \quad \text{and} \quad \bar{z}(v,t) := z(t) w(t) A_H h(v)^{\phi}. \tag{4.4}$$

The individual wage rate is proportional to individual human capital,

$$\bar{w}(v,t) := w(t)\bar{h}(v,t), \tag{4.5}$$

which is given by[29]

$$\bar{h}(v,t) := \begin{cases} 0 & \text{for } 0 \leq u < e(v), \\ A_H h(v)^{\phi} e(v) & \text{for } u \geq e(v), \end{cases} \qquad 0 \leq \phi \leq 1. \tag{4.6}$$

The pension benefit formula from the model of Chapter 3 needs to be modified to account for the actual years at work instead of the retirement age alone. Furthermore, we assume that the benefit not only depends on the length of the working period (as in Eq. (3.5)) but also on the wage rate earned. The pension benefit of our model is calculated as a certain percentage of lifetime wage income. We abstract from an early eligibility age and set

$$\bar{p}(v, e(v), R(v)) := \vartheta \int_{e(v)}^{R(v)} \bar{w}(v, v + s)\, ds \tag{4.7}$$

where $\vartheta > 0$ is referred to as the accrual rate of the pension scheme. Due to the small open economy assumption, the market wage is time-constant, $w(t) \equiv w$, and the pension benefit simplifies to

$$\bar{p}(v, e(v), R(v)) = \vartheta[R(v) - e(v)] w A_H h(v)^{\phi} e(v).$$

[29] Notice that this expression differs slightly from Eq. (2.9) regarding the inequality signs. Defining $\bar{h}$ in this way does not cause any analytical differences but makes Eq. (4.2) more sensible: An individual receives the education subsidy while $u \in [0, e)$, then starts working and earns wage income while $u \in [e, R)$ and receives a pension thereafter. By contrast, (2.9) implies $\bar{h}(v, e(v)) = 0$ and hence $\bar{w}(v, e(v)) = 0$. According to (4.2), at $t = e(v)$, the agent would neither receive an education subsidy (as $I(e(v), e(v)) = 0$) nor a wage.

At time t, an individual of cohort v chooses his time path for consumption $\bar{c}(v,\tau)$ (for $\tau \in [t,\infty)$), length of schooling $e(v)$, and retirement age $R(v)$ in order to maximize $\Lambda(v,t)$ given in (4.1) subject to (4.2)–(4.7), taking the level of financial assets in the planning period, $\bar{a}(v,t)$, as given. Additionally, we require a No-Ponzi-Game condition (see footnote 10) as well as some further regularity conditions.[30]

Notice that both the schooling and the retirement decision are irreversible. Once at work, $e(v)$ is no longer a choice variable. Analogously, retired individuals cannot re-enter the labor market.

To analytically solve the dynamic optimization problem, we proceed in a three-step fashion:

1. Choose the optimal consumption path $\left\{\bar{c}^*(v,\tau,e(v),R(v))\right\}_{\tau=t}^{\infty}$ conditioning on given levels of $e(v)$ and $R(v)$.
2. Obtain the optimal retirement age $R^*(v,e(v))$ for a fixed education time span $e(v)$.
3. Calculate the optimal length of schooling $e^*(v)$ and substitute it into the above outcomes to pin down $R^*(v)$ and the trajectory of consumption.

Step 1: Consumption. The optimal consumption path for the combined model is essentially identical to the one found in Chapters 2 and 3. The consumption Euler equation reads

$$\frac{\dot{\bar{c}}^*(v,\tau,e(v),R(v))}{\bar{c}^*(v,\tau,e(v),R(v))} = \sigma(r-\theta), \tag{4.8}$$

and the time path of consumption is given by

$$\bar{c}^*(v,\tau,e(v),R(v)) = e^{\sigma(r-\theta)(\tau-t)}\,\bar{c}^*(v,t,e(v),R(v)), \quad \tau \geq t, \tag{4.9}$$

$$\bar{c}^*(v,t,e(v),R(v)) = \frac{\bar{a}(v,t) + \overline{li}(v,t,e(v),R(v))}{\Delta(u,r')}. \tag{4.10}$$

where $r' := r - \sigma(r-\theta)$. The only difference is the definition of lifetime income, which reads

$$\begin{aligned}\overline{li}(v,t,e(v),R(v)) := {} & wA_H h(v)^{\phi} e^{ru+M(u)} \\ & \left[s_E \int_u^{\max\{u,e(v)\}} e^{-[rs+M(s)]}\,ds + (1-t_L)e(v) \int_{\max\{u,e(v)\}}^{\max\{u,R(v)\}} e^{-[rs+M(s)]}\,ds \right. \\ & \left. + \vartheta\big[R(v)-e(v)\big]e(v) \int_{\max\{u,R(v)\}}^{\infty} e^{-[rs+M(s)]}\,ds - \int_t^{\infty} z(\tau) e^{-[r(\tau-v)+M(\tau-v)]}\,d\tau \right].\end{aligned} \tag{4.11}$$

As $\bar{a}(v,t)$ is predetermined in t and lifetime income does not depend on consumption, Eq. (4.10) pins down (conditional) optimal consumption at time t. The path for $\tau > t$ is given in Eq. (4.9). Notice that the initial level of consumption depends on the mortality process, while the optimal consumption growth rate does not. Yet unknown is the level of lifetime income associated with the individual's optimal decision. Once we know $e^*(v)$ and $R^*(v)$, this can be easily calculated from (4.11).

Further procedure. We proceed in the same way as in Chapter 3 and form a concentrated lifetime utility function by substituting the optimal consumption decision into (4.1),[31]

$$\bar{\Lambda}(v,t) := e^{\theta u+M(u)} \left[\int_t^{\infty} U(\bar{c}^*(v,\tau,e,R)) e^{-[\theta(\tau-v)+M(\tau-v)]}\,d\tau - \int_u^R D(s) e^{-[\theta s+M(s)]}\,ds \right]. \tag{4.12}$$

[30] In particular, we require remaining schooling time to be non-negative, $v-t+e(v) \geq 0$, remaining time in the workforce to be non-negative, $v-t+R(v) \geq 0$, and the retirement age to exceed the age of labor market entry, $R(v)-e(v) > 0$. While the last condition will be fulfilled in any sensible calibration, the others clearly depend on the agent's age in the planning period.

[31] To simplify notation, we omit the cohort index of e and R from this point on.

To obtain optimal schooling time and retirement age, we maximize $\bar{\Lambda}$ with respect to e and R subject to (4.10), which represents the lifetime budget constraint once consumption is chosen optimally, see the discussion at (2.15). Fortunately, we do not need a Lagrangian approach at this point because of the simple form of the constraint.

In the following, we implicitly assume that the maximizing agent is young enough to have both schooling and retirement under his control. Dynamic consistency ensures that the agent's optimal decisions are independent from the planning period.

Step 2: Retirement. The optimal retirement age can be found by differentiating $\bar{\Lambda}$ with respect to R and setting the derivative equal to zero,

$$\frac{\partial \bar{\Lambda}(v,t)}{\partial R} = e^{\theta u + M(u)} \left[\int_t^\infty U'(\bar{c}^*(v,\tau,e,R)) \frac{\partial \bar{c}^*(v,\tau,e,R)}{\partial R} e^{-[\theta(\tau-v)+M(\tau-v)]}\, d\tau - D(R) e^{-[\theta R + M(R)]} \right] \stackrel{!}{=} 0.$$

This is the same first order condition as in Chapter 3, and it can be inferred from Section A.2.2 that the expression can be simplified to

$$U'(\bar{c}^*(v,t,e,R)) \frac{\partial \overline{li}(v,t,e,R)}{\partial R} = D(R) e^{-\theta(R-u)-[M(R)-M(u)]}. \tag{4.13}$$

In optimum, the marginal disutility of work due to postponing retirement (right-hand side) equals the marginal utility that can be financed by the additional wage income (left-hand side).

For convenience, we rewrite (4.13) in terms of the *transformed retirement age* introduced in Chapter 3,

$$S(u,R) = e^{ru+M(u)} \int_0^R e^{-[rs+M(s)]}\, ds, \quad 0 \le u \le R,$$

to obtain[32]

$$U'(\bar{c}^*(v,t,e,R)) \frac{d\overline{li}(v,t,e,R)}{dS} = D(R) e^{(r-\theta)(R-u)}. \tag{4.14}$$

If an optimal retirement age exists, it has to lie on the upward sloping part of the lifetime income function because both U' and $D(R)$ are positive. It can be shown that, for $R > e$,

$$\begin{aligned} \frac{d\overline{li}}{dS} = \frac{\partial \overline{li}}{\partial R}\frac{\partial R}{\partial S} &= (1-t_L)\bar{w}(v, v+R) - \bar{p}(e,R) + \frac{\partial \bar{p}(e,R)}{\partial R}\Delta(R,r) \\ &= (1-t_L) w A_H h(v)^\phi e \left[1 - \frac{\vartheta(R-e)}{1-t_L} + \frac{\vartheta}{1-t_L}\Delta(R,r) \right]. \end{aligned} \tag{4.15}$$

The marginal effect of postponing retirement on lifetime income equals additional wage income minus forgone pension benefits in period $v+R$, augmented by the present value of changes in future pension benefits.

Eq. (4.14) implicitly defines a unique optimal retirement age, $R^*(v,e)$, which is strictly greater than e if $D(e)$ is sufficiently low. The two expressions determining the optimal retirement decision are depicted in Figure 3 as functions of R. Observe that the left-hand side of (4.14) exhibits a positive jump at $R = e$, i.e. when the individual enters the labor market. Clearly, for $R < e$, the agent has no wage earnings and hence $d\overline{li}/dS = 0$. For $R \ge e$, the expression is continuous and decreasing in R. As the right-hand side is continuous, positive and increasing, a unique equilibrium point exists if and only if $D(e)$ is low enough.

If disutility at labor market entry, $D(e)$, is too high, there is no intersection between the lines in Fig. 3 and no inner optimum exists. In this case, the individual wants to "skip" working and retire straightaway, resulting in $R^*(v,e) = e$.

[32]Notice that, although u enters the optimality condition, the optimal decision is in fact independent from the agent's age in t. Hence the retirement decision is indeed dynamically consistent. The argumentation is the same as in Section A.2.2.

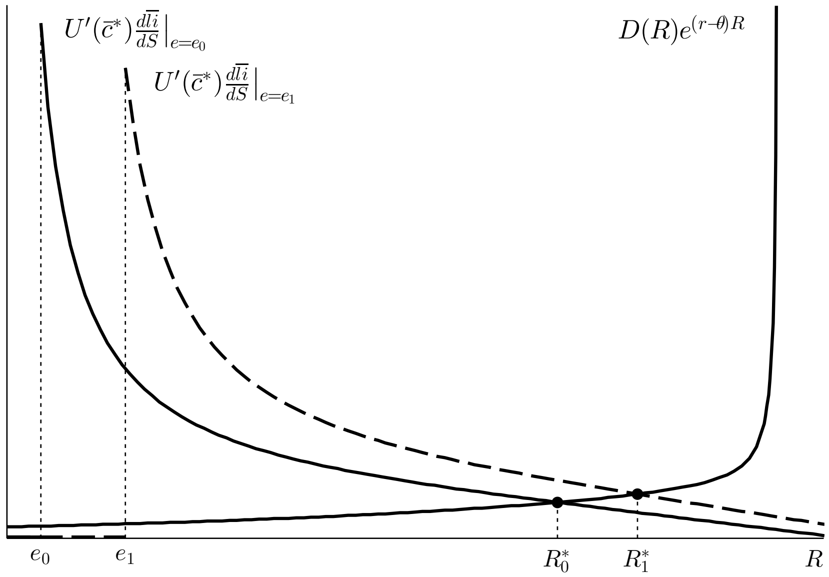

Figure 3: Determining the optimal retirement age R^* conditional on e

As noted before, the optimal retirement age fulfills $d\overline{li}/dS > 0$. However, it turns out to be convenient to assume that in optimum a slightly stronger condition holds:

Assumption 4.1 *Let the* net replacement rate *of the pension system be defined as*

$$NRR(e, R) := \frac{\bar{p}(e, R)}{(1 - t_L)\bar{w}(v, v + R)} = \frac{\vartheta(R - e)}{1 - t_L}. \tag{4.16}$$

At the optimal retirement age, the replacement rate does not exceed unity,

$$NRR(e(v), R^*(v, e)) \leq 1. \tag{4.17}$$

In the vast majority of OECD countries, the net replacement rate of the average earner is indeed less than one (OECD, 2011, p. 125). The only exceptions are the Netherlands, Iceland, Hungary and Greece. However, the net replacement rates of these countries are still close to unity and condition (4.17) is not a necessary condition for the following results to hold.

Notice that, for a given schooling time, the conditions determining the optimal retirement decision (4.14)–(4.15) in the combined model are essentially the same as (3.10)–(3.11) found in Chapter 3.

Step 3: Education. The optimal schooling length can now be obtained by substituting the optimal retirement rule, $R^*(v, e)$, into the first order condition with respect to e, which reads[33]

$$\frac{\partial \bar{\Lambda}(v, t)}{\partial e} = e^{\theta u + M(u)} \int_t^\infty U'(\bar{c}^*(v, \tau, e, R)) \frac{\partial \bar{c}^*(v, \tau, e, R)}{\partial e} e^{-[\theta s + M(s)]}\, d\tau \stackrel{!}{=} 0.$$

Because of (4.9) and $U' > 0$, this is equivalent to

$$\frac{\partial \bar{c}^*(v, t, e, R)}{\partial e} = \frac{1}{\Delta(u, r')} \frac{\partial \overline{li}(v, t, e, R)}{\partial e} \stackrel{!}{=} 0. \tag{4.18}$$

Thus, individuals choose the education time which maximizes their remaining lifetime income.

[33] Equivalently, we could form another concentrated utility function controlling for both optimal consumption and retirement, and maximize it with respect to e.

Straightforward differentiation yields

$$\begin{aligned}\frac{\partial \overline{li}}{\partial e} &= e^{ru+M(u)}\Bigg\{\big[s_E(v,v+e)-(1-t_L)\bar{w}(v,v+e)\big]e^{-[re+M(e)]}\\ &\qquad + (1-t_L)\int_e^R \frac{\partial \bar{w}(v,v+s)}{\partial e}e^{-[rs+M(s)]}\,ds + \frac{\partial \bar{p}(e,R)}{\partial e}\int_R^\infty e^{-[rs+M(s)]}\,ds\Bigg\} \qquad (4.19)\\ &= wA_H h(v)^\phi e^{r(u-e)+M(u)-M(e)}\big[s_E-(1-t_L)e+(1-t_L)\Delta_1(e,r,R)+\vartheta(R-2e)\Delta_2(e,r,R)\big]\end{aligned}$$

where Δ_1 and Δ_2 are "truncated" Δ-functions defined as[34]

$$\begin{aligned}\Delta_1(x,\lambda,C) &:= e^{\lambda x+M(x)}\int_x^C e^{-[\lambda\alpha+M(\alpha)]}\,d\alpha, \quad C\geq x\geq 0,\\ \Delta_2(x,\lambda,C) &:= e^{\lambda x+M(x)}\int_C^\infty e^{-[\lambda\alpha+M(\alpha)]}\,d\alpha, \quad C\geq x\geq 0.\end{aligned}$$

Therefore, the first order condition can be rewritten as

$$\Delta_1(e,r,R)+\frac{\vartheta(R-2e)}{1-t_L}\Delta_2(e,r,R)=e-\frac{s_E}{1-t_L}. \qquad (4.20)$$

The right-hand side represents the immediate cost of postponing labor market entry, i.e. forgone earnings minus the effective education subsidy. However, longer education implies higher wages in the future and a change in pension benefits, which is captured by the left-hand side. In optimum, marginal gains equal marginal costs.

Let us consider R as a parameter for the following paragraphs. For any $R>0$, Eq. (4.20) implicitly defines a unique and positive optimal schooling length, $e^*(R)<R$. To see this, notice that, by Proposition A.2, the left-hand side is positive at $e=0$, strictly decreasing in e, and negative at $e=R$.[35] The right-hand side is negative at $e=0$, strictly increasing and positive at $e=R$ (since $R\gg s_E/(1-t_L)$ in any sensible calibration). This yields a unique intersection, which satisfies $0<e^*(R)<R$.

Next, we want to compare the optimal education level determined by (4.20) to the one found in Chapter 2. Therefore, we use the identity $\Delta_1(e,r,R)=\Delta(e,r)-\Delta_2(e,r,R)$ to rewrite the condition as

$$\Delta(e,r)=e-\frac{s_E}{1-t_L}+\left[1-\frac{\vartheta(R-2e)}{1-t_L}\right]\Delta_2(e,r,R). \qquad (4.21)$$

If there was no possibility to retire in our model, i.e. $R=\infty$, Eq. (4.21) is identical to Eq. (2.18) because $\lim_{R\to\infty} R\cdot\Delta_2(e,r,R)=0$. For $R<\infty$, there is an additional term capturing the effect of longer schooling on pension benefits. In particular, the term in square brackets compares the marginal effect of e on the wage rate to its effect on the pension benefit. Under Assumption 4.1, this term is positive and thus decreasing the marginal benefit of education and reducing optimal schooling time.

Graphically, this is illustrated by Figure 4. As R rises from R_0 to R_1, the curve gets flatter and optimal schooling length increases. For $R\to\infty$, the curve approaches the dotted line, which corresponds to the setup of Chapter 2 and yields the highest education time spans.

At first sight, this result may seem surprising, but from an individual point of view, returns to schooling are lower in presence of a pension scheme because the pension benefit typically lies below the wage rate earned at the labor market. But even a replacement rate of 100% would not be sufficient to replicate the same education level as in a world without pensions.

[34] See Proposition A.2 in the Appendix for important properties of these functions.

[35] Indeed, the derivative of the left-hand side fulfills $\frac{\partial\Delta_1}{\partial e}-\frac{2\vartheta}{1-t_L}\Delta_2+\frac{\vartheta(R-2e)}{1-t_L}\frac{\partial\Delta_2}{\partial e}\leq\frac{\partial\Delta_1}{\partial e}+\frac{\partial\Delta_2}{\partial e}-\frac{2\vartheta}{1-t_L}\Delta_2$ under Assumption 4.1. The expression to the right is strictly negative because of $\frac{\partial\Delta_1}{\partial e}+\frac{\partial\Delta_2}{\partial e}=\frac{\partial\Delta}{\partial e}\leq 0$.

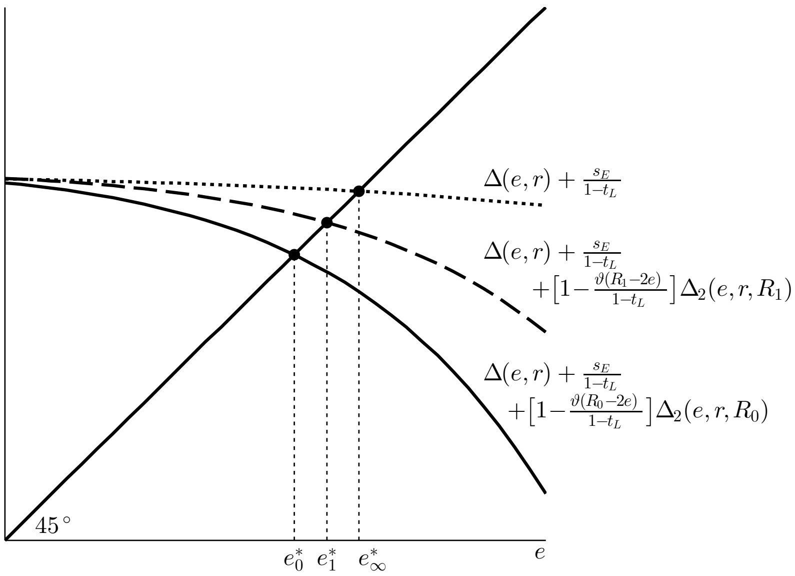

Figure 4: Determining the optimal schooling length e^* conditional on R

The reason is that, if R is exogenously given and finite, every period of additional schooling decreases working life by the same extent. Although wage rates rise with education, the period shortens during which wage is earned and pension claims are accumulated. Analytically, observe that the elasticity of wages with respect to education is $\eta_w(e,R) := \frac{\partial \bar{w}}{\partial e}\frac{e}{\bar{w}} = 1$ because the relationship is linear. Whereas the elasticity of pension benefits is

$$\eta_p(e,R) := \frac{\partial \bar{p}}{\partial e}\frac{e}{\bar{p}} = \frac{R-2e}{R-e} < 1.$$

Using these expressions, Eq. (4.21) can be written as

$$\Delta(e,r) = e - \frac{s_E}{1-t_L} + \big[\eta_w - \eta_p(e,R)\cdot NRR(e,R)\big]\Delta_2(e,r,R).$$

As mentioned above, $NRR = 1$ is not sufficient to establish the same level of education as in the absence of a retirement possibility. Since $\eta_w - \eta_p$ is positive, the marginal benefit of schooling is still lower. In order to achieve the same level of schooling, the replacement rate needs to be sufficiently greater than unity to compensate the individual for the lower returns to schooling of the pension scheme.

However, this result crucially depends on our way of modeling individual human capital, which is not subject to any depreciation. Therefore, an individual aged 30 has the same working productivity as a 90-year-old. For a productivity profile that is decreasing at older ages, the incentive to invest in human capital is likely to be higher in a world with retirement possibility because then a pension benefit formula like (4.7) allows to achieve a higher lifetime income.

So far, R was merely treated as a parameter. In Step 2, however, we showed that the retirement age chosen by an individual is in fact a function of his education time span, $R^*(v,e)$. The optimal schooling length can thus be found by plugging $R^*(v,e)$ defined by Eq. (4.14) into Eq. (4.21), which yields

$$\Delta(e,r) = e - \frac{s_E}{1-t_L} + \left[1 - \frac{\vartheta(R^*(v,e)-2e)}{1-t_L}\right]\Delta_2(e,r,R^*(v,e)). \tag{4.22}$$

If a solution $e^*(v)$ to (4.22) exists, it is used to pin down optimal retirement, $R^*(v) := R^*(v,e^*(v))$. The optimal consumption path is then found by inserting these values into (4.9)–(4.11).

4.2 Per capita plans

Per capita variables at time t are calculated as the integral of individual variables multiplied by the cohort weights $l(v,t)$. For instance, per capita consumption is defined as

$$c(t) := \int_{-\infty}^{t} \bar{c}(v,t)l(v,t)\,dv.$$

By differentiating $c(t)$ with respect to time and noting (4.8) as well as (2.2), we obtain per capita consumption growth,

$$\dot{c}(t) = b(t)\bar{c}(t,t) + \sigma(r-\theta)c(t) - n(t)c(t) - \int_{-\infty}^{t} \bar{c}(v,t)l(v,t)m(t-v,\psi_m(v))\,dv.$$

The first term represents consumption of newborns in t, the second captures consumption growth of all surviving individuals. The third term corrects for population growth and the last term for consumption of those individuals passing away in t.

Per capita human capital is given by

$$h(t) := \int_{-\infty}^{t} I(t-v,R^*(v))\bar{h}(v,t)l(v,t)\,dv \tag{4.23}$$

and reflects average individual human capital in the *working* population since $\bar{h}(v,t)=0$ for $t-v<e^*(v)$. Notice that, as $e^*(v)$ and $R^*(v)$ depend on the cohort-index, various cohorts can enter or leave the labor market simultaneously. Therefore, a general expression for $\dot{h}(t)$ is hard to obtain.

Per capita financial assets, $a(t) := \int_{-\infty}^{t} \bar{a}(v,t)l(v,t)\,dv$, accumulate according to

$$\dot{a}(t) = [r-n(t)]a(t) + \omega(t) - g(t) - c(t) - z(t)$$

where $\omega(t) := \int_{-\infty}^{t} I(t-v,R^*(v))\bar{w}(v,t)l(v,t)\,dv$ is per capita wage income before taxes and $g(t) := \int_{-\infty}^{t} \bar{g}(v,t)l(v,t)\,dv$ are per capita net tax payments. Individual net taxes, $\bar{g}(v,t)$, are given as

$$\bar{g}(v,t) := \bar{z}(v,t) - I(u,e^*(v))\bar{s}_E(v,t) + I(u,R^*(v))t_L\bar{w}(v,t) - [1-I(u,R^*(v))]\bar{p}(v,e^*(v),R^*(v))$$

where again $u=t-v$.

4.3 Firms

Firms are perfectly competitive and use both physical and human capital to produce a homogeneous good, $Y(t)$, which is traded internationally. We assume a Cobb-Douglas production function with constant returns to scale,

$$Y(t) = A_Y K(t)^{\alpha} H(t)^{1-\alpha}.$$

A_Y is a constant technology parameter, $K(t) := L(t)k(t)$ is the aggregate physical capital stock and $H(t) := L(t)h(t)$ is the aggregate human capital stock. The former is subject to depreciation and evolves according to $\dot{K}(t) = I(t) - \delta K(t)$ where $I(t)$ is gross investment and δ is the depreciation rate.

The rental rate of capital, r, is time-invariant and given internationally. Profit maximization yields the usual factor demand equations:

$$\frac{\partial Y(t)}{\partial K(t)} = \alpha A_Y \left(\frac{h(t)}{k(t)}\right)^{1-\alpha} = r+\delta$$

and

$$\frac{\partial Y(t)}{\partial H(t)} = (1-\alpha)A_Y \left(\frac{h(t)}{k(t)}\right)^{-\alpha} = w(t).$$

As the world interest rate pins down the capital labor ratio, $k(t)/h(t)$, the wage rate is indeed time invariant, $w(t) \equiv w$. In particular,

$$w = (1-\alpha)A_Y \left(\frac{\alpha A_Y}{r+\delta}\right)^{\alpha/(1-\alpha)}. \tag{4.24}$$

Furthermore, physical capital is proportional to human capital at each point in time,

$$k(t) = \left(\frac{\alpha A_Y}{r+\delta}\right)^{1/(1-\alpha)} h(t). \tag{4.25}$$

Since the production function exhibits constant returns to scale, this relation implies that per capita output, $y(t) := Y(t)/L(t)$, is also proportional to the per capita human capital stock,

$$y(t) = \frac{w}{1-\alpha}\, h(t). \tag{4.26}$$

From the definition of $\dot{K}$ we derive the evolution of the per capita stock of physical capital as

$$\dot{k}(t) = i(t) - [\delta + n(t)]k(t) \tag{4.27}$$

where $i(t) := I(t)/L(t)$ is per capita investment.

4.4 Government

Although a period-by-period budget constraint would be easier to implement, we stick to the setup of the Heijdra and Romp papers and allow the government to run up debts as long as a solvency condition is met.

Remember from (4.3) that $\bar{d}(v,t)$ is individual stock of domestic government bonds. From the government's point of view, bonds are liabilities which accumulate according to

$$\dot{d}(t) = [r - n(t)]d(t) - g(t)$$

where $d(t) := \int_{-\infty}^{t} \bar{d}(v,t)l(v,t)\,dv$ is per capita government debt. The solvency condition reads $\lim_{\tau\to\infty} d(\tau)e^{-[r(\tau-t)-N(t,\tau)]} = 0$, and therefore the intertemporal budget constraint of the government is

$$d(t) = \int_{t}^{\infty} g(\tau)e^{-[r(\tau-t)-N(t,\tau)]}\,d\tau. \tag{4.28}$$

All current liabilities must be covered by future primary surpluses, using the net interest rate, $r - n(t)$, for discounting. Holding s_E and t_L constant, the lump sum tax $z(t) \equiv z$ is chosen to exactly meet this constraint. Therefore, perfect tax smoothing is feasible.

All individual assets not invested domestically will affect the economy's current account by

$$\dot{f}(t) = [r - n(t)]f(t) + y(t) - c(t) - i(t) \tag{4.29}$$

where $f(t) := \int_{-\infty}^{t} \bar{f}(v,t)l(v,t)\,dv$ is per capita stock of foreign bonds. The last three terms in the above equation represent the net exports of the economy.

5 Comparative static effects

In order to gain a better insight into the complex comparative static effects of the model introduced in Chapter 4, we take a step back and analyze the two essential parts of the model in isolation, namely

1. endogenous schooling under an exogenously given retirement age $R = R_0$,
2. endogenous retirement with a fixed education time span $e = e_0$.

Although these specifications may at first sight seem identical to the situations studied in Chapters 2 and 3, there are some important differences:

While in the model of Chapter 2 schooling was also the only endogenous variable, retirement was not possible at all. This can be interpreted as $R_0 = \infty$. In the following, we consider an exogenous but *finite* retirement age $R_0 < \infty$. Indeed, for $R_0 \to \infty$, the results presented in Section 5.1 coincide with those of Sections 2.2–2.3.

The model of Chapter 3 took the efficiency profile of workers as exogenously given and only incorporated agents older than biological age 20. By contrast, the sub-model studied in Section 5.2 considers the whole population (and hence also captures non-working young individuals) and replaces $E(u)$ by the human capital formation rule of (4.6) using $e(v) = e_0$. Therefore, it now also features a human capital externality, which was absent in the model of Chapter 3.

Nevertheless, we expect the response of individual decisions to parameter changes in the two sub-models to be similar to the results of Sections 2.2–2.3 and Sections 3.2–3.3. We will try to highlight similarities and differences where applicable. Then, in Section 5.3, we discuss the full model and show that the behavioral response is in fact a combination of the effects found in the two sub-models. The last section of this chapter is devoted to the comparative static effects of the aggregate economy, in particular to changes in per capita human capital, $h(t)$.

5.1 Determinants of schooling under exogenous retirement

We assume that the retirement age is exogenously set by the government and kept constant at $R(v) = R_0 < \infty$. The individual still maximizes $\Lambda(v, t)$ given in (4.1) subject to the asset accumulation equation (4.2). However, he can now only decide on two variables, consumption and schooling time, and takes the retirement age as given.

It can easily be verified that optimal consumption conditioning on education is given by (4.9)–(4.11) where $R(v) = R_0$. The schooling decision is found by maximizing the concentrated utility function (4.12) with respect to e, which yields (4.20). For convenience, we restate the resulting first order condition here in the form

$$F(e, R_0) := \Delta_1(e, r, R_0) + \frac{\vartheta(R_0 - 2e)}{1 - t_L}\Delta_2(e, r, R_0) - e + \frac{s_E}{1 - t_L} = 0. \tag{5.1}$$

Notice that none of the terms on the right-hand side depends on the cohort index v. Hence each cohort chooses the same length of schooling provided that the mortality process (incorporated into the Δ-function) is also cohort-independent.

5.1.1 Policy shocks

First, we want to assess how individuals respond to parametric policy reforms. We consider changing the mandatory retirement age, R_0, the benefit accrual rate, ϑ, the education subsidy, s_E, and the labor income tax, t_L.

Retirement shocks. The effect of an unanticipated change in the retirement age on schooling time can be calculated from

$$\frac{\partial e^*}{\partial R_0} = -\frac{F_R(e^*, R_0)}{F_e(e^*, R_0)}. \tag{5.2}$$

By using Proposition A.2(vii), we find

$$F_R(e, R_0) = \frac{\vartheta}{1-t_L}\Delta_2(e,r,R_0) + \left[1 - \frac{\vartheta(R_0-2e)}{1-t_L}\right]\frac{\partial\Delta_1(e,r,R_0)}{\partial R_0}.$$

Since $\partial\Delta_1(e,r,R_0)/\partial R_0 > 0$, this expression is certainly positive under Assumption 4.1. This assumption also ensures

$$F_e(e, R_0) = \frac{\partial\Delta_1(e,r,R_0)}{\partial e} - \frac{2\vartheta}{1-t_L}\Delta_2(e,r,R_0) + \frac{\vartheta(R_0-2e)}{1-t_L}\frac{\partial\Delta_2(e,r,R_0)}{\partial e} - 1 < 0$$

because $\partial\Delta_1/\partial e < 0$ and $\partial\Delta_2/\partial e \in (0,1)$ by Proposition A.2(vi).

Combining these results shows that schooling length increases if the mandatory retirement age is raised, $\partial e^*/\partial R_0 > 0$. The reason is that the marginal gain from education is higher when time at the labor market is longer.

The response to a higher accrual rate can be calculated as

$$\frac{\partial e^*}{\partial\vartheta} = \xi_0\frac{R_0-2e^*}{1-t_L}\Delta_2(e^*,r,R_0) \tag{5.3}$$

where $\xi_0 := -1/F_e(e^*, R_0) > 0$. Optimal education time rises provided that time in the workforce exceeds time in schooling, i.e. $R_0 - e^* > e^*$, and decreases otherwise. This change in sign stems from the elasticity of pension benefits with respect to schooling, which amounts to

$$\eta_p(e, R_0) = \frac{\partial\bar{p}}{\partial e}\frac{e}{\bar{p}} = \frac{R_0-2e}{R_0-e}.$$

Higher education increases pension benefits as long as $e < R_0/2$. Beyond this point, the gain in future pension benefits through higher wages is more than offset by the loss of pension claims due to less years of wage accumulation. Clearly, a higher replacement rate (through a rise in ϑ) induces higher education levels only if the individual exhibits a gain in pension benefits through delaying labor market entry, i.e. if $\eta_p > 0$. Whereas in the opposite situation where η_p is negative, a higher replacement rate further increases the opportunity cost of education in pension accumulation, tempting the individual to reduce schooling time.

Notice that condition (5.1) does not rule out that $\eta_p < 0$ at the optimal education level e^*. It can be shown, however, that η_p is positive if the education subsidy is sufficiently low. To see this, we use Proposition A.2(iv) to obtain

$$0 = F(e^*, R_0) < (R_0 - 2e^*)\left[1 + \frac{\vartheta}{1-t_L}\Delta_2(e^*,r,R_0)\right] + \frac{s_E}{1-t_L}. \tag{5.4}$$

For $s_E = 0$, this inequality reduces to

$$0 < (R_0 - 2e^*)\left[1 + \frac{\vartheta}{1-t_L}\Delta_2(e^*,r,R_0)\right],$$

which is true if and only if $R_0 - 2e^* > 0$ and hence $\eta_p(e^*, R_0) > 0$. In an economy without education subsidies, a schooling level associated with a negative pension benefit elasticity cannot be optimal.

For a positive education subsidy, $s_E > 0$, inequality (5.4) could hold even if $R_0 - 2e^* \leq 0$. The higher s_E, the higher the probability of observing an optimum with negative elasticity, $\eta_p < 0$. For the rest of the discussion we proceed under the following assumption:

Assumption 5.1 *The effective education subsidy, $s_E/(1-t_L)$, is sufficiently low to ensure $\eta_p(e^*, R_0) > 0$.*

As noted above, this is equivalent to $R_0 - 2e^* > 0$, and thus we find $\partial e^*/\partial\vartheta > 0$.

Other fiscal shocks. As in the model without the possibility to retire, the education subsidy positively affects individual human capital formation,

$$\frac{\partial e^*}{\partial s_E} = \frac{\xi_0}{1-t_L} > 0.$$

However, the effect of a labor tax increase is in general ambiguous,

$$\frac{\partial e^*}{\partial t_L} = \frac{\xi_0}{1-t_L}\left[\frac{s_E}{1-t_L} + \frac{\vartheta(R_0-2e^*)}{1-t_L}\Delta_2(e^*,r,R_0)\right].$$

Although both the effective education subsidy, $s_E/(1-t_L)$, and the pension replacement rate rise, the sign of the elasticity of pension benefits with respect to schooling, η_p, is again crucial. Under Assumption 5.1, a higher labor tax rate postpones labor market entry.

5.1.2 Demographic shocks

Both a fertility shock and an unanticipated change of the disutility function, $D(s)$, do not influence individual behavior. In order to analyze changes in longevity, we again rewrite the mortality function as $m(u,\psi_m)$ where the parameter ψ_m captures old-age mortality. As all Δ-functions depend on the specific mortality process, these are also affected by a change in ψ_m.

Assumption 5.1 ensures

$$\frac{\partial e^*}{\partial \psi_m} = \xi_0\left[\frac{\partial \Delta_1(e^*,r,R_0;\psi_m)}{\partial \psi_m} + \frac{\vartheta(R_0-2e^*)}{1-t_L}\frac{\partial \Delta_2(e^*,r,R_0;\psi_m)}{\partial \psi_m}\right] > 0 \tag{5.5}$$

since $\partial\Delta_i/\partial\psi_m > 0$ by Proposition A.3(iv)–(v). Expected length of retirement increases when old-age mortality decreases. Therefore, receiving a high pension becomes more important, and individuals are encouraged to increase their level of education.

5.2 Determinants of retirement under exogenous schooling

In contrast to the previous section, we know assume that the retirement decision is endogenous and that the government sets a mandatory education time span, $e(v) = e_0$. Therefore, individual human capital accumulation is beyond the agents' control. Individuals maximizes lifetime utility $\Lambda(v,t)$ given in (4.1) subject to the asset accumulation equation (4.2) with respect to consumption and retirement age.

Optimal consumption conditioning on retirement is given by (4.9)–(4.10) where $e(v) = e_0$. The optimal retirement age is found by maximizing the concentrated utility function (4.12) with respect to R, which yields (4.14). For convenience, we restate this condition here in the form

$$G(e_0,R) := U'(\bar{c}^*(v,t,e_0,R))\frac{d\overline{li}(v,t,e_0,R)}{dS} - D(R)e^{(r-\theta)(R-u)} = 0 \tag{5.6}$$

where

$$\frac{d\overline{li}(v,t,e_0,R)}{dS} = (1-t_L)wA_H h(v)^{\phi} e_0\left[1 - \frac{\vartheta(R-e_0)}{1-t_L} + \frac{\vartheta}{1-t_L}\Delta(R,r)\right]. \tag{5.7}$$

5.2.1 Policy shocks

Again, we consider the behavioral response of retirement to several policy reforms. In particular, we regard changes in mandatory length of schooling, e_0, the education subsidy, s_E, the labor income tax, t_L, and the accrual rate of the pension system, ϑ.

Schooling shocks. Whether longer mandatory schooling is able to increase retirement ages depends on various factors. Analytically, the response to a change in e_0 is given by

$$\frac{dR^*}{de_0} = -\frac{G_e(e_0, R^*)}{G_R(e_0, R^*)}. \tag{5.8}$$

To begin with, it can be shown that

$$G_R(e_0, R) = \frac{U''(\bar{c}^*)}{\Delta(u, r')} \frac{\partial \overline{li}}{\partial R} \frac{d\overline{li}}{dS} + U'(\bar{c}^*) \frac{\partial}{\partial R} \left[\frac{d\overline{li}}{dS} \right] - \left[D'(R) + (r - \theta) D(R) \right] e^{(r-\theta)(R-u)} \tag{5.9}$$

is negative at $R = R^*$. As noted before, lifetime income must be upward sloping at the optimal retirement age. Hence $\partial\overline{li}/\partial R$ and $\partial\overline{li}/\partial S$ are both positive but are multiplied by $U'' < 0$, which renders the first term negative. The second term is also negative as can be easily seen from (5.7) and noting Prop. A.1(ii).

The partial derivative of G with respect to e is more difficult to handle. We find

$$G_e(e_0, R) = \frac{U''(\bar{c}^*)}{\Delta(u, r')} \frac{\partial \overline{li}}{\partial e} \frac{d\overline{li}}{dS} + U'(\bar{c}^*) \frac{\partial}{\partial e} \left[\frac{d\overline{li}}{dS} \right] \tag{5.10}$$

and notice that the sign of G_e determines the sign of dR^*/de_0 because of (5.8)–(5.9).

The first term captures the income effect of longer schooling coming from a change in lifetime income. In general, this could be positive or negative, depending on the initial level of e_0. If $\partial\overline{li}/\partial e|_{e=e_0} < 0$, the individual perceives a loss in lifetime income through longer schooling and is therefore encouraged to postpone retirement for compensation. In case the government chooses education such that $\partial\overline{li}/\partial e \geq 0$ at $e = e_0$, lifetime income increases through the education reform and the individual is tempted to retire earlier. Notice that if the education level was endogenous, individuals would always choose e_0 such that lifetime income is maximized, i.e. $\partial\overline{li}/\partial e|_{e=e_0} = 0$, see (4.18). In this case, the first term in (5.10) cancels.

However, education not only affects lifetime income but also the marginal gain of postponing retirement through a higher wage rate. Under Assumption 4.1, this effect is found to be positive,

$$\frac{\partial}{\partial e} \left[\frac{d\overline{li}}{dS} \right] = (1 - t_L) w A_H h(v)^{\phi} \left[1 - \frac{\vartheta(R - 2e_0)}{1 - t_L} + \frac{\vartheta}{1 - t_L} \Delta(R, r) \right] > 0.$$

Thus, we observe an ambiguous income effect and a positive substitution effect when mandatory schooling is raised. The effect is unambiguously positive if $\partial\overline{li}/\partial e|_{e=e_0} \leq 0$ because this renders the first term in (5.10) non-negative. Indeed, this case seems to cover the most relevant situations:

If the education time span was also under the agent's control (as in the combined model), the level is chosen such that $\partial\overline{li}/\partial e|_{e=e_0} = 0$. However, individuals do not consider the positive externality in human capital accumulation. Human capital is not only accumulated for one's own sake but also contributes to the marginal schooling productivity of future generations. Therefore, per capita human capital could be expected to be too low from a social point of view. The socially optimal schooling time would then be located somewhere on the downward sloping part of the lifetime income function, $\partial\overline{li}/\partial e|_{e=e_0} < 0$.

In brief, when the initial level of schooling, e_0, is at least as high as the individual optimal level, an unanticipated increase in mandatory schooling always induces later retirement. However, this does not say anything about whether or not an agent's time in the workforce, $R^* - e_0$, increases.

Other fiscal shocks. If the government leaves e_0 unchanged but instead raises the education subsidy, s_E, lifetime income increases, $\partial\overline{li}/\partial s_E > 0$, while the marginal effect of postponing retirement remains unchanged, $\partial[\frac{d\overline{li}}{dS}]/\partial s_E = 0$. Hence individuals retire earlier due to the income effect,

$$\frac{\partial R^*}{\partial s_E} = \xi_1 \frac{U''(\bar{c}^*)}{\Delta(u, r')} \frac{\partial \overline{li}}{\partial s_E} \frac{d\overline{li}}{dS} < 0$$

where $\xi_1 := -1/G_R(e_0, R^*) > 0$.

A higher labor tax rate, t_L, both lowers lifetime income and the marginal effect of postponing retirement as the additional wage income falls. The first effect tends to increase retirement ages, while the second promotes early retirement. The total effect is therefore ambiguous, the same result as in Section 3.3.

A pension reform through an increase in the accrual rate, ϑ, also yields an ambiguous effect on individual retirement. On the one hand, lifetime income increases and encourages agents to retire earlier. On the other hand, the sign of the substitution effect is not clear,

$$\frac{\partial}{\partial \vartheta}\left[\frac{d\overline{li}}{dS}\right] = (1-t_L)wA_H h(v)^{\phi}\left[-\frac{R^*-2e_0}{1-t_L} + \frac{1}{1-t_L}\Delta(R^*,r)\right] \gtreqless 0.$$

According to Assumption 5.1, the first term in square brackets is negative, while the second term is positive. Under the new regime, the foregone pension payment during an additional working period is higher. Hence there is an incentive to retire earlier. On the other hand, the marginal gain from postponing retirement is also higher, which creates an incentive to postpone retirement.

Observe that the term in square brackets is positive for $R^* = e_0$, declining in R^* and turns negative for high levels of R^*. Thus, the higher the optimal retirement age in the initial state, the more likely will the income effect dominate and cause retirement ages to decrease when the pension system gets more generous.

This result is in line with the findings of Section 3.3. Due to the very general pension benefit formula of Heijdra and Romp (2009b), we could explicitly distinguish between effects coming from a change in the *level* and those coming from a change in the *slope* of the pension benefit formula. These effects were both unambiguous but differed in sign. In the model framework of Chapter 4, however, the pension benefit is determined by Eq. (4.7). Varying ϑ changes both level and slope of the formula, which renders the impact on retirement ambiguous.

5.2.2 Demographic shocks

Concerning the impact of demographic shocks, we again observe that changes in fertility do not affect individual decision making. It remains to specify the effects of productive and biological aging. Notice that, unlike in Chapter 3, we do not study changes in the efficiency profile since this is now determined by e_0.

A decrease in disutility of work at older ages, $\partial D(u,\psi_D)/\partial\psi_D \leq 0$ with strict inequality around $u = R^*$, encourages individuals to work longer,

$$\frac{\partial R^*}{\partial \psi_D} = -\xi_1 \frac{\partial D(R^*,\psi_D)}{\partial \psi_D} e^{(r-\theta)(R^*-u)} > 0. \tag{5.11}$$

This expression is basically identical to Eq. (3.13).

A reduction in old-age mortality, $\partial M(u,\psi_D)/\partial\psi_m \leq 0$, has the same complex effects on retirement as described in the last part of Section 3.2.2. Analytically, we find

$$\frac{\partial R^*}{\partial \psi_m} = \xi_1 \frac{U''(\bar{c}^*)}{\Delta(u,r';\psi_m)}\left[\frac{\partial\overline{li}}{\partial\psi_m} - \bar{c}^*\frac{\partial\Delta(u,r';\psi_m)}{\partial\psi_m}\right]\frac{d\overline{li}}{dS} + \xi_1 U'(\bar{c}^*)\frac{\partial}{\partial\psi_m}\left[\frac{d\overline{li}}{dS}\right]. \tag{5.12}$$

The last term is unambiguously positive,

$$\frac{\partial}{\partial\psi_m}\left[\frac{d\overline{li}}{dS}\right] = wA_H h(v)^{\phi}\vartheta\frac{\partial\Delta(R^*,r;\psi_m)}{\partial\psi_m} > 0.$$

Gains in longevity increase the marginal effect of postponing retirement because (ceteris paribus) benefits are collected over a longer period.

The term in square brackets represents the effect of ψ_m on optimal consumption, $\partial\bar{c}^*/\partial\psi_m$. Eq. (4.11) reveals that is not clear how decreasing old-age mortality affects lifetime income. Under the reasonable assumption that the lump sum tax is time-constant, $z(\tau) \equiv z$, and somewhat lower than the pension benefit, the effect on lifetime income is likely to be positive.

However, we expect this gain to be small because significant changes in the mortality function only happen at relatively high ages, which are heavily discounted. At the same time, the planning horizon (remaining life expectancy) increases. Therefore, a possible gain in lifetime income will probably not be high enough to actually afford a higher level of consumption over the entire lifespan.

Altogether, the response of individual retirement ages to a mortality shock is driven by (i) a change in lifetime income, (ii) a longer planning horizon and (iii) a higher marginal effect from postponing retirement. While the first effect may be moderately positive, thereby decreasing the agent's willingness to work, the other two effects certainly encourage the individual to postpone retirement. Thus, $\partial R^*/\partial \psi_m > 0$ seems to be reasonable in realistic scenarios.

5.3 Determinants of schooling and retirement in the full model

After analyzing the comparative static effects on schooling and retirement in isolation, we now again consider both variables as endogenous. As shown in Chapter 4, the optimal individual decisions regarding education and retirement are determined by the first order conditions

$$F(e, R) = 0,$$
$$G(e, R) = 0,$$

where F is given in (5.1) and G is given in (5.6). Figure 5 shows these two curves in the (e, R)-plane, using the parametrization given in Table 1 of Chapter 6. Remember that $F(e, R) = 0$ by itself defines $e^*(R)$, i.e. the optimal length of schooling conditioning on retirement at age R. Whereas $G(e, R) = 0$ implicitly defines $R^*(e)$, i.e. the optimal retirement age given a schooling period of e years. In the full model, both conditions have to hold simultaneously and the (unique) intersection of $F = 0$ and $G = 0$ determines the optimal solution (e^*, R^*). Notice that only points above the 45° line are feasible since we require $R > e$.

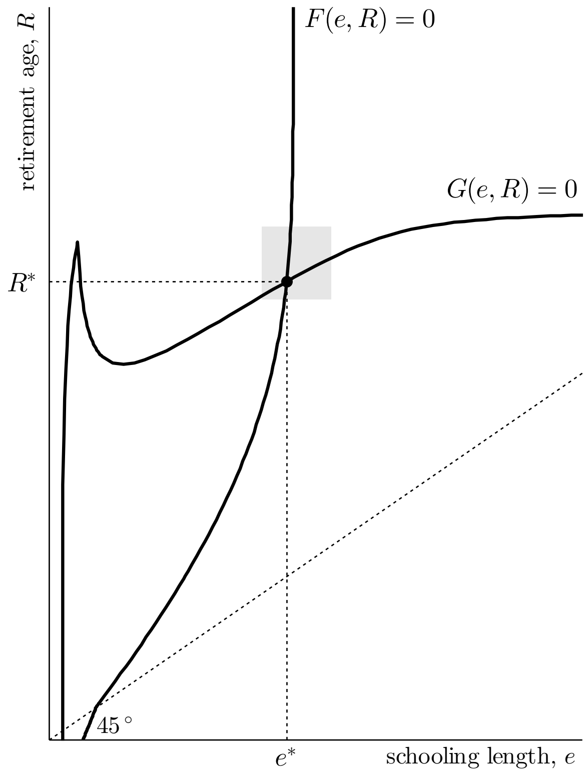

Figure 5: System of first order conditions determining (e^*, R^*)

Next, we observe how the optimum reacts to a parameter variation. Even if a shock affects only one of the first order conditions, the optimal level of *both* choice variables will change because R^* and e^* are interdependent through the equilibrium system.

The implicit function theorem can be used to analytically calculate the response of (e^*, R^*) to a change in an arbitrary parameter ξ. If a neighborhood of (e^*, R^*, ξ) exists where (i) all partial derivatives of (F, G) exist and are continuous, and (ii) the Jacobian matrix of (F, G) (in the following denoted by J) is invertible, i.e. $\det J \neq 0$, we have

$$\begin{pmatrix} \dfrac{\partial e^*}{\partial \xi} \\ \dfrac{\partial R^*}{\partial \xi} \end{pmatrix} = -\begin{pmatrix} F_e & F_R \\ G_e & G_R \end{pmatrix}^{-1} \begin{pmatrix} F_\xi \\ G_\xi \end{pmatrix} = \frac{F_\xi}{\det J} \begin{pmatrix} -G_R \\ G_e \end{pmatrix} + \frac{G_\xi}{\det J} \begin{pmatrix} F_R \\ -F_e \end{pmatrix} \tag{5.13}$$

where all partial derivatives are evaluated at (e^*, R^*, ξ).[36]

The effect of ξ on (e^*, R^*) is actually a combination of the effects found in the two sub-models. This can be seen from rewriting the right-hand side as

$$\begin{pmatrix} \dfrac{\partial e^*}{\partial \xi} \\ \dfrac{\partial R^*}{\partial \xi} \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ \dfrac{\partial R^*}{\partial e_0} \end{pmatrix} \left.\frac{\partial e^*}{\partial \xi}\right|_{R_0 = R^*} + \lambda \begin{pmatrix} \dfrac{\partial e^*}{\partial R_0} \\ 1 \end{pmatrix} \left.\frac{\partial R^*}{\partial \xi}\right|_{e_0 = e^*} \tag{5.14}$$

where $\lambda := (1 - \frac{\partial e^*}{\partial R_0} \cdot \frac{\partial R^*}{\partial e_0})^{-1}$. The first term captures the effect of ξ on optimal schooling, e^*, if the retirement age is held fixed at its initial level, $R_0 = R^*$. Whereas the second term measures the effect of ξ on optimal retirement, R^*, if length of schooling stays constant at $e_0 = e^*$. The multiplicative factor λ accounts for the interaction between e^* and R^*. We can establish the following Lemma:

Lemma 5.2 *Let* $\lambda := (1 - \frac{\partial e^*}{\partial R_0} \cdot \frac{\partial R^*}{\partial e_0})^{-1}$*, then* $\lambda > 1$.

Proof. We show that $0 < \frac{\partial e^*}{\partial R_0} \frac{\partial R^*}{\partial e_0} < 1$. As the previous sections already revealed that both factors are positive, the first inequality is clear. In order to prove the second inequality, we substitute (5.2) and (5.8) to arrive at the equivalent condition $\det J = F_e G_R - F_R G_e > 0$ where $\det J$ is the Jacobian of (F, G). It can be shown that in optimum the Jacobian matrix of (F, G) is related to the Hessian of $\bar{\Lambda}$ (defined in (4.12)) according to[37]

$$J = \begin{pmatrix} F_e & F_R \\ G_e & G_R \end{pmatrix} = \begin{pmatrix} \mathbb{C}_1 \bar{\Lambda}_{ee} & \mathbb{C}_1 \bar{\Lambda}_{eR} \\ \mathbb{C}_2 \bar{\Lambda}_{Re} & \mathbb{C}_2 \bar{\Lambda}_{RR} \end{pmatrix} = \begin{pmatrix} \mathbb{C}_1 & 0 \\ 0 & \mathbb{C}_2 \end{pmatrix} d^2 \bar{\Lambda} \quad \text{where} \quad \mathbb{C}_1, \mathbb{C}_2 > 0. \tag{5.15}$$

If the second-order condition of maximization holds, $d^2\bar{\Lambda}$ is negative definite around the optimum, and therefore $\det J = \mathbb{C}_1 \mathbb{C}_2 \det d^2 \bar{\Lambda} > 0$ according to Sylvester's criterion. □

Noting $\lambda > 1$ in Eq. (5.14) now lets us obtain conditions under which a change in ξ yields an unambiguous effect on individual decisions.

Corollary 5.3 *The sign of either* $\frac{\partial e^*}{\partial \xi}$ *or* $\frac{\partial R^*}{\partial \xi}$ *is unambiguous if (a) in both sub-models the effect of* ξ *is unidirectional on the respective endogenous variable, i.e.*

$$\operatorname{sgn}\left(\left.\frac{\partial e^*}{\partial \xi}\right|_{R_0 = R^*} \right) = \operatorname{sgn}\left(\left.\frac{\partial R^*}{\partial \xi}\right|_{e_0 = e^*} \right), \tag{5.16}$$

or (b) one of the two signs is zero.

[36] This standard result can be found in any sound undergraduate textbook on analysis, for instance Heuser (1989).

[37] The relationships $F(e, R) = \mathbb{C}_1 \bar{\Lambda}_e(e, R)$ with $\mathbb{C}_1^{-1} := U'(\bar{c}^*)(1 - t_L) w A_H h(v)^\phi > 0$ and $G(e, R) = \mathbb{C}_2 \bar{\Lambda}_R(e, R)$ with $\mathbb{C}_2 := \frac{\partial R}{\partial S} > 0$ can be inferred from Section 4.1. Evaluating the partial derivatives of these expressions at $(e, R) = (e^*, R^*)$ yields Eq. (5.15). Notice that the "constants" $\mathbb{C}_1$ and $\mathbb{C}_2$ actually depend on e and R themselves, but their derivatives can be neglected because $\bar{\Lambda}_e = \bar{\Lambda}_R = 0$ holds in optimum.

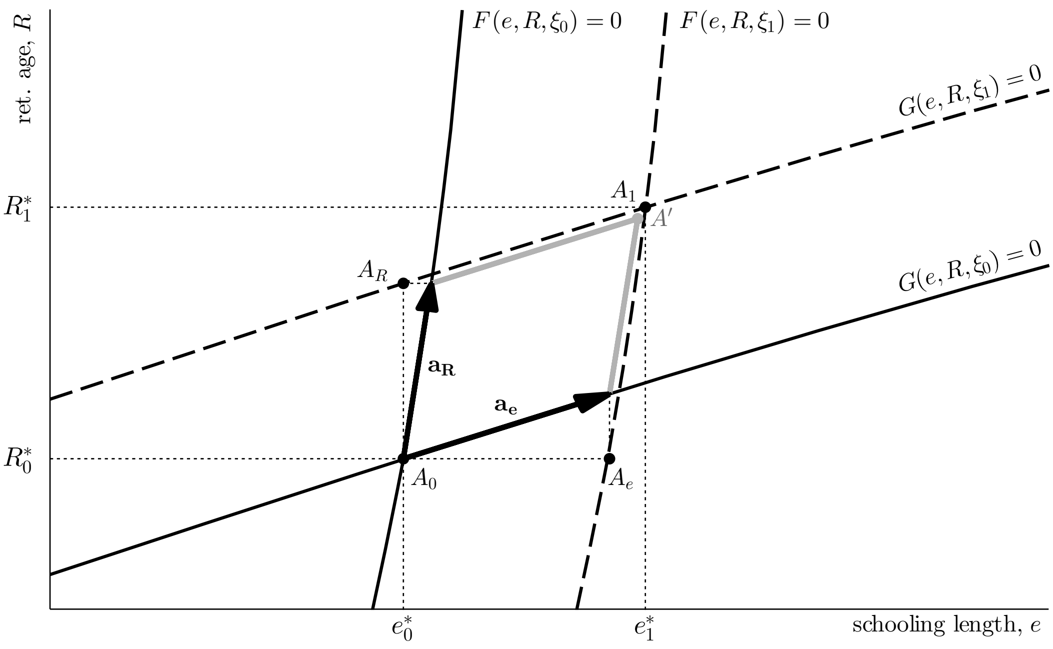

Figure 6: Response to a parameter change with $F_\xi > 0$, $G_\xi > 0$

Whenever this condition holds, the following results can be established:

(i) The signs of $\frac{\partial e^*}{\partial \xi}$ *and* $\frac{\partial R^*}{\partial \xi}$ *are identical. In case (a), the sign of* $\frac{\partial e^*}{\partial \xi}$ *and* $\frac{\partial R^*}{\partial \xi}$ *is equal to the sign in* (5.16)*. Whereas in case (b), the sign of the effects is equal to the sign of the non-zero term appearing in* (5.16)*.*

(ii) The effect of a shock in ξ *on individual decisions is stronger in the full model than in the sub-models,*[38]

$$\left|\frac{\partial e^*}{\partial \xi}\right| > \left|\frac{\partial e^*}{\partial \xi}\Big|_{R_0=R^*}\right| \quad \text{and} \quad \left|\frac{\partial R^*}{\partial \xi}\right| > \left|\frac{\partial R^*}{\partial \xi}\Big|_{e_0=e^*}\right|.$$

To put it less formally, an analytically unambiguous response of individual decisions to a parameter change can certainly be observed if e^* and R^* either both increase or both decrease in the sub-models. As a result, in the full model both variables are then also found to increase or decrease, respectively. In all other cases, the sign of these effects will in general depend on the parametrization.

Figure 6 tries to graphically illustrate Eq. (5.14) and the results of Corollary 5.3. It is a detail of Fig. 5 and shows the area colored in grey. Whereas ξ could in general be any model parameter, the particular situation depicted in the figure represents a decrease in old-age mortality, i.e. an increase in $\xi = \psi_m$.

We assume that for $\xi = \xi_0$ individuals choose the education-retirement-combination (e_0^*, R_0^*). Earlier, we found that under reasonable assumptions both $e^*(R)$ and $R^*(e)$ will increase if the parameter changes to $\xi_1 > \xi_0$. Graphically, this corresponds to an upwards shift of $G = 0$ and a shift to the right of $F = 0$.

In the sub-model of Section 5.1, education is the only endogenous variable and determined by $F(e, R) = 0$ evaluated in $R = R_0^*$. As a result, the optimal decision moves from A_0 to A_e, where $F(e, R, \xi_1) = 0$ and $R = R_0^*$ intersect. Analogously, in the second sub-model presented in Section 5.2, retirement is endogenous while e is exogenously given. The optimal retirement age is found by solving $G(e_0^*, R) = 0$. The new combination chosen by the individuals is A_R.

It should be clear that in the full model, where $F = G = 0$ needs to hold, the new optimal point is A_1. In line with Corollary 5.3(i), A_1 exceeds A_0 in terms of both education and retirement. Moreover, due

[38]To avoid confusion, notice that $\left|\frac{\partial e^*}{\partial \xi}\Big|_{R_0=R^*}\right|$ refers to the absolute value of $\frac{\partial e^*}{\partial \xi}\Big|_{R_0=R^*}$.

to the shape of the curves, A_1 exhibits longer schooling than A_e. It also exhibits a higher retirement age than A_R. This illustrates statement (ii) of Corollary 5.3.

There is also a graphical interpretation of Eq. (5.14). The two vectors plotted in Fig. 6 correspond to

$$\mathbf{a_e} := \left(1, \frac{\partial R^*}{\partial e_0}\right)' \frac{\partial e^*}{\partial \xi}\bigg|_{R_0=R^*} \quad \text{and} \quad \mathbf{a_R} := \left(\frac{\partial e^*}{\partial R_0}, 1\right)' \frac{\partial R^*}{\partial \xi}\bigg|_{e_0=e^*}.$$

Hence (5.14) is equivalent to

$$\left(\frac{\partial e^*}{\partial \xi}, \frac{\partial R^*}{\partial \xi}\right)' = \overrightarrow{A_0A_1} = \lambda(\mathbf{a_e} + \mathbf{a_R}).$$

According to the parallelogram rule, $\mathbf{a_e} + \mathbf{a_R} = \overrightarrow{A_0A'}$. The remaining distance between A' and A_1 results from the scaling factor $\lambda > 1$, which controls for the positive interaction between e and R.

5.3.1 Policy shocks

In Sections 5.1 and 5.2 we found the following comparative static effects:

$$\frac{\partial e^*}{\partial \vartheta}\bigg|_{R_0} > 0, \quad \frac{\partial e^*}{\partial s_E}\bigg|_{R_0} > 0, \quad \frac{\partial e^*}{\partial t_L}\bigg|_{R_0} > 0,$$
$$\frac{\partial R^*}{\partial \vartheta}\bigg|_{e_0} \gtreqless 0, \quad \frac{\partial R^*}{\partial s_E}\bigg|_{e_0} < 0, \quad \frac{\partial R^*}{\partial t_L}\bigg|_{e_0} \gtreqless 0.$$

In all cases, the assumptions of Corollary 5.3 are violated and hence we cannot easily determine the effect on education or retirement in the full model. How individuals respond to policy reforms will in general depend on the particular parametrization.

5.3.2 Demographic shocks

Fortunately, we find that demographic shocks lead to unambiguous responses of schooling times and retirement ages. As a change in the fertility rate, b, does not influence individual decisions in the two sub-models, the same applies in the full model.

A decrease in disutility of work at older ages through a higher ψ_D was found to have a positive effect on retirement for exogenous education time. As the FOC for education is independent from work-specific disutility, $\partial F/\partial \psi_D = 0$ and hence $\frac{\partial e^*}{\partial \psi_D}\big|_{R_0=R^*} = 0$. Therefore, Eq. (5.14) reduces to

$$\frac{\partial e^*}{\partial \psi_D} = \lambda \frac{\partial e^*}{\partial R_0} \frac{\partial R^*}{\partial \psi_D}\bigg|_{e_0=e^*} > 0,$$
$$\frac{\partial R^*}{\partial \psi_D} = \lambda \frac{\partial R^*}{\partial \psi_D}\bigg|_{e_0=e^*} > 0,$$

where $\frac{\partial R^*}{\partial \psi_D}\big|_{e_0=e^*}$ is Eq. (5.11) evaluated at $e_0 = e^*$. Since this term is positive, both education and retirement will increase in the full model. Furthermore, the effect of ψ_D on retirement is the same as if education is exogenous apart from a scaling factor $\lambda > 1$, which captures the interconnection between e^* and R^*. Optimal schooling changes only because of the altered retirement age since the respective FOC does not depend on ψ_D. In particular, the effect on schooling can be written

$$\frac{\partial e^*}{\partial \psi_D} = \frac{\partial e^*}{\partial R_0} \frac{\partial R^*}{\partial \psi_D},$$

which is the effect on retirement scaled by a term measuring the dependence of e^* on the retirement age.

Next, we study shocks in old-age mortality. We discovered effects of population aging on individual behavior in both sub-models. In the case of exogenous retirement, we found a positive effect on education time spans. Whereas in the model with exogenous schooling, the effect was generally ambiguous, but we argued that individuals are likely to postpone retirement. Thus, a mortality shock in the full model increases both education time and retirement age,

$$\frac{\partial e^*}{\partial \psi_m} = \lambda \left.\frac{\partial e^*}{\partial \psi_m}\right|_{R_0=R^*} + \lambda \frac{\partial e^*}{\partial R_0} \left.\frac{\partial R^*}{\partial \psi_m}\right|_{e_0=e^*} > 0,$$
$$\frac{\partial R^*}{\partial \psi_m} = \lambda \left.\frac{\partial R^*}{\partial \psi_m}\right|_{e_0=e^*} + \lambda \frac{\partial R^*}{\partial e_0} \left.\frac{\partial e^*}{\partial \psi_m}\right|_{R_0=R^*} > 0,$$

where $\left.\frac{\partial e^*}{\partial \psi_m}\right|_{R_0=R^*}$ is Eq. (5.5) evaluated at $R_0 = R^*$ and $\left.\frac{\partial R^*}{\partial \psi_m}\right|_{e_0=e^*}$ is Eq. (5.12) evaluated at $e_0 = e^*$.

5.4 Comparative static effects of the aggregate economy

So far, we only considered the rearrangement of individual decisions following parameter changes. Of course, an altered individual behavior also brings about changes in the evolution of the economy as a whole. As human capital formation is the driving force in the economy (remember that the individual education decision depends on the human capital stock of former generations if $\phi > 0$), this section explores comparative static effects of per capita human capital.

In Eq. (4.23), $h(t)$ was defined as the average individual human capital in the working population,

$$h(t) = \int_{-\infty}^{t} I(t-v, R^*(v))\bar{h}(v,t)l(v,t)\, dv \tag{5.17}$$

where $\bar{h}(v,t)$ is given in Eq. (4.6) and evaluated at $e(v) = e^*(v)$. However, this expression turns out to be hardly analytically tractable. First, individual optimal decisions may vary between cohorts because both schooling and retirement depend on the cohort index v. As a result, there may be dates at which several generations enter or leave the labor market, while at some dates the workforce may not change at all. Second, and more importantly, $\bar{h}(v,t)$ itself depends on former realizations of $h(t)$ due to the externality in schooling productivity.

Therefore, we restrict our attention to the long-term effects of shocks on per capita human capital. In a demographic and economic steady state, the age distribution remains constant, i.e. $l(v,t) = l(u)$, and all cohorts take the same decisions, i.e. $e^*(v) \equiv e^*$ and $R^*(v) \equiv R^*$. Hence Eq. (5.17) reduces to

$$\hat{h}^{1-\phi} = A_H \cdot e^* \cdot \hat{x} \tag{5.18}$$

where $\hat{x}$ is the steady state participation rate, defined as

$$\hat{x} := \int_{e^*}^{R^*} l(u)\, du = b \int_{e^*}^{R^*} e^{-[\hat{n}(b,\psi_m)u + M(u,\psi_m)]}\, du, \tag{5.19}$$

and the long-run population growth rate, $\hat{n}(b, \psi_m)$, is given in Eq. (2.21).

For $\phi \in [0, 1)$, the steady state level of per capita human capital, $\hat{h}$, is unique and given by (5.18)–(5.19). Notice that these expressions are basically identical to (2.26)–(2.27), which determined steady state human capital in absence of retirement. As in Chapter 2, we do not consider the case $\phi = 1$ that gives rise to endogenous growth.

The following discussion comprises the long-run effects on $\hat{h}$ in the model of Chapter 4. Different to the analysis of individual responses in the preceding sections, we focus on the full model specification straightaway.

Fertility shock. A birth rate shock affects human capital solely via the altered age distribution. Optimal individual behavior remains unchanged. According to Proposition A.5, the elasticity of per capita human capital with respect to the birth rate equals

$$\frac{\partial \ln \hat{h}^{1-\phi}}{\partial \ln b} = \frac{\partial \ln \hat{x}}{\partial \ln b} = \frac{\bar{u} - \bar{u}_x}{\bar{u}}.$$

The sign of the effect depends on the difference in mean age between the total population and the workforce, $\bar{u} - \bar{u}_x$. The reason is that a higher birth rate increases the mass of young cohorts in the demographic steady state. If the workforce is (on average) younger than the total population, i.e. $\bar{u}_x < \bar{u}$, it will absorb a comparatively larger part of these additional young agents, and the participation rate rises. Whereas in the opposite situation, where $\bar{u}_x > \bar{u}$, relatively more young people are engaged in education rather than in working. Therefore, higher fertility will increase the *dependency ratio* of the population, while the participation rate falls.

Pure schooling shock. Similar to Section 2.2.3, we first consider the partial effect of a schooling shock before turning to the long-term impact of aging shocks. This corresponds to the effect of a change in mandatory schooling, e_0, on steady state human capital in the sub-model with endogenous retirement.

Differentiating (5.18)–(5.19) with respect to e^* yields

$$\frac{\partial \hat{h}^{1-\phi}}{\partial e^*} = A_H \left[\hat{x} + e^* \frac{\partial \hat{x}}{\partial e^*} \right].$$

We replace the optimal schooling level with expression (4.20) and substitute $\hat{x} = l(e^*)\Delta_1(e^*, \hat{n}, R^*)$ to arrive at

$$\frac{\partial \hat{h}^{1-\phi}}{\partial e^*} = A_H l(e^*) \left[\underbrace{\Delta_1(e^*, \hat{n}, R^*) - \Delta_1(e^*, r, R^*)}_{>0} - \frac{\vartheta(R-2e)}{1-t_L}\Delta_2(e^*, r, R^*) - \frac{s_E}{1-t_L} \right].$$

In absence of social transfers, i.e. $s_E = 0$ and $\vartheta = 0$, human capital increases because the underbraced term is positive for $r > \hat{n}$ by Proposition A.2(v). However, if either education subsidies or pensions are paid, the impact of a schooling shock could eventually turn negative.

This is very similar to the effect found in Chapter 2, see Eq. (2.28) and the discussion thereafter. Indeed, both expressions coincide in the limiting case $R^* = \infty$, while we face an additional negative term if the optimal retirement age is finite. In this case, Assumption 2.2 is no longer a sufficient condition for the effect on human capital to be positive.[39]

However, it is still reasonable to assume $\partial \hat{h}/\partial e^* > 0$ in order to avoid "over-educating" of the model population. Indeed, if we consider per capita human capital as a function of e,

$$\hat{h}(e)^{1-\phi} = A_H e \int_e^{R^*} l(u)\, du,$$

we find an inverted U-shape with $\hat{h} = 0$ for $e = 0$ and $e = R^*$, and a unique interior maximum in between, $e_{\max} \in [0, R^*)$. Therefore, any human capital level in $(0, \hat{h}(e_{max}))$ can be achieved by two different choices of schooling, one lying at the upward sloping branch of $\hat{h}$ and one lying at the downward sloping part.

From this it should be clear that the response of per capita human capital to a schooling shock is negative if and only if the individually optimal length of schooling, e^*, happens to lie beyond $e_{\max}$. However, the same level of human capital could also be achieved with a lower education time, $e' < e_{\max}$, that exhibits a positive slope, i.e. $\partial \hat{h}/\partial e' > 0$. Thus, per capita human capital could be raised by decreasing e^*; individuals are over-investing in education. Such an economically unfavorable situation may arise if the welfare state is too generous, i.e. if the effective schooling subsidy, $s_E/(1-t_L)$, or the pension replacement rate is too high.

[39] While it is no longer sufficient, it remains necessary since

$$\Delta_1(e^*, \hat{n}, R^*) - \Delta_1(e^*, r, R^*) - \tfrac{\vartheta(R-2e)}{1-t_L}\Delta_2(e^*, r, R^*) =$$
$$\Delta(e^*, \hat{n}) - \Delta(e^*, r) - \Big[\underbrace{\Delta_2(e^*, \hat{n}, R^*) - \Delta_2(e^*, r, R^*)}_{>0 \text{ (Prop. A.2(v))}} + \underbrace{\tfrac{\vartheta(R-2e)}{1-t_L}}_{>0 \text{ (Ass. 5.1)}} \Delta_2(e^*, r, R^*) \Big] < \Delta(e^*, \hat{n}) - \Delta(e^*, r).$$

We preclude educationally inefficient allocations in the long run and proceed under the more restrictive conditions of Assumption 5.4, which ensure $\partial\hat{h}/\partial e^* > 0$.

Assumption 5.4 *The steady state net interest rate, $r - \hat{n}$, is sufficiently high and the social transfer system is designed in a way that ensures*

$$\Delta_1(e^*, \hat{n}, R^*) - \Delta_1(e^*, r, R^*) > \frac{\vartheta(R-2e)}{1-t_L}\Delta_2(e^*, r, R^*) + \frac{s_E}{1-t_L}.$$

Pure retirement shock. If we instead consider a shock solely affecting retirement ages, while education time spans remain unchanged, human capital rises due to a higher participation rate,

$$\frac{\partial\hat{h}^{1-\phi}}{\partial R^*} = A_H e^* \frac{\partial\hat{x}}{\partial R^*} = A_H e^* l(R^*) > 0.$$

This corresponds to the effect of raising the external retirement age in the sub-model where only schooling is endogenous.

Productive aging. Armed with these results, we can consider changes in working disutility of older workers. Under Assumption 5.4, the effect on per capita human capital is positive,

$$\frac{\partial\hat{h}^{1-\phi}}{\partial\psi_D} = \frac{\partial\hat{h}^{1-\phi}}{\partial e^*}\frac{\partial e^*}{\partial\psi_D} + \frac{\partial\hat{h}^{1-\phi}}{\partial R^*}\frac{\partial R^*}{\partial\psi_D} > 0,$$

since both schooling time and retirement age increase when disutility falls (see the previous section).

Biological aging. The preceding section showed that lower old-age mortality encourages individuals to prolong education time and postpone retirement. While these two effects increase per capita human capital, there is a third–potentially ambiguous–effect stemming from the altered age distribution. According to Proposition A.6, the partial effect of ψ_m on the share of workers in the population is

$$\frac{\partial\hat{x}}{\partial\psi_m} = \mathbb{C}\left[\frac{\int_{e^*}^{R^*} M_\psi(u,\psi_m)\lambda(u)\,du}{\int_0^\infty M_\psi(u,\psi_m)l(u)\,du} - \frac{\bar{u}_x}{\bar{u}}\right] \tag{5.20}$$

where $\mathbb{C}$ is a positive constant and $\lambda(u) := l(u)/\hat{x}$ is the cohort's share among the workforce.

The expression in square brackets essentially measures at which ages the mortality gains occur. The first term compares the average marginal effect of ψ_m between working population and total population, while the second compares the respective mean ages. As the distribution of marginal longevity gains in the population is extremely right skewed (significant gains only happen at old ages), the first term will be comparatively small for moderate levels of R^*.

For the special case $e^* = 0$, we already showed in Proposition A.6(ii) that $\partial\hat{x}/\partial\psi_m \leq 0$ regardless of the individual retirement age. Unfortunately, such a general statement cannot be established for $e^* > 0$.[40] Compared to the situation without schooling, the first term in (5.20) is higher. As the effect of ψ_m on mortality is increasing with age, the average marginal effect of aging is higher when integrating over an older part of the population. However, the same argument holds for the mean age, and thus both terms in (5.20) are increasing in e^*.

[40] However, Proposition A.6(iv) also applies in this case. If $M_\psi(u,\psi_m)$ is linear in u, the first term reduces to $\bar{u}_x/\bar{u}$ and $\partial\hat{x}/\partial\psi_m = 0$. Therefore, the remaining discussion only considers $\partial^2 m(u,\psi_m)/\partial u\partial\psi_m < 0$.

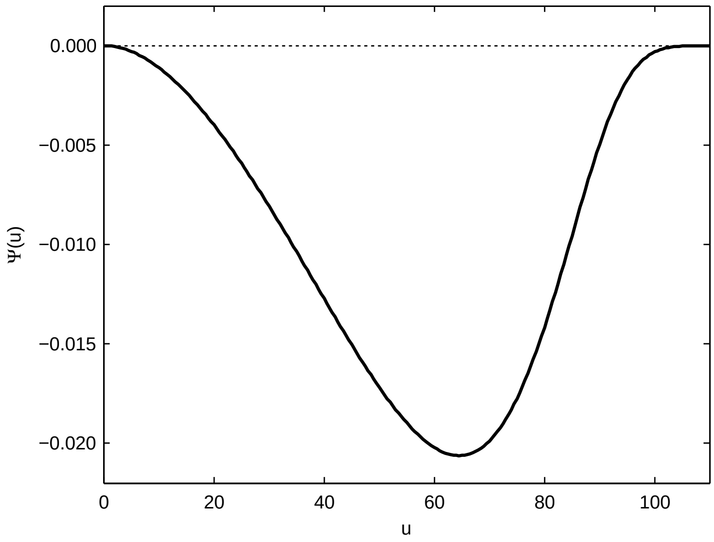

Figure 7: Shape of the Ψ-function

As shown in the proof of Proposition A.6(iii), the effect of a mortality shock depends on the sign of

$$\frac{\partial \hat{x}}{\partial \psi_m} = \Psi(R) - \Psi(e)$$

where Ψ is defined in Eq. (A.13) in the Appendix. As $\Psi(0) = 0$ and $\Psi(R) < 0$, we find $\partial \hat{x}/\partial \psi_m < 0$ for $e = 0$. By means of continuity, the effect on the participation rate will remain negative as long as e^* is "sufficiently" small compared to R^*. Admittedly, this is a pure analytical finding without much significance for practical applications.

Therefore, we want to illustrate the resulting effects on the participation rate for various combinations of (e, R) using the model parametrization of Heijdra and Romp (2009a). As argued above, the sign of $\partial \hat{x}/\partial \psi_m$ is determined by the Ψ-function, which is shown in Figure 7. Its minimum lies at $u = 64.51$, and therefore the participation rate will decrease in response to a mortality shock for any reasonable levels of e and R. Hence assuming $\partial \hat{x}/\partial \psi_m < 0$ in realistic scenarios seems defensible.

Putting things together, the effect of a mortality shock on per capita human capital is unclear,

$$\frac{\partial \hat{h}^{1-\phi}}{\partial \psi_m} = \underbrace{\frac{\partial \hat{h}^{1-\phi}}{\partial e^*} \frac{\partial e^*}{\partial \psi_m}}_{>0} + \underbrace{\frac{\partial \hat{h}^{1-\phi}}{\partial R^*} \frac{\partial R^*}{\partial \psi_m}}_{>0} + A_H e^* \underbrace{\frac{\partial \hat{x}}{\partial \psi_m}}_{<0} \gtreqless 0. \tag{5.21}$$

Although individuals intend to increase education and postpone retirement, the resulting demographic change is likely to decrease the population weight of working generations, thereby rendering the effect on human capital ambiguous.

Notice that the respective effect is lower if we hold either e or R fixed, like in the sub-models of Sections 5.1 and 5.2. While the size of the negative demographic effect in Eq. (5.21) is the same in any specification, one of the positive terms is missing. Furthermore, Corollary 5.3(ii) indicates that the individual response in the remaining endogenous variable is weaker than in the case of two decision variables. As a result,

$$\left.\frac{\partial \hat{h}^{1-\phi}}{\partial \psi_m}\right|_{e_0=e^*} < \frac{\partial \hat{h}^{1-\phi}}{\partial \psi_m} \quad \text{and} \quad \left.\frac{\partial \hat{h}^{1-\phi}}{\partial \psi_m}\right|_{R_0=R^*} < \frac{\partial \hat{h}^{1-\phi}}{\partial \psi_m}. \tag{5.22}$$

6 Transitional dynamics

In the previous chapter, our focus was to understand the long-run effects of various shocks and–where analytically possible–to determine the sign of individual and aggregate shock responses. We considered fertility shocks, changes in the age profiles of aging and disutility as well as policy reforms. Unfortunately, we found a lot of ambiguous results, most prominently for the effect of aging on human capital.

In a next step, we determine the whole dynamic transition path of the economy. Therefore, we computationally implemented the model using the mathematical software MATLAB®. All files are available from the author upon request.

6.1 Parametrization

The base parametrization of our model is almost identical to Heijdra and Romp (2009a) in order to allow for comparability of results. The exact parameter values are reported in Table 1. Two parameters were modified to obtain more sensible results:[41]

In our framework, $\theta = 0.03$ would cause a negative per capita asset stock in the steady state. The reason is that life-cycle savings are negative throughout the first decades, turn positive during working years and peak in retirement until they fall back to zero as time approaches infinity.[42] On the aggregate, for $\theta = 0.03$, old generations' savings are not high enough to compensate for younger generations' debts. A lower level of θ induces individuals to act more foresightedly and maintain a higher stock of assets at any point in time. Hence young individuals have lower debt levels, and older individuals have higher savings. A rather small reduction to $\theta = 0.025$ is already sufficient to render the per capita asset stock positive. Another deviation results from the initial level of government debt, d_0. While Heijdra and Romp (2009a) assume a small government surplus, we decided to start at a debt level of zero. A lump sum tax rate of $z = 2.63$ is required to cover education and pension expenditures out of tax earnings, see column (1) in Table 2.

Our model also features variables which are not part of Heijdra and Romp (2009a). The accrual rate of the pension system is set to $\vartheta = 0.0181$, which is in line with the Austrian or Dutch pension scheme, see OECD (2011, p.111). For disutility of work, D, we assume the same functional form that Heijdra and Reijnders (2012) use for human capital depreciation, $D(u) = D_0 + D_1/(\bar{u} - u)$. Disutility rises slightly for lower ages but increases rapidly after age 60 and reaches a vertical asymptote at $\bar{u} = 72$, see Figure 11.

The parameters D_0 and ϑ were calibrated to replicate the steady state education and retirement choices $e^* = 22.3$ and $R^* = 62.5$. As a result, individuals devote 40.2 years of their expected lifetime of 76.61 years to working. The net replacement rate of the pension scheme is 85.6%, while the education subsidy amounts to 27.1% of the net wage. Due to the pyramid shape of the age distribution–see Figure 8, it happens that government spending for education and pensions are almost identical. Column (1) in Table 2 provides a complete overview of all important initial steady state variables.

Demography		Individuals		Firms		Macroeconomy		Government	
μ_0	$5.733 \cdot 10^{-4}$	θ	*0.025*	A_Y	1	r	0.055	d_0	*0*
μ_1	$3.118 \cdot 10^{-5}$	σ	1	α	0.3	A_H	1	t_L	0.15
μ_2	0.095	D_0	0.263926	δ	0.07	ϕ	0.3	s_E	5.1457
b	0.0212	D_1	0.648					ϑ	0.0181
		$\bar{u}$	72						

Table 1: Parameter values

[41] You may also notice that Chapter 5.2 in Heijdra and Romp (2009a) lists a different value for the education subsidy, namely $s_E = 4.915$. After we could not reproduce the steady state values of the paper with the parametrization mentioned there, Ward Romp kindly provided his original MATLAB® files. These revealed that actually $s_E = 5.1457$ was used in the computation.

[42] In particular, individual assets are negative during schooling and begin to rise afterwards. Everything else unchanged, $\theta = 0.03$ implies that individual assets remain negative until age 43.3. For $\theta = 0.025$, the asset stock already turns positive at age 36.7.

		(1) ISS	(2) Baby bust			(3) Longevity			(4) Productivity			(5) Education			(6) Pension		
			EDU	RET	FULL	EDU	RET	FULL	EDU	RET	FULL	EDU	RET	FULL	EDU	RET	FULL
Individual plans:																	
e	length of schooling	22.30	22.30	22.30	22.29	22.46	22.30	22.53	22.30	22.30	22.34	23.36	22.30	23.40	22.26	22.30	22.33
R	retirement age	62.50	62.50	62.41	62.41	62.50	63.24	63.43	62.50	63.05	63.09	62.50	62.07	63.07	62.50	63.42	63.45
$R–e$	years at work	40.20	40.20	40.11	40.11	40.04	40.94	40.90	40.20	40.75	40.75	39.14	39.77	39.67	40.24	41.12	41.12
U	lifetime utility	92.61	93.70	93.66	93.65	95.16	95.39	95.44	92.67	93.16	93.16	93.89	94.25	94.19	93.57	94.46	94.46
Effective earnings/taxes:																	
$\bar{s}_E$	eff. education subsidy	14.28	14.59	14.58	14.58	14.07	14.14	14.17	14.28	14.34	14.34	17.22	17.08	17.29	14.28	14.37	14.38
$\bar{w}$	eff. wage rate	61.89	63.23	63.18	63.16	61.41	61.27	62.04	61.89	62.13	62.28	65.13	61.70	65.54	61.78	62.29	62.40
$\bar{p}$	eff. pension benefits	45.03	46.01	45.87	45.86	44.51	45.40	45.93	45.03	45.83	45.94	46.14	44.41	47.06	40.49	41.73	41.80
$\bar{z}$	eff. lump sum tax	7.30	7.26	7.27	7.26	7.25	7.25	7.29	7.30	7.15	7.16	8.83	8.41	8.79	6.71	6.40	6.41
Per capita plans:																	
h	human capital	28.24	30.33	30.26	30.25	26.89	27.31	27.53	28.24	28.61	28.66	28.68	27.95	29.10	28.22	28.86	28.89
x	participation rate, %	46.48	48.87	48.78	48.79	44.60	45.40	45.20	46.48	46.91	46.88	44.85	46.15	45.23	46.54	47.20	47.17
c	consumption	31.50	37.63	37.57	37.56	32.64	32.91	32.96	31.50	31.98	31.98	32.73	33.06	33.06	32.42	33.35	33.34
li	lifetime income	534.67	541.67	540.62	540.45	542.67	547.20	552.63	534.67	542.03	542.97	539.64	523.29	546.72	529.66	543.69	544.36
a	assets	65.64	111.25	111.27	111.42	104.65	105.37	101.08	65.64	67.51	66.61	84.21	106.83	83.34	88.26	91.84	91.14
Firms:																	
k	physical capital	98.63	105.94	105.67	105.64	93.91	95.37	96.15	98.63	99.93	100.10	100.16	97.61	101.64	98.57	100.80	100.92
i	investment	8.23	7.96	7.94	7.94	7.95	8.07	8.13	8.23	8.33	8.35	8.35	8.14	8.48	8.22	8.41	8.42
y	output	41.10	44.14	44.03	44.02	39.13	39.74	40.06	41.10	41.64	41.71	41.73	40.67	42.35	41.07	42.00	42.05
Government:																	
z	lump sum tax rate	2.63	2.56	2.56	2.56	2.65	2.64	2.65	2.63	2.57	2.57	3.17	3.04	3.14	2.42	2.29	2.29
g/y	primary deficit, % of GDP	0.00	2.67	2.99	3.00	2.62	2.85	2.84	0.00	0.07	0.04	0.04	0.36	–0.12	0.00	0.30	0.28
d/y	public debt, % of GDP	0.00	–53.53	–54.40	–54.47	–64.76	–51.88	–51.64	0.00	–1.57	–1.07	–0.99	–8.77	2.92	0.01	–7.22	–6.81
f/y	net foreign assets, % of GDP	–80.28	65.56	67.12	67.60	92.21	77.05	63.94	–80.28	–76.29	–79.22	–37.22	31.44	–46.12	–25.11	–14.12	–16.43
	education spending, % of GDP	14.10	11.00	11.02	11.02	14.50	14.26	14.30	14.10	13.97	13.98	17.41	17.05	17.26	14.09	13.89	13.90
	pension spending, % of GDP	14.17	18.62	18.70	18.70	17.14	16.57	16.47	14.17	13.76	13.74	14.30	14.49	13.88	12.75	12.14	12.12
Pension scheme:																	
RR	replacement rate	72.8%	72.8%	72.6%	72.6%	72.5%	74.1%	74.0%	72.8%	73.8%	73.8%	70.8%	72.0%	71.8%	65.5%	67.0%	67.0%
NRR	net replacement rate	85.6%	85.6%	85.4%	85.4%	85.3%	87.2%	87.1%	85.6%	86.8%	86.8%	83.3%	84.7%	84.5%	77.1%	78.8%	78.8%
Demography:																	
n	population growth rate	0.0134		0.0051			0.0146			0.0134			0.0134			0.0134	
E_0	life expectancy at birth	76.61		76.61			82.29			76.61			76.61			76.61	
E_{60}	life expectancy at age 60	21.21		21.21			26.01			21.21			21.21			21.21	

Table 2: Initial steady state and long-run effects

6.2 Individual and aggregate response to demographic and fiscal shocks

The economy is assumed to reside in its initial steady state until $t = 0$. At this time, a discrete parameter change happens which pushes the economy out of the equilibrium. The economy is then driven by the dynamics specified in Chapter 4 and will eventually converge to a new equilibrium in the long run.

To numerically compute the transition path of the economy, we apply the following iterative procedure:

1. Solve for the initial steady state.
2. Compute the population growth path $n(t)$ for $t \geq 0$ using Eq. (2.3).
3. Compute the equilibrium path until convergence to a new steady state.
   (i) Obtain the individual decisions taking the lump sum tax z as given.
   (ii) Use these values in (4.23) to solve for the trajectory of human capital $h(t)$ for $t \geq 0$.
   (iii) Compute the lump sum tax z which balances the government budget (4.28), and use this as new value of z in (i). Iterate until convergence of z.

Determining the initial steady state is straightforward. For sake of completeness we report the algebraic system determining the steady state values in Section A.4.1 of the Appendix. Calculating the time paths of $n(t)$ and $h(t)$, however, requires to solve a Volterra integral equation. A numerically efficient way to obtain the population growth rate was presented by Romp (2007) and is discussed in Section A.4.2 of the Appendix.

Although the adjustment path of human capital, $h(t)$, is obtained in a similar way, some further considerations are necessary: First, we need to determine how pre-shock generations ($v < 0$) adjust their education and retirement behavior. Per definition, agents with $v \leq -R_0$ are already retired and cannot re-enter the workforce. Working generations, i.e. $v \in (-R_0, -e_0]$, will alter their retirement decision, while generations who are still in education, i.e. $v \in (-e_0, 0)$, will adjust both length of schooling and retirement age. In a next step, we obtain the optimal decision of post-shock generations. This can be tricky because for $v > 0$, $e(v)$ and $R(v)$ in general depend on $h(v)$–the variable we intend to compute. However, if we postulate log-utility, i.e. $\sigma = 1$, it can be shown that $h(v)$ cancels out of the first order condition for retirement, Eq. (4.14). Hence all generations born after the shock choose the same levels of education and retirement, which simplifies the problem significantly. Once we have determined all these values, the time path of $h(t)$ can be computed from Eq. (4.23) using numerical integration techniques.

However, the solution so obtained will in general not satisfy the government budget constraint (4.28). Altered individual behavior affects earnings and expenditures of the government and requires an adjustment of the lump sum tax, z, in order to keep the budget in balance. Individual decisions in turn depend on the current level of z. Hence we implement an iterative procedure to find levels of $e(v)$ and $R(v)$ which are (i) individually optimal and (ii) consistent with a balanced government budget.

Further technical details on how to computationally solve for the equilibrium path of the economy can be found in Section A.4.3 in the Appendix.

In the following, we discuss the adjustments in individual and aggregate variables following demographic and fiscal shocks. In particular, we want to highlight the difference in outcomes when controlling for both education and retirement ("full model") compared to the case where only one of these variables is endogenous. For convenience, we will refer to the sub-models presented in Section 5.1 and Section 5.2 as "education model" and "retirement model", respectively.

6.2.1 Baby bust

We begin our analysis with the effects of a baby bust. At $t = 0$, the birth rate shrinks from $b_0 = 0.0212$ to $b_1 = 0.0159$, which is a reduction of 25%. The dynamics of the population growth rate are depicted in Fig. 2a. It features a downward jump at impact and then non-monotonically converges to a new equilibrium growth rate of $n_1 = 0.0051$. Although individual decisions are not directly affected by the fertility shock (see the discussion in the previous chapters), Figure 9b reveals that the retirement age is slightly falling in the full model and in the retirement model. The reason is that the retirement decision depends on lifetime income, which increases as the equilibrium lump sum tax rate lowers from $z_0 = 2.63$ to $z_1 = 2.56$, see column (2) in Table 2. In the full model, the tax cut also implies a decrease in education time spans. However, the adjustments in e and R are quantitatively so small that the time paths of aggregate variables differ only unremarkably between the three model specifications.

As depicted in panel (d) of Fig. 9, the participation rate increases until $t = 22.3$. Due to the baby bust, the mass of newborns is comparatively smaller after $t = 0$. Hence the share of students gradually decreases and $x(t)$ rises. In contrast to Fig. 5 in Heijdra and Romp (2009a), the participation rate does not stabilize in the following but declines as quickly as it has increased before. The reason is that after $t = 22.3$, post-shock generations start to enter the workforce, while the (comparatively larger) pre-shock generations are retiring. This steadily increases the old-age dependency ratio until all post-shock generations have retired. After $t = 62.5$, the dependency ratio falls again because the retired pre-shock cohorts gradually die out. In the long run, the participation rate converges to a level which exceeds the initial steady state level by 2.3 to 2.4 percentage points, see column (2) in Table 2. This is due to the altered long-run age distribution of the economy. Fig. 8a shows that the share of young generations in the new equilibrium (dashed line) is significantly lower than in the initial steady state (solid line). This demographic effect allows for a higher participation rate, even though education and retirement choices remain *de facto* unchanged.

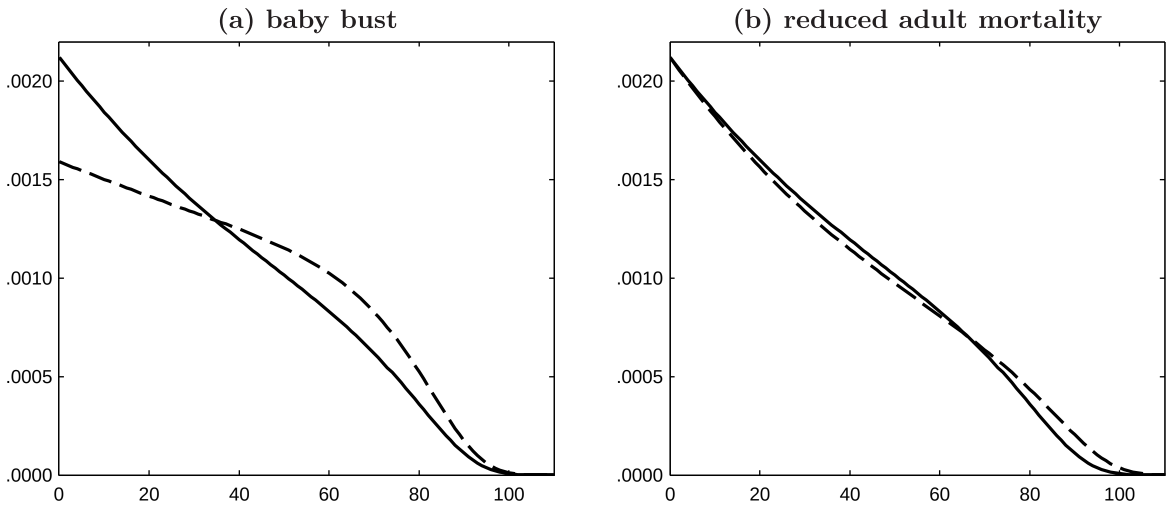

Figure 8: Long-term age distribution, $l(u)$

Except for a scaling factor, the time path of per capita human capital, $h(t)$, shown in panel (e), is almost identical to $x(t)$ because education levels are hardly affected by the fertility shock. Comparing lifetime utility, $\Gamma(v)$, between the cohorts reveals the same qualitative shape since lifetime consumption of cohort v is proportional to $h(v)$. It also turns out that post-shock cohorts ($v \geq 0$) are clearly better of in terms of utility than pre-shock cohorts.

Last but not least, the government maintains a long-run surplus of almost 55% of GDP. The reason is that, in the short run, government spending on education decreases because of the lower share of young individuals. During the first decades, government tax revenues exceed expenditures, and thereby a surplus is generated. In the long run, however, the share of retirees rises significantly–see Fig. 8a. The rise in pension spending exceeds the decline in education spending, generating a long-run primary deficit of 3% of GDP, which stabilizes government debt at -55% of GDP.

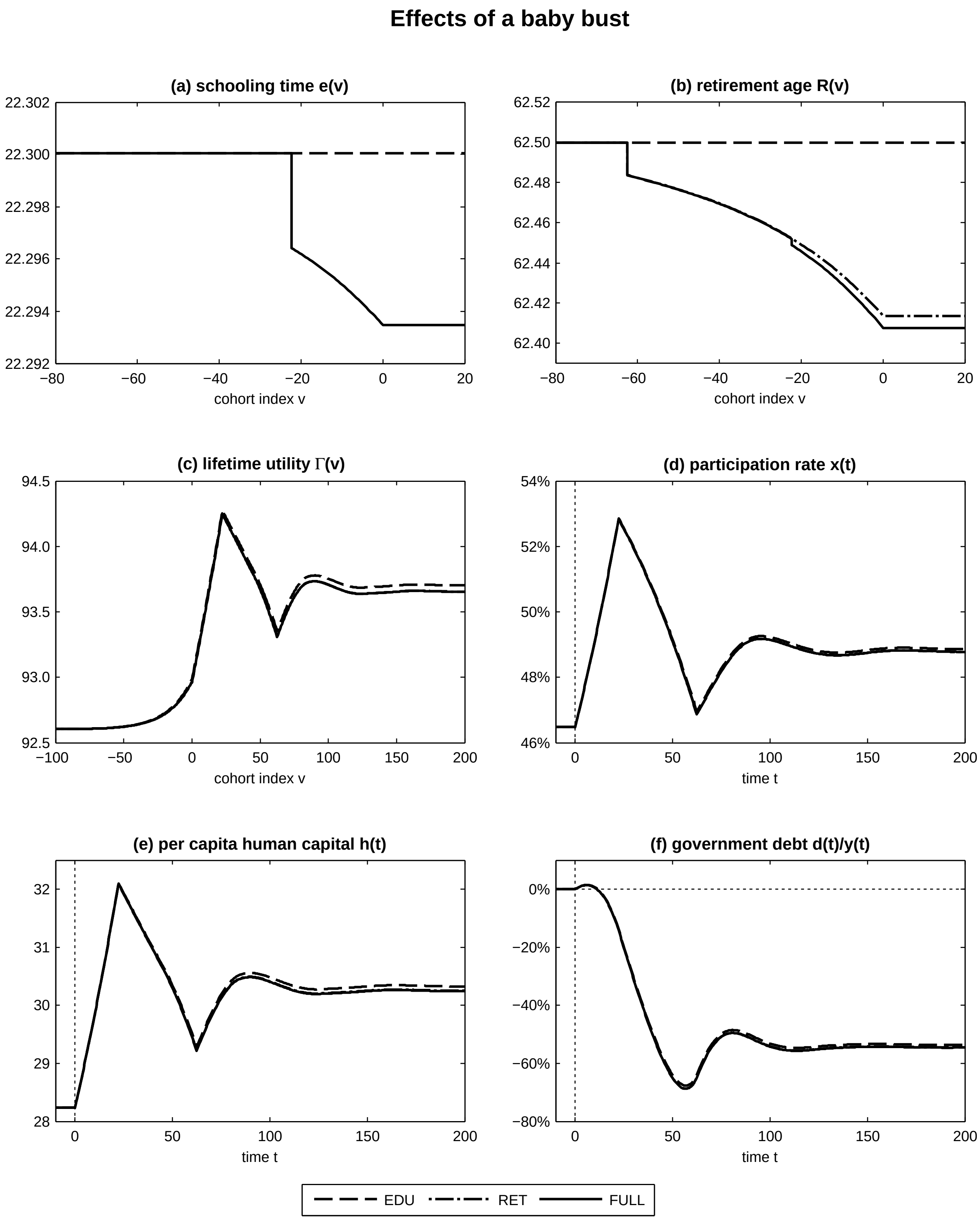

Figure 9: Effects of a baby bust

6.2.2 Biological aging

Having observed the effect of fertility shocks, we now turn to the impact of population aging. Particularly, we study a 50% decrease in μ_1 (from $\mu_{10} = 3.118 \cdot 10^{-5}$ to $\mu_{11} = 1.559 \cdot 10^{-5}$) combined with a moderate increase in μ_2 of 1.8% (from $\mu_{20} = 0.095$ to $\mu_{21} = 0.09671$). As a result, life expectancy at birth rises by 5.68 years to 82.29 years, and remaining expected lifetime at age 60 rises by 4.8 years. Hence the parameter change indeed reduced mainly *adult* mortality. This can also be seen from Figure 8b. In the long run, the age distribution slowly shifts from the initial steady state (represented by the solid line) to a new demographic steady state (represented by the dashed line). The share of cohorts between age 20 and age 60 moderately decreases, while the population features a substantially higher share of over-70-year-olds in the new equilibrium.

In line with Heijdra and Romp (2009a), we only study the effect of embodied mortality shocks, i.e. the mortality process of pre-shock generations is unaffected.[43] During the transition, the population growth rate follows the pattern shown in Fig. 2b and increases slightly from $n_0 = 0.0134$ to $n_1 = 0.0146$ in the long run. In discussing the individual and aggregate effects of population aging, it is recommendable to study the three model specifications in separation. While the transitional dynamics can be inferred from Figure 10, the long-run effects are reported in column (3) of Table 2.

Education model. All paths corresponding to the sub-model with endogenous schooling and exogenous retirement are indicated by a dashed line in Fig. 10. The retirement age is held fixed at $R_0 = 60.5$ in order to replicate the same initial steady state as in the full model.

Due to the embodied character of the shock, the education decision of pre-shock generations remains *de facto* unchanged. The general equilibrium effect coming from a higher z only insignificantly increases optimal schooling. Although generations born after the shock face a significantly higher life expectancy, there is only a moderate increase in the education time span from $e_0 = 22.3$ to $e_1 = 22.46$.

The participation rate adjusts accordingly: It remains almost unchanged at the beginning and then drops sharply as no individuals enter the labor market during $t = 22.3$ and $t = 22.46$.[44] Afterwards, there is again a constant inflow of generations into the labor market and the participation rate rises. However, it falls again once the post-shock generations start to retire. Due to their lower mortality rate, retiring generations after $t = 60.5$ (completely made up by post-shock individuals) are comparatively larger than before (where only individuals born before the shock retired). Therefore, the old-age dependency ratio first rises steadily but then stabilizes as pre-shock generations gradually die out. In the long run, $x(t)$ falls from 46.48% to 44.6%. Per capita human capital $h(t)$ qualitatively follows the same path and decreases from 28.24 to 26.89 in the long run.

Retirement model. Next, we study the effects of the same mortality shock in the framework where individuals adjust retirement ages while length of schooling stays fixed at $e_0 = 22.3$. In Fig. 10, the time paths corresponding to this specification are represented by dash-dotted lines.

As old-age life expectancy rises, after-work income becomes more important and individuals are willing to postpone retirement in order to accumulate higher pension claims. Individuals born after the shock retire 9 months (i.e. 0.75 years) later than pre-shock generations.

The participation rate mimics the dynamics of the previous model, except for the timing of the jump.[45] Since the age of labor market entry is the same for both pre- and post-shock generations, there is now no jump at $t = e_0$. However, retirement ages differ and nobody retires between $t = 62.5$ and $t = 63.24$. During this period the participation rate increases rapidly. Thereafter, the first post-shock cohorts retire and the participation rate falls again (the same reasoning applies as above). However, the participation

[43] Although this might not be a realistic assumption, Section A.4.2 shows that this convention simplifies the practical solution of the model significantly.

[44] Notice that this abrupt drop-off is due to the discrete parameter change. A continuous change in parameters would result in much smoother trajectories.

[45] Although we sloppily refer to a "jump" in $x(t)$, the variable actually evolves continuously with time. What we mean is a sharp increase (or decrease) of the variable, not a discrete jump in the sense of a discontinuity. The same holds for $h(t)$.

rate converges to a 0.8 percentage points higher long-run level than in the model where schooling is the endogenous variable.

The reason is simple: In the education model, years spent in the workforce, $R - e$, decrease since individuals increase education while the retirement age remains unchanged. Whereas in the retirement model, R increases and e stays constant, implying longer working periods. Hence, on the aggregate, a higher share of the population is working compared to the education model. Nevertheless, the long-run participation rate is still below its initial steady state level. This is indeed due to the change in the age distribution which follows the mortality shock.

The same reasoning applies to per capita human capital. In the long run, $h(t)$ converges to a higher level than in the model with endogenous schooling.

Full model. If individuals can adjust education *and* retirement, rising longevity will increase both variables. Due to the positive feedback effect between e and R (see Corollary 5.3), education and retirement levels of post-shock cohorts lie above the levels which would be chosen in case of only one endogenous variable. This is illustrated by the solid lines in panels (a) and (b) of Fig. 10. Education increases moderately by 0.23 years to $e_1 = 22.53$, while the retirement age increases by almost one year to $R_1 = 63.43$. The effect of a change in old-age mortality on retirement ages is about 4 times higher than its effect on education times. Thus, the number of years spent in the workforce increases.

Nevertheless, individuals spend the major part of their additional lifetime in retirement. Column (3) in Table 2 indicates that from the total gain of 5.68 years, only 0.23 years are spent on schooling and 0.7 years are dedicated to working. Consequently, the expected retirement period increases by 4.75 years.

Not surprisingly, the time path of the participation rate, $x(t)$, is a combination of the dynamics found in the two sub-models. For the first two decades, $x(t)$ remains unchanged until a rapid decline happens between $t = 22.3$ and $t = 22.53$. In the following, the share of workers increases again because pre-shock cohorts are gradually replaced by larger post-shock cohorts. Between $t = 62.5$ and $t = 63.43$, no generations retire and $x(t)$ rises sharply. Although the upwards jump *per se* is higher than in the retirement model, it is too low to reach the same level. At any time, the participation rate of the full model falls short of the participation rate of the specification where only retirement is endogenous.

However, the picture is different for per capita human capital, $h(v)$. Although $x(t)$ is smaller in the full model, the increase in $e(v)$ is sufficiently high to reach a higher level of human capital in the long run (remember $\hat{h}^{1-\phi} = A_H \cdot e^* \cdot \hat{x}$). Panel (e) of Fig. 10 shows that after $t = 50$, per capita human capital of the full model outperforms the trajectories of both sub-models.

As $h(t)$ determines the stock of physical capital and output, this is indeed a striking result: Examining the impact of longevity in a model framework which does not control for both education and retirement at the same time tends to over-estimate the negative impact of aging on the economy. In particular, we find that the steady state level of per capita output decreases by 4.8% and 3.3% in the two sub-models, while the reduction amounts to only 2.5% when adjustments in both variables are considered. In case the individuals could neither adjust education nor retirement, the output level would even drop by 5% due to the demographic change. This confirms our theoretical findings of Section 5.4. We showed in Eq. (5.22) that following an increase in ψ_m, the full model features the highest steady state level of per capita human capital among all model specifications considered.[46]

Interestingly, government debt converges to a long-run level of -51.9% of GDP (in the full model), although public expenditures for education and pensions increase in the long run. The reason is that the government acts farsightedly and anticipates this development with a one-off increase in the lump-sum tax from $z_0 = 2.63$ to $z_1 = 2.65$. Although this seems minor, it is sufficient to generate a government surplus during the first periods. Due to the embodied character of the shock, pre-shock generations only marginally react to the tax increase. After $t = 62.5$, the primary surplus even rises because the participation rate rapidly increases, which generates more revenue from the labor tax. In the long run, however, the national budget is dominated by the higher pension spending. The long-run primary deficit equals 2.84% of GDP.

[46] However, the comparative static analysis of Section 5.4 disregarded the necessary adjustment of the lump tax z.

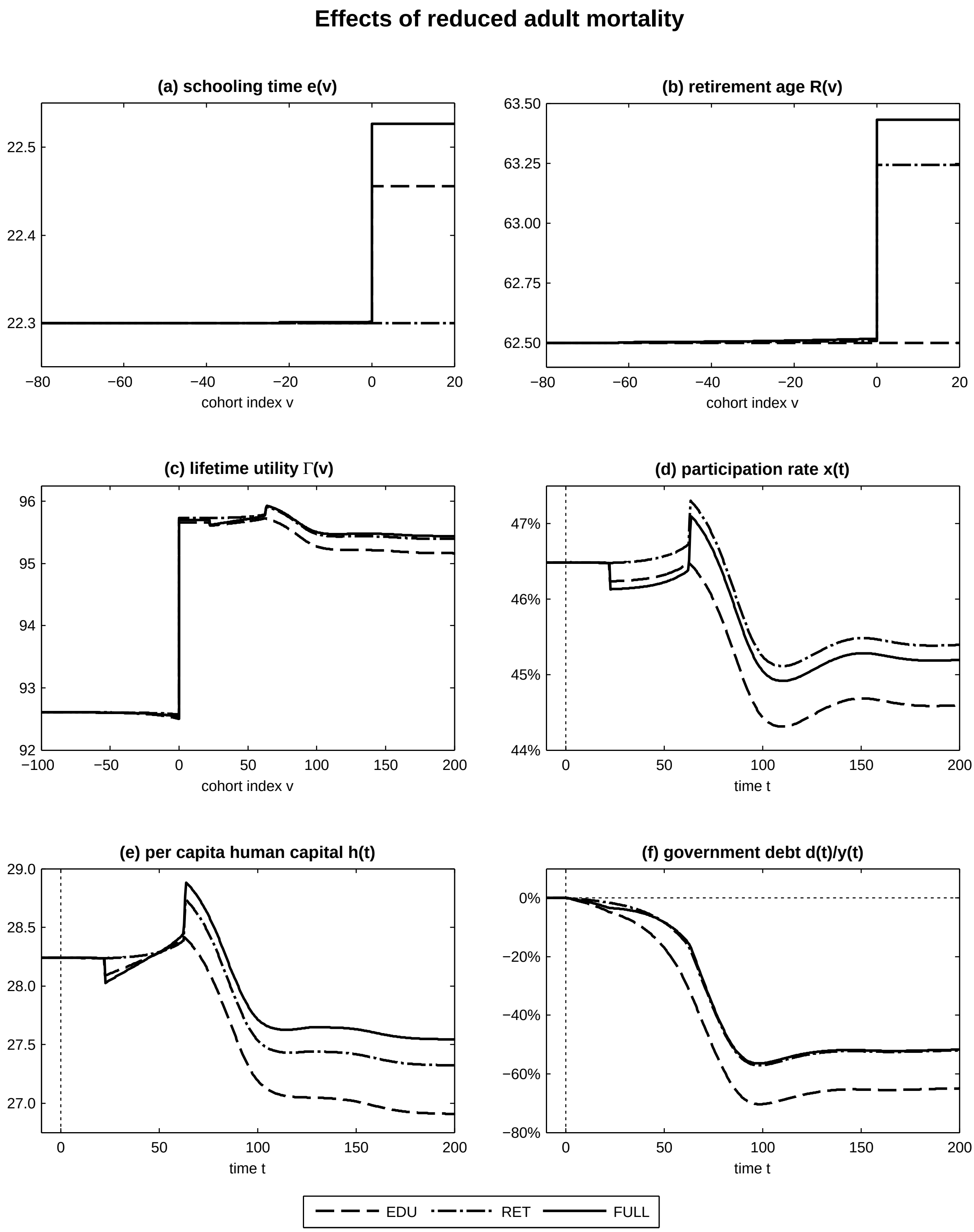

Figure 10: Effects of reduced adult mortality

6.2.3 Productive aging

Besides slower biological aging, we also consider slower economic aging through lower working disutility of older employees. In particular, we study an increase of $\bar{u}$ from 72 to 77 years. This results in a shift of the disutility function to the left as can be seen from Figure 11. Unlike the embodied mortality shock, we assume that all (non-retired) generations immediately benefit from a decrease in disutility of work. The effects on individual and aggregate variables are depicted in Figure 12; long-run effects can be found in column (4) of Table 2.

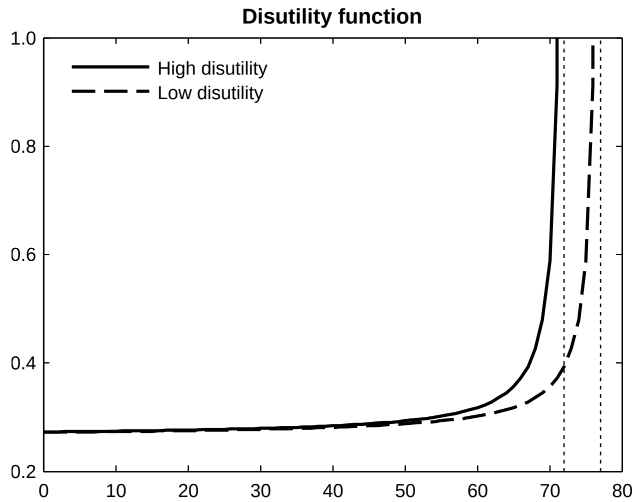

Figure 11: Disutility function, $D(u)$

As noted earlier, the FOC for education, Eq. (4.20), is independent from working disutility. Therefore, when schooling is the only endogenous variable, nothing changes apart from lifetime utility. Conversely, in the retirement model, individuals tend to postpone retirement. Additional time in the labor force ranges from 0.5 to 0.6 years with agents who are in their thirties at $t = 0$ delaying retirement the most. The participation rate rises quickly for $t \in [0, 0.55]$ since there are no exits from the labor force. Thereafter, $x(t)$ varies only slightly and stays at a level of 46.91% after $t = 63.05$. Notice that–unlike for the fertility and mortality shock–$x(t)$ remains constant once all individuals in the workforce choose the same levels of e and R, i.e. when all pre-shock generations have retired. The cyclical fluctuations of $x(t)$ in Fig. 9 and 10 are actually driven by changes in the age distribution, which is not affected in this case.

The evolution of human capital is very similar to $x(t)$ for the first 22.3 periods. Then, the human capital externality starts to work: Cohorts entering the labor market after $t = 22.3$ were born at a time v where $h(v)$ was significantly higher than in the initial steady state. Therefore, they exhibit a higher individual level of human capital. This causes the slope of $h(t)$ to change abruptly at $t = 22.3$. Due to the intergenerational spillover effect, human capital keeps on rising even after the participation rate is already constant and converges to a level of 28.61 in the long run.

As for the mortality shock, the long-run level of $h(t)$ is higher if we allow agents to adjust both education and retirement. While working generations choose the same retirement age as in the retirement model, younger individuals are also eager to moderately increase length of schooling. At the same time, they choose a slightly higher retirement age in order to spend the same total years in the workforce than before. However, the participation rate of the full model is somewhat lower. Although the length of working periods is identical to the retirement model, some individuals enter the workforce at a higher age. The pyramid shape of the age distribution implies that cohorts with higher ages have less weight in the population. Therefore, the participation rate is lower in the full model than in the retirement model.

Nevertheless, the increase in schooling time again compensates for the lower $x(t)$ after about 23 years, resulting in a higher per capita stock of human capital in the long run. Quantitatively, the difference is negligible. The retirement model shows an increase in $h(t)$ of 1.31% compared to the initial steady state, while the improvement is 1.49% in the full model.

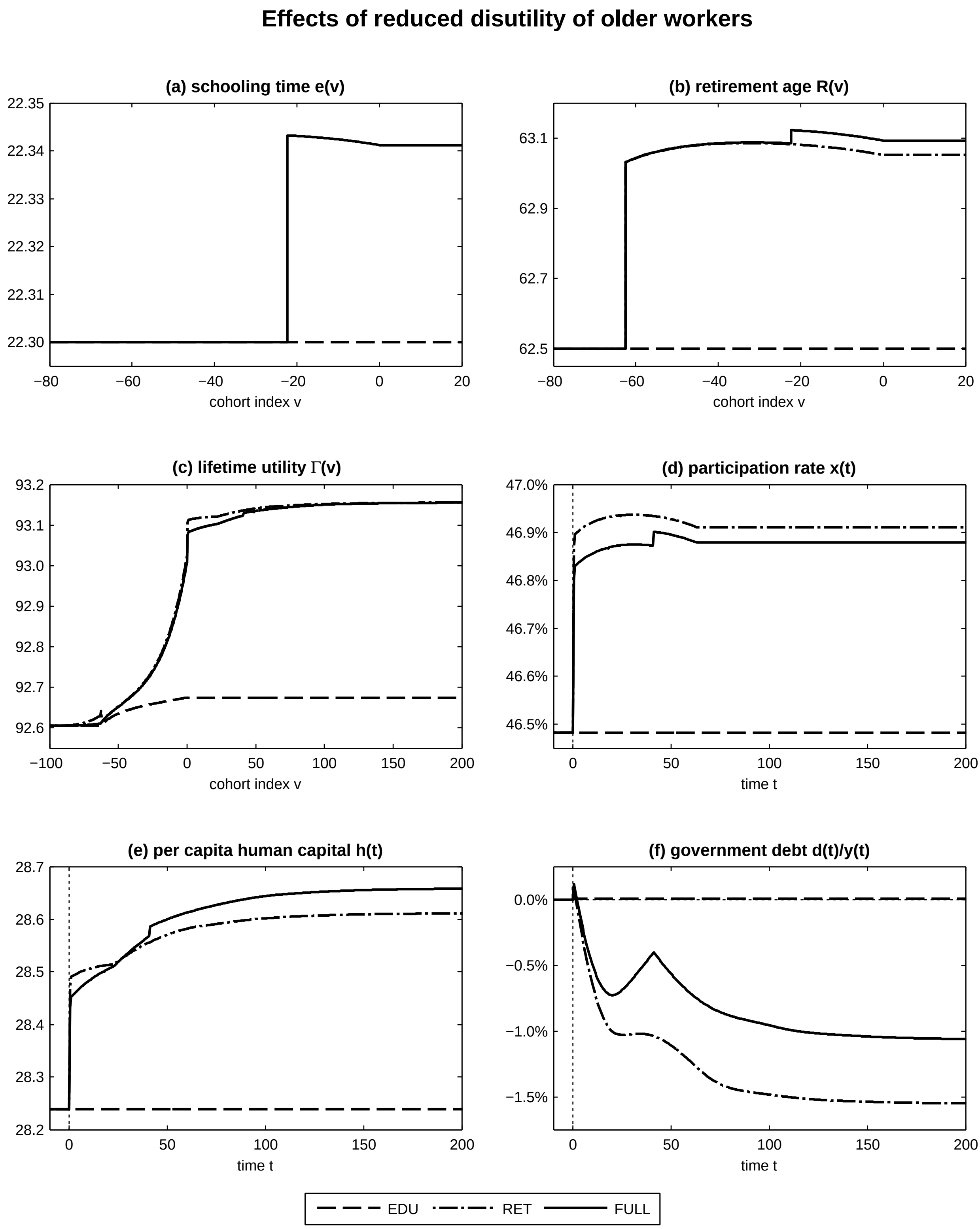

Figure 12: Effects of reduced disutility of older workers

6.2.4 Education reform

In line with Heijdra and Romp (2009a), we study a 20% increase in the education subsidy from $s_E = 5.1457$ to $s_E = 6.17484$. The time paths of individual and aggregate variables are shown in Figure 13; long-run effects are reported in column (5) of Table 2.

As observed in Chapter 5, this reform promotes schooling in the education model. Education time spans rise by approximately one year from $e_0 = 22.3$ to $e_1 = 23.36$. Furthermore, we analytically determined a decrease in R in the retirement sub-model. However, panel (b) in Fig. 13 reveals a slightly different picture: Post-shock cohorts indeed retire 0.43 years earlier than in the initial steady state. Whereas working generations and part of the cohorts in education are actually willing to postpone retirement. Those having just completed schooling delay their retirement the most, by approximately 4 months. The reason is that, in the long run, education spending in percent of GDP rises from 14.1% to 17% due to the reform. To balance the budget, an increase of the lump sum tax from $z = 2.63$ to $z = 3.04$ is required. Agents want to (partly) compensate this loss in lifetime income by choosing a higher retirement age. Individuals who have just entered the labor market at the time of the reform will adjust retirement the most. This is because they cannot gain from the increase in s_E anymore but perceive the higher tax rate for a comparatively long period of time. Cohorts born after $t = -22.3$ still benefit from the reform, while cohorts born before have faced the lower tax rate for a longer period of their lives. This is also reflected in cohorts' lifetime utility depicted in panel (c) of Fig. 13.

Interestingly, if agents are allowed to adjust both education and retirement, *all* generations will postpone retirement. The response of pre-shock workers is similar to the retirement model. Whereas pre-shock students and post-shock cohorts retire at least one year later than in this sub-model. Compared to the initial steady state, retirement is postponed by 0.57 to 1.2 years. The reason is that these generations increase their education time spans due to the higher subsidy (by about 1.1 years–the same magnitude as in the education model). While prolonged schooling increases individual wage, it reduces the period of wage earnings at the same time. Since the pension replacement rate is less than one, agents are eager to retire later. However, the response of retirement ages is weaker than the increase in education time spans. As a result, in the long run, the period cohorts spend in the workforce shortens by half a year from 40.2 to 39.67 years.

The shorter working periods are also reflected in the evolution of the participation rate. In the education model, $x(t)$ drops sharply at the time point of the educational reform and remains constant thereafter. Whereas in the retirement model, $x(t)$ actually rises at impact and later falls when the last pre-shock worker has retried. The participation rate of the full model evolves somewhere in between, with an immediate drop-off after $t = 0$ and an upwards jump stemming from the increase in retirement ages of educating generations.

While the retirement model generates the highest participation rate, the trajectory of $h(t)$ is the lowest in the long run because the other two models feature higher education levels. Until $t = 40.2$, per capita human capital evolves similarly in the education model and in the full model. Then, the additional increase in retirement ages pushes human capital even further, and in the long run $h(t)$ converges to a level which is 3.1% higher than in the initial steady state. By contrast, the increase in $h(t)$ only amounts to 1.56% if retirement ages are held fixed. Hence, by neglecting the interaction between education and retirement, we underestimate the aggregate effect of the education reform by almost 100%.

The evolution of government debt also highly depends on the model specification. In both sub-models, the lump sum tax increase is not sufficient to finance the higher expenditures. For the resulting primary deficits to be feasible, a negative government debt level is required in the long run. In the full model, however, there is a small long-run government surplus of 0.12% of GDP, which allows the government to have a moderate debt level of 2.9%.

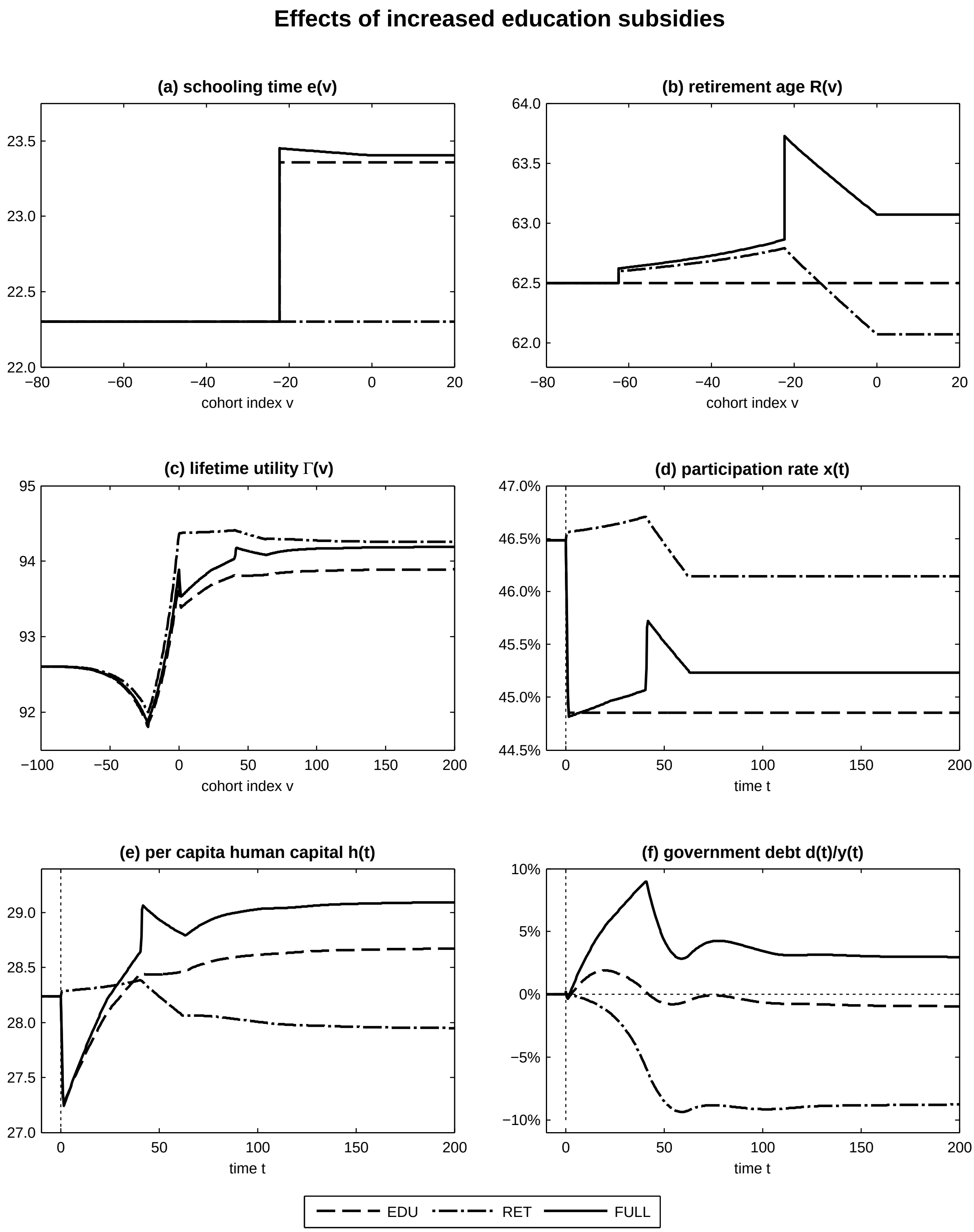

Figure 13: Effects of higher education subsidies

6.2.5 Pension reform

Last but not least, we study the effects of a pension reform. The governments reduces the pension accrual rate by 10% from $\vartheta = 0.0181$ to $\vartheta = 0.01629$.[47] If individual decisions remained constant, the net replacement rate would accordingly decrease from 85.6% to 77%. However, the reform induces an adjustment in individual behavior. The effects on individual and aggregate variables are depicted in Figure 14; long-run effects are given in column (6) of Table 2.

While the adjustments in schooling length are unremarkable, retirement ages increase significantly. Individuals who were on the brink of retiring before the reform postpone retirement by almost 1.5 years. Younger individuals adjust their behavior less severely but still retire at least 0.92 years later than in the initial steady state. The difference between the retirement model and the full model is negligible.

Due to the small response of schooling, aggregate variables of the education model remain roughly at their initial steady state levels. The only exception is lifetime utility, $\Gamma(v)$, which rises for young and future generations since the pension reform allows a tax reduction from $z = 2.63$ to $z = 2.42$.

The decrease in lump sum taxes is even higher when retirement is endogenous. The lower pension expenditures are then accompanied by an increase of the period during which individuals are net contributors to the tax scheme. In particular, working periods increase from 40.2 to 41.12 years in the long run.

The evolution of the participation rate, $x(t)$, mimics those of the cohort specific retirement age, $R(v)$. It rises rapidly from 46.48% to over 47.5% at the time point of the pension reform and then gradually falls back to a level of 47.17%. Although the participation rate is slightly higher in the retirement model, per capita human capital again falls short of the full model–though the difference is quantitatively negligible. For both specifications, $h(t)$ increases sharply at impact and remains roughly constant thereafter. In the long run, human capital exceeds its initial steady state level by approximately 2%.

Due to the altered individual behavior, the net replacement rate amounts to 78.8% in the long run. Hence a 10% decrease in ϑ lowers the replacement rate by only 8%. Nevertheless, pension spending relative to GDP still decreases by 2 percentage points. Because of the high participation ratio in the first decades after shock, the government runs a surplus and the debt level becomes negative. Hence in the long run it is feasible to have a small primary deficit of 0.3%.

[47] For simplicity, we assume that this reform also affects existing pensions. Hence the pension benefits for retirees drop by 10% after $t = 0$. This results in a moderate reduction of lifetime utility of these cohorts, see panel (c) in Fig. 14.

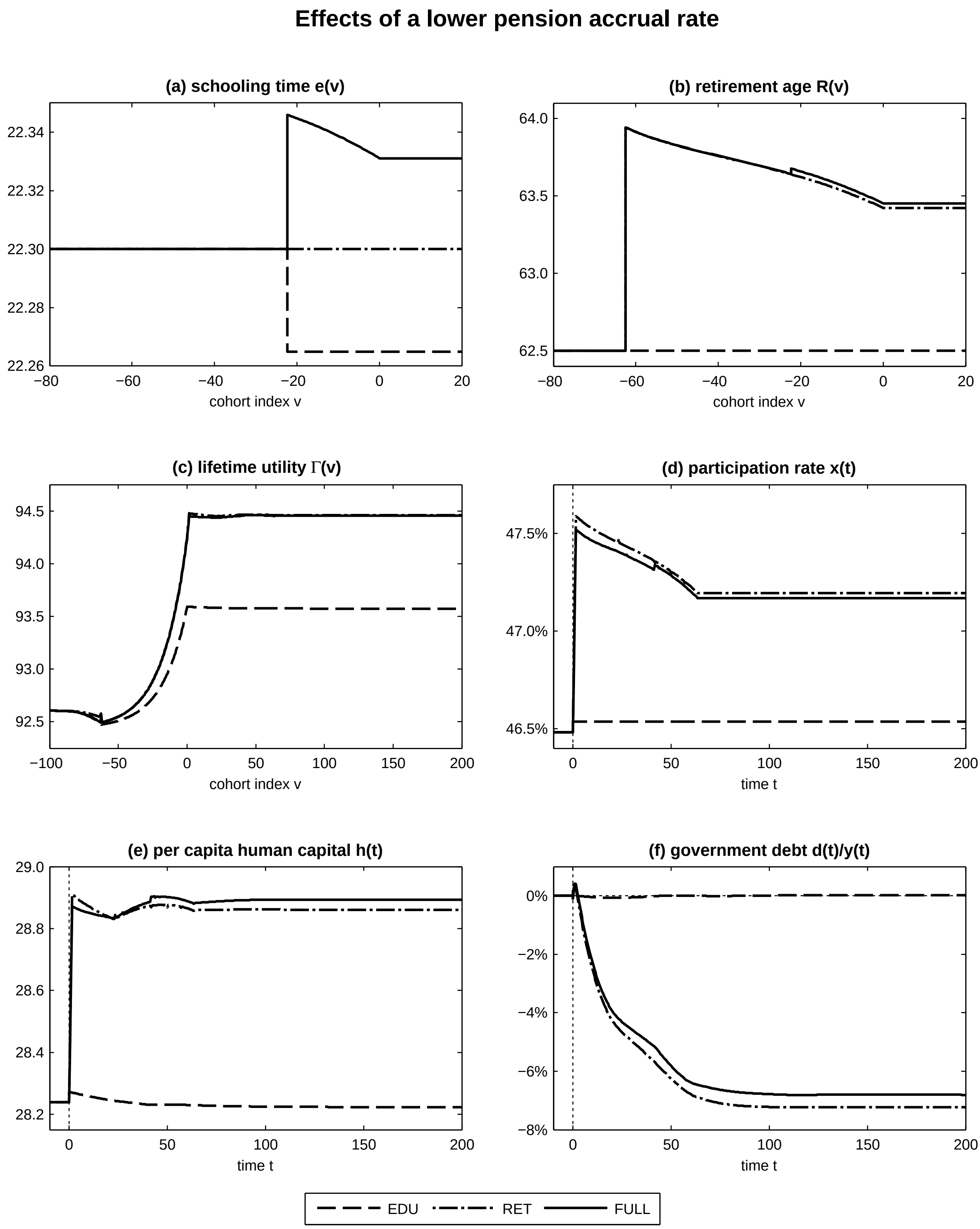

Figure 14: Effects of a lower pension accrual rate

7 Conclusion

In this thesis, we built a continuous time overlapping generations model where individuals are allowed to choose both length of schooling and retirement age endogenously. The two papers Heijdra and Romp (2009a) and Heijdra and Romp (2009b) served as a starting point of our analysis. While the first considers endogenous human capital accumulation but neglects any possibility to retire, the latter allows for retirement but takes human capital as exogenously given. By controlling for education and retirement at the same time, we found that the two decisions are positively interrelated. Longer education time spans encourage later retirement because the period devoted to work (and accumulation of pension claims) shortens. Contrariwise, later retirement increases length of schooling since a high wage income becomes more important. Therefore, even if a parameter shock seems to only affect the schooling or the retirement decision, the other variable will adjust as well.

Concerning the effects of population aging, we assumed that mortality shocks are embodied. Cohorts born before the shock face the old mortality process and will therefore not adjust their decisions.[48] By contrast, post-shock individuals respond to a reduction in adult mortality with longer schooling and later retirement. Because of the aforementioned interaction, the size of these effects is quantitatively higher if we control for both decision variables at the same time instead of considering only one of them. Due to later retirement combined with a later labor market entry, the effect of lower adult mortality on an individual's working period is analytically ambiguous. In our quantitative model, however, we find that post-shock generations work 0.7 years longer than pre-shock generations. In view of the underlying increase in life expectancy by 5.7 years, this effect is indeed only moderate. The major part of the additional lifetime is spent in retirement. The expected retirement period extends by 4.75 years, which accounts for 83% of the total gain in life expectancy. A similar result was found by Heijdra and Reijnders (2012). They argue that slower biological aging is not enough to bring about significantly longer working periods. If it is accompanied by slower depreciation of human capital, however, retirement periods may even become shorter. Since we use a flat efficiency profile without accounting for working experience or depreciation of skills, we are not able to replicate this finding in our framework.

On the aggregate level, the model we presented features a realistic demography with an age-dependent mortality profile. A mortality shock not only implies altered individual decisions but also a changing age distribution. In order to assess how joint modeling of education and retirement affects the aggregate economy, we solved the model for three different specifications: One where individuals adjust length of schooling but continue to retire at the initial retirement age, one where retirement can be revised but education time spans remain fixed, and one where the individual has both variables under control. Comparing the model outcomes, we found that using a framework which does not control for both education and retirement at the same time tends to overestimate the negative impact of aging on the macroeconomy. In particular, long-run per capita output decreases by 4.8% or 3.3% compared to the initial steady state if either education or retirement is endogenous. Whereas the reduction amounts to only 2.5% when adjustments in both variables are considered. Although the quantitative dimension of this result seems negligible, it gives a hint that the negative economic effect of population aging may be overestimated by more than 30% in common models where only retirement is endogenous.

Similar results can be obtained concerning the impact of other parameter shocks. Yet, the quantitative difference between the full model specification and the most appropriate restricted specification is not always substantial. Especially the quantitative effects of a pension reform seem to be equally well captured by a model that neglects human capital accumulation. Surprisingly, the analogous claim for education reforms is not true at all. Considering a 20% increase of the education subsidy, we found that whether retirement is treated as exogenous or endogenous significantly affects the model outcomes. The reason is that individuals not only want to increase schooling but also intend to postpone retirement due to less years of wage accumulation. As a result, the aggregate effect of the reform is much higher when this increase in retirement ages is taken into account. In particular, the gain in long-run output is nearly twice as high in the full model specification (+3.0%) as in the setting where only adjustments in schooling are considered (+1.6%).

[48] To be precise, there is a quantitatively small adjustment due to the general equilibrium effect.

In light of these results, we are warned that ignoring either the education or the retirement decision in a life-cycle overlapping generations model may result in wrong expectations about the quantitative impact of demographic shocks and policy reforms. Since common models studying the effects of aging usually only consider retirement and models regarding human capital accumulation only consider schooling as endogenous, authors should be aware of potential interactions between these two decision variables. Hence–similar to van Groezen et al. (2003), Sinn (2004) or Cigno and Werding (2007), who stress that pension and family policy are twins that should not be regarded in isolation–our work calls for a twin hypothesis of education and retirement.

However, our model relies on a number of assumptions that may influence the outcomes more or less severely. Most importantly, all taxes and subsidies are indexed to the cohort-specific marginal schooling productivity, $A_H h(v)^{\phi}$. Thereby, the tax system corrects for different levels of per capita human capital at birth, $h(v)$, between the generations. An individual born at a date v where $h(v)$ is comparatively high requires less years of schooling to achieve a certain level of individual human capital because of a high schooling productivity. However, the tax system compensates this comparative advantage by levying higher taxes on cohort v. Since net taxes are assumed to be proportional to $A_H h(v)^{\phi}$, per capita human capital cancels out of the first order condition for schooling. In case of log-utility, the retirement decision is independent from $h(v)$ as well. As a result, the optimal levels of schooling and retirement are the same for all cohorts, i.e. $e^*(v) \equiv e^*$ and $R^*(v) \equiv R^*$. Consequently, the dynamics of per capita human capital do not influence individual decision making. Regardless whether the economy is in a steady state or on a transition path, all new-born generations choose the same levels of education and retirement. This significantly simplifies the system dynamics, and the equilibrium path of $h(t)$ can be (relatively) easily computed by solving a nonlinear Volterra integral equation. Giving up the indexing assumption would not only significantly increase the computational effort necessary to practically solve the model but also raise the question of existence and stability of a solution.

Apart from assessing the role of the tax system, some further points are left for future research: First of all, we assumed that only cohorts born after the shock are affected by mortality changes. More realistic transitional dynamics can be obtained if living generations also exhibit gains in longevity. Furthermore, we consider relieving the assumption that individual human capital remains constant after schooling. In reality, working experience increases an individual's intrinsic value, while aging causes gradual loss of skills and abilities. As indicated above, Heijdra and Reijnders (2012) show that improvements in economic aging, i.e. slower depreciation of innate abilities, have a much stronger impact on the timing of retirement than improvements in biological aging. Therefore, incorporating human capital depreciation could be a promising extension–also because it brings about more realistic life-cycle income profiles.

Finally, an extension of the model to allow for education-specific mortality rates seems to be insightful. There is a well-documented correlation between education and adult mortality, see e.g. Grossman and Kaestner (1997) for an overview of studies. In Austria, for instance, life expectancy at age 65 ranges from 16.6 years for individuals with lower education (compulsory school) to 19.3 years for high-school and university graduates, see Statistics Austria (2009). The magnitude of these estimates suggests to link the mortality process to the individual level of education. To our knowledge, Rojas (2004) is the only work controlling for education-specific mortality rates in a similar setting. He assumes that life expectancy between skilled and unskilled workers differs by 2.55 years. However, his discussion leaves the role of the mortality differential in individual decision making unclear. Apart from mortality, educational attainment is also likely to affect disutility of work. High skilled jobs are typically less physically demanding, which facilitates later retirement. One way to assess the impact of these effects on individual behavior is to allow for heterogeneous learning abilities as proposed by Echevarría and Iza (2011). This, together with education-specific mortality and disutility profiles, would enable us, for example, to study the impact of longevity improvements which are only beneficial for part of the population (like medical treatment which only top-earners can afford).

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List of Figures

A Mathematical appendix

A.1 The demographic discount function

Proposition A.1 *Let $\Delta(x,\lambda)$ be defined as*

$$\Delta(x,\lambda) := e^{\lambda x + M(x)} \int_x^\infty e^{-[\lambda\alpha + M(\alpha)]}\, d\alpha, \quad x \geq 0,$$

where $M(x) = \int_0^x m(\alpha)\, d\alpha$ and assume that the mortality rate $m(.)$ is non-decreasing, i.e. $m'(\alpha) \geq 0$ for all $\alpha \geq 0$, and $\lambda + m(\alpha) > 0$ for some α. Then the following properties hold:

(i) Δ *is decreasing in* λ, $\frac{\partial\Delta(x,\lambda)}{\partial\lambda} = -e^{\lambda x + M(u)} \int_x^\infty (\alpha - x) e^{-[\lambda\alpha + M(\alpha)]}\, d\alpha < 0$.

(ii) Δ *is non-increasing in* x, $\frac{\partial\Delta(x,\lambda)}{\partial x} = (\lambda + m(x))\Delta(x,\lambda) - 1 \leq 0$.

(iii) Δ *is strictly positive,* $\Delta(x,\lambda) > 0$.

(iv) $\lim_{\lambda\to\infty} \Delta(x,\lambda) = 0$

(v) For $m'(\alpha) > 0$ and $m''(\alpha) \geq 0$, the inequality in (ii) is strict and $\lim_{x\to\infty} \Delta(x,\lambda) = 0$.

Proof. See Heijdra and Romp (2008, p.119). □

Proposition A.2 *Let the truncated Δ-functions be defined as*

$$\Delta_1(x,\lambda,C) := e^{\lambda x + M(x)} \int_x^C e^{-[\lambda\alpha + M(\alpha)]}\, d\alpha, \quad C \geq x \geq 0,$$
$$\Delta_2(x,\lambda,C) := e^{\lambda x + M(x)} \int_C^\infty e^{-[\lambda\alpha + M(\alpha)]}\, d\alpha, \quad C \geq x \geq 0.$$

Under the assumptions of Proposition A.1, the following properties can be established:

(i) $\Delta_1(x,\lambda,C) + \Delta_2(x,\lambda,C) = \Delta(x,\lambda)$

(ii) The limits for $C \to \infty$ are $\lim_{C\to\infty} \Delta_1(x,\lambda,C) = \Delta(x,\lambda)$ *and* $\lim_{C\to\infty} \Delta_2(x,\lambda,C) = 0$.

(iii) For $C < \infty$, both functions are strictly positive with $0 < \Delta_i < \Delta \leq 1/(\lambda + m(x))$.

(iv) For $C > x$, $\Delta_1(x,\lambda,C) < C - x$.

(v) For $C > x$, Δ_1 and Δ_2 are decreasing in λ, $\frac{\partial\Delta_i(x,\lambda,C)}{\partial\lambda} < 0$.

(vi) The marginal effects with respect to x are

$$\frac{\partial\Delta_1(x,\lambda,C)}{\partial x} = (\lambda + m(x))\Delta_1(x,\lambda,C) - 1 < 0$$

and

$$\frac{\partial\Delta_2(x\lambda,C)}{\partial x} = (\lambda + m(x))\Delta_2(x,\lambda,C) \in (0,1).$$

(vii) The marginal effects with respect to C are $\frac{\partial\Delta_1(x,\lambda,C)}{\partial C} = -\frac{\partial\Delta_2(x,\lambda,C)}{\partial C} = e^{-\lambda(C-x)-[M(C)-M(x)]} > 0$.

Proof. (i)–(iii) follow directly from the definition. (v)–(vii) are found by straightforward differentiation. The signs and boundaries in (vi) follow from (iii). To show (iv), observe that $\lambda x + M(x) < \lambda\alpha + M(\alpha)$ for $\alpha > x$ and therefore

$$\Delta_1(x,\lambda,C) = \int_x^C e^{\lambda x + M(x) - [\lambda\alpha + M(\alpha)]}\, d\alpha < \int_x^C 1\, d\alpha = C - x.$$

□

Proposition A.3 *Let* $\Delta(x,\lambda;\psi)$ *be defined as*

$$\Delta(x,\lambda;\psi) := e^{\lambda x + M(x,\psi)} \int_x^\infty e^{-[\lambda\alpha + M(\alpha,\psi)]}\, d\alpha, \quad x \geq 0, \tag{A.1}$$

where $M(x,\psi) = \int_0^x m(\alpha,\psi)\, d\alpha$ *and the mortality function fulfills Assumption 2.1. Then the results of Proposition A.1 apply mutatis mutandis. Furthermore, the following properties can be established:*

(i) M *is non-increasing in* ψ, $\frac{\partial M(x,\psi)}{\partial \psi} = \int_0^x \frac{\partial m(\alpha,\psi)}{\partial \psi}\, d\alpha \leq 0$.

(ii) The effect of ψ *on* M *is non-increasing in age,* $\frac{\partial^2 M(x,\psi)}{\partial\psi\partial x} = \frac{\partial m(x,\psi)}{\partial \psi} \leq 0$, *and strictly decreasing at some* x.

(iii) Δ *is increasing in* ψ, $\frac{\partial \Delta(x,\lambda;\psi)}{\partial \psi} = e^{\lambda x + M(x,\psi)} \int_x^\infty \left[\frac{\partial M(x,\psi)}{\partial \psi} - \frac{\partial M(\alpha,\psi)}{\partial \psi}\right] e^{-[\lambda\alpha + M(\alpha,\psi)]} d\alpha > 0$.

Let $\Delta_1(x,\lambda,C;\psi)$ *and* $\Delta_2(x,\lambda,C;\psi)$ *be defined analogously. Then, for* $C > x$, *we observe:*

(iv) Δ_1 *is increasing in* ψ, $\frac{\partial \Delta_1(x,\lambda,C;\psi)}{\partial \psi} = e^{\lambda x + M(x,\psi)} \int_x^C \left[\frac{\partial M(x,\psi)}{\partial \psi} - \frac{\partial M(\alpha,\psi)}{\partial \psi}\right] e^{-[\lambda\alpha + M(\alpha,\psi)]} d\alpha > 0$.

(v) Δ_2 *is increasing in* ψ, $\frac{\partial \Delta_2(x,\lambda,C;\psi)}{\partial \psi} = e^{\lambda x + M(x,\psi)} \int_C^\infty \left[\frac{\partial M(x,\psi)}{\partial \psi} - \frac{\partial M(\alpha,\psi)}{\partial \psi}\right] e^{-[\lambda\alpha + M(\alpha,\psi)]} d\alpha > 0$.

Proof. (i) and (ii) follow from differentiation and noting Assumption 2.1(iii). (iii)–(v) are then obtained by straightforward differentiation and noting the signs in (i) and (ii). □

A.2 Solving the individual maximization problem

A.2.1 Optimal consumption

In this section, we obtain the individual consumption decision which maximizes (2.4) subject to (2.6) using optimal control techniques (see Acemoglu (2008) among others). The according present-value Hamiltonian reads[49]

$$H(\bar{a},\bar{c},\lambda) = U(\bar{c}(v,\tau))e^{-[\theta(\tau-t)+M(\tau-v)]} + \lambda(\tau)\big\{[r + m(\tau-v)]\bar{a}(v,\tau) + \bar{w}(v,\tau) - \bar{g}(v,\tau) - \bar{c}(v,\tau)\big\}.$$

By Theorem 7.9 in Acemoglu (2008), the necessary conditions for an optimum are

$$H_{\bar{c}}(\bar{a},\bar{c},\lambda) = 0, \tag{A.2}$$

$$H_{\bar{a}}(\bar{a},\bar{c},\lambda) = -\dot{\lambda}, \tag{A.3}$$

$$H_{\lambda}(\bar{a},\bar{c},\lambda) = \dot{\bar{a}}.$$

From Eq. (A.2) we get

$$\lambda(\tau) = U'(\bar{c})e^{-[\theta(\tau-t)+M(\tau-v)]}, \tag{A.4}$$

which we differentiate with respect to time to arrive at

$$\dot{\lambda}(\tau) = -[\theta + m(\tau-v)]\lambda(\tau) + U''(\bar{c})e^{-[\theta(\tau-t)+M(\tau-v)]}\dot{\bar{c}}. \tag{A.5}$$

From the evolution of the costate in (A.3) we know

$$\dot{\lambda}(\tau) = -\lambda(\tau)[r + m(\tau-v)]. \tag{A.6}$$

[49]Notice that the term $e^{M(t-v)}$ is a constant and can be neglected in the optimization problem.

Combining (A.5) and (A.6) and substituting (A.4) yields after some basic algebra

$$\frac{\dot{\bar{c}}(v,\tau)}{\bar{c}(v,\tau)} = -\frac{U'(\bar{c})}{U''(\bar{c})\bar{c}}(r-\theta) = \sigma(r-\theta)$$

where the last equality follows from using the CES felicity function (2.5).

Since the Hamiltonian is concave and the NPG condition needs to hold for any admissible trajectory of $\bar{a}(v,\tau)$, the necessary conditions are also sufficient according to Theorem 7.11 in Acemoglu (2008).

A.2.2 Optimal retirement

The optimal retirement age is found by maximizing the concentrated utility function (3.8) with respect to $R(v)$. The first order condition reads

$$\frac{\partial\bar{\Lambda}(v,t)}{\partial R} = e^{\theta u+M(u)}\left[\int_t^\infty U'(\bar{c}^*(v,\tau,R))\frac{\partial\bar{c}^*(v,\tau,R)}{\partial R}e^{-[\theta(\tau-v)+M(\tau-v)]}\,d\tau - D(R)e^{-[\theta R+M(R)]}\right] \stackrel{!}{=} 0.$$

Remembering the optimal consumption decision (3.6)–(3.7), one can easily verify that

$$\frac{\partial\bar{c}^*(v,\tau,R)}{\partial R} = e^{\sigma(r-\theta)(\tau-t)}\,\frac{\partial\bar{c}^*(v,t,R)}{\partial R} = \frac{e^{\sigma(r-\theta)(\tau-t)}}{\Delta(u,r')}\frac{\partial\overline{li}(v,t,R)}{\partial R}$$

and, by taking advantage of the particular form of the felicity function,

$$U'(\bar{c}^*(v,\tau,R)) = e^{-(r-\theta)(\tau-t)}\,U'(\bar{c}^*(v,t,R)).$$

This implies

$$e^{\theta u+M(u)}\int_t^\infty U'(\bar{c}^*(v,\tau,R))\frac{\partial\bar{c}^*(v,\tau,R)}{\partial R}e^{-[\theta(\tau-v)+M(\tau-v)]}\,d\tau =$$
$$U'(\bar{c}^*(v,t,R))\frac{\partial\overline{li}(v,t,R)}{\partial R}\left[\frac{e^{\theta u+M(u)}}{\Delta(u,r')}\int_t^\infty e^{(\sigma-1)(r-\theta)(\tau-t)-\theta(\tau-v)-M(\tau-v)}\,d\tau\right],$$

and the expression in squared brackets is equal to one. Therefore, the FOC is equivalent to

$$U'(\bar{c}^*(v,t,R))\frac{\partial\overline{li}(v,t,R)}{\partial R} = D(R)e^{-\theta(R-u)-[M(R)-M(u)]},$$

which resembles Eq. (3.9).

It remains to show that the optimal retirement decision is independent from the agent's age, i.e. that decisions are dynamically consistent. For this purpose, we consider the formulation of the FOC which uses the transformed retirement age:

$$U'(\bar{c}^*(v,t,R))\frac{d\overline{li}(v,t,R)}{dS} = D(R)e^{(r-\theta)(R-u)}.$$

$d\overline{li}/dS$ is given in Eq. (3.11) and does not depend on u. Moreover, marginal utility fulfills $U'(\bar{c}^*(v,t,R)) = U'(\bar{c}^*(v,v,R))e^{-(r-\theta)u}$, and therefore individual age, u, cancels out of the equation. As a result, the optimal retirement age is indeed independent from the agent's age in the planning period.

A.3 Comparative static effects

This section contains several propositions which allow us to determine the comparative static effects of per capita variables. The following Lemma will prove useful:

Lemma A.4 *Consider the function*

$$x_\varphi(b,\psi) := \int_e^R \varphi(u)l(u)\,du = b\int_e^R \varphi(u)e^{-[\hat{n}(b,\psi)u+M(u,\psi)]}\,du \tag{A.7}$$

and the transformed cohort weights $\lambda_\varphi(u) := \varphi(u)l(u)/x_\varphi$. *Then the following results can be established:*

(i) The elasticity of x_φ *with respect to* b *is*

$$\frac{\partial \ln x_\varphi}{\partial \ln b} = \frac{\partial x_\varphi}{\partial b}\frac{b}{x_\varphi} = \frac{1}{\bar{u}}\left[\bar{u} - \int_e^R u\lambda_\varphi(u)\,du\right]$$

where $\bar{u} := \int_0^\infty ul(u)\,du$ *is the mean age of the population.*

(ii) The partial derivative with respect to ψ *can be written*

$$\frac{\partial x_\varphi}{\partial \psi} = -x_\varphi\int_0^\infty M_\psi(u,\psi)l(u)\,du\cdot\left[\frac{\int_e^R M_\psi(u,\psi)\lambda_\varphi(u)}{\int_0^\infty M_\psi(u,\psi)l(u)\,du} - \frac{\int_e^R u\lambda_\varphi(u)\,du}{\int_0^\infty ul(u)\,du}\right].$$

Proof. (i) Straightforward partial differentiation yields

$$\frac{\partial x_\varphi}{\partial b} = \int_e^R \varphi(u)e^{-[\hat{n}u+M(u)]}\,du - b\frac{\partial \hat{n}}{\partial b}\int_e^R u\varphi(u)e^{-[\hat{n}u+M(u)]}\,du = \frac{x_\varphi}{b} - \frac{\partial \hat{n}}{\partial b}\int_e^R u\varphi(u)l(u)\,du.$$

Furthermore, using (2.21) in (2.22) gives

$$\frac{\partial \hat{n}}{\partial b} = -\frac{\Delta(0,\hat{n})}{b\,\partial\Delta(0,\hat{n})/\partial\hat{n}} = -\left[b^2\frac{\partial\Delta(0,\hat{n})}{\partial\hat{n}}\right]^{-1} = \left[b\int_0^\infty ul(u)\,du\right]^{-1} = \frac{1}{b\bar{u}}$$

where the third equality follows from Prop. A.1(i). Now, by substituting this result into the expression above, we get

$$\frac{\partial x_\varphi}{\partial b} = \frac{x_\varphi}{b} - \frac{\int_e^R u\varphi(u)l(u)\,du}{b\bar{u}} = \frac{x_\varphi}{b\bar{u}}\left[\bar{u} - \frac{\int_e^R u\varphi(u)l(u)\,du}{x_\varphi}\right] = \frac{x_\varphi}{b\bar{u}}\left[\bar{u} - \int_e^R u\lambda_\varphi(u)\,du\right].$$

(ii) Partial differentiation with respect to ψ yields

$$\begin{aligned}\frac{\partial x_\varphi}{\partial \psi} &= -b\int_e^R \varphi(u)e^{-[\hat{n}u+M(u)]}\left[\frac{\partial\hat{n}}{\partial\psi}u + M_\psi(u,\psi)\right]\,du\\ &= -\int_e^R M_\psi(u,\psi)\varphi(u)l(u)\,du - \frac{\partial\hat{n}}{\partial\psi}\int_e^R u\varphi(u)l(u)\,du.\end{aligned} \tag{A.8}$$

By noting (2.23), Prop. A.1(i) and Prop. A.3(iii), we obtain

$$\frac{\partial\hat{n}}{\partial\psi} = -\frac{\partial\Delta(0,\hat{n},\psi)/\partial\psi}{\partial\Delta(0,\hat{n};\psi)/\partial\hat{n}} = -\frac{\int_0^\infty M_\psi(u,\psi)l(u)\,du}{\int_0^\infty ul(u)\,du}. \tag{A.9}$$

Plugging this expression into Eq. (A.8) and collecting terms completes the proof. □

The next two propositions present the comparative static effects of the steady state participation rate with respect to fertility and mortality shocks for the various models presented in this thesis:

Proposition A.5 *The elasticity of the steady state participation rate, $x := \int_e^R l(u)\,du$, with respect to the birth rate equals*

$$\frac{\partial \ln x}{\partial \ln b} = \frac{\partial x}{\partial b}\frac{b}{x} = \frac{\bar{u} - \bar{u}_x}{\bar{u}} < 1 \tag{A.10}$$

where $\bar{u} := \int_0^\infty u l(u)\,du$ and $\bar{u}_x := \int_e^R u\,l(u)\,du / \int_e^R l(u)\,du$ are the mean ages of the total population and the workforce, respectively. For the particular models, we observe:

(i) If schooling time is non-zero, $e > 0$, and the agents never retire, $R = \infty$, as in Heijdra and Romp (2009a), the effect is negative, $\frac{\partial x}{\partial b} < 0$.

(ii) If schooling time is zero, $e = 0$, and there is a finite retirement age, $R < \infty$, as in Heijdra and Romp (2009b), the effect is positive, $\frac{\partial x}{\partial b} > 0$.

(iii) Otherwise, the effect ambiguous in general.

Proof. Using Lemma A.4(i) with $\varphi(u) \equiv 1$ directly gives Eq. (A.10). Scenario (i) implies that all but the youngest e cohorts are in the labor market and hence $\bar{u}_x > \bar{u}$. Results (ii) and (iii) are obtained in a similar manner. □

Proposition A.6 *The partial effect of a mortality shock on the steady state participation rate is*

$$\frac{\partial x}{\partial \psi} = -x \int_0^\infty M_\psi(u,\psi) l(u)\,du \cdot \left[\frac{\int_e^R M_\psi(u,\psi)\lambda(u)\,du}{\int_0^\infty M_\psi(u,\psi) l(u)\,du} - \frac{\bar{u}_x}{\bar{u}} \right] \tag{A.11}$$

where $\bar{u}$ and $\bar{u}_x$ are defined as in Proposition A.5 and $\lambda(u) := l(u)/x$ is the share of the cohort with age u among the workforce. For the particular models, we observe:

(i) If schooling time is non-zero, $e > 0$, and the agents never retire, $R = \infty$, as in Heijdra and Romp (2009a), the effect is non-negative, $\frac{\partial x}{\partial \psi} \geq 0$.

(ii) If schooling time is zero, $e = 0$, and there is a finite retirement age, $R < \infty$, as in Heijdra and Romp (2009b), the effect is non-positive, $\frac{\partial x}{\partial \psi} \leq 0$.

(iii) Otherwise, the effect is ambiguous in general. However, it will be negative if e is sufficiently small compared to R.

(iv) In all cases, $\frac{\partial x}{\partial \psi} = 0$ if and only if $\partial^2 m(u,\psi)/\partial u \partial \psi = 0$.

Proof. Using Lemma A.4(ii) with $\varphi(u) \equiv 1$ directly gives Eq. (A.11). To show (i), we follow the outline presented in Lemma 2 of Heijdra and Romp (2009a) and define an auxiliary function

$$\Xi(e) := -\int_e^\infty M_\psi(u,\psi) l(u)\,du - \frac{\partial \hat{n}}{\partial \psi} \int_e^\infty u l(u)\,du.$$

Notice that this resembles the right hand-side of Eq. (A.8) for $\varphi(u) \equiv 1$ and $R = \infty$, considered as a function of e. By substituting Eq. (A.9), we observe $\Xi(0) = 0$ and $\lim_{e\to\infty} \Xi(e) = 0$.

Potential extreme values of Ξ coincide with the roots of

$$\Xi'(e) := \left[M_\psi(e,\psi) + \frac{\partial \hat{n}}{\partial \psi} e \right] l(e). \tag{A.12}$$

The first term, $M_\psi(e,\psi)$, is non-positive, non-increasing and concave in e (see Prop. A.3(i)–(ii) and Assumption 2.1(iv)), while the second term rises linearly. Being a linear combination of concave functions, the expression in square brackets is concave itself. As $\Xi'(0) = 0$ and $l(u) > 0$ for all $u \geq 0$, concavity implies that once Ξ' is negative, it remains negative forever. If $\Xi'(e)$ should be zero for some $e > 0$, it can either stay zero or get negative, but never turn positive.

We can now think of the following possibilities regarding the slope of $\Xi'(e)$ at $e = 0$:

(a) If $\Xi''(0) < 0$, $\Xi'(e)$ is falling and hence negative near the origin since $\Xi'(0) = 0$. As mentioned above, this implies $\Xi'(e) < 0$ for all $e < \infty$. This would cause the original function, Ξ, to be ever decreasing, which clearly violates $\lim_{e\to\infty} \Xi(e) = 0$.

(b) If $\Xi''(0) = 0$, we have to consider two alternatives: (i) There is a level of $e > 0$ with $\Xi''(e) < 0$. Then case (a) applies. (ii) $\Xi''(e) = 0$ for all $e < \infty$. Together with $\Xi'(0) = 0$, this implies $\Xi'(e) = 0$ for all $e < \infty$, which is only possible if $M_\psi(e, \psi)$ is linear in e, i.e. if $\partial^2 m(u, \psi)/\partial u \partial \psi = 0$.

(c) If $\Xi''(0) > 0$, Ξ' is increasing and therefore positive near the origin. However, as $\lim_{e\to\infty} \Xi(e) = 0$, there must be a threshold $\bar{e} > 0$ after which Ξ' is negative. $\bar{e}$ is unique and determined by the (positive) root of (A.12). As $\Xi''(\bar{e}) < 0$, Ξ features a maximum at $e = \bar{e}$.

To sum up, if $\partial^2 m(u, \psi)/\partial u \partial \psi = 0$, Ξ is constant and equal to zero. Otherwise, Ξ features a single maximum at some positive level $\bar{e} > 0$. Since $\Xi(0) = 0$ and $\Xi(e)$ goes to zero as e approaches infinity, differentiability ensures $\Xi(e) > 0$ for all $e > 0$.

Point (ii) is shown analogously. We use Eq. (A.8) with $\varphi(u) \equiv 1$ and $e = 0$, and consider this expression as a function of R,

$$\Psi(R) := -\int_0^R M_\psi(u, \psi) l(u)\, du - \frac{\partial \hat{n}}{\partial \psi} \int_0^R u l(u)\, du. \tag{A.13}$$

Again, $\Psi(0) = 0$ and $\lim_{R\to\infty} \Psi(R) = 0$. Moreover, $\Psi'(R) = -\Xi'(R)$ and therefore the above discussion applies to Ψ with the exactly reserved arguments. Particularly, Ψ is non-positive, with $\Psi \equiv 0$ if and only if $\partial^2 m(u, \psi)/\partial u \partial \psi = 0$. Otherwise, Ψ is strictly negative in $(0, \infty)$ with a single minimum at some positive level $\bar{R} > 0$.

(iii) In the case $e > 0$ and $R < \infty$, we can form a similar auxiliary function,

$$\Phi(e, R) := -\int_e^R M_\psi(u, \psi) l(u)\, du - \frac{\partial \hat{n}}{\partial \psi} \int_e^R u l(u)\, du.$$

By linearity of the integral, we have $\Phi(e, R) = \Psi(R) - \Psi(e)$ where Ψ is defined in (A.13). In the case of $\partial^2 m(u, \psi)/\partial u \partial \psi = 0$, we find $\Phi(e, R) = 0$ because of $\Psi(R) = \Psi(e) = 0$. However, if the effect of ψ on mortality rates is increasing with age, both terms are negative and the sign of the difference generally depends on the particular levels of e and R. Nevertheless, continuity ensures that for any given R, $\Phi(e, R)$ is negative for sufficiently low levels of e since $\Phi(0, R) < 0$. □

A.4 Obtaining a numerical solution

To numerically compute the transitional dynamics, we apply the following iterative procedure:[50]

1. Solve for the initial steady state variables.
2. Compute the population growth path $n(t)$ for $t \in [0, T]$ using Eq. (2.3).
3. Compute the equilibrium path for $t \in [0, T]$ until convergence to a new steady state.
   (i) Obtain the individual decisions taking the lump sum tax z as given.
   (ii) Use these values in (4.23) to solve for the trajectory of human capital accumulation.
   (iii) Compute the lump sum tax z which balances the government budget (4.28), and use this as new value of z in (i). Iterate until convergence of z.

[50] The terminal date T must be set high enough to ensure convergence. In particular we used $T = 300$.

A.4.1 Solving for the steady state

Essentially, the system of equations determining the steady state breaks down into three parts: As the interest rate, r, is exogenously given, the market wage rate, w, can easily be computed from Eq. (4.24). Also, the equilibrium population growth rate, $\hat{n}$, can be calculated straightforward from (2.21) by finding the unique root of $b\,\Delta(0,\hat{n}) - 1 = 0$.

All other variables are highly interdependent and need to be computed simultaneously. First of all, notice that in a steady state all cohorts choose the same levels of education and retirement. Therefore, it suffices to solve the first order conditions (4.14) and (4.20) for a newborn, $v = 0$. Lifetime income, and thus consumption and assets, are age-dependent but exhibit the same age-profile for any cohort.

The following nonlinear system yields the steady state values of education, e, retirement, R, and initial consumption, $\bar{c}_0$:

$$0 = \hat{\bar{c}}_0^{-1/\sigma} A_H \hat{h}^\phi we[1 - t_L - \vartheta(R-e) + \vartheta\Delta(R,r)] - D(R)e^{(r-\theta)R}, \tag{A.14a}$$

$$0 = (1-t_L)\Delta_1(e,r,R) + \vartheta(R-2e)\Delta_2(e,r,R) + s_E, \tag{A.14b}$$

$$\hat{\bar{c}}_0 = \hat{\bar{li}}_0/\Delta(0,r'), \tag{A.14c}$$

$$\hat{h} = \left[A_H e\, b\, \Pi(0,e,R,\hat{n})\right]^{1/(1-\phi)}, \tag{A.14d}$$

$$\hat{\bar{li}}_0 = A_H \hat{h}^\phi w\left[s_E\, \Pi(0,0,e,r) + (1-t_L)e\,\Pi(0,e,R,r) + \vartheta(R-e)e\,\Pi(0,R,\infty,r) - \hat{z}\Delta(0,r)\right], \tag{A.14e}$$

$$\hat{z} = s_E\, b\,\Pi(0,0,e,\hat{n}) - t_L e\, b\,\Pi(0,e,R,\hat{n}) + \vartheta(R-e)e\, b\,\Pi(0,R,\infty,\hat{n}) + (r-\hat{n})\frac{d_0}{wA_H\hat{h}^\phi}, \tag{A.14f}$$

where[51]

$$\Pi(x,a,b,\lambda) := e^{\lambda x + M(x)} \int_a^b e^{-[\lambda s + M(s)]}\, ds.$$

The first three lines simply represent the steady state equivalents of the first order conditions (4.10), (4.14) and (4.20) from a newborn's perspective. However, these equations also depend on $\hat{h}$ and $\hat{\bar{li}}_0$, which in turn depend on $\hat{z}$. Eq. (A.14d) defines steady state per capita human capital and resembles (5.18)–(5.19). Eq. (A.14e) is the steady state formulation of (4.11) and determines lifetime income for a newborn. Eq. (A.14f) follows from (4.28) and ensures that the government budget constraint is met.

Once we know $\hat{\bar{c}}_0$, the complete age profile of consumption follows from $\hat{\bar{c}}(u) = e^{\sigma(r-\theta)u}\,\hat{\bar{c}}_0$. The pattern of lifetime income, $\hat{\bar{li}}(u)$, can directly be computed from Eq. (4.11), and the savings profile is given by

$$\hat{\bar{a}}(u) = \Delta(u,r')\hat{\bar{c}}(u) - \hat{\bar{li}}(u).$$

The according per capita variables, $\hat{c}$, $\hat{li}$ and $\hat{a}$, can be obtained by straightforward numerical integration of the above age-profiles using the weights of the steady state age distribution, $l(u)$.

Per capita levels of physical capital, $\hat{k}$, and economic output, $\hat{y}$, are obtained by plugging $\hat{h}$ into (4.25) and (4.26), respectively. Eq. (4.27) is used to pin down per capita investment, $\hat{\imath}$, since $\dot{k} = 0$ in a steady state. Finally, the per capita stock of net foreign assets, $\hat{f}$, is calculated from (4.29) by setting $\dot{f} = 0$.

A.4.2 Computing the population growth rate

We assume that the economy resides in its initial steady state until $t = 0$. Therefore, the population growth rate is $n(t) = \hat{n} =: \hat{n}_0$ for $t < 0$. However, we observed in Section 2.2.1 that changes in either fertility or mortality will affect the growth rate of the population. To calculate $n(t)$ for $t > 0$, we use

[51] Notice that $\Delta(x,\lambda) = \Pi(x,x,\infty,\lambda)$, $\Delta_1(x,\lambda,C) = \Pi(x,x,C,\lambda)$ and $\Delta_2(x,\lambda,C) = \Pi(x,C,\infty,\lambda)$. Hence the Π-function is used very often in the numerical computation of the model. For the G-M mortality process, an efficient way to calculate Π exists, which makes use of the incomplete Gamma function, see the Appendix of Heijdra and Romp (2008) for further details.

Eq. (2.3) which is restated here for convenience,

$$1 = \int_{-\infty}^{t} b(v) e^{-[N(v,t)+M(t-v,\psi_m(v))]} \, dv, \tag{A.15}$$

and follow the discussion in Romp (2007).

Let b_0 and b_1 be the birth rate before and after the shock, respectively. Moreover, let ψ_0 be the shape parameter of the mortality function for all pre-shock generations ($v < 0$), while all cohorts born after the shock ($v \geq 0$) face a mortality function with parameter ψ_1. The convention that mortality shocks are embodied is in line with Heijdra and Romp (2009a) and Heijdra and Romp (2009b). Although probably not realistic, this assumption simplifies the numerical computation of the model significantly.

In particular, for $t \geq 0$, the integral in (A.15) breaks into two parts:

$$1 = b_0 \int_{-\infty}^{0} e^{-[N(v,t)+M(t-v,\psi_0)]} \, dv + b_1 \int_{0}^{t} e^{-[N(v,t)+M(t-v,\psi_1)]} \, dv.$$

The first term captures the share of pre-shock generations in the population, while the second one is the share of cohorts born in $[0, t]$. By noting $N(v, t) = N(0, t) - N(0, v)$ and defining $N(t) := N(0, t)$, we can rewrite this equation as

$$\begin{aligned} e^{N(t)} &= b_0 \int_{-\infty}^{0} e^{N(v)-M(t-v,\psi_0)} \, dv + b_1 \int_{0}^{t} e^{N(v)-M(t-v,\psi_1)} \, dv \\ &= b_0 e^{-M(t,\psi_0)} \Delta(t, \hat{n}_0; \psi_0) + b_1 \int_{0}^{t} e^{N(v)-M(t-v,\psi_1)} \, dv \end{aligned}$$

where we used $N(v) = \hat{n}_0 v$ for $v < 0$ and the definition of Δ in going from the first to the second line.

In the long run, the population growth rate will converge towards a level $\hat{n}_1$ which can be calculated from $b_1 \Delta(0, \hat{n}_1; \psi_1) = 1$. To bring the integral equation in a more stationary form, we multiply both sides by $e^{-\hat{n}_1 t}$ and arrive at

$$f(t) = g(t) + \int_{0}^{t} k(t-v) f(v) \, dv \tag{A.16}$$

where $f(t) := e^{N(t)-\hat{n}_1 t}$, $g(t) := b_0 e^{-M(t,\psi_0)-\hat{n}_1 t} \, \Delta(t, \hat{n}_0; \psi_0)$ and $k(t-v) := b_1 e^{-[\hat{n}_1(t-v)+M(t-v,\psi_1)]}$.

This is a so-called *renewal equation*, i.e. a Volterra equation of the second kind with a convolution type kernel. We employ an algorithm proposed by Linz (1985) to solve (A.16) with respect to $f(t)$. The basic idea is to approximate f on a grid $\{t_n : n = 0, 1, \dots\}$ with constant step size, $t_n = nh$. The integral on the right-hand side is replaced by the finite sum of a numerical integration rule (in the simplest case the trapezoidal rule):

$$f(t_n) = g(t_n) + h \sum_{i=0}^{n} w_{ni} k(t_n - t_i) f(t_i) \, dv.$$

The algorithm is initialized with $f(0) = g(0)$ and then iterates forward in time. Notice that the sum at the right-hand side depends on the values $f(0), \dots, f(t_{n-1})$, which have already been computed before t_n, but also on $f(t_n)$ itself. Nevertheless, the equation can easily be solved for $f(t_n)$.

Having obtained $\{f(t_n) : n = 0, 1, \dots T\}$ for a sufficiently high T to ensure convergence, we still need to deduce the path of the population growth rate, $n(t)$. Noting the definition of f, we find that $n(t) = d \ln f(t)/dt + \hat{n}_1$. In practice, the derivative is approximated by a simple difference quotient.

In deriving our numerical results, we use the composite Simpson's rule for computing the time path of $n(t)$ for $t = 1, \dots, 300$. Simpson's rule is a three point integration formula and therefore has a higher accuracy than the trapezoidal rule. However, initiation of the algorithm would require that f is known at two nodes, while actually we only know $f(0)$. Although a nifty way to bypass this problem is presented in Linz (1985, Ch. 7.6), we simply use the trapezoidal rule to obtain $f(t_1)$ and proceed with Simpson's rule thereafter.

A.4.3 Computing the new equilibrium path

Step 1: Calculating the individual response. In order to compute the path to the new equilibrium, we first obtain the altered individual decisions. Altogether, the population breaks into four broad groups of generations which are specified in Table 3.

description	cohort indices, v	age at $t = 0$, u_0	choice variables	initial condition
pre-shock retirees	$v \leq -R_0$	$R_0 \leq u_0$	$\bar{c}$	$a(v, 0) = \hat{\hat{a}}(u_0)$
pre-shock workers	$-R_0 < v \leq -e_0$	$e_0 \leq u_0 < R_0$	R, $\bar{c}$	$a(v, 0) = \hat{\hat{a}}(u_0)$
pre-shock students	$-e_0 < v < 0$	$0 < u_0 < e_0$	e, R, $\bar{c}$	$a(v, 0) = \hat{\hat{a}}(u_0)$
post-shock cohorts	$0 \leq v$	$u_0 \leq 0$	e, R, $\bar{c}$	$a(v, v) = 0$

Table 3: Overview of the different groups of generations

For easier understanding, we denote the age of cohort v at $t = 0$ with $u_0 := -v$. If the cohort is already retired when the shock happens, i.e. $u_0 \geq R_0$, individuals can only change their remaining consumption path, $\{\bar{c}(v, t)\}_{t=0}^{\infty}$. Cohorts which are in the workforce at $t = 0$, i.e. $u_0 \in [e_0, R_0)$, can additionally change their retirement age, $R(v)$, but cannot re-enter education. Whereas cohorts still in education, $u_0 \in (0, e_0)$, can re-arrange all three decisions.

Notice that within these groups, cohorts are the same in the sense that they have the same variables under control. However, they will in general not decide on the same optimal levels. The reason is that the re-arrangement of decisions is conditional on individual assets at $t = 0$ and the impact of a shock on remaining lifetime income varies with age. Consider, for instance, an increase in the education subsidy. A student who has just started his education collects the high subsidy during almost his whole schooling period. Whereas an older student who was just about to enter the labor market had received the lower subsidy for a long period of time. To make up for this comparative disadvantage, the older student is willing to adjust schooling length more pronouncedly.

In the following, we discuss how to practically obtain the altered education and retirement decisions. Since only pre-shock students and pre-shock workers can re-arrange e and/or R, we build a grid for v spanning from $-R_0$ to 0 and evaluate education and retirement decisions at the points of this grid. It seems reasonable to ensure that $-e_0$ is part of the grid as it separates the two groups. For pre-shock students we solve the first order conditions (4.14) and (4.20) to obtain $e_S(v)$ and $R_S(v)$ for all grid points in $[-e_0, 0]$. Because pre-shock workers have already completed schooling and spent e_0 years in education, we solve (4.14) with $e = e_0$ to obtain $R_W(v)$ for all grid points in $[-R_0, -e_0]$. Notice that in general $R_S(-e_0) \neq R_W(-e_0)$ because students additionally have e to respond to the shock and therefore will adjust R in a different way. In their decision making, both groups of generations take their asset stock at $t = 0$, $\hat{\hat{a}}(u_0)$, and the lump sum tax z as given. While physical wealth is predetermined, z is adjusted in Step 3 in order to balance the (new) government budget constraint.

It remains to determine the optimal behavior of post-shock cohorts ($v \geq 0$). Observe that the FOC for retirement depends on per capita human capital at birth, $h(v)$. For cohorts born before the shock, this is equal to $\hat{h}$, the initial steady state of h. However, for cohorts born after the shock, $h(v)$ is determined by (4.23) and will vary with v. Since $e(v)$ and $R(v)$ are functions of $h(v)$, individual behavior is in general cohort-dependent for all $v \geq 0$. Moreover, the level of $h(v)$ considered in the decision making process of cohort v incorporates the optimal decisions of a variety of older generations. Solving such a system is numerically intensive and may as well cause stability problems.

However, we find that the solving procedure can be simplified by assuming log-utility, i.e. $\sigma = 1$. In this case, $h(v)^{\phi}$ cancels out of (4.14) and renders $e(v)$ and $R(v)$ cohort-independent for all post-shock generations. To see this, notice that although the economy is hit by an unanticipated shock, the optimal choices of cohorts born *after* the shock are time-consistent. Therefore, for $v \geq 0$, optimal decisions can be determined by evaluating the first order conditions at the cohort's date of birth, $t = v$. In this case, consumption is proportional to $h(v)^{\phi}$. Under $\sigma = 1$, $U'(\bar{c}) = 1/\bar{c}$ and therefore human capital cancels out of (4.14) when $U'(\bar{c}^*)$ is multiplied with $d\overline{li}/dS$ (because this is proportional to $h(v)^{\phi}$ itself). Hence both first order conditions are independent from the cohort index and yield the same optimal decisions $e^*(v) \equiv e_1$ and $R^*(v) \equiv R_1$ for all $v \geq 0$.

Putting things together, we obtain education and retirement decisions as functions of the cohort index:

$$e^*(v) = \begin{cases} e_0 & \text{for } v \leq -e_0, \\ e_S(v) & \text{for } v \in (-e_0, 0), \\ e_1 & \text{for } v \geq 0, \end{cases} \quad \text{and} \quad R^*(v) = \begin{cases} R_0 & \text{for } v \leq -R_0, \\ R_W(v) & \text{for } v \in (-R_0, e_0], \\ R_S(v) & \text{for } v \in (-e_0, 0), \\ R_1 & \text{for } v \geq 0. \end{cases}$$

In practice, we use cubic spline interpolation for $e_S(v)$, $R_S(v)$ and $R_W(v)$ to approximate $e^*(v)$ and $R^*(v)$ for all v that are not part of the grid.

Step 2: Computing the time path of per capita human capital. Per capita human capital is indeed the crucial variable in our model. Once we have computed $h(t)$, it is easy to obtain $k(t)$ and $y(t)$. It is also necessary for calculating individual consumption, $\bar{c}(v, t)$, and hence per capita consumption, $c(t)$.[52] The remaining per capita variables, $a(t)$, $g(t)$, $d(t)$, $i(t)$ and $f(t)$, are also functions of $h(t)$ and are fully determined after this variable has been calculated.

Per capita human capital was defined in Eq. (4.23) as

$$h(t) := \int_{-\infty}^{t} I(t - v, R^*(v))\bar{h}(v, t)l(v, t)\, dv$$

with $\bar{h}(v, t)$ given in Eq. (4.6).

For $t < 0$, the economy is in a steady state, and $h(t) = \hat{h}$. For $t \geq 0$, the integral breaks into three parts:

$$h(t) = \int_{-R_0}^{-e_0} I(t-v, R_W(v))\bar{h}(v,t)l(v,t)dv + \int_{-e_0}^{0} I(t-v, R_S(v))\bar{h}(v,t)l(v,t)dv + \int_{0}^{t} I(t-v, R_1)\bar{h}(v,t)l(v,t)dv.$$

Substituting $\bar{h}(v, t) = [1 - I(t - v, e(v))]A_H h(v)^{\phi} e(v)$ and $l(v, t)$ from (2.2) yields

$$h(t) = h_{SW}(t) + A_H b_1 e_1 \int_{\max\{0, t-R_1\}}^{\max\{0, t-e_1\}} h(v)^{\phi} e^{-[N(v,t)+M(t-v,\psi_1)]}\, dv \tag{A.17}$$

where

$$\begin{aligned} h_{SW}(t) := {} & A_H b_0 e_0 \hat{h}^{\phi} \int_{-R_0}^{-e_0} I(t - v, R_W(v)) e^{-[N(v,t)+M(t-v,\psi_0)]}\, dv \\ & + A_H b_0 \hat{h}^{\phi} \int_{-e_0}^{0} \big[I(t - v, R_S(v)) - I(t - v, e_S(v))\big] e_S(v) e^{-[N(v,t)+M(t-v,\psi_0)]}\, dv. \end{aligned}$$

Eq. (A.17) is again a Volterra equation of the second kind, but now of nonlinear type (since we rule out $\phi = 1$) and with a more general kernel of the form $k(v, t)$. Nevertheless, the same iterative procedure as for solving (A.16) could be employed. However, we use a slightly different approach and switch from the cohort domain to the age domain, which gives

$$h(t) = h_{SW}(t) + A_H b_1 e_1 \int_{\min\{t, e_1\}}^{\min\{t, R_1\}} h(t - u)^{\phi} e^{-[N(t-u,t)+M(u,\psi_1)]}\, du. \tag{A.18}$$

Although (A.17) and (A.18) are equivalent, the latter is more convenient to implement because the limits of integration are bounded. For $t \in [0, e_1]$, the integral term is zero, and hence $h(t) = h_{SW}(t)$. For $t \in (e_1, R_1]$, integration starts at $u = e_1$ and ends at $u = t$. For all $t > R_1$, integration is carried out on the fixed interval $[e_1, R_1]$.

[52] Notice that even under log-utility, lifetime income and consumption depend on $h(v)$. Using $\sigma = 1$ only makes education and retirement independent from human capital.

Notice that, in contrast to (A.16), the right-hand side of (A.18) does not depend on $h(t)$ but only on past values of human capital. There is a time-span of e_1 periods between a cohort's birth and its significance for the human capital stock.

Therefore, the numerical implementation is straightforward: We choose a grid with constant step size, $G := \{t_n := nh : n = 0, 1, \dots\}$, and ensure that $e_1 \in G$. It is essential to have an equidistant grid because the computation of $h(t_n)$ requires the level of human capital at the dates $\{t_n - t_k : k \leq n\}$. For the algorithm to be feasible, these points need to be part of the grid as well. For $t_n \in [0, e_1]$, we have already observed that $h(t) = h_{SW}(t)$. In practice, h_{SW} is evaluated by straightforward application of MATLAB®'s numerical integration procedure. For $t_n > e_1$, the integral term in (A.18) is nonzero and approximated using Simpson's rule. As both 0 and e_1 are part of the grid, $t_n - e_1 \in G$ for all n and we can indeed use $u = e_1$ as the first point in Simpson's rule. As long as $t_n \leq R_1$, the quadrature rule makes use of the points $\{e_1, \dots, t_n\} \subset G$.

However, a problem arises for $t_n > R_1$ when the integral must be computed for $u \in [e_1, R_1]$. Let $t_R := \max\{t_n \in G : t_n \leq R_1\}$ be the highest grid-point which is less than or equal to R_1. In general, $t_R \neq R_1$ since R_1 is not part of the grid and therefore also $t_n - R_1 \notin G$. It follows that $h(t_n - R_1)$ is unknown and the upper limit of the integral cannot be used as evaluation point in the numerical integration rule. To overcome this problem, we form an approximation $\tilde{h}(t_n - R_1)$ by cubic interpolation from $\{h(0), \dots, h(t_{n-1})\}$. Simpson's rule is then evaluated at the (not equidistantly spaced) points $\{e_1, \dots, t_R, R_1\}$ where $\tilde{h}(t_n - R_1)$ is used instead of the unknown real value.

Step 3: Balancing the government budget. Having calculated $h(t)$, we can pin down all remaining per capita variables. However, we have ignored the solvency condition of the government so far. As discussed in Section 4.4, the current public debt must be covered by future surpluses, i.e.

$$d_0 = d(0) = \int_0^\infty g(\tau) e^{-r\tau + N(\tau)} \, d\tau$$

where d_0 is per capita government debt in the initial steady state. By substituting $g(\tau)$ and applying some basic algebra, it can be shown that this equation can be rewritten

$$\begin{aligned} d_0 = &A_H w b_1 \left[z\Delta(0, r; \psi_1) - \tilde{g}_F \right] \int_0^\infty h(v)^\phi e^{-rv + N(v)} \, dv \\ &+ A_H w b_0 \hat{h}^\phi \left[z \int_0^\infty \Delta(u, r; \psi_0) e^{-[\hat{n}_0 u + M(u, \psi_0)]} \, du - \int_0^\infty \tilde{g}(u) e^{(r - \hat{n}_0)u} \, du \right]. \end{aligned} \tag{A.19}$$

For simplification, $\tilde{g}_F$ denotes the present value of net taxes (labor tax minus education subsidies and pensions) for a post-shock individual at the date of his birth, and $\tilde{g}(u)$ is the present value of remaining net taxes for an individual aged u at $t = 0$. Conditioning on the levels of $\tilde{g}_F$, $\tilde{g}(u)$ and $h(v)$, the tax level which balances the government budget can easily be computed from (A.19).

However, this value will in general differ from the level of z which was considered in the individual decision making process of Step 1. To determine a tax level which is consistent with both individual optimal behavior and a balanced budget, an iterative root-finding procedure is necessary. In each step of the algorithm, individual decisions and the time path of human capital have to be re-computed.

In practice, the integral terms are again approximated by numerical integration rules. The infinite integration range turns out to be unproblematic as the integrands approach zero very quickly. We use the same grid as above together with the composite Simpson's rule to evaluate these expressions. The upper limit of integration is replaced by the greatest grid point, which is around 300.

Altogether, the numerical algorithm presented in this section requires less than a minute to compute $n(t)$, $e(v)$, $R(v)$, $h(t)$ and the equilibrium level of z.[53]

[53] An Intel Core i5-3210M processor with two 2.5 GHz cores was used for the computation. All MATLAB® codes are available from the author upon request.