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Restricted permutations on multisets

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RESTRICTED PERMUTATIONS ON MULTISETS

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May 2011

DECLARATION

I herewith declare that I have completed the present thesis independently, making use only of the specified literature and aids. Sentences or parts of sentences quoted literally are marked as quotations; identification of other references with regard to the statement and scope of the work is quoted. The thesis in this form or in any other form has not been submitted to an examination body and has not been published.

Vienna, May 2011

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ABSTRACT

The area of pattern avoidance is a young and rapidly growing field of discrete mathematics. Its roots can be discerned both in computer science within the theory of sorting algorithms and in a far-reaching generalization of the concept of inversions in permutations.
The objects analysed within pattern avoidance are so-called "restricted permutations". Permutations are the objects at the heart of discrete mathematics and algebra that can either be seen as linear orderings or as bijective maps from a set onto itself. Restricted permutations are permutations with a special inherent structure, namely such a structure that does not allow for certain patterns to appear. A pattern is to be understood as a subsequence of a permutation in which the elements lie in a certain, pre-defined order to each other. The central problem of pattern avoidance can be formulated as follows: How many n-permutations are there that avoid a given pattern? For patterns of length three it is relatively easy to find an answer to this question. Surprisingly enough, the integer sequence obtained is the very well-known sequence of Catalan numbers. For longer patterns, this problem turns out to be a lot more complicated. Results achieved so far are presented. Further to this, a more general result that received a lot of attention, the so-called Stanley-Wilf-conjecture, is recorded here along with its detailed proof.
In this thesis our interest focusses on permutations on multisets, i.e. sets in which elements may appear more than once. It is only a few years ago that one ventured into this natural extension of the field of pattern avoidance. The two main articles within this young field dealing with enumeration questions for multiset-permutations avoiding (multiset)-patterns are presented. Following this, we show that several methods used for ordinary permutations can successfully be extended to multiset-permutations. With the help of generating trees and recursions, accompanied by generating functions and the Kernel method, we close a gap in the study of 122-avoiding permutations. In all cases where in addition to 122 an ordinary pattern of length three is avoided, we manage to develop closed enumeration formulae. Again, well-known sequences emerge, e.g. (generalized) Catalan and Fibonacci numbers. In some special cases, we provide additional insight by constructing bijections to other objects enumerated by the same sequences.
An appendix to this thesis contains an excerpt of a series of riddles that we call "MOUNTAINOUS PATTERNS" and that hopefully give a playful insight into the fascinating topic of pattern avoidance.

ZUSAMMENFASSUNG

Die Mustervermeidung ist ein junges und schnell wachsendes Teilgebiet der diskreten Mathematik. Ihr Ursprung kann einerseits in den Computerwissenschaften, genauer gesagt in der Theorie der Sortieralgorithmen, andererseits in einer weitreichenden Verallgemeinerung des Konzepts der Inversion einer Permutation gefunden werden.
Zentrales Untersuchungsobjekt der Mustervermeidung sind sogenannte "restricted permutations" (= eingeschränkte Permutationen), wobei Permutationen jene elementaren Bausteine der diskreten Mathematik und Algebra sind, die sowohl als lineare Ordnungen als auch als Bijektionen von einer endlichen Menge auf sich selbst aufgefasst werden können. Die Einschränkung besteht darin, dass nur Permutationen, die eine gewisse Struktur aufweisen, nämlich eine, die ein bestimmtes Muster (oder mehrere Muster) nicht erlaubt, zugelassen sind. Unter einem Muster verstehen wir dabei eine Teilfolge einer Permutation, in der die Elemente in einer fest vorgegebenen Ordnungsrelation zueinander stehen. Die Frage, die im Mittelpunkt der Mustervermeidung steht, ist jene nach der Anzahl von Permutationen der Länge n, welche ein vorgegebenes Muster vermeiden. Für Muster mit drei Elementen ist die Beantwortung dieser Frage einfach und man erhält überraschenderweise eine wohlbekannte Zahlenfolge als Antwort: die der Catalan-Zahlen. Für längere Muster stellt sich diese Fragestellung jedoch als viel komplizierter heraus. Bekannte Resultate sowie Abzählformeln werden vorgestellt und die stark beachtete Stanley-Wilf-Vermutung, die eine allgemeine Aussage über die Anzahl an eingeschränkten Permutationen trifft, wird zusammen mit ihrem Beweis anschaulich präsentiert.
Ausführlich beschäftigt sich diese Arbeit mit Permutationen auf Multimengen, also Mengen in denen Elemente öfter als ein Mal vorkommen dürfen. Dieses Gebiet wurde bisher weit weniger erforscht als jenes der eingeschränkten Muster auf gewöhnlichen Mengen. Es konnte hier gezeigt werden, dass sich viele der bei gewöhnlichen Permutationen verwendeten Methoden auch auf Multimengen-Permutationen übertragen lassen. Mithilfe von erzeugenden Bäumen und Rekursionen, fallweise erzeugenden Funktionen und der Kernel Methode, ist es in dieser Arbeit gelungen die Untersuchung jener Permutationen, die das Muster 122 vermeiden zu vervollständigen. Für alle Fälle, in denen noch ein weiteres, gewöhnliches Muster der Länge drei vermieden wird, konnten geschlossene Abzählformeln bewiesen werden. Auch hier treten wieder bekannte Zahlenfolgen auf, etwa (verallgemeinerte) Catalan-Zahlen oder Fibonacci-Zahlen. Darüber hinaus war es in einigen speziellen Fällen möglich, Bijektionen zu anderen Objekten, die durch dieselben Folgen abgezählt werden, zu konstruieren.
In einem Anhang dieser Arbeit wird ein Auszug einer Serie von mathematischen Rätseln, die wir MOUNTAINOUS PATTERNS nennen, vorgestellt. Dieser ermöglicht hoffentlich einen spielerischen Einblick in die faszinierende Welt der Mustervermeidung.

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First of all I would like to thank my supervisor Alois Panholzer who introduced me to the captivating area of pattern avoidance and guided me throughout the development of my thesis. He encouraged me to work independently and develop my own results but also told me when it was time to stop thinking about new problems to be solved and start writing. My thanks also go to Michael Drmota, Bernhard Gittenberger and Reinhard Winkler for "infecting" me with their love for discrete mathematics and (linear) algebra.

It was Karl Sigmund who made me realize at an early age that there is "something" called a mathematician. It is mainly thanks to him, my school teacher François "Nono" Novitzky and my parents (who kept me busy on our weekend drives with their highway-riddles) that my interest in mathematics was awoken and kept alive. I am grateful to my mother and father who enabled and encouraged me to pursue this interest by choosing to study mathematics. They ensured that I never lost courage and didn't even think of giving up.

I want to thank my mother Ingela Bruner-Newton for her great help with proof-reading my thesis and for the challenging discussions that arose from it. Throughout my studies she was an incredible mentor and coach. What I appreciate most is her capability of giving me honest feedback and very constructive criticism. My father Gerhard Bruner taught me how important patience and the respect for others are.

My thanks also go to Martin Lackner for his great (mathematical) interest in my thesis, above all for his appreciation of my work, giving me self-confidence and teaching me how much fun it is to be two who speak the same language of mathematics. I strongly hope that the topic of pattern avoidance will lead to some joint work.

Kiwi, my four-legged companion, took me for many walks during which I developed the main input for my ideas.

Last but not least, I wish to thank Hans Lackner who, without knowing it, inspired me to think about pattern avoidance problems as mathematical riddles and who gave the impulse to MOUNTAINOUS PATTERNS, a playful introduction for newcomers. In this context, my thanks also go to my father for his creative input.

CONTENTS

1 INTRODUCTION

The area of pattern avoidance is a fairly young and rapidly growing field of discrete mathematics. In 2003, Kitaev and Mansour gave a survey [29] of this field including more than two hundred papers[1]. Since then, pattern avoidance has continued to grow and remains a prolific field of research. The objects analysed within this field are so-called "restricted permutations". Permutations are the objects at the heart of discrete mathematics and algebra that can either be seen as linear orderings or as bijective maps from a set onto itself. Restricted permutations are permutations with a special inherent structure, namely such a structure that does not allow for certain patterns to appear.

Let us give an example in a first attempt to describe what is meant when saying that a permutation contains or avoids a certain pattern. This example was inspired by [29]. Consider the word PERMUTATION and impose the lexicographic order on the letters of this word (The letter A is the smallest letter, then comes E, etc.). It then corresponds to the sequence of numbers 61739808254. We introduce the following terms:

- the sub-word MAIN corresponds to a 3124-pattern, whereas PAIN and RAIN correspond to a 4123-pattern since P and R come after N and RUIN corresponds to a 3412-pattern;
  PERMUTATION therefore *contains* the patterns 3124, 4123 and 3412,
- PUT corresponds to a 132- and PUTT to a 1322-pattern;
  PERMUTATION *contains* the patterns 132 and 1322,
- PERMUTATION *avoids* the patterns 54321 and 112,
  since no sub-words can be found where the letters relate in the same way to each other as those in SOLID (= 54321) or in EEL (=112).

A formal description of the concept of avoiding or containing patterns in permutations on ordinary sets or multisets will be given in the Chapters 3 and 4.

The central question in pattern avoidance is the following:

How many permutations of length n are there
that avoid a given pattern?

Another related area, namely the one of *pattern counting*, tries to determine the number of permutations of length n where a certain pattern occurs a fixed number of times.

A first fact within pattern avoidance is that the integer sequences that arise in this context are often very well-known sequences, e.g. those of Catalan or Fibonacci numbers. This is puzzling since there is a priori no connection to other objects counted by these sequences. However, it is quite often possible to find bijections to these objects providing a much deeper understanding of restricted permutations.

1 A new book entitled *Patterns in permutations and words* surveying the topic of avoiding respectively containing patterns and written by Sergey Kitaev is about to appear in July 2011 at Springer's.

Another interesting fact is that the language of pattern avoidance has proven to be apt to describe other seemingly unrelated problems. These are for instance: Kazhdan-Lusztig polynomials, singularities of Schubert varieties, Chebyshev polynomials and rook polynomials for a rectangular board. See [29] for references. Another area where pattern avoidance emerges is that of sorting algorithms. This is less surprising since the analysis of stack sortable permutations can be seen as one possible origin of the concept of pattern avoidance.

1.1 SOME HISTORY

A very first result within pattern avoidance-theory can be found as early as 1915 in Percy MacMahon's study of so-called lattice permutations in [33]. The author showed that permutations that can be written as the disjoint union of two decreasing subsequences (these correspond exactly to the 123-avoiding permutations) are counted by the Catalan numbers. Another early result is the so-called Erdös-Szekeres Theorem (see [19]) published in 1935 and stating the following: any sequence of $(nk+1)$ distinct real numbers contains either a decreasing subsequence of length $(k+1)$ or an increasing subsequence of length $(n+1)$.
However, these two results remained isolated and it was only at the end of the 1960's that the notion of pattern avoidance was picked up again when Donald Knuth investigated stack-sortable permutations in [30]. He showed that permutations can be sorted with one stack iff they avoid the pattern 231 and that these permutations are counted by Catalan numbers. In the early 1970's, Robert Tarjan analysed sorting networks and Vaughan Pratt characterised permutations sortable with a deque with the help of pattern avoidance. In the late 1970's, Doron Rotem established a correspondence between binary trees and certain permutations in [43] and D. G. Rogers analysed ascending sequences in permutations in [42].
After this first computer theoretical interest for pattern avoidance questions the topic remained dormant for several years. It was in 1985, when Rodica Simion and Frank Schmidt published the first systematic study of *Restricted Permutations* [46] that the area of pattern avoidance started to flourish for good. Their paper focussed on enumeration problems, counting even and odd permutations avoiding a given pattern of length three and permutations avoiding pairs of patterns of length three. They also gave the first bijective proof of the fact that there is exactly the same number of 123- and 132-avoiding permutations. With this publication, the foundation was laid for pattern avoidance as a branch of enumerative combinatorics.
Irrespective of the computer theoretical origins of pattern avoidance, one must note that the notion of "patterns in a permutation" simply generalizes the concept of inversions in permutations from pairs of entries to k-tuples of entries. If inversions enable us to say how *disturbed* the elements of a permutation are in comparison to the natural order $12\dots n$, the study of patterns in a permutation give much more detailed information about the order of the elements. The emergence of pattern avoidance is therefore also both plausible and comprehensible from within permutation theory.

1.2 STRUCTURE OF THE THESIS

Chapter 2 starts with a definition of two of the terms used in the title of this thesis: permutations and multisets. For permutations, several different interpretations and possibilities for a graphical representation are described. For multisets, a combinatorial motivation for this concept is given and several enumerative questions are treated. An introduction to the subsequently used methods - generating functions, generating trees and the Kernel method - follows. The definitions given here are all accompanied by explanatory examples.
Chapter 3 starts with a formal definition of the notion of containing, respectively avoiding a pattern in a permutation, making it clear what is meant by the term *restricted permutation*. This definition is accompanied by a possible graphical interpretation of pattern avoidance. First important results for patterns of length three are then presented together with their proofs and a first illustration of the methods introduced in the preceding chapter is given. This classification of patterns of length three is followed by an overview over the case of patterns of length four. Finally, a very strong and general result, the Stanley-Wilf conjecture is presented. Roughly speaking, it states that there are considerably less permutations avoiding a given pattern than permutations in total. Its proof is given in several steps.
In Chapter 4 the notion of pattern avoidance is extended to permutations on multisets. The two main articles published within this area are treated. First the results of Albert, Aldred et.al. who studied multiset-permutations avoiding one or more ordinary patterns of length three in [1] are presented. Then the work of Heubach and Mansour who considered multiset-patterns of length three instead of ordinary patterns in [25] is presented. In this article, one specific problem, namely the one of $(112, 122)$-avoiding permutations remains open. We give a very satisfying answer to this question for regular multisets. This result is enriched by a bijective correspondence to Dyck words.
Chapter 5 contains new results. We pursue our work and take a closer look at permutations on regular multisets avoiding the pattern 122 and some other, ordinary pattern of length three. For two of the six arising pairs of patterns, we prove Wilf-equivalence. For the remaining five equivalence classes, exact enumeration formulae are proven. In the case of $(122, 123)$-avoiding permutations, we additionally construct a bijection to well-known lattice paths.
In Chapter 6 we finally sum up our results in a summary table and present possible directions for further research .
An appendix to this thesis contains an excerpt of a series of riddles that we call "MOUNTAINOUS PATTERNS". It is an attempt to make the topic of pattern avoidance accessible to a broad public that does not necessarily have a mathematical background. MOUNTAINOUS PATTERNS explain what pattern avoidance is about and offer an illustration of the results presented in this thesis. The representation of permutation matrices with the help of square (or rectangular in the case of multisets) grids introduced in Chapter 3 is used to construct several pattern avoidance-riddles, ranging from easy to extremely difficult. I very much hope that these riddles give a playful insight into the fascinating topic of pattern avoidance.

2 PRELIMINAIRIES

2.1 PERMUTATIONS

Permutations (from latin *permutare*, to (inter)change) are objects at the heart of discrete mathematics and algebra. They also arise within computer science, see Knuth's example of the *Kernel method* in Section 2.5. Their importance and omnipresence is certainly due to the fact that there are several different ways of defining and interpreting permutations. This section gives a first introduction to this topic by presenting some of the possible definitions, interpretations and graphical representations of permutations, and introducing some of the commonly studied permutation statistics. For a deeper study of permutations (seen as combinatorial objects), the book *Combinatorics of permutations* [12] written by Miklós Bóna can be highly recommended.
Informally, a permutation on a given set is a rearrangement of its elements into a certain order. For instance, an anagram of a word is a permutation of its letters[1]. For instance:

PERMUTATION ~ IMPORTUNATE ~ TRAUMPOETIN[2]

A first possible definition of permutations as combinatorial objects would be to see permutations as an ordered listing of the elements of a given set. For sets with n elements, we will always choose the set $[n] := \{1, 2, 3, \ldots, n\}$ to simplify matters but every other set with n elements would do equally well.

Definition 2.1.1. *A sequence $p_1 p_2 p_3 \ldots p_n$ containing all the elements of the set $[n]$ exactly once is called a* ***permutation*** *of $[n]$ or an n-permutation.*

The basic observation contained in the following theorem is probably the best-known fact about permutations.

Theorem 2.1.2. *The number of n-permutations is equal to*

$$n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n.$$

Proof. When constructing a permutation on $[n]$, we have n possibilities for the first element p_1, $(n-1)$ possibilities for p_2, etc. until two possibilities remain for the element p_{n-1} and the last element left over must be p_n. □

This first definition already suggests two possible representations of permutations. The first one, the so-called **two-line representation**, describes a permutation with the help of a $(2 \times n)$-matrix. In the first row, the elements from 1 to n are placed in increasing order, representing the position in the sequence. In the second row, the elements p_1 to p_n are placed. Thus,

1 Note that for most words one or more letters occur more than once. Thus their anagrams correspond to permutations on a multiset, see the next section for definitions.
2 German for "dream poet".

when reading this matrix column by column, we read (i, p_i). See Figure 2.1 for an example. As one can see, all the necessary information about the described permutation lies in the second row of the matrix. We can therefore eliminate the first row, obtaining a $(1 \times n)$-matrix respectively an n-vector, the so-called **one-line representation**. See again Figure 2.1 for the same example.
These two representations also enable the visualization of permutations with the help of graphs, as can be seen in Figure 2.1. On the one hand, the two-line representation can directly be translated into the language of bipartite graphs. For a permutation p on $[n]$ the associated graph G_p is the bipartite graph with vertex set $([n], [n])$ and where $e = (i, j)$ is an edge iff $p_i = j$ in p (i is an element of the first set of n elements, j is an element of the second one). On the other hand, the one-line representation can be illustrated with the help of the plot of the function that maps i to p_i (where $i \in [n]$), i.e. by marking the points (i, p_i) in the plane. This visualization contains information about the "ups" and "downs" in the permutation, such as descents, ascents, left-to-right-minima and so on (see the following definitions). Representing permutations in this way shows that they can also be seen as maps from the set $[n]$ to itself. This leads to the following definition:

Definition 2.1.3. *A **permutation** on the set* $[n]$ *is a bijective map*

$$f : [n] \to [n].$$

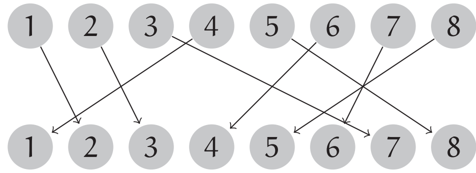
This definition is certainly not in contradiction with the first definition we made, since we simply have $p_i = f(i)$. To the contrary, it enriches our understanding of what permutations really are. Regarding permutations as bijective functions enables us to consider the composition of two permutations, i.e. performing two rearrangements of the set $[n]$ in succession. Obviously $f = f_2 \circ f_1$ is again a bijective map from the set $[n]$ to itself if f_1 and f_2 were bijections from $[n]$ to $[n]$. We can also define the inverse of a permutation f, simply as to be the unique map f^{-1} for which it holds that $f \circ f^{-1} = f^{-1} \circ f = id_{[n]}$. These observations show that the set of all permutations on $[n]$ together with the operation $\circ$ of function composition and the identity map as neutral element forms a group, the so-called **symmetric group** of degree n, denoted by $\mathcal{S}_n$ or $Sym([n])$. In general, $Sym(S)$ denotes the group on the set of all bijective maps from S to itself, where S is a non-empty set. The following result is fundamental in group theory.

Theorem 2.1.4 (Cayley's theorem)**.** *Every group* G *is isomorphic to a subgroup of a symmetric group, i.e. a group of permutations. In particular, if* G *is finite, it is isomorphic to a subgroup of* S_n *with* $n = |G|$.

A proof of Cayley's theorem can be found in any algebra-textbook, see e.g. [26].
In the context of permutations as bijective maps, the cycle decomposition of permutations plays an important role. Let us define the following:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 7 & 1 & 8 & 4 & 6 & 5 \end{pmatrix}$$

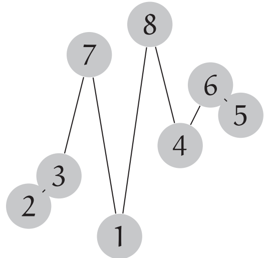
two-line-representation



bipartite graph

$$\begin{pmatrix} 2 & 3 & 7 & 1 & 8 & 4 & 6 & 5 \end{pmatrix}$$

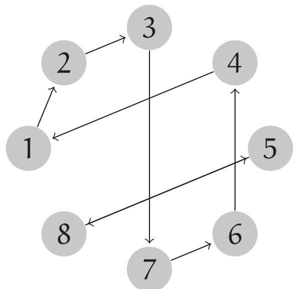
one-line-representation



plot of the permutation as a function

(764123) (85)

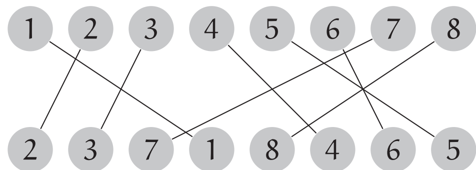
cycle-representation



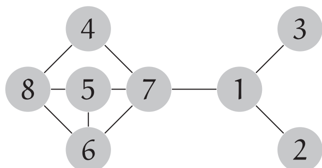
functional graph of the permutation

(12) — (13) — (17)
(47) — (48)
(56) — (57) — (58)
(67) — (68)

list of all inversions



Every intersection corresponds to an edge in



the permutation graph

Figure 2.1: Several ways of representing the permutation $p = (23718465)$ with the help of graphs.

Definition 2.1.5. *Let* $i_1, i_2, \dots, i_k$ *be distinct elements of* $[n]$. *Then* $(i_1 i_2 \dots i_k)$ *denotes the permutation that maps* $i_1 \mapsto i_2$, $i_2 \mapsto i_3$, ..., $i_{k-1} \mapsto i_k$ *and* $i_k \mapsto i_1$, *and leaves all the other elements of* $[n]$ *fixed. Such a permutation* $(i_1 i_2 \dots i_k)$ *is called a* ***cycle*** *of length* k; *a cycle of length two is called a* ***transposition***. *Two cycles* $\sigma = (i_1 i_2 \dots i_k)$ *and* $\tau = (j_1 j_2 \dots j_l)$ *are said to be* ***disjoint*** *if the sets* $\{i_1, i_2, \dots, i_k\}$ *and* $\{j_1, j_2, \dots, j_l\}$ *are disjoint.*

Note that cycles of length k are group elements of order k.
With the help of this definition, we can see that every permutation can be written as the product of disjoint cycles. However, the cycle representation is not unique, for instance $(58)(764123) = (412376)(85)$. Indeed, there are k different ways of writing the same cycle of length k and changing the order among the cycles does not either change the permutation since the cycles are assumed to be disjoint. Now, if we impose that the first element within a cycle is also its largest and if we arrange the cycles in increasing order of their first elements, we do obtain a unique way of writing a permutation using the cycle notation. This is the so-called **canonical cycle representation**, see Figure 2.1 for our running example.
The cycle representation of a permutation p also suggests a new way of visualizing p with the help of graphs. Indeed, the **functional graph** (not to be confounded with the graph of the function used earlier) describes precisely the cycle structure of p. It is the simple directed graph with vertex set $[n]$ and an edge from i to j iff p maps i to j, i.e. iff $p_i = j$. See again Figure 2.1.
Another unique way of describing a permutation is by listing all its inversions.

Definition 2.1.6. *A pair of entries* (p_i, p_j) *is called an* ***inversion*** *in the permutation* p, *if* $i < j$ *but* $p_i > p_j$.

A pair (p_i, p_j) being an inversion means that the *natural order* $123 \dots n$ is disturbed at the positions i and j. Thus the permutation $n(n-1)\dots 21$ where the natural order has been reversed contains all $\binom{n}{2}$ possible inversions on the set $[n]$. As can easily be seen, a permutation p is already fully described by the complete list of its inversions. See Figure 2.1 for the list of inversions in our running example. The corresponding graph-representation is the so-called **permutation graph** and is probably the most commonly used graph in connection with permutations. It is the simple undirected graph on n vertices where (k, l) is an edge iff (k, l) is an inversion in the permutation. This graph can be obtained in the following way: consider the two-line representation of the given permutation and draw a line between the element i in the first row and the element i in the second row for all $i \in [n]$. Every intersection of such lines corresponds to an edge in the permutation graph: if the k-line and the l-line intersect for some $k, l \in [n]$, draw an edge between k and l in the graph. See Figure 2.1. This graph contains information about how *disturbed* the order is in a given permutation: many edges correspond to a high degree of disturbance, every cycle corresponds to a subset of elements where the order has been completely inverted and so on.

Another completely different approach to permutations can be made via certain binary matrices, namely permutation matrices.

Definition 2.1.7. *A* ***permutation matrix*** *is a square binary matrix that has exactly one entry 1 in each row and each column and 0's everywhere else.*

There are two different possibilities of translating an n-permutation p into its corresponding $n \times n$-permutation matrix and vice-versa:

$$A_p(j,i) = \begin{cases} 1 & \text{if } p_i = j, \\ 0 & \text{otherwise.} \end{cases}, \qquad B_p(i,j) = \begin{cases} 1 & \text{if } p_i = j, \\ 0 & \text{otherwise.} \end{cases}.$$

Note the following: In A_p the i-th column corresponds to $\mathbf{e}_{\mathbf{p_i}}$, in B_p the i-th row corresponds to $\mathbf{e}_{\mathbf{p_i}}{}^T$, where $\mathbf{e_j}$ is the j-th vector in the standard basis of $\mathbb{R}^n$. B_p is therefore simply the transpose of A_p. One can also easily check that $A_pB_p = B_pA_p = I$, the inverse of a permutation matrix is thus its transpose.

Example 2.1.8. In our running example, the two possible corresponding permutation matrices are given by:

$$A_p = \begin{pmatrix} 0&0&0&1&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&1&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \end{pmatrix}, \quad B_p = \begin{pmatrix} 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0 \\ 1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&1&0&0&0 \end{pmatrix}$$

Most commonly the matrix A_p is used to represent the permutation p, since the associated map $f : S_n \to \mathbb{R}^{n \times n}$ defined by $f(p) = A_p$ is a homomorphism (contrarily, the map g defined by $g(p) = B_p$ is an anti-homomorphism).
For even more possibilities of representing permutations see the Section *Geometric representations of permutations* in [49].

Finally, here follows a list of definitions that shall be useful in the subsequent chapters.

Definition 2.1.9.

- *For a given permutation* $p = p_1p_2 \dots p_n$ *its* ***reverse*** p^r *is the permutation* $p_np_{n-1} \dots p_2p_1$*, i.e. the permutation* p *read from right to left.*
- *The* ***complement*** p^c *of* p *is the permutation in which the* i*-th entry is* $n + 1 - p_i$.
- *The position* i *in* p *is called a* ***descent*** *of* p*, if* $p_i > p_{i+1}$.
- *Similarly, the position* i *is called an* ***ascent****, if* $p_i < p_{i+1}$.
- *An entry* p_i *is called* ***left-to-right-minimum*** *if, for all* $j < i$, $p_i < p_j$. *Similarly we define left-to-right maxima, right-to-left minima and right-to-left maxima.*

2.2 MULTISETS AND PERMUTATIONS ON MULTISETS

In informal terms, a multiset is an (unordered) collection of objects in which, in opposition to ordinary sets, elements may occur more than once. For instance, $\{a, a, b, c, c, c\}$ denotes the multiset with the three distinct elements a, b and c, where a occurs twice, b once and c three times. We may also write $\left\{a^2, b, c^3\right\}$ for this multiset. Multisets arise naturally in many different areas of mathematics. For example, the prime factorization of an integer leads to the multiset of its prime factors:

$$180 = 2^2 \cdot 3^2 \cdot 5^1 \text{ leads to } P_{180} = \{2, 2, 3, 3, 5\}.$$

Other simple examples can be found within the field of probability theory, when one wants to keep track of the outcomes of a repeated experiment. For instance, let us imagine an experiment where a die is thrown ten times in a row and we would like to describe how often every side of the die turned up but are not interested in the order of the outcomes. Then the multiset of outcomes could possibly be $\{1, 2, 2, 2, 4, 4, 5, 5, 5, 6\}$. Richard Dedekind is thought to be the first to utilize the concept of multisets in his short monograph *Was sind und was sollen die Zahlen?* [18] in 1888. The term *multiset* however is much younger, it was introduced by the Dutch mathematician de Bruijn in the 1970's. Earlier, these objects were referred to as heaps, bags or also weighted sets.
This section gives a short introduction to multisets and extends the concept of permutations on sets to permutations on multisets, following the presentation in Section 1.2 of Stanley's *Enumerative combinatorics 1* [49]. For a comprehensive survey over the field of multiset theory and an overview of the applications of multisets in mathematics, physics and computer science, see e.g. [9] and [52].

Let us start with a formal definition of multisets:

Definition 2.2.1. *A* ***multiset*** M *on a set* S *is a pair* (S, m)*, where* $m : S \to \mathbb{N}$ *is a map from* S *to the set of natural numbers. For* $s \in S$*, one regards* $m(s)$ *as the number of repetitions of* s *and one calls* $m(s) = m_s$ *the* ***multiplicity*** *of* s *in* M*. A multiset is called* ***regular*** *if* $m_s = m$ *for all* $s \in S$ *and some* $m \in \mathbb{N}^*$*, i.e. if all elements occur the same number of times. For finite sets* $S = \{x_1, x_2, \ldots, x_n\}$ *respectively* $S = \{1, 2, \ldots, n\}$*, the integer* $\sum_{i \in [n]} m(x_i)$ *respectively* $\sum_{i \in [n]} m_i$ *is called the* ***cardinality***, ***size*** *or* ***number of elements*** *of* M*, denoted by* $|M|$*. If* $|M| = k$*, we call* M *a* k***-multiset****.*

For finite multisets, we write $M = \{1^{m_1}, 2^{m_2}, \ldots, n^{m_n}\}$. Sometimes we shall also write $M = [n]_{m_1 m_2 \ldots m_n}$. This notation reduces to $M = [n]_m$ for regular multisets $M = \{1^m, \ldots, n^m\}$.

The first problem that arises within multiset theory is to count the number of multisets of a given length k on a set S with n elements (respectively the set $[n]$). For this purpose we slightly modify the definition of binomial coefficients in the following way:

- The *binomial coefficient*

  $$\binom{n}{k} = \frac{n!}{k!(n-k)!},^{3}$$

  reads "n choose k" and denotes the number of subsets of $[n]$ with k elements. Such subsets are also called k*-compositions of* $[n]$ *without repetitions*.

- The *multiset coefficient*[4]

  $$\left(\!\binom{n}{k}\!\right),$$

  reads "n multichoose k" and counts the number of k*-compositions of* $[n]$ *with repetitions*, i.e. the number of possibilities of choosing k elements amongst n elements, disregarding the order and allowing repeated elements.

It is clear that a k-composition of $[n]$ with repetitions is a multiset on $[n]$ with k elements.

Theorem 2.2.2. *For all integers* n *and* k *it holds that*

$$\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}.$$

Proof. This result can easily be shown geometrically, using the idea of "stars and bars" (or "balls in boxes"), see [49] for this and other combinatorial proofs of this identity. There are $\binom{n+k-1}{k}$ different sequences consisting of k stars and $(n-1)$ vertical bars. An example of such a sequence for $k=5$ and $n=8$ is given by:

$$\star \mid\mid \star \mid\mid\mid \star\star \mid \star \mid$$

The $(n-1)$ bars divide the k stars into n compartments (some of these compartments may be empty). The number of dots in the i-th compartment shall then correspond to m_i, the multiplicity of i. If $m_i = 0$, this simply means that the element i does not occur in the multiset. Thus the sequences with k stars and $(n-1)$ bars correspond to the k-multisets on $[n]$. In the example given above, the corresponding multiset would be $\{1,3,6,6,7\}$. □

We shall now give a new interpretation of binomial coefficients that will lead to their generalization to multinomial coefficients. This will then suggest a generalization of the concept of permutations from ordinary sets to multisets.

- The *binomial coefficient*

  $$\binom{n}{k}$$

  is the number of possibilities of assigning every element of the set $[n]$ to one of two categories so that the first category C_1 has exactly k elements and the second category C_2 has $(n-k)$ elements.

3 See e.g. [49] for several proofs of this identity.
4 Not to be confounded with the multinomial coefficient that will be introduced later on.

- The *multinomial coefficient*

$$\binom{n}{a_1, a_2, \ldots, a_m},^5$$

where $a_1 + a_2 + \ldots a_m = n$, denotes the number of ways of assigning each element of $[n]$ to one of m categories $C_1, C_2, \ldots, C_m$ so that there are exactly a_i elements in the category C_i.

The following picture is commonly used to illustrate the meaning of multinomial coefficients: the elements of $[n]$ correspond to n distinguishable balls and the categories $C_1, C_2, \ldots, C_m$ correspond to m distinguishable boxes. Then $\binom{n}{a_1, a_2, \ldots, a_m}$ corresponds to the number of possibilities of placing the balls into the boxes so that the i-th box contains exactly a_i balls.
Multinomial coefficients can easily be computed using binomial coefficients:

Theorem 2.2.3. *For every sequence* $a_1, a_2, \ldots, a_m$ *of positive integers with* $\sum_{i=1}^{m} a_i = n$ *it holds that*

$$\binom{n}{a_1, a_2, \ldots, a_m} = \frac{n!}{a_1! a_2! \ldots a_m!}. \tag{2.1}$$

Proof. This proof is again taken from Section 1.2 in [49]. There are $\binom{n}{a_1}$ ways of placing a_1 elements of $[n]$ into the first category, then there are $\binom{n-a_1}{a_2}$ ways of placing a_2 of the remaining $(n - a_1)$ elements into the second category, $\binom{n-a_1-a_2}{a_3}$ ways of placing a_3 of the remaining $(n - a_1 - a_2)$ elements into the third category, etc. This leads to:

$$\begin{aligned} \binom{n}{a_1, a_2, \ldots, a_m} &= \binom{n}{a_1}\binom{n-a_1}{a_2}\cdots\binom{n-a_1-\ldots-a_{m-1}}{a_m} \\ &= \frac{n!}{a_1!(n-a_1)!}\frac{(n-a_1)!}{a_2!(n-a_1-a_2)!}\cdots \\ &\quad \cdots\frac{(n-a_1-\ldots-a_{m-1})!}{a_m!(n-a_1-\ldots-a_{m-1}-a_m)!}, \end{aligned}$$

which reduces to the identity (2.1). □

Multinomial coefficients can also be interpreted in terms of *permutations on multisets*. In analogy to Definition 2.1.1, we have the following:

Definition 2.2.4. *A* ***multiset-permutation*** *of the* k*-element multiset* $M = ([n], m)$ *is a sequence* $p_1 p_2 p_3 \ldots p_k$ *containing every element* i *of* $[n]$ *exactly* m_i*-times.*

We can also say that a permutation of a multiset M is a linear ordering of its "elements".

Example 2.2.5. There are 12 permutations on the multiset $M = \{1, 2, 3, 3\}$:

1233, 1323, 1332, 2133, 2313, 2331, 3123, 3132, 3213, 3231, 3312, 3321.

5 For the case $m = 2$ that corresponds to binomial coefficients we write $\binom{n}{k}$ instead of $\binom{n}{k, n-k}$.

In the following, we shall write $\mathcal{S}_M$ for the set of all permutations on the multiset M and S_M for its cardinality. In the special case of regular multisets, i.e. $M = [n]_m$, we write $\mathcal{S}_{n,m}$ and $S_{n,m}$ instead. Then the following result is clear:

Theorem 2.2.6. *For a multiset* $M = \{1^{m_1}, 2^{m_2}, \ldots, n^{m_n}\}$ *with* k *elements it holds that*

$$S_M = \binom{k}{m_1, m_2, \ldots, m_n}.$$

Proof. If the element i appears in the j-th position of the permutation, i.e. $p_j = i$, then we put the element $j \in [n]$ into the i-th category. □

In analogy to Definition 2.1.3, we can also define permutations on multisets as certain maps.

Definition 2.2.7. *A* ***multiset-permutation*** *of the* k*-element multiset* $M = ([n], m)$ *is a surjective map* $f : [k] \to [n]$ *where* $|\{j \in [k] : f(j) = i\}| = m_i$ *for all* $i \in [n]$.

As we can see from the above definition, we have lost injectivity. In particular, this means that we loose the group structure on the set of all permutations on a given multiset. Thus all group theoretical results are no longer valid for permutations on multisets and concepts such as the cycle representation are no longer possible. Describing permutations with the help of the list of all inversions no longer makes sense either. This means that permutations on multisets can no longer be represented with the help of their functional graph or their permutation graph. However, the other graphical representations described in the previous section are still possible, as shown in Figure 2.2 for the multiset-permutation $p = (31421214)$ on the 8-element multiset $\{1, 1, 1, 2, 2, 3, 4, 4\}$.
Again, permutations on multisets can be described with the help of binary matrices. Generalizing the ideas of the previous section, we see that a multiset-permutation p of the k-element multiset $M = ([n], m)$ can be represented by the $n \times k$-matrix A given by:

$$A_p(j, i) = \begin{cases} 1 & \text{if } p_i = j, \\ 0 & \text{otherwise.} \end{cases}$$

This means that every column of A has exactly one entry equal to 1 and the $i - th$ row has exactly m_i entries equal to 1. Analogeously, we can define the transpose matrix B_p.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 1 & 4 & 2 & 1 & 2 & 1 & 4 \end{pmatrix}$$

two-line-representation

$$\begin{pmatrix} 3 & 1 & 4 & 2 & 1 & 2 & 1 & 4 \end{pmatrix}$$

one-line-representation

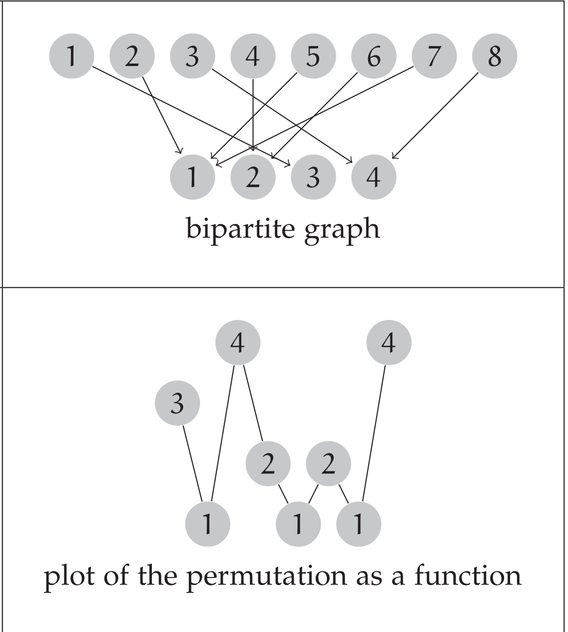


Figure 2.2: Two ways of representing the multiset-permutation $p = (31421214)$ with the help of graphs.

Example 2.2.8. In our running example, the two possible corresponding permutation matrices are given by:

$$A_p = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad B_p = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

2.3 GENERATING FUNCTIONS

If the answer to a certain (most often combinatorial) question is a sequence $a_0, a_1, a_2, \dots$ and we want to know how this sequence can be computed, generating functions can be a very powerful tool. In the ideal case they help to determine an explicit formula for a_n or, if this is not possible, often lead to a recurrence formula. In his book *generatingfunctionolgy* [56] Herbert Wilf describes generating functions in the following way:

> A generating function is a clothesline on which we hang up a sequence of numbers for display.

Definition 2.3.1. *Given a sequence $a_1, a_2, a_3, \dots$ its **(ordinary) generating function** is the formal power series*

$$A(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Example 2.3.2. We shall give a simple example to illustrate this concept.
Problem: How many full binary trees are there with $(n+1)$ leaves? Or equivalently, how many expressions with n pairs of correctly matched parentheses are there? This sequence of numbers shall be denoted by $(a_n)_{n\in\mathbb{N}}$.
Recursion: An expression with $(n+1)$ pairs of parentheses can be written in the following way:

$$(\underbrace{(\qquad\qquad)}_{k \text{ pairs}}\underbrace{(\qquad\qquad\qquad)}_{(n-k) \text{ pairs}}),$$

where $0 \leqslant k \leqslant n$. This leads to the simple recurrence formula:

$$a_{n+1} = \sum_{k=0}^{n} a_k a_{n-k}, \text{ with } a_0 = 1.$$

Generating function: The generating function $A(x) = \sum_{n=0}^{\infty} a_n x^n$ of this sequence will help us find an explicit formula for a_n. Note that $\sum_{k=0}^{n} a_k a_{n-k}$ is the coefficient of x^n in $A(x) \cdot A(x) = A^2(x)$.

$$\begin{aligned}
\sum_{n=0}^{\infty} a_{n+1}x^{n+1} &= \sum_{n=0}^{\infty} x^{n+1}\left(\sum_{k=0}^{n} a_k a_{n-k}\right) \\
A(x) - a_0x^0 &= x \cdot \sum_{n=0}^{\infty} x^{n}\left(\sum_{k=0}^{n} a_k a_{n-k}\right) \\
A(x) - 1 &= x \cdot A^2(x)
\end{aligned}$$

Solving this quadratic equation leads to two solutions for $A(x)$:

$$A_{1,2}(x) = \frac{1 \pm \sqrt{1-4x}}{2x}.$$

$A_1(x)$ cannot be the solution we are interested in since it cannot be expanded into a power series at $x = 0$. Therefore, a_n is the coefficient of x^n in $A_2(x)$:

$$\begin{aligned}
a_n &= [x^n]\frac{1-\sqrt{1-4x}}{2x} = -\frac{1}{2}[x^n]\frac{\sqrt{1-4x}}{x} \\
&= -\frac{1}{2}[x^{n+1}]\sqrt{1-4x} = -\frac{1}{2}\binom{1/2}{n+1}(-4)^{n+1} \\
&= (-1)^n 2^{2n+1}\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(\frac{1}{2}-n\right)}{(n+1)!} \\
&= 2^n\frac{1\cdot 1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{(n+1)!} = \frac{2^n(2n)!}{(n+1)!2\cdot 4\cdot\cdots\cdot 2n} \\
&= \frac{(2n)!}{(n+1)!n!} = \frac{1}{n+1}\binom{2n}{n}.
\end{aligned}$$

This is the sequence of Catalan numbers that will be introduced in Section 3.2.2.

For a sequence of numbers that depends on more than one parameter, i.e. a sequence $s(n_1, n_2, \ldots, n_m)$ where $n_1, n_2, \ldots, n_m$ are integers, we introduce the multivariate generating function:

$$A(x_1, x_2, \ldots, x_m) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_m=0}^{\infty} s(n_1, n_2, \ldots, n_m) {x_1}^{n_1} {x_2}^{n_2} \ldots {x_m}^{n_m}.$$

A remarkable tool for solving certain kinds of functional equations is the Lagrange Inversion Formula (LIF). The type of functional equations that LIF can help us solve is

$$u = t\phi(u), \tag{2.2}$$

where ϕ is a given function in u and we want to determine u as a function of t.

Theorem 2.3.3 (The Lagrange Inversion Formula)**.** *Let $f(u)$ and $\phi(u)$ be formal power series in u with $\phi(0) \neq 0$. Then there is a unique formal power series $u = u(t)$ that satisfies (2.2). The value $f(u(t))$ of f at that root $u = u(t)$, when expanded in a power series in t at $t = 0$, satisfies*

$$[t^n]f(u(t)) = \frac{1}{n}[u^{n-1}]\left(f'(u)\phi(u)^n\right).$$

For more details and proofs see e.g. Chapter 5 in [56] or Section 5.4 in [50]. The notation used here is borrowed from [56].

2.4 GENERATING TREES

When proving enumeration formulae for n-permutations avoiding certain patterns, we will often use an enumeration technique called *generating trees* that was introduced in the study of Baxter permutations and has been systematized by Julian West in [55] in the context of pattern avoidance.

Definition 2.4.1. *A **generating tree** is a rooted, labelled tree having the property that the labels of the children of any given node m can be determined by the label of m itself.*

Therefore, every generating tree can uniquely be described by a recursive definition, the so-called **rewriting rule**, consisting of:

- the label of its root (corresponds to the basis of an induction),
- a set of succession rules, explaining how to derive the number of children and their labels for any given parent node, when the parent's label is given (corresponds to the induction step).

Example 2.4.2 (The full binary tree)**.** This is the simplest example of a generating tree: every node has exactly two children. The only information held by the label of a node is how many children it has and this number is the same for all nodes. The rewriting rule is therefore given by:

Root: (2)
Rule: $(2) \rightarrow (2)(2)$.

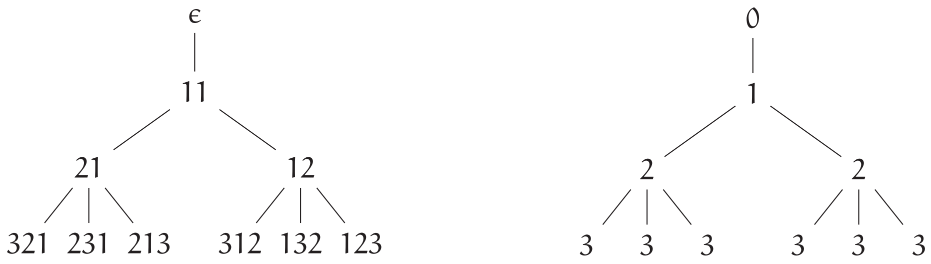


Figure 2.3: Generating tree of permutations up to the level 3. In the tree on the left-hand side nodes are labelled by the permutations themselves, in the tree on the right-hand side nodes are labelled by the length of the permutations.

Example 2.4.3 (The full k-ary tree)**.** Similarly as in the preceding example, the rewriting rule of the full k-ary tree (every node has exactly k children) is given by:

$$\begin{aligned} &\text{Root:} \quad (k) \\ &\text{Rule:} \quad (k) \quad \rightarrow \underbrace{(k)(k)\ldots(k)}_{k \text{ times}} = (k)^k. \end{aligned}$$

Example 2.4.4 (The generating tree of unrestricted permutations)**.** Permutations without any restrictions can be described with the help of a simple generating tree (see [14]), nodes on the n-th level corresponding to n-permutations. The branch leading to a specific node reflects the choices made in the construction of the permutation. See Figure 2.3. The root is indexed by the empty permutation ϵ and a node labelled by a certain permutation p of length n has $(n+1)$ children respectively labelled by the $(n+1)$ permutations that can be obtained by inserting the element $(n+1)$ into p. This tree is obviously isomorphic to the tree where the nodes are labelled by the length of the permutations. This tree is given by the rewriting rule:

$$\begin{aligned} &\text{Root:} \quad (0) \\ &\text{Rule:} \quad (n) \quad \rightarrow (n+1)^{n+1}. \end{aligned}$$

Generally, one is interested in counting the number of nodes on the n-th level (the nodes that are at the distance n of the root) and in some cases one also wants to know their distribution by labels. This can be done with the help of (multivariate) generating functions. In the next chapter we will see how this technique can be applied to restricted permutations and give a first non-trivial example of a generating tree, namely the one of 123-avoiding permutations. See the second proof of Theorem 3.2.5. For a survey of generating trees arising in the context of pattern avoidance in permutations, see [55]. Simple examples are the Catalan tree, the Fibonacci tree and the Schröder tree. For more refined examples, where generating trees with two labels were used, see the work of Bousquet-Mélou [14].

2.5 THE KERNEL METHOD

When using multivariate generating functions, it is not seldom the case that one obtains (systems of) functional equations that do not seem to have a unique solution at first sight because they involve "too many variables". In this case, the so called *Kernel method* which uses the fact that it is a priori known that a power series representation of the involved generating functions must be possible, can be of great help. It was first used as a "trick" in the 1970's and was probably discovered by several persons independently. One possible origin can be found in Exercise 2.2.1-4 of Knuth's *Fundamental algorithms* [30], the first volume of *The Art of computer programming*, and will be presented shortly. Some other examples of applications of the *Kernel trick* can e.g. be found in [40]. It is mainly thanks to Bousquet-Mélou, Flajolet, Petkovšek et.al. that this trick was extended to a method, see [15] and [7] for the special case of generating functions that arise when using generating trees. We shall give a brief overview of the results that will be needed in the following chapters, for details and proofs see the original literature.

Knuth's Exercise 2.2.1-4 reads as follows: Find a simple formula for a_n, the number of permutations on n elements that can be obtained with a stack[6].
A *stack* is a linear list for which all insertions and deletions are made at one end of the list. Stacks are also called LIFO-(last-in-first-out-)lists. We can represent stacks with the help of a railway switching network as shown in Figure 2.4. On the right side we imagine n railroad cars, numbered from 1 to n from left to right - this corresponds to the input. On the left hand side stands the output, a reordering of the railroad cars corresponding to a permutation of n elements.
With this simple switching network two types of operations are possible:

- S: move a car from the input into the stack,
- X: move a car from the stack into the output.

Applying a sequence of operations consisting of n S's and n X's to the input defines the output, a permutation of the n elements. Note however that it is neither possible to perform the operation S if the input side is empty nor to perform the operation X if the stack is empty. Therefore the "admissible" sequences of n S's and n X's are exactly those where the number of X's never exceeds the number of Y's when the sequence is read from left to right[7].
The solution presented in [30] solves a more general problem namely *Bertrand's ballot problem*[8] which is the following question: In an election with n votes where candidate S receives m votes more than candidate X, what is the probability that S is strictly ahead of X throughout the entire count? To answer this question we define $g(n, m)$ to be the number

6 The following Exercise 2.2.1-5 shows that the permutations that can be obtained with the help of a stack are exactly the 231-avoiding permutations. See Section 3.1 for the definition of pattern avoidance. The two Exercises 2.2.1-4 and 2.2.1-5 can therefore be seen as the birthplace of both the kernel method and the field of pattern avoidance.

7 Such sequences are also called Dyck words, see Definition 4.4.4. For another way to count such words, using an idea of Désiré André, see Section 4.4.

8 Named after Joseph Louis François Bertrand who introduced it in [8] in 1887.

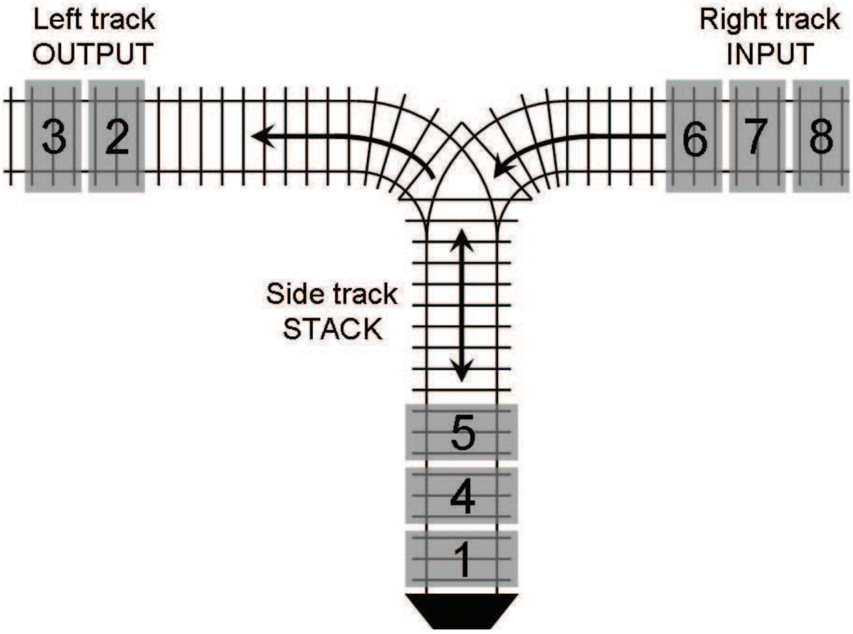


Figure 2.4: A stack of length 8 represented as a railway switching network with 8 railway cars.

of sequences of length n consisting of the letters S and X in which the number of X's never exceeds the number of S's when counting from the left and in which there are m more S's than X's in total. Then the number we are interested in is $g(2n, 0) = a_n$.
Obviously $g(n, m) = 0$ if $(m + n)$ is an odd number and $g(0, m) = \delta_{0,m}$. One easily obtains the following recurrence relation for these numbers:

$$\begin{aligned} g(n+1, m) &= g(n, m-1) + g(n, m+1) \text{ for all } n \geqslant 0, m \geqslant 1, \\ g(n+1, 0) &= g(n, 1) \text{ for all } n \geqslant 0. \end{aligned} \tag{2.3}$$

The bivariate generating function of this sequence is defined as follows:

$$G(u) := G(t, u) = \sum_{n,m\geqslant 0} g(n, m) u^m t^n.$$

Applying the recurrence relation (2.3) to this generating function yields:

$$\begin{aligned} G(u) &= t \sum_{m,n\geqslant 0} g(n+1, m) u^m t^n + \sum_{m\geqslant 0} g(0, m) u^m \cdot 1 \\ &= t \left(\sum_{n\geqslant 0} g(n, 1) t^n + \sum_{\substack{m\geqslant 1 \\ n\geqslant 0}} \left(g(n, m-1) + g(n, m+1) \right) u^m t^n \right) + 1 \\ &= t \left(G(0) + uG(t, u) + \frac{1}{u} \left(G(t, u) - uG(0) - G(0) \right) \right) + 1 \\ &= t \left(uG(t, u) + \frac{1}{u} \left(G(t, u) - G(0) \right) \right) + 1 \end{aligned}$$

Rewriting this equation finally gives:

$$G(u) = \frac{tG(0) - u}{t(u^2 + 1) - u}. \tag{2.4}$$

The above equation involves the two unknown functions $G(u)$ and $G(0)$ ("two variables"), it is therefore a priori not clear how it is supposed to be solved. Setting $u = 0$ does not lead anywhere, but we can have a closer look at the denominator $K(t, u) = t(u^2 + 1) - u$ which will be called the *kernel* of (2.4). $K(u)$ has two roots, namely:

$$u_{1,2}(t) = \frac{1 \pm \sqrt{1 - 4t^2}}{2t}.$$

The trick is now the following: we have to find some value of $G(0)$ such that $G(u)$ given by (2.4) has a power series expansion in t and u ($G(u)$ was defined that way). One notes that $\lim_{t \to 0} u_2(t) = 0$ and $u_2(t)$ can be expanded into a power series. Therefore, for any fixed t, $u = u_2(t)$ causes the denominator $K(t, u)$ to vanish. Since there may not be any singularities, we may pick $G(0)$ so that the numerator also vanishes when $u = u_2(t)$. One concludes that $G(0) = \frac{u_2(t)}{t}$.

One can now derive a simple form for all the coefficients $g(n, m)$ of $G(t, u)$. For the coefficients $g(2n, 0)$ of $G(0)$ this can be done in the same way as demonstrated in Section 2.3 and we obtain:

$$a_n = g(2n, 0) = \frac{1}{n+1}\binom{2n}{n},$$

which again are the Catalan numbers.

We shall now briefly give an outline of the *Kernel method* in the way it was elaborated by Bousquet-Mélou, Flajolet et.al. in [7] for the case of generating functions of generating trees. The authors analysed different types of generating trees, depending on structural properties of their rewriting rules, leading to different types of generating functions: rational, algebraic and transcendent ones. The cases in which we will use the Kernel method always belong to the category of generating trees given by rewriting rules of the *factorial form*. The rewriting rules we are going to deal with all have the special form:

Root: (r_0)

Rule: $(m) \quad \to [k, m-1] \cup (m + A),$

where k is an integer with $1 \leqslant k \leqslant m$ and A[9] is a k-element (multi)set with elements in $\mathbb{Z}$. By $[k, m-1] \cup (m + A)$ we mean the nodes $(k)(k + 1) \dots (m-2)(m-1)(m+a_1) \dots (m+a_k)$ where the a_i are the elements of A. Note that a node labelled with m then has exactly m children. We denote by $f(n, m)$ the number of nodes at level n with label m, respectively the number of branches of length n ending at a node labelled with m. The multivariate generating function we are going to consider is $F(t, u) = \sum_{n,m \geqslant 0} f(n, m) u^m t^n$. Using the rewriting rule given above, one obtains the following functional equation for $F(t, u)$:

$$F(t, u)\left(1 + \frac{t}{1-u} - t\sum_{i=1}^{k} u^{a_i}\right) = 1 + \frac{tF(t, 1)}{1-u}. \tag{2.5}$$

9 If we consider the branches of the generating tree in question as walks over the integer half-line, A specifies the allowed supplementary jumps, see [7] for an explanation of these terms.

Multiplying Equation (2.5) by $(1-u)$ leads to an equation of the type $K(t,u)F(t,u) = R(t,u)$, where $K(t,u)$ is the *kernel* of the equation:

$$K(t,u) = (1-u) + t - t(1-u)\sum_{i=1}^{k} u^{a_i}. \tag{2.6}$$

As in Knuth's example, we now want to couple t and u in a way that the left-hand side of (2.5) vanishes. The kernel is a polynomial of degree $a+1$ in u, where a is the maximal element of A and thus admits $a+1$ solutions which are algebraic functions of t. These solutions can be expanded into Puiseux series (series involving fractional exponents) around any point. Expanded around 0 they can be classified as follows:

- u_0 is a power series in t with constant term 1,
- $u_1, \ldots, u_a$ are Laurent series in $t^{1/a}$.

Note that $F(t,u)$ is a series in t with polynomial coefficients in u (at every level, there is only a finite number of nodes). Therefore the first root u_1, having no negative exponents, may be substituted for u in $F(t,u)$. If we do this in Equation (2.5), the right hand side vanishes, giving us a linear equation for $F(t,1)$. Once $F(t,1)$ is known, we can also determine $F(t,u)$ and have solved the problem.

A first application of the Kernel method can be found in the second proof of Theorem 3.2.5.

3 RESTRICTED PERMUTATIONS

This chapter gives an overview over the most important results in the field of restricted permutations and provides the necessary basis for the results concerning permutations on multisets that will be presented in the following chapters. Starting with the definition of pattern avoidance and a graphical illustration of this concept, we shall then turn to enumerative questions. The case of patterns of length three shall be treated exhaustively and an overview of the main results for patterns of length four given. Finally, we shall end this chapter with a more general result, the so-called Stanley-Wilf conjecture that will be presented together with its entire proof.

3.1 DEFINITION: PATTERN AVOIDANCE

In the previous chapter we defined inversions of permutations, see Definition 2.1.6. These were pairs (p_i, p_j) of elements with $i < j$ that disturbed the natural order $1, 2, \ldots, n-1, n$. This means that these pairs could occur anywhere in the permutation but always had one thing in common: the element to the left was always larger than the one to the right. In other words, the elements p_i and p_j related to each other in the same way as the elements 2 and 1 (or e.g. 5 and 3, 19 and 2 etc.). This concept can now be generalized to k-tuples of elements of a permutation and leads to the following definition:

Definition 3.1.1. *Let* $q = (q_2 q_2 \ldots q_k) \in S_k$ *be a permutation of length* $k \leqslant n$. *We say that the permutation* $p = (p_1 p_2 \ldots p_n) \in S_n$ ***contains*** q ***as a pattern*** *if we can find* k *entries* $p_{i_1}, p_{i_2}, \ldots, p_{i_k}$ *with* $i_1 < i_2 < \ldots < i_k$ *such that* $p_{i_a} < p_{i_b} \Leftrightarrow q_a < q_b$, *i.e. if we can find a subsequence of* p *that is order-isomorphic to* q. *If there is no such subsequence we say that* p ***avoids the pattern*** q.

Example 3.1.2. The permutation $p = 23718465$ contains the pattern 312, since the entries 714 (or several other examples) form a 312-pattern. As a matter of fact, this permutation contains all possible patterns of length three. This is different for patterns of length four: p contains the pattern 2134 as is shown by the entries 3146, but p avoids the pattern 4321 since it contains no decreasing subsequence of length four.

Recalling the Definition 2.1.7 of permutation matrices in Section 2.1, it can easily be seen that the definition of pattern avoidance is directly translated from permutations to permutation matrices. With the help of the following definition we get a much more *graphical* understanding of what it means that a permutation p avoids or contains a certain pattern q.

Definition 3.1.3. *Let* $P \in \{0,1\}^{n \times n}$ *and* $Q \in \{0,1\}^{k \times k}$ *be two permutation matrices with* $k < n$. *We then say that* P ***contains*** Q *if there is a submatrix* $\tilde{Q}$

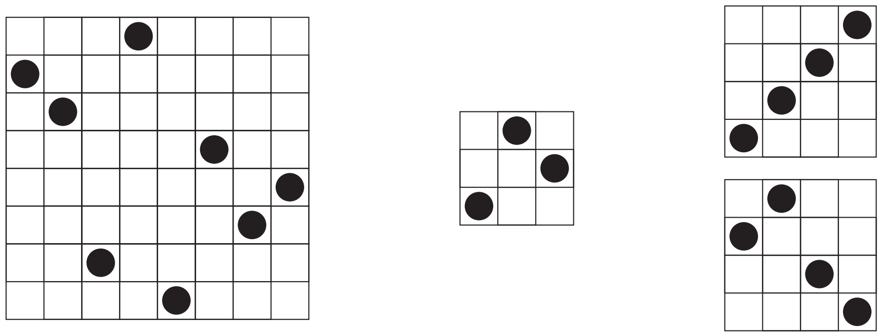


Figure 3.1: Permutation matrices and pattern avoidance visualized with the help of square grids. From left to right: 23718465, 312, 4321 (top) and 2134 (bottom).

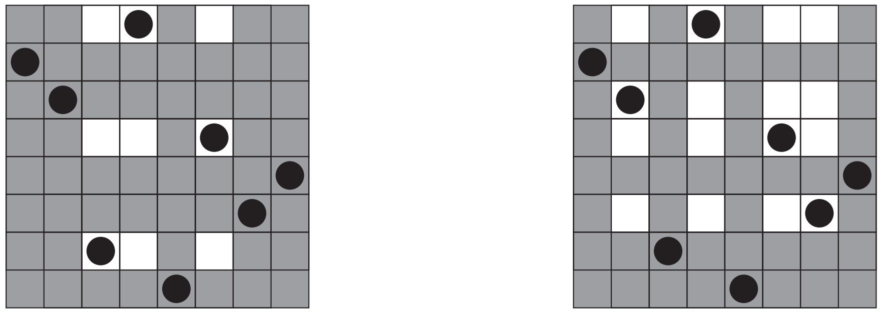


Figure 3.2: Containing patterns in permutation matrices: representation with the help of square grids. Deleting the rows and columns marked in gray leads to the desired patterns (on the left hand side the pattern 312, on the right hand side the pattern 2134.

of P *with size* $k \times k$ *that is a copy of* Q*, i.e. it holds* $Q_{i,j} = 1 \iff \tilde{Q}_{i,j} = 1$*. If there is no such submatrix* $\tilde{Q}$*, we say that* P ***avoids*** Q*.*

In other words, a permutation matrix P contains a matrix-pattern Q if, by deleting some rows and some columns of P, we obtain a smaller matrix that looks like Q. Let us demonstrate this on a simple example.

Example 3.1.4. We represent an $n \times n$-permutation matrix P by a square grid in which we place a black pawn in the i-th row and the j-th column if $P_{i,j} = 1$. All the other squares remain empty. Then the permutation $p = 23718465$ of the example we gave above is represented by the 8×8-grid in Figure 3.1. In this same figure we have also represented the patterns 312, 2134 and 4321. It is now very easy to understand *graphically* which patterns are contained in and which patterns are avoided by p. Figure 3.2 shows which rows and columns have to be deleted in order to obtain copies of the patterns 312 and 2134. That no copy of the filled grid representing 4321 can be found, is obvious.

For inversions, i.e. 21-patterns, there are several strong results, amongst others one can describe the number of permutations with a given number of inversions (see Section 2.1 in [12]). For patterns in general, the question "How many permutations of length n are there that contain a give pattern q m times?" appears to be much more difficult and we concentrate on the case where $m = 0$. The central question in pattern avoidance will therefore be:

"How many permutations of length n avoid the pattern q?"

Definition 3.1.5. *We denote by* $\mathcal{S}_n(q) = \{p \in S_n | p$ *avoids* $q\}$ *the set of all n-permutations avoiding* q *and by* $S_n(q) = |\mathcal{S}_n(q)|$ *its cardinality. For sets* T *of permutations we write* $\mathcal{S}_n(T) = \{p \in S_n | p$ *avoids* t *for all* $t \in T\}$ *for the set of all* n*-permutations avoiding all elements of* T.

Obviously the only permutation avoiding the pattern 12 is $n(n-1)\dots 21$ and the only one avoiding 21 is $12\dots(n-1)n$. Thus $S_n(12) = S_n(21) = 1 \ \forall n \in \mathbb{N}$.

3.2 AVOIDING PATTERNS OF LENGTH THREE

In the previous section we saw that $S_n(q)$ was the same for all patterns q of length 2. We therefore say that the patterns 12 and 21 are equivalent. In the following definition we shall generalize this concept to patterns of arbitrary length.

Definition 3.2.1. *Two permutations* $q, q' \in \mathcal{S}_k$ *are said to be* ***Wilf[1]-equivalent*** *or to lie in the same* ***Wilf-class*** *if* $S_n(q) = S_n(q')$ *holds for all* $n \in \mathbb{N}$. *We then write* $q \sim q'$ *or* $T \sim T'$ *for sets of patterns* T, T' *for which* $S_n(T) = S_n(T')$ *holds for all integers.*

For patterns of length three, we will show that all six possible patterns are Wilf-equivalent.

First note that if p avoids the pattern q, then its reverse p^r avoids the reverse pattern q^r and its complement p^c avoids q^c (recall Definition 2.1.9 in Chapter 1). Since

$$(123)^r = (123)^c = (321),$$

$$(132)^r = (231), (132)^c = (312) \text{ and } ((132)^r)^c = (213),$$

we obtain

$$S_n(123) = S_n(321) \text{ and}$$

$$S_n(132) = S_n(231) = S_n(312) = S_n(213) \forall n \in \mathbb{N}.$$

We can even show that these two Wilf-equivalence classes are identical and thus $S_n(q) = S_n(132)$ for all patterns q of length three.

3.2.1 *One Wilf-equivalence class for single patterns of length three*

Theorem 3.2.2. $S_n(123) = S_n(132)$ *holds for all integers* n.

Proof. Several proofs have been given for this first non-trivial result in pattern avoidance. We shall follow the proof given in [12] which uses the idea of Simion and Schmidt [46]. We shall construct a bijective map f

1 named after Herbert S. Wilf, the well-known combinatorialist and graph theorist, author of e.g. *generatingfunctionology* [56] and co-author of *A=B* [39]. Together with Neil Calkin, Wilf was the founder of *The Electronic Journal of Combinatorics*. See Section 3.4 for the Stanley-Wilf-conjecture that was postulated by H. Wilf and R. Stanley.

from the set of all 132-avoiding permutations to the set of all 123-avoiding permutations.
Given a 132-avoiding permutation p, f keeps all the left-to-right-minima (see Definition 2.1.9) fixed. The remaining elements are filled into the free positions in decreasing order. Note that the sequence of left-to-right-minima always forms a decreasing subsequence, since $p_i < p_j$ for two left-to-right-minima with $i < j$ would be in contradiction with the fact that $p_k < p_j \; \forall k < j$. Thus $f(p)$ consists of two decreasing subsequences, one of which is the sequence of left-to-right-minima, and the other is the decreasing sequence into which the remaining elements were arranged and therefore $f(p)$ is always 123-avoiding.
The obtained permutation $f(p)$ is the only 123-avoiding permutation with the same set and positions of left-to-right-minima as p. First note that the left-to-right-minima of p and $f(p)$ are the same, even though the other entries have possibly[2] been moved. The map f simply rearranges the k entries that are not left-to-right-minima following the rule: if you see a pair of such entries that is not in decreasing order, then swap them. This algorithm terminates at most after $\binom{k}{2}$ steps, since all the possible pairs have by then been checked. Every time two elements are swapped, a smaller entry moves to the right and a larger one to the left, and thus no new left-to-right-minimum can be created. Now note that placing two of the k remaining entries in increasing order would necessarily create a 123-pattern. Indeed, if two elements x and y that are not left-to-right-minima were to form a 12-pattern, then wxy would be a 123-pattern, where w is the left-to-right-minimum closest to x on the left.
In order to prove that f is a bijection, we describe its inverse g. Given a 123-avoiding permutation q, we hold its left-to-right-minima fixed and write the remaining elements from left to right in the following way. At each free position, place the smallest element not yet placed that is larger than the closest left-to-right-minimum on the left of the given position. The obtained permutation $g(q)$ can never contain a 132-pattern. Indeed, if we could find a 132-pattern in it, we could find one that starts with a left-to-right-minimum. By construction, the elements that are larger than any given left-to-right-minimum are written in increasing order and thus a 132-pattern is impossible.
Note again that $g(q)$ is the only 132-avoiding permutation with the given set and positions of left-to-right-minima. If two elements $x < y$ that are larger than the left-to-right-minimum w were placed in decreasing order, then wyx would form a 132-pattern.
This yields $g(f(p)) = p$, i.e. f is indeed a bijection. □

Example 3.2.3. We illustrate the above proof by giving a simple example. Let's start with $p = \underline{7}89\underline{4}5\underline{2}6\underline{1}$ - it can easily be seen that p is 132-avoiding. We have underlined the left-to-right-minima. The remaining elements $8, 9, 5$ and 6 are reordered in decreasing order and thus $f(p) = q = \underline{7}98\underline{4}6\underline{2}5\underline{1}$. The left-to-right-minima are again underlined and the same (and in the same positions) as those of p.
Now, to reconstruct p from q, we do the following. In the first empty

2 It is possible that $p = f(p)$, consider e.g. the case of monotone decreasing permutations of the form $p = n\,(n-1)\ldots 321$

slot between 7 and 4, we place the smallest of the two elements that are larger than 7, i.e. 8. In the next slot we have to place the 9, since this is the only remaining element larger than 7. Between 4 and 2, place the smallest element not yet placed that is larger than 4, that is 5. In the last remaining empty slot, place the last element, i.e. 6. We thus obtain $g(q) = 78945261 = p$.

3.2.2 *Catalan numbers count n-permutations avoiding any single pattern of length three*

We have seen that $S_n(q)$ is equal for all patterns q of length three. Therefore we now only need to compute $S_n(q)$ for one of these patterns.

Definition 3.2.4. *For* $n \in \mathbb{N}$, $c_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{n!(n+1)!}$ *is called the* n*-th Catalan number.*

The Catalan numbers form a very interesting and omnipresent sequence in enumerative mathematics. Leonard Euler was the first to study this sequence in the 18th century when he searched for the number of different ways to divide a polygon into triangles. Later on, the Catalan numbers were studied by Eugéne Charles Catalan[3] who established the connection to parenthesized expressions when he was studying the "Towers of Hanoi"-problem and whose name this sequence carries since then. An immense variety of different combinatorial objects are enumerated by the Catalan numbers; in his "Catalan Addendum" of [50], Richard Stanley currently presents a list of 190 different combinatorial interpretations of Catalan numbers. A few of them are:

- the number of different ways a convex polygon with $n+2$ sides can be partitioned into triangles
- the number of expressions with n pairs of correctly matched parentheses
- the number of Dyck words of length $2n$
- the number of Dyck paths from $(0,0)$ to $(2n,0)$
- the number of full binary trees (every node has either 0 or 2 children) with $n+1$ leaves
- the number of binary trees (every node has 0, 1 or 2 children) on n nodes
- the number of stack-sortable permutations (see Section 2.3) of length n

Other objects that are counted by the Catalan numbers are 132-avoiding permutations.

Theorem 3.2.5. *For all* $n \in \mathbb{N}$ *it holds that* $S_n(132) = c_n$.

Numerous proofs of this result or of the equivalent result, namely that q-avoiding permutations where q is any other permutation of length three are counted by the Catalan numbers, have been given over the last decades. We shall present two of them here.

The first proof of this kind was given in 1973 by Donald Knuth in one of the exercises in [30], where he showed that a permutation is stack-sortable

3 French-Belgian mathematician, 1814-1894.

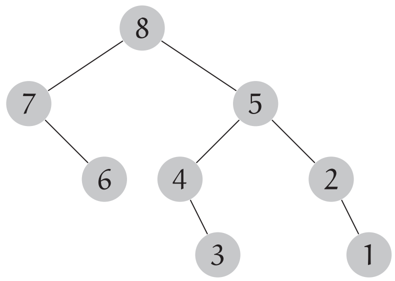


Figure 3.3: The binary graph corresponding to the 132-avoiding permutation 76843521.

if and only if it is 231-avoiding. 132-avoiding permutations can easily be shown to correspond to Dyck words respectively Dyck paths (or equivalently to lattice paths) in a bijective way (see the first chapter in Stanley's enumerative combinatorics 2 [50] as well as the work of Krattenthaler [31]). Another possibility is to describe the connection between binary trees and 231-avoiding permutations (see e.g. the work of Jani and Rieper [27]). With the proof of Theorem 3.2.5 given below, this bijection is pretty straightforward and therefore we shall merely illustrate it with the help of an example. See Figure 3.3.
Note that binary trees can directly be translated into Dyck paths using a depth-first traversal of the trees (also see [50]): moving away from the root corresponds to a NE-step (respectively a letter X in a Dyck word) and moving back towards the root corresponds to a SE-step (respectively a letter Y). Another possibility is to use the structure of 231-avoiding permutations to establish a recurrence relation for the numbers $S_n(321)$ that turns out to be exactly the recursion defining the Catalan numbers. Finally, the methods presented in the Sections 2.4 and 2.5 can be utilized. These two proofs are presented in the following.

First proof of Theorem 3.2.5. See e.g. Theorem 4.6 in [12]. Set $a_n = S_n(132)$. For a given 132-avoiding permutation $p \in \mathcal{S}_n$, suppose the entry n is in the i-th position, i.e. $p(i) = n$. Then all the entries to the left of n must be larger than all the entries to the right of n. Otherwise, i.e. if there was an element x to the left and an element y to the right of n such that $x < y$, a 132-pattern would be created by the sequence xny. Therefore the set of entries to the left of the i-th position must be $\{n-i+1, n-i+2, \dots, n-1\}$ and the set of the entries to the right of n must be $\{1, 2, \dots, n-i\}$. Since p is a 132-avoiding permutation, neither the elements to the left of n nor the elements to the right of n may form a 132-pattern. Therefore there are a_{i-1} allowed possibilities for the order of the elements to the left of n and a_{n-i} possibilities for the order of those to the right. Since n can take any position i between 1 and n, we obtain:

$$a_n = \sum_{i=1}^{n} a_{i-1} a_{n-i}. \tag{3.1}$$

This recursion defines the Catalan numbers as was shown in Section 2.3. Therefore $a_n = \frac{1}{n+1}\binom{2n}{n} = c_n$ which finishes the proof.

□

Second proof of Theorem 3.2.5. Here we carry out the proof sketched by Bousquet-Mélou in [14] for the equivalent result $S_n(123) = c_n$. We make use of the methods presented in the preceding chapter: *generating trees* and the *Kernel method*. It might seem far-fetched to use these tools to prove a result that allows such a simple proof as just seen but it provides a first insight into the methods that will be used later on for more difficult problems.

In Section 2.4 we saw that generating trees can be used to describe permutations without restrictions. The same can be done for permutations avoiding a certain pattern (or a certain set of patterns). It will then be clear that it does not make a difference whether the permutations are on ordinary sets or an (regular) multisets, allowing us to use this method in the following two chapters.

For the tree of permutations avoiding a certain set T of patterns, the root will again be the empty permutation ϵ that avoids all permutations. The level n in this tree should correspond to all T-avoiding permutations of length n (or on n different letters for regular multisets). For a given permutation p at the n-th level, its children are all T-avoiding permutations on the set $[n+1]$ that can be obtained by inserting the element $(n+1)$ somewhere in p. A priori it is neither clear how many children a given permutation has nor how the labelling of the nodes should be done. This will always depend on the set of patterns that are to be avoided.

In the case of 123-avoiding permutations, we observe the following: given an n-permutation p, the element $(n+1)$ may be introduced anywhere before the first 12-pattern. Thus, if a_p denotes the position of the first ascent in p (recall Definition 2.1.9), the element $(n+1)$ may be inserted before the positions $1, 2, \ldots, a_p + 1$. This suggests a labelling of the nodes with $a_p + 1$. A node with label r then has r children. If the element $(n+1)$ is inserted before the first position, i.e. right at the beginning of p, no new ascent is produced and the position of the first ascent will be r. This yields a child with label $r+1$. If $(n+1)$ is placed in any other allowed gap, i.e. before the position i where $i \in \{2, \ldots, r\}$, $i-1$ will certainly be the position of the first ascent in the new permutation (since $(n+1)$ is the largest element). We thus obtain r children with respective labels $2, \ldots, r$. The rewriting rule of the generating tree of 123-avoiding permutations (see Figure 3.4) is therefore:

$$\begin{aligned} &\text{Root:} \quad (1) \\ &\text{Rule:} \quad (r) \quad \to (r+1)(2)\ldots(r). \end{aligned} \tag{3.2}$$

Using the notation introduced by Bousquet-Mélou et.al. in [7] and presented in Section 2.5, we have $k = 2$ and $A = \{0, 1\}$.

We are now interested in counting the number of nodes at each level. We therefore introduce the bivariate generating function

$$S(u) := S(t,u) = \sum_{p \in \mathcal{S}(123)} t^{l(p)} u^{a_p+1} = \sum_{n,r \geqslant 0} s(n,r) t^n u^r,$$

where $n = l(p)$ denotes the length of p or rather its height in the generating tree and $r = a_p + 1$ is its label; $s(n,r)$ is the number of permutations at height n and with label r. Thus we want to compute the coefficients of t

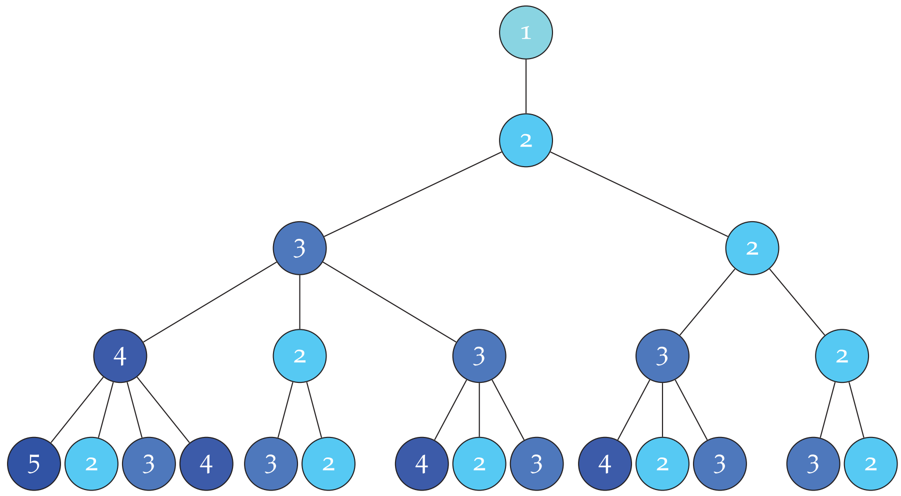


Figure 3.4: Nodes labelled by $a_p + 1$, where a_p is the position of the first ascent.

in $S(1) = S(t, 1)$. With the help of the rewriting rule (3.2) we obtain the following functional equation:

$$\begin{aligned} S(u) &= u + t \sum_{n,r \geqslant 0} t^n (u^2 + \ldots + u^r + u^{r+1}) \\ &= u + tu^2 \frac{S(u) - S(1)}{u-1}. \end{aligned}$$

This equation can equivalently be written as

$$S(u) = \frac{tu^2 S(1) - u^2 - u}{tu^2 - u + 1} = \frac{u(tuS(1) - u - 1)}{tu^2 - u + 1}, \tag{3.3}$$

its kernel being $K(t, u) = tu^2 - u + 1$. This kernel could also have been obtained directly using the formula in Equation (2.6). Note the similarity to the Kernel $tu^2 - u + t$ in Knuth's example presented in Section 2.5. Again, $K(t, u)$ has two roots, namely $u_{1,2}(t) = \frac{1 \pm \sqrt{1-4t}}{2t}$. The second root $u_2(t)$ can be expanded into a power series at $t = 0$ and may therefore be plugged into (3.3), yielding:

$$K(t, u_2(t)) \cdot S(u_2(t)) = u_2(t) \left(tu_2(t) S(1) - u_2(t) - 1 \right) = 0.$$

In particular, this means that $S(1) = u_2(t) = \frac{1-\sqrt{1-4t}}{2t}$. As we know from Section 2.3 this is the generating function of Catalan numbers. We have thus shown that $[x^n]S(1) = c_n$, providing another proof of the result $S_n(123) = c_n$. □

Corollary 3.2.6. *Let* q *be any permutation of length three. Then* $S_n(q) = c_n$ *for all integers* n.

Proof. Follows directly from the Theorems 3.2.2 and 3.2.5. □

3.2.3 *Avoiding more than one pattern of length three*

In [46] Simion and Schmidt also analyse the case of avoiding more than one pattern of length three. For all subsets $T \subseteq S_3$ with two or more elements they were able to determine the number of permutations avoiding all

Symmetry class T	$S_n(T)$
$\overline{\{123, 132\}}$	2^{n-1}
$\overline{\{123, 321\}}$	$\begin{cases} n & \text{if } n = 1 \text{ or } n = 2 \\ 4 & \text{if } n = 3 \text{ or } n = 4 \\ 0 & \text{if } n \geqslant 5 \end{cases}$
$\overline{\{132, 213\}}$	2^{n-1}
$\overline{\{132, 231\}}$	2^{n-1}
$\overline{\{132, 312\}}$	2^{n-1}
$\overline{\{132, 321\}}$	$\binom{n}{2} + 1$
$\overline{\{123, 132, 213\}}$	F_{n+1}
$\overline{\{123, 132, 231\}}$	n
$\overline{\{123, 132, 312\}}$	n
$\overline{\{123, 132, 312\}}$	n
$\overline{\{132, 213, 231\}}$	n
$\lvert T\rvert = 3, 4, 5, T \supset \{123, 321\}$	0 if $n \geqslant 5$
$\lvert T\rvert = 4, 5, T \not\supset \{123, 321\}$	2, if $n \geqslant 4$
$\mathcal{S}_3$	0, if $n \geqslant 3$

Figure 3.5: Multiple restrictions of length three

elements of T. Their results are presented in Figure 3.5. The statement for $n \geqslant 5$ and sets of restriction patterns containing both the pattern 123 and the pattern 321 follows from the so-called Erdős-Szekeres theorem (see [19]) which states the following: any sequence of $nk + 1$ distinct real numbers contains either a decreasing subsequence of length $k + 1$ or an increasing subsequence of length $n + 1$. All the other proofs are presented in [46].

3.3 Results on patterns of length four

After obtaining very satisfying results for patterns of length three, we shall now turn to patterns of length four. In this case, as we will see from the following results, finding an exact formula is unfortunately a lot more difficult in this case; for one Wilf-class great progress has been done in the last decade but there is still no explicit formula known yet. The results presented here will be given without proofs, we refer to the original literature instead.

3.3.1 *Three Wilf-equivalence classes for patterns of length four*

In total, there are 24 different patterns of length four. This number can be reduced significantly by utilising the several equivalences that exist for symmetry reasons. As we did in the previous section for patterns of length three, we can take account of reverses and complements and thus only need to consider permutations where the first element is smaller than the last one and where the first element is either 1 or 2. We can also drop 2314 (since $(1423)^r = 3241$ and $3241^c = 2314$). The remaining nine patterns are then the following:

$$1234, 1243, 1324, 1342, 1423, 1432, 2134, 2143, 2413.$$

Since the inverse of a permutation matrix (recall Definition 2.1.7) is its transpose, it holds that if p contains the pattern q, then the inverse p^{-1} of p contains the inverse pattern q^{-1}. We can therefore eliminate the pattern 1423 (since $(1423)^{-1} = 1342$).

The next reduction of equivalence classes can be done by using a more general result of Backelin, West and Xin [6].

Theorem 3.3.1. *Let* k *be a positive integer and let* q *be a permutation on the set* $\{k+1, k+2, \ldots, k+r\}$. *Then the following equivalence holds:*

$$(123\ldots kq) \sim (k(k-1)\ldots 1q).$$

With the help of Theorem 3.3.1 we can now eliminate the patterns 2134, 2143, 1432 and 1243. See chapter 4.4. in [12] for details.
We now have four patterns of length four left, namely 1234, 1324, 1342 and 2413. The next result of Stankova [47] that eliminates one last pattern received much attention.

Theorem 3.3.2. *The patterns* 1342 *and* 2413 *are Wilf-equivalent.*

Stankova proved her result in the equivalent form of $S_n(4132) = S_n(3142)$. In fact, she proved the stronger result that the generating trees of 4132-avoiding and 3142-avoiding permutations are isomorphic.

We thus have three classes of Wilf-equivalent patterns of length four: 1234, 1342 and 1324. The task of computing the values for $S_n(q)$ numerically has been completed by Julian West for $n \leqslant 10$ in his thesis [54]. Taking a look at the first eight values, we can observe a significant difference to patterns of length three:

- $S_n(1234) = 1, 2, 6, 23, 103, \underline{513}, \underline{2761}, 15767$
- $S_n(1342) = 1, 2, 6, 23, 103, \underline{512}, 2740, 15485$
- $S_n(1324) = 1, 2, 6, 23, 103, \underline{513}, \underline{2762}, 15793$

As the underlined numbers $S_6(q)$ respectively $S_7(q)$ show, the number of n-permutations avoiding a certain pattern q of length four is no longer independent of the pattern. This means that there are some patterns of length four that are easier to avoid than others. Another interesting observation that can be made is that the monotone pattern 1234 does not seem to play a special role: it is neither the easiest nor the hardest to avoid. From this numerical data several questions arise:

- Do we have $S_n(1234) < S_n(1324)$ for all $n \geqslant 7$?
- Do we have $S_n(1342) < S_n(1234)$ for all $n \geqslant 6$?
- In general, if $S_n(q_1) < S_n(q_2)$ for some integer n, is it true that $S_N(q_1) < S_N(q_2)$ for all $N > n$?
- What makes a certain pattern q_1 easier to avoid than another pattern q_2?

The first two questions will be answered in the following subsections. The answer to the third question is "no" for the general case. The first counterexample was given by Stankova and West in [48] for two patterns of length five and $n = 12$.
The last and most fundamental question remains unanswered until now, showing that the field of pattern avoidance is still far from being exhaustively investigated.

3.3.2 *Avoiding the monotone pattern 1234*

The pattern 1234 is a so-called monotone pattern and therefore a more general result provides a very good exponential upper bound.

Definition 3.3.3. *A k-permutation of the type $123\ldots(k-1)k$ is called* ***monotone****.*

Theorem 3.3.4. *For all positive integers n and for $k > 2$, we have*

$$S_n(123\ldots k) \leqslant (k-1)^{2n}.$$

A proof of this result can e.g. be found in Section 4.3 in [12].
For the case $n = 3$ we get $S_n(123) < 4^n$ which agrees with our results from Section 3.2 where we saw that $S_n(123) = c_n$. In fact, no better constant that 4 can be found (cf. Stirling's formula) and this is not by coincidence. The following, stronger result of Regev [41] shows that for monotone patterns no better exponential bound than $\left((k-1)^2\right)^n$ can be found.

Theorem 3.3.5. *For all integers n, the sequence $S_n(123\ldots k)$ asymptotically equals*

$$\lambda_k \frac{(k-1)^{2n}}{n^{(k^2-2k)/2}},$$

with a certain constant λ_k.

See Theorem 4.11 in [12] or [41] for the expression of the constant λ_k. From Theorem 3.3.5 it can directly be seen that $\lim_{n\to\infty} \sqrt[n]{S_n(1234)} = (k-1)^2$.

Using symmetric functions, Ira Gessel [22] proved the following exact formula for the numbers $S_n(1234)$.

Theorem 3.3.6. *For all integers n it holds that*

$$S_n(1234) = 2 \cdot \sum_{k=0}^{n} \binom{2k}{k} \binom{n}{k}^2 \frac{3k^2 + 2k + 1 - n - 2nk}{(k+1)^2(k+2)(n-k+1)}.$$

A few years later Gessel managed to find an alternative form of this formula, eliminating the summands on the right hand side of the equation that are not always non-negative.

Theorem 3.3.7. *For all integers* n *it holds that*

$$S_n(1234) = \frac{1}{(n+1)^2(n+2)} \sum_{k=0}^{n} \binom{2k}{k}\binom{n+1}{k+1}\binom{n+2}{k+1}.$$

As Bóna noted in [12], the division in this second formula suggests that it is probably difficult to find a direct combinatorial proof for this result.

3.3.3 *Avoiding the pattern 1342*

For the pattern $q = 1342$, Bóna was capable of giving an exact formula for $S_n(q)$. It is given by the following theorem.

Theorem 3.3.8. *For all positive integers* n*, the number of* n*-permutations avoiding the pattern* 1342 *is equal to:*

$$\begin{aligned} S_n(1342) &= (-1)^{n-1} \cdot \frac{7n^2 - 3n - 2}{2} \\ &\quad + 3\sum_{i=2}^{n} (-1)^{n-i} \cdot 2^{i+1} \cdot \frac{(2i-4)!}{i!(i-2)!} \cdot \binom{n-i+2}{2}. \end{aligned}$$

This result can be proven by showing that 1342-avoiding permutations bijectively correspond to certain labelled trees, namely $\beta(0,1)$-trees. See [17] for a definition and 4.4.2 [12] for a complete proof. Other objects that bijectively correspond to 1342-avoiding permutations are rooted bicubic maps. These planar maps, in which each vertex has degree three and the underlying graph is bipartite, have been studied earlier by William Tutte, see e.g. [53].

From the formula for $S_n(1342)$ given in Theorem 3.3.8 one can deduce an exponential upper bound. Indeed, whenever $n \geqslant 8$, the second summand is larger than the whole right hand side due to the alternating signs. Now, thanks to Stirling's formula, we know that $\frac{(2n-4)!}{n!(n-2)!} < \frac{8^{n-2}}{n^{2.5}}$ which proves the following result.

Corollary 3.3.9. *For all integers* n *it holds that* $S_n(1342) < 8^n$.

We can then also prove the following asymptotic result.

Corollary 3.3.10. *We have*

$$\lim_{n\to\infty} \sqrt[n]{S_n(1342)} = 8.$$

One can show that $S_n(1342) < S_n(1234)$ for all $n \geqslant 6$ and with Corollary 3.3.10 together with Theorem 3.3.5 one sees the stronger result that the two sequences $S_n(1342)$ and $S_n(1234)$ are not even equal in the logarithmic sense, since:

$$\lim_{n\to\infty} \sqrt[n]{S_n(1342)} = 8 \neq 9 = \lim_{n\to\infty} \sqrt[n]{S_n(1234)}.$$

3.3.4 *Avoiding the pattern 1324*

In [11] Bóna gave the first example in which a pattern was shown to be more restrictive than another pattern of the same length.

Theorem 3.3.11. *For all $n \geqslant 7$ it holds that*

$$S_n(1234) < S_n(1324).$$

This proof is carried in three steps: first a classification of all n-permutations is defined where two permutations are said to be in the same class if their left-to-right-minima respectively right-to-left-maxima (recall Definition 2.1.9) are the same and in the same positions. In a next step, Bóna shows that each class contains exactly one 1234-avoiding permutation and at least one 1324-avoiding permutation. Finally, he shows that for every integer $n \geqslant 7$, one specific class of n-permutations can be found that contains two 1324-avoiding permutations.

It even holds that the sequences $S_n(1234)$ and $S_n(1324)$ are not asymptotically equal.

Several new results on 1324-avoiding permutations have been published in the last few years (see e.g. [36]) but no explicit formula for $S_n(1324)$ has been found yet. However, we do have an exponential upper bound for $S_n(1324)$.

Theorem 3.3.12. *For all integers n, we have $S_n(1324) < c^n$ with $c = 288$.*

The constant $c = 288$ is certainly not the best possible one, but it has been shown in [2] that the limit of $\sqrt[n]{S_n(4231)} = \sqrt[n]{S_n(1324)}$ is at least 9.47. This is an interesting result, firstly since it shows that the class of 1324- respectively 4231-avoiding permutations has the largest such limit among all Wilf-classes of permutations avoiding a single permutation of length four and secondly because it refutes a conjecture stated by Arratia in [3]. He had conjectured that the limit of the sequence $\sqrt[n]{S_n(q)}$, the so-called Stanley-Wilf limit, cannot exceed $(k-1)^2$ if q is a pattern of length k.

3.4 THE STANLEY-WILF CONJECTURE

For patterns of length three, we obtained very satisfying and surprisingly simple results, see Section 3.2. Unfortunately, as we have seen in the previous section, finding explicit formulae for the number of n-permutations avoiding a certain pattern q of length four is a lot more difficult and not yet an entirely solved problem. For longer patterns, nearly no exact formulae are known yet.
In this context it is interesting to note that the computational complexity of a closely related problem has been studied. In [16] the following result was shown.

Theorem 3.4.1. *For a given n-permutation p the so-called pattern matching problem, i.e. the problem of deciding whether an arbitrary pattern q is contained in p or not, is NP-complete. Moreover, counting the number of matchings of q into p is #P-complete.*

Pattern class $\overline{q}$	Exact formula for $S_n(q)$	Exponential upper bound	Asymptotics $\lim_{n\to\infty} \sqrt[n]{S_n(q)}$
$\overline{132}$	$c_n = \frac{1}{n+1}\binom{2n}{n}$	4^n	4
$\overline{1234}$	Theorem 3.3.6 and 3.3.7	9^n	9
$\overline{1342}$	Theorem 3.3.8	8^n	8
$\overline{1324}$	?	288^n	c, where $9.47 < c < 288$
$\overline{123\ldots k}$	?	$(k-1)^{2n}$	$(k-1)^2$

Figure 3.6: Exponential upper bounds for $S_n(q)$ and asymptotic results for $\sqrt[n]{S_n(q)}$.

Note that under standard complexity-theoretical assumptions, namely $P \neq NP$, this implies that the pattern matching problem cannot be solved in polynomial time and hence this theorem provides evidence for the high inherent complexity of the problem.

However, a powerful result providing an upper bound for all q-avoiding permutations does exist. Indeed, no matter what q is, the number of n-permutations avoiding q is very small compared to $n!$, the total number of n-permutations. This is the claim of the so-called Stanley-Wilf-conjecture, formulated by Richard Stanley and Herbert Wilf in an oral communication in 1990 (It is difficult to find an exact reference; one of the first written formulations of the conjecture can be found in [10].) and proven nearly fifteen years later by Adam Marcus and Gábor Tardos. This section presents the conjecture as well as an interesting equivalent conjecture and the spectacular story of its proof, following the presentation of Miklós Bóna in [12]. A very entertaining and interesting version of this story was told by Doron Zeilberger in the lecture *How Adam Marcus and Gábor Tardos Divided and Conquered the Stanley-Wilf Conjecture (An Étude in paramathematics)* and can be found in [57].

In the previous two sections we have already found exponential upper bounds for patterns of length three and four as well as a more general result for monotone patterns. The Stanley-Wilf conjecture is thus already proven for these special cases. We sum up the results in Figure 3.6.

3.4.1 *Two equivalent conjectures*

Conjecture 3.4.2 (Stanley-Wilf-conjecture, 1990). *Let q be an arbitrary pattern. Then there exists a constant c_q so that for all positive integers it holds that*

$$S_n(q) \leqslant c_q^n. \qquad (3.4)$$

Note that this is quite an ambitious conjecture since it postulates that the number of q-avoiding n-permutations does not grow faster than exponentially whereas the total number of n-permutations grows super-exponentially, cf. the Stirling formula that states that $n!$ is asymptotically equal to $\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$.

The following conjecture that appears to claim an even stronger result, can be proven to be equivalent to the Stanley-Wilf-conjecture.

Conjecture 3.4.3. *Let q be an arbitrary pattern. Then the limit*

$$\lim_{n\to\infty} \sqrt[n]{S_n(q)}$$

exists and is finite.

Proof. The first proof of the equivalence of these two conjectures was given in 1999 by Richard Arratia in [3] and we shall present the idea of his proof here. See also [34].
First note that it is obvious that conjecture 3.4.3 implies conjecture 3.4.2. To prove the implication in the other direction, we show that $S_n(q)S_m(q) \leqslant S_{n+m}(q)$ for all patterns q and all integers n.
We show this inequality by injectively constructing, from a given n-permutation and a given m-permutation that avoid the pattern q, a $(m+n)$-permutation that equally avoids q. Without any loss of generality, we may assume that k, the maximal element of q, precedes the minimal element 1 in the pattern (if this is not the case, you simply need to consider reverse permutations). Now let p_1 and p_2 be permutations on the set $[m]$ and $[n]$ respectively, both avoiding the pattern q. Let p_3 be the result of adding m to every element of p_2. We thus obtain a permutation p_3 on the set $\{m, m+1, \dots, m+n\}$. Consider the concatenation p of p_1 and p_3, then clearly $p \in \mathcal{S}_{n+m}$ and p avoids q (by construction, all the elements in the p_1-part of p are smaller than all the elements in the p_3-part and therefore a pattern in which the largest element precedes the smallest one cannot be found).
This shows that the sequence $(S_n(q))_{n\in\mathbb{N}}$ is *super-multiplicative*, or equivalently that the sequence $(-\log(S_n(q)))_{n\in\mathbb{N}}$ is *sub-additive* (note that $S_n(q) \geqslant 1$ for all integers and all patterns with the above restriction, since e.g. the identity permutation avoids every pattern in which the largest element stands in front of the smallest). A sequence a_n is called sub-additive if $a_{m+n} \leqslant a_m + a_n$ holds for all integers m, n. We can therefore apply Fekete's Lemma [20] (see also Lemma 1.2.1 in [51]) which states the following: for a given sub-additive sequence of real numbers a_n, the limit $\lim_{n\to\infty} \frac{a_n}{n}$ exists and is equal to $\inf \frac{a_n}{n}$. This means that the limit of the sequence $\sqrt[n]{S_n(q)}$ exists and is equal to its supremum; the exponential upper bound for $S_n(q)$ guarantees that it is indeed finite. □

Remark 3.4.4. So far, in every case in which the limit of $\sqrt[n]{S_n(q)}$could be computed, this limit was an integer. See again Figure 3.6. That this should hold in the general case would therefore be a tempting conjecture. Unfortunately, this is not true. In [13] Miklós Bóna proved that $\lim_{n\to\infty} \sqrt[n]{S_n(12453)} = 9 + 4\sqrt{2}$ so we may not even expect these limits to be rational.

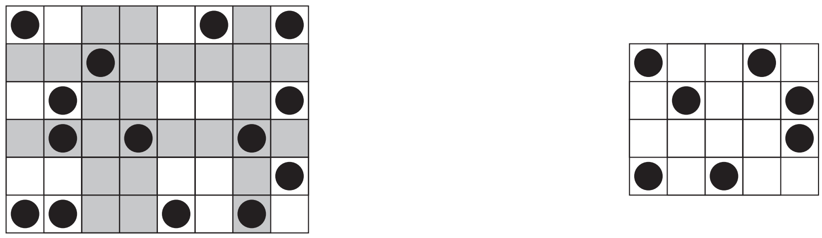


Figure 3.7: Pattern avoidance in binary matrices visualized with the help of rectangular grids. The matrix P that contains the pattern Q represented on the right hand side is visualized on the left hand side.

3.4.2 *The Füredi-Hajnal-conjecture*

In 1992, Füredi and Hajnal presented another conjecture concerning pattern avoidance involving binary matrices in [21] (originally, they formulated this conjecture as a question). Let us therefore extend the notion of pattern avoidance from permutation matrices (recall Definition 2.1.7) to arbitrary 0-1-matrices in the following way.

Definition 3.4.5. *Let* P *and* Q *be matrices with entries in* $\{0,1\}$ *and let* Q *have the dimension* $m \times n$*. We say that the matrix* P ***contains*** *the matrix* Q *as a pattern, if there is a submatrix* $\tilde{Q}$ *of* P*, so that* $\tilde{Q}_{i,j} = 1$ *whenever* $Q_{i,j} = 1$ *for* $i \leqslant m$ *and* $j \leqslant n$*. If there is no such submatrix* $\tilde{Q}$*, we say that* P ***avoids*** Q.

This means that P contains Q as a pattern, if, by deleting some rows and some columns, one can obtain a matrix $\tilde{Q}$ with the same size as Q that has a 1-entry everywhere where Q has a 1-entry. Note that $\tilde{Q}$ must not necessarily have its 0-entries in the same places as in Q: $\tilde{Q}$ may have more 1-entries than Q but not less. Also note that in the case where a submatrix of size $m \times n$ containing only 1-entries can be found, the matrix P contains every possible $m \times n$-pattern.

Example 3.4.6. Let us give an example for pattern avoidance in binary matrices using the same graphical representation as in Section 3.1, i.e. a $k \times l$-binary matrix P is represented by a rectangular grid with k rows and l columns, in which a black pawn is placed in the i-th row and the j-th column, if $P_{i,j} = 1$. In the example presented in Figure 3.7 we can see that the 6×8-matrix P contains the 4×5-pattern Q, since deleting the rows and columns marked in gray in P leads to a binary matrix of the same size as Q that has *more* 1-entries that Q in the sense defined above.

Conjecture 3.4.7. *Let* Q *be any permutation matrix. We define* $f(n, Q)$ *as the maximal number of* 1*-entries that a* Q*-avoiding* $n \times n$*-matrix* P *can have. Then there exists a constant* d_Q *so that*

$$f(n, Q) \leqslant d_Q \cdot n.$$

3.4.3 *Füredi-Hajnal implies Stanley-Wilf*

We shall present here the argument given by Martin Klazar in [28] to prove that the Füredi-Hajnal conjecture implies the Stanley-Wilf conjecture. In order to establish a connection between pattern avoidance in matrices and pattern avoidance in permutations Klazar takes an elegant detour via pattern avoidance in bipartite graphs. We therefore need to introduce this new notion of avoidance and containment.

Definition 3.4.8. *Let* $P([n],[n'])$ *and* $Q([m],[m'])$ *be simple bipartite graphs, where* $m < n$ *and* $m' < n'$*. Then we say that* P ***contains*** Q ***as an ordered subgraph*** *if two order preserving injections* $f : [n] \to [m]$ *and* $f' : [n'] \to [m']$ *can be found so that if* vv' *is an edge of* Q*, then* $f(v)f'(v')$ *is an edge of* P*.*

Note again that - as in the definition of pattern avoidance on matrices - the following holds: if $f(v)f'(v')$ is an edge of P, vv' does not necessarily have to be an edge of Q.
As seen in Section 2.1 and in Figure 2.1, every n-permutation p naturally defines a corresponding bipartite graph G_p on $([n],[n])$. In the following, G_p will always stand for the bipartite graph corresponding to the permutation p. With this observation the following two results are immediate consequences of the definitions made above.

Theorem 3.4.9. *If a simple bipartite graph* P *on* $([n],[n'])$ *avoids the* $([m],[m'])$*-graph* Q*, then the adjacency matrix*[4] $A(P)$ *of* P *must avoid the adjacency matrix of* Q*.*

Theorem 3.4.10. *If the permutation* p *contains the permutation* q *as a pattern, then* G_p *contains* G_q *as an ordered subgraph. Reversly, if* p *avoids* q*,* G_p *will also avoid* G_q*.*

Note that not every simple bipartite graph corresponds to a permutation. There are therefore more simple bipartite graphs on $([n],[n])$ that avoid a given graph-pattern G_q than there are n-permutations p so that the corresponding bipartite graph G_p avoids G_q.

We now hold in hands the required tools to prove the following result:

Theorem 3.4.11. *If the Füredi-Hajnal conjecture is true, then the Stanley-Wilf conjecture is also true.*

Proof. If $G_n(q)$ is the number of simple bipartite graphs on $([n],[n])$ avoiding the graph G_q, then it follows from Theorem 3.4.10 that $S_n(q) \leqslant G_n(q)$. We therefore need to show that the Füredi-Hajnal conjecture implies that for every graph-pattern G_q, there is a constant c_q so that $G_n(q) \leqslant c_q^n$ for all $n \in \mathbb{N}$.
In the following, let P be a simple bipartite graph on $([n],[n])$ that avoids G_q. Assume that Conjecture 3.4.7 is true. Then, following Theorem 3.4.9, the adjacency matrix $A(P)$ of P can have at most $c_q \cdot n$ entries equal to 1 and this means that G_p can have at most $d_q \cdot n = d_{A(G_q)} \cdot n$ edges. What we are going to show now is that this leaves at most an exponential number of possibilities for the graph P.
For this purpose, let us gradually contract the initial graph to a bipartite graph with merely two vertices, reducing the graph to half the size and keeping track of how much information gets lost in every step. Starting

4 The binary matrix for which $a_{i,j} = 1$ iff (i,j) is an edge in the corresponding graph.

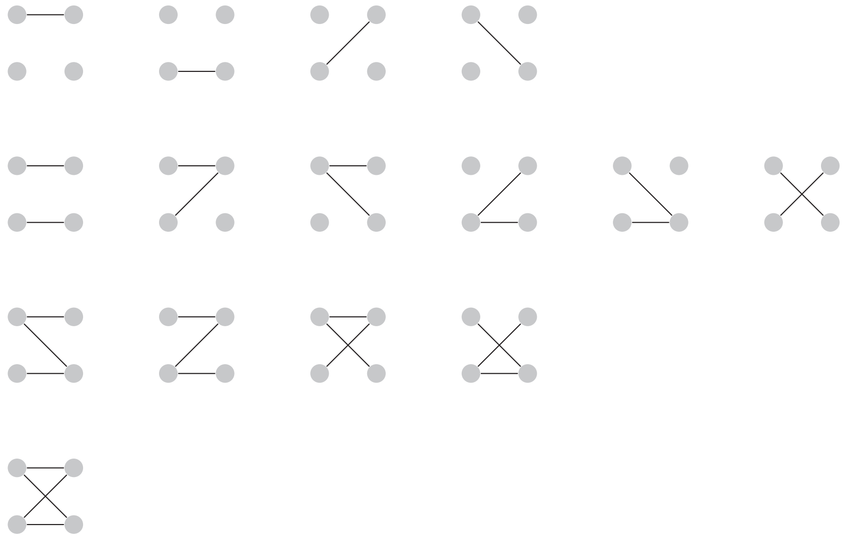


Figure 3.8: The 15 possible graphs leading, after contraction, to a bipartite graph with two vertices and one edge.

with P, we construct the smaller bipartite graph P_1 on the vertex set $[\lceil \frac{n}{2} \rceil], [\lceil \frac{n}{2} \rceil]$ in the following way.
If $i \in [\lceil \frac{n}{2} \rceil]$ and $j \in [\lceil \frac{n}{2} \rceil]$ are two vertices in P_1, then let (i, j) be an edge if there is at least one edge between the sets of vertices $\{2i-1, 2i\}$ and $\{2j-1, 2j\}$ in P. It is clear that this contraction passes on the G_q-avoiding property from P to P_1.

Now how many different graphs P can lead to the same contracted graph P_1? As can be seen in Figure 3.8, there are exactly fifteen simple bipartite graphs on $([2], [2])$ that lead to the bipartite graph with two vertices and one edge. Therefore, since P_1 may not have more than $d_q \cdot \lceil \frac{n}{2} \rceil$ edges, there are at most $15^{d_q \cdot \lceil \frac{n}{2} \rceil}$ different graphs P that can lead to P_1. We thus obtain

$$G_n(q) \leqslant 15^{d_q \cdot \lceil \frac{n}{2} \rceil} \cdot G_{\lceil \frac{n}{2} \rceil}(q).$$

By iterating this argument until we have $G_1(q) \leqslant 2$ on the right hand side, we finally get

$$G_n(q) \leqslant 15^{2 d_q n}.$$

Thus we have proven $G_n(q) \leqslant c_q^n$ with $c_q = 15^{2d_q}$. □

3.4.4 *Proof of the Füredi-Hajnal-conjecture*

We shall now present the last missing and thus crucial element in the proof of the Stanley-Wilf conjecture, namely the proof of the Füredi-Hajnal conjecture. It was given in [35] in the year 2004 by Gábor Tardos and his PHD student Adam Marcus. In the lecture [57] about the proof, Doron Zeilberger makes it clear that:

> "Adam Marcus, the co-prover, [...] is a mere (mathematical) epsilon, a first-year grad student. [...] Neither Stanley, nor Wilf,

> nor any one of the many skilled enumerators that tried very hard to prove it, succeeded. What made it even more amazing and frustrating was the fact that the Marcus-Tardos proof may be presented in half a page. Of course, it is gorgeous!"

The following presentation of the proof will however be longer than half a page since it will provide all the details given in [35]. The main ideas are well-known to combinatorialists and remain simple: the first idea is to reduce the considered matrix that avoids a certain pattern to a matrix of smaller size that still avoids the same pattern. This process will then be iterated. The second idea will be to apply the pigeon hole principle when counting the numbers of ones that the considered matrix may contain. Let us now follow the argumentation of Marcus and Tardos.

Let Q be a $k \times k$-permutation matrix and P a $n \times n$-binary matrix that avoids the pattern Q and has exactly $f(n, Q)$ elements equal to 1. Moreover assume that n is a multiple of k^2, the other cases will be dealt with later. With this assumption, we can decompose P into $\left(\frac{n}{k^2} \cdot \frac{n}{k^2}\right)$ square blocks each of the size $k^2 \times k^2$. For $(i, j) \in \left[\frac{n}{k^2}\right] \times \left[\frac{n}{k^2}\right]$ we denote by $S_{i,j}$ the submatrix (block) in the i-th row and the j-th column of blocks. Thus $S_{i,j}$ consists of the intersection of the rows $(i-1)k^2+1, (i-1)k^2+2, \dots, ik^2$ and of the columns $(j-1)k^2+1, (j-1)k^2+2, \dots, jk^2$. We shall now contract the matrix P into a smaller matrix B of size $\frac{n}{k^2} \times \frac{n}{k^2}$, keeping track of where (in which blocks) entries equal to 1 can be found and where not. The matrix $B = (b_{i,j})$ is defined as follows:

$$b_{i,j} = \begin{cases} 0, & \text{if all entries of } S_{i,j} \text{ are zero} \\ 1, & \text{if there is at least one entry equal to 1 in } S_{i,j}. \end{cases}$$

Figure 3.9 shows how the block decomposition of the matrix P and the contracted matrix B hang together. The following holds:

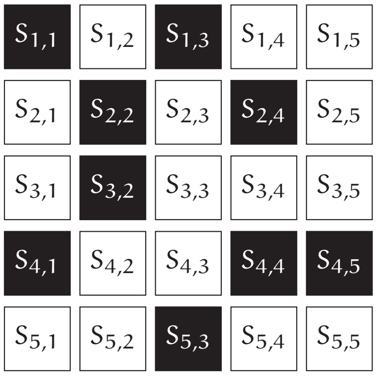
Lemma 3.4.12. B *avoids* Q.

Proof. With the definition made above, it is clear that the property of avoiding the pattern-matrix Q is inherited by the matrix B from P. Indeed, if B did not avoid the pattern Q, we could find a copy Q_C of Q in B. Then, considering the fact that Q_C is a permutation matrix, we could take a 1-entry of each block of P that defined a 1-entry of B and obtain a copy of Q in P. □

We continue by characterizing blocks that contain "many" 1-entries. The following definition will show in what sense this is meant.

Definition 3.4.13. *A block* $S_{i,j}$ *of* P *is called* ***wide*** *if it contains a* 1*-entry in at least* k *different columns. Analogously, a block is called* ***tall*** *if it contains a* 1*-entry in at least* k *different rows.*

The next crucial step consists in noticing that a Q-avoiding matrix P may not contain "too many" of these wide or tall blocks.



$$B = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Figure 3.9: Example of a decomposition of P into 5×5 blocks and the corresponding contracted matrix B. Black boxes correspond to submatrices containing at least one entry 1, white boxes correspond to blocks where all entries are 0.

Lemma 3.4.14. *For every* $j \in \left[\frac{n}{k^2}\right]$*, the column of blocks* $C_{i,j} = \left\{S_{i,j} | i \in \left[\frac{n}{k^2}\right]\right\}$ *of the matrix* P *contains less than* $k\binom{k^2}{k}$ *wide blocks. Similarly, for every* $i \in \left[\frac{n}{k^2}\right]$*, the row of blocks* $R_{i,j} = \left\{S_{i,j} | 1 \leqslant j \leqslant \frac{n}{k^2}\right\}$ *of the matrix* P *contains less than* $k\binom{k^2}{k}$ *tall blocks.*

Proof. For columns of blocks: Assume that there are $k\binom{k^2}{k}$ or more wide blocks. We then obtain a contradiction to the fact that P is Q-avoiding by applying the pigeonhole principle[5]. Indeed, for every wide block, there are $\binom{k^2}{k}$ different ways of choosing the k rows in which the 1-entries stand. We thus have $\binom{k^2}{k}$ "pigeonholes" (= choices of k columns) where $k\binom{k^2}{k}$ or more "pigeons" (= the k columns with 1-entries of wide blocks) are to be placed. Thus, there is at least one pigeonhole that contains at least k pigeons. This means that there is (at least) one selection of columns $c_1 < c_2 < \ldots < c_k$ and a selection of (at least) k blocks $S_{d_1,j}, S_{d_2,j}, \ldots, S_{d_k,j}$, where $1 \leqslant d_1 < \ldots < d_k \leqslant n/k^2$, with ones in these columns. Then, with the help of these k blocks, a copy of Q can after all be constructed in P: if $q_{r,s}$ is a 1-entry in Q, then pick any 1-entry in column c_s of $S_{d_r,j}$.
For rows of blocks, the argument is obviously exactly the same. □

We have now seen that the matrix P may neither contain too many non-zero blocks (see Lemma 3.4.12) nor too many wide or tall blocks (see Lemma 3.4.14). Putting this information about P together, we get the following recursive result:

Lemma 3.4.15. *For a given* $k \times k$*-permutation matrix* Q *and a positive multiple* n *of* k^2*, we have*

$$f(n, Q) \leqslant (k-1)^2 f\left(\frac{n}{k^2}, Q\right) + 2k^3 \binom{k^2}{k} n.$$

Proof. There are three types of non-zero blocks that have to be considered:

5 The *pigeonhole principle* states that if n items (=pigeons) are put into m pigeonholes with $n > m$, then at least one pigeonhole must contain more than one item. A more general version states that if n objects are to be placed into m boxes, then at least one box must hold at least $\lceil n/m \rceil$ objects.

- B_w ... the set of wide blocks.
  Every B_w-block can contain at most k^4 entries equal to 1 and by Lemma 3.4.14 we know that
  $$|B_w| \leqslant \frac{n}{k^2} k \binom{k^2}{k}.$$
- B_t ... the set of tall blocks.
  Every B_t-block can contain at most k^4 entries equal to 1 and by Lemma 3.4.14 we know that
  $$|B_t| \leqslant \frac{n}{k^2} k \binom{k^2}{k}.$$
- B_n ... the set of non-zero blocks, that are neither wide nor tall.
  Every B_n-block can contain at most $(k-1)^2$ entries equal to 1 and by Lemma 3.4.12 we know that
  $$|B_n| \leqslant f\left(\frac{n}{k^2}, Q\right).$$

We can now bound the number of 1-entries in P by summing the estimates of the number of 1-entries in these three categories of blocks and obtain:

$$\begin{aligned} f(n, Q) &\leqslant k^4 |B_w| + k^4 |B_t| + (k-1)^2 |B_n| \\ &\leqslant 2k^3 \tbinom{k^2}{k} n + (k-1)^2 f\left(\tfrac{n}{k^2}, Q\right). \end{aligned}$$

□

We now merely need to consider the case where n is not a multiple of k^2 and can then prove the Füredi-Hajnal conjecture by induction over n.

Theorem 3.4.16. *For all permutation matrices* Q *of size* $k \times k$ *it holds that*

$$f(n, Q) \leqslant 2k^4 \binom{k^2}{k} n.$$

Proof. The base cases (when $n \leqslant k^2$) are trivial since P then has at most k^4 entries.

Now assume the hypothesis is true for all integers less than n, and we shall prove it for n. Let n' be the largest integer less than or equal to n which is divisible by k^2. In the worst case we fill the part of P that cannot be partitioned into $k^2 \times k^2$-blocks with ones, which adds at the most $2k^2 n$ ones. Together with Lemma 3.4.15 we then have:

$$\begin{aligned} f(n, Q) &\leqslant f(n', Q) + 2k^2 n \\ &\leqslant (k-1)^2 f\left(\frac{n'}{k^2}, Q\right) + 2k^3 \binom{k^2}{k} n' + 2k^2 n. \end{aligned}$$

Applying the induction hypothesis leads to:

$$\begin{aligned} f(n, Q) &\leqslant (k-1)^2 \left[2k^4 \binom{k^2}{k} \frac{n'}{k^2}\right] + 2k^3 \binom{k^2}{k} n' + 2k^2 n \\ &\leqslant 2k^2((k-1)^2 + k + 1) \binom{k^2}{k} n \\ &\leqslant 2k^4 \binom{k^2}{k} n, \end{aligned}$$

where the last inequality holds for all $k \geqslant 2$. □

We now directly obtain:

Corollary 3.4.17. *For an arbitrary pattern* q *of length* k *and all integers* n *it holds that*

$$S_n(q) \leqslant c_q^n,$$

where $c_q = 15^{2k^4\binom{k^2}{k}}$.

This constant c_q can certainly be improved. For example, for patterns of length three, we had obtained that the best possible result was $c_q = 4$, whereas Corollary 3.4.17 suggests $c_q = 15^{27216}$. Note however the interesting fact that c_q does not depend on the structure of the permutation itself, but only on its length.

4 PATTERN AVOIDANCE ON MULTISETS

In this chapter we are going to generalize the concept of pattern avoidance in permutations on ordinary sets to permutations on multisets. We shall present the results of Albert, Aldred et. al. [1] and Heubach and Mansour [25] who studied multiset-permutations avoiding ordinary, respectively multiset-patterns of length three. We then close a gap in [25], providing enumeration formulae for permutations on multisets avoiding the patterns 112 and 122 simultaneously.

4.1 RESTRICTED PERMUTATIONS ON MULTISETS: DEFINITIONS

Here we consider permutations on multisets (recall Definition 2.2.4) and shall anew define what we mean when saying that a permutation avoids or contains a certain pattern.

Definition 4.1.1. *Let* $q = (q_2 q_2 \dots q_{\tilde{l}})$ *be a permutation of length* $\tilde{l} = \sum_{i=1}^{\tilde{n}} \tilde{m}_i$ *on the multiset* $[\tilde{n}]_{\tilde{\mathbf{m}}} = [\tilde{n}]_{\tilde{m}_1 \tilde{m}_2 \dots \tilde{m}_{\tilde{n}}} = \left\{1^{\tilde{m}_1}, 2^{\tilde{m}_2}, \dots, \tilde{n}^{\tilde{m}_{\tilde{n}}}\right\}$. *We say that the multiset-permutation* $p = (p_1 p_2 \dots p_l)$ *of length* $l = \sum_{i=1}^{n} m_i$ *on* $[n]_{\mathbf{m}} = [n]_{m_1 m_2 \dots m_n}$, *where* $n \geqslant \tilde{n}$, ***contains*** q ***as a pattern*** *if we can find* $\tilde{l}$ *entries* $p_{i_1}, p_{i_2}, \dots, p_{i_{\tilde{l}}}$ *with* $i_1 < i_2 < \dots < i_{\tilde{l}}$ *so that* $p_{i_a} < p_{i_b} \Leftrightarrow q_a < q_b$. *If there is no such subsequence, we say that* p ***avoids the pattern*** q.

This definition might seem a bit cumbersome because of the bulky multiset-notations, but it is simply the natural extension of the definition 3.1.1 of pattern avoidance from ordinary sets to multisets. Note that the considered pattern q may either be an ordinary permutation on a set or a *multiset-pattern*, i.e. a permutation on a multiset. Let us give a simple example to illustrate this concept.

Example 4.1.2. Consider the permutation $p = 433321412531$ on the multiset $[5]_{3,2,4,2,1} = \{1,1,1,2,2,3,3,3,3,4,4,5\}$. It contains the pattern 1111 (and consequently the patterns 111 and 11), since the element 3 occurs four times. Another, more interesting pattern of length four contained in p is for instance 1121, as can be seen by considering the entries 3353 or 1131. It also contains the pattern 211 as shown by the entries 322, 311 or 211, but avoids the pattern 122 as can easily be seen .
Note that the condition $p_{i_a} < p_{i_b} \Leftrightarrow q_a < q_b$ in Definition 4.1.1 guarantees that $p_{i_a} = p_{i_b} \Leftrightarrow q_a = q_b$. This means that for a certain pattern to be contained in a permutation, repetitions in the pattern have to be represented by repetitions in the permutation. Here this means for instance that the elements 123 do not represent the pattern 122.

Remark 4.1.3. In the same way as for ordinary permutations (see Figures 3.1 and 3.2), we can represent permutations on multisets with the help of (no longer square but) rectangular grids in which black pawns have been placed. As seen in Section 2.2, permutations on multisets can also be represented by binary matrices. For a permutation p on the multiset

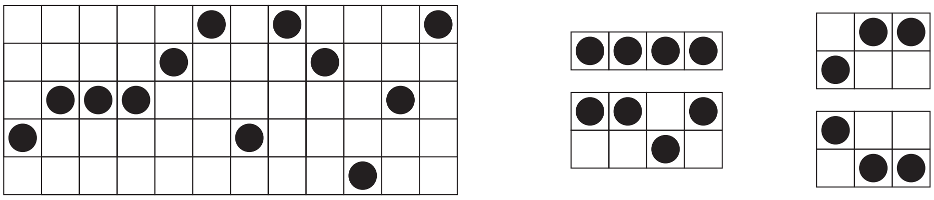


Figure 4.1: Multiset-permutation matrices visualized with the help of rectangular grids. From left to right: $p = 433321412531$, 1111 (top) and 1121 (bottom), 211 (top) and 122 (bottom).

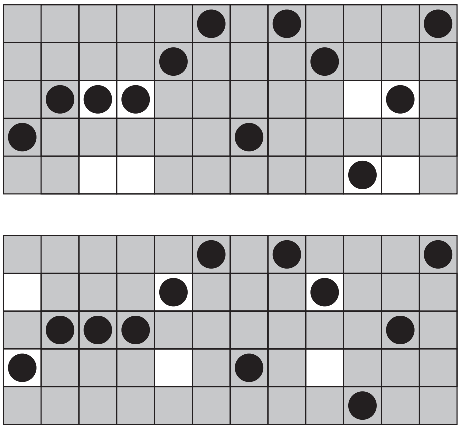


Figure 4.2: Containing patterns in multiset-permutation matrices: representation with the help of rectangular grids. Deleting the rows and columns marked in gray leads to the desired patterns (on the top the pattern 1121, on the bottom the pattern 211).

$[n]_{m_1 m_2 \dots m_n}$, draw a grid of size $n \times l$ where $l = \sum_{i=1}^{n} m_i$ is the length of p and place a pawn in the i-th row and the j-th column if $P_{i,j} = 1$ in the matrix P belonging to p, i.e. if $p_j = i$. See Figure 4.1 for the grid-representation of the permutation $p = 433321412531$ and the patterns 1111, 1121, 211 and 122 of the previous example. Again, containing a certain pattern-grid means that by deleting some rows and some columns of the grid representing the permutation p, we obtain a smaller grid that looks exactly like the pattern-grid. See Figure 4.2 for the patterns 1121 and 211.

We introduce the following notations for the number of permutations on multisets avoiding a certain pattern or a set of patterns. For regular multisets $[n]_m$, i.e. where $m_i = m$ for all $i \in [n]$, we adapt the notation for permutations on ordinary sets and write

$$\mathcal{S}_{n,m}(q)$$

for the set of all permutations on $[n]_m$ avoiding the pattern q and

$$S_{n,m}(q) = |\mathcal{S}_{n,m}(q)|$$

for its cardinality. Analogously, for a set of patterns T, we denote by $\mathcal{S}_{n,m}(T)$ the set of permutations on $[n]_m$ avoiding all patterns in T simultaneously and by $S_{n,m}(T)$ its cardinality.
For non-regular multisets, i.e. $m_i \neq m_j$ for some $i \neq j$, the multiplicities

m_i play a crucial role in determining the number of permutations avoiding a certain pattern and we therefore write

$$\mathcal{T}(m_1, m_2, \ldots, m_n)$$

for the set of all permutations on the multiset $\{1^{m_1}, 2^{m_2}, \ldots, n^{m_n}\}$ avoiding the set of patterns T and

$$T(m_1, m_2, \ldots, m_n) = |\mathcal{T}(m_1, m_2, \ldots, m_n)|$$

for its cardinality, using the notation introduced in [1]. Note that T may consist of a single pattern, i.e. $T = \{q\}$. Also note that

$$\mathcal{S}_{n,m}(T) = \mathcal{T}(\underbrace{m, m, \ldots, m}_{n-\text{times}}).$$

As for permutations on regular sets, the case of patterns of length two can be treated very easily. There are now three different patterns of length two: 11, 12 and 21. We obtain the following trivial results:

Theorem 4.1.4. *If* $m_i \geqslant 2$ *for some* $i \in [n]$ *and* $T = \{11\}$,

$$T(m_1, m_2, \ldots, m_n) = 0$$

for all $n \in \mathbb{N}$. *Otherwise* $T(1, 1, \ldots, 1) = S_n(T) = n!$.

Proof. Let $i \in [n]$ be such that $m_i \geqslant 2$. Then the subsequence (ii) will always form a 11-pattern. On the other hand, if $m_i = 1$, i.e. in the case of ordinary sets, no 11-pattern is possible. □

Theorem 4.1.5. *Set* $T_1 = \{12\}$ *and* $T_2 = \{21\}$. *Then*

$$T_1(m_1, m_2, \ldots, m_n) = T_2(m_1, m_2, \ldots, m_n) = 1$$

holds for all $n \in \mathbb{N}$ *and all multisets* $\{1^{m_1}, 2^{m_2}, \ldots, n^{m_n}\}$.

Proof. The only permutation avoiding the pattern 12 is the permutation p in which identical elements are grouped into blocks and these n blocks are placed in decreasing order. Similarly, the only permutation avoiding the pattern 21 is the permutation q in which these blocks are placed in increasing order. □

In the following two sections, we shall turn to permutations on multisets avoiding patterns (or sets of patterns) of length three. We first consider ordinary patterns and then multiset-patterns.
Note that we may assume $n \geqslant 3$ in the following since it holds that $T(m_1) = 1$ and $T(m_1, m_2) = \binom{m_1+m_2}{m_1}$ (the total number of permutations on the multiset $\{1^{m_1}\}$ respectively $\{1^{m_1}, 2^{m_2}\}$), whenever T is a set of permutations of length three.

4.2 AVOIDING ORDINARY PATTERNS OF LENGTH THREE

This section presents the results of Albert, Aldred, Atkinson, Handley and Holton who studied permutations on multisets avoiding ordinary patterns of length three in [1]. The authors were able to derive explicit enumeration formulae for almost all pattern-sets, but in some cases (B_1, B_4 and C_1) only recurrences, both for the actual values and for the associated multivariate generating functions, could be found.
If not specified, all the following results originate from [1] where details and proofs can be found.

4.2.1 *Single patterns of length three*

For ordinary permutations, Section 3.2.1 showed that all patterns of length three were Wilf-equivalent. This was done by establishing a bijection between the two symmetry classes $\overline{123}$ and $\overline{132}$, see Theorem 3.2.2.
For permutations on multisets the situation is a bit different: when considering symmetry classes we may use reverse permutations but not complements. Indeed, if p is a permutation on the multiset $\{1^{m_1}, 2^{m_2}, \dots, n^{m_n}\}$ avoiding a pattern q, then p^r is a permutation on the same multiset and avoids the pattern q^r. Therefore $T(m_1, m_2, \dots, m_n) = T^r(m_1, m_2, \dots, m_n)$, where $T = \{q\}$ and $T^r = \{q^r\}$. Unfortunately this does not work for complementary permutations since p^c is a permutation on the multiset $\{1^{m_n}, \dots, (n-1)^{m_2}, n^{m_1}\}$ which, in general[1], is different from the multiset $\{1^{m_1}, 2^{m_2}, \dots, n^{m_n}\}$. This observation leaves us with three symmetry classes of permutations of length three:

123, 132 and 213.

In [38] Amy Myers constructed bijections between the set of all 123-avoiding and the set of 132-avoiding multiset-permutations as well as between the set of all 123-avoiding and the set of 213-avoiding multiset-permutations. This was done by using the idea of Simion and Schmidt [46] and constructing lattices whose covering relations are labelled with the elements of $[n]$ in a way that the different maximal chains bijectively correspond to the 123-(respectively 132- or 213-)avoiding permutations. These bijections then prove the following result that had earlier been indicated by Savage and Wilf in [45].

Theorem 4.2.1. *The number of permutations on a given multiset avoiding a pattern* q *of length three is independent of* q.

We therefore merely need to consider one equivalence class of permutations of length three.

Theorem 4.2.2. *For* $A = \{123\}$, $m_2 \geqslant 1$ *and* $n \geqslant 2$ *the following holds:*

$$A(m_1, \dots, m_n) = A(m_1 + m_2, m_3, \dots, m_n) + \sum_{j=0}^{m_1 - 1} A(m_1 - j, m_2 - 1, \dots, m_n).$$

1 In the case of regular multisets, these two multisets are identical and it is thus permitted to consider complements.

Class name	Representative
B_1	$\{123, 132\}$
B_2	$\{123, 231\}$
B_3	$\{123, 321\}$
B_4	$\{132, 231\}$
B_5	$\{132, 312\}$
B_6	$\{132, 321\}$

Figure 4.3: Equivalence classes of pairs of ordinary patterns of length three.

For $m_2 = 0$*,* $A(m_1, m_2, \dots, m_n) = A(m_1, m_3, \dots, m_n)$*.*

In [5] it was shown that the multivariate generating function

$$G(x_1, \dots, x_n) = \sum_{m_1=0} \cdots \sum_{m_n=0} A(m_1, \dots, m_n) {x_1}^{m_1} \dots {x_n}^{m_n}$$

is a symmetric function and is explicitly given by

$$G(x_1, \dots, x_n) = \sum_{i=1}^{n} \frac{{x_i}^{n-1}(1 - x_i)^{n-2}}{\prod\limits_{\substack{1 \leqslant j \leqslant n \\ j \neq i}} \left[(x_i - x_j)(1 - x_i - x_j)\right]}.$$

For instance (see [45]),

$$G(x_1, x_2) = \frac{1}{1 - x_1 - x_2}$$

and

$$G(x_1, x_2, x_3) = \frac{1 - x_1 - x_2 - x_3 + x_1x_2 + x_1x_3 + x_2x_3}{(1 - x_1 - x_2)(1 - x_1 - x_3)(1 - x_2 - x_3)}.$$

4.2.2 *Pairs of patterns of length three*

Figure 4.3 shows the classes of pairs of patterns of length three that shall be considered in this section.

The equivalence class $\overline{\{123, 132\}}$

Theorem 4.2.3.

$$\begin{aligned} B_1(m_1, m_2, \dots, m_n) &= B_1(m_1 + m_2, m_3, \dots, m_n) \qquad (4.1) \\ &+ \sum_{j=1}^{m_2} \binom{m_1 + j - 1}{j} B_1(m_2 - j, m_3, \dots, m_n). \end{aligned}$$

Using the recurrence (4.1), $B_1(m_1, m_2, \dots, m_n)$ can be computed in stages, in a total of $O(N^2)$ steps where N denotes the length $\sum_i m_i$ of the permutation. For the special case $n = 3$ (4.1) can be simplified to:

$$B_1(m_1, m_2, m_3) = \binom{m_1 + m_2 + m_3}{m_3} - \binom{m_2 + m_3}{m_3} + \binom{m_1 + m_2 + m_3}{m_2}.$$

Theorem 4.2.4. *If*

$$b_1(x_1,\ldots,x_n) = \sum_{m_1=0} \cdots \sum_{m_n=0} B_1(m_1,\ldots,m_n) {x_1}^{m_1} \ldots {x_n}^{m_n}$$

denotes the multivariate generating function for B_1*, it holds that:*

$$\begin{aligned} b_1(x_1,\ldots,x_n) &= \frac{x_1}{x_1-x_2} b_1(x_1,x_3,\ldots,x_n) \\ &\quad - \left(\frac{x_2}{x_1-x_2} - \frac{1-x_2}{1-x_1-x_2} + \frac{1}{1-x_1} \right) b_1(x_2,\ldots,x_n). \end{aligned}$$

The equivalence class $\overline{\{123, 231\}}$

Theorem 4.2.5.

$$B_2(m_1,m_2,\ldots,m_n) = \binom{m_1+m_2+\ldots+m_n}{m_1} + \sum_{2 \leqslant i<j \leqslant n} m_i m_j.$$

The equivalence class $\overline{\{123, 321\}}$

This is again a special case of the Theorem of Erdös and Szekeres (recall Section 3.2.3), stating that $B_3(m_1,\ldots,m_n) = 0$ if $n \geqslant 5$. In the other non-trivial cases we obtain:

Theorem 4.2.6. *For permutations on multisets with three distinct elements*

$$B_3(m_1,m_2,m_3) = (m_2+1)\binom{m_1+m_3}{m_1},$$

for permutations on multisets with four distinct elements

$$B_3(m_1,m_2,m_3,m_4) = 2\binom{m_1+m_4}{m_1}.$$

The equivalence class $\overline{\{132, 213\}}$

Theorem 4.2.7.

$$\begin{aligned} B_4(m_1,m_2,\ldots,m_n) &= m_1 B_4(1,m_3,\ldots,m_n) - m_1 B_4(m_3,\ldots,m_n) \\ &\quad + \sum_{j=0}^{m_2} \binom{m_1+j-1}{j} B_4(m_2-j,m_3,\ldots,m_n). \end{aligned}$$

Note that $\{(132^c)^r, (213^c)^r\} = \{(312)^r, (231)^r\} = \{213, 132\} = B_4$ which implies

$$B_4(m_1,m_2\ldots,m_n) = B_4(m_n,m_{n-1},\ldots,m_1).$$

The equivalence class $\overline{\{132, 231\}}$

Theorem 4.2.8.

$$B_5(m_1,\ldots,m_n) = \binom{m_1+m_2}{m_1} \prod_{i=3}^{n} (m_i+1).$$

The equivalence class $\overline{\{132, 312\}}$

Theorem 4.2.9.

$$B_6(m_1, \ldots, m_n) = \sum_{i=1}^{n-1} \binom{N}{N_i} - \sum_{i=1}^{n-2} \binom{N - m_{i+1}}{N_i},$$

where $N_i = \sum_{j=1}^{i} m_j$ *and* $N = \sum_{j=1}^{n} m_j$ *denotes the length of the permutation.*

4.2.3 *Avoiding more than two patterns of length three*

For the sake of completeness we present the results of Albert, Aldred et.al. for sets of patterns of length three containing more than two elements. These are summed up in Figure 4.4. For all cases except for $C_1 = \{123, 132, 213\}$ explicit enumeration formulae could be found. For C_1 a similar recursion as for the case B_1 was however established, stating that

$$C_1(m_1, \ldots, m_n) = \sum_{j=0}^{m_{n-1}} \binom{m_n - 1 + j}{j} C_1(m_1, \ldots, m_{n-1} - j).$$

This allows $C_1(m_1, \ldots, m_n)$ to be computed in $O(\sum_i m_{i-1} m_i)$ steps.

4.3 AVOIDING MULTISET-PATTERNS OF LENGTH THREE

In [25] Heubach and Mansour studied compositions avoiding a single pattern or a pair of patterns of length three on the alphabet $\{1, 2\}$ and then deduced results for permutations on multisets avoiding these patterns. A composition $c = (c_1, c_2, \ldots, c_l)$ of an integer k into l parts is an l-tuple of positive integers for which $c_1 + c_2 + \ldots + c_l = k$. Two l-tuples involving the same elements but differing in their order define different compositions, implying that compositions correspond to ordered partitions. See e.g. Chapter 2 in [12] for an introduction to the topic of compositions and partitions.

There is an apparent connection between compositions and permutations on multisets. Indeed, a permutation $p = p_1 p_2 \ldots p_l$ of length $l = \sum_{i=1}^{n} m_i$ on the multiset $\{1^{m_1}, 2^{m_2}, \ldots, n^{m_n}\}$ defines a composition of $k = \sum_{i=1}^{n} i m_i$ into l parts (involving the elements $1, 2, \ldots, n$). Therefore results for pattern avoidance in compositions can be helpful when trying to enumerate permutations on multisets avoiding certain patterns. In the following, we shall nevertheless only present Heubach and Mansour's results for permutations on multisets since the results for compositions are not directly relevant for us. The authors found explicit enumeration formulae in all cases except for one: the pair of patterns 112 and 122. This exception will be treated in the next section.

Details and proofs of all the results in this section can be found in [25]. Note however that the notation employed there differs from the one used here and introduced in Section 4.1.

Class name	Representative	Enumeration formula
C_1	$\{123, 132, 213\}$	no explicit formula
C_2	$\{123, 132, 231\}$	$\binom{m_1+m_2}{m_1} + \sum_{i=3}^{n} m_i$
C_3	$\{123, 132, 312\}$	$\binom{m_1+m_2+\ldots+m_n}{m_n}$
C_4	$\{123, 132, 321\}$	$\begin{cases} \binom{m_1+m_3}{m_1} + m_2 & \text{if } n = 3, \\ 1 & \text{if } n = 4, \\ 0 & \text{if } n \geqslant 5. \end{cases}$
C_5	$\{123, 231, 312\}$	$\sum_{i=1}^{n} m_i$
C_6	$\{132, 213, 231\}$	$\binom{m_1+m_2}{m_1} + \sum_{i=3}^{n} m_i$
D_1	$\{123, 132, 213, 231\}$	$\binom{m_1+m_2}{m_2}$
D_2	$\{123, 132, 231, 312\}$	$m_n + 1$
D_3	$\{132, 213, 231, 312\}$	2
D_4	$\{123, 132, 213, 321\}$	$\begin{cases} m_2 + 1 & \text{if } n = 3, \\ 1 & \text{if } n = 4, \\ 0 & \text{if } n \geqslant 5. \end{cases}$
D_5	$\{123, 231, 132, 321\}$	$\begin{cases} 2 & \text{if } n = 3, \\ 0 & \text{if } n \geqslant 4. \end{cases}$
D_6	$\{123, 213, 231, 321\}$	$\begin{cases} \binom{m_1+m_3}{m_1} & \text{if } n = 3, \\ 0 & \text{if } n \geqslant 4. \end{cases}$
E_1	$\{123, 132, 213, 231, 312\}$	1
E_2	$\{123, 132, 213, 231, 321\}$	$\begin{cases} 1 & \text{if } n = 3, \\ 0 & \text{if } n \geqslant 4. \end{cases}$

Figure 4.4: Overview over the results in [1] for sets of more than two patterns of length three.

4.3.1 *Single patterns of length 3*

There are seven patterns of length three on the alphabet $\{1,2\}$. Considering reverse permutations[2] eliminates two of these patterns and leaves us with the following five symmetry classes:

$$\tilde{A}_1 = \overline{\{111\}},\ \tilde{A}_2 = \overline{\{112\}},\ \tilde{A}_3 = \overline{\{121\}},\ \tilde{A}_4 = \overline{\{221\}} \text{ and } \tilde{A}_5 = \overline{\{212\}}.$$

The equivalence class $\overline{\{111\}}$

Theorem 4.3.1.

$$\tilde{A}_1(m_1,\dots,m_n) = \begin{cases} \frac{(m_1+m_2+\dots+m_n)!}{m_1!\dots m_n!} & \text{if } m_i \leqslant 2 \text{ for all } i, \\ 0 & \text{otherwise.} \end{cases}$$

The equivalence classes $\overline{\{112\}}$ *and* $\overline{\{121\}}$

Theorem 4.3.2.

$$\tilde{A}_2(m_1,\dots,m_n) = \tilde{A}_3(m_1,\dots,m_n) = \prod_{i=2}^{n} (m_i + \dots + m_n + 1)$$

The equivalence classes $\overline{\{221\}}$ *and* $\overline{\{212\}}$

Theorem 4.3.3.

$$\tilde{A}_4(m_1,\dots,m_n) = \tilde{A}_5(m_1,\dots,m_n) = \prod_{i=1}^{n-1} (m_1 + \dots + m_i + 1)$$

4.3.2 *Pairs of patterns of length three*

We now consider pairs of multiset-patterns of length three. In total, $\binom{7}{2} = 21$ such pairs can be formed but this number can again be reduced for symmetry reasons. Indeed, if a permutation p avoids the pair of patterns (q_1, q_2), then its reverse p^r obviously avoids the pair of patterns $({q_1}^r, {q_2}^r)$. Heubach and Mansour also showed that $\{111, 112\}$ and $\{111, 121\}$ as well as $\{111, 212\}$ and $\{111, 221\}$ are Wilf-equivalent; otherwise no equivalences could be found. The remaining eleven pairs of patterns are presented in Figure 4.5.

The equivalence class $\overline{\{111, 112\}}$

Theorem 4.3.4.

$$\tilde{B}_1(m_1,\dots,m_n) = \begin{cases} 0 & \text{if } m_i \geqslant 3 \text{ for some } i, \\ \prod_{i=2}^{n} (m_i + \dots + m_n + 1) & \text{otherwise.} \end{cases}$$

2 As in Section 4.2 we may not consider complements, since the complement of a given multiset-permutation is generally not a permutation on the same multiset.

Class name	Representative
$\tilde{B}_1$	$\{111, 112\}$
$\tilde{B}_2$	$\{111, 212\}$
$\tilde{B}_3$	$\{112, 121\}$
$\tilde{B}_4$	$\{112, 122\}$
$\tilde{B}_5$	$\{112, 211\}$
$\tilde{B}_6$	$\{112, 212\}$
$\tilde{B}_7$	$\{112, 221\}$
$\tilde{B}_8$	$\{121, 212\}$
$\tilde{B}_9$	$\{122, 212\}$
$\tilde{B}_{10}$	$\{122, 221\}$
$\tilde{B}_{11}$	$\{122, 121\}$

Figure 4.5: Equivalence classes of pairs of multiset-patterns of length three.

The equivalence class $\overline{\{111, 212\}}$

Theorem 4.3.5.

$$\tilde{B}_2(m_1, \ldots, m_n) = \begin{cases} 0 & \text{if } m_i \geqslant 3 \text{ for some } i, \\ \prod_{i=1}^{n-1} (m_1 + \ldots + m_i + 1) & \text{otherwise.} \end{cases}$$

The equivalence class $\overline{\{112, 121\}}$

Theorem 4.3.6.

$$\tilde{B}_3(m_1, \ldots, m_n) = \prod_{i=1}^{n-1} b_i, \quad \text{where } b_i = \begin{cases} (m_i + \ldots m_n) & \text{if } m_i = 1, \\ 1 & \text{otherwise.} \end{cases}$$

The equivalence class $\overline{\{112, 211\}}$

Theorem 4.3.7.

$$\tilde{B}_5(m_1, \ldots, m_n) = \prod_{i=1}^{n-1} b_i, \quad \text{where } b_i = \begin{cases} (m_i + \ldots m_n) & \text{if } m_i = 1, \\ 1 & \text{if } m_i = 2, \\ 0 & \text{if } m_i \geqslant 3. \end{cases}$$

The equivalence class $\overline{\{122, 212\}}$

Theorem 4.3.8.

$$\tilde{B}_9(m_1, \ldots, m_n) = \prod_{i=2}^{n} c_i, \quad \text{where } c_i = \begin{cases} (m_1 + \ldots m_i) & \text{if } m_i = 1, \\ 1 & \text{otherwise.} \end{cases}$$

The equivalence class $\overline{\{122, 221\}}$

Theorem 4.3.9.

$$\tilde{B}_{10}(m_1, \dots, m_n) = \prod_{i=2}^{n} c_i, \quad \textit{where } c_i = \begin{cases} (m_1 + \dots m_i) & \textit{if } m_i = 1, \\ 1 & \textit{if } m_i = 2, \\ 0 & \textit{if } m_i \geqslant 3. \end{cases}$$

The equivalence class $\overline{\{112, 212\}}$

Theorem 4.3.10.

$$\tilde{B}_6(m_1, \dots, m_n) = \sum_{\substack{Q \sqcup P = \\ \{2,\dots,n\}}} \tilde{B}_6(m_{q_1}, \dots, m_{q_s}) \cdot \tilde{B}_6(m_{p_1}, \dots, m_{p_{n-s-1}})$$

where $Q \sqcup P$ *denotes the disjoint union of the two sets* $Q = \{q_1, \dots, q_s\}$ *and* $P = \{p_1, \dots, p_{n-1-s}\}$*. The number of permutations on the empty set is defined to be equal to* 1*.*

The equivalence class $\overline{\{122, 121\}}$

Theorem 4.3.11.

$$\tilde{B}_{11}(m_1, \dots, m_n) = \sum_{\substack{Q \sqcup P = \\ \{1,\dots,n-1\}}} \tilde{B}_{11}(m_{q_1}, \dots, m_{q_s}) \cdot \tilde{B}_{11}(m_{p_1}, \dots, m_{p_{n-s-1}})$$

where $Q \sqcup P$ *denotes the disjoint union of the two sets* $Q = \{q_1, \dots, q_s\}$ *and* $P = \{p_1, \dots, p_{n-1-s}\}$*. The number of permutations on the empty set is defined to be equal to* 1*.*

The equivalence class $\overline{\{121, 212\}}$

Theorem 4.3.12.

$$\tilde{B}_8(m_1, \dots, m_n) = n!.$$

The equivalence class $\overline{\{112, 221\}}$

Theorem 4.3.13.

$$\tilde{B}_7(m_1, \dots, m_n) = \begin{cases} 0 & \textit{if } \exists i, j \textit{ s.t. } m_i, m_j \geqslant 2, \\ n! & \textit{otherwise.} \end{cases}$$

4.4 closing a gap: avoiding the patterns 112 and 122

In this section we are going to close the gap in the work of Heubach and Mansour [25], deducting enumeration formulae for permutations on multisets avoiding both the pattern 112 and the pattern 122. We differ from their work by restricting ourselves to regular multisets, i.e. multisets where every element occurs the same number of times (see Section 2.2). This restriction seems to be necessary in order to obtain explicit formulae

for the number of permutations avoiding the considered two patterns. Nevertheless, we end this section with some remarks on the case of permutations on general multisets.
The methods used in this section have been presented in the Sections 2.3, 2.4 and 2.5. The idea of the proof of our main Theorem 4.4.2 will be to describe the $\{112, 122\}$-avoiding permutations by means of their generating tree. This will then lead to an equation for their generating function that we will be able to solve with the help of the Kernel method. In addition to this proof, we also provide a bijection from $(112, 122)$- avoiding permutations to Dyck words giving further insight to our result.

First, it is clear that $S_{n,1}(112, 122) = n!$ since all permutations on the set $[n]$ avoid both patterns (every element occurs only once and thus a pattern 112 or 122 is impossible).

Theorem 4.4.1. *For* $m > 2$, $S_{n,m}(112, 122) = 2^{n-1}$

Proof. By induction over n.
For $n = 1$ it is clear that $S_{1,m}(112, 122) = 1 = 2^0$, since there is only a single permutation on $[1]_m$ and it obviously avoids both patterns.
Suppose you are given a permutation p in $S_{n,m}$ which avoids 112 and 122 and want to insert the element $(n + 1)$ m-times in order to produce an element of $S_{n+1,m}$ avoiding the given patterns. First observe that the permutation p has to have at least two n's at its beginning, otherwise it will be of the form $p_i \ldots n \ldots n \ldots$ or $np_j \ldots n \ldots n \ldots$ where p_i and p_j are smaller than n and thus $(p_i nn)$ respectively $(p_j nn)$ forms a 122-pattern.
When inserting the first $(n + 1)$ there are two possibilities: it can be placed either at the beginning of the permutation or directly after the first n. Otherwise, i.e. if it were to be placed somewhere after the second n, the permutation would be of the form $nn \ldots (n + 1) \ldots$ and contain a 112-pattern. For the second, third and any following $(n + 1)$ there is no other choice than to place them at the beginning of the permutation, we would otherwise create a 122-pattern. Thus, using the induction hypothesis, we obtain $S_{n+1,m}(112, 122) = 2 \cdot S_{n,m}(112, 122) = 2 \cdot 2^{n-1} = 2^n$. □

Theorem 4.4.2. *For* $m = 2$, $S_{n,m}(112, 122) = c_n$, *where* c_n *denotes the* n*-th Catalan number.*

Proof. We shall first describe the generating tree of 112- and 122-avoiding permutations and then derive the coefficients of its generating function using the *Kernel method*.
Generating tree: Starting with a permutation $p \in S_{n,m}$ which avoids 112 and 122, we determine where the two new elements $(n + 1)$ are permitted to be inserted.
We note that the first $(n + 1)$ may only be inserted before the first element of $[n]$ occurs the second time. The new permutation would otherwise be of the form $\ldots p_i \ldots p_i \ldots (n + 1) \ldots$ and $(p_i p_i (n + 1))$ would form a 112-pattern. Let us denote by r_p the position of the first repetition in the permutation p, i.e. $r_p = \min\{r \in [2n] | \exists i < r : p(i) = p(r)\}$. For the empty permutation ϵ we set $r_\epsilon := 1$. Now, if $r_p = i$, $(n + 1)$ can be placed in front of any of the i-first elements and thereby yield i different permutations.
We see that the second $(n + 1)$ may only be placed at the beginning of the

permutation, otherwise a 122-pattern would be created.
For example, if $p = 2121$, $\nu_p = 3$ and we have three choices for the placing of the first element 3, namely 32121, 23121 and 21321. The second element 3 must be placed at the beginning and p gives us three new permutations in $S_{3,2}$ which avoid both patterns 112 and 122: $p' = 332121$, $p'' = 2323121$ and $p''' = 321321$ with $\nu_{p'} = 2$, $\nu_{p''} = 3$ and $\nu_{p'''} = 4$.
When placing two new elements in a permutation p with $r_p = i$, we obtain one permutation $\tilde{p}$ each with $r_{\tilde{p}} = j$ for all $j \in \{2, \dots, i+1\}$. We can sum up these results in the generating tree of 112- and 122-avoiding permutations. Its nodes are labelled by the position of the first repetition, i.e. with r_p. See Figures 4.6 and 4.7.
The generating tree of these permutations can be described by the simple rewriting rule:

$$\begin{aligned} &(1) \\ &(r) \longrightarrow (2)(3)\dots(r)(r+1) \end{aligned} \tag{4.2}$$

Using the notation introduced by Bousquet-Mélou et.al. in [7] and presented in Section 2.5, we have $k = 2$ and $A = \{0, 1\}$.
Generating function: Let

$$S(u,v) = \sum_{p \in S_{n,2}(112,122)} u^{l_p} v^{r_p} = \sum_{n,r \geqslant 0} s(n,r) u^n v^r$$

be the associated generating function, counting the nodes of the tree by their height (the root is at height 0) and their label, respectively counting the 112- and 122-avoiding permutations by their length l_p and the position r_p of their first repetition. We want to show that

$$[u^n]\, S(u,1) = [u^n] \sum_{n,r \geqslant 0} s(n,r) u^n = c_n.$$

Using the rewriting rule (4.2), we obtain:

$$\begin{aligned} S(u,v) &= v + \sum_{\substack{n \geqslant 0 \\ r \geqslant 0}} s(n,r) u^{n+1} \left(v^2 + v^3 + \dots + v^r + v^{r+1}\right) \\ &= v + uv^2 \sum_{\substack{n \geqslant 0 \\ r \geqslant 0}} s(n,r) u^n \frac{1 - v^r}{1 - v} \\ &= v + uv^2 \frac{S(u,1) - S(u,v)}{1 - v}. \end{aligned}$$

Rewriting this equation gives:

$$S(u,v) = \frac{v\,(v - 1 - uvS(u,1))}{v - 1 - uv^2}. \tag{4.3}$$

This equation can be solved using the *Kernel method*. Let us write (4.3) in the follwoing way:

$$K(u,v)S(u,v) = v\,(v - 1 - uvS(u,1))\,, \tag{4.4}$$

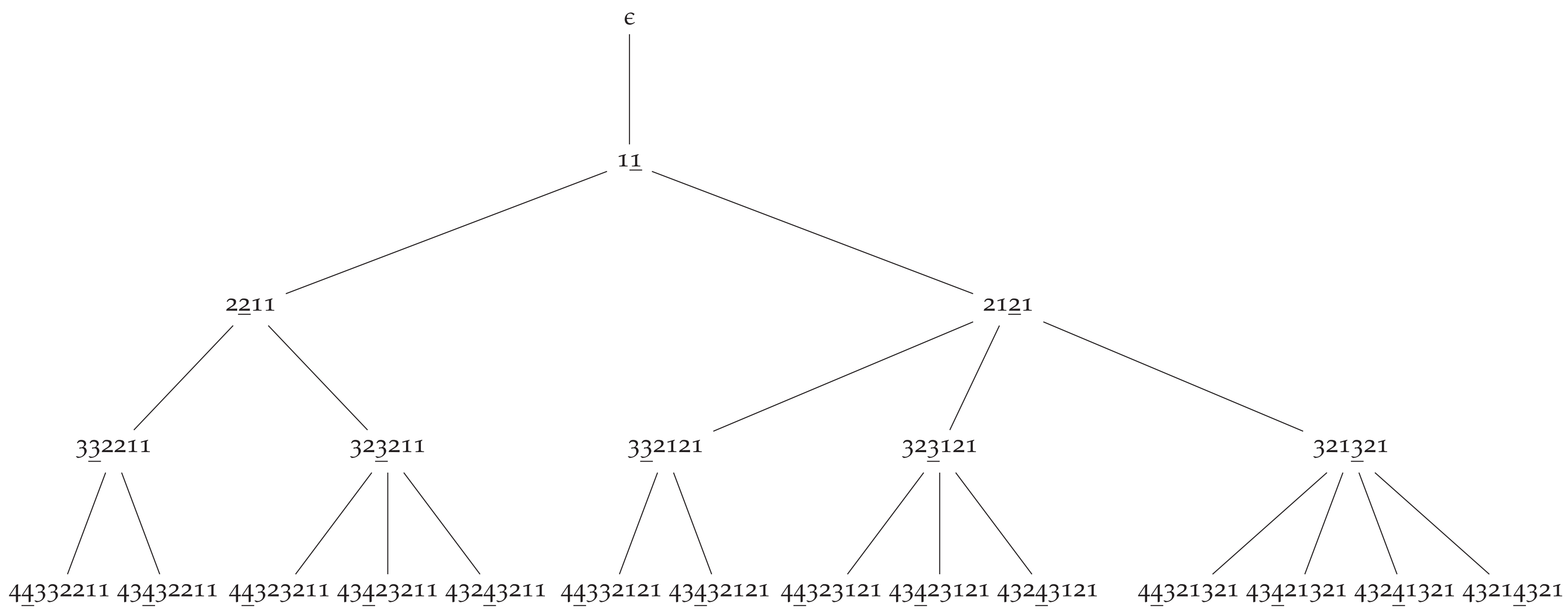


Figure 4.6: Generating tree of 112- and 122-avoiding permutations on regular multisets with $m = 2$. The first repetition is underlined.

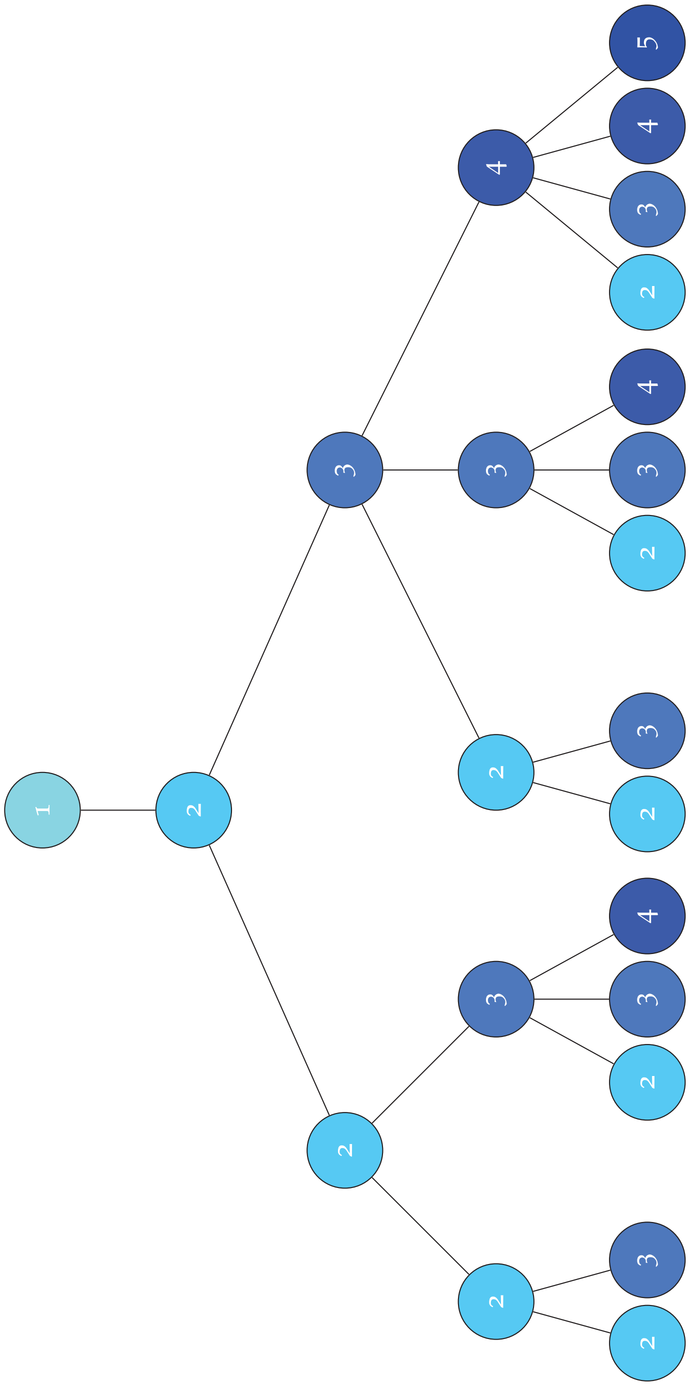


Figure 4.7: Nodes labelled by the position r_p of the first repetition.

where $K(u,v) = v - 1 - uv^2$ is the *kernel* of equation (4.3). This kernel could also have been obtained directly using the formula in Equation (2.6). The roots of $K(u,v)$ are:

$$v_{1,2}(u) = \frac{1 \pm \sqrt{1-4u}}{2u}.$$

Note that $\lim_{u\to 0} v_1(u) \to \infty$ and $\lim_{u\to 0} v_2(u) \to 1$, $v_2(u)$ therefore is the only of the two roots of $K(u,v)$ that can be expanded into a power series in u around 0 (such a root must exist, see [7]). Now recall from Section 2.3 that this root may be plugged into $S(u,v)$ since $S(u,v)$ is a series in u with polynomial coefficients in v. We then obtain that the right-hand side of equation (4.4) must vanish for $v = v_2(u)$. In particular, this means that $v_2(u) - 1 - uv_2(u)S(u,1) = 0$, implying $S(u,1) = v_2(u)$. As is well-known and was shown in the proof of Theorem 3.2.5, this is the generating function of Catalan numbers:

$$S(u,1) = \frac{1-\sqrt{1-4u}}{2u} = \sum_{n\geqslant 1} \frac{1}{n+1}\binom{2n}{n} u^n.$$

□

Remark 4.4.3. Now that we know that $S_{n,2}(112,122) = c_n$, we can also prove this result by giving a bijection from $\mathcal{S}_{n,2}(112,122)$ to the set of Dyck words of the length $2n$.

Definition 4.4.4. *A Dyck word of the length* $2n$ *is a string consisting of* n *X's and* n *Y's such that no initial segment of the string has more Y's than X's. We denote by* $\mathcal{D}_n$ *the set of all Dyck words of length* $2n$.

For example the Dyck words of length 6 are:

XXXYYY, XXYXYY, XXYYXY, XYXXYY, XYXYXY

It is well-known that Dyck words are counted by the Catalan numbers. For example, Dyck words can be counted in the following way: first we count the number of words consisting of n X's and n Y's and then subtract the number of these words which are not Dyck words. In total there are $\binom{2n}{n}$ words built of n X's and n Y's, since we have to choose n among $2n$ positions where the X's are to be placed and fill the remaining n positions with Y's. Now we can use a bijection given in [4] to count the number of such words that are not Dyck words. If you are given a word with n X's and n Y's that is not a Dyck word, pick the first Y for which the Dyck-condition is no longer fulfilled and change all the X's occurring after this Y into Y's and vice-versa. For example, XXYY$\underline{\text{Y}}$XYYXY is turned into XXYY$\underline{\text{Y}}$YXXYX. Using this transformation, we get exactly all the words containing $(n-1)$ X's and $(n+1)$ Y's and there are $\binom{2n}{n-1}$ such words. Hence, the number of Dyck words of the length $2n$ is equal to $\binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n} = c_n$.

Theorem 4.4.5. *The elements of* $\mathcal{S}_{n,2}(112,122)$ *can be identified in a bijective way with those of* $\mathcal{D}_n$.

Proof. Let's start with a Dyck word of length $2n$. The X's shall correspond to the first set of elements $1, \dots, n$ and the Y's to the second set. Replace the sequence of X's by the decreasing sequence $n, n-1, \dots, 2, 1$ and then do the same for the sequence of Y's.
For a given Dyck word $w \in \mathcal{D}_n$, let $\mathrm{occ}(i) := k$, if the i-th letter of w occurs at the position i for the k-th time. Then the function described above is given by:

$$\begin{aligned} F: \mathcal{D}_n &\longrightarrow \mathcal{S}_{n,2} \\ w &\longmapsto p = f(1)f(2)\dots f(2n), \text{where} \\ f(i) &= n - \mathrm{occ}(i) + 1 \end{aligned}$$

For example $F(XXYXYY) = 323121$.
We still have to show that $p = F(w)$ indeed avoids the patterns 112 and 122 for every Dyck word w. p is formed by two decreasing subsequences $(n, n-1, \dots, 2, 1)$ and therefore all the elements to the left of the first occurrence of every $i \in [n]$ must be larger than i. This means that a 122-pattern is impossible. On the other hand, all the elements to the right of the second occurrence of every $i \in [n]$ must be smaller than i. Thus a 112-pattern is also impossible.

Now let us construct a Dyck word from a given element of $\mathcal{S}_{n,2}(112, 122)$. This is easy: the first occurrences of all elements of $[n]$ are transformed into X's and the second ones into Y's. With $\mathrm{occ}(i)$ defined in the same way as above, this means:

$$\begin{aligned} G: \mathcal{S}_{n,2}(112, 122) &\longrightarrow \mathcal{W}_{2n} \\ p &\longmapsto w = g(1)g(2)\dots g(2n), \text{where} \\ g(i) &= \begin{cases} X & \text{if } \mathrm{occ}(i) = 1 \\ Y & \text{if } \mathrm{occ}(i) = 2 \end{cases} \end{aligned}$$

We now need to show that this function G maps $\mathcal{S}_{n,2}(112, 122)$ into $\mathcal{D}_n$ as subset of $\mathcal{W}_{2n}$, the set of words with n X's and n Y's. We prove this by induction over n.
The case $n = 1$ gives us the only possible permutation 11 with $G(11) = XY$ which obviously is a (the only) Dyck word of length two.
Suppose we are given an element $p \in \mathcal{S}_{n,2}(112, 122)$ and its corresponding Dyck word $w = G(p)$. We already know where the two elements $(n+1)$ may be introduced: one $(n+1)$ somewhere to the left of the first repetition in p and another one at the beginning of p. Translated into Dyck words this means: place an X at the beginning of the word and a Y somewhere to the left of the first Y in w. For the new Dyck word $\tilde{w}$ this means that for every initial segment ending before the newly introduced Y there is one X more than in w. Thus for all initial segments ending before this Y, there are certainly more X's then Y's. For every other initial segment, an additional X and an additional Y have been introduced, thus, since w was a Dyck word itself, $\tilde{w}$ is also a Dyck word.
Summing this up, $F: \mathcal{D}_n \to \mathcal{S}_{n,2}(112, 122)$ and $G = F^{-1}$ by construction.

Thus F is a bijection between Dyck words of length $2n$ and permutations on $[n]_2$ avoiding 112 and 122 simultaneously. □

Remark 4.4.6. We have found a closed formula for the number of permutations on *regular multisets* avoiding the two patterns 112 and 122 simultaneously but the case of multisets in general still remains unsolved. The fact that we obtained three different enumeration formulae - one for $m = 1$, one for $m = 2$ and one for the case $m \geqslant 3$ - for the number of 112- and 122-avoiding permutations indicates that no such formula can be found in the general case when using the methods presented here. Indeed, if we try to proceed in the same way as in the proof of Theorem 4.4.2 and want to construct the generating tree of these permutations we cannot derive succession rules that are independent of the level a node lies in: the number of children (and their labels) of a node at the level i depends on the multiplicity m_{i+1}. However, the case where all multiplicities m_i, $i \in \mathbb{N}$, are larger or equal to 3 is easy since all the observations made in the proof of Theorem 4.4.1 are still valid, implying that $\tilde{B}_4(m_1, \ldots, m_n) = 2^{n-1}$.

5 AVOIDING THE PATTERN 122

5.1 AN OVERVIEW OF THE RESULTS OF THIS CHAPTER

In this chapter we are going to take a closer look at permutations on multisets avoiding the pattern 122. Inspired by the work of Kuba and Panholzer [32], who classified 212-avoiding permutations[1] additionally avoiding a set of patterns of length three and derived enumeration formulae for all these Wilf-equivalence classes, we shall complete a similar task for 122-avoiding permutations. We consider permutations avoiding simultaneously the pattern 122 and some other pattern of length three. For patterns on the multiset $\{1, 1, 2\}$ respectively $\{1, 2, 2\}$ this has been done in the previous chapter, presenting the results of Heubach and Mansour [25]. What has not been done yet, is considering the case of permutations on multisets avoiding a pair of patterns of length three, where one pattern is a multiset-pattern and the other one is a regular pattern. Here we will therefore investigate permutations avoiding simultaneously the pattern 122 and an ordinary pattern of length three and give enumeration formulae for these permutations. In this entire chapter, we shall consider permutations on regular multisets, i.e. multisets where $m_i = m$ for all $i \in [n]$ (recall the definitions in Section 4.1). All the results concerning permutations avoiding the pattern 122 are presented in Figure 5.1.
Considering reverses and complements of permutations on regular multisets[2] implies that the patterns 122, 221, 211 and 112 are equivalent. Similarly, the patterns 212 and 121 are equivalent. These are all non-trivial multiset-patterns of length three. Thus we achieve, together with the work of Kuba and Panholzer [32], a full classification of permutations on multisets avoiding a pair of patterns of length three, where one pattern is a multiset-pattern and the other one is a regular pattern.

5.2 $\{122, 123\}$ AND $\{122, 132\}$ ARE WILF-EQUIVALENT

In Chapter 3 we saw that the patterns 123 and 132 were Wilf-equivalent for permutations on sets. We shall now use Simion and Schmidt's idea [46] of the proof of Theorem 3.2.2 to show the following result:

Theorem 5.2.1. *For all $n \in \mathbb{N}$ and all $m \in \mathbb{N}$ it holds that*

$$S_{n,m}(122, 123) = S_{n,m}(122, 132).$$

1 These permutations are also known as generalized Stirling permutations. The notion of Stirling permutations was introduced by Gessel and Stanley in [23].
2 Note that in the case of regular multisets, we may not only consider reverses but also complements.

Forbidden pattern	$m = 1$	$m \geqslant 2$	Proof in Section
122		$\prod_{k=1}^{n-1}(k \cdot m + 1)$	4.3.1 [25]
122, 111		$\begin{cases} 0 & \text{if } m \geqslant 3, \\ \prod_{k=1}^{n-1}(k \cdot m + 1) & \text{otherwise.} \end{cases}$	4.3.2 [25]
122, 112		$\begin{cases} 2^{n-1} & \text{if } m \geqslant 3, \\ c_n = \frac{1}{n+1}\binom{2n}{n} & \text{if } m = 2. \end{cases}$	4.4
122, 121	$\lvert \mathcal{S}_n \rvert = n!$	$c_n = \frac{1}{n+1}\binom{2n}{n}$	4.3.2 [25]
122, 211		0	4.3.2 [25]
122, 212		1	4.3.2 [25]
122, 221		$\begin{cases} 1 & \text{if } m = 2, \\ 0 & \text{otherwise.} \end{cases}$	4.3.2 [25]
122, 123		$c_{m,n} = \frac{1}{m \cdot n+1}\binom{(m+1) \cdot n}{n}$	5.2 and 5.3
122, 132		$c_{m,n} = \frac{1}{m \cdot n+1}\binom{(m+1) \cdot n}{n}$	5.2 and 5.3
122, 231		$s_n = 2 \cdot s_{n-1} + s_{n-2}$ explicit formula in Th. 5.4.4	5.4
122, 213	$c_n = \frac{1}{n+1}\binom{2n}{n}$	$s_n = m \cdot s_{n-1} + s_{n-2}$ explicit formula in Th. 5.5.3	5.5
122, 312		$(n-1) \cdot m + 1$	5.6
122, 321		$\begin{cases} 1 & \text{for } n = 1, \\ m+1 & \text{for } n = 2, \\ 0 & \text{for all } n \geqslant 3. \end{cases}$	5.7

Figure 5.1: 122-avoiding permutations on regular multisets.

First we need to extend the definition of left-to-right-minima respectively left-to-right-maxima from permutations on sets to permutations on multisets.

Definition 5.2.2. *Given a permutation p on a (not necessarily regular) multiset, we call p_i a **left-to-right-minimum** if $p_i \leqslant p_j$ holds for all $j < i$. Analogously p_i is called **left-to-right-maximum** if $p_i \geqslant p_j$ for all $j < i$.*

Proof of Theorem 5.2.1. We use the same map f as in the proof of Theorem 3.2.2 and show that it is a bijection from $\mathcal{S}_{n,m}(122, 123)$ to $\mathcal{S}_{n,m}(122, 132)$. Recall the definition of f and of its inverse g:

- f keeps all the left-to-right-minima fixed, the remaining elements are placed in decreasing order,
- its inverse g also keeps the left-to-right-minima fixed, the remaining elements are placed in the following way: at each free position, place the smallest element not yet placed that is larger than the closest left-to-right-minimum on the left of the given position.

We have already seen that for a given 132-avoiding permutation, $f(p)$ is the only 123-avoiding permutation with the same set and positions of left-to-right-minima as p and reversely, for a given 123-avoiding permutation q, $g(q)$ is the only 132-avoiding permutation with the same set and positions of left-to-right-minima as q. Now, in order to prove that f is a bijection from $\mathcal{S}_{n,m}(122, 123)$ to $\mathcal{S}_{n,m}(122, 132)$, we only need to show that for a 132- and 122-avoiding permutation p its image $f(p)$ is not only 123- but also 122-avoiding and reversely, that $g(q)$ is indeed 122-avoiding for a 123- and 122-avoiding permutation q.

Note that for any 122-avoiding permutation on the regular multiset $[n]_m$ the following elements will always be left-to-right-minima (in this order):

$$\underbrace{n, \ldots, n}_{(m-1)-\text{times}}, \underbrace{n-1, \ldots, n-1}_{(m-1)-\text{times}}, \ldots, \underbrace{2, \ldots, 2}_{(m-1)-\text{times}}, \underbrace{1, \ldots, 1}_{(m-1)-\text{times}}.$$

This can easily be seen by induction over n. For $n = 1$ the induction hypothesis is trivially true. Suppose you are given a 122-avoiding permutation on $[n-1]_m$ and want to introduce the element n m-times in order to produce a 122-avoiding permutation on $[n]_m$. As seen earlier, one element n may occur anywhere in p and all the remaining n's have to be placed at the beginning of the permutation. This block of $(m-1)$ n's will of course be a block of left-to-right-minima. The previous left-to-right-minima will stay left-to-right-minima since all elements of p are smaller than n which proves the induction hypothesis. Since the maps f and g keep the left-to-right-minima fixed, this observation is also true for $f(p)$ and $g(q)$, if p and q are 122-avoiding permutations.

Now let p respectively q be 122- and 123-avoiding permutations and suppose $f(p)$ (respectively $g(q)$) contains a 122-pattern formed by some entries xyy. With the remark made above it follows that at least one of the two y's must be a left-to-right-minimum. On the one hand, x can impossibly also be a left-to-right-minimum since $x < y$. On the other hand, if x is one of the entries that have been rearranged by g, it cannot have been placed to the left of y. To be placed to the left of y, x would have to be larger than the closest left-to-right-minimum on the left. But all left-to-right-minima

w on the left of y are larger than y and thus also larger than x. Therefore $f(p)$ (respectively $g(q)$) avoids the pattern 122.
This finishes the proof. □

5.3 AVOIDING THE PATTERNS 122 AND 123

For 122- and 123-avoiding permutations we prove the following result:

Theorem 5.3.1. *For* $m \geqslant 1$ *and* $n \in \mathbb{N}$ *it holds that*

$$S_{n,m}(122, 123) = \frac{1}{m \cdot n + 1} \binom{(m+1) \cdot n}{n}.$$

Remark 5.3.2. Note that the numbers $c_{m,n} = \frac{1}{m \cdot n+1}\binom{(m+1)\cdot n}{n}$ can be seen as one of many possible generalizations of the well-known Catalan numbers that have been introduced in Chapter 2. Indeed, $c_{1,n} = \frac{1}{n+1}\binom{2n}{n}$ and $c_{m,n}$ reduces to the n-th Catalan number for $m = 1$. These generalized Catalan numbers can be seen as special cases of the so-called Rothe[3] numbers that are defined in the following way: $A_n(a, b) = \frac{a}{a+bn}\binom{a+bn}{n}$ (see e.g. [24]). One easily sees that $A_n(1, m+1) = c_{m,n}$.

Proof. For $m = 1$, i.e. permutations on sets, the pattern 122 will always be avoided and therefore $S_{n,1}(122, 123) = S_n(123) = c_n = c_{1,m}$ as shown in Theorem 4.4.2. In the following we therefore always assume $m \geqslant 2$. As in the proof of Theorem 4.4 we shall first describe the generating tree of 122- and 123-avoiding permutations and then derive the coefficients of its generating function using the kernel method.
Generating tree: Let p be a $[n]_m$-permutation avoiding the patterns 122 and 123, and let us try to insert the element $n+1$ m-times into p without producing one of the forbidden patterns. In order to avoid 122, $(m-1)$ occurences of $(n+1)$ have to be inserted at the beginning of p, the last $(n+1)$ may be placed anywhere. The $m-1$ $(n+1)$'s placed at the beginning will never produce a 123-pattern, so the only restriction we have is that the last $(n+1)$ may not be placed after an increasing subsequence of length 2. In other words, if a_p is the position of the first ascent in p, i.e. $a_p = \min\{i \in [n \cdot m] : p_i < p_{i+1}\}$, then $(n+1)$ may not be inserted after the (a_p+1)-th position. For the empty permutation ϵ we set $a_\epsilon = 0$ and for any permutation p with no ascents $a_p = nm$.
For example ($n = 2$ and $m = 3$), if $p = 221211$, $a_p = 3$. Two 3's must be placed at the beginning of p and the third 3 may be inserted in front of one of the first four elements of p. We obtain four new permutations $\tilde{p}$ in $S_{3,3}$ avoiding the patterns 122 and 123: $\tilde{p}_1 = 333221211$, $\tilde{p}_2 = 332321211$, $\tilde{p}_3 = 332231211$ and $\tilde{p}_4 = 332213211$ where $a_{\tilde{p}_1} = 6$, $a_{\tilde{p}_2} = 3$, $a_{\tilde{p}_3} = 4$ and $a_{\tilde{p}_4} = 5$.
In general, if $a_p = i$, the last element $(n+1)$ may be inserted in front of any of the first $i+1$ elements of p yielding $i+1$ new permutations $\tilde{p}$. If the last $(n+1)$ is inserted at the beginning of p (i.e. $\tilde{p}$ starts with a block of m $(n+1)$'s), no new ascents are inserted and $a_{\tilde{p}} = a_p + m$. If the position (in

3 August Friedrich Rothe (1773-1842) was one of the first to investigate the properties of these sequences in [44].

p) where the last $(n+1)$ is inserted is $j > 1$, then we create a new ascent at $(m-1)+(j-1)$ since $p_{j-1}\ (n+1)$ will always form a 12-pattern and therefore $a_{\tilde{p}} = m+j-2$.
We can sum up these results by translating them into the language of generating trees. Let $\mathcal{T}_m$ be the generating tree of 122- and 123-avoiding permutations where the nodes have been labelled with a_p+1. The Figures 5.2 and 5.3 show $\mathcal{T}_2$ and Figure 5.4 shows the general case $\mathcal{T}_m$. For the general case the rewriting rule is given by:

$$\begin{array}{ll} \text{Root:} & (1) \\ & (a) \quad \longrightarrow (a+m)(m+1)(m+2)\dots(a+m-1) \end{array}$$

Note the similarity to the rewriting rule of the generating tree of 122- and 112-avoiding permutations that are counted by the Catalan numbers (see the proof of Theorem 4.4.2). Using the notation introduced by Bousquet-Mélou et.al. in [7] and presented in Section 2.5, we have $k = m+1$ and $A = \{0, 1, \dots, m\}$.
Generating function: Let

$$S_m(u,v) = \sum_{p \in S_{n,m}(122,123)} u^{l_p} v^{a_p} = \sum_{n,a \geqslant 0} s_m(n,a) u^n v^a$$

be the associated generating function, counting the nodes of the tree by their height and their label, respectively counting the 122- and 123-avoiding permutations by their length l_p and the position a_p of their first ascent. We want to show that

$$[u^n]\, S_m(u,1) = [u^n] \sum_{n,a \geqslant 0} s_m(n,a) u^n = c_{m,n} \text{ for all } m,n \in \mathbb{N}.$$

Using the rewriting rule given above, we obtain:

$$\begin{aligned} S_m(u,v) &= v + \sum_{\substack{n \geqslant 0 \\ a \geqslant 0}} s_m(n,a) u^{n+1} \left(v^{a+m} + v^{m+1} + \dots + v^{a+m-1}\right) \\ &= v + uv^{m+1} \sum_{\substack{n \geqslant 0 \\ a \geqslant 0}} s(n,a) u^n \frac{1-v^a}{1-v} \\ &= v + uv^{m+1} \frac{S_m(u,1) - S_m(u,v)}{1-v} \end{aligned}$$

Rewriting this equation gives:

$$S_m(u,v) = \frac{v(1-v+uv^m S_m(u,1))}{1-v+uv^{m+1}}. \tag{5.1}$$

This equation can be solved using the *Kernel method*. Let us write (5.1) in the following way:

$$K_m(u,v) S_m(u,v) = 1 - v + uv^m S_m(u,1). \tag{5.2}$$

$K_m(u,v) = 1-v+uv^{m+1}$ is the kernel of equation (5.1). Again, this kernel could also have been obtained directly using the formula in Equation (2.6).

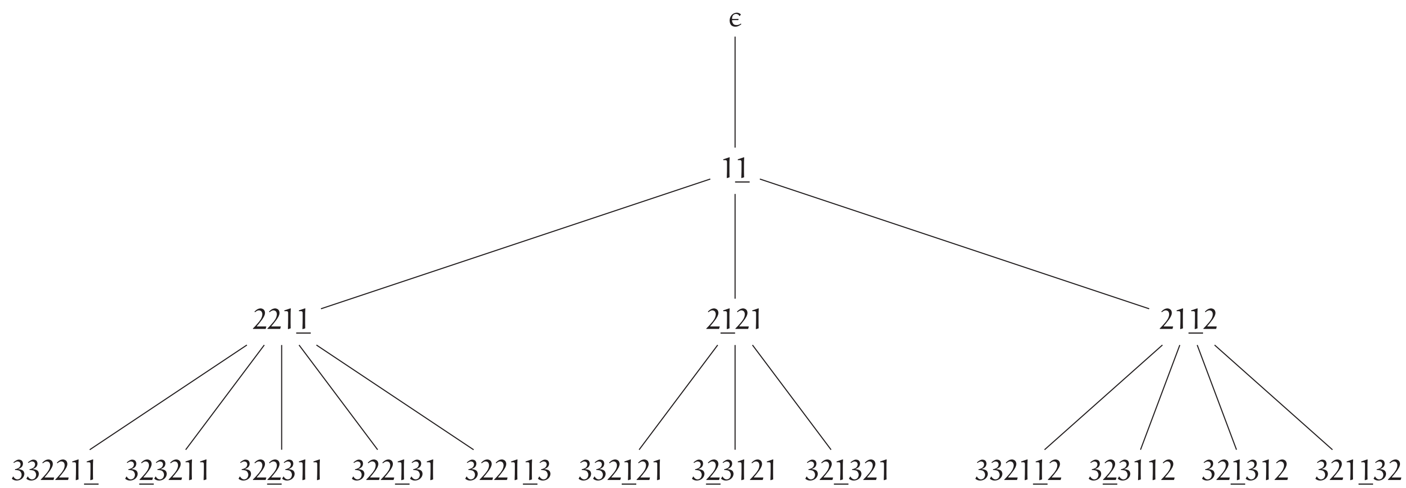


Figure 5.2: $\mathcal{T}_2$: Generating tree of 112- and 123-avoiding permutations on regular multisets with m=2. The first ascent is underlined. New elements have to be introduced before the underlined position.

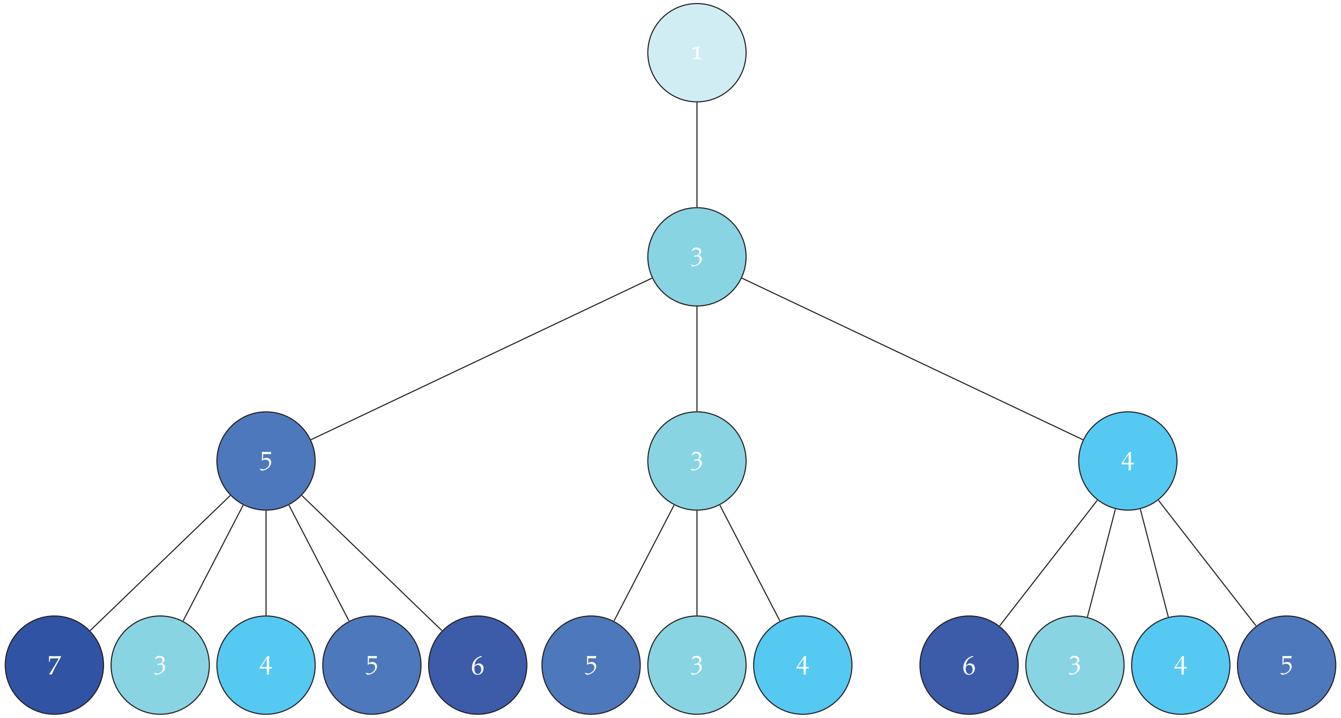


Figure 5.3: $\mathcal{T}_2$ with nodes labelled by $a_p + 1$, where a_p is the position of the first ascent.

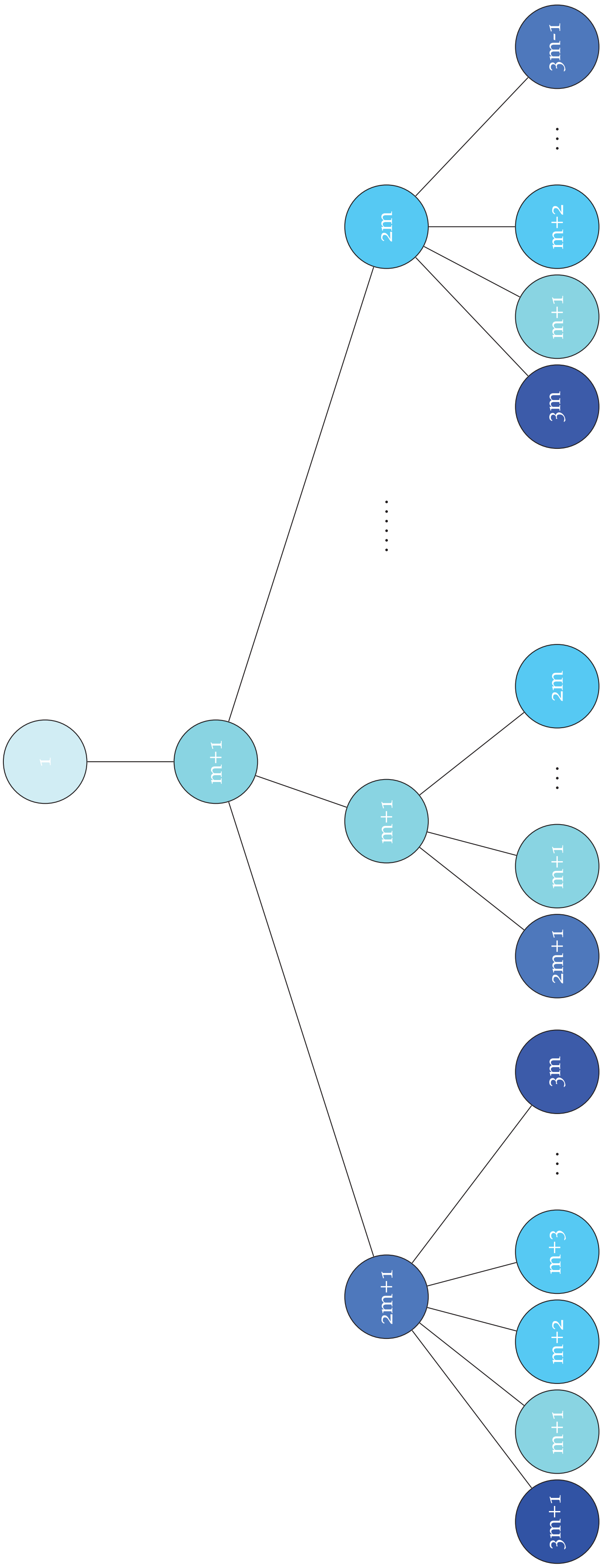


Figure 5.4: $\mathcal{T}_m$ with nodes labelled by $a_p + 1$, the position of the first ascent in p. A node labelled with a has exactly a children.

As stated by Bousqet-Mélou et. al. in [7], this polynomial of degree $m+1$ has exactly one root $v_0(u)$ that can be expanded to a power series in u around 0, the other m roots can be expanded to Laurent series in $u^{1/m}$ around 0. Now, recall from Section 2.5 that this root may be plugged into $S_m(u,v)$, since $S_m(u,v)$ is a series in u with polynomial coefficients in v. We then obtain that the right-hand side of equation (5.2) must vanish for $v = v_0(u)$. In particular, this means that $1 - v_0(u) + uv_0(u)^m S_m(u,1) = 0$, implying $S_m(u,1) = v_0(u)$.

Let us now develop $v_0(u)$ into a power series, using *Lagrange's Inversion Formula* (LIF, see Section 2.3). From [7] it is known that the constant term is 1 and we can therefore write $v_0(u) = 1 + z$, where z is a power series in u with constant term 0. $v_0(u)$ being a root of $1 - v + uv^{m+1}$, this leads to $-z + u(1+z)^{m+1} = 0$, implying

$$u = \frac{z}{\phi(z)}, \text{where } \phi(z) = (1+z)^{m+1}.$$

Note that $\phi_0 = 1$ and $f(z) = v_0(u) = 1 + z$ is a power series in z and LIF can be applied. Noting that

$$(\phi(z))^n = (1+z)^{(m+1)n} = \sum_{k \geqslant 0} \binom{(m+1)n}{k} z^k \text{ leads to}$$

$$\begin{aligned}
\left[u^0\right] f(z) &= \left[z^0\right] f(z) = 1, \\
[u^n] f(z) &= \frac{1}{n} \left[z^{n-1}\right] f'(z) \left(\phi(x)\right)^n \\
&= \frac{1}{n} \binom{(m+1)n}{n-1} \text{ for } n \geqslant 1.
\end{aligned}$$

Putting this together with the remarks made above, we conclude that

$$[u^n] S_m(u,1) = \frac{1}{n} \binom{(m+1)n}{n-1} \text{ for } n \geqslant 1.$$

To finish this proof, note that

$$\frac{1}{n} \binom{(m+1)n}{n-1} = \frac{((m+1)n)!}{n!(mn+1)!} = \frac{1}{mn+1} \binom{(m+1)n}{n} = c_{m,n}.$$

□

Corollary 5.3.3. *For* $m \geqslant 1$ *and* $n \in \mathbb{N}$ *it holds that*

$$S_{n,m}(122, 132) = \frac{1}{m \cdot n + 1} \binom{(m+1) \cdot n}{n}.$$

With the help of the generating tree used in the proof above we can also give a bijective proof of Theorem 5.3.1. Indeed, we can show that $(122, 123)$-avoiding multiset-permutations bijectively correspond to certain lattice paths.

Definition 5.3.4. *A **lattice path** is a path in the integer plane consisting of connected horizontal and vertical line segments. It can be described by a sequence of points* $P_1 = (x_1, y_1), P_2 = (x_2, y_2), \ldots, P_k = (x_k, y_k)$ *with integer coordinates and for all* $i \in [k-1]$ *it either holds that* P_{i+1} *is obtained from* P_i *by*

moving one step to the right, i.e. $x_{i+1} = x_i + 1$ *and* $y_{i+1} = y_i$*, or by moving one step up, i.e.* $x_{i+1} = x_i$ *and* $y_{i+1} = y_i + 1$.
For given integers a, b *and* n *with* $n \geqslant 0$*,* $a, b \geqslant 1$*,* $\mathcal{P}_n(a, b)$ *denotes the set of all lattice paths from* $(0, 0)$ *to* $(a + bn, n)$ *not touching the line* $\Delta : y = \frac{x-a}{b}$ *except at the endpoint.*

For an example of a lattice path in $\mathcal{P}_4(1, 3)$, see Figure 5.7.

Remark 5.3.5. It is obvious that Dyck words can also be interpreted as lattice paths from the origin $(0, 0)$ to (n, n) that lie above the line $y = x$ and may touch it, by translating a letter X into a step up and a letter Y into a step to the right and vice-versa. Such lattice paths are also called Dyck paths. By adding one step to the right at the end, we obtain a lattice path from $(0, 0)$ to $(n + 1, n)$ that does not touch the line $\Delta : y = x - 1$ except at the end. Thus Dyck words of length $2n$ can bijectively be identified with paths in $\mathcal{P}_n(1, 1)$. We already know from Section 4.4 that Dyck words of length $2n$ are counted by the Catalan numbers. Thus we obtain $|\mathcal{P}_n(1, 1)| = c_n = A_n(1, 2)$.

The following result generalizes the remark made above. For a proof see e.g. [37].

Theorem 5.3.6. *For integers* a, b *and* n *with* $n \geqslant 0$*,* $a, b \geqslant 1$*, it holds that*

$$|\mathcal{P}_n(a, b)| = A_n(a, b + 1),$$

where $A_n(a, b)$ *is the generalized Catalan number introduced in Remark 5.3.2.*

We are now going to show the following result:

Theorem 5.3.7. *The elements of* $\mathcal{P}_n(1, m)$ *can bijectively be identified with permutations on the multiset* $[n]_m$ *that avoid the patterns* 122 *and* 123 *simultaneously. This implies*

$$S_{n,m}(122, 123) = |\mathcal{P}_n(1, m)| = A_n(1, m + 1) = c_{n,m}.$$

Proof. First we shall bijectively identify a $(122, 123)$-avoiding $[n]_m$-permutations with a certain sequence of integers of length n. Then we shall do the same for all lattice paths in $\mathcal{P}_n(1, m)$.
Recall the definition of the generating tree of $(122, 123)$-avoiding permutations made in the first proof of Theorem 5.3.1. In this tree, every branch of length n corresponds to a unique permutation that avoids both mentioned patterns. The branch corresponding to a given permutation defines a sequence of length n, namely the sequence of the labels of the nodes defining the branch. This sequence is well-defined and two different n-permutations cannot correspond to the same sequence of integers since, for any given node, each child has a different label. For an example, see Figure 5.5. It can easily be checked that the sequence 14777 in the tree $\mathcal{T}_3$ of $(122, 123)$-avoiding permutations corresponds to the permutation $p = (443322421311) \in \mathcal{S}_{4,3}(122, 123)$.
Now, what kind of sequences can be obtained in this way? The first element of the sequence is always 1, the second one always $(m + 1)$ (these two elements could thus be omitted in the sequence since they bear no information). The third element can be one of the following: $(m + 1), (m +$

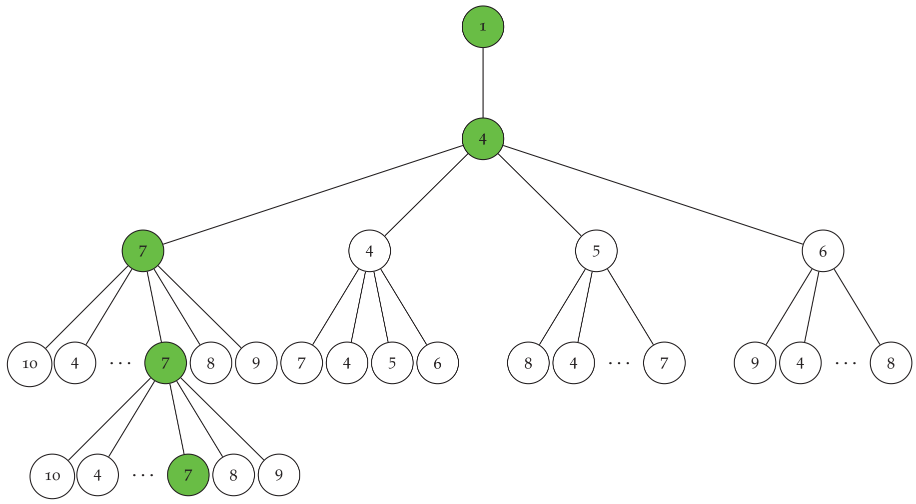


Figure 5.5: In the generating tree of 112- and 123-avoiding permutations on regular multisets with m=3, the sequence 14777 corresponds to the permutation (443322421311).

$2), \ldots, (2m+1)$. In general, if the i-th element in the sequence is a_i, the next element a_{i+1} can be one of the following $(m+1), (m+2), \ldots, (a_i + m)$. The map $f : \mathcal{S}_{n,m}(122, 123) \to \mathcal{B}_{n,m}$ where

$$\mathcal{B}_{n,m} := \{(1, m+1, a_3 \ldots, a_n, a_{n+1}) : m+1 \leqslant a_i \leqslant a_{i-1} + m\}$$

described above is then a bijection.
In order to establish a bijection between the mentioned lattice paths and $(122, 123)$-avoiding permutations, we construct a bijection $g : \mathcal{P}_n(1, m) \to \mathcal{B}_{n,m}$. For this purpose we are going to label the points in the plane through which an allowed lattice path may lead. The label at a certain point shall correspond to the number of different choices that can be made at this point: is one allowed to take an up-step and how many side-steps may one take? This number can easily be calculated with the help of the equation of Δ. For any point (x_i, y_i) with $y_i < n$ lying above the "forbidden" line the number of steps that can be made to the right is $m \cdot y_i - x_i$. Take note of the following for $y_i = n$: the line must be touched at the end which allows for one more side-step but no further up-steps may be taken. So in total, the number of possible steps at the point (x_i, y_i) is equal to $(m \cdot y_i - x_i) + 1$. For instance, in the example of lattice paths from $(0, 0)$ to $(13, 4)$, i.e. $m = 3$ and $n = 4$, shown in Figure 5.6 there are $5 = (3 \cdot 2 - 2) + 1$ possibilities at the point $(2, 2)$: one can take one step up; or one, two, three or four steps to the side. A fifth step to the side would touch the line and is therefore not allowed.
A lattice path from $(0, 0)$ to $(1 + mn, n)$ then defines a sequence of integers as shown in the following. We start at the point $(0, 0)$ which has the label 1, therefore the first element of our sequence is 1. Then we walk along the lattice path and every time an up-step is taken, we note the label of the point that is reached by this up-step. This means that the second element of our sequence must always be $(m + 1)$. For the example leading to the sequence 14777, see Figure 5.7. The labels at level i, i.e. the labels of points with y-coordinate equal to i, range from 1 to $im + 1$ but not all these values

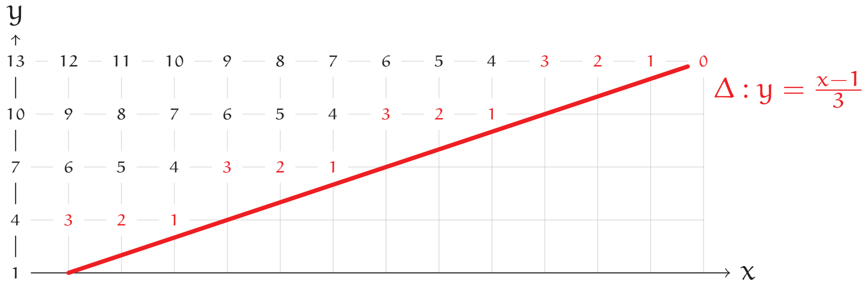


Figure 5.6: Labelled lattice for paths from $(0,0)$ to $(13,4)$ not touching the line Δ except at the endpoint. The points marked in red cannot be reached by an up-step.

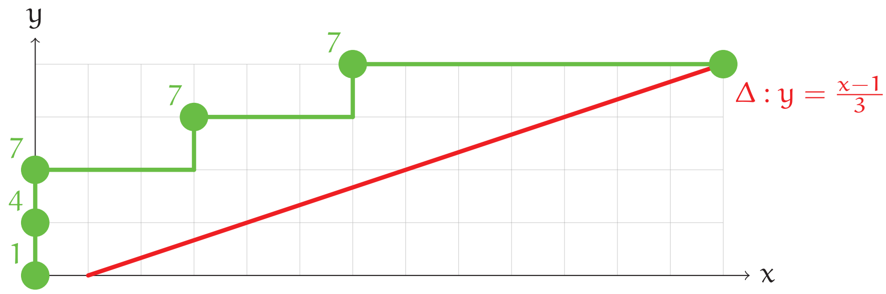


Figure 5.7: In $\mathcal{P}_4(1,3)$ the lattice path marked in green corresponds to the sequence 14777.

can appear in the sequence. On the one hand, not all "allowed" points can be reached by up-steps. In Figure 5.7 these points are emphasized by the use of red labels. In this specific case, one notes that the points that cannot be reached by up-steps are those with labels $1, 2$ or $3 = m$. For the general case, one can easily check that these points are those with labels between 1 and m. On the other hand, one is only allowed to make right-steps and not left-steps and therefore all positions that lie to the left of the steps made so far can no longer be reached. This means that if the label at the i-th level is a_i, the positions labelled with $a_i + m + 1, a_i + m + 2, \ldots, m \cdot i + 1$ cannot be reached at the next level. Thus the sequences obtained in the way described above have the following property: the first two elements are always 1 and $(m+1)$, the $(i+1)$-th element $(i \geqslant 3)$ can take any value between $(m+1)$ and $a_{i-1} + m$. Thus the obtained sequences are again in $\mathcal{B}_{n,m}$. It is clear that conversely any sequence in $\mathcal{B}_{n,m}$ can uniquely be identified with a lattice path in $\mathcal{P}_n(1,m)$, making the map g to a bijection and finishing this proof. □

5.4 AVOIDING THE PATTERNS 122 AND 231

Recall from the previous chapters that if a permutation p avoids a certain pattern q, then its complementary permutation p^c avoids the pattern q^c. Therefore $S_n(q) = S_n(q^c)$ for all patterns q and all $n \in \mathbb{N}$. This holds not only for single permutations, but also for sets of permutations and for permutations on regular[4] multisets. From $(122)^c = 211$ and $(231)^c = 213$ it therefore follows that $S_{m,n}(122, 231) = S_{m,n}(211, 213)$ for all $m, n \in \mathbb{N}$. We will show that $S_{m,n}(211, 213)$ satisfies the following recurrence relation.

Theorem 5.4.1. *Denote* $S_{m,n}(211, 213)$ *by* $s_m(n)$. *Then it holds that*

$$s_m(n) = 2s_m(n-1)) + s_m(n-2)$$

for all $m \in \mathbb{N}$ *and all* $n \geqslant 3$. $s_m(1) = 1$ *and* $s_m(2) = m + 1$.

Proof. As in previous proofs of this type, suppose you are given a permutation $p \in \mathcal{S}_{n-1,m}(211, 213)$ and want to introduce the element n m-times in order to generate a new permutation $\tilde{p} \in [n]_m$ that avoids the patterns 211 and 213.

We will start with the case $m = 2$, it will then be easy to generalize the obtained results.

On the one hand, the new elements may not be placed before the *first* occurrence of any element in order not to create a 211-pattern. Since this rule has also been followed when constructing the permutation p, the first occurrences of all elements $1, 2, \ldots, (n-1)$ appear in increasing order. Therefore, if no n is placed *before* the first occurrence of $n-1$, all n's will end up *behind* all first occurrences and no 211-pattern will be created. Let us denote this position by o_p, i.e. $o_p = \max\left\{i \in [(n-1)\,m] : p_j \neq n-1\ \forall j < i\right\}$ for $p \in \mathcal{S}_{n-1,m}$.

On the other hand, no new element is allowed to be placed after a decreasing subsequence of length 2, otherwise a 213-pattern is created. Let us denote the position of the first descent in a permutation $p \in \mathcal{S}_{n-1,m}$ by d_p, i.e. $d_p = \min\{i \in [(n-1)\,m] : p_i > p_{i+1}\}$ for $p \in \mathcal{S}_{n-1,m}$. If there is no such $i \in [(n-1)\,m]$, i.e. there are no descents in p, we set $d_p = (n-1)\,m$. Therefore, in order to create a 211- and 213-avoiding permutation $\tilde{p}$, the n-elements must be placed to the right of the first occurrence of $(n-1)$ and to the left of the first decreasing subsequence of length 2 in p. In Figure 5.8 the generating tree $\mathcal{T}_2$ of 211- and 213-avoiding permutations is given for $m = 2$ and $n = 1, 2, 3$.

The number of permutations $\tilde{p}$ generated from p simply depends on the distance $a_p := d_p + 1 - o_p$ between the first occurrence of the largest element in p and the first descent. a_p corresponds to the number of positions where new elements are allowed to be placed. For example, if $a_p = 1$ as in $12\underline{32}31$, there is only one possibility of placing the n-elements. Note that if the difference $d_p - o_p$ is negative or zero, no new elements can be introduced. We now claim the following:

Claim 1. *For* $m = 2$ *and* a_p *as defined above, it holds that:*

- a_p *is always equal to* 1 *or to* 2*,*
- *if* $a_p = 1$ *one new permutation* $\tilde{p}$ *is produced with* $a_{\tilde{p}} = 2$*,*
- *if* $a_p = 2$ *two new permutations* $\tilde{p}$ *are produced with* $a_{\tilde{p}} = 2$ *and one new one with* $a_{\tilde{p}} = 1$.

Proof of Claim 1. If $a_p = 1$, there is an element $i \in [(n-1)\,m]$ so that $o_p = i$ and $d_p + 1 = i + 1$. The new n-elements must be introduced

4 Note that p^c is a permutation on $[n]_m$ iff p is a permutation on $[n]_m$. This does not hold for multisets in general. For instance, 122333 is a permutation on the multiset $\{1, 2, 2, 3, 3, 3\}$ but its complement 322111 is a permutation on the multiset $\{1, 1, 1, 2, 2, 3\}$

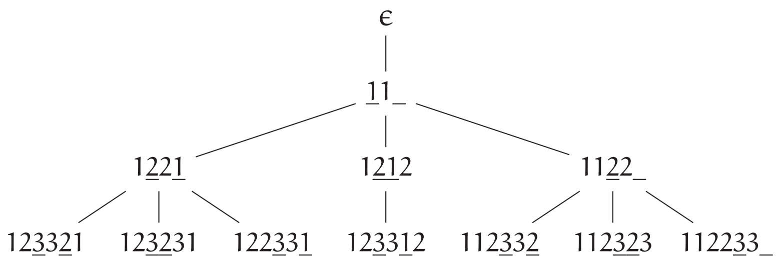


Figure 5.8: Generating tree $\mathcal{T}_2$ of 211- and 213-avoiding permutations on regular multisets with m=2. The position of the first occurrence of the largest element and of the position after the first descent are underlined. When moving from one level to the next one, the new elements are neither allowed to be placed to the left of the first nor to the right of the second underlined element.

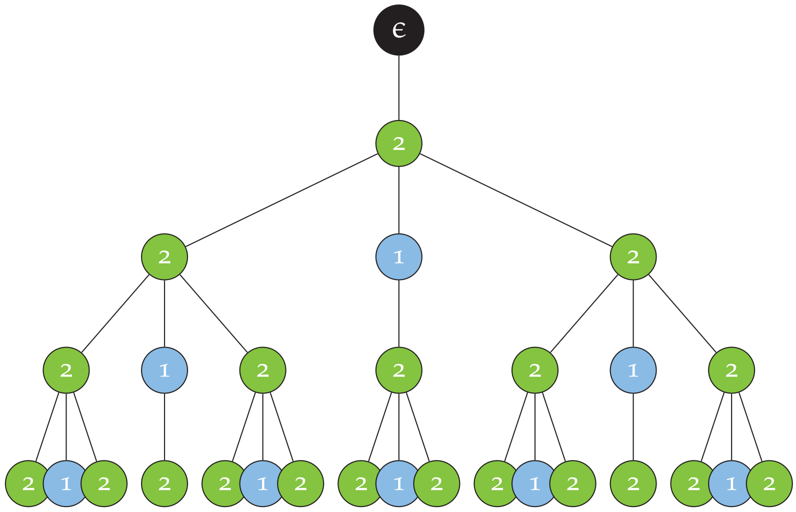


Figure 5.9: $\mathcal{T}_2$ with nodes labelled by a_p, the distance between the first occurrence of the largest element and the first descent.

between the positions i and $i+1$ and therefore the permutation $\tilde{p}$ is of the form $\ldots p_i n n p_{i+1} \ldots$ respectively $\ldots p_i n n$ if $i = (n-1)m$. Then $o_{\tilde{p}} = i+1$ and $d_{\tilde{p}} = i+2$ since the first decreasing subsequence of length 2 in $\tilde{p}$ is formed by (np_{i+1}). Thus $a_{\tilde{p}} = 2$.
Now, if $a_p = 2$ there is an $i \in [(n-1)m-1]$ so that $o_p = i$ and $d_p = i+1$ and there are three possibilities for placing the 2 n's. The permutation $\tilde{p}$ can then be of the form $\ldots p_i n n p_{i+1} p_{i+2} \ldots$ (respectively $\ldots p_i n n p_{i+1}$ if $i = (n-1)m-1$) or $\ldots p_i n p_{i+1} n p_{i+2} \ldots$ (respectively $\ldots p_i n p_{i+1} n$) or $\ldots p_i p_{i+1} n n p_{i+2} \ldots$ (respectively $\ldots p_i p_{i+1} n n$). If two n's are placed next to each other as in the first and the third case, $a_{\tilde{p}} = 2$ follows as above. If the two n's are separated by another element p_{i+1}, $o_{\tilde{p}} = i+1$ and $d_{\tilde{p}} = i+1$ since the first descent is (np_{i+1}), and $a_{\tilde{p}} = 1$.
For $n = 1$ and $p = 11$, it is clear that $a_p = 2$ and thus we have showed that a_p is equal to 1 or to 2 for every permutation p. □

We can collect these results for $m = 2$ in $\mathcal{T}_2$, the generating tree of 211- and 213-avoiding permutations where the nodes are labelled by a_p. See Figure 5.9.

Let us now show similar results for the general case $m \geqslant 2$. First note that new elements may not be introduced before the $(m-1)$-th occurrence of the largest element (this corresponds to the first occurrence for $m = 2$) otherwise a 211-pattern will be created. We shall therefore slightly change the definition of o_p to

$$o_p = \max\left\{i \in [(n-1)\,m] : \left|\left\{j < i : p_j = n-1\right\}\right| = m-2\right\}$$

for $p \in \mathcal{S}_{n-1,m}$. Again, the number of possible positions for the n-elements is equal to $a_p = d_p + 1 - o_p$.
We shall prove the following:

Claim 2. *For $m \geqslant 2$ and a_p as defined above, it holds that:*

- *a_p is always equal to 1 or 2 or is negative,*
- *if a_p is negative no new permutation is produced,*
- *if $a_p = 1$ one new permutation $\tilde{p}$ is produced with $a_{\tilde{p}} = 2$,*
- *if $a_p = 2$ two new permutations $\tilde{p}$ are produced with $a_{\tilde{p}} = 2$, one new one with $a_{\tilde{p}} = 1$ and $(m-2)$ new ones with $a_{\tilde{p}} < 0$.*

Proof of Claim 2. In the first case clearly no new permutation can be produced since $a_p < 0$ means that the first descent is to the left of or at the same position as the $(m-1)$-th occurrence of the largest element, thus every insertion of an n-element would produce a 211- or a 213-pattern.
In the second case there is only a single possibility for placing the new elements where $o_{\tilde{p}} = i + m - 1$ and $d_{\tilde{p}} = i + m + 1$. Thus $a_{\tilde{p}} = 2$.
Now, if the n-elements may be placed in two different positions, there are $m+1$ possibilities leading to 211- and 213-avoiding permutations $\tilde{p}$: place 0 to m elements in the first and the remaining elements in the second position. This leads to three different cases: placing m elements in the first or m elements in the second position leads, as in the case $m = 2$, to two new permutations with $a_{\tilde{p}} = 2$. Placing exactly one element in the first position leads to one new permutation with $a_{\tilde{p}} = 1$. If at least one element is placed in the first and at least two elements are placed in the second position (there are $(m-2)$ such possibilities), we have the following situation: $\tilde{p}$ is of the form $\ldots p_i \ldots n p_{i+1} n \ldots n n p_{i+2}$. Thus the first 21-pattern is given by $n p_{i+1}$, whereas the $(m-1)$-th occurrence of n will always be to the right of p_{i+1} and therefore $a_{\tilde{p}} < 0$.
To finish this, note again that for $n = 1$ and $p = 11\ldots m$, $a_p = 2$. □

We can sum up these general results in $\mathcal{T}_m$, the generating tree of 211- and 213-avoiding permutations, where the nodes are labelled by a_p. See Figure 5.10. It is constructed by applying the following rewriting rule:

$$\begin{aligned} &\text{Root: } (\epsilon) \\ &\quad (\epsilon) \longrightarrow (2) \\ &\quad (2) \longrightarrow (2)(1)\underbrace{(N)\ldots(N)}_{(m-2)-\text{times}}(2) \\ &\quad (1) \longrightarrow (2) \\ &\quad (N) \longrightarrow \emptyset \end{aligned} \tag{5.3}$$

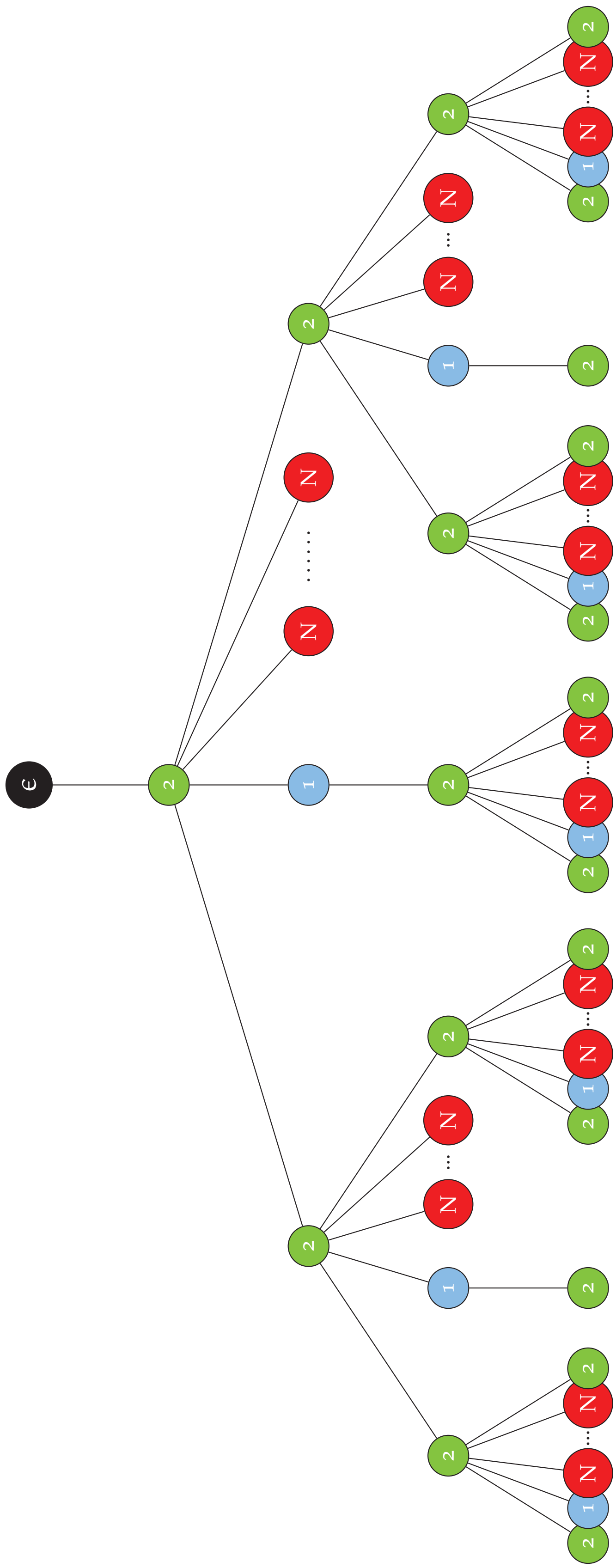


Figure 5.10: $\mathcal{T}_m$ with nodes labelled by a_p, the distance between the $(m-1)$-th occurrence of the largest element and the first descent. The red N-nodes, corresponding to permutations with negative a_p are repeated $(m-2)$-times in every "block" so that every 2-node has exactly $m+1$ children.

Let $s_m(n,i)$ be the number of nodes in the generating tree $\mathcal{T}_m$ labelled with i at the level n. Then $S_n,m(211,213) = s_m(n) = s_m(n,N) + s_m(n,1) + s_m(n,2)$ is the number $S_n,m(211,213)$ of permutations on $[n]_m$ avoiding the patterns 211 and 213.
Translating the rewriting rule (5.3) into recurrence relations we obtain, for $n \geqslant 2$:

$$\begin{aligned}
s_m(n,N) &= (m-2)\cdot s_m(n,1)\\
s_m(n,1) &= s_m(n-1,2)\\
s_m(n,2) &= 2s_m(n-1,2) + s_m(n-1,1)
\end{aligned}$$

Applying these recurrence relations to the definition of $s_m(n)$ leads to the following for $n \geqslant 3$:

$$\begin{aligned}
s_m(n) &= (m-1)\cdot s_m(n,1) + s_m(n,2)\\
&= (m-1)\cdot s_m(n-1,2) + 2s_m(n-1,2) + s_m(n-1,1)\\
&= \underbrace{2\,(m-1)\cdot s_m(n-1,1) + 2s_m(n-1,2)}_{=2s_m(n-1)}\\
&\quad + \underbrace{(m-1)\cdot s_m(n-1,2) - (2m-3)\cdot s_m(n-1,1)}_{\substack{=2(m-1)s_m(n-2,2)+(m-1)s_m(n-2,1)-(2m-3)s_m(n-2,2)\\ =(m-1)s_m(n-2,1)+s_m(n-2,2)}}\\
&= 2s_m(n-1) + s_m(n-2) \qquad (5.4)
\end{aligned}$$

Following the rewriting rule (5.3), the initial values are given by: $s_m(1) = 1$ and $s_m(2) = m+1$.

□

Remark 5.4.2. The sequence defined by the recurrence relation $s(n) = 2s(n-1) + s(n)$ has a certain similarity to the well-known Fibonacci numbers $F_n = F_{n-1} + F_{n-2}$ with initial values $F_0 = 0$ and $F_1 = 1$. Both the sequence of Fibonacci numbers and the sequence $S_{m,n}(211,213)$ can be seen as special cases of the sequences given by the recursion $F_n = k\cdot F_{n-1} + F_{n-2}$. It is a well-known fact that Fibonacci numbers can explicitly be computed using the so-called Binet-formula $F_n = \frac{\varphi^n-(1-\varphi)^n}{\sqrt{5}} = \frac{\varphi^n-(-1/\varphi)^n}{\sqrt{5}}$, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. We shall derive a similar explicit formula for the sequence $S_{m,n}(211,213)$.

Theorem 5.4.3. *Denote* $S_{m,n}(211,213)$ *by* $s_m(n)$*. Then, for* $m \geqslant 2$ *and* $n \geqslant 1$*, it holds that*

$$s_m(n) = \frac{1}{4}\left(\left(2-m\sqrt{2}\right)\left(1-\sqrt{2}\right)^{n-1} + \left(2+m\sqrt{2}\right)\left(1+\sqrt{2}\right)^{n-1}\right).$$

Proof. Instead of $s_m(n)$ we consider the sequence $a_m(n) = s_m(n+1)$ defined for $n \geqslant 0$ and let $A_m(x) = \sum_{n\geqslant 0} a_m(n)x^n$ be its generating function. Using the recurrence relation (5.4) and the initial values for $s_m(n)$ respectively $a_m(n)$ we obtain:

$$A_m(x) = 2x\,(A_m(x)-1) + x^2A_m(x) + 1 + (m+1)\,x. \qquad (5.5)$$

Rearranging equation (5.5) and partial fraction decomposition lead to

$$A_m(x) = \frac{1+(m-1)\,x}{1-2x-x^2} = \frac{1}{4}\left(\frac{c_1}{x+1+\sqrt{2}} + \frac{c_2}{x+1-\sqrt{2}}\right)$$

where $c_1 = 2+2\sqrt{2}-m\left(2+\sqrt{2}\right)$ and $c_2 = 2-2\sqrt{2}-m\left(2-\sqrt{2}\right)$.

Then $A_m(x) = \frac{1}{4}\sum_{n\geqslant 0}\left[\frac{c_1}{1+\sqrt{2}}\left(-\frac{1}{1+\sqrt{2}}\right)^n + \frac{c_2}{1-\sqrt{2}}\left(-\frac{1}{1-\sqrt{2}}\right)^n\right]x^n.$

Noting that $\frac{1}{1+\sqrt{2}} = \sqrt{2}-1$ and that $\frac{1}{1-\sqrt{2}} = -\sqrt{2}-1$ finally leads to

$$\begin{aligned} s_m(n) &= a_m(n-1) = [x^{n-1}]\,A_m(x) \\ &= \frac{1}{4}\left[\left(2-m\sqrt{2}\right)\left(1-\sqrt{2}\right)^{n-1} + \left(2+m\sqrt{2}\right)\left(1+\sqrt{2}\right)^{n-1}\right] \end{aligned}$$

for all $n \geqslant 1$. □

Corollary 5.4.4. *For $m \geqslant 2$ and $n \geqslant 1$ $S_{m,n}(122,231)$ satisfies the recurrence relation*

$$s(n) = 2s(n-1)+s(n-2), \textit{ where } s(1) = 1, s(2) = m+1 \textit{ and it holds that}$$

$$s(n) = \frac{1}{4}\left(\left(2-m\sqrt{2}\right)\left(1-\sqrt{2}\right)^{n-1} + \left(2+m\sqrt{2}\right)\left(1+\sqrt{2}\right)^{n-1}\right).$$

5.5 AVOIDING THE PATTERNS 122 AND 213

Theorem 5.5.1. *Denote $S_{m,n}(211,213)$ by $s_m(n)$. Then it holds that*

$$s_m(n) = ms_m(n-1)) + s_m(n-2)$$

for all $m \in \mathbb{N}$ and all $n \geqslant 3$. $s_m(1) = 1$ and $s_m(2) = m+1$.

Proof. We proceed in the same way as before. Suppose you are given a permutation $p \in \mathcal{S}_{n,m}(122,213)$ and want to introduce the element $(n+1)$ m-times in order to generate a new permutation $\tilde{p} \in [n+1]_m$ that avoids the patterns 211 and 213. From Section **??** and the proof of Theorem 5.4.1 we already know that $(m-1)$ elements have to be placed at the beginning of p and the remaining $(n+1)$ has to be placed somewhere to the left of the position d_p+1 (d_p is the position of the first descent in the permutation p, see again the proof of Theorem 5.4.1). For the same reason there must be $(m-1)$ or m n's at the beginning of p. In the first case $d_p+1 = m$ and in the second case $d_p+1 = m+1$. If $d_p+1 = m$, the single element $(n+1)$ can be placed in m different positions, the one at the beginning of p leading to a permutation $\tilde{p}$ with $d_{\tilde{p}}+1 = m+1$ and all the other positions leading to permutations with $d_{\tilde{p}}+1 = m$. Analogously, if $d_p+1 = m+1$, the single element $(n+1)$ can be placed in $(m+1)$ different positions, leading to one permutation with $d_{\tilde{p}}+1 = m+1$ and m permutations with $d_{\tilde{p}}+1 = m$. Note that $d_p+1 = m+1$ for $p = 1^m$. The generating tree $\mathcal{T}_m$ of 122- and 213-avoiding permutations (see Figure 5.11) with nodes labelled by d_p+1 can therefore be constructed by applying the following rewriting rule.

$$\begin{array}{lll} \text{Root:} & (\epsilon) & \\ & (\epsilon) & \longrightarrow (m+1) \\ & (m+1) & \longrightarrow (m+1)\underbrace{(m)\dots(m)}_{m-\text{times}} \\ & (m) & \longrightarrow (m+1)\underbrace{(m)\dots(m)}_{(m-1)-\text{times}} \end{array} \tag{5.6}$$

Let $s(n,i)$ be the number of nodes in the generating tree $\mathcal{T}_m$ in 5.11 labelled with i at the level n. Then $S_n, m(122,213) = s_m(n) = s(n,m) + s(n,m+1)$ is the number of permutations on $[n]_m$ avoiding the patterns 122 and 213. Translating the rewriting rule (5.6) into recurrence relations we obtain, for $n \geqslant 2$:

$$\begin{aligned} s(n,m) &= (m-1)\,s(n-1,m) + ms(n-1,m+1), \\ s(n,m+1) &= s(n-1,m) + s(n-1,m+1), \\ s(1,m) &= 0, \\ s(1,m+1) &= 1. \end{aligned}$$

For $n \geqslant 3$, applying these recurrence relations to the definition of $s(n)$ leads to:

$$\begin{aligned} s(n) &= s(n,m) + s(n,m+1) \\ &= ms(n-1,m) + ms(n-1,m+1) + s(n-1,m+1) \\ &= ms(n-1) + s(n-2,m) + s(n-2,m+1) \\ &= ms(n-1) + s(n-2) \end{aligned}$$

and the initial values $s(1) = 1$, $s(2) = m+1$.

□

Remark 5.5.2. This sequence defined by the recurrence relation $s_n = ms_{n-1} + s_{n-2}$ can be seen as a generalization of Fibonacci numbers, cf. Remark 5.4.2. Also in this general case, an explicit formula can be given.

Theorem 5.5.3. *Denote $S_{m,n}(211,213)$ by $s_m(n)$. Then, for $m \geqslant 2$ and $n \geqslant 1$, it holds that*

$$\begin{aligned} s_m(n) = \frac{2^{-n}}{\sqrt{m^2+4}} \Bigg(& \left(2+\sqrt{m^2+4}+m\right)\left(m+\sqrt{m^2+4}\right)^{n-1} \\ & + \left(-2+\sqrt{m^2+4}-m\right)\left(m-\sqrt{m^2+4}\right)^{n-1}\Bigg). \end{aligned}$$

Proof. As in the proof of Theorem 5.4.3 we consider the sequence $b_m(n) = s_m(n+1)$ defined for $n \geqslant 0$ instead of $S_{m,n}(211,213) = s_m(n)$ and let $B_m(x) = \sum_{n\geqslant 0} b_m(n)x^n$ be its generating function. Using the recurrence

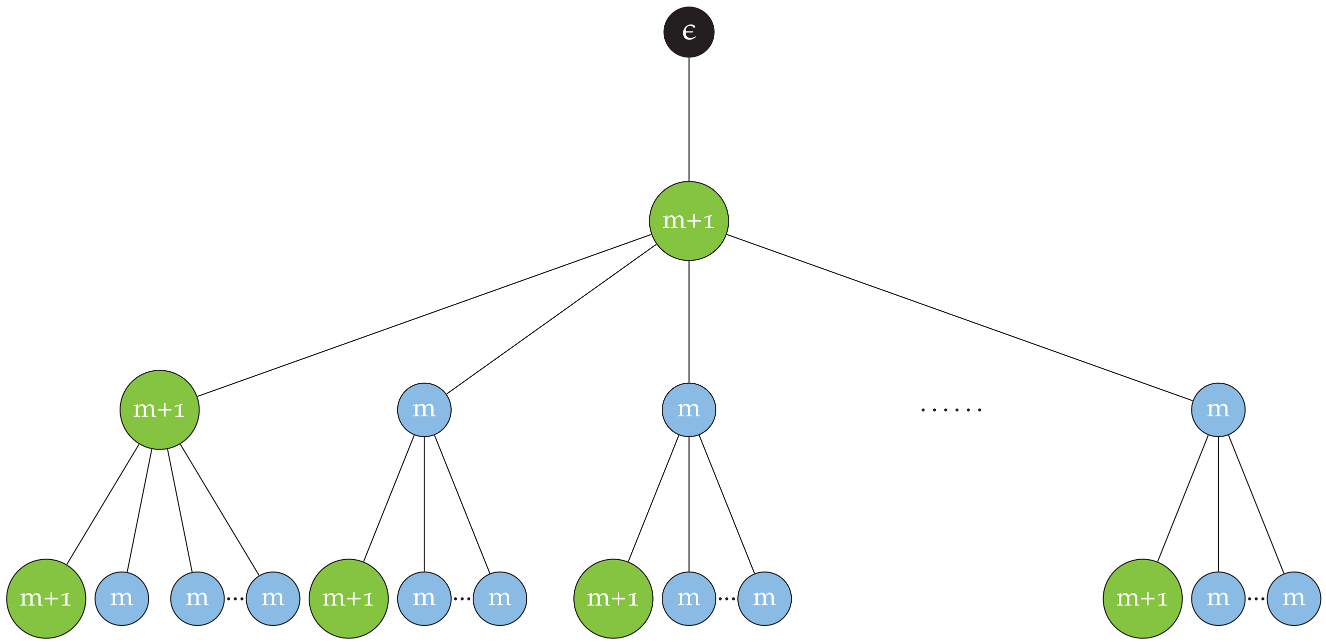


Figure 5.11: Generating tree of 122- and 213-avoiding permutations with nodes labelled by $d_p + 1$, where d_p is the position of the first descent. The dots between two small blue m-nodes correspond to $(m-3)$ small blue m-nodes, so that a big green $(m+1)$-node has exactly $(m+1)$ children and a small blue m-node has exactly m children.

relation (5.6) and the initial values for $s_m(n)$ respectively $b_m(n)$ we obtain:

$$B_m(x) = mx\,(B_m(x) - 1) + x^2 B_m(x) + 1 + (m+1)\,x. \tag{5.7}$$

Rearranging equation (5.7) and partial fraction decomposition leads to

$$B_m(x) = \frac{1+x}{1-mx-x^2} = \frac{1}{2\sqrt{m^2+4}}\left(\frac{c_1}{x+\frac{m-\sqrt{m^2+4}}{2}} + \frac{c_2}{x+\frac{m+\sqrt{m^2+4}}{2}}\right)$$

where $c_1 = -2 - \sqrt{m^2+4} + m$ and $c_2 = 2 - \sqrt{m^2+4} - m$.

$$\text{Then } B_m(x) = \frac{1}{2\sqrt{m^2+4}} \sum_{n \geqslant 0} x^n \left[\frac{c_1}{\frac{m-\sqrt{m^2+4}}{2}}\left(-\frac{1}{\frac{m-\sqrt{m^2+4}}{2}}\right)^n + \frac{c_2}{\frac{m+\sqrt{m^2+4}}{2}}\left(-\frac{1}{\frac{m+\sqrt{m^2+4}}{2}}\right)^n\right]$$

Noting that $\frac{2}{m-\sqrt{m^2+4}} = -\frac{m+\sqrt{m^2+4}}{2}$ and that $\frac{2}{m+\sqrt{m^2+4}} = -\frac{m-\sqrt{m^2+4}}{2}$ finally leads to

$$\begin{aligned} s_m(n) &= b_m(n-1) = \left[x^{n-1}\right] B_m(x) \\ &= \frac{2^{-n}}{\sqrt{m^2+4}}\left[\left(2+\sqrt{m^2+4}+m\right)\left(m+\sqrt{m^2+4}\right)^{n-1} \right. \\ &\quad \left. + \left(-2+\sqrt{m^2+4}-m\right)\left(m-\sqrt{m^2+4}\right)^{n-1}\right] \end{aligned}$$

for all $n \geqslant 1$. □

5.6 AVOIDING THE PATTERNS 122 AND 312

Theorem 5.6.1. $S_{n,m}(122, 312) = (n-1) \cdot m + 1$ *for all* $n \in \mathbb{N}, m \geqslant 2$.

Proof. We prove this statement by induction over n. For $n = 1$, the only possible permutation 1^m obviously avoids both patterns and $S_{1,m} = (1-1) \cdot m + 1 = 1$ is true for all $m \in \mathbb{N}$.
Now suppose you are given a permutation p on $[n-1]_m$, where $n > 1$, that avoids both the patterns 122 and 312 and want to introduce the new element n m-times in order to obtain a permutation $\tilde{p} \in \mathcal{S}_{n,m}(122, 312)$. On the one hand we know that at least $(m-1)$ n's have to be introduced at the beginning of p if a 122-pattern should be avoided. On the other hand, in order to produce a 312-avoiding $\tilde{p}$, the n-entries may not be introduced before a 12-pattern, i.e. before an ascending subsequence of length 2. Thus, only permutations with no ascents can lead to new 122- and 312-avoiding permutations. This means that the only permutation p on $[n-1]_m$ that can generate elements of $\mathcal{S}_{n,m}(122, 312)$ is:

$$p = \underbrace{n-1, \ldots, n-1}_{m-times}, \underbrace{n-2, \ldots, n-2}_{m-times}, \ldots, \underbrace{2, \ldots, 2}_{m-times}, \underbrace{1, \ldots, 1}_{m-times}.$$

Place one n at any position of p and all the others at its beginning and you will obtain a permutation $\tilde{p} \in \mathcal{S}_{n,m}(122, 312)$. For placing the single n, there are $(n-1) \cdot m + 1$ possibilities, since the length of p is $(n-1) \cdot m$.

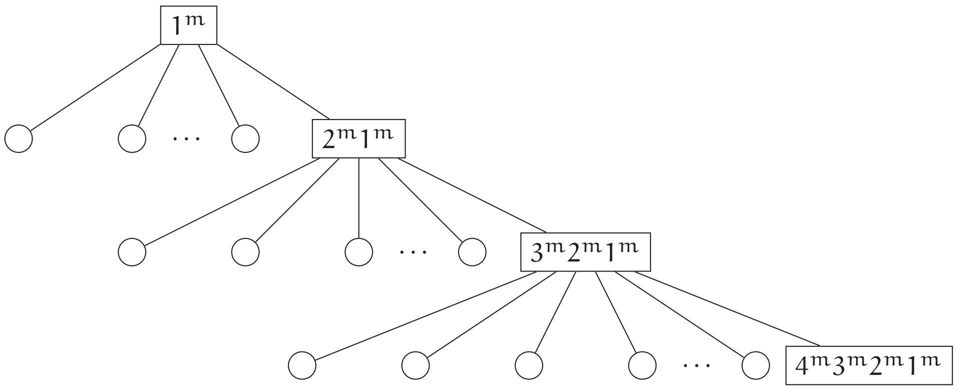


Figure 5.12: Generating tree of 122- and 312-avoiding permutations. Round nodes do not have any children, whereas rectangular nodes at the n-th level have exactly $nm+1$ children - nm round nodes and one rectangular one.

Thus, for all natural numbers n and m there are $(n-1)\cdot m+1$ permutations on $[n]_m$ avoiding the given patterns. This can be illustrated by drawing the generating tree of 122- and 312-avoiding permutations, see Figure 5.12. □

5.7 AVOIDING THE PATTERNS 122 AND 321

Theorem 5.7.1. *For* $m \geqslant 2$

$$S_{n,m}(122,321) = \begin{cases} 1 & \textit{for } n=1, \\ m+1 & \textit{for } n=2, \\ 0 & \textit{for all } n \geqslant 3. \end{cases}$$

Proof. For $n \leqslant 2$, all permutations on $[n]_m$ avoid the pattern 321 and thus $S_{n,m}(122,321) = S_{n,m}(122)$. With Theorem 4.3.3 or 4.3.2 we obtain $S_{1,m}(122,321) = 1$ and $S_{2,m}(122,321) = m+1$.
As seen for 122-avoiding permutations in the proof of Theorem 5.2.1, the elements $n^{m-1}, (n-1)^{m-1}, \ldots, 2^{m-1}, 1^{m-1}$ will always appear in this order from left to right (and are left-to-right-minima). Thus, for $n \geqslant 3$ and $m \geqslant 2$, we can always find a $n\,(n-1)\,(n-2)$-subsequence and every $p \in \mathcal{S}_{n,m}$ contains the pattern 321. □

6 CONCLUSION

In the last section of Chapter 4 and in Chapter 5 we developed new results concerning permutations on regular multisets avoiding the pattern 122. These results are summarized in Figure 6.1. They deliver an interesting contribution to the study of restricted permutations on multisets since they complete the classification of permutations on multisets avoiding a pair of patterns of length three, where one pattern is a multiset-pattern and the other one is a regular pattern, which was started by Kuba and Panholzer in [32]. Our results also lay the base for several questions opening doors to future research. We group these questions around three sets of conclusions.

1: A first remark than can be made is that the methods utilised here (generating trees, generating functions and the Kernel method) had in the past been used in the context of pattern avoidance in ordinary permutations. In this thesis, they were successfully applied to the study of permutations on (regular) multisets. From this, there arise a number of questions: Can other methods used in pattern avoidance also be used for multiset-pattern avoidance? Can the methods used in this thesis also be applied to other multiset-pattern-problems? Could one carry on the task of enumerating permutations on multisets avoiding a pair (q_1, q_2) of patterns of length three where q_1 is a pattern on a multiset and q_2 is an ordinary patterns with the use of the same methods? Could these methods even be applied to the study of multiset-permutations avoiding longer patterns?

2: Another interesting fact is that we were capable of developing "nice" enumeration formulae in all cases. For all considered pairs of patterns, we did not only obtain recursions or functional equations for the generating functions but also developed explicit formulae for the number of permutations with n distinct elements avoiding these patterns. This was for instance not the case in the study of multiset-permutations avoiding one or more ordinary pattern of length three in [1]. This raises the following question: Is it due to the fact that all the permutations considered here avoid the pattern 122 that we obtained such satisfying results? Or is it due to the fact that we restricted our analysis to regular multisets? Can the method of generating trees be successfully applied to permutations on not necessarily regular multisets or do other methods have to be used in general?

3: The sequences obtained in our research were not just any unknown integer sequences: in one case we obtained the sequence of Catalan numbers, in two Wilf-equivalent cases we obtained generalized Catalan numbers and in two other cases, generalized Fibonacci numbers were the answer. The sequence of Catalan numbers is one of the most-studied integer sequences and nearly two hundred examples of objects counted by them are known do date. This enabled us to give bijective proofs in the case of ordinary and generalized Catalan numbers. This had already been encountered in pattern avoidance: the very first result within this field

<table>
<tr><th>Forbidden patterns</th><th>$m = 1$</th><th>$m \geqslant 2$</th><th>Proof in Section</th></tr>
<tr><td>122, 112</td><td>$|\mathcal{S}_n| = n!$</td><td>$\begin{cases} 2^{n-1} & \text{if } m \geqslant 3, \\ c_n = \frac{1}{n+1}\binom{2n}{n} & \text{if } m = 2. \end{cases}$
Extra: Bijection to Dyck words</td><td>4.4</td></tr>
<tr><td>122, 123</td><td rowspan="6">$c_n = \frac{1}{n+1}\binom{2n}{n}$</td><td>$c_{m,n} = \frac{1}{m \cdot n+1}\binom{(m+1) \cdot n}{n}$
Extra: Bijection to lattice paths</td><td>5.2 & 5.3</td></tr>
<tr><td>122, 132</td><td>$c_{m,n} = \frac{1}{m \cdot n+1}\binom{(m+1) \cdot n}{n}$
Extra: Bijection to lattice paths</td><td>5.2 & 5.3</td></tr>
<tr><td>122, 231</td><td>$s_n = 2 \cdot s_{n-1} + s_{n-2}$
explicit formula in Th. 5.4.4</td><td>5.4</td></tr>
<tr><td>122, 213</td><td>$s_n = m \cdot s_{n-1} + s_{n-2}$
explicit formula in Th. 5.5.3</td><td>5.5</td></tr>
<tr><td>122, 312</td><td>$(n-1) \cdot m + 1$</td><td>5.6</td></tr>
<tr><td>122, 321</td><td>$\begin{cases} 1 & \text{for } n = 1, \\ m+1 & \text{for } n = 2, \\ 0 & \text{for all } n \geqslant 3. \end{cases}$</td><td>5.7</td></tr>
</table>

Figure 6.1: New results obtained in this thesis concerning 122-avoiding permutations on regular multisets.

[33] was that 123-avoiding permutations are counted by Catalan numbers. One of the many proofs given by Knuth in [30] for the equivalent result that 231-avoiding are counted by Catalan numbers was a bijection from stack-sortable, i.e. 231-avoiding, permutations to Dyck words. Fibonacci numbers had also appeared earlier, e.g. in the work of Simion and Schmidt [46] who considered permutations avoiding more than one pattern of length three. This raises the questions: How come Catalan and Fibonacci numbers pop up again and again within the field of pattern avoidance? In general, how come Catalan numbers arise in so many different fields of mathematics? What makes them apt to describe such a multitude of objects that are of (mathematical) interest?

Beyond the three sets of questions arising from our own research, we wish to draw the attention to the Stanley-Wilf conjecture which we presented in Chapter 3. Recall that this far-reaching result stated that the number $S_n(q)$ can be bounded by ${c_q}^n$ where c_q is a constant that depends only on the length of the pattern q. It would be very tempting to extend this result to the number of multiset-permutations avoiding a given pattern and to prove that it is exponentially bounded. Unfortunately this cannot be true in the case when the forbidden pattern itself is a permutation on a multiset. This can for instance be seen with the number of k-Stirling permutations (these are 212-avoiding permutations on the multiset $[n]_k$) which is equal to $n!k^n\binom{n-1+\frac{1}{k}}{n}$ and thus grows super-exponentially, as stated in [32]. For patterns on ordinary sets however, it seams plausible that the various elements of the proof of Stanley-Wilf may be extended to multiset-permutations if slight adaptations are made. It would certainly be worthwhile to investigate this question.

A

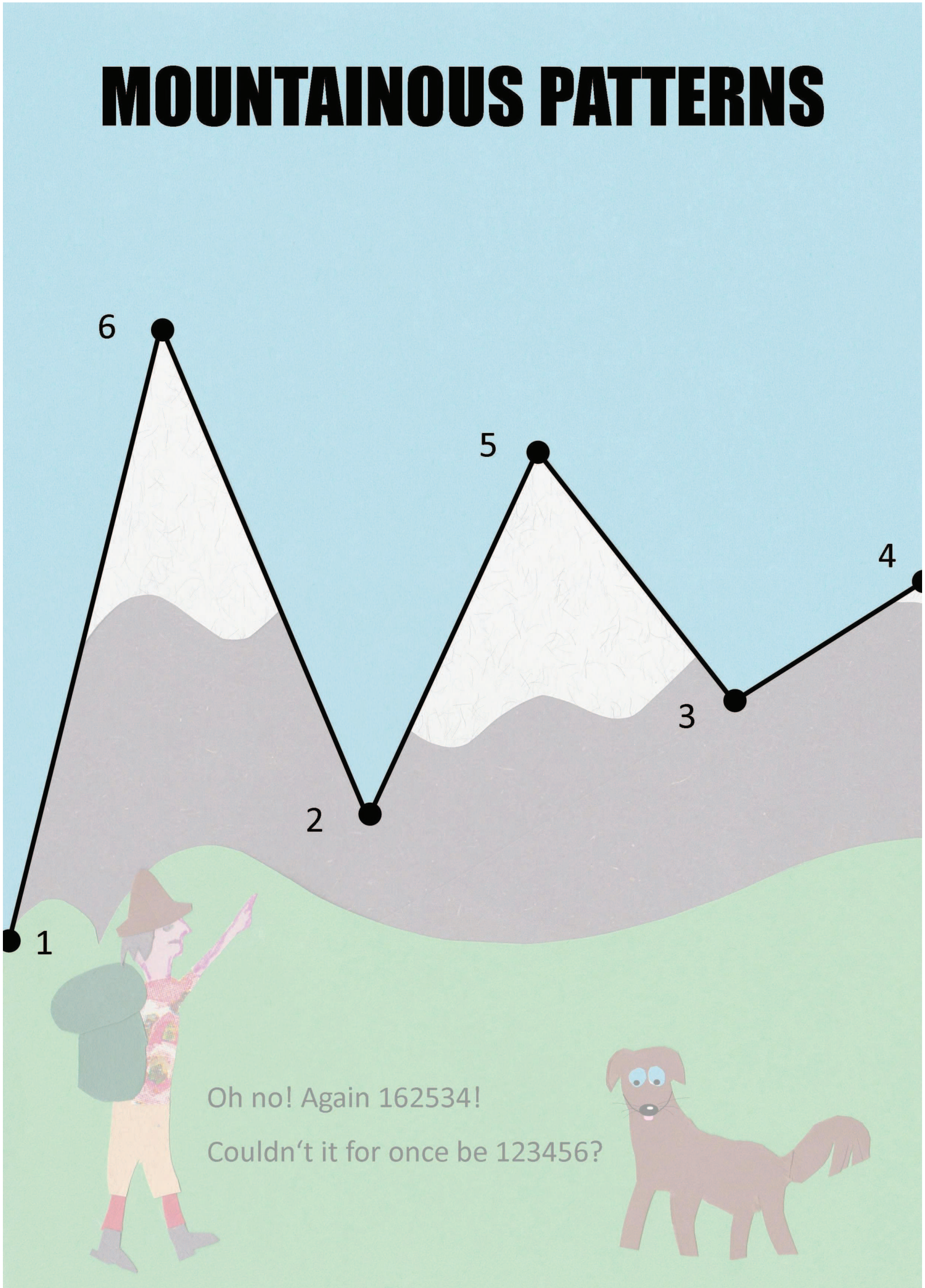
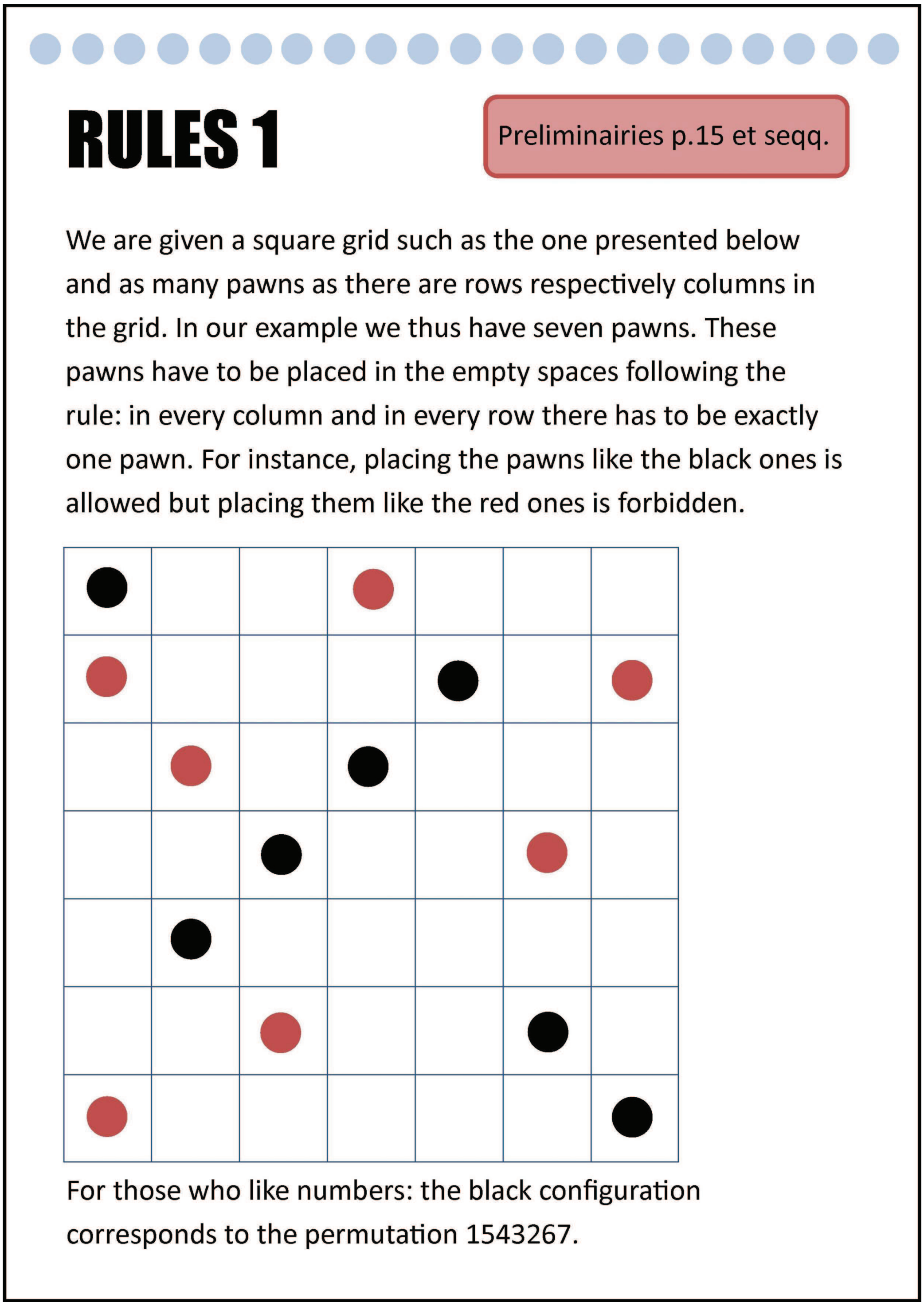


Figure A.1: Cover of the riddle-booklet entitled "Mountainous patterns".

The booklet entitled MOUNTAINOUS PATTERNS is an attempt to make the topic of this thesis accessible to a broad public that does not necessarily have a mathematical background. MOUNTAINOUS PATTERNS is a series of riddles explaining what pattern avoidance is about and offering an illustration of some results presented in this thesis. I very much hope that these riddles give newcomers a playful insight into the fascinating topic of pattern avoidance.
The representation of permutation matrices with the help of square (or rectangular in the case of multisets) grids introduced in Chapter 3 is used to construct more than forty pattern avoidance-riddles, ranging from easy to extremely difficult. All riddles were given the name of an Austrian park, hill or mountain, indicating the difficulty level. The booklet is divided into three parts. In the first part, pawns have to be placed in a square grid following certain rules. This corresponds to the Chapters 2 and 3 and illustrates pattern avoidance in permutations. In the second part, the square grids are replaced by rectangular ones. This corresponds to the Chapters 2, 4 and 5 where ordinary permutations have been replaced by permutations on multisets. In the last part, solutions to all the riddles are given.
In this appendix we merely provide a brief excerpt of this booklet, presenting the following pages:

- the cover (see Figure A.1),
- the rules of the game for the first part of riddles (see Figures A.2, A.3 and A.4),
- Schönbrunn, a riddle leading to the result that there are exactly $n!$ permutations of length n (see Figure A.5),
- Sophienalpe, where occurrences of the pattern 12 have to be found and counted (see Figure A.6),
- Hirschenkogel, in which the pattern 123 must be avoided (see Figure A.7),
- Planspitze, a riddle in which the two patterns 123 and 321 have to be avoided simultaneously and the Erdős-Szekeres-Theorem is illustrated (see Figure A.8),
- Gahns, where occurrences of the multiset-patterns 111 and 1111 have to be found and counted in a multiset-permutation (see Figure A.9),
- Dachstein, in which the patterns 122 and 132 must be avoided, illustrating one of the new results of this thesis (see Figure A.10) and
- a page with solutions to the riddles Schönbrunn, Sophienalpe, Planspitze and Dachstein (see Figure A.11).

In case you should be interested in the entire booklet MOUNTAINOUS PATTERNS, please send an email to ml.bruner@gmail.com. Please also do so in case you find any mistakes, have ideas for new riddles or want to give me feedback.



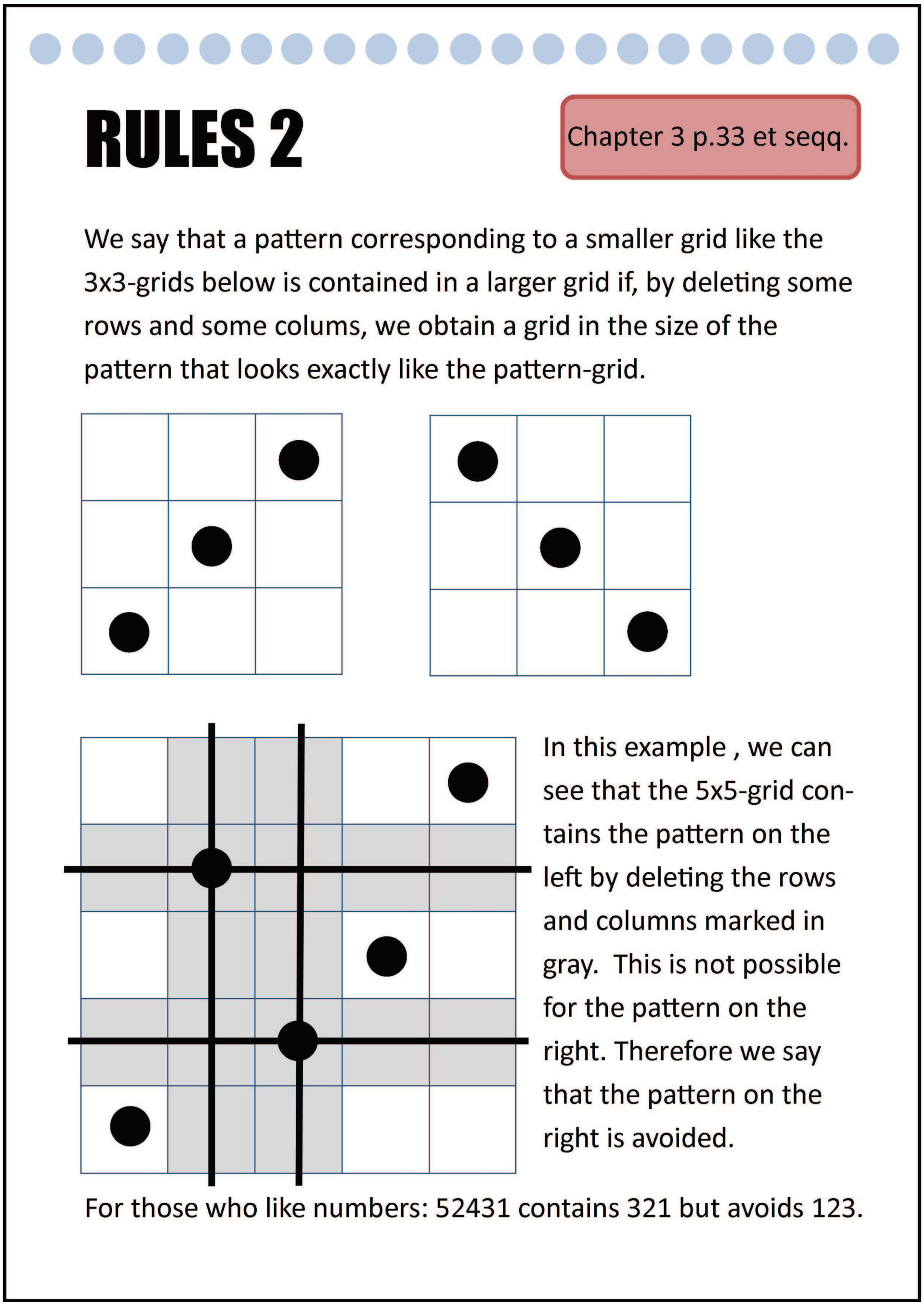
RULES 1

Preliminairies p.15 et seqq.

We are given a square grid such as the one presented below and as many pawns as there are rows respectively columns in the grid. In our example we thus have seven pawns. These pawns have to be placed in the empty spaces following the rule: in every column and in every row there has to be exactly one pawn. For instance, placing the pawns like the black ones is allowed but placing them like the red ones is forbidden.

For those who like numbers: the black configuration corresponds to the permutation 1543267.

Figure A.2: First page of the rules for the first part of the booklet.



RULES 2

Chapter 3 p.33 et seqq.

We say that a pattern corresponding to a smaller grid like the 3x3-grids below is contained in a larger grid if, by deleting some rows and some colums, we obtain a grid in the size of the pattern that looks exactly like the pattern-grid.

In this example , we can see that the 5x5-grid contains the pattern on the left by deleting the rows and columns marked in gray. This is not possible for the pattern on the right. Therefore we say that the pattern on the right is avoided.

For those who like numbers: 52431 contains 321 but avoids 123.

Figure A.3: Second page of the rules for the first part of the booklet.

RULES 3

No reference

The level of difficulty is specified with the help of stars:

☆☆☆☆☆	EASY-PEASY
★☆☆☆☆	WARM-UP
★★☆☆☆	HALF-WAY
★★★☆☆	HEAD-SCRATCHER
★★★★☆	TOUGH NUT
★★★★★	BRAIN-KILLER

If you need a hint or want to read up on the theoretical background of MOUNTAINOUS PATTERNS, the chapters, definitions, theorems or pages of my thesis referred to in the red boxes in the top right corner of every page can be helpful.

I hope you will enjoy solving these riddles and get a playful insight into the exciting field of pattern avoidance.

GOOD LUCK!

Vienna, May 2011 Marie-Louise Bruner

Figure A.4: Third page of the rules for the first part of the booklet.

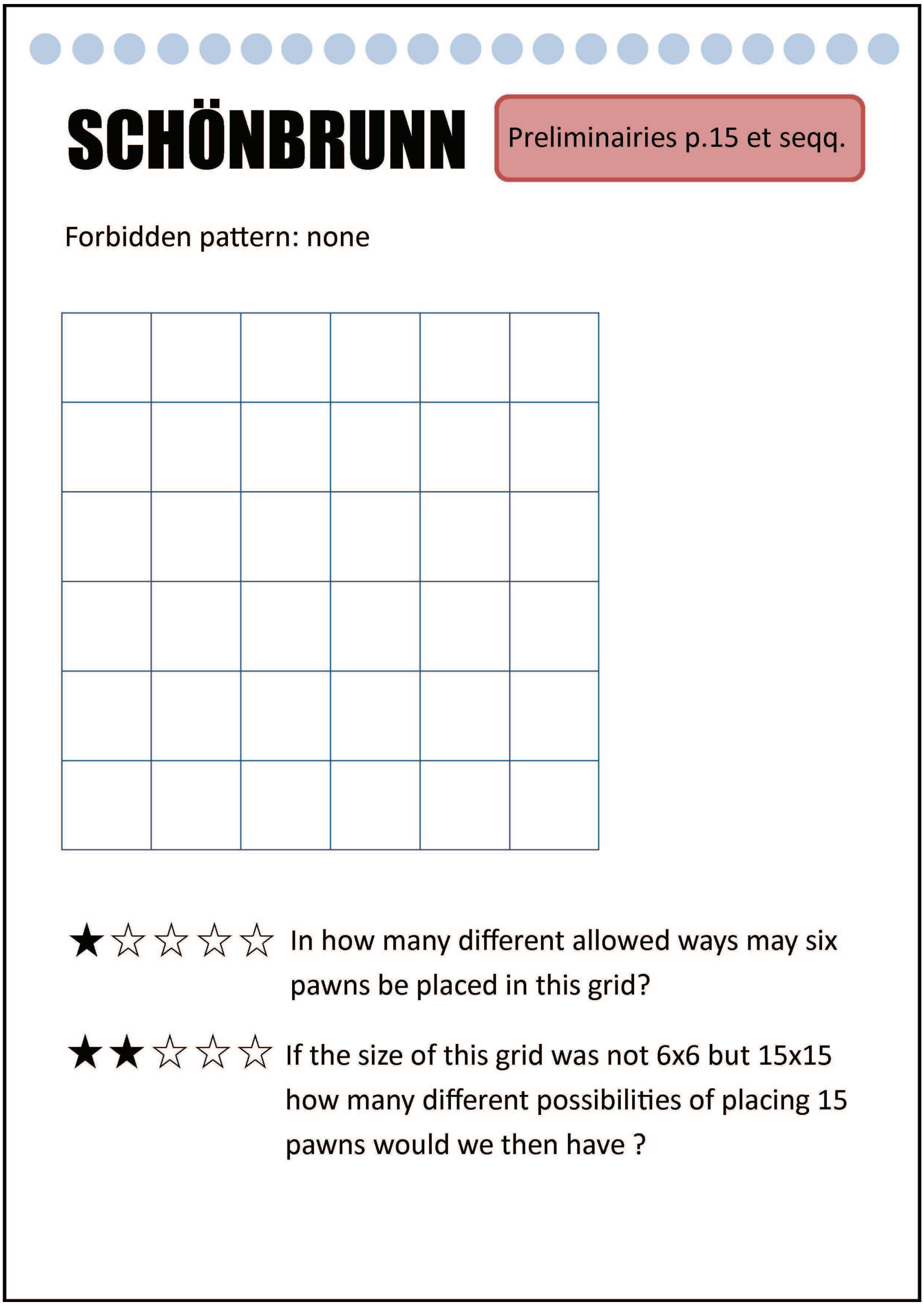


Figure A.5: Schönbrunn, a riddle teaching basics about permutations.

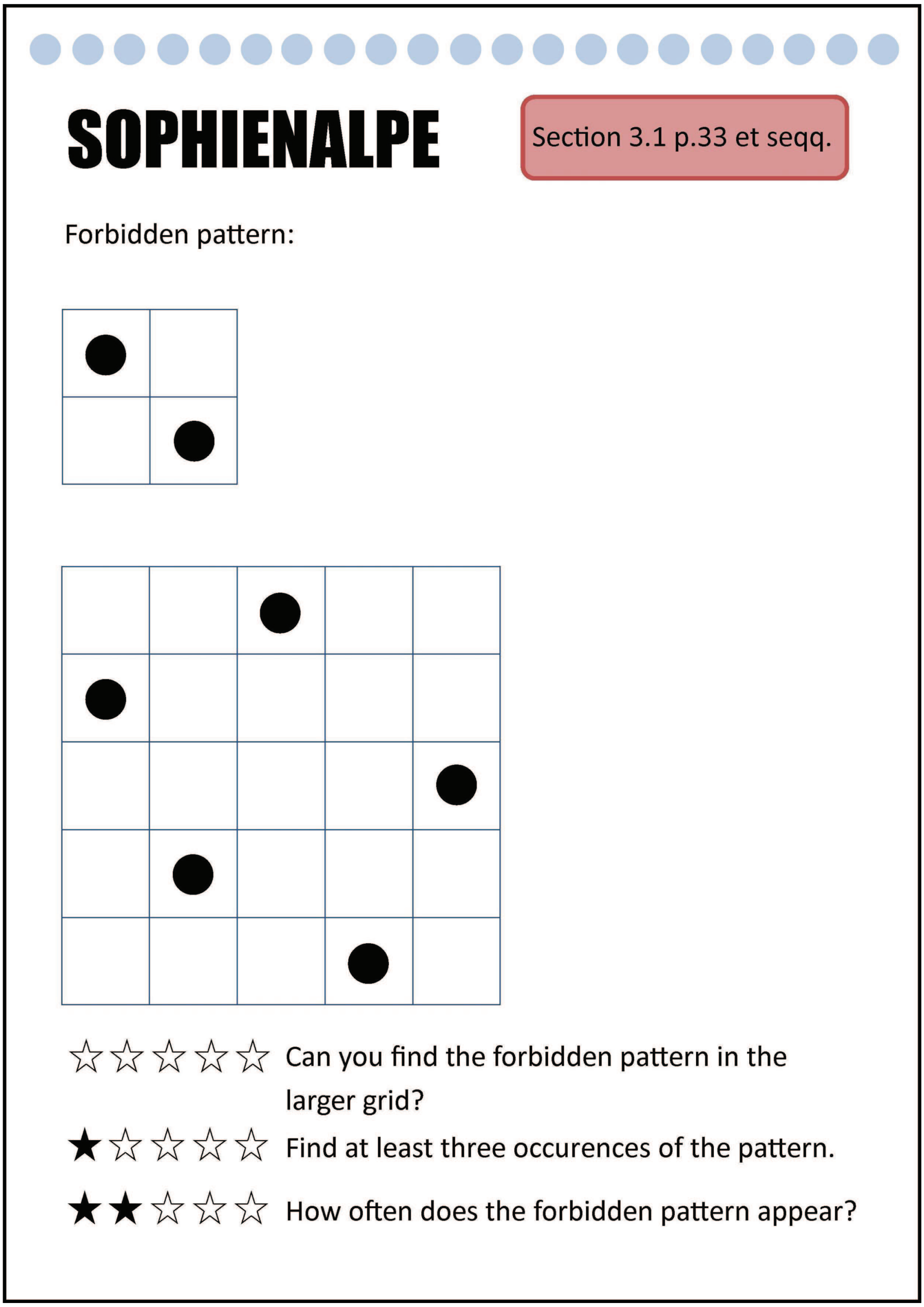


Figure A.6: Sophienalpe gives an introduction to recognizing and counting patterns of length two.

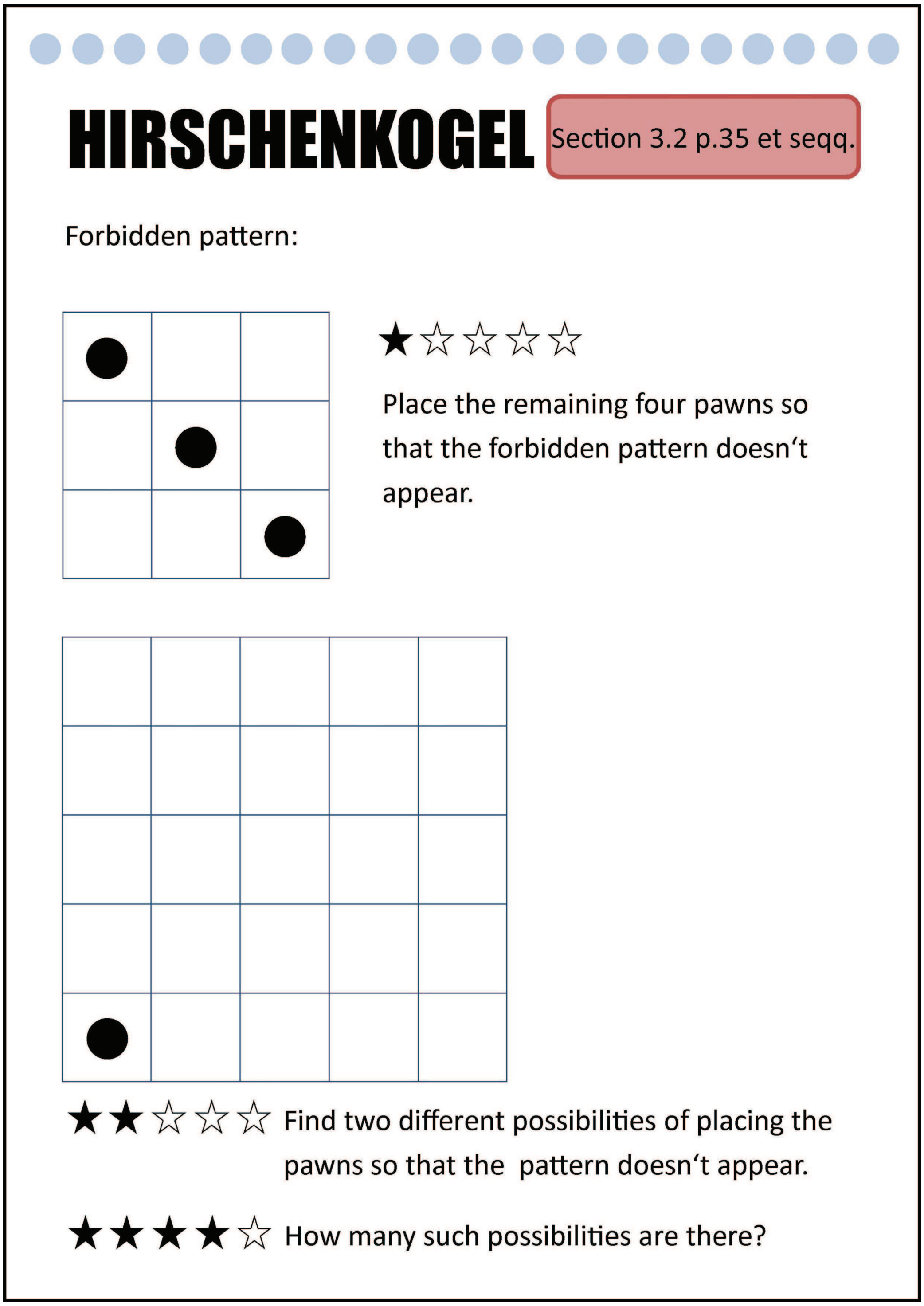


Figure A.7: Hirschenkogel, a riddle in which the pattern 123 must be avoided.

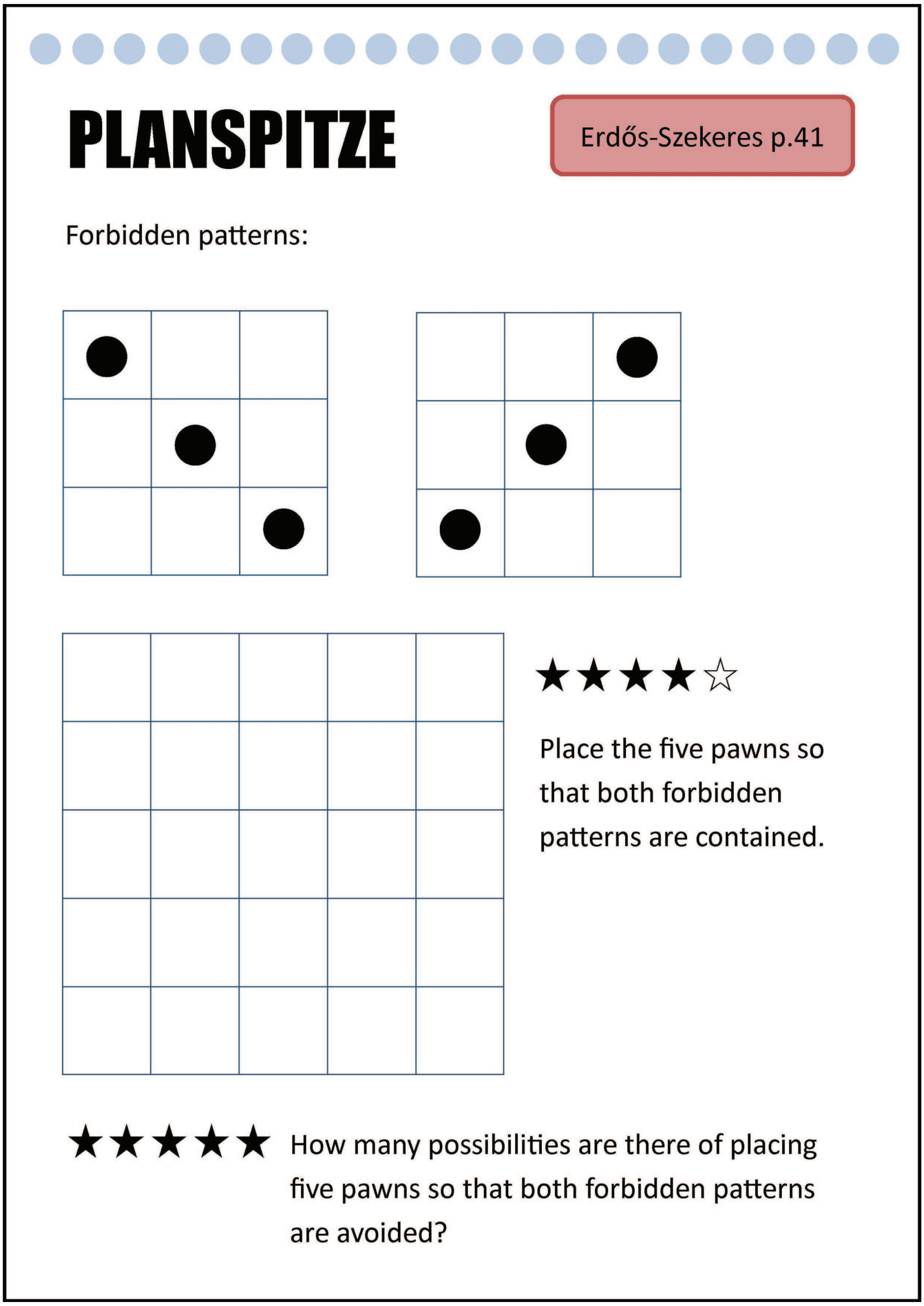


Figure A.8: Planspitze illustrates a special case of the Erdős-Szekeres-Theorem.

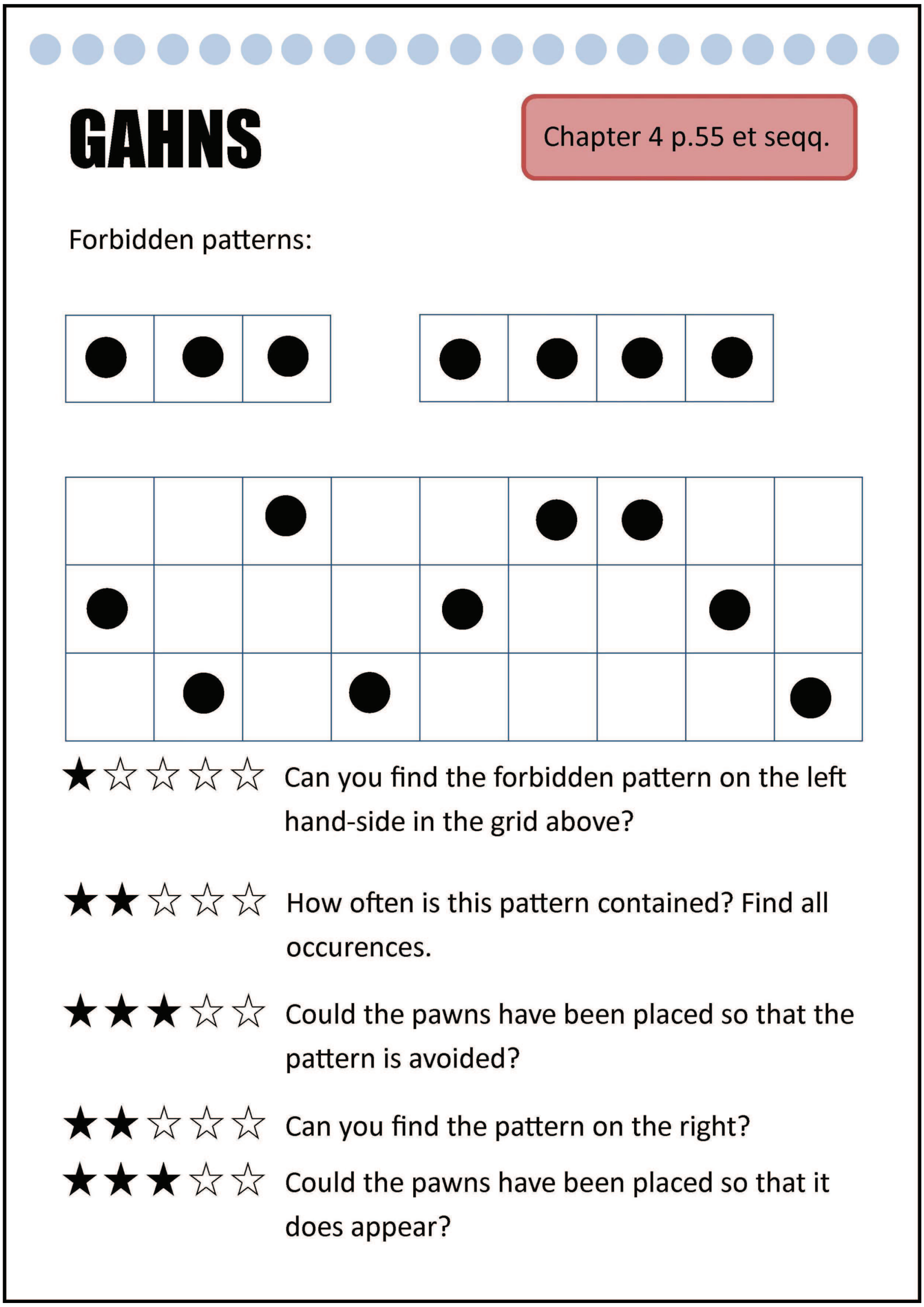


Figure A.9: Gahns gives an introduction to the concept of containing or avoiding a multiset-pattern.

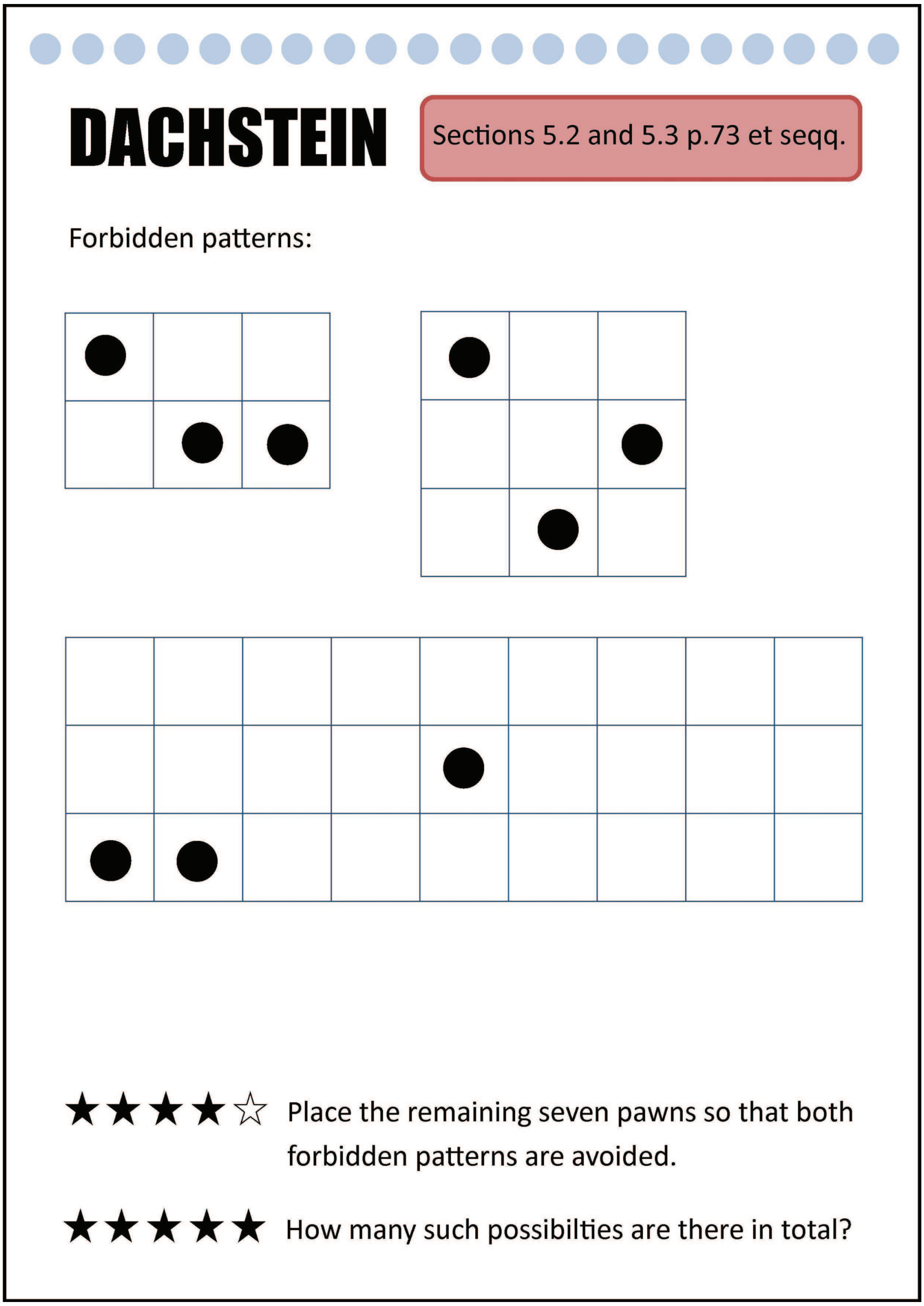
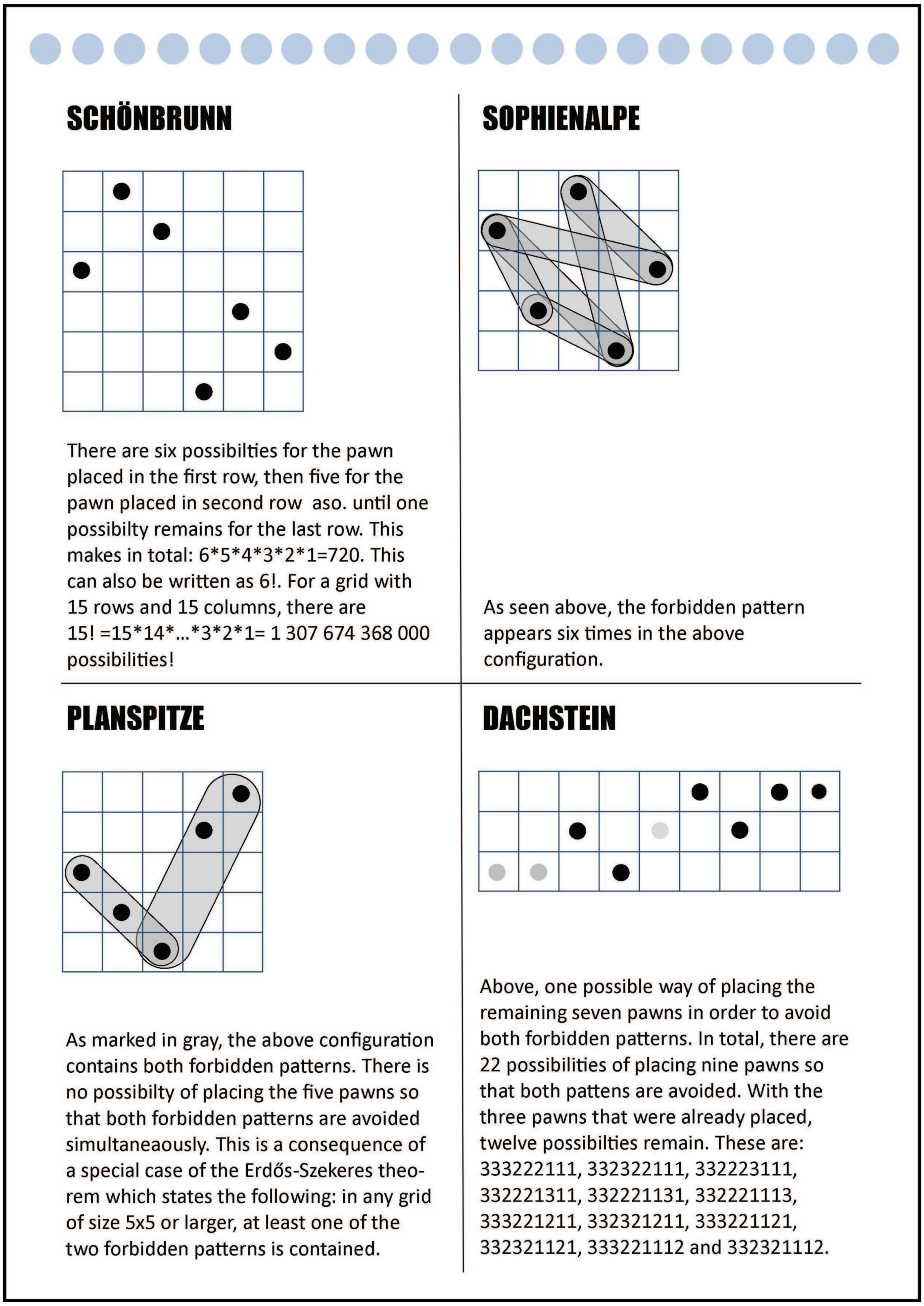


Figure A.10: Dachstein provides insight into one of the new results of this thesis.



SCHÖNBRUNN

There are six possibilties for the pawn placed in the first row, then five for the pawn placed in second row aso. until one possibilty remains for the last row. This makes in total: 6*5*4*3*2*1=720. This can also be written as 6!. For a grid with 15 rows and 15 columns, there are 15! =15*14*...*3*2*1= 1 307 674 368 000 possibilities!

SOPHIENALPE

As seen above, the forbidden pattern appears six times in the above configuration.

PLANSPITZE

As marked in gray, the above configuration contains both forbidden patterns. There is no possibilty of placing the five pawns so that both forbidden patterns are avoided simultaneaously. This is a consequence of a special case of the Erdős-Szekeres theorem which states the following: in any grid of size 5x5 or larger, at least one of the two forbidden patterns is contained.

DACHSTEIN

Above, one possible way of placing the remaining seven pawns in order to avoid both forbidden patterns. In total, there are 22 possibilities of placing nine pawns so that both pattens are avoided. With the three pawns that were already placed, twelve possibilties remain. These are: 333222111, 332322111, 332223111, 332221311, 332221131, 332221113, 333221211, 332321211, 333221121, 332321121, 333221112 and 332321112.

Figure A.11: An example of a solution-sheet.

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