TECHNISCHE UNIVERSITÄT WIEN
Vienna University of Technology

FAM
Financial & Actuarial Mathematics

Diplomarbeit

Aggregation ganzzahliger Risiken mit Copula-induzierter Abhängigkeitsstruktur

Zur Erlangung des akademischen Grades

Diplom-Ingenieur

Im Rahmen des Masterstudiums

Finanz- und Versicherungsmathematik
E 066 405

Eingereicht von

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Wien, 27.02.2019

Unterschrift Verfasser

Unterschrift Betreuer

TECHNISCHE UNIVERSITÄT WIEN
Vienna University of Technology

FAM
Financial & Actuarial Mathematics

Diploma Thesis

Aggregation of Integer-Valued Risks with Copula-Induced Dependency Structure

Carried out for the purpose of obtaining the degree of

Master of Science

Within the master's program

Financial and Actuarial Mathematics
E 066 405

Submitted by

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Vienna, 27.02.2019

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Acknowledgement

I am pleased to have the opportunity to thank my supervisor Dr. Uwe Schmock for his friendly, patient and professional assistance during the writing of this diploma thesis. Without his dedicated support and help this work would not be available in its present form. I am particularly grateful for his valuable comments on the results in the 4th chapter of this thesis and for helping me to improve my mathematical skills.

Special thanks go to Sandra, who often gave me valuable advice during my studies. Thank you very much for your work at FAM – it is highly appreciated by all students.

I would also like to express my gratitude to my family: On the one hand, my parents Juliane and Wilhelm, who enabled me to study mathematics. On the other hand, my wife Jane, who constantly supported me during my time at university. Acknowledgements go to my grandmother Maria, who has been giving me energy for learning with her excellent meals. Many thanks also to my sister Claudia for proofreading my thesis.

Furthermore, I would like to thank my employer ING Austria – especially my team leader Jan Jelovsek – both for the opportunities given to me and for the enormous support during my master's studies.

Last but not least I would like to mention my university colleagues Berkay, Filip and Mario, with whom I spent many amusing hours during my studies.

Martin Schmidt

Abstract

In this diploma thesis we investigate a portfolio of d integer-valued risks and calculate the distribution of the aggregate loss S, which is the sum of these. To generalize the popular assumption of independence used in practice, we model the dependency structure of the individual risks using copulas, allowing for a wide range of flexibility. After a rather detailed introduction to copula theory, the main part of this thesis starts with a formula for the distribution function of S. In addition, a recursion formula for the probability mass function of S is provided. Bounds on the distribution of S determined by the Rearrangement Algorithm serve to quantify the model risk caused by feasible scenarios of dependency. To illustrate the theoretical considerations, the final chapter contains a multitude of numerical examples in which, besides the distribution and probability mass function, common risk measures such as Value-at-Risk and Expected Shortfall for S are calculated under various dependency structures.

Keywords: Copula, Dependent Random Variables, Sum of Random Variables, Discrete Risk Aggregation, Value-at-Risk, Expected Shortfall, Rearrangement Algorithm.

Zusammenfassung

In der vorliegenden Diplomarbeit betrachten wir ein Portfolio d ganzzahliger Risiken und berechnen die Verteilung des aggregierten Schadens S, welcher die Summe dieser ist. Um die in der Praxis gängige Annahme unabhängiger Risiken zu verallgemeinern, modellieren wir die Abhängigkeitsstruktur der einzelnen Risiken mittels Copulas, wodurch wir erheblich an Flexibilität gewinnen. Nach einer Einführung in die Copula-Theorie beweisen wir im Hauptteil dieser Arbeit eine Formel zur Berechnung der Verteilungsfunktion des Gesamtschadens S. Darüber hinaus wird eine Rekursionsformel für die Wahrscheinlichkeitsfunktion von S aufgestellt. Die mittels Rearrangement Algorithm errechneten Schranken für die Verteilung von S dienen der Quantifizierung des Modellrisikos, welches durch unterschiedliche Abhängigkeitsszenarien verursacht wird. Um die theoretischen Aspekte dieser Arbeit zu veranschaulichen, enthält das letzte Kapitel eine Vielzahl numerischer Beispiele, in denen neben der Verteilung und Wahrscheinlichkeitsfunktion auch Risikomaße wie Value-at-Risk und Expected Shortfall für S unter verschiedenen Abhängigkeitsstrukturen berechnet werden.

Schlagworte: Copula, Abhängige Zufallsvariablen, Summe von Zufallsvariablen, Diskrete Risikoaggregation, Value-at-Risk, Expected Shortfall, Rearrangement Algorithm.

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Chapter 1

Introduction

Risk aggregation is becoming increasingly important in the banking and insurance sector for the calculation of the total economic capital. The solvency capital requirement for insurance companies under the regulation framework Solvency II is calculated using a standard formula that corresponds to the Value-at-Risk of an insurance company's total own funds at a confidence level of 99.5%, for example. The overall risk to which an insurance company is exposed is subdivided into individual risk modules and a capital requirement is calculated for each module. To determine the total solvency capital requirement, the capital requirements of the individual risks are aggregated using correlation matrices assuming a multivariate normal distribution.[1]

Let $d \in \mathbb{N},\ d \geq 2$ and consider a portfolio of d $\mathbb{N}_0$-valued risks $(X_1, \dots, X_d)$. In this diploma thesis we investigate the distribution of the aggregate loss $S = X_1 + \dots + X_d$. If the individual risks are independent of each other, the distribution of S can be calculated by means of convolution. Since independence rarely corresponds to reality in practice, we will model the dependence structure of $(X_1, \dots, X_d)$ by an arbitrary copula C, allowing for more flexibility.

The structure of this thesis is as follows: Chapter 2 summarizes copula theory in a rather detailed manner. The focus is on Sklar's theorem, which can be seen as the main theorem of copula theory and which we will prove in its general form. It is worth mentioning that beside well-known copulas we will also briefly introduce the less widespread concept of asymmetric copulas. Chapter 3 discusses measures of dependency, mainly linear correlation and rank correlation measures like Kendall's tau or Spearman's rho. In the main part of this thesis, Chapter 4, we prove a formula for the distribution of the sum of $\mathbb{N}_0$-valued random variables with copula-induced dependency structure in arbitrary dimension. In addition, we provide a recursion formula for the calculation of the probability mass function of the aggregate loss S. As the concept of copulas linking discrete marginal

[1] Details on the underlying assumptions on the standard formula in Solvency II can be found in [11].

distributions is slightly less natural than for continuous margins[2], the obtained results can be seen as a generalization of [17, Section 2]. Pointwise sharp bounds on the distribution of S, which serve to quantify the model risk caused by feasible scenarios of dependency, are computed using the Rearrangement Algorithm, which we will briefly present. Chapter 5 introduces common risk measures of the financial industry, Value-at-Risk and Expected Shortfall. Bounds on these are calculated by slight adaptation of the Rearrangement Algorithm. To illustrate the theoretical considerations, numerical results for specified marginal distributions – mainly Poisson and negative binomial distribution – and copulas are presented in Chapter 6. Besides the distribution and probability mass function, we will also calculate Value-at-Risk and Expected Shortfall under several dependence scenarios. As we will see, the comonotonicity copula maximizes the Expected Shortfall, whereas this statement cannot be made in the case of Value-at-Risk. From the examples it can be further concluded that the model risk increases with increasing confidence level.

All calculations in this diploma thesis were performed using R version 3.5.1 on a 64-bit machine running Windows 10 with 16 GB RAM. The respective scripts can be found in the appendix to this thesis.

[2] cf. [16]

Chapter 2

Elementary Copula Theory

In this chapter we will deal with the fundamentals of copula theory. In the first section we will commit ourselves to an appropriate notation which we shall use throughout this thesis. Subsequently, we will give essential results of probability theory with a focus on (multivariate) distribution functions. Copulas are defined in the third section and their most important properties are listed. At the heart of this chapter is Sklar's theorem, which can be seen as one of the main results of copula theory. Furthermore, bounds are specified between which each copula moves. To conclude the first part of this master thesis, we will provide examples of copulas.

2.1 Notational Conventions

At first, we will specify some basic notational conventions that will be convenient throughout this thesis.

- A probability space is a triplet $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is nonempty, $\mathcal{F}$ is a σ-algebra of subsets of Ω and $\mathbb{P}$ denotes a probability measure on $\mathcal{F}$.

- By $\mathbb{N}$ we denote the set of natural numbers $\{1, 2, \dots\}$ and we define $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. $\mathbb{R}$ is the set of real numbers. In addition, let $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\} = [-\infty, +\infty]$ be the extended real numbers. Moreover, for $d \in \mathbb{N}$, $\mathbb{R}^d$ (respectively, $\overline{\mathbb{R}}^d$) denotes the set of all d-dimensional column vectors with entries in $\mathbb{R}$ (respectively, $\overline{\mathbb{R}}$).

- For all pairs of points $x = (x_1, \dots, x_d)^\top, y = (y_1, \dots, y_d)^\top \in \mathbb{R}^d$ (respectively, $\overline{\mathbb{R}}^d$) we define a partial order by

$$x < y \iff x_i < y_i \quad \forall i = 1, \ldots, d.$$

Analogously, $x \leq y$ is defined.

- Given $x, y \in \mathbb{R}^d$ (respectively, $\overline{\mathbb{R}}^d$) with $x < y$, we define a d-dimensional left-open rectangle $(x, y]$ by the Cartesian product

$$(x, y] = \bigtimes_{i=1}^{d} (x_i, y_i].$$

Similar definitions can be given for d-dimensional closed, open or right-open rectangles.

2.2 Preliminaries on (Multidimensional) Distribution Functions

Definition 2.1. Given a random vector $(X_1, \ldots, X_d)^\top : \Omega \to \mathbb{R}^d$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, its *$d$-dimensional (multivariate) distribution function* $F : \mathbb{R}^d \to [0, 1]$ is defined by[3]

$$F(x_1, \ldots, x_d) = \mathbb{P}[X_1 \leq x_1, \ldots, X_d \leq x_d], \quad x_1, \ldots, x_d \in \mathbb{R}.$$

By setting the values at infinity as the corresponding limits, the domain of F can be uniquely extended to $\overline{\mathbb{R}}^d$.

Each (multivariate) distribution function F has the following properties:[4]

(DF1) F is a non-decreasing function, i.e. $F(x) \leq F(y)$ for all $x, y \in \mathbb{R}^d$ with $x \leq y$.

(DF2) F is right-continuous in each argument, i.e. for all points $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d) \in \mathbb{R}^{d-1}$ the function $\mathbb{R} \ni t \mapsto F(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_d)$ is continuous on the right.

(DF3) $\lim_{x_1, \ldots, x_d \to \infty} F(x_1, \ldots, x_d) = 1.$

(DF4) For $i = 1, \ldots, d$ and $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d \in \mathbb{R}$ it holds that $\lim_{x_i \to -\infty} F(x_1, \ldots, x_d) = 0.$

(DF5) For points $a = (a_1, \ldots, a_d)$ and $b = (b_1, \ldots, b_d)$ in $\mathbb{R}^d$ with $a \leq b$ and $j = 1, \ldots, d$,

$$\sum_{i_1=1}^{2} \cdots \sum_{i_d=1}^{2} (-1)^{i_1 + \cdots + i_d} F(x_{1,i_1}, \ldots, x_{d,i_d}) \geq 0, \quad x_{j,1} = a_j, \; x_{j,2} = b_j.$$

[3] cf. [10, Definition 1.2.9]
[4] cf. [9, Theorem 1.2.1, Definition 1.2.2] and [20, p. 27]

Remark 2.2. Note that by [22, Lemma 6.59], characteristics (DF1) and (DF2) imply that F is a right-continuous function on $\mathbb{R}^d$.

Conversely, for every function $F : \mathbb{R}^d \to [0,1]$ satisfying conditions (DF1)–(DF5), there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a random vector X defined on it, such that F is the distribution function of X.[5]

Definition 2.3. Let $d \geq 2$ and let F be the d-dimensional distribution function of a random vector $(X_1, \ldots, X_d)^\top$. For a subset $\kappa = \{i_1, \ldots, i_k\}$ of $I = \{1, \ldots, d\}$ of size k, $k = 1, \ldots, d-1$, we define the *κ-marginal distribution function* $F_\kappa : \mathbb{R}^k \to [0,1]$ of the random vector $(X_{i_1}, \ldots, X_{i_k})^\top$ by[6]

$$F_\kappa(x_{i_1}, \ldots, x_{i_k}) = \lim_{x_i \to \infty \forall i \in I \setminus \kappa} F(x_1, \ldots, x_d), \quad (x_{i_1}, \ldots, x_{i_k}) \in \mathbb{R}^k,$$

where $x_{i_\ell} = x_j$ for $i_\ell = j$ ($\ell = 1, \ldots, k$ and $j \in \kappa$). If $\kappa = \{i\}$, for $i \in I$ we call $F_i(x_i) \coloneqq F_\kappa(x_i)$ a *univariate marginal distribution function* of F.

A generalized inverse distribution function can be assigned to each univariate distribution function. Under additional conditions (compare for Remark 2.5), this generalized inverse agrees with the standard inverse.

Definition 2.4. For a univariate distribution function $F : \mathbb{R} \to [0,1]$, its *generalized inverse function (respectively, lower quantile function)* $F^\leftarrow : [0,1] \to \overline{\mathbb{R}}$ is given by[7]

$$F^\leftarrow(y) \coloneqq \inf\{x \in \mathbb{R} : F(x) \geq y\}, \quad y \in [0,1].$$

By convention, $\inf \varnothing = +\infty$ and $\inf \mathbb{R} = -\infty$.

Remark 2.5. In the case of a continuous and strictly increasing distribution function F, the generalized inverse $F^\leftarrow$ coincides with the ordinary inverse F^{-1} on $\mathrm{Ran}(F)$, where $\mathrm{Ran}(F) \coloneqq \{F(x) : x \in \mathbb{R}\}$ denotes the image (range) of F.[8]

In the following lemma we will list important properties of the generalized inverse function. For a proof, interested readers are referred to [12], where the authors explain numerous facts about generalized inverses and deal with various fallacies.

Lemma 2.6. *Let F be a univariate distribution function on $\mathbb{R}$ with limits $F(-\infty) \coloneqq \lim_{x \to -\infty} F(x) = 0$ and $F(+\infty) \coloneqq \lim_{x \to +\infty} = 1$. Further, let $F^\leftarrow$ denote the generalized inverse function of F. Then for all $x \in \mathbb{R}$ and $y \in [0,1]$:*[9]

(GI1) *$F^\leftarrow$ is increasing.*

(GI2) *$F^\leftarrow(y) \in \mathbb{R} \Rightarrow F^\leftarrow$ is left-continuous at y and permits a limit from the right.*

[5] cf. [10, Theorem 1.2.13]
[6] following closely [10, Definition 1.2.15]
[7] cf. [12, Definition 2.1]
[8] cf. [12, Remark 2.2(1)]
[9] cf. [12, Proposition 2.3]

(GI3) *F is continuous* $\iff$ *$F^{\leftarrow}$ is strictly increasing.*

(GI4) $F(x) \geq y \iff F^{\leftarrow}(y) \leq x.$

(GI5) $F^{\leftarrow}(F(x)) \leq x$. *If F is strictly increasing* $\Rightarrow F^{\leftarrow}(F(x)) = x.$

(GI6) $F^{\leftarrow}(y) < \infty \Rightarrow F(F^{\leftarrow}(y)) \geq y.$

(GI7) *If* $y \in \mathrm{Ran}(F) \cup \{\inf \mathrm{Ran}(F), \sup \mathrm{Ran}(F)\} \Rightarrow F(F^{\leftarrow}(y)) = y.$

The subsequent lemma can be useful when working with generalized inverse functions. It states that for a given univariate distribution function F, $\{x \in \mathbb{R} : F^{\leftarrow}(F(x)) \neq x\}$ is a null set with respect to the corresponding probability measure:[10]

Lemma 2.7. *For a random variable X on $(\Omega, \mathcal{F}, \mathbb{P})$ with univariate distribution function F,*

$$F^{\leftarrow}(F(X)) = X \quad \mathbb{P}\text{-}a.s. \tag{2.1}$$

Theorem 2.8. *Let F be a univariate distribution function and let $F^{\leftarrow}$ denote its generalized inverse function. Then the following statements apply:*[11]

(a) Quantile Transformation: *If U is a standard uniformly distributed random variable, then $\mathbb{P}[F^{\leftarrow}(U) \leq x] = F(x)$ holds for all x in $\mathbb{R}$.*

(b) Distributional Transform: *If F is continuous and a random variable X has distribution function F, then $F(X)$ is standard uniformly distributed.*

Proof. For proving part (a) we use property (GI4) and obtain

$$\mathbb{P}[F^{\leftarrow}(U) \leq x] = \mathbb{P}[U \leq F(x)] = F(x), \quad x \in \mathbb{R},$$

as U is standard uniformly distributed.

For part (b) let $u \in (0,1)$. We infer

$$\begin{aligned}\mathbb{P}[F(X) \leq u] &\overset{\text{(GI3)}}{=} \mathbb{P}[F^{\leftarrow}(F(X)) \leq F^{\leftarrow}(u)] \overset{(2.1)}{=} \mathbb{P}[X \leq F^{\leftarrow}(u)] = F(F^{\leftarrow}(u)) \\ &\overset{\text{(GI7)}}{=} u,\end{aligned}$$

which completes the proof. ❑

Among other results, Theorem 2.8 will be used for a proof of the first part of Sklar's theorem in the case of continuous univariate marginal distributions. It describes how to generate random samples from any given probability distribution with known generalized inverse function and is usually used for Monte Carlo methods.

[10] cf. [25, Proposition A.4]

[11] cf. [25, Proposition 5.2]

2.3 Definition and Basic Properties

Broadly speaking, the multivariate distribution function F of a random vector $X = (X_1, \ldots, X_d)^\top$ contains two kinds of information:[12]

(i) the univariate marginal distributions $F_1, \ldots, F_d$ and

(ii) the dependency structure among the components.

In most cases, knowing the univariate marginal distributions $F_1, \ldots, F_d$ is not enough in order to determine F. Additional understanding of how the margins are coupled is required. This information can be obtained by means of a copula of X. In general, it can be said that it is sufficient to know the copula and the marginal distributions to specify the underlying multivariate distribution function.[13] This leads us to the definition of copulas:[14]

Definition 2.9. Let $d \in \mathbb{N}, d \geq 2$ and let $(U_1, \ldots, U_d)^\top$ denote a random vector on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that the random variable U_i is standard uniformly distributed for all $i = 1, \ldots, d$. A *d-dimensional copula* C is a multivariate distribution function on the d-dimensional unit cube with univariate standard uniform marginals,

$$\begin{aligned} C : [0,1]^d &\to [0,1] \\ (u_1, \ldots, u_d) &\mapsto \mathbb{P}[U_1 \leq u_1, \ldots, U_d \leq u_d]. \end{aligned}$$

The following key characteristics can be concluded from the definition above, as C is a multivariate distribution function:[15]

(C1) Each copula C is *increasing* in each component:

$$C(u) \leq C(v) \quad \forall u, v \in [0,1]^d, \quad u \leq v.$$

(C2) Each copula C is *grounded*, meaning that for all $(u_1, \ldots, u_d) \in [0,1]^d$:

$$C(u_1, \ldots, u_d) = 0, \quad \text{if } u_i = 0 \text{ for at least one } i = 1, \ldots, d.$$

(C3) Each copula C has *standard uniform marginal distributions*:

$$C(1, \ldots, 1, u_i, 1, \ldots, 1) = u_i \quad \forall u_i \in [0,1], \quad i = 1, \ldots, d.$$

(C4) Each copula C is *d-increasing*, meaning that for $a = (a_1, \ldots, a_d)$ and $b = (b_1, \ldots, b_d)$ with $0 \leq a_j < b_j \leq 1$ we have that for all $j = 1, \ldots, d$:

$$\sum_{i_1=1}^{2} \cdots \sum_{i_d=1}^{2} (-1)^{i_1 + \cdots + i_d} C(u_{1,i_1}, \ldots, u_{d,i_d}) \geq 0, \quad u_{j,1} = a_j,\ u_{j,2} = b_j.$$

[12] cf. [25, p. 184]
[13] cf. [24, p. 4]
[14] cf. [24, Definition 1.1]
[15] cf. [25, p. 185]

Remark 2.10. At first glance, the last condition seems very abstract. Property (C4) stems from the inclusion–exclusion principle and guarantees that for a random vector $(U_1, \dots, U_d)^\top$ with distribution function C, the probability $\mathbb{P}[a_1 < U_1 \leq b_1, \dots, a_d < U_d \leq b_d]$ is not negative.[16]

Any function $C : [0,1]^d \to [0,1]$ fulfilling (C1)–(C4) is a copula.[17] The continuity – and thus the right-continuity – results from the uniformly distributed marginal distributions. More generally, one can also conclude that every copula C is a Lipschitz continuous function. To confirm this assertion, we will first prove the following lemma:[18]

Lemma 2.11. *Let F denote a multivariate distribution function with univariate marginal distributions $F_1, \dots, F_d$. Then for all pairs of points $(x_1, \dots, x_d)$ and $(y_1, \dots, y_d)$ in $\mathbb{R}^d$,*

$$|F(x_1, \dots, x_d) - F(y_1, \dots, y_d)| \leq \sum_{i=1}^{d} |F_i(x_i) - F_i(y_i)| \, . \tag{2.2}$$

Proof. Let F be the multivariate distribution function of a random vector X on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with univariate marginals $F_1, \dots, F_d$ and let $(x_1, \dots, x_d)$ and $(y_1, \dots, y_d)$ denote arbitrary points in $\mathbb{R}^d$. Using the triangle inequality, it follows that

$$\begin{aligned} |F(x_1, \dots, x_d) - F(y_1, \dots, y_d)| \leq & \, |F(x_1, \dots, x_d) - F(y_1, x_2, \dots, x_d)| \\ & + |F(y_1, x_2, \dots, x_d) - F(y_1, y_2, x_3, \dots, x_d)| \\ & + \dots \\ & + |F(y_1, \dots, y_{d-1}, x_d) - F(y_1, \dots, y_d)| \, . \end{aligned}$$

For any $i = 1, \dots, d$ we can now consider the term

$$F(y_1, \dots, y_i, x_{i+1}, \dots, x_d) - F(y_1, \dots, y_{i-1}, x_i, \dots, x_d).$$

Assuming $x_i < y_i$, property (DF1) yields

$$F(y_1, \dots, y_i, x_{i+1}, \dots, x_d) - F(y_1, \dots, y_{i-1}, x_i, \dots, x_d) \geq 0.$$

Further,

$$\begin{aligned} & F(y_1, \dots, y_i, x_{i+1}, \dots, x_d) - F(y_1, \dots, y_{i-1}, x_i, \dots, x_d) \\ & = \mathbb{P}[X_1 \leq y_1, \dots, X_i \leq y_i, X_{i+1} \leq x_{i+1}, \dots, X_d \leq x_d] \\ & \quad - \mathbb{P}[X_1 \leq y_1, \dots, X_{i-1} \leq y_{i-1}, X_i \leq x_i, \dots, X_d \leq x_d] \\ & = \mathbb{P}[X_1 \leq y_1, \dots, x_i < X_i \leq y_i, \dots, X_d \leq x_d] \\ & \leq \mathbb{P}[x_i < X_i \leq y_i] = F_i(y_i) - F_i(x_i). \end{aligned}$$

In the opposite case $x_i \geq y_i$ the signs turn around and in total we have that

$$|F(y_1, \dots, y_i, x_{i+1}, \dots, x_d) - F(y_1, \dots, y_{i-1}, x_i, \dots, x_d)| \leq |F_i(y_i) - F_i(x_i)| \, .$$

Applying this partial result for all $i = 1, \dots, d$, we obtain (2.2). ❑

[16] cf. [24, p. 8]
[17] cf. [10, Theorem 1.4.1]
[18] cf. [20, Lemma 8.2]

As a direct consequence of the preceding lemma, we receive the following:[19]

Theorem 2.12. *A function $C : [0,1]^d \to [0,1]$ satisfying (C1)–(C4) (in particular, each copula) is globally Lipschitz-continuous with Lipschitz constant 1 with respect to the ℓ_1-norm,*

$$|C(u_1,\dots,u_d) - C(v_1,\dots,v_d)| \le \sum_{i=1}^{d} |u_i - v_i|, \quad u,v \in [0,1]^d.$$

Proof. As each copula has standard uniform marginal distributions, the statement follows directly by application of Lemma 2.11. ❑

2.4 Sklar's Theorem

In this section, we shall deal with the theorem of Sklar[20], which can be seen as the main theorem of copula theory. In essence, it states that any dependency structure – however complicated – can be described by means of a copula, which provides a better understanding of dependency. Furthermore, Sklar's theorem permits a flexible modelling of dependence, since a multivariate distribution can be constructed from arbitrary univariate marginal distributions and copulas.[21]

Theorem 2.13 (**Sklar**[22]). *Let F denote a multivariate distribution function with univariate marginals $F_1,\dots,F_d$. Then there exists a d-dimensional copula $C : [0,1]^d \to [0,1]$, such that for all $x_1,\dots,x_d \in \mathbb{R}$ it holds that*

$$F(x_1,\dots,x_d) = C(F_1(x_1),\dots,F_d(x_d)). \tag{2.3}$$

If $F_1,\dots,F_d$ are continuous, then C is unique. Conversely, if C is a copula and $F_1,\dots,F_d$ are univariate distribution functions, then the function F defined via (2.3) is a d-dimensional distribution function with one-dimensional margins $F_1,\dots,F_d$.

Remark 2.14. As the concept of copulas is slightly different for multivariate discrete or mixed distributions, we will prove the first part of the previous theorem – existence and uniqueness of copulas – for the case of continuous marginal distributions first. Afterwards, we will return to the general case.

Proof of Theorem 2.13. Let F be the distribution function of a random vector $(X_1,\dots,X_d)^\top$ on the probability space $(\Omega,\mathcal{F},\mathbb{P})$ with continuous univariate marginals $F_1,\dots,F_d$. For $i = 1,\dots,d$ we define

$$U_i = F_i(X_i).$$

[19] cf. [10, Theorem 1.5.1]
[20] cf. [34]
[21] cf. [25, p. 186]
[22] cf. [24, Theorem 1.2]

From Theorem 2.8(b) we already know that $U_i \sim U(0,1)$ for all $i = 1, \dots, d$. Define C to be the distribution function of the random vector $(U_1, \dots, U_d)^\top$. For $x_1, \dots, x_d \in \mathbb{R}$,

$$\begin{aligned}
F(x_1, \dots, x_d) &= \mathbb{P}[X_1 \le x_1, \dots, X_d \le x_d] \\
&= \mathbb{P}[F_1^{\leftarrow}(F_1(X_1)) \le x_1, \dots, F_d^{\leftarrow}(F_d(X_d)) \le x_d] && \text{Lemma 2.7} \\
&= \mathbb{P}[F_1(X_1) \le F_1(x_1), \dots, F_d(X_d) \le F_d(x_d)] && \text{(GI3),(GI5)} \\
&= \mathbb{P}[U_1 \le F_1(x_1), \dots, U_d \le F_d(x_d)] \\
&= C(F_1(x_1), \dots, F_d(x_d)).
\end{aligned}$$

Then, by construction, C is a copula, as it is the distribution function of a random vector with standard uniform marginal distributions. Under the use of (GI7) and $u_i := F_i(x_i) \in (0,1)$ for $x_i \in \mathbb{R}$, $i = 1, \dots, d$,

$$C(u_1, \dots, u_d) = F(F_1^{\leftarrow}(u_1), \dots, F_d^{\leftarrow}(u_d)),$$

which enables us to express C explicitly by F and the margins $F_1, \dots, F_d$ and thus proves uniqueness on $[0,1]^d$, since the boundary values of C are uniquely determined by (C2) and (C3).

For the reverse statement let C be a copula and let $F_1, \dots, F_d$ denote arbitrary univariate distribution functions that do not necessarily have to be continuous. Then there exists a random vector $(U_1, \dots, U_d) \sim C$ such that $U_i \sim U(0,1)$ for all $i = 1, \dots, d$. Applying Theorem 2.8(a) yields that the random variable

$$X_i := F_i^{\leftarrow}(U_i)$$

has distribution function F_i for $i = 1, \dots, d$. Let $x_1, \dots, x_d$ take values in $\mathbb{R}$. Using (GI4),

$$\begin{aligned}
\mathbb{P}[X_1 \le x_1, \dots, X_d \le x_d] &= \mathbb{P}[F_1^{\leftarrow}(U_1) \le x_1, \dots, F_d^{\leftarrow}(U_d) \le x_d] \\
&= \mathbb{P}[U_1 \le F_1(x_1), \dots, U_d \le F_d(x_d)] \\
&= C(F_1(x_1), \dots, F_d(x_d)) \\
&= F(x_1, \dots, x_d),
\end{aligned}$$

which proves the theorem, as we have constructed a random vector $(X_1, \dots, X_d)$ with multivariate distribution function F and one-dimensional marginals $F_1, \dots, F_d$. ❑

As a direct consequence of the proof of Sklar's theorem we can infer the following corollary, which allows us to determine the copula for a given multivariate distribution function with known marginals.[23]

Corollary 2.15. *Let F be a d-dimensional distribution function on $\mathbb{R}^d$ with continuous univariate margins $F_1, \dots, F_d$. Then for all $u_1, \dots, u_d$ in $(0,1)$,*

$$C(u_1, \dots, u_d) = F(F_1^{\leftarrow}(u_1), \dots, F_d^{\leftarrow}(u_d))$$

is the copula of F. The boundary values of C are uniquely determined by properties (C2) and (C3).

[23] cf. [25, p. 187]

Example 2.16 shows how to find a bivariate distribution function for a pair of random variables with given marginals and copula. Contrariwise, Example 2.17 demonstrates how to construct a copula for a given 2-dimensional distribution function of a random vector:

Example 2.16. Let X_1, X_2 denote exponentially distributed random variables with expectations $\mathbb{E}[X_1] = \frac{1}{a}$ and $\mathbb{E}[X_2] = \frac{1}{b}$, $a, b > 0$. It is standard knowledge that the univariate marginal distributions of X_1 and X_2 are given by

$$F_1(x_1) = \begin{cases} 1 - e^{-ax_1} & \text{for } x_1 \geq 0, \\ 0 & \text{otherwise}, \end{cases}$$
$$F_2(x_2) = \begin{cases} 1 - e^{-bx_2} & \text{for } x_2 \geq 0, \\ 0 & \text{otherwise}. \end{cases}$$

Suppose that the dependency structure of (X_1, X_2) is given by a bivariate copula[24]

$$C(u_1, u_2) = u_1 u_2, \quad u_1, u_2 \in [0, 1].$$

From Sklar's theorem it follows that the 2-dimensional distribution function F of the random vector (X_1, X_2) is given by

$$F(x_1, x_2) = C(F_1(x_1), F_2(x_2)) = \begin{cases} (1 - e^{-ax_1})(1 - e^{-bx_2}) & \text{for } x_1, x_2 \geq 0, \\ 0 & \text{otherwise}. \end{cases}$$

Example 2.17. Let

$$F(x_1, x_2) = \left(1 + e^{-x_1} + e^{-x_2}\right)^{-1}, \quad x_1, x_2 \in \mathbb{R},$$

denote the bivariate distribution function of a random vector (X_1, X_2).[25] Its margins are given by

$$F_1(x_1) = \lim_{x_2 \to +\infty} F(x_1, x_2) = \left(1 + e^{-x_1}\right)^{-1},$$
$$F_2(x_2) = \lim_{x_1 \to +\infty} F(x_1, x_2) = \left(1 + e^{-x_2}\right)^{-1}.$$

Obviously, F_1 and F_2 are continuous on $\mathbb{R}$ as compositions of continuous functions. As they are also strictly increasing on $\mathbb{R}$, the generalized inverse function coincides with the standard inverse and for u_1, u_2 in $(0, 1)$ we have[26]

$$F_1^{-1}(u_1) = \ln\left(\frac{u_1}{1 - u_1}\right),$$
$$F_2^{-1}(u_2) = \ln\left(\frac{u_2}{1 - u_2}\right).$$

Corollary 2.15 states that

$$C(u_1, u_2) = F(F_1^{-1}(u_1), F_2^{-1}(u_2))$$

[24] Details on examples of copulas will be discussed in Section 2.7.
[25] A proof that F is a bivariate distribution function is given in the appendix (Lemma A1).
[26] Here, $\ln(\cdot)$ denotes the natural logarithm.

$$= \left(1 + \frac{1-u_1}{u_1} + \frac{1-u_2}{u_2}\right)^{-1}, \quad u_1, u_2 \in (0,1),$$

is the copula of F. By Theorem 2.12 and properties (C2) and (C3) we can conclude that C is unique and well defined on $[0,1]^2$.

Remark 2.18. A noteworthy characteristic for copulas of random vectors is, that in the case of continuous univariate marginal distributions of the vector-components, the copula is invariant under strictly increasing transformations. Evidence of this claim can be found in [25, Proposition 5.6.], among others.

2.4.1 Generalization of the Distributional Transform

For the purpose to prove the remainder of Sklar's theorem for non-continuous univariate margins we will deal with a generalization of Theorem 2.8(b). To this end, let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a suitable probability space.

Definition 2.19. Let F denote the one-dimensional distribution function of a real-valued random variable X on $(\Omega, \mathcal{F}, \mathbb{P})$. In addition, let V denote a standard uniformly distributed random variable on the same probability space, such that V is independent of X. For[27] $v \in (0,1)$ and $x \in \mathbb{R}$ we define a *modified distribution function* by[28]

$$F(x; v) = \mathbb{P}[X < x] + v\,\mathbb{P}[X = x]. \tag{2.4}$$

The *generalized distributional transform* of the random variable X is then given by

$$U := F(X; V). \tag{2.5}$$

For ease of notation let

$$F(x_-) := \lim_{y \nearrow x} F(y) = \mathbb{P}[X < x]$$

denote the left-sided limit at $x \in \mathbb{R}$ of a univariate distribution function F. For $x \in \mathbb{R}$ and $v \in (0,1)$, (2.4) can then be rewritten as

$$F(x; v) = F(x_-) + v\,(F(x) - F(x_-)).$$

Remark 2.20. In the case of a continuous distribution function F, $F(x; v) = F(x)$ is valid for all $v \in (0,1)$ and Theorem 2.8(b) holds, meaning that the random varible U as given in (2.5) is standard uniformly distributed.[29] The aim will be to show that this statement can be adapted to the case of discrete or mixed distributions as well.

[27] Note that $v = 1$ would be admissibly, but the modified distribution function would then coincide with the ordinary distribution function which does not require generalization.
[28] cf. [33, Definition 1.2]
[29] cf. [32, p. 2]

Lemma 2.21. *Let V and X denote independent random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ with $V \sim U(0,1)$ and $X \sim F$, where F denotes an arbitrary univariate distribution function on $\mathbb{R}$. Fix $u \in (0,1)$ and let $q_u := \inf\{x \in \mathbb{R} : F(x) \geq u\}$. Then for $(x, v) \in (q_u, \infty) \times (0,1)$ the following holds:*[30]

$$F(x; v) \leq u \iff F(x) = F(q_u) = u. \tag{2.6}$$

Proof. Fix $u \in (0,1)$ and let $q_u := \inf\{x \in \mathbb{R} : F(x) \geq u\}$ denote the lower u-quantile of F.

For the implication " $\Rightarrow$ " notice that

$$u \geq F(x; v) = vF(x) + (1-v)F(x_-) \overset{\text{(DF1)}}{\geq} F(x_-) \overset{\text{(DF1)}}{\geq} F(q_u) \overset{\text{(DF2)}}{\geq} u.$$

Consequently,

$$F(q_u) = F(x_-) = u$$

and together with

$$F(x_-) = F(x_-) + v(F(x) - F(x_-)) \iff F(x) - F(x_-) = 0$$

the statement is proved.

For the reverse implication " $\Leftarrow$ " let $x > q_u$ and $F(x) = F(q_u) = u$. As F is a non-decreasing function we infer that

$$F(x) = F(x_-) = u.$$

Hence,

$$F(x; v) = F(x_-) + v(F(x) - F(x_-)) = u + v(u - u) = u \leq u$$

which proves (2.6). ❑

Proposition 2.22 (Generalized Distributional Transform[31]). *As defined in (2.5) let U denote the generalized distributional transform of a random variable X with univariate distribution function F. Then the following statements hold:*

(a) $U \sim U(0,1)$.

(b) $\mathbb{P}[F^{\leftarrow}(U) = X] = 1$.

Proof. In order to prove part (a) let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a probability space supporting two independent random variables $X \sim F$ and $V \sim U(0,1)$ fulfilling the requirements of Definition 2.19. Further, we set $U := F(X; V)$. For a fixed u in $(0,1)$ let

$$q_u := \inf\{x \in \mathbb{R} : F(x) \geq u\}$$

[30] cf. [24, p. 14]
[31] cf. [32, Proposition 2.1] and [24, Lemma 1.4]

be the lower u-quantile of F. To prove that U is a standard uniformly distributed random variable we need to show that $\mathbb{P}[U \leq u] = u$ holds.

As $F(x; v) \leq F(x)$ for all $x \in \mathbb{R}$ and $v \in (0,1)$ we can infer that

$$(-\infty, q_u) \times (0,1) \subseteq \{(x,v) \in \mathbb{R} \times (0,1) : F(x; v) < u\}. \tag{2.7}$$

As a result of (2.6) and (2.7) the following applies:

$$\begin{aligned}
\{F(X;V) \leq u\} &= \{F(X;V) \leq u, X < q_u\} \\
&\quad \cup \{F(X;V) \leq u, X = q_u\} \\
&\quad \cup \{F(X;V) \leq u, X > q_u\} \\
&= \{X < q_u\} \\
&\quad \cup \{F(X;V) \leq u, X = q_u\} \\
&\quad \cup \{F(X) = F(q_u) = u, X > q_u\}.
\end{aligned}$$

Obviously $\{F(X) = F(q_u) = u, X > q_u\}$ corresponds to flat pieces of the distribution function F and thus

$$\mathbb{P}[F(X) = F(q_u) = u, X > q_u] = 0.$$

Therefore we can conclude that

$$\begin{aligned}
\mathbb{P}[U \leq u] &= \mathbb{P}[X < q_u] + \mathbb{P}[F(X;V) \leq u, X = q_u] \\
&= \mathbb{P}[X < q_u] + \mathbb{P}[F(q_u;V) \leq u]\, \mathbb{P}[X = q_u] \\
&= F(q_{u_-}) + \mathbb{P}[F(q_{u_-}) + V(F(q_u) - F(q_{u_-})) \leq u]\, \mathbb{P}[X = q_u].
\end{aligned}$$

- Case 1: $\mathbb{P}[X = q_u] = 0$.
  It holds that

  $$\mathbb{P}[U \leq u] = F(q_{u_-}) = F(q_u) = \mathbb{P}[X \leq q_u] = u.$$

- Case 2: $\mathbb{P}[X = q_u] > 0$.
  As V is standard uniformly distributed it follows that

  $$\mathbb{P}[U \leq u] = F(q_{u_-}) + \mathbb{P}\left[V \leq \frac{u - F(q_{u_-})}{F(q_u) - F(q_{u_-})}\right](F(q_u) - F(q_{u_-})) = u.$$

In summary, it follows that U is a standard uniformly distributed random variable.

For part (b) we use the definition of $U = F(X;V)$ and can conclude that $F(X_-) \leq U \leq F(X)$ holds $\mathbb{P}$-a.s. Moreover, we know that for $x \in \mathbb{R}$ with $F(x) \neq F(x_-)$ and $u \in (F(x_-), F(x)]$ we have that $F^{\leftarrow}(u) = x$. Thus by Lemma 2.7 we have that

$$\mathbb{P}[F^{\leftarrow}(U) = X] = 1,$$

which finalizes the proof. ❑

2.4.2 Completion of the Proof of Sklar's Theorem

Proposition 2.22 allows us to generalize the proof of the first part of Theorem 2.13 for arbitrary univariate marginal distributions:[32]

Let F denote the multidimensional distribution function of a random vector $X = (X_1, \dots, X_d)^\top$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with univariate margins $F_1, \dots, F_d$. In addition, let V denote a standard uniformly distributed random variable on the same probability space such that V is independent of X. For $i = 1, \dots, d$ we set

$$U_i := F(X_i; V).$$

Proposition 2.22 states that for $i = 1, \dots, d$ the random variables U_i are standard uniformly distributed and $X_i = F_i^{\leftarrow}(U_i)$ holds $\mathbb{P}$-a.s. If we define C to be the d-dimensional distribution function of the random vector $(U_1, \dots, U_d)$ we can infer that

$$\begin{aligned} F(x_1, \dots, x_d) &= \mathbb{P}[X_1 \le x_1, \dots, X_d \le x_d] \\ &= \mathbb{P}[F_1^{\leftarrow}(U_1) \le x_1, \dots, F_d^{\leftarrow}(U_d) \le x_d] && \text{Proposition 2.22(b)} \\ &= \mathbb{P}[U_1 \le F_1(x_1), \dots, U_d \le F_d(x_d)] && \text{(GI4)} \\ &= C(F_1(x_1), \dots, F_d(x_d)). \end{aligned}$$

Thus, (2.3) holds and C is the copula of F and consequently Theorem 2.13 is proven in its general form.

Remark 2.23. It should be noted that there are also other ways to prove Sklar's theorem in the case of non-continuous marginal distributions. Hence, interested readers are referred to [10, Section 2.3] where the authors present three proofs which make use of different properties of copulas.

2.4.3 Lack of Uniqueness of Copulas in the Case of Non-Continuous Univariate Marginals

To see that there is lack of uniqueness of the underlying copula in the case of discrete or mixed marginals, we will give an example in this subsection.[33] Furthermore, we will show that even in the case of non-continuous univariate marginal distributions we have unambiguousness, at least on the image of the one-dimensional margins.

Example 2.24. Consider a bivariate random vector (X, Y) on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with distribution as given in Table 2.1. For $i = 1, 2$ and $j = 1, 2, 3$, the entries p_{ij} correspond to the probabilities $\mathbb{P}[X = x_i, Y = y_j]$.

[32] cf. [32, Theorem 2.2]
[33] cf. [25, Example 5.5]

$X \backslash Y$	$y_1 = 0$	$y_2 = 1$	$y_3 = 2$
$x_1 = 0$	$p_{11} = \frac{1}{6}$	$p_{12} = \frac{1}{3}$	$p_{13} = \frac{1}{6}$
$x_2 = 1$	$p_{21} = 0$	$p_{22} = \frac{1}{6}$	$p_{23} = \frac{1}{6}$

Table 2.1

The probability mass functions are thus given by

$$\mathbb{P}[X = 0] = \frac{2}{3}, \quad \mathbb{P}[X = 1] = \frac{1}{3} \quad \text{and}$$
$$\mathbb{P}[Y = 0] = \frac{1}{6}, \quad \mathbb{P}[Y = 1] = \frac{1}{2}, \quad \mathbb{P}[Y = 2] = \frac{1}{3}.$$

The univariate distribution functions of X and Y can be specified as follows:

$$F_X(x) = \begin{cases} 0 & \text{for } x \in (-\infty, 0), \\ \frac{2}{3} & \text{for } x \in [0, 1), \\ 1 & \text{for } x \in [1, \infty) \end{cases} \quad \text{and} \quad F_Y(y) = \begin{cases} 0 & \text{for } y \in (-\infty, 0), \\ \frac{1}{6} & \text{for } y \in [0, 1), \\ \frac{2}{3} & \text{for } y \in [1, 2), \\ 1 & \text{for } y \in [2, \infty). \end{cases}$$

From Sklar's theorem 2.13 it follows that the 2-variate joint distribution F of (X, Y) is of the form

$$F(x, y) = C(F_X(x), F_Y(y)), \quad x, y \in \mathbb{R},$$

for some copula C. As $\operatorname{Ran}(F_X) = \{0, \frac{2}{3}, 1\}, \operatorname{Ran}(F_Y) = \{0, \frac{1}{6}, \frac{2}{3}, 1\}$, due to characteristics (C2) and (C3), the only restrictions we have for C are

$$C\left(\frac{2}{3}, \frac{1}{6}\right) = \frac{1}{6} \quad \text{and} \quad C\left(\frac{2}{3}, \frac{2}{3}\right) = \frac{1}{2}. \tag{2.8}$$

Therefore, the copula C is not unique on $[0, 1]^2$, since each bivariate copula fulfilling (2.8) is suitable for representing the 2-dimensional distribution of the random vector (X, Y).

The subsequent theorem guarantees uniqueness of copulas on the image of the univariate distribution functions when they are not all continuous:[34]

Theorem 2.25. *Let F denote the multivariate distribution function of a random vector $X = (X_1, \ldots, X_d)^\top$ on $\mathbb{R}^d$ with univariate marginals $F_1, \ldots, F_d$. Then a copula C of X is uniquely determined on* $\operatorname{Ran}(F_1) \times \cdots \times \operatorname{Ran}(F_d)$.

Proof. Let F denote a d-variate joint distribution function of a random vector $X = (X_1, \ldots, X_d)^\top$ on $\mathbb{R}^d$ and let $F_1, \ldots, F_d$ denote its univariate marginals. For $x_1, \ldots, x_d \in \mathbb{R}$ we define

$$y_i = F_i(x_i), \quad i = 1, \ldots, d.$$

[34]cf. [10, Lemma 2.2.9]

Further, let C denote a copula of X. By Sklar's theorem 2.13 it holds that

$$F(x_1, \dots, x_d) = C(y_1, \dots, y_d), \quad x_1, \dots, x_d \in \mathbb{R}.$$

If we now assume that there exists a second copula $\tilde{C}$ of X, then the following applies for $x_1, \dots, x_d \in \mathbb{R}$:

$$|C(y_1, \dots, y_d) - \tilde{C}(y_1, \dots, y_d)| = |F(x_1, \dots, x_d) - F(x_1, \dots, x_d)| = 0.$$

Hence, C is unique on $\operatorname{Ran}(F_1) \times \cdots \times \operatorname{Ran}(F_d)$. ❑

2.5 Fréchet–Hoeffding Bounds

In this section we will give bounds on copulas. While we will see that the upper bound is itself a copula, the lower one is a copula only in dimension $d = 2$.

Definition 2.26. We define multivariate functions $W, M : [0,1]^d \to [0,1]$ by

$$W(u_1, \dots, u_d) = \max \left\{ \sum_{i=1}^{d} u_i + 1 - d, 0 \right\}, \tag{2.9}$$

$$M(u_1, \dots, u_d) = \min \left\{ u_1, \dots, u_d \right\}, \tag{2.10}$$

where $u_1, \dots, u_d \in [0,1]$.

Theorem 2.27 (**Fréchet–Hoeffding Bounds**[35]). *For every copula C we have the bounds*

$$W(u_1, \dots, u_d) \leq C(u_1, \dots, u_d) \leq M(u_1, \dots, u_d), \quad u_1, \dots, u_d \in [0,1].$$

Proof. To prove the first inequality let $u_1, \dots, u_d \in [0,1]$ and notice that for a random vector $(U_1, \dots, U_d)$ with univariate standard uniform marginal distributions,

$$\begin{aligned} C(u_1, \dots, u_d) &= \mathbb{P}[U_1 \leq u_1, \dots, U_d \leq u_d] \\ &\geq 1 - \sum_{i=1}^{d} \mathbb{P}[U_i > u_i] \\ &= 1 - \sum_{i=1}^{d} (1 - \mathbb{P}[U_i \leq u_i]) \\ &= 1 - d + \sum_{i=1}^{d} u_i. \end{aligned}$$

Since $C(u_1, \dots, u_d) \geq 0$ by definition, we have that

$$C(u_1, \dots, u_d) \geq W(u_1, \dots, u_d).$$

[35] cf. [25, Theorem 5.7]

An alternative way to prove the first inequality would be to use Theorem 2.12: we know that for $u_1, \ldots, u_d \in [0,1]$,

$$|C(1,\ldots,1) - C(u_1,\ldots,u_d)| = 1 - C(u_1,\ldots,u_d) \leq \sum_{i=1}^{d}(1-u_i) = d - \sum_{i=1}^{d} u_i,$$

which can be rewritten as

$$1 - d + \sum_{i=1}^{d} u_i \leq C(u_1,\ldots,u_d).$$

The second inequality follows from combining characteristics (C1) and (C3), as for $u_1, \ldots, u_d \in [0,1]$ we have

$$C(u_1,\ldots,u_d) \leq C(1,\ldots,1,u_i,1,\ldots,1) = u_i$$

for all $i = 1, \ldots, d$. ❑

In a similar way we come to a comparable result for multivariate distribution functions:[36]

Corollary 2.28. *Every multivariate distribution function F on $\mathbb{R}^d$ with univariate marginal distributions $F_1, \ldots, F_d$ fulfils*

$$\max\left\{\sum_{i=1}^{d} F_i(x_i) + 1 - d, 0\right\} \leq F(x_1,\ldots,x_d) \leq \min\left\{F_1(x_1),\ldots,F_d(x_d)\right\},$$

for $x_1, \ldots, x_d \in \mathbb{R}$.

As mentioned at the beginning of this section we will now provide a proof that the upper bound in Theorem 2.27 is a copula for all $d \in \mathbb{N}, d \geq 2$, while the lower bound is a copula only for dimension $d = 2$. For this purpose we consider the subsequent proposition:[37]

Proposition 2.29. *Consider the functions M and W from Definition 2.26. Then the following statements hold:*

(a) *M is a copula on $[0,1]^d$ for all $d \in \mathbb{N}, d \geq 2$.*

(b) *W is a copula only on $[0,1]^2$.*

Proof. In intention of proving part (a), let U denote a univariate standard uniformly distributed random variable on a suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and consider the random vector $(U, \ldots, U)$ of length $d \geq 2$. For $u_1, \ldots, u_d \in [0,1]$,

$$\mathbb{P}[U \leq u_1, \ldots, U \leq u_d] = \mathbb{P}[U \leq \min\{u_1,\ldots,u_d\}] = \min\{u_1,\ldots,u_d\}.$$

[36] cf. [25, Remark 5.8]
[37] cf. [10, Example 1.3.3, Example 1.3.5]

Consequently, $M(u_1, \ldots, u_d) = \min\{u_1, \ldots, u_d\}$ is a copula, as it is the distribution function of a random vector with univariate standard uniform marginals.

To see that W is a copula for dimension $d = 2$ we consider the random vector $(U, 1-U)$ where U denotes a standard uniformly distributed random variable.[38] We observe that for $u_1, u_2 \in [0,1]$,

$$\mathbb{P}[U \leq u_1, 1-U \leq u_2] = \mathbb{P}[1-u_2 \leq U \leq u_1] = \max\{u_1 + u_2 - 1, 0\}.$$

Thus W is a copula for dimension $d = 2$.

We will show explicitly[39] that W is not a copula for $d \geq 3$, since the d-increasing property is not met. Let d be arbitrary in $\mathbb{N}$, $d \geq 3$ and consider the hypercube $[\frac{1}{2}, 1]^d \subset [0,1]^d$. Further, let $u_{i,1} = \frac{1}{2}, u_{i,2} = 1$ for $i = 1, \ldots, d$. We can infer that

$$\sum_{i_1=1}^{2} \cdots \sum_{i_d=1}^{2} (-1)^{i_1+\cdots+i_d} \left(u_{1,i_1} + \cdots + u_{d,i_d} + 1 - d\right)^+ = 1 - \frac{d}{2},$$

since the summand is differing from 0 exactly when $\sum_{k=1}^{d} i_k \geq 2d-1$. Evidently, $1 - \frac{d}{2} < 0$ for $d \geq 3$ and thus W can not be a copula for dimensions greater than 2, as characteristic (C4) would be not fulfilled. ❑

2.6 Comonotonicity and Countermonotonicity

This brief section is dedicated to perfect dependence (not to be confused with perfect linear dependence) between random variables.

Comonotonicity between random variables means that they are perfectly positively dependent on each other:[40]

Definition 2.30. Random variables $X_1, \ldots, X_d$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ are called *comonotonic* if they can be coupled by means of the Fréchet–Hoeffding upper bound copula as given in (2.10).

Let $X_1, \ldots, X_d$ denote arbitrary random variables on a probability space with univariate marginals $F_1, \ldots, F_d$. The above definition states that we can represent the joint distribution function F of $(X_1, \ldots, X_d)$ according to the theorem of Sklar as $M(F_1(x_1), \ldots, F_d(x_d))$, $x_1, \ldots, x_d \in \mathbb{R}$.

The proof of the subsequent lemma is omitted. Instead, interested readers are referred to [25, Proposition 5.16].

[38] A more analytical proof of W being a copula for $d = 2$, where the properties (C1)–(C4) are verified, can be found in the appendix (cf. Lemma A2).
[39] cf. [10, p. 27]
[40] cf. [25, Definition 5.15]

Lemma 2.31. *Let $f_1, \ldots, f_d$ denote monotonically increasing functions. Then random variables $X_1, \ldots, X_d$ are comonotonic if and only if $(X_1, \ldots, X_d) \stackrel{\mathrm{d}}{=} (f_1(Z), \ldots, f_d(Z))$ holds for some random variable Z.*[41]

Similarly, contramonotonicity is defined by the lower Fréchet–Hoeffding barrier. Since this is a copula only for dimension $d = 2$ (c.f. proof of Proposition 2.29(b)) the following applies:[42]

Definition 2.32. Two random variables X and Y are called *countermonotonic* if they allow the lower Fréchet–Hoeffding bound W as copula.

As in the case of comonotonicity the following characterization holds:[43]

Lemma 2.33. *Let f_1 denote a monotonically increasing function and let f_2 be a monotonically decreasing function. Then random variables X and Y are countermonotonic if and only if $(X, Y) \stackrel{\mathrm{d}}{=} (f_1(Z), f_2(Z))$ holds for some random variable Z.*

Remark 2.34. For random variables with continuous marginal distributions, there are further characterizations of co- and contramonotonicity, which we will not discuss in this master thesis. Interested readers are referred to [25, Section 5.1.6].

2.7 Examples of Copulas

In this section, we will give some examples of copulas. We distinguish between:[44]

(i) *Fundamental Copulas*, which describe fundamental dependency structures,

(ii) *Implicit Copulas*, which are determined from known multivariate distributions under the use of Sklar's theorem,

(iii) *Explicit Copulas*, which are given by simple and explicit formulas, mostly according to a mathematical principle of construction.

In addition, we will briefly introduce the concept of asymmetric multivariate copulas.

Remark 2.35. For a detailed description of all common and not yet so well-known copulas we can refer to the work of Joe [20, Chapter 3 and 4], where the author also addresses the interesting theory of vine copulas.

[41]Here, $\stackrel{\mathrm{d}}{=}$ denotes equality in distribution.
[42]cf. [25, Definition 5.18]
[43]cf. [25, Proposition 5.19]
[44]cf. [25, p. 189]

2.7.1 Fundamental Copulas

As already stated above, fundamental copulas describe essential dependence structures. These include comonotonicity (perfect positive dependency), countermonotonicity (perfect negative dependency) and independence (no dependency).

Definition 2.36. The *independence copula* $\Pi : [0,1]^d \to [0,1]$ is given by[45]

$$\Pi(u_1, \ldots, u_d) \coloneqq \prod_{i=1}^{d} u_i, \quad u_1, \ldots, u_d \in [0,1].$$

Clearly, Π is actually a copula, as for independent and standard uniformly distributed random variables $U_1, \ldots, U_d$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ it holds that

$$\mathbb{P}[U_1 \le u_1, \ldots, U_d \le u_d] = \prod_{i=1}^{d} \mathbb{P}[U_i \le u_i] = \prod_{i=1}^{d} u_i, \quad u_1, \ldots, u_d \in [0,1].$$

Thus, Π is a copula in the sense of Definition 2.9.

Definition 2.37. The *comonotonicity copula* $M : [0,1]^d \to [0,1]$ is given by the Fréchet–Hoeffding upper bound (2.10):[46]

$$M(u_1, \ldots, u_d) \coloneqq \min\{u_1, \ldots, u_d\}, \quad u_1, \ldots, u_d \in [0,1].$$

That M is actually a copula has been already proved in Proposition 2.29(a). If we consider a standard uniformly distributed random variable U and set $U_1 = U, \ldots, U_d = U$, then M is the distribution function of the random vector $(U_1, \ldots, U_d)$.

Definition 2.38. The *countermonotonicity copula* $W : [0,1]^2 \to [0,1]$ is given by the Fréchet–Hoeffding lower bound (2.9) for dimension $d = 2$:[47]

$$W(u_1, u_2) \coloneqq \max\{u_1 + u_2 - 1, 0\}, \quad u_1, u_2 \in [0,1].$$

Proposition 2.29(b) states that W is a copula for $d = 2$. For a standard uniformly distributed random variable U, W is the distribution function of the random vector $(U, 1-U)$.

2.7.2 Implicit Copulas

Implicit copulas are obtained from known multivariate distributions using Sklar's theorem. In the following we will define the Gaussian and the t-copula,

[45] cf. [24, Example 1.1]
[46] cf. [24, Example 1.2]
[47] cf. [24, Example 1.3]

which are certainly two of the most well-known implicit copulas:

Definition 2.39. Let Φ denote the distribution function of a univariate standard normal distribution and let Φ_P^d be the distribution function of a d-variate normal distribution with correlation matrix[48] P and mean 0.[49] Then the d-dimensional *Gaussian copula* $C_P^{\mathrm{Ga}} : [0,1]^d \to [0,1]$ is given as[50]

$$C_P^{\mathrm{Ga}}(u_1, \dots, u_d) \coloneqq \Phi_P^d(\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_d)), \quad u_1, \dots, u_d \in [0,1].$$

There are the following special cases for the Gaussian copula C_P^{Ga}:[51]

(i) If $P = I_d$, where I_d denotes the d-dimensional identity matrix, then $C_{I_d}^{\mathrm{Ga}}$ coincides with the independence copula Π.

(ii) If $P = J_d$, where J_d denotes a $d \times d$ matrix consisting entirely of ones, then $C_{J_d}^{\mathrm{Ga}}$ coincides with the comonotonicity copula M.

(iii) If $d = 2$ and $P = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, then C_P^{Ga} coincides with the countermonotonicity copula W. Especially in the bivariate case the Gaussian copula interpolates between countermonotonicity, independence and comonotonicity.

Definition 2.40. Let t_ν be the distribution function of a univariate standard t-distribution with ν degrees of freedom, $\nu > 0$. By $t_{\nu,P}$ we denote the multivariate distribution function of a d-variate t-distribution with correlation matrix P and $\nu > 0$ degrees of freedom.[52] Then the d-dimensional t-copula $C_{\nu,P}^t : [0,1]^d \to [0,1]$ is given as[53]

$$C_{\nu,P}^t(u_1, \dots, u_d) \coloneqq t_{\nu,P}(t_\nu^{-1}(u_1), \dots, t_\nu^{-1}(u_d)), \quad u_1, \dots, u_d \in [0,1].$$

As a special case, $P = J_d$ corresponds to comonotonicity. Unlike to the Gaussian copula, for $P = I_d$ we do not obtain independence.

Remark 2.41. Note that both, the multidimensional normal and t-distribution are described in detail in [25, Chapter 3]. For the implementation of the Gaussian and the t-copula in R, in particular for the evaluation of the multivariate distribution functions, we refer interested readers to [3] and [4] at this point.

[48] A correlation matrix is a symmetric and positive semi-definite matrix with entries 1 in the diagonal.
[49] Here, 0 denotes the origin in $\mathbb{R}^d$.
[50] cf. [20, p. 163]
[51] cf. [25, p. 191]
[52] cf. [25, Example 3.7]
[53] cf. [25, p. 191]

2.7.3 Explicit Copulas

We will define common explicit copulas for the general d-dimensional case:

Definition 2.42. For a real-valued parameter $\alpha > 0$, the d-variate *Clayton copula* $C_\alpha^{\mathrm{Cl}} : [0,1]^d \to [0,1]$ is defined as[54]

$$C_\alpha^{\mathrm{Cl}}(u_1, \ldots, u_d) = \max\left\{\left(\sum_{i=1}^{d} u_i^{-\alpha} - d + 1\right)^{-1/\alpha}, 0\right\}, \quad u_1, \ldots, u_d \in (0,1].$$

By taking the limit $\alpha \searrow 0$, the resulting Clayton copula coincides with the independence copula. For $\alpha \nearrow +\infty$ one obtains the comonotonicity copula.[55]

Definition 2.43. For a real-valued parameter $\alpha \geq 1$ the d-dimensional *Gumbel copula* $C_\alpha^{\mathrm{Gu}} : [0,1]^d \to [0,1]$ is given by[56]

$$C_\alpha^{\mathrm{Gu}}(u_1, \ldots, u_d) \coloneqq \exp\left(-\left(\sum_{i=1}^{d}(-\ln u_i)^\alpha\right)^{1/\alpha}\right), \quad u_1, \ldots, u_d \in (0,1].$$

Special cases of the Gumbel copula are for $\alpha = 1$ independence and for $\alpha \nearrow +\infty$ comonotonicity.[57]

Definition 2.44. For a real-valued parameter $\alpha > 0$ the d-dimensional *Frank copula* $C_\alpha^{\mathrm{Fr}} : [0,1]^d \to [0,1]$ is given by[58]

$$C_\alpha^{\mathrm{Fr}}(u_1, \ldots, u_d) \coloneqq -\frac{1}{\alpha} \ln\left(1 - (1 - \mathrm{e}^{-\alpha})^{1-d} \prod_{i=1}^{d}(1 - \mathrm{e}^{-\alpha u_i})\right),$$

where $u_1, \ldots, u_d \in (0,1]$.

For $\alpha \searrow 0$ one obtains independence as a special case.[59]

All listed copulas in this subsection belong to the family of Archimedean copulas, for whose concept we refer interested readers to [24, Chapter 2] and [25, Section 5.4], for instance.

2.7.4 Asymmetric Copulas

All copulas presented in the previous Subsection 2.7.3 are symmetric, i.e. $C(u_1, \ldots, u_d) = C(u_{\pi(1)}, \ldots, u_{\pi(d)})$ for all $u_1 \ldots, u_d \in [0,1]$ and permutations

[54] cf. [10, Example 6.5.17]
[55] cf. [10, Example 6.5.17]
[56] cf. [10, Example 6.5.16]
[57] cf. [10, Example 6.5.16]
[58] cf. [23, Example 2]
[59] cf. [10, Example 6.5.18]

$\pi : \{1, \ldots, d\} \to \{1, \ldots, d\}$. Thus, in this subsection we will briefly introduce the concept of asymmetric copulas. We will mainly get our information from [23], thus we will avoid explicit references to this paper.

Although copulas offer the possibility of flexible dependence-modelling, most known copulas usually only have one or two parameters and consequently a limited variety of shapes. In [23] the author addresses this problem by introducing a methodology of obtaining parametric families of copulas with a flexible number of parameters. As the author states, the main advantage over known copulas is that asymmetric copulas can – in many cases – be better fitted to data since they can be used to model a large number of feasible dependence structures.

One of the main results in [23] is as follows:

Theorem 2.45. *For $k \in \mathbb{N}$, $i = 1, \ldots, d$ and $j = 1, \ldots, k$ let $g_{j,i} : [0,1] \to [0,1]$ denote strictly increasing functions with the exception that $g_{j,i}$ can also correspond to the 1-function. Suppose that $\prod_{j=1}^{k} g_{j,i}(v) = v$ for all $v \in [0,1]$ and $i = 1, \ldots, d$. Further, let $g_{j,i}(0) = \lim_{v \searrow 0} g_{j,i}(v)$ for $j = 1, \ldots, k$ and $i = 1, \ldots, d$. If $C_1, \ldots, C_k : [0,1]^d \to [0,1]$ are d-variate copulas, then the function*

$$\begin{aligned} \tilde{C} : [0,1]^d &\to [0,1] \\ (u_1, \ldots, u_d) &\mapsto \prod_{j=1}^{k} C_j(g_{j,1}(u_1), \ldots, g_{j,d}(u_d)) \end{aligned}$$

is a d-dimensional copula as well.

For the choice of the functions $g_{j,i}$ the author gives four examples which meet the requirements of the theorem above.

Example 2.46. The copula $\tilde{C}^{\mathrm{Cl}}_{\theta,\alpha} : [0,1]^d \to [0,1]$ given as

$$\tilde{C}^{\mathrm{Cl}}_{\theta,\alpha}(u_1, \ldots, u_d) = \max\left\{ \left(\sum_{i=1}^{d} u_i^{-\alpha\theta_i} - d + 1 \right)^{-1/\alpha}, 0 \right\} \prod_{i=1}^{d} u_i^{1-\theta_i},$$

where $u_1, \ldots, u_d \in (0,1]$, is an extension of the Clayton copula with $d+1$ parameters $\alpha > 0$ and $\theta = (\theta_1, \ldots, \theta_d) \in [0,1]^d$.

Remark 2.47. The aim of this subsection was to give a brief and concise introduction to the theory of asymmetric copulas. In [23] further generalizations of the copulas presented in Subsection 2.7.3 can be found. Interested readers may turn to this paper for further information.

Chapter 3

Measures of Dependency

As we have already seen in Chapter 2, the dependence structure between random variables is described by means of copulas. Since these are multi-dimensional mathematical objects and therefore not easily explainable (especially to non-mathematicians), the idea would be to depict dependency by a single number in the interval $[-1, 1]$, although this inevitably leads to a massive loss of information. Consequently, we will deal with classical dependency measures in this chapter. Amongst others, these include linear correlations and rank correlation measures. On the following pages we will discuss advantages and disadvantages of the respective approach and give examples.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

3.1 Covariance and Pearson's Correlation Coefficient

The content of the following definition is one of the foundations of probability theory and can be found in many textbooks:[60]

Definition 3.1. Let X and Y denote arbitrary random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{E}[X^2] < \infty$ and $\mathbb{E}[Y^2] < \infty$. We define the *covariance* between X and Y as

$$\begin{aligned}\mathrm{Cov}[X, Y] &= \mathbb{E}\left[(X - \mathbb{E}[X])\,(Y - \mathbb{E}[Y])\right] \\ &= \mathbb{E}\left[XY\right] - \mathbb{E}\left[X\right]\mathbb{E}\left[Y\right].\end{aligned}$$

Further, we denote the *variance* of the random variable X as

$$\mathrm{Var}[X] \coloneqq \mathrm{Cov}[X, X].$$

[60] cf. [22, Definition 15.7]

Provided that additionally $\mathrm{Var}[X], \mathrm{Var}[Y] \in (0, +\infty)$, the *Pearson correlation coefficient* between X and Y is given by

$$\mathrm{Corr}[X, Y] := \frac{\mathrm{Cov}[X, Y]}{\sqrt{\mathrm{Var}[X]\mathrm{Var}[Y]}}. \tag{3.1}$$

We will say that X and Y are *uncorrelated* if $\mathrm{Cov}[X, Y] = 0$ holds.

Remark 3.2. In the literature, the correlation operator is often referred to as ρ. To counteract confusion with Spearman's Rho, we leave it at the notation given in the definition above.

Pearson's correlation measures linear dependence and takes values in $[-1, 1]$ which can be easily verified using the Cauchy–Schwarz inequality[61]: Let X and Y denote square integrable random variables and define

$$\tilde{X} = X - \mathbb{E}[X], \quad \tilde{Y} = Y - \mathbb{E}[Y].$$

Then it obviously holds that $\tilde{X}$ and $\tilde{Y}$ are square integrable and

$$\mathrm{Cov}[X, Y] = \mathbb{E}[\tilde{X}\tilde{Y}] \le \sqrt{\mathbb{E}[\tilde{X}^2]}\sqrt{\mathbb{E}[\tilde{Y}^2]} = \sqrt{\mathrm{Var}[X]}\sqrt{\mathrm{Var}[Y]}.$$

For independent square integrable random variables X and Y it holds that $\mathrm{Cov}[X, Y] = 0$. The inverse of this statement generally does not apply. Further, $\mathrm{Corr}[X, Y]$ takes one of the boundary values -1 or 1 if and only if X and Y are perfectly linearly dependent, i.e. $Y = a + bX$ almost surely for arbitrary $a \in \mathbb{R}$ and $b \neq 0$. Another important property is that the correlation coefficient as defined in (3.1) is invariant under strictly increasing linear transformations. However, it should be noted that this does in general not apply to non-linear transformations.[62]

A useful formula for calculating the covariance is given by Hoeffding's Covariance Identity, which we will use in this thesis without proof. Interested readers are referred to [25, Lemma 5.24].

Lemma 3.3 (**Hoeffding's Covariance Identity**). *Let X and Y denote square integrable random variables with joint distribution function F and let F_X and F_Y denote the univariate distribution functions of X and Y, respectively. Then the following formula applies:*

$$\mathrm{Cov}[X, Y] = \int_{\mathbb{R}} \int_{\mathbb{R}} (F(x, y) - F_X(x)F_Y(y)) \, dx \, dy.$$

3.1.1 Pitfalls in the Use of Linear Correlations

We will now turn to two known pitfalls in the use of linear correlations. For each we will give a counterexample and correct the statement.

[61] For the Cauchy–Schwarz inequality we refer the reader to [22, Theorem 13.4].
[62] cf. [25, p. 202]

Pitfall I: The multidimensional distribution of a random vector can be described through univariate marginal distributions and pairwise correlations.[63]

Counterexample to Pitfall I: Let U_1 and U_2 denote standard uniformly distributed random variables and let $\operatorname{Corr}[U_1, U_2] = \rho \in [-1, 1]$. For $\lambda \in [0, 1]$ let[64]

$$F^\lambda(u_1, u_2) := \lambda M(u_1, u_2) + (1 - \lambda) W(u_1, u_2), \quad u_1, u_2 \in [0, 1]$$

denote a possible 2-dimensional distribution function of the random vector (U_1, U_2). For the calculation of the correlation between U_1 and U_2 we use Lemma 3.3 to calculate $\operatorname{Cov}[U_1, U_2]$, as $\operatorname{Var}[U_1] = \operatorname{Var}[U_2] = \frac{1}{12}$ is generally known. By simple calculation:

$$\begin{aligned}
\operatorname{Cov}[U_1, U_2] &= \int_0^1 \int_0^1 \lambda M(u_1, u_2) + (1 - \lambda) W(u_1, u_2) - u_1 u_2 \; du_1 \, du_2 \\
&= \lambda \int_0^1 \int_0^1 \min\{u_1, u_2\} \; du_1 \, du_2 \\
&\quad + (1 - \lambda) \int_0^1 \int_0^1 \max\{u_1 + u_2 - 1, 0\} \; du_1 \, du_2 - \frac{1}{4} \\
&= 2\lambda \int_0^1 \int_{u_2}^1 u_2 \; du_1 \, du_2 + (1 - \lambda) \int_0^1 \int_{1-u_2}^1 u_1 + u_2 - 1 \; du_1 \, du_2 - \frac{1}{4} \\
&= \frac{\lambda}{3} + \frac{1 - \lambda}{6} - \frac{1}{4} \\
&= \frac{1}{12}(2\lambda - 1).
\end{aligned}$$

Consequently, the following applies:

$$\operatorname{Corr}[U_1, U_2] = 2\lambda - 1 \overset{!}{=} \rho \iff \lambda = \frac{1 + \rho}{2}.$$

Thus $F^{\frac{1+\rho}{2}}(u_1, u_2)$ is a joint distribution of (U_1, U_2) with given marginals and correlation.

In order to show that Pitfall I is not correct, we will construct a second two-dimensional distribution function with standard uniformly distributed margins and correlation ρ. For this, let $u_1, u_2, \lambda \in [0, 1]$ and let

$$\begin{aligned}
G_1^\lambda(u_1, u_2) &:= \lambda M(u_1, u_2) + (1 - \lambda) \Pi(u_1, u_2), \\
G_2^\lambda(u_1, u_2) &:= \lambda W(u_1, u_2) + (1 - \lambda) \Pi(u_1, u_2)
\end{aligned}$$

denote bivariate distribution functions of (U_1, U_2). Then, similarly to the above considerations,

$$G^\rho(u_1, u_2) := \begin{cases} G_1^\rho(u_1, u_2) & \text{for } \rho \in [0, 1], \\ G_2^{-\rho}(u_1, u_2) & \text{for } \rho \in [-1, 0) \end{cases}$$

[63] cf. [25, Fallacy 1]

[64] That F is actually a multivariate distribution function follows by simple recalculation. It even applies that convex combinations between distribution functions on $\mathbb{R}$ (respectively, $\mathbb{R}^d$) are again distribution functions, cf. [10, p. 16].

is a multivariate distribution function with given marginals and correlation.

Obviously, $F^{\frac{1+\rho}{2}}$ and G^ρ do not match[65], showing that multidimensional distributions can not be determined solely by marginal distributions and correlations in general.

Correction of Pitfall I: As we already discussed at the beginning of Section 2.3, multivariate distribution functions are determined by one-dimensional marginal distributions and copulas. It should also be noted that different dependency structures can give the same linear correlation.[66]

In the following we will see that the set of attainable linear correlations between random variables does not necessarily coincide with the whole interval $[-1, 1]$. Especially in risk management it is therefore important to have a deeper knowledge of dependence that is going beyond simple linear correlations.

Pitfall II: Given two random variables X and Y with marginals F_1 and F_2 and $\rho \in [-1, 1]$, there always exists a possibility to construct a bivariate distribution function F with the given margins F_1 and F_2 and linear correlation ρ.[67]

The next counterexample can be found in a similar form in [13, Example 4].

Counterexample to Pitfall II: Let X and Y denote two square integrable random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with univariate distribution functions F_X and F_Y respectively. For $x, y \in (-\infty, 0)$ let

$$F_X(x) = F_Y(y) = 0 \tag{3.2}$$

and

$$\sup\{x \in \mathbb{R} : F_X(x) < 1\} = \sup\{y \in \mathbb{R} : F_Y(y) < 1\} = +\infty. \tag{3.3}$$

If we additionally assume that $\mathrm{Corr}[X, Y] = -1$, then $Y = a + bX$ $\mathbb{P}$-a.s. for some $a \in \mathbb{R}$ and $b < 0$. Hence, for $y < 0$ it holds that

$$\begin{aligned} F_Y(y) = \mathbb{P}[Y \leq y] = \mathbb{P}[a + bX \leq y] &= \mathbb{P}\left[X \geq \frac{y-a}{b}\right] \\ &\geq \mathbb{P}\left[X > \frac{y-a}{b}\right] = 1 - F_X\left(\frac{y-a}{b}\right) \\ &\overset{(3.3)}{>} 0, \end{aligned}$$

which contradicts condition (3.2). Consequently, a negative linear correlation of -1 between X and Y is impossible to achieve.

The question that arises now is how Pitfall II can be corrected to a valid statement. To this end we first need the following definition:[68]

[65] For instance, for $\rho \in [0, 1]$ we have that $F^{\frac{1+\rho}{2}}(\frac{1}{3}, \frac{1}{3}) = \frac{1+\rho}{6}$ and $G_1^\rho(\frac{1}{3}, \frac{1}{3}) = \frac{2\rho-1}{9}$.

[66] cf. [25, p. 203]

[67] cf. [25, Fallacy 2]

[68] cf. [25, Definition A.1]

Definition 3.4. Two random variables X and Y on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ are of the *same type* if there exist constants $a \in \mathbb{R}$ and $b > 0$ such that

$$Y \stackrel{\mathrm{d}}{=} a + bX$$

holds.

The following result provides information on the maximum possible interval for linear correlations between pairs of random variables. A proof of this theorem is not given in this master thesis, interested readers are referred to [25, Theorem 5.25] and [13, Theorem 4], respectively.

Theorem 3.5 (Attainable linear correlations[69]). *Let X and Y denote square integrable random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, i.e. $\mathbb{E}[X^2] < \infty$ and $\mathbb{E}[Y^2] < \infty$. Further, let $\mathrm{Var}[X] > 0$ and $\mathrm{Var}[Y] > 0$. Then the following statements apply:*

(a) *The attainable linear correlations form a closed interval $[\rho_{\min}, \rho_{\max}]$, such that $\rho_{\min} \in [-1, 0)$ and $\rho_{\max} \in (0, 1]$.*

(b) *The minimum correlation $\rho_{\min}$ is reached if and only if the random variables X and Y are countermonotonic. The maximum correlation $\rho_{\max}$ is reached if and only if the random variables X and Y are comonotonic.*

(c) *$\rho_{\min} = -1$ if and only if X and $-Y$ are of the same type. Analogously, $\rho_{\max} = 1$ if and only if X and Y are of the same type.*

Correction of Pitfall II: Pitfall II is corrected by the theorem of attainable linear correlations.

The subsequent example is in a similar way widely used in the literature because of the closed formulas for the linear correlation barriers $\rho_{\min}$ and $\rho_{\max}$:[70]

Example 3.6. Let X and Y denote log-normal distributed random variables, i.e. $\ln(X) \sim \mathcal{N}(0, \sigma_X^2)$ and $\ln(Y) \sim \mathcal{N}(0, \sigma_Y^2)$, where $\sigma_X, \sigma_Y > 0$. Further, let Z be a standard normally distributed random variable, i.e. $Z \sim \mathcal{N}(0, 1)$. Then $X = \mathrm{e}^{\sigma_X Z}$ and $Y = \mathrm{e}^{\sigma_Y Z}$ hold. From Section 2.6 we know that X and Y are comonotonic as the exponential function is strictly increasing. Thus, by Theorem 3.5(b) we can infer that

$$\rho_{\max} = \mathrm{Corr}[\mathrm{e}^{\sigma_X Z}, \mathrm{e}^{\sigma_Y Z}].$$

The variance of X and Y can be calculated as follows:

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \mathrm{e}^{\sigma_X^2}(\mathrm{e}^{\sigma_X^2} - 1), \quad \mathrm{Var}[Y] = \mathrm{e}^{\sigma_Y^2}(\mathrm{e}^{\sigma_Y^2} - 1).$$

For the covariance between X and Y follows:

$$\mathrm{Cov}[X, Y] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = \mathrm{e}^{\frac{\sigma_X^2 \sigma_Y^2}{2}}(\mathrm{e}^{\sigma_X \sigma_Y} - 1).$$

[69] cf. [25, Theorem 5.25]

[70] cf. [13, Example 5]

Thus, the maximum attainable linear correlation between X and Y is given by

$$\rho_{\max} = \frac{e^{\sigma_X \sigma_Y} - 1}{\sqrt{(e^{\sigma_X^2} - 1)(e^{\sigma_Y^2} - 1)}}, \quad \sigma_X, \sigma_Y > 0.$$

For calculating $\rho_{\min}$ we use the fact that $X = e^{\sigma_X Z}$ and $Y = e^{-\sigma_Y Z}$ are countermonotonic and obtain in an analogous way as above

$$\rho_{\min} = \frac{e^{-\sigma_X \sigma_Y} - 1}{\sqrt{(e^{\sigma_X^2} - 1)(e^{\sigma_Y^2} - 1)}}, \quad \sigma_X, \sigma_Y > 0.$$

3.2 Rank Correlation Measures

In this section we will get to know rank correlation measures, which – assuming continuous marginal distributions – depend only on the underlying copula. As the name suggests, rank correlations are calculated based on the rank of the data.[71]

3.2.1 Kendall's Tau ρ_τ

Before starting with a definition of Kendall's tau we need the two fundamentals below:

Definition 3.7. Let $p = (x, y)$ and $\tilde{p} = (\tilde{x}, \tilde{y})$ denote arbitrary points in $\mathbb{R}^2$. We call p and $\tilde{p}$ *concordant* if $(x - \tilde{x})(y - \tilde{y}) > 0$. Otherwise, if $(x - \tilde{x})(y - \tilde{y}) < 0$, p and $\tilde{p}$ will be called *discordant*.[72]

Definition 3.8. Let X and Y denote arbitrary random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with distribution functions F_X and F_Y. We call Y an *independent copy* of X if X and Y are independent and $F_X = F_Y$ holds.[73]

Literally, for a pair (X, Y) of random variables with an independent copy $(\tilde{X}, \tilde{Y})$, Kendall's tau can be described as the probability of concordance minus the probability of discordance between these pairs:[74]

Definition 3.9. Let X and Y denote random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $(\tilde{X}, \tilde{Y})$ denote an independent copy of (X, Y). *Kendall's tau* ρ_τ between X and Y is then defined as

$$\begin{aligned}\rho_\tau(X, Y) &= \mathbb{P}[(X - \tilde{X})(Y - \tilde{Y}) > 0] - \mathbb{P}[(X - \tilde{X})(Y - \tilde{Y}) < 0] \\ &= \mathbb{E}[\operatorname{sign}((X - \tilde{X})(Y - \tilde{Y}))],\end{aligned}$$

where $\operatorname{sign}(\cdot)$ denotes the sign-function.

71 cf. [25, p. 206]
72 cf. [20, p. 55]
73 cf. [21, p. 369]
74 cf. [25, Definition 5.27]

As already mentioned in the introduction to this section, we are now looking at a representation of Kendall's tau ρ_τ by the means of copulas. A detailed proof of this statement can be found in [27, Theorem 5.1.1]:

Theorem 3.10. *Let X and Y denote random variables with continuous univariate marginal distributions, coupled by a copula C. Then Kendall's tau can be expressed as*

$$\rho_\tau(X,Y) = 4\int_0^1\int_0^1 C(u_1,u_2)\ dC(u_1,u_2) - 1.$$

Example 3.11. For two random variables X and Y, coupled by a bivariate Clayton copula with parameter $\alpha \geq -1,\ \alpha \neq 0$, we have that[75]

$$\rho_\tau(X,Y) = \frac{\alpha}{\alpha+2}.$$

For two random variables X and Y, coupled by a bivariate Gumbel copula with parameter $\alpha \geq 1$ we have that

$$\rho_\tau(X,Y) = 1 - \frac{1}{\alpha}.$$

3.2.2 Spearman's Rho ρ_S

As can be inferred from the definition below, Spearman's rho corresponds to the ordinary linear correlation between the probability-transformed random variables.

Definition 3.12. Let X and Y denote arbitrary random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ with one-dimensional distribution functions F_X and F_Y. Spearman's rho ρ_S between X and Y is given by[76]

$$\rho_S(X,Y) = \mathrm{Corr}[F_X(X), F_Y(Y)].$$

The equivalent of Theorem 3.10 for Spearman's rho is as follows:[77]

Theorem 3.13. *Let X and Y denote random variables with continuous univariate marginal distributions, coupled by a copula C. Then the integral representation of Spearman's rho is given by*

$$\rho_S(X,Y) = 12\int_0^1\int_0^1 \left(C(u_1,u_2) - u_1u_2\right)\ du_1du_2.$$

[75] cf. [25, p. 222]
[76] cf. [25, Definition 5.28]
[77] cf. [13, Theorem 3]

Chapter 4

Sum of $\mathbb{N}_0$-Valued Dependent Random Variables

In this chapter we investigate the distribution of the aggregate financial loss $S = X_1, \ldots, X_d$ of $\mathbb{N}_0$-valued risks. We assume that the dependence structure of $(X_1, \ldots, X_d)$ is modelled by copulas and prove a formula for the distribution of S. In addition, we provide a recursion formula for the computation of the probability mass function of S, which could be used in the recursion formula of Panjer for the calculation of the total loss amount, for example. Pointwise sharp bounds on the distribution of S are obtained by application of the Rearrangement Algorithm, which will be introduced briefly. These bounds serve to quantify the model risk caused by feasible scenarios of dependency.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a probability space.

4.1 Setting

Let $d \in \mathbb{N}$, $d \geq 2$ and let $X_1, \ldots, X_d$ denote $\mathbb{N}_0$-valued random variables such that $X_i \sim F_i$ for given univariate distribution functions F_i, $i = 1, \ldots, d$. For example, one can think about the random variables X_i representing claim sizes in an insurer's portfolio or credit losses in banking. Suppose further, that the dependency structure of the portfolio $(X_1, \ldots, X_d)$ is given by a d-dimensional copula C.

The main objective of this chapter will be to examine the distribution and probability mass function of the aggregated portfolio

$$S := \sum_{i=1}^{d} X_i. \tag{4.1}$$

4.2 Distribution Function of the Sum S

For ease of notation we define

$$\mathcal{J}_n^d = \{j = (j_1, \ldots, j_d) \in \mathbb{N}_0^d : \ j_1 + \cdots + j_d \leq n\}, \quad n \in \mathbb{N}_0$$

and

$$\mathcal{I}^d = \{i = (i_1, \ldots, i_d) \in \{0, 1\}^d\}.$$

For $i \in \mathcal{I}^d$ let

$$\operatorname{sign}(i) := (-1)^{\sum_{k=1}^{d} i_k}.$$

From the properties of copulas we already know that for any $n_1, \ldots, n_d \in \mathbb{N}_0$ and $\mathbb{N}_0$-valued random variables $X_1, \ldots, X_d$ with univariate distribution functions $F_1, \ldots, F_d$ such that $X_i \sim F_i$ for $i = 1, \ldots, d$ the following applies:[78]

$$\mathbb{P}[X_1 = n_1, \ldots, X_d = n_d] = \sum_{i \in \mathcal{I}^d} \operatorname{sign}(i) \ C(F_1(n_1 - i_1), \ldots, F_d(n_d - i_d)).$$

For the distribution function of the aggregate loss S we can thus infer the following:[79]

Lemma 4.1. *Let $n \in \mathbb{N}_0$ and $\mathcal{J}_n^d$, $\mathcal{I}^d$ as given above. Let $X_1, \ldots, X_d$ denote $\mathbb{N}_0$-valued random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $X_i \sim F_i$ for univariate distribution functions $F_1, \ldots, F_d$ and $i = 1, \ldots, d$. For every copula C of the random vector $(X_1, \ldots, X_d)$ it holds for S as defined in (4.1):*

$$\mathbb{P}[S \leq n] = \sum_{j \in \mathcal{J}_n^d} \sum_{i \in \mathcal{I}^d} \operatorname{sign}(i)\, C(F_1(j_1 - i_1), \ldots, F_d(j_d - i_d)). \tag{4.2}$$

It should be taken into account that for an evaluation of (4.2) with arbitrary $n \in \mathbb{N}_0$,

$$2^d \sum_{k=0}^{n} \binom{k + d - 1}{d - 1}$$

terms have to be summed up.[80]

In this section we will focus on finding more efficient formulas for the distribution function of S. We will start with a reformulation of [17, Theorem 2.2] to the case of $\mathbb{N}_0$-valued random variables:

[78] cf. [6, p. 5]
[79] This approach is due to the definition of the multivariate discrete distribution function.
[80] This follows immediately using (4.6) below.

Proposition 4.2. *Let $n \in \mathbb{N}_0$ and $\mathcal{J}_n^d$, $\mathcal{I}^d$ as given above. Let $X_1, \dots, X_d$ denote $\mathbb{N}_0$-valued random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $X_i \sim F_i$ for univariate distribution functions $F_1, \dots, F_d$ and $i = 1, \dots, d$. For every copula C of the random vector $(X_1, \dots, X_d)$ it holds for S as defined in (4.1):*[81]

$$\mathbb{P}[S \leq n] =$$
$$\sum_{j \in \mathcal{J}_n^{d-1}} \sum_{i \in \mathcal{I}^{d-1}} \operatorname{sign}(i)\; C\left(F_1(j_1 - i_1), \dots, F_{d-1}(j_{d-1} - i_{d-1}), F_d\left(n - \sum_{k=1}^{d-1} j_k\right)\right). \tag{4.3}$$

Proof. Let $n_1, \dots, n_{d-1} \in \mathbb{N}_0$. Under the prerequisites of Proposition 4.2 it holds that

$$\begin{aligned}
&\mathbb{P}[X_1 \leq n_1, \dots, X_{d-1} \leq n_{d-1}, S \leq n] \\
&= \sum_{j_1=0}^{n_1} \mathbb{P}[X_1 = j_1, X_2 \leq n_2, \dots, X_{d-1} \leq n_{d-1}, X_2 + \dots + X_d \leq n - j_1] \\
&= \sum_{j_1=0}^{n_1} \Big(\mathbb{P}[X_1 \leq j_1, X_2 \leq n_2, \dots, X_{d-1} \leq n_{d-1}, X_2 + \dots + X_d \leq n - j_1] \\
&\quad - \mathbb{P}[X_1 \leq j_1 - 1, X_2 \leq n_2, \dots, X_{d-1} \leq n_{d-1}, X_2 + \dots + X_d \leq n - j_1] \Big) \\
&= \sum_{j_1=0}^{n_1} \sum_{i_1=0}^{1} (-1)^{i_1} \mathbb{P}[X_1 \leq j_1 - i_1, X_2 \leq n_2, \dots, X_{d-1} \leq n_{d-1}, \\
&\qquad\qquad X_2 + \dots + X_d \leq n - j_1] \\
&= \sum_{j_1=0}^{n_1} \sum_{j_2=0}^{n_2} \sum_{i_1=0}^{1} (-1)^{i_1} \mathbb{P}[X_1 \leq j_1 - i_1, X_2 = j_2, X_3 \leq n_3, \dots, X_{d-1} \leq n_{d-1}, \\
&\qquad\qquad X_3 + \dots + X_d \leq n - j_1 - j_2] \\
&= \sum_{j_1=0}^{n_1} \sum_{j_2=0}^{n_2} \sum_{i_1=0}^{1} \sum_{i_2=0}^{1} (-1)^{i_1 + i_2} \mathbb{P}[X_1 \leq j_1 - i_1, X_2 \leq j_2 - i_2, X_3 \leq n_3, \dots, \\
&\qquad\qquad X_{d-1} \leq n_{d-1}, X_3 + \dots + X_d \leq n - j_1 - j_2] \\
&= \dots \\
&= \sum_{j_1=0}^{n_1} \dots \sum_{j_{d-1}=0}^{n_{d-1}} \sum_{i \in \mathcal{I}^{d-1}} \operatorname{sign}(i) \mathbb{P}[X_1 \leq j_1 - i_1, \dots, X_{d-1} \leq j_{d-1} - i_{d-1}, \\
&\qquad\qquad X_d \leq n - j_1 - \dots - j_{d-1}].
\end{aligned} \tag{4.4}$$

As X_d is a $\mathbb{N}_0$-valued random variable and only takes values greater than or equal to zero we can set $n_1, \dots, n_{d-1} = n$ in (4.4) and subsequently we get the distribution of S:

[81] For dimension $d = 2$ the proof is following closely [5, Proposition 4.1].

$$\begin{aligned}
&\mathbb{P}[S \le n] \\
&= \sum_{j_1=0}^{n} \cdots \sum_{j_{d-1}=0}^{n} \sum_{i \in \mathcal{I}^{d-1}} \operatorname{sign}(i)\, \mathbb{P}[X_1 \le j_1 - i_1, \dots, X_{d-1} \le j_{d-1} - i_{d-1}, \\
&\qquad\qquad\qquad\qquad X_d \le n - j_1 - \cdots - j_{d-1}] \\
&= \sum_{j \in \mathcal{J}_n^d} \sum_{i \in \mathcal{I}^{d-1}} \operatorname{sign}(i)\, \mathbb{P}[X_1 \le j_1 - i_1, \dots, X_{d-1} \le j_{d-1} - i_{d-1}, \\
&\qquad\qquad\qquad X_d \le n - j_1 - \cdots - j_{d-1}], \qquad (4.5)
\end{aligned}$$

where the last equality follows again from the fact that for $j \in \{0, 1, \dots, n\}^d \setminus \mathcal{J}_n^d$ we have that

$$\mathbb{P}[X_d \le n - j_1 - \cdots - j_{d-1}] = 0,$$

since the random variable X_d does not take negative values.

From Sklar's theorem 2.13 we know that we can rewrite the probability in (4.5) as follows:

$$\begin{aligned}
&\mathbb{P}[X_1 \le j_1 - i_1, \dots, X_{d-1} \le j_{d-1} - i_{d-1}, X_d \le n - j_1 - \cdots - j_{d-1}] \\
&= C\left(F_1(j_1 - i_1), \dots, F_{d-1}(j_{d-1} - i_{d-1}), F_d\left(n - \sum_{k=1}^{d-1} j_k\right)\right),
\end{aligned}$$

which completes the proof. ❑

The major advantage of (4.3) over (4.2) is that for the calculation of the distribution function of S using (4.3) fewer terms, namely

$$2^{d-1} \sum_{k=0}^{n} \binom{k+d-2}{d-2},$$

have to be summed up. Especially for large portfolios, this conclusion can make a big difference in the runtime of the numerical calculation.

In the proposition below we will now prove that the computational effort can be reduced even further.[82] To this end we first define

$$\bar{\mathcal{J}}_n^d = \{j = (j_1, \dots, j_d) \in \mathbb{N}_0^d : \ j_1 + \cdots + j_d = n\}, \quad n \in \mathbb{N}_0.$$

Obviously,

$$\mathcal{J}_n^d = \bigcup_{k=0}^{n} \bar{\mathcal{J}}_k^d, \quad n \in \mathbb{N}_0. \tag{4.6}$$

To ease notation, we set

$$c_n := \sum_{j \in \bar{\mathcal{J}}_n^d} C(F_1(j_1), \dots, F_d(j_d)), \quad n \in \mathbb{N}_0, \tag{4.7}$$

[82] I am grateful to my supervisor Dr. Uwe Schmock for providing me with a direct proof of Proposition 4.3. An earlier version of the proof can be found in the appendix to this thesis.

with the convention that $c_n = 0$ if $n < 0$.

Proposition 4.3. *Let $X_1, \ldots, X_d$ denote $\mathbb{N}_0$-valued random variables with univariate distribution functions $F_1, \ldots, F_d$, respectively. Then for all $n \in \mathbb{N}_0$ and every copula C of the random vector $(X_1, \ldots, X_d)$ it holds that*

$$\mathbb{P}[S \leq n] = \sum_{k=0}^{\min\{d-1,n\}} (-1)^k \binom{d-1}{k} c_{n-k}. \tag{4.8}$$

Proof. Let

$$\bar{\mathcal{I}}_l^d := \{i = (i_1, \ldots, i_d) \in \mathcal{I}^d : i_1 + \cdots + i_d = l\}, \quad l \in \mathbb{N}_0,\ l \leq d.$$

Using the representation from Lemma 4.1, we have that

$$\begin{aligned}
\mathbb{P}[S \leq n] &= \sum_{j \in \mathcal{J}_n^d} \sum_{i \in \mathcal{I}^d} \operatorname{sign}(i) C(F_1(j_1 - i_1), \ldots, F_d(j_d - i_d)) \\
&= \sum_{k=0}^{d} \sum_{l=0}^{n} (-1)^k \sum_{j \in \bar{\mathcal{J}}_l^d} \sum_{i \in \bar{\mathcal{I}}_k^d} C(F_1(j_1 - i_1), \ldots, F_d(j_d - i_d)) \\
&= \sum_{k=0}^{d} \sum_{l=0}^{n} (-1)^k \binom{d}{k} c_{l-k},
\end{aligned}$$

where the last equality can be justified as follows: For every pair $(i, \tilde{j}) \in \bar{\mathcal{I}}_k^d \times \bar{\mathcal{J}}_{l-k}^d$ there exists $j := \tilde{j} + i$ such that $(i, j) \in \bar{\mathcal{I}}_k^d \times \bar{\mathcal{J}}_l^d$. Conversely, for every pair $(i, j) \in \bar{\mathcal{I}}_k^d \times \bar{\mathcal{J}}_l^d$ there exists $\tilde{j} := j - i$ such that $(i, \tilde{j}) \in \bar{\mathcal{I}}_k^d \times \bar{\mathcal{J}}_{l-k}^d$ or at least one component of $\tilde{j}$ is negative, hence $C(F_1(\tilde{j}_1), \ldots, F_d(\tilde{j}_d)) = 0$. Furthermore, $\bar{\mathcal{I}}_k^d$ contains $\binom{d}{k}$ elements.

An index inversion in the first step and an index shift in the second step further results in

$$\begin{aligned}
\sum_{k=0}^{d} \sum_{l=0}^{n} (-1)^k \binom{d}{k} c_{l-k} &= \sum_{k=0}^{d} \sum_{l=0}^{n} (-1)^k \binom{d}{k} c_{n-(k+l)} \\
&= \sum_{k=0}^{d} \sum_{l=k}^{n} (-1)^k \binom{d}{k} c_{n-l}.
\end{aligned}$$

By interchanging the sums we obtain

$$\sum_{k=0}^{d} \sum_{l=k}^{n} (-1)^k \binom{d}{k} c_{n-l} = \sum_{l=0}^{n} \sum_{k=0}^{\min\{d,l\}} (-1)^k \binom{d}{k} c_{n-l}.$$

Using the algebraic identity

$$\sum_{k=0}^{\min\{d,l\}} (-1)^k \binom{d}{k} = \begin{cases} 0 & \text{for } l \geq d, \\ (-1)^l \binom{d-1}{l} & \text{otherwise} \end{cases}$$

we can conclude that

$$\mathbb{P}[S \leq n] = \sum_{l=0}^{\min\{d-1,n\}} (-1)^l \binom{d-1}{l} c_{n-l},$$

which completes the proof. ❑

Regarding numerical efficiency, for an evaluation of (4.8)

$$\sum_{k=n-\min\{d-1,n\}}^{n} \binom{k+d-1}{d-1}$$

terms have to be summed up.

Remark 4.4. For the special case of the distribution of the sum of comonotonic random variables, i.e. $C = M$, we refer interested readers to [7].

4.3 Calculation of Sharp Bounds on the Distribution Function of S[83]

In this section we provide an algorithm – the Rearrangement Algorithm – for the calculation of pointwise sharp bounds on the distribution function of S. The underlying theoretical framework was developed by Puccetti and Rüschendorf in 2011 in [29], to which interested readers are referred. Compared to existing bounds in the literature, such as those described in [8], for example, they are more accurate and relatively straightforward to determine.

Let $d \in \mathbb{N}$, $d \geq 2$ and let $X_1, \dots, X_d$ denote $\mathbb{N}_0$-valued random variables with univariate distributions $F_1, \dots, F_d$. By S we denote the sum of this d random variables. The main objective in [29] was to obtain sharp bounds on the probability

$$\mathbb{P}[S \geq n], \quad n \in \mathbb{N}_0.$$

Due to the relationship

$$\mathbb{P}[S \geq n] = 1 - \mathbb{P}[S < n], \quad n \in \mathbb{N}_0,$$

we can relate the results of Puccetti and Rüschendorf to our specific problem of finding bounds on the distribution of S.

If we define

$$m(n) = \inf\left\{\mathbb{P}[S > n] : X_i \sim F_i,\ i = 1, \dots, d\right\},$$

$$M(n) = \sup\left\{\mathbb{P}[S \geq n] : X_i \sim F_i,\ i = 1, \dots, d\right\},$$

[83] This section is mainly based on the results from [29], thus we will avoid explicit references to this paper. Only the results taken from other sources will be marked as such.

then for $n \in \mathbb{N}_0$ we can observe that:

$$m(n) \leq \mathbb{P}[S > n] \iff m(n) \leq 1 - \mathbb{P}[S \leq n] \iff 1 - m(n) \geq \mathbb{P}[S \leq n]$$

and

$$M(n) \geq \mathbb{P}[S \geq n] \iff 1 - M(n) \leq \mathbb{P}[S < n] \iff 1 - M(n+1) \leq \mathbb{P}[S \leq n]$$

holds.

For ease of notation we set

$$p_n^{\mathrm{L}} := 1 - M(n+1) \tag{4.9}$$

and

$$p_n^{\mathrm{U}} := 1 - m(n), \tag{4.10}$$

such that $p_n^{\mathrm{L}} \leq \mathbb{P}[S \leq n] \leq p_n^{\mathrm{U}}$.

We will now state exemplary the algorithm to compute the lower bound p_n^{L} of the distribution of S. For a detailed proof on the functionality we refer interested readers to the original paper [29].

Definition 4.5. For $N \in \mathbb{N}$ two vectors $a = (a_1, \dots, a_N)$, $b = (b_1, \dots, b_N) \in \mathbb{R}^N$ are *oppositely ordered*, if $(a_j - a_k)(b_j - b_k) \leq 0$ holds for all $j, k = 1, \dots, N$.

Definition 4.6. Let $N \in \mathbb{N}$. For a $(N \times d)$-matrix $Y = (y_{i,j}), i = 1, \dots, N,\ j = 1, \dots, d$, with entries in $\mathbb{R}$ we define the minimum, respectively maximum, row sum as

$$\operatorname{minRS}(Y) = \min_{i=1,\dots,N} \sum_{j=1}^{d} y_{i,j},$$

$$\operatorname{maxRS}(Y) = \max_{i=1,\dots,N} \sum_{j=1}^{d} y_{i,j}.$$

Algorithm 4.7 (Computation of p_n^{L}).

Step 1: a: Fix $n \in \mathbb{N}_0$, $N \in \mathbb{N}$, a tolerance level $\epsilon > 0$ and d univariate distributions $F_1, \dots, F_d$ of random variables $X_1, \dots, X_d$.

b: Set $L = 0$ and $R = 1$.

Step 2: Repeat the following:

a: Set $\alpha = \dfrac{L+R}{2}$

b: For $i = 1, \dots, N$ and $j = 1, \dots, d$ define a matrix $X = (x_{i,j})$ as

$$x_{i,j} = F_j^{\leftarrow}\left(\alpha + \frac{(1-\alpha)(i-1)}{N}\right).$$

c: For $j = 1, \ldots, d$, rearrange the j-th column of X to make it oppositely ordered to the row sums of the other columns. This gives a matrix Y.

d: Repeat c. until

$$|\text{minRS}(Y) - \text{minRS}(X)| = 0$$

and set $\tilde{n} = \text{minRS}(Y)$.

e: If $|\tilde{n} - n - 1| = 0$ or $|L - R| < \epsilon$ then break and return α. Otherwise do the following:

ea: If $\tilde{n} < n + 1$ then set

$$R = 2\alpha - L,$$
$$L = \alpha.$$

eb: Else set

$$L = 2\alpha - R,$$
$$R = \alpha.$$

ec: Return to Step 2a.

In the course of this master thesis the algorithm above was implemented in R. The program can be found in the appendix (cf. Program R5).

4.4 Recursion for the Probability Mass Function of S

In this section we provide a recursion formula for calculating the probability mass function of the sum of $\mathbb{N}_0$-valued random variables whose dependence is described by an arbitrary copula. In the given setting (see Section 4.1) we define

$$p_n = \mathbb{P}[S = n], \quad n \in \mathbb{N}_0,$$

with the convention that $p_n = 0$ if $n < 0$. Together with c_n as given in (4.7) we obtain the subsequent result:[84]

Theorem 4.8. *The following recursion formula applies for all $n \in \mathbb{N}_0$:*

$$p_n = c_n - \sum_{k=1}^{n} \binom{k+d-1}{d-1} p_{n-k}, \quad p_0 = c_0. \tag{4.11}$$

[84] I am grateful to my supervisor Dr. Uwe Schmock for providing me with an abbreviated proof of Theorem 4.8. An earlier version of the proof can be found in the appendix to this thesis.

Proof. From Sklar's Theorem 2.13 we know that

$$\begin{aligned} C(F_1(j_1),\dots,F_d(j_d)) &= \mathbb{P}(X_1 \le j_1,\dots,X_d \le j_d) \\ &= \sum_{k=0}^{n} \sum_{\substack{i\in\bar{\mathcal{J}}^d_{n-k} \\ i\le j}} \mathbb{P}(X_1 = i_1,\dots,X_d = i_d), \quad j \in \bar{\mathcal{J}}^d_n. \end{aligned}$$

Note that given an $i \in \bar{\mathcal{J}}^d_{n-k}$, every $j \in \bar{\mathcal{J}}^d_n$ with $i \le j$ determines a unique $l := j - i \in \bar{\mathcal{J}}^d_k$ and vice versa. Furthermore, $\bar{\mathcal{J}}^d_k$ contains $\binom{k+d-1}{d-1}$ elements. Hence, using these results,

$$\begin{aligned} c_n &= \sum_{j\in\bar{\mathcal{J}}^d_n} \sum_{k=0}^{n} \sum_{\substack{i\in\bar{\mathcal{J}}^d_{n-k} \\ i\le j}} \mathbb{P}(X_1 = i_1,\dots,X_d = i_d) \\ &= \sum_{k=0}^{n} \sum_{i\in\bar{\mathcal{J}}^d_{n-k}} \sum_{\substack{j\in\bar{\mathcal{J}}^d_n \\ i\le j}} \mathbb{P}(X_1 = i_1,\dots,X_d = i_d) \\ &= \sum_{k=0}^{n} \binom{k+d-1}{d-1} p_{n-k}. \end{aligned}$$

Rearranging the above equality proves the claim. ❏

4.5 Numerical Speed-Up for the Calculation of the Probability Mass Function of S

Although the recursion formula for the calculation of the probability mass function of S as given in (4.11) is straightforward to implement, there could be numerical difficulties regarding stability and efficiency – especially for sparse univariate marginal distributions or large dimensions d combined with large n. The aim of this section is to provide the reader with two additional computation methods for the probability mass function of S, which in many cases turn out to outperform the recursion from Theorem 4.8.[85] The first one can be derived from Proposition 4.3:[86]

Proposition 4.9. *Let $X_1,\dots,X_d$ denote $\mathbb{N}_0$-valued random variables with univariate distribution functions $F_1,\dots,F_d$, respectively. Then for all $n \in \mathbb{N}_0$ and every copula C of the random vector $(X_1,\dots,X_d)$ it holds that*

$$\mathbb{P}[S = n] = \sum_{k=0}^{\min\{d,n\}} (-1)^k \binom{d}{k} c_{n-k}. \tag{4.12}$$

[85]It should be taken into account that this statement is made on the basis of an implementation in R.

[86]Special thanks to my supervisor Dr. Uwe Schmock for pointing out this result to me.

Proof. For $n = 0$, (4.12) reduces to $\mathbb{P}[S = 0] = c_0$, which is a true statement. For arbitrary $n \in \mathbb{N}$ we have[87]

$$\begin{aligned}
\mathbb{P}[S = n] &= \mathbb{P}[S \leq n] - \mathbb{P}[S \leq n-1] \\
&= \sum_{k=0}^{\min\{d-1,n\}} (-1)^k \binom{d-1}{k} c_{n-k} - \sum_{k=0}^{\min\{d,n\}-1} (-1)^k \binom{d-1}{k} c_{n-1-k} \\
&= \sum_{k=0}^{\min\{d-1,n\}} (-1)^k \binom{d-1}{k} c_{n-k} + \sum_{k=1}^{\min\{d,n\}} (-1)^k \binom{d-1}{k-1} c_{n-k} \\
&= c_n + \sum_{k=1}^{\min\{d-1,n\}} (-1)^k \left[\binom{d-1}{k} + \binom{d-1}{k-1}\right] c_{n-k} + (-1)^d c_{n-d} \mathbb{1}_{n \geq d} \\
&= c_n + \sum_{k=1}^{\min\{d-1,n\}} (-1)^k \binom{d}{k} c_{n-k} + (-1)^d c_{n-d} \mathbb{1}_{n \geq d} \\
&= \sum_{k=0}^{\min\{d,n\}} (-1)^k \binom{d}{k} c_{n-k},
\end{aligned}$$

which completes the proof. ❑

In the course of this master thesis, formula (4.12) was implemented in R and can be found in the appendix (Program R2).

For the second method we need the concept of copula densities:[88]

Definition 4.10. If the probability measure associated with a copula C is absolutely continuous with respect to the Lebesgue measure on $[0,1]^d$, then by Radon–Nikodým there exists an almost everywhere unique function $c : [0,1]^d \to [0,\infty)$ such that

$$C(u_1, \ldots, u_d) = \int_0^{u_1} \ldots \int_0^{u_d} c(v_1, \ldots, v_d)\, dv_d \ldots dv_1,$$

where $u_1, \ldots, u_d \in [0,1]$. In this case C is called absolutely continuous with *copula density* c.

Given that we have a copula that meets the requirements of Definition 4.10, i.e. we have a copula C with density c, we can conclude that

$$\begin{aligned}
\mathbb{P}[S = n] &= \sum_{j \in \bar{\mathcal{J}}_n^d} \mathbb{P}[X_1 = j_1, \ldots, X_d = j_d] \\
&= \sum_{j \in \bar{\mathcal{J}}_n^d} \sum_{i \in \mathcal{I}^d} \operatorname{sign}(i) C(F_1(j_1 - i_1), \ldots, F_d(j_d - i_d))
\end{aligned}$$

[87] Here, $\mathbb{1}$ denotes the indicator function.
[88] cf. [24, p. 12]

$$= \sum_{j \in \bar{\mathcal{J}}_n^d} \int_{F_1(j_1-1)}^{F_1(j_1)} \cdots \int_{F_d(j_d-1)}^{F_d(j_d)} c(v_1, \ldots, v_d)\, dv_d \ldots dv_1. \tag{4.13}$$

For an implementation of (4.13) in R we refer to [2] and Program R4. For copula densities of Archimedean copulas interested readers may turn to [26] for further information.

To conclude this section, we present a brief runtime comparison for random variables $X_i \sim \mathrm{Poi}(5)$, $i = 1, \ldots, 3$, between (4.11), (4.12) and (4.13) for a Gaussian copula with correlation matrix

$$P = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}$$

for various $n \in \mathbb{N}_0$ in R. In order to have a reasonable comparison, we challenge the recursive formula for $P[S = n]$ with a calculation of $\mathbb{P}[S = 0], \ldots, \mathbb{P}[S = n]$ using (4.12) and (4.13), respectively.[89] The results are presented in Table 4.1 below:

[89]The reason for this is that for $n \in \mathbb{N}$ an evaluation of $P[S = n]$ using (4.11) also returns the probabilities $\mathbb{P}[S = k]$ for $k = 0, \ldots, n-1$, which is due to the recursive representation.

n	evaluation of (4.12) in seconds (cummulative)	evaluation of (4.13) in seconds (cummulative)	evaluation of (4.11) in seconds
0	0.01	0.01	0.01
1	0.02	0.05	0.01
2	0.03	0.09	0.02
3	0.04	0.13	0.03
4	0.07	0.19	0.05
5	0.10	0.28	0.07
6	0.16	0.41	0.15
7	0.24	0.56	0.30
8	0.33	0.76	0.63
9	0.43	1.01	1.27
10	0.56	1.30	1.39
11	0.71	1.63	2.43
12	0.89	2.02	3.07
13	1.09	2.47	4.95
14	1.33	2.98	6.57
15	1.58	3.56	10.07
16	1.88	4.21	13.08
17	2.21	5.06	21.14
18	2.59	5.92	24.30
19	3.01	6.88	41.26
20	3.48	7.94	73.10

Table 4.1: Runtime comparison between (4.11), (4.12) and (4.13) for a Gaussian copula in seconds using R

Chapter 5

Risk Measures for an Aggregated Portfolio

The fifth chapter focuses on common risk measures of the financial industry – Value-at-Risk and Expected Shortfall. These are defined first and a connection to the calculation of these for an aggregated portfolio S is given. Finally, we deal with a slight adapted Rearrangement Algorithm to calculate sharp bounds on Value-at-Risk and Expected Shortfall. The results are then presented in the next Chapter 6.

5.1 Value-at-Risk

Despite its known shortcomings, the Value-at-Risk is one of the most widely used risk measures in the financial industry and is also applied in the Basel II and Solvency II framework.[90] The idea is to calculate the maximum possible portfolio loss which, given a certain confidence level, is not exceeded:[91]

Definition 5.1. At a given confidence level $\alpha \in (0, 1)$, the *Value-at-Risk* (VaR) of a random variable S is the smallest value $s \in \mathbb{R}$ where the distribution function F_S of S reaches or exceeds the value α for the first time:

$$\mathrm{VaR}_\alpha(S) \coloneqq \inf\{s \in \mathbb{R} : F_S(s) \geq \alpha\}.$$

By convention, we can set $\inf \varnothing = +\infty$ and $\inf \mathbb{R} = -\infty$, so that the VaR is even well-defined for $\alpha \in [0, 1]$.

Remark 5.2. In other words, the VaR can simply be described as the quantile of a distribution function, cf. Definition 2.4.

[90] cf. [25, p. 37]
[91] cf. [25, Definition 2.10]

Remark 5.3. Typically, in risk management one works with losses, i.e. the right tails of the distribution. Common levels for α thus are for example $\alpha = 0.95$ or $\alpha = 0.995$, as required under Solvency II.

The key shortcoming of the VaR is that it is not a subadditive risk measure.[92] This means that for a portfolio of d risks $X_1, \dots, X_d$,

$$\mathrm{VaR}_\alpha(X_1 + \cdots + X_d) > \mathrm{VaR}_\alpha(X_1) + \cdots + \mathrm{VaR}_\alpha(X_d)$$

is possible, which contradicts the economic idea of portfolio diversification.

5.2 Bounds on the Value-at-Risk[93]

We will dedicate this section to bounds on VaR. For this purpose we consider a portfolio of d $\mathbb{N}_0$-valued risks $X = (X_1^C, \dots, X_d^C)$ which are coupled by an arbitrary copula C. Furthermore, we assume that we know the marginal distributions $F_1, \dots, F_d$ of $X_1^C, \dots, X_d^C$. By

$$S := \sum_{i=1}^{d} X_i^C$$

we denote the sum of the individual risks.

The lower and upper bounds of the VaR can be defined as follows:

Definition 5.4. Denote by $\mathcal{C}_d$ the set of all d-dimensional copulas. Then we have that the lower, respectively upper, bound on the VaR at a given confidence level $\alpha \in (0, 1)$ for S is given by:

$$\mathrm{VaR}_\alpha^L(S) = \inf\{\mathrm{VaR}_\alpha(X_1^C + \cdots + X_d^C) : C \in \mathcal{C}_d\}, \tag{5.1}$$

$$\mathrm{VaR}_\alpha^U(S) = \sup\{\mathrm{VaR}_\alpha(X_1^C + \cdots + X_d^C) : C \in \mathcal{C}_d\}. \tag{5.2}$$

In practice, it is difficult to explicitly determine the barriers defined above, which is mainly due to the non-subadditivity of the VaR. In general, the comonotonicity copula M is not a solution for (5.2). Also, for $d = 2$ the countermonotonicity copula W does not solve (5.1). For interested readers, counterexamples related to these deceiving problems are provided in [15, Section 1.3].

In order to calculate the VaR-bounds, the rearrangement algorithm (RA) originally introduced in [29] was slightly adapted by Embrechts, Puccetti and Rüschendorf in [15]. This algorithm, which we will describe below, calculates VaR_α^L and VaR_α^U for arbitrary one-dimensional distribution functions $F_1, \dots, F_d$ and $\alpha \in (0, 1)$. The pseudo-code for the calculation of (5.1) is as follows:

[92]cf. [15, p. 2]

[93]This section is mainly based on the results in [15], thus we will avoid explicit references to this paper. Only the results taken from other sources will be marked as such.

Algorithm 5.5 (Computation of VaR_α^L).

Step 1: Fix $N \in \mathbb{N}$, a tolerance level $\epsilon > 0$, a confidence level $\alpha \in (0,1)$ and d univariate distributions $F_1, \dots, F_d$ of random variables $X_1, \dots, X_d$.

Step 2: For $i = 1, \dots, N$ and $j = 1, \dots, d$ define matrices $\underline{X} = (\underline{x}_{i,j})$ and $\overline{X} = (\overline{x}_{i,j})$ as:

$$\underline{x}_{i,j} = F_j^{\leftarrow}\left(\frac{\alpha(i-1)}{N}\right), \quad \overline{x}_{i,j} = F_j^{\leftarrow}\left(\frac{\alpha i}{N}\right).$$

Step 3: Randomly permutate the columns of $\underline{X}$ and $\overline{X}$.

Step 4: For $j = 1, \dots, d$, rearrange the j-th column of $\underline{X}$ to make it oppositely ordered to the sum of the other columns. This results in a matrix $\underline{Y}$.

Step 5: Repeat step 4. until

$$|\mathrm{maxRS}(\underline{X}) - \mathrm{maxRS}(\underline{Y})| = 0.$$

Set $\underline{X}^* = \underline{Y}$.

Step 6: Apply steps 4 and 5 to the matrix $\overline{X}$ until a matrix $\overline{X}^*$ is found.

Step 7: It holds that

$$\mathrm{maxRS}(\underline{X}^*) \leq \mathrm{VaR}_\alpha^L(X_1 + \cdots + X_d) \leq \mathrm{maxRS}(\overline{X}^*).$$

For demonstration purposes the algorithms for the computation of VaR_α^L and VaR_α^U are implemented in R (cf. Program R6). Numerical results for concrete examples are presented in Chapter 6.

5.3 Expected Shortfall

Closely related to the Value-at-Risk is the Expected Shortfall , which is defined as follows:[94]

Definition 5.6. For a random variable S with $\mathbb{E}[|S|] < \infty$ the Expected Shortfall (ES) at a confidence level $\alpha \in (0,1)$ is given as

$$\mathrm{ES}_\alpha(S) := \frac{1}{1-\alpha} \int_\alpha^1 \mathrm{VaR}_u(S)\, du.$$

The definition above means that the Expected Shortfall at a confidence level $\alpha \in (0,1)$ corresponds to an average over all values of Value-at-Risk

[94] cf. [25, Definition 2.15]

with confidence levels greater than α. Thus, for a random variable S fulfilling the requirements of the prior definition it immediately follows that $\mathrm{ES}_\alpha(S) \geq \mathrm{VaR}_\alpha(S)$ holds for all $\alpha \in (0,1)$.

The main advantage of ES over VaR is that ES is a subadditive risk measure, i.e. for random variables $X_1, \ldots, X_d$ it holds that

$$\mathrm{ES}_\alpha(X_1 + \cdots + X_d) \leq \mathrm{ES}_\alpha(X_1) + \cdots + \mathrm{ES}(X_d). \tag{5.3}$$

5.4 Bounds on the Expected Shortfall[95]

In this section we provide an algorithm for the computation of sharp bounds on the Expected Shortfall for a sum of random variables. We refer keen readers to [28] for details and a proof of the functionality.

We are interested in determining

$$\mathrm{ES}^L_\alpha(S) = \inf\{\mathrm{ES}_\alpha(X_1 + \cdots + X_d) : X_i \sim F_i,\ i = 1, \ldots, d\}, \tag{5.4}$$

$$\mathrm{ES}^U_\alpha(S) = \sup\{\mathrm{ES}_\alpha(X_1 + \cdots + X_d) : X_i \sim F_i,\ i = 1, \ldots, d\}. \tag{5.5}$$

From the subadditivity of the Expected Shortfall (5.3) the upper bound follows instantly: For random variables $X_1, \ldots, X_d$ we have that the upper bound (5.5) is given by

$$\mathrm{ES}^U_\alpha(X_1 + \cdots + X_d) = \frac{1}{1-\alpha} \sum_{i=1}^{d} \int_\alpha^1 \mathrm{VaR}_u(X_i)\, du.$$

For the rest of this section we will thus concentrate on the calculation of a lower bound. The algorithm for this problem is based on the Rearrangement Algorithm and works as follows:

Algorithm 5.7 (Computation of ES^L_α).

Step 1: Fix $N \in \mathbb{N}$ and a confidence level $\alpha \in (0,1)$ such that $\alpha N \in \mathbb{N}$, a tolerance level $\epsilon > 0$ and d univariate distributions $F_1, \ldots, F_d$ of random variables $X_1, \ldots, X_d$.

Step 2: For $i = 1, \ldots, N$ and $j = 1, \ldots, d$ define matrices $\underline{X} = (\underline{x}_{i,j})$ and $\overline{X} = (\overline{x}_{i,j})$ as:

$$\underline{x}_{i,j} = F_j^{\leftarrow}\left(\frac{i-1}{N}\right), \quad \overline{x}_{i,j} = F_j^{\leftarrow}\left(\frac{i}{N}\right).$$

[95] This section is based on the results in [28], thus we will avoid explicit references to this paper.

Step 3: Randomly permutate the columns of $\underline{X}$ and $\overline{X}$.

Step 4: For $j = 1, \ldots, d$, rearrange the j-th column of $\underline{X}$ to make it oppositely ordered to the sum of the other columns. This results in a matrix $\underline{Y} = (\underline{y})_{i,j}$.

Step 5: Repeat Step 4. until

$$\frac{1}{(1-\alpha)N}\left(\sum_{i=\alpha N+1}^{N}\sum_{j=1}^{d}\left|\underline{x}_{i,j} - \underline{y}_{i,j}\right|\right) < \epsilon.$$

Set $\underline{X}^* = (\underline{x}^*_{i,j}) = \underline{Y}$.

Step 6: Apply steps 4 and 5 to the matrix $\overline{X}$ until a matrix $\overline{X}^* = (\overline{x}^*_{i,j})$ is found.

Step 7: It holds that $\mathrm{ES}^L_\alpha(X_1 + \cdots + X_d)$ is between

$$\frac{1}{(1-\alpha)N}\sum_{i=\alpha N+1}^{N}\sum_{j=1}^{d}\underline{x}^*_{i,j} \quad \text{and} \quad \frac{1}{(1-\alpha)N}\sum_{i=\alpha N+1}^{N}\sum_{j=1}^{d}\overline{x}^*_{i,j}.$$

For demonstration purposes the algorithms for the computation of ES^L_α and ES^U_α are implemented in R (cf. Program R6). Numerical results for concrete examples are given in the next Chapter 6.

Chapter 6

Examples

In this section we will specify discrete univariate margins and then calculate the distribution, probability mass function and risk measures for S. The focus will be on the Poisson and negative binomial distribution, although the Bernoulli, the binomial and an arbitrary distribution on $\mathbb{N}_0$ will also be covered.

6.1 Distribution and Probability Mass Functions of S

The lower and upper bounds on the distribution of S are calculated according to the results in Section 4.3 using $N = 10^5$ and tolerance $\epsilon = 10^{-8}$. Note, that by a Gaussian copula with parameter ρ, i.e. C_ρ^{Ga}, we denote a Gaussian copula with correlation matrix such that all pairwise correlations coincide to $\rho \in [-1, 1]$. The same holds for the t-copula with $\nu > 0$ degrees of freedom, $C_{\nu,\rho}^{\mathrm{t}}$. If we consider correlation matrices P of another form, this will be denoted by C_P^{Ga} and $C_{\nu,P}^{\mathrm{t}}$, respectively.

6.1.1 Bernoulli-Distributed Margins

Definition 6.1. A random variable X on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ follows a Bernoulli distribution with parameter $p \in [0, 1]$, $X \sim \mathrm{Ber}(p)$, if[96]

$$\mathbb{P}[X = 0] = 1 - p, \quad \mathbb{P}[X = 1] = p.$$

[96] cf. [22, Example 6.29]

Example 6.2. Let $d = 2$ and consider Bernoulli-distributed random variables $X_1 \sim \text{Ber}(0.2)$ and $X_2 \sim \text{Ber}(0.7)$. Considering the three fundamental copulas and a convex combination of these we obtain the following results:

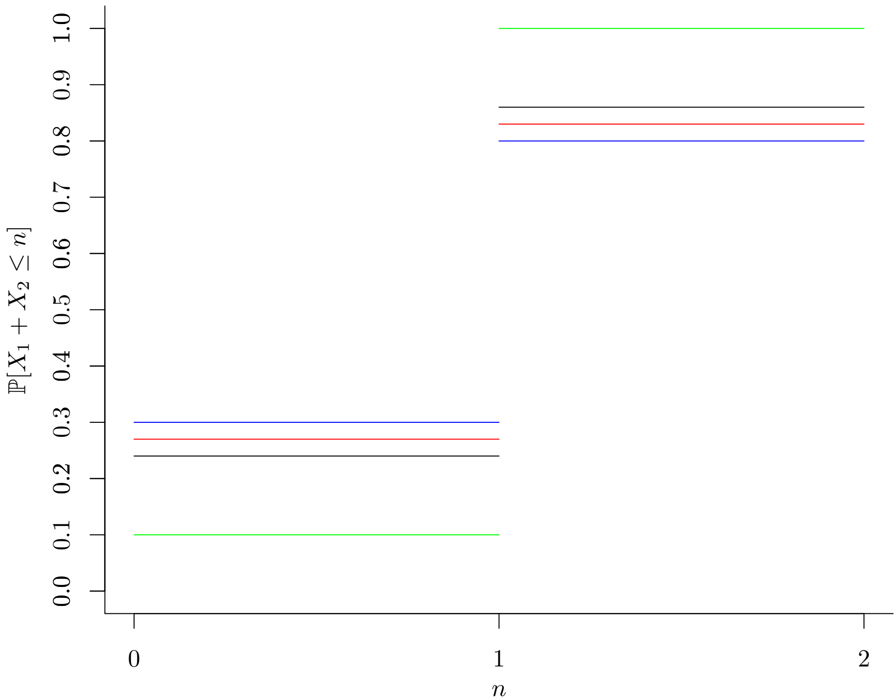

Figure 6.1: Distribution function of S for $X_1 \sim \text{Ber}(0.2)$ and $X_2 \sim \text{Ber}(0.7)$ coupled by fundamental copulas. Green: W, Black: Π, Red: $0.5M + 0.5\Pi$, Blue: M

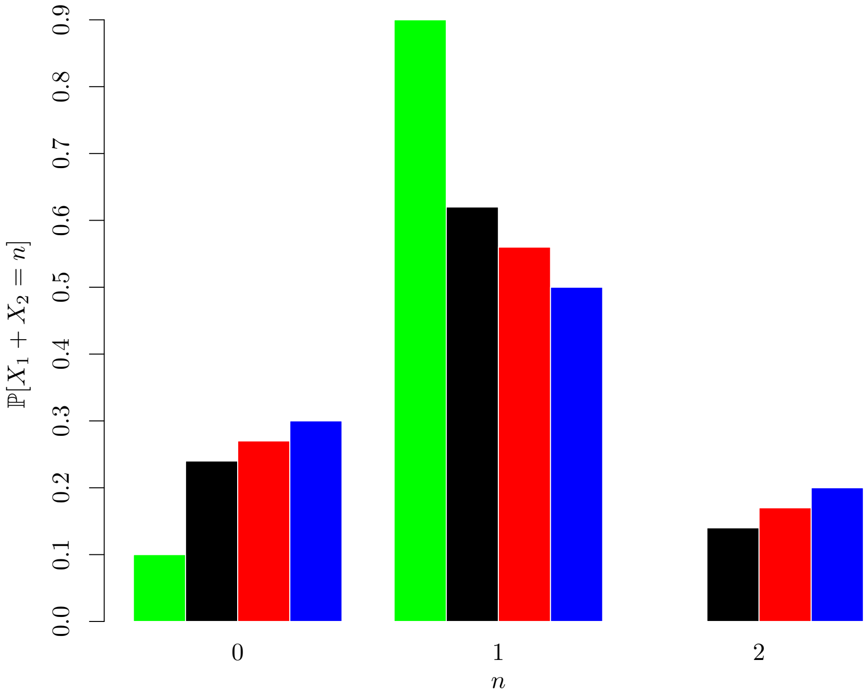

Figure 6.2: Probability mass function of S for $X_1 \sim \text{Ber}(0.2)$ and $X_2 \sim \text{Ber}(0.7)$ coupled by fundamental copulas. Green: W, Black: Π, Red: $0.5M + 0.5\Pi$, Blue: M

Example 6.3. Let $d = 2$ and consider Bernoulli-distributed random variables $X_1 \sim \mathrm{Ber}(0.2)$ and $X_2 \sim \mathrm{Ber}(0.7)$. We assume that the dependence structure is given by a bivariate Gaussian copula C_ρ^{Ga} and obtain the subsequent results:

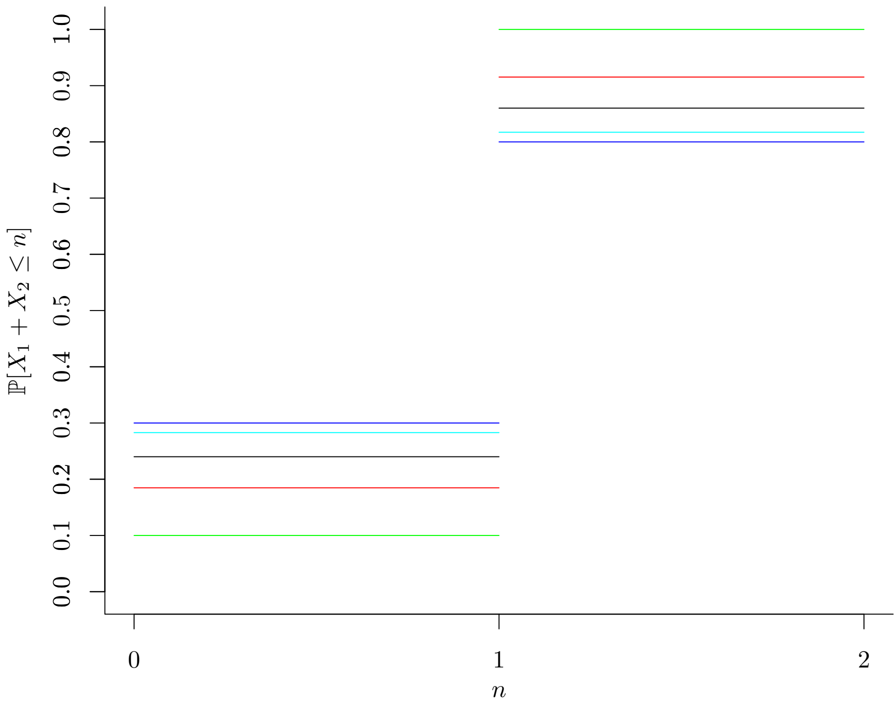

Figure 6.3: Distribution function of S for $X_1 \sim \mathrm{Ber}(0.2)$ and $X_2 \sim \mathrm{Ber}(0.7)$ coupled by a Gaussian copula. Green: C_{-1}^{Ga}, Red: $C_{-0.5}^{\mathrm{Ga}}$, Black: C_0^{Ga}, Cyan: $C_{0.5}^{\mathrm{Ga}}$, Blue: C_1^{Ga}

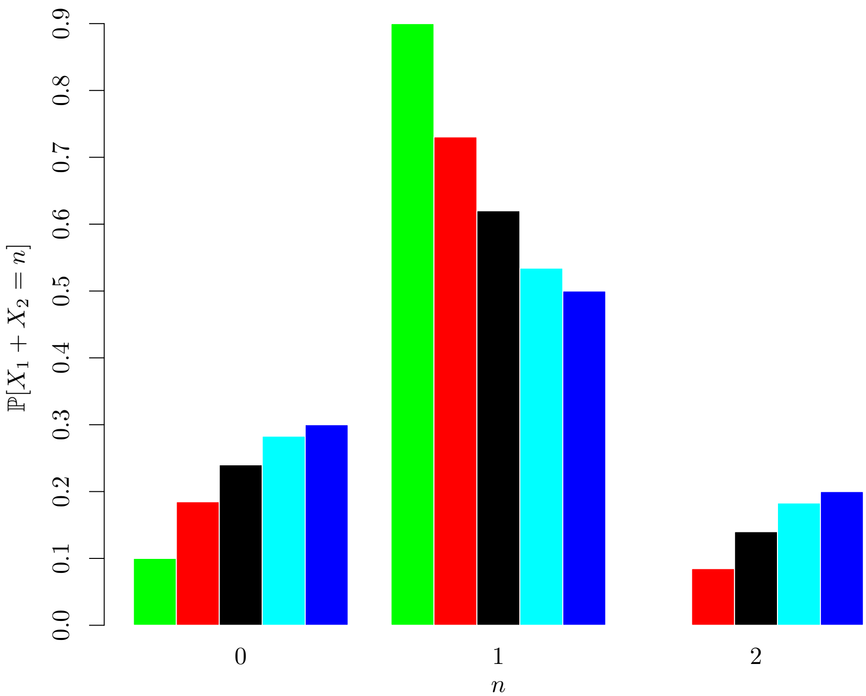

Figure 6.4: Probability mass function of S for $X_1 \sim \text{Ber}(0.2)$ and $X_2 \sim \text{Ber}(0.7)$ coupled by a Gaussian copula. Green: C^{Ga}_{-1}, Red: $C^{\text{Ga}}_{-0.5}$, Black: C^{Ga}_{0}, Cyan: $C^{\text{Ga}}_{0.5}$, Blue: C^{Ga}_{1}

At the end of this subsection we will deal with the Clayton copula C_α^{Cl}:

Example 6.4. Let $d = 2$ and consider Bernoulli-distributed random variables $X_1 \sim \mathrm{Ber}(0.2)$ and $X_2 \sim \mathrm{Ber}(0.7)$. Assume that the random variables are coupled by a Clayton copula C_α^{Cl}. We will chose the parameter α in such a way, that Kendall's tau ρ_τ takes the following values (cf. Example 3.11):

α	$\rho_\tau(X_1, X_2)$
-1	-1
$-\frac{2}{3}$	$-\frac{1}{2}$
2	$\frac{1}{2}$
20	$\frac{10}{11}$

As can be seen from the graphics below, the distribution of (X_1, X_2) under a Clayton copula with parameter $\alpha = -1$ corresponds to countermonotonicity. For $\alpha \to +\infty$ we observe comonotonicity.

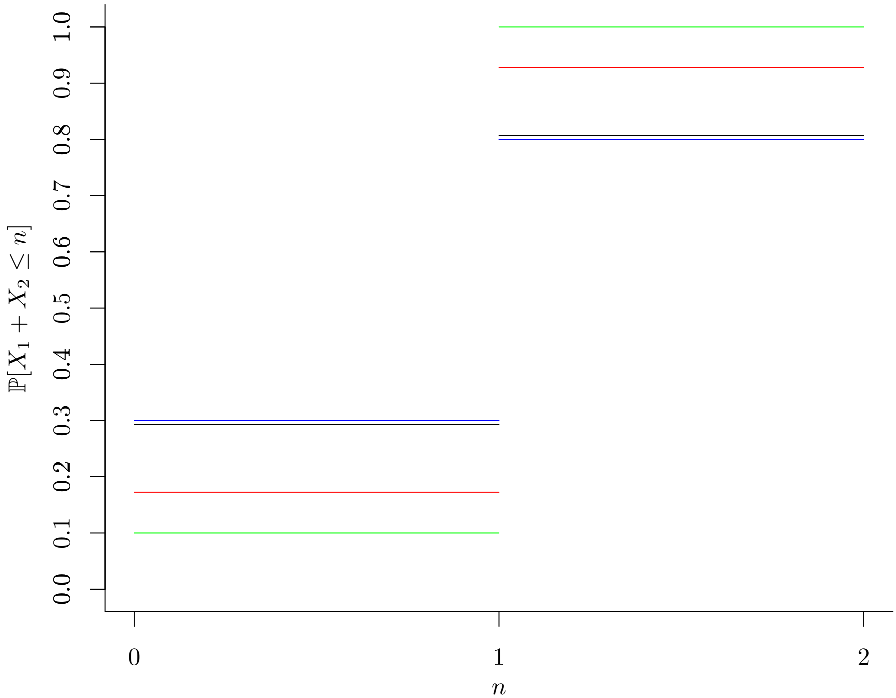

Figure 6.5: Distribution function of S for $X_1 \sim \mathrm{Ber}(0.2)$ and $X_2 \sim \mathrm{Ber}(0.7)$ coupled by a Clayton copula. Green: C_{-1}^{Cl}, Red: $C_{-\frac{2}{3}}^{\mathrm{Cl}}$, Black: C_2^{Cl}, Blue: C_{20}^{Cl}

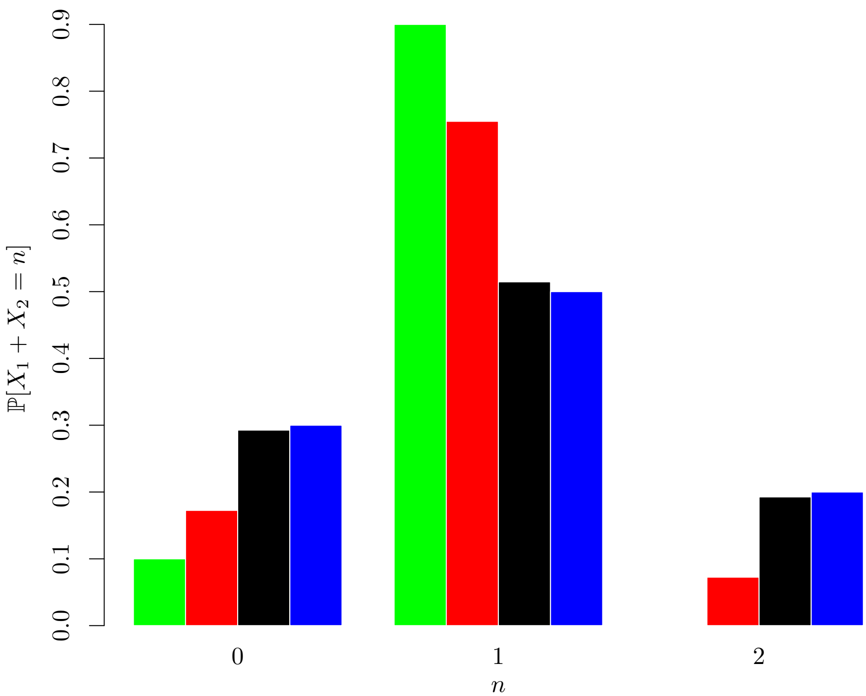

Figure 6.6: Probability mass function of S for $X_1 \sim \text{Ber}(0.2)$ and $X_2 \sim \text{Ber}(0.7)$ coupled by a Clayton copula. Green: C^{Cl}_{-1}, Red: $C^{\text{Cl}}_{-\frac{2}{3}}$, Black: C^{Cl}_{2}, Blue: C^{Cl}_{20}

6.1.2 Binomial-Distributed Margins

Definition 6.5. A random variable X on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to follow a binomial distribution with parameters $n \in \mathbb{N}$ and $p \in [0,1]$, $X \sim \text{Bin}(n,p)$, if[97]

$$\mathbb{P}[X = k] = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, \ldots, n.$$

For $n = 1$ we get a Bernoulli distribution with parameter p as a special case.

Example 6.6. Let $d = 4$ and consider binomial-distributed random variables $X_i \sim \text{Bin}(n_i, p_i)$, where $n_i = 10$ and $p_i = \frac{i}{20}$, $i = 1, \ldots, 4$. If we assume that $(X_1, \ldots, X_4)$ is coupled by fundamental copulas we obtain the following results:

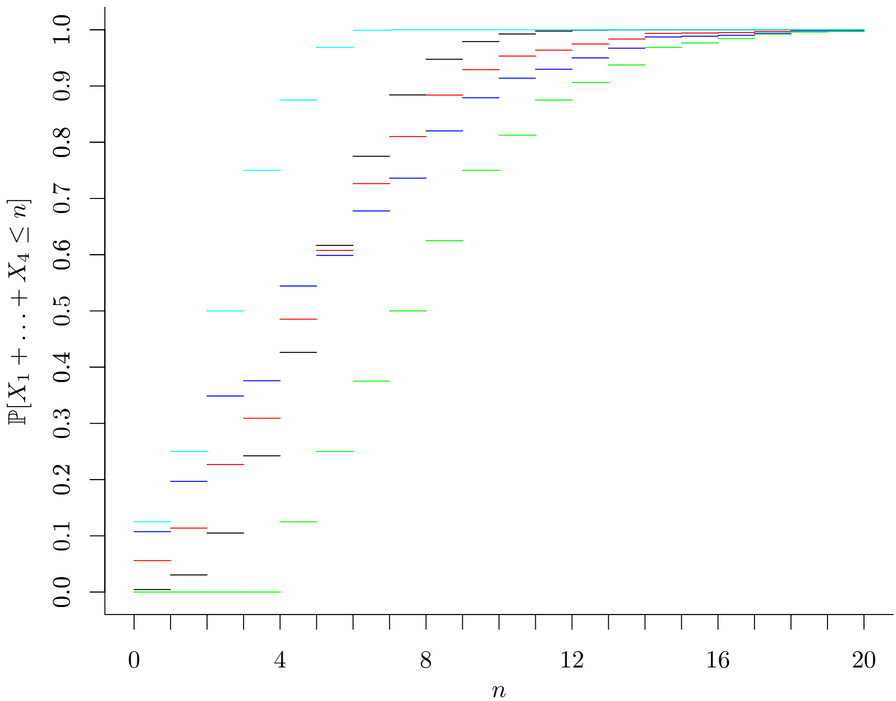

Figure 6.7: Distribution function of S for $X_i \sim \text{Bin}(n_i, p_i)$, where $n_i = 10$ and $p_i = \frac{i}{20}$, $i = 1, \ldots, 4$, coupled by fundamental copulas. Green: lower bound, Black: Π, Red: $0.5M + 0.5\Pi$, Blue: M, Cyan: upper bound

[97]cf. [22, Example 6.31]

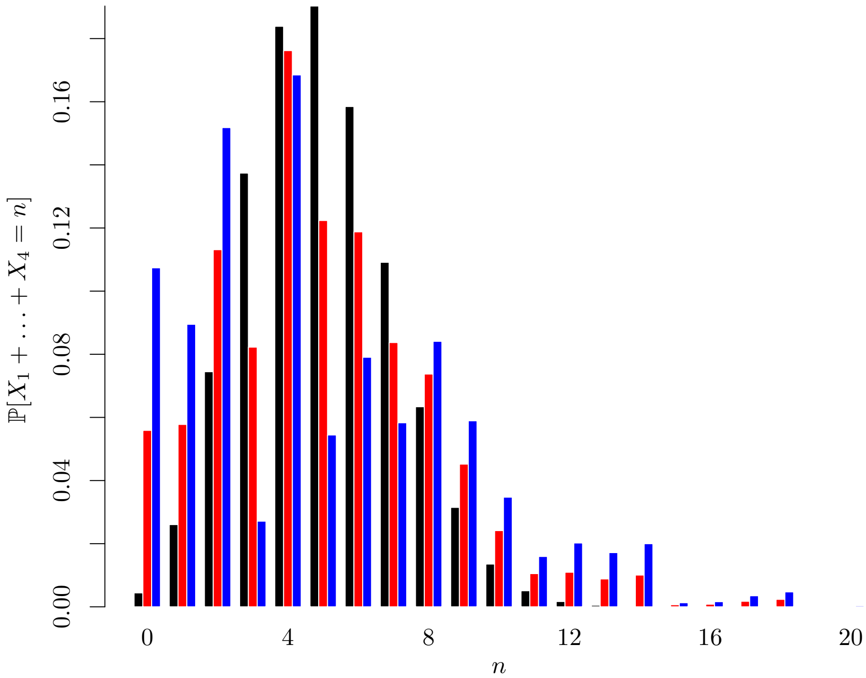

Figure 6.8: Probability mass function of S for $X_i \sim \text{Bin}(n_i, p_i)$, where $n_i = 10$ and $p_i = \frac{i}{20}$, $i = 1, \ldots, 4$, coupled by fundamental copulas. Black: Π, Red: $0.5M + 0.5\Pi$, Blue: M

Example 6.7. Let $d = 4$ and consider binomial-distributed random variables $X_i \sim \mathrm{Bin}(n_i, p_i)$, where $n_i = 10$ and $p_i = \frac{i}{20}$, $i = 1, \ldots, 4$. If we assume that $(X_1, \ldots, X_4)$ is coupled by a Gaussian copula we obtain the subsequent results:

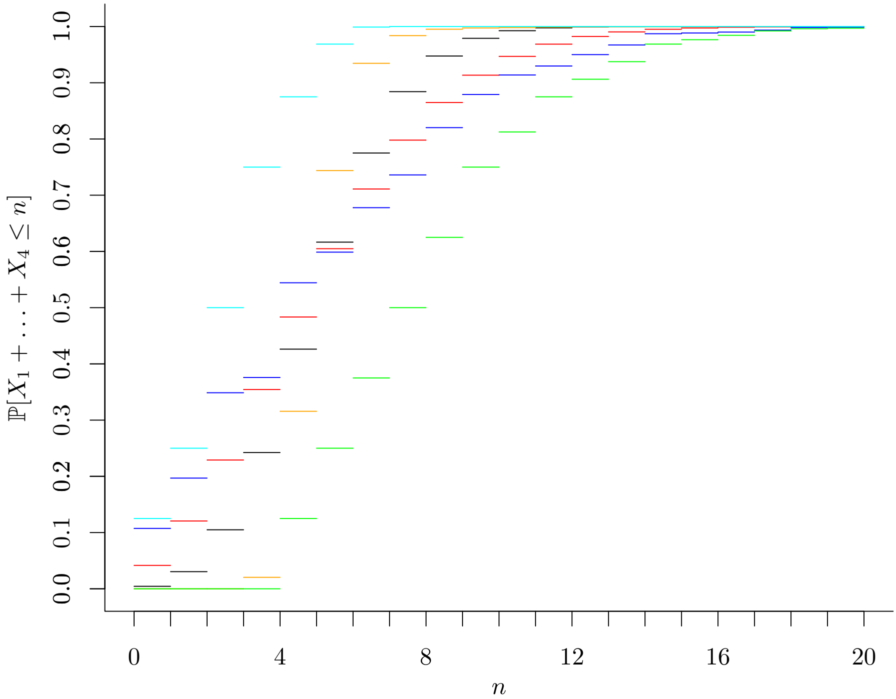

Figure 6.9: Distribution function of S for $X_i \sim \mathrm{Bin}(n_i, p_i)$, where $n_i = 10$ and $p_i = \frac{i}{20}$, $i = 1, \ldots, 4$, coupled by a Gaussian copula. Green: lower bound, Orange: $C^{\mathrm{Ga}}_{-\frac{1}{3}}$, Black: C^{Ga}_{0}, Red: $C^{\mathrm{Ga}}_{0.5}$, Blue: C^{Ga}_{1}, Cyan: upper bound

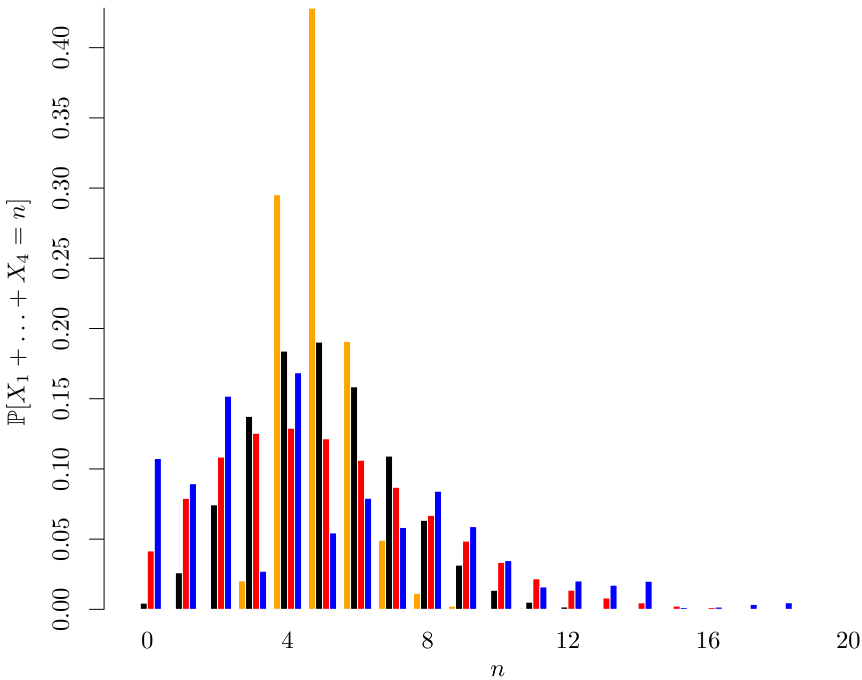

Figure 6.10: Probability mass function of S for $X_i \sim \text{Bin}(n_i, p_i)$, where $n_i = 10$ and $p_i = \frac{i}{20}$, $i = 1, \ldots, 4$, coupled by a Gaussian copula. Orange: $C^{\text{Ga}}_{-\frac{1}{3}}$, Black: C^{Ga}_{0}, Red: $C^{\text{Ga}}_{0.5}$, Blue: C^{Ga}_{1}

6.1.3 Poisson-Distributed Margins

Definition 6.8. A random variable X on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ follows a Poisson distribution with parameter $\lambda > 0$, $X \sim \mathrm{Poi}(\lambda)$, if[98]

$$\mathbb{P}(X = n) = \mathrm{e}^{-\lambda} \frac{\lambda^n}{n!}, \quad n \in \mathbb{N}_0.$$

For independent random variables $X_i \sim \mathrm{Poi}(\lambda_i)$, $i = 1, \ldots, d$, it follows immediately using generating functions that

$$X_1 + \cdots + X_d \sim \mathrm{Poi}(\lambda_1 + \cdots + \lambda_d), \quad \lambda_1, \ldots, \lambda_d > 0,$$

holds, which we also observe when considering the random variables coupled by the independence copula.

Example 6.9. Let $d = 3$ and consider Poisson-distributed random variables $X_1 \sim \mathrm{Poi}(3)$, $X_2 \sim \mathrm{Poi}(5)$ and $X_3 \sim \mathrm{Poi}(8)$. We obtain the following results for fundamental copulas:

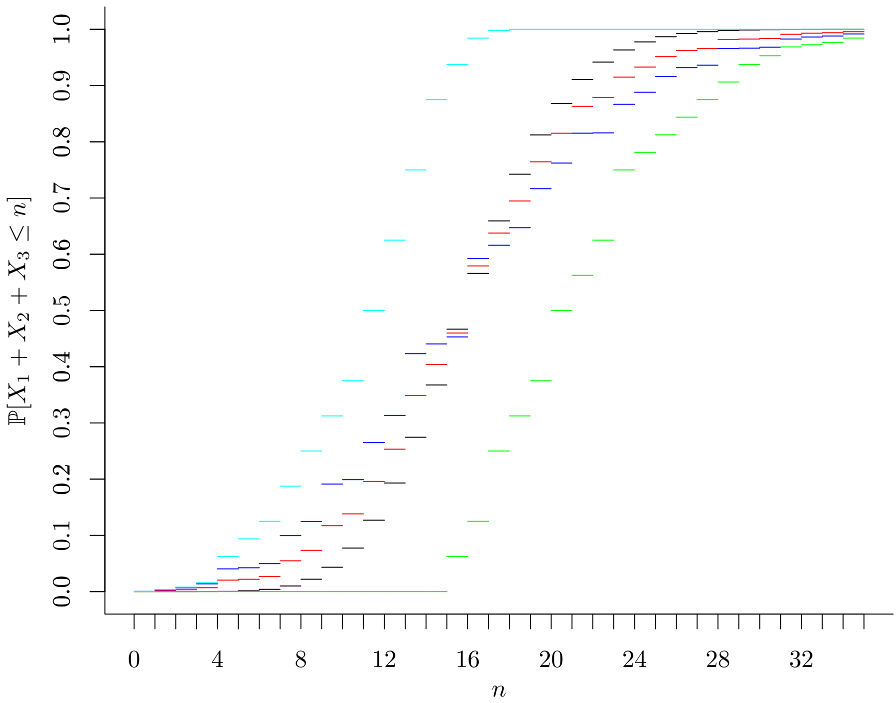

Figure 6.11: Distribution function of S for $X_1 \sim \mathrm{Poi}(3)$, $X_2 \sim \mathrm{Poi}(5)$ and $X_3 \sim \mathrm{Poi}(8)$ coupled by fundamental copulas. Green: lower bound, Black: Π, Red: $0.5M + 0.5\Pi$, Blue: M, Cyan: upper bound

[98]cf. [22, Example 6.32]

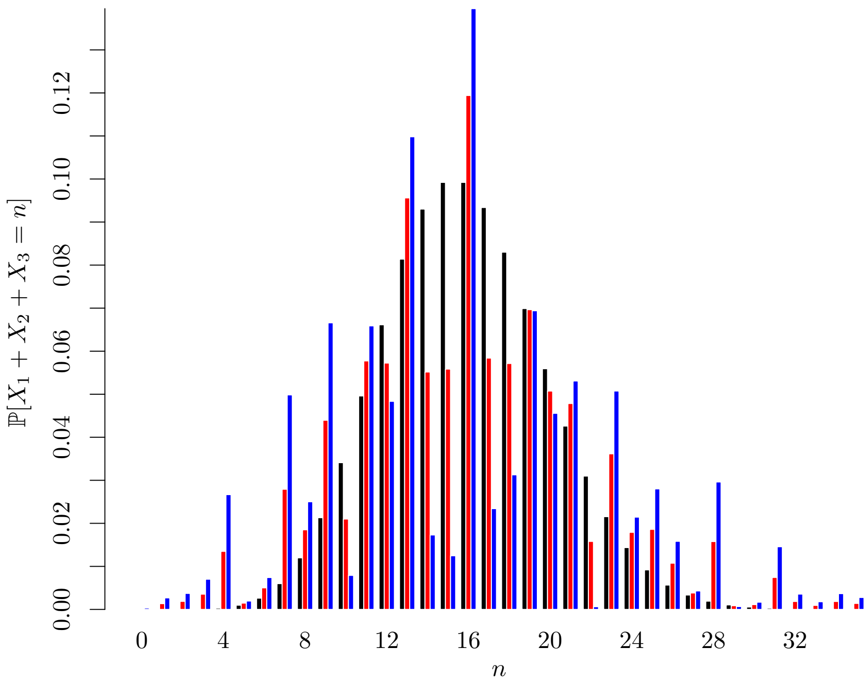

Figure 6.12: Probability mass function of S for $X_1 \sim \text{Poi}(3)$, $X_2 \sim \text{Poi}(5)$ and $X_3 \sim \text{Poi}(8)$ coupled by fundamental copulas. Black: Π, Red: $0.5M + 0.5\Pi$, Blue: M

Example 6.10. Let $d = 3$ and consider Poisson-distributed random variables $X_1 \sim \mathrm{Poi}(3)$, $X_2 \sim \mathrm{Poi}(5)$ and $X_3 \sim \mathrm{Poi}(8)$.

For independent random variables X_1 and X_2 it holds that $X_1 + X_2 \overset{d}{=} X_3$. So if we try to minimize the variance of the sum $X_1 + X_2 + X_3$ we can use

$$V = \begin{pmatrix} 1 & 0 & -\sqrt{\frac{3}{8}} \\ 0 & 1 & -\sqrt{\frac{5}{8}} \\ -\sqrt{\frac{3}{8}} & -\sqrt{\frac{5}{8}} & 1 \end{pmatrix},$$

where the entries of the correlation matrix V are obtained by simple calculation under positive semi-definite constraints.

We obtain the following results for a Gaussian copula:

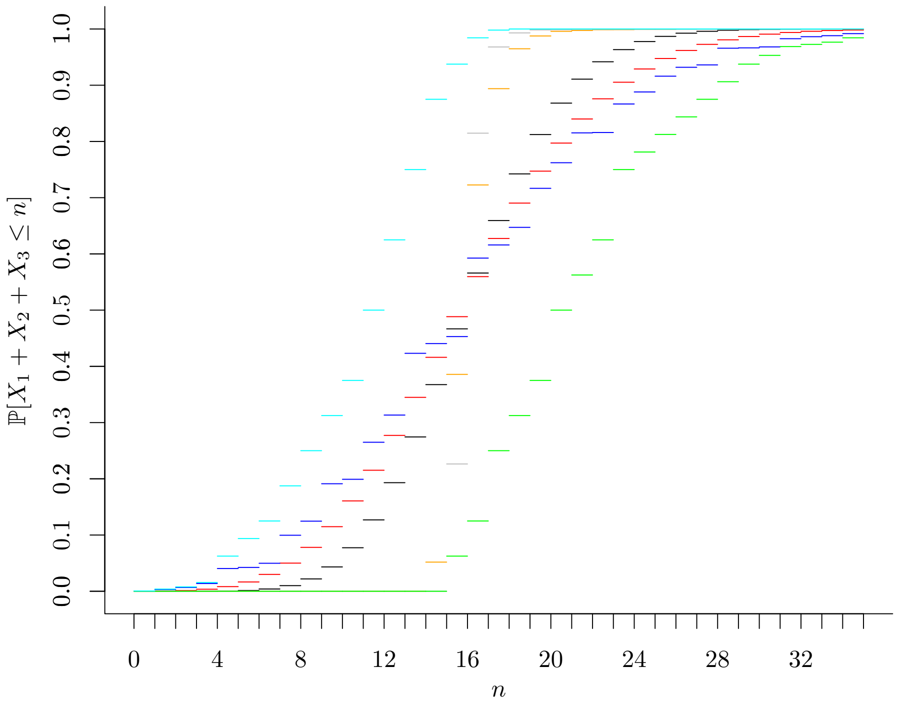

Figure 6.13: Distribution function of S for $X_1 \sim \mathrm{Poi}(3)$, $X_2 \sim \mathrm{Poi}(5)$ and $X_3 \sim \mathrm{Poi}(8)$ coupled by a Gaussian copula. Green: lower bound, Orange: $C^{\mathrm{Ga}}_{-0.5}$, Black: C^{Ga}_{0}, Red: $C^{\mathrm{Ga}}_{0.5}$, Blue: C^{Ga}_{1}, Grey: C^{Ga}_{V}, Cyan: upper bound

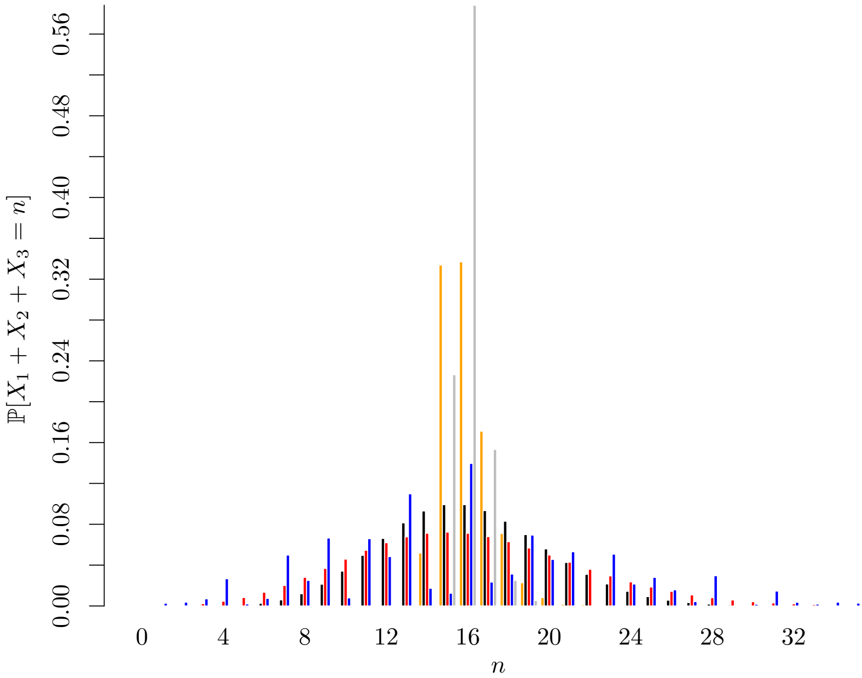

Figure 6.14: Probability mass function of S for $X_1 \sim \text{Poi}(3)$, $X_2 \sim \text{Poi}(5)$ and $X_3 \sim \text{Poi}(8)$ coupled by a Gaussian copula. Orange: $C^{\text{Ga}}_{-0.5}$, Black: C^{Ga}_{0}, Red: $C^{\text{Ga}}_{0.5}$, Blue: C^{Ga}_{1}, Grey: C^{Ga}_{V}

Example 6.11. Let $d = 3$ and consider Poisson-distributed random variables $X_1 \sim \text{Poi}(3)$, $X_2 \sim \text{Poi}(5)$ and $X_3 \sim \text{Poi}(8)$. Using the same correlation matrices as in Example 6.10, we obtain the following results for a t-copula with $\nu = 1$:

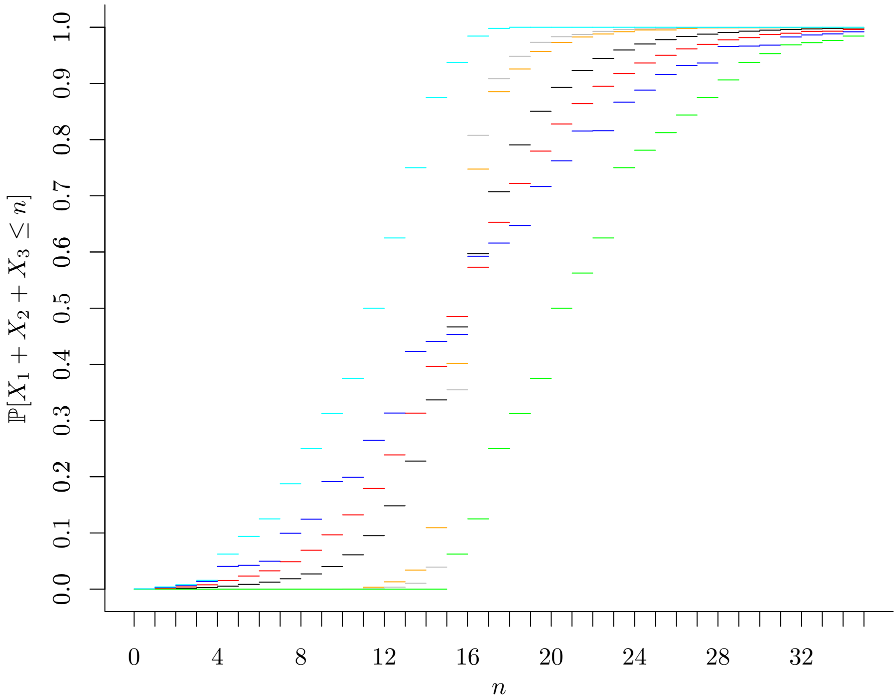

Figure 6.15: Distribution function of S for $X_1 \sim \text{Poi}(3)$, $X_2 \sim \text{Poi}(5)$ and $X_3 \sim \text{Poi}(8)$ coupled by a t-copula with $\nu = 1$. Green: lower bound, Orange: $C^{\text{t}}_{1,-0.5}$, Black: $C^{\text{t}}_{1,0}$, Red: $C^{\text{t}}_{1,0.5}$, Blue: $C^{\text{t}}_{1,1}$, Grey: $C^{\text{t}}_{1,V}$, Cyan: upper bound

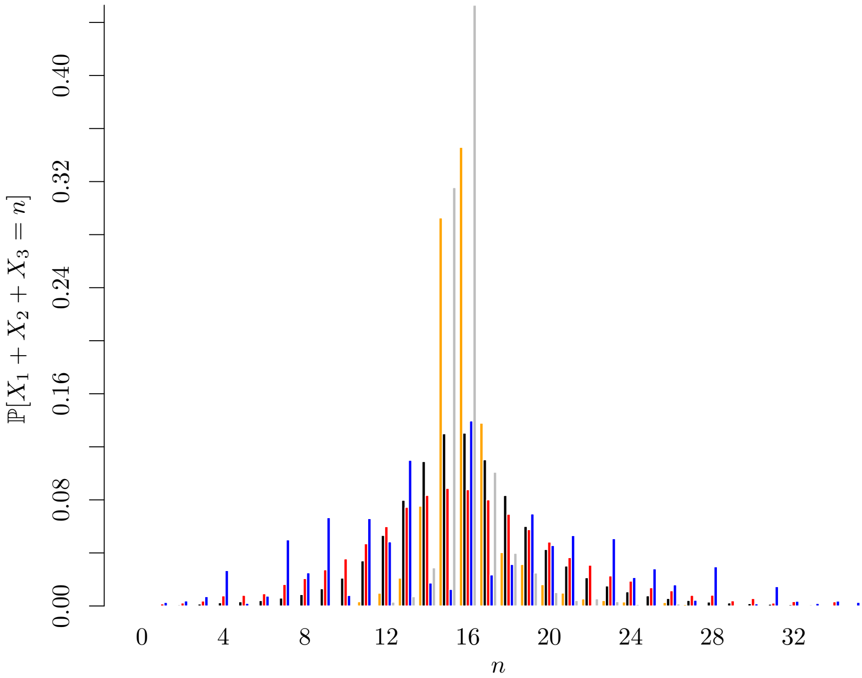

Figure 6.16: Probability mass function S for $X_1 \sim \mathrm{Poi}(3)$, $X_2 \sim \mathrm{Poi}(5)$ and $X_3 \sim \mathrm{Poi}(8)$ coupled by a t-copula with $\nu = 1$. Orange: $C^{\mathrm{t}}_{1,-0.5}$, Black: $C^{\mathrm{t}}_{1,0}$, Red: $C^{\mathrm{t}}_{1,0.5}$, Blue: $C^{\mathrm{t}}_{1,1}$, Grey: $C^{\mathrm{t}}_{1,V}$

Example 6.12. Let $d = 3$ and consider Poisson-distributed random variables $X_1 \sim \text{Poi}(3)$, $X_2 \sim \text{Poi}(5)$ and $X_3 \sim \text{Poi}(8)$. Using the same correlation matrices as in Example 6.10, we obtain the following results for a t-copula with $\nu = 3$:

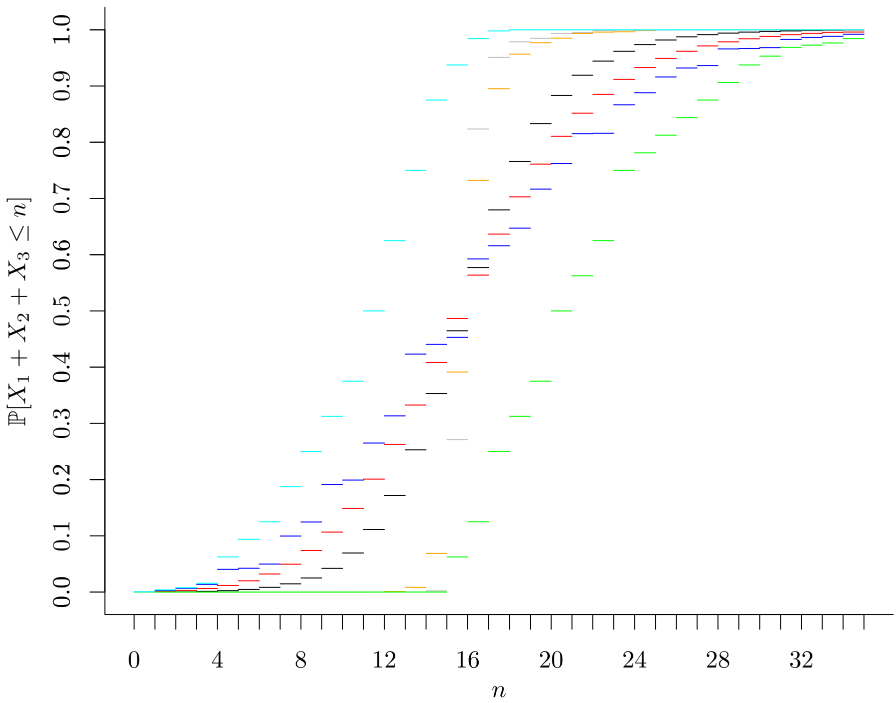

Figure 6.17: Distribution function of S for $X_1 \sim \text{Poi}(3)$, $X_2 \sim \text{Poi}(5)$ and $X_3 \sim \text{Poi}(8)$ coupled by a t-copula with $\nu = 3$. Green: lower bound, Orange: $C^{\text{t}}_{3,-0.5}$, Black: $C^{\text{t}}_{3,0}$, Red: $C^{\text{t}}_{3,0.5}$, Blue: $C^{\text{t}}_{3,1}$, Grey: $C^{\text{t}}_{3,V}$, Cyan: upper bound

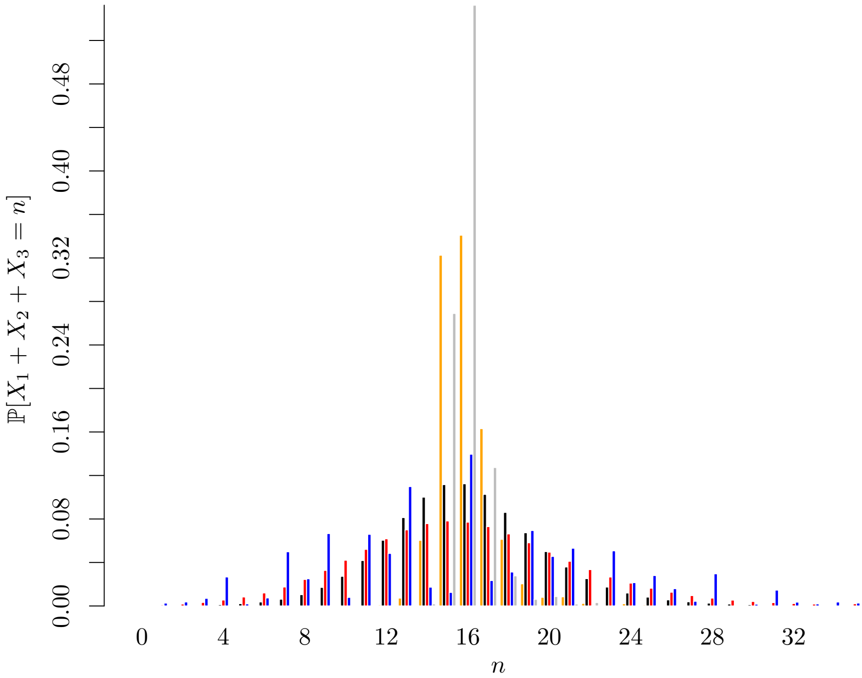

Figure 6.18: Probability mass function S for $X_1 \sim \text{Poi}(3)$, $X_2 \sim \text{Poi}(5)$ and $X_3 \sim \text{Poi}(8)$ coupled by a t-copula with $\nu = 3$. Orange: $C^{\mathrm{t}}_{3,-0.5}$, Black: $C^{\mathrm{t}}_{3,0}$, Red: $C^{\mathrm{t}}_{3,0.5}$, Blue: $C^{\mathrm{t}}_{3,1}$, Grey: $C^{\mathrm{t}}_{3,V}$

Example 6.13. Let $d = 3$ and consider Poisson-distributed random variables $X_1 \sim \text{Poi}(3)$, $X_2 \sim \text{Poi}(5)$ and $X_3 \sim \text{Poi}(8)$. For a Gumbel copula we obtain the following results:

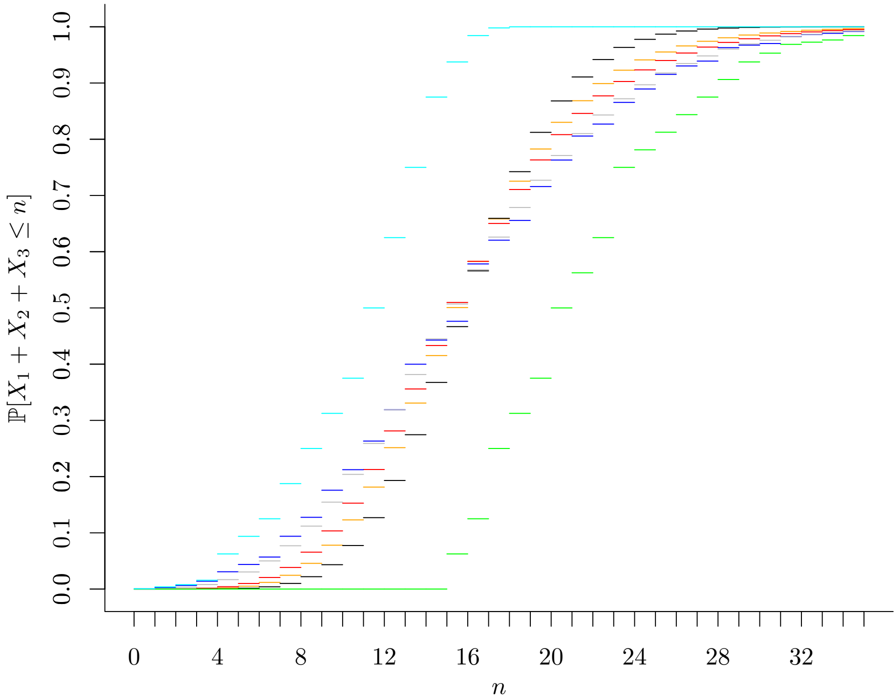

Figure 6.19: Distribution function of S for $X_1 \sim \text{Poi}(3)$, $X_2 \sim \text{Poi}(5)$ and $X_3 \sim \text{Poi}(8)$ coupled by a Gumbel copula. Green: lower bound, Black: C_1^{Gu}, Orange: $C_{1.25}^{\text{Gu}}$, Red: $C_{1.5}^{\text{Gu}}$, Grey: C_3^{Gu}, Blue: C_{10}^{Gu}, Cyan: upper bound

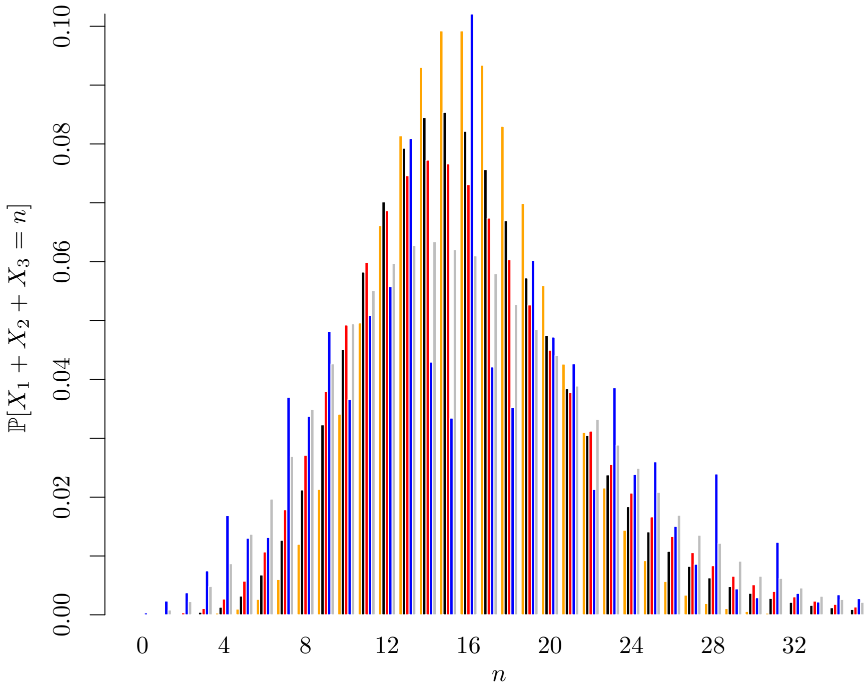

Figure 6.20: Probability function of S for $X_1 \sim \mathrm{Poi}(3)$, $X_2 \sim \mathrm{Poi}(5)$ and $X_3 \sim \mathrm{Poi}(8)$ coupled by a Gumbel copula. Black: C_1^{Gu}, Orange: $C_{1.25}^{\mathrm{Gu}}$, Red: $C_{1.5}^{\mathrm{Gu}}$, Grey: C_3^{Gu}, Blue: C_{10}^{Gu}

Example 6.14. Let $d = 3$ and consider Poisson-distributed random variables $X_1 \sim \mathrm{Poi}(3)$, $X_2 \sim \mathrm{Poi}(5)$ and $X_3 \sim \mathrm{Poi}(8)$. For a Frank copula we obtain the following results:

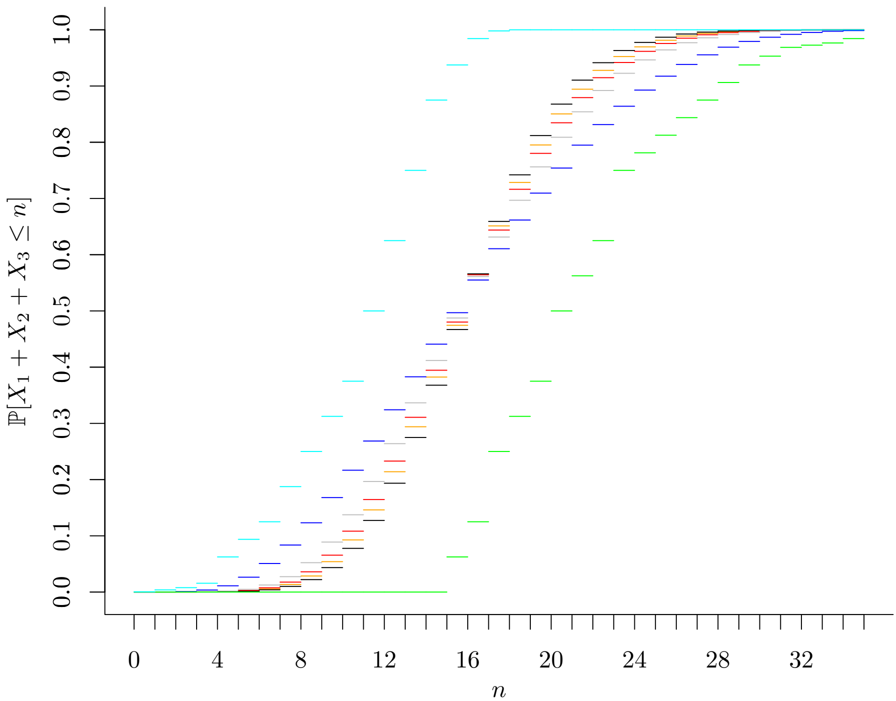

Figure 6.21: Distribution function of S for $X_1 \sim \mathrm{Poi}(3)$, $X_2 \sim \mathrm{Poi}(5)$ and $X_3 \sim \mathrm{Poi}(8)$ coupled by a Frank copula. Green: lower bound, Black: $C_{0.01}^{\mathrm{Fr}}$, Orange: $C_{0.5}^{\mathrm{Fr}}$, Red: C_1^{Fr}, Grey: C_2^{Fr}, Blue: C_{10}^{Fr}, Cyan: upper bound

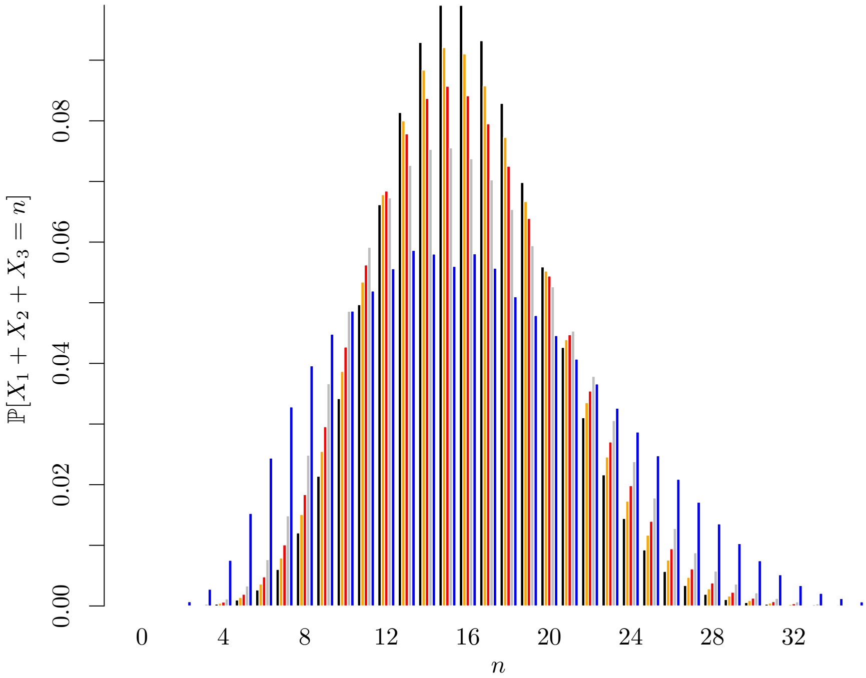

Figure 6.22: Probability mass function of S for $X_1 \sim \mathrm{Poi}(3)$, $X_2 \sim \mathrm{Poi}(5)$ and $X_3 \sim \mathrm{Poi}(8)$ coupled by a Frank copula. Black: $C^{\mathrm{Fr}}_{0.01}$, Orange: $C^{\mathrm{Fr}}_{0.5}$, Red: C^{Fr}_{1}, Grey: C^{Fr}_{2}, Blue: C^{Fr}_{10}

6.1.4 Negative Binomial-Distributed Margins

Definition 6.15. A random variable X on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ follows a negative binomial distribution with parameters $n \in \mathbb{N}$ and $p \in (0,1]$, $X \sim \mathrm{NB}(n,p)$, if[99]

$$\mathbb{P}[X = k] = \binom{n+k-1}{n-1} p^n (1-p)^k, \quad k \in \mathbb{N}_0,$$

holds.

Example 6.16. Let $d = 3$ and consider negative binomial-distributed random variables $X_1 \sim \mathrm{NB}(5, 0.9)$, $X_2 \sim \mathrm{NB}(5, 0.7)$ and $X_3 \sim \mathrm{NB}(0.3)$. For fundamental copulas we obtain the following results:

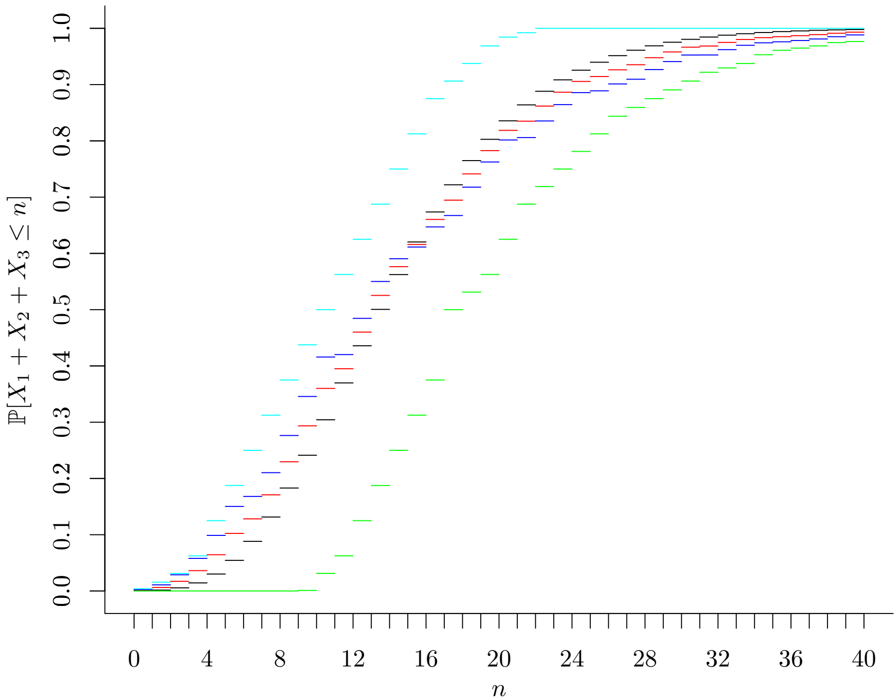

Figure 6.23: Distribution function of S for $X_1 \sim \mathrm{NB}(5, 0.9)$, $X_2 \sim \mathrm{NB}(5, 0.7)$ and $X_3 \sim \mathrm{NB}(5, 0.3)$ coupled by fundamental copulas. Green: lower bound, Black: Π, Red: $0.5M + 0.5\Pi$, Blue: M, Cyan: upper bound

[99] cf. [21, Example 1.105]

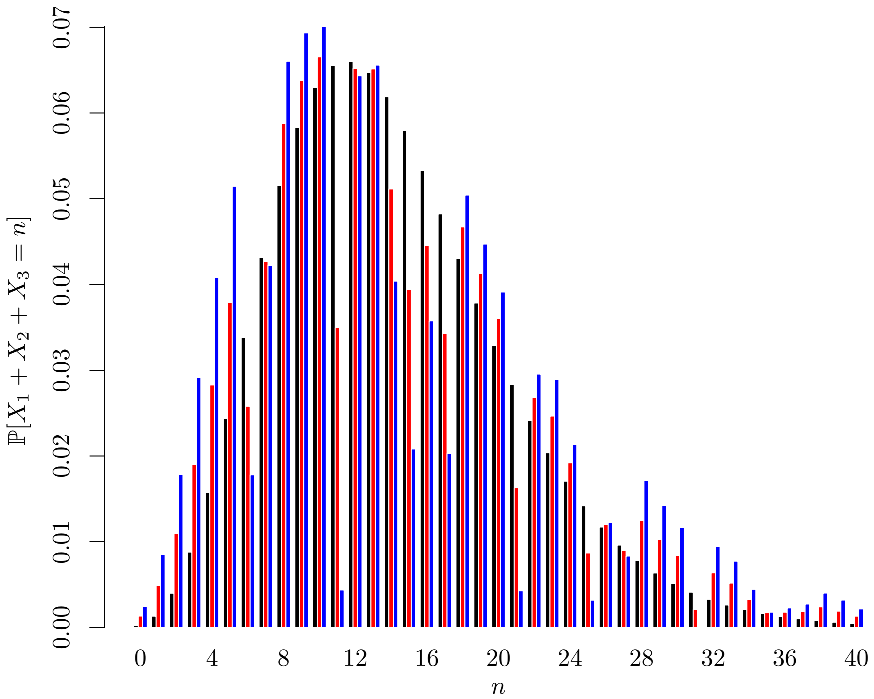

Figure 6.24: Probability mass function of S for $X_1 \sim \mathrm{NB}(5, 0.9)$, $X_2 \sim \mathrm{NB}(5, 0.7)$ and $X_3 \sim \mathrm{NB}(5, 0.3)$ coupled by fundamental copulas. Black: Π, Red: $0.5M + 0.5\Pi$, Blue: M

Example 6.17. Let $d = 3$ and consider negative binomial-distributed random variables $X_1 \sim \mathrm{NB}(5, 0.9)$, $X_2 \sim \mathrm{NB}(5, 0.7)$ and $X_3 \sim \mathrm{NB}(5, 0.3)$. We obtain the following results using a Gaussian copula:

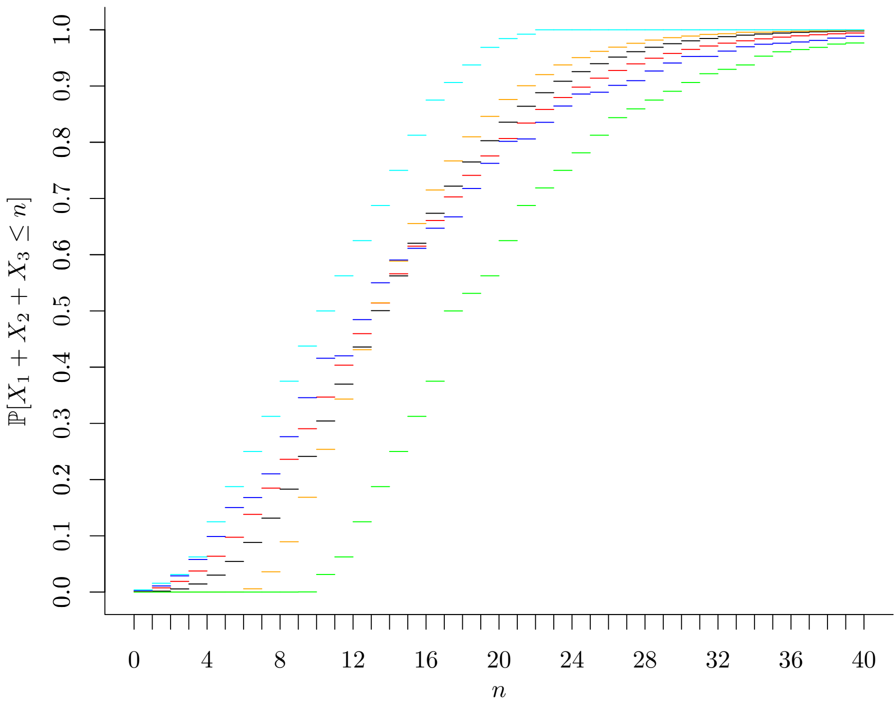

Figure 6.25: Distribution function of S for $X_1 \sim \mathrm{NB}(5, 0.9)$, $X_2 \sim \mathrm{NB}(5, 0.7)$ and $X_3 \sim \mathrm{NB}(5, 0.3)$ coupled by a Gaussian copula. Green: lower bound, Orange: $C^{\mathrm{Ga}}_{-0.5}$, Black: C^{Ga}_{0}, Red: $C^{\mathrm{Ga}}_{0.5}$, Blue: C^{Ga}_{1}, Cyan: upper bound

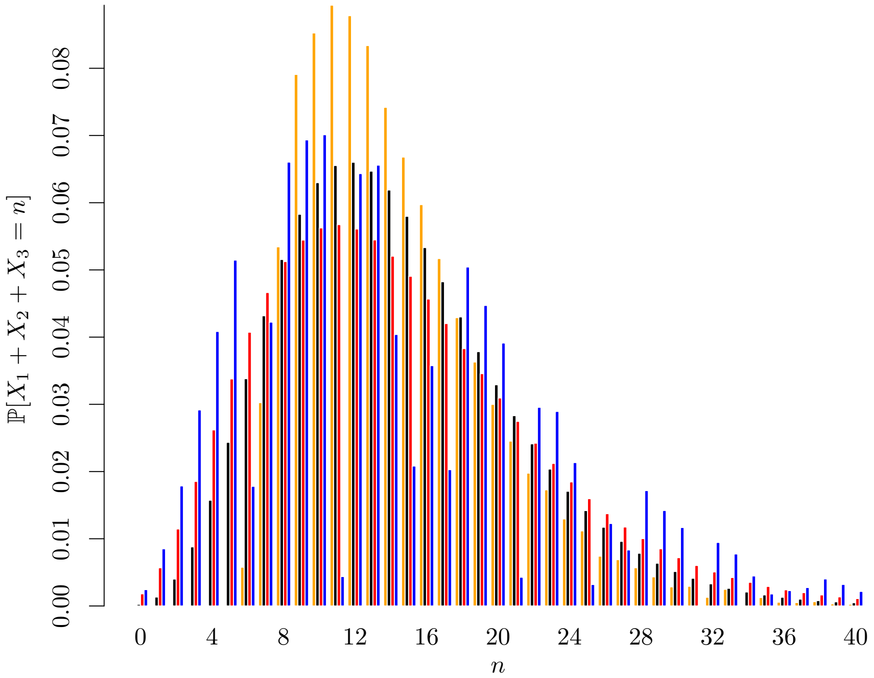

Figure 6.26: Probability mass function of S for $X_1 \sim \text{NB}(5, 0.9)$, $X_2 \sim \text{NB}(5, 0.7)$ and $X_3 \sim \text{NB}(5, 0.3)$ coupled by a Gaussian copula. Orange: $C^{\text{Ga}}_{-0.5}$, Black: C^{Ga}_{0}, Red: $C^{\text{Ga}}_{0.5}$, Blue: C^{Ga}_{1}

Example 6.18. Let $d = 3$ and consider negative binomial-distributed random variables $X_1 \sim \mathrm{NB}(5, 0.9)$, $X_2 \sim \mathrm{NB}(5, 0.7)$ and $X_3 \sim \mathrm{NB}(5, 0.3)$. Considering a t-copula with $\nu = 1$ we obtain the following results when considering the same correlation matrices as in Example 6.17:

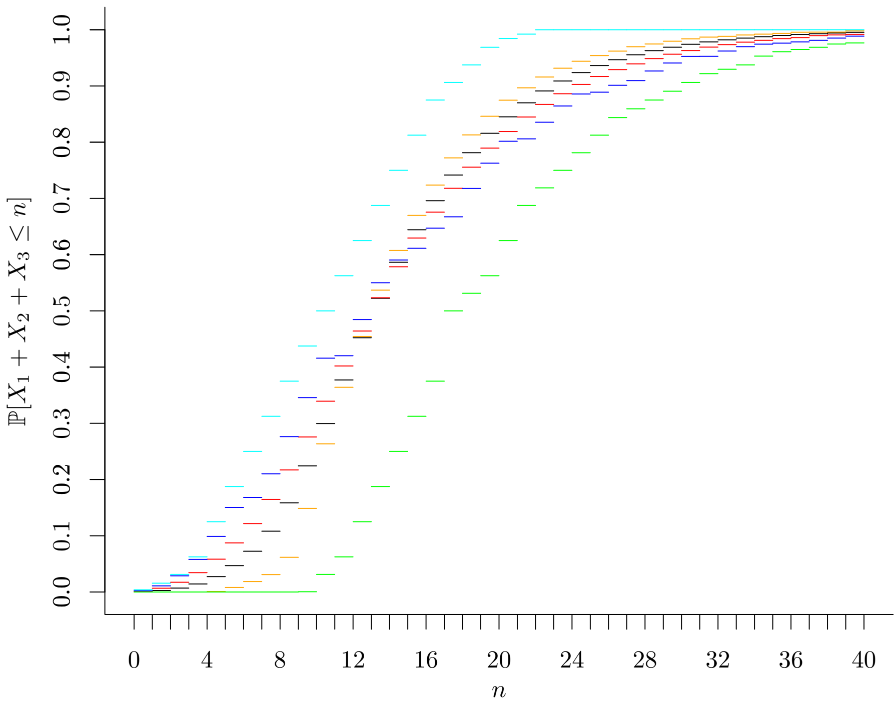

Figure 6.27: Distribution function of S for $X_1 \sim \mathrm{NB}(5, 0.9)$, $X_2 \sim \mathrm{NB}(5, 0.7)$ and $X_3 \sim \mathrm{NB}(5, 0.3)$ coupled by a t-copula with $\nu = 1$. Green: lower bound, Orange: $C^{\mathrm{t}}_{1,-0.5}$, Black: $C^{\mathrm{t}}_{1,0}$, Red: $C^{\mathrm{t}}_{1,0.5}$, Blue: $C^{\mathrm{t}}_{1,1}$, Cyan: upper bound

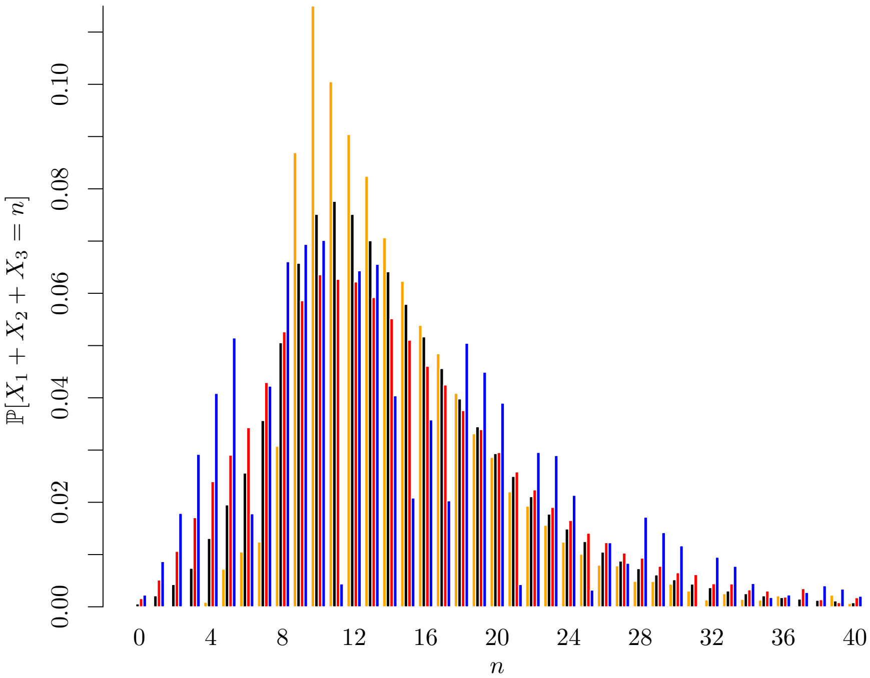

Figure 6.28: Probability mass function of S for $X_1 \sim \mathrm{NB}(5, 0.9)$, $X_2 \sim \mathrm{NB}(5, 0.7)$ and $X_3 \sim \mathrm{NB}(5, 0.3)$ coupled by a t-copula with $\nu = 1$. Orange: $C^{\mathrm{t}}_{1,-0.5}$, Black: $C^{\mathrm{t}}_{1,0}$, Red: $C^{\mathrm{t}}_{1,0.5}$, Blue: $C^{\mathrm{t}}_{1,1}$

Example 6.19. Let $d = 3$ and consider negative binomial-distributed random variables $X_1 \sim \mathrm{NB}(5, 0.9)$, $X_2 \sim \mathrm{NB}(5, 0.7)$ and $X_3 \sim \mathrm{NB}(5, 0.3)$. For a Gumbel copula we obtain the following results:

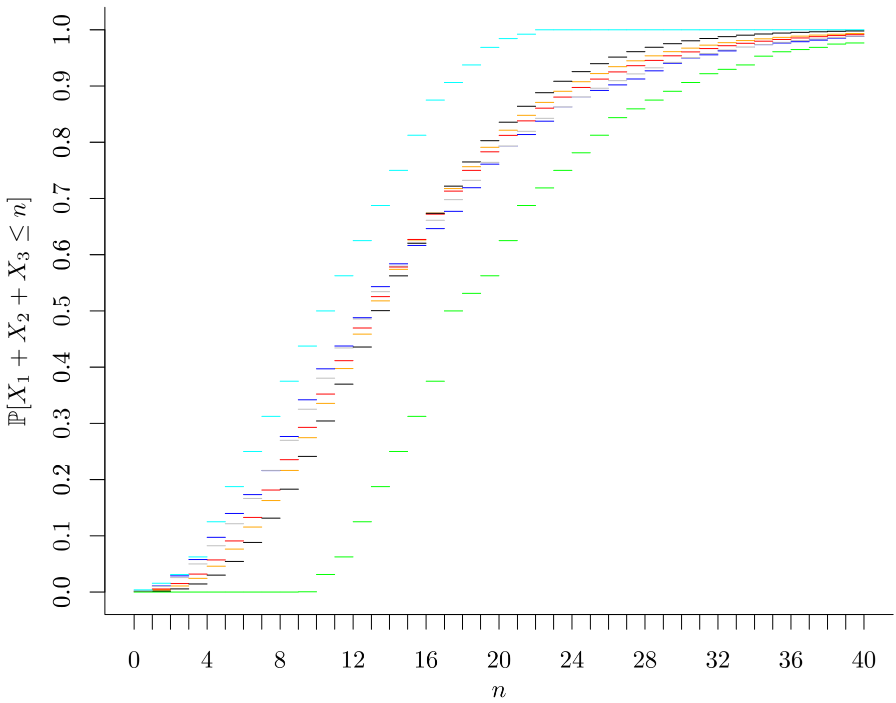

Figure 6.29: Distribution function of S for $X_1 \sim \mathrm{NB}(5, 0.9)$, $X_2 \sim \mathrm{NB}(5, 0.7)$ and $X_3 \sim \mathrm{NB}(5, 0.3)$ coupled by a Gumbel copula. Green: lower bound, Black: C_1^{Gu}, Orange: $C_{1.25}^{\mathrm{Gu}}$, Red: $C_{1.5}^{\mathrm{Gu}}$, Grey: C_3^{Gu}, Blue: C_{10}^{Gu}, Cyan: upper bound

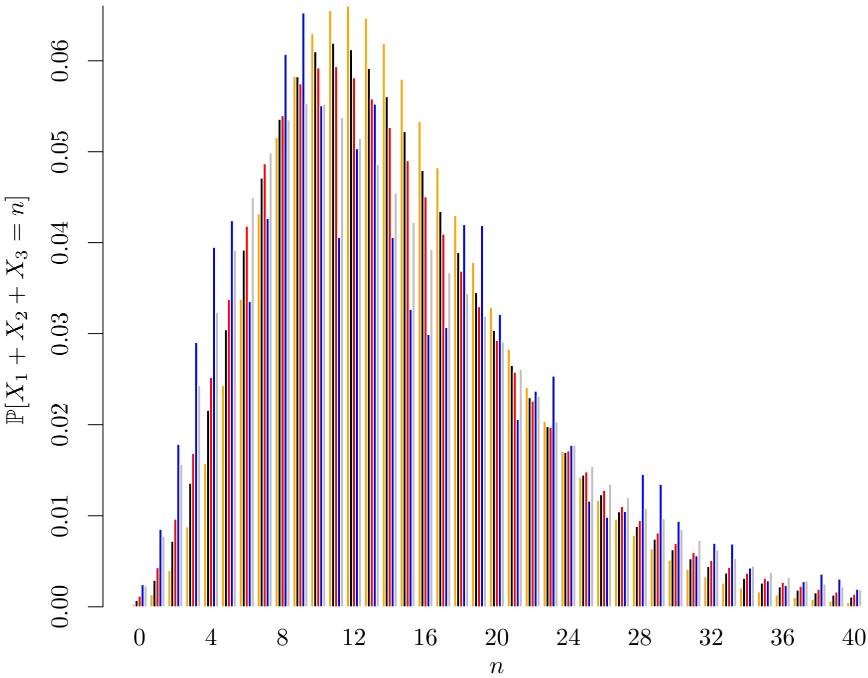

Figure 6.30: Probability mass function of S for $X_1 \sim \mathrm{NB}(5, 0.9)$, $X_2 \sim \mathrm{NB}(5, 0.7)$ and $X_3 \sim \mathrm{NB}(5, 0.3)$ coupled by a Gumbel copula. Black: C_1^{Gu}, Orange: $C_{1.25}^{\mathrm{Gu}}$, Red: $C_{1.5}^{\mathrm{Gu}}$, Grey: C_3^{Gu}, Blue: C_{10}^{Gu}

Example 6.20. Let $d = 3$ and consider negative binomial-distributed random variables $X_1 \sim \mathrm{NB}(5, 0.9)$, $X_2 \sim \mathrm{NB}(5, 0.7)$ and $X_3 \sim \mathrm{NB}(5, 0.3)$. For a Frank copula we obtain the following results:

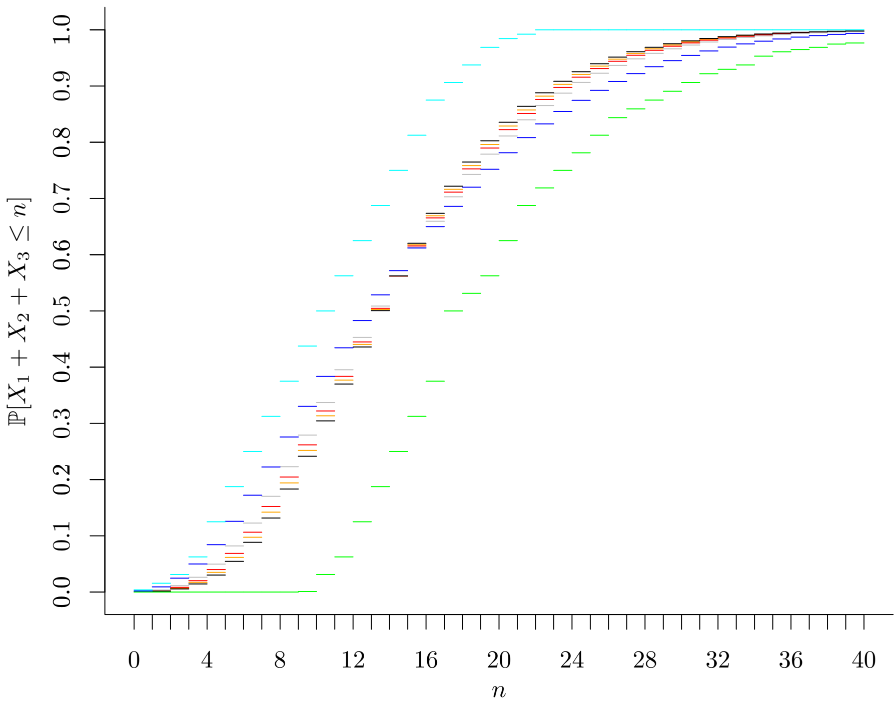

Figure 6.31: Distribution function of S for $X_1 \sim \mathrm{NB}(5, 0.9)$, $X_2 \sim \mathrm{NB}(5, 0.7)$ and $X_3 \sim \mathrm{NB}(5, 0.3)$ coupled by a Frank copula. Green: lower bound, Black: $C^{\mathrm{Fr}}_{0.01}$, Orange: $C^{\mathrm{Fr}}_{0.5}$, Red: C^{Fr}_{1}, Grey: C^{Fr}_{2}, Blue: C^{Fr}_{10}, Cyan: upper bound

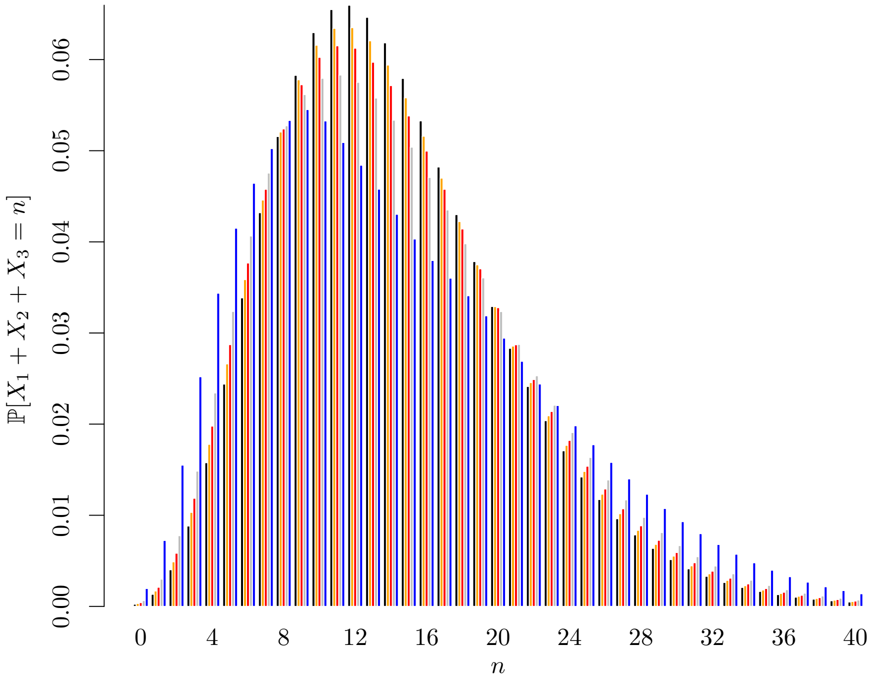

Figure 6.32: Probability mass function of S for $X_1 \sim \mathrm{NB}(5, 0.9)$, $X_2 \sim \mathrm{NB}(5, 0.7)$ and $X_3 \sim \mathrm{NB}(5, 0.3)$ coupled by a Frank copula. Black: $C^{\mathrm{Fr}}_{0.01}$, Orange: $C^{\mathrm{Fr}}_{0.5}$, Red: C^{Fr}_{1}, Grey: C^{Fr}_{2}, Blue: C^{Fr}_{10}

6.1.5 Arbitrary Distribution on $\mathbb{N}_0$

Consider an insurance company with a portfolio of $d = 4$ policies. By random variables X_i we denote the individual claim size in millions of Euro, $i = 1, \ldots, 4$. We are interested in modelling the aggregate claim amount, which is given by $S = X_1 + \cdots + X_4$, under various dependence scenarios. For this we assume that we know the marginal distributions $F_1, \ldots, F_4$ of $X_1, \ldots, X_4$, which are given as follows:

$$\mathbb{P}[X_1 = 0] = 0.90, \quad \mathbb{P}[X_1 = 1] = 0.05, \quad \mathbb{P}[X_1 = 3] = 0.03, \quad \mathbb{P}[X_1 = 5] = 0.02$$

$$\mathbb{P}[X_2 = 0] = 0.80, \quad \mathbb{P}[X_2 = 2] = 0.16, \quad \mathbb{P}[X_2 = 4] = 0.04$$

$$\mathbb{P}[X_3 = 0] = 0.65, \quad \mathbb{P}[X_3 = 1] = 0.20, \quad \mathbb{P}[X_3 = 2] = 0.10, \quad \mathbb{P}[X_3 = 6] = 0.05$$

$$\mathbb{P}[X_4 = 0] = 0.70, \quad \mathbb{P}[X_4 = 1] = 0.15, \quad \mathbb{P}[X_4 = 4] = 0.05, \quad \mathbb{P}[X_4 = 5] = 0.10$$

If we consider the dependency described by fundamental copulas (i.e. independence, comonotonicity and a convex combination of these), we obtain the subsequent results:

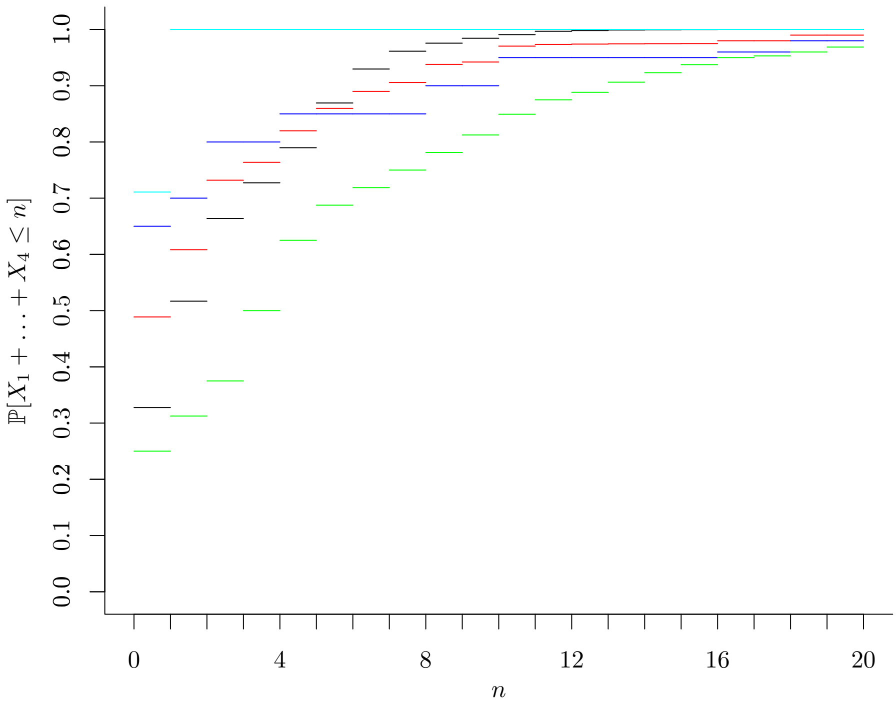

Figure 6.33: Distribution function of S for X_i as defined above, $i = 1, \ldots, 4$, coupled by fundamental copulas. Green: lower bound, Black: Π, Red: $0.5M + 0.5\Pi$, Blue: M, Cyan: upper bound

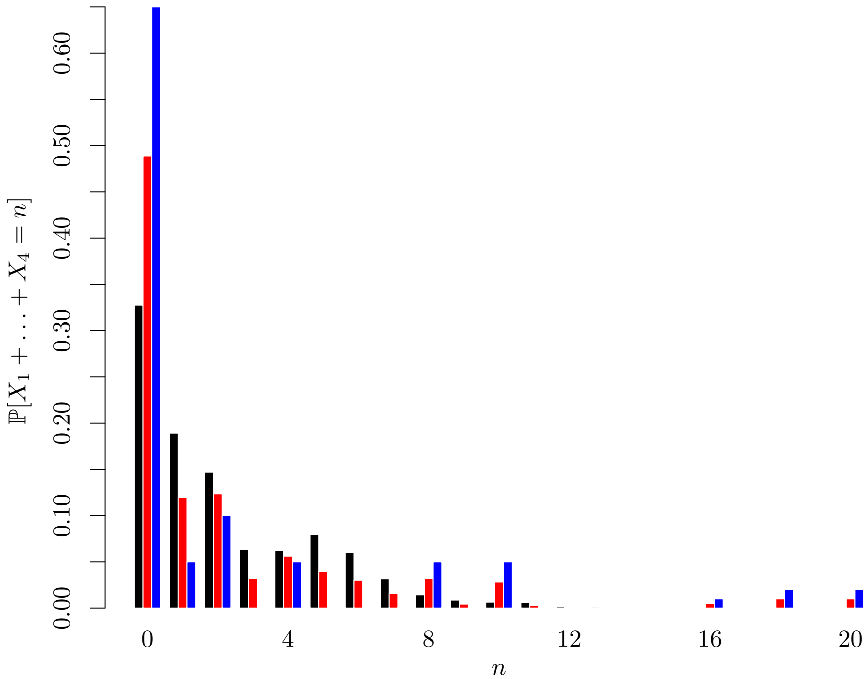

Figure 6.34: Probability mass function of S for X_i as defined above, $i = 1, \ldots, 4$, coupled by fundamental copulas. Black: Π, Red: $0.5M + 0.5\Pi$, Blue: M

For the next illustration we apply a Gaussian copula to the insurance portfolio described above. We obtain the following results:[100]

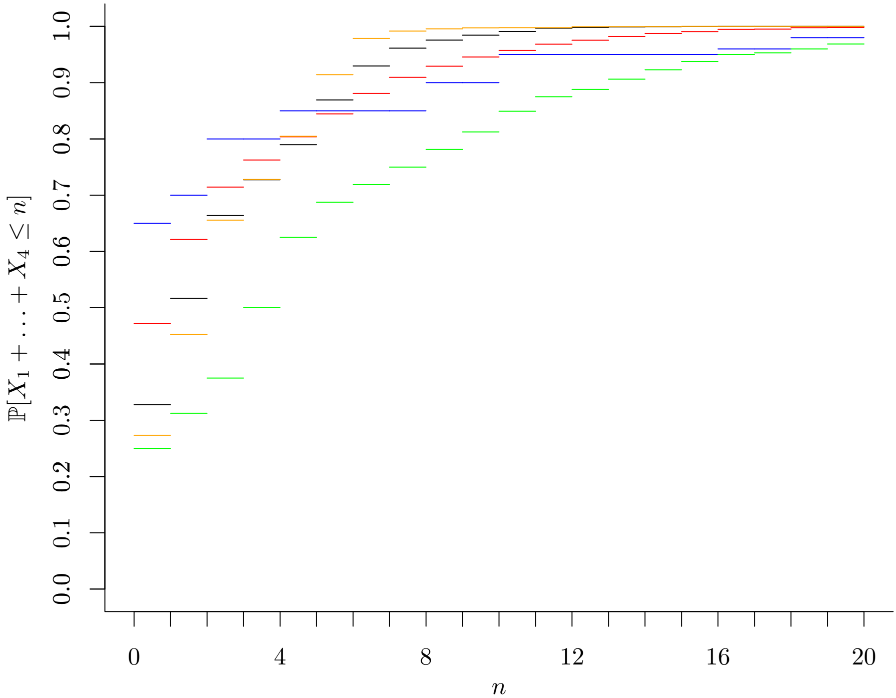

Figure 6.35: Distribution function of S for X_i as defined above, $i = 1, \ldots, 4$, coupled by a Gaussian copula. Green: lower bound, Orange: $C^{\mathrm{Ga}}_{-\frac{1}{3}}$, Black: C^{Ga}_{0}, Red: $C^{\mathrm{Ga}}_{0.5}$, Blue: C^{Ga}_{1}

[100]Note that since the upper bound on the distribution function of S already reaches values close to 1 for $n = 1$, a plot of it is omitted.

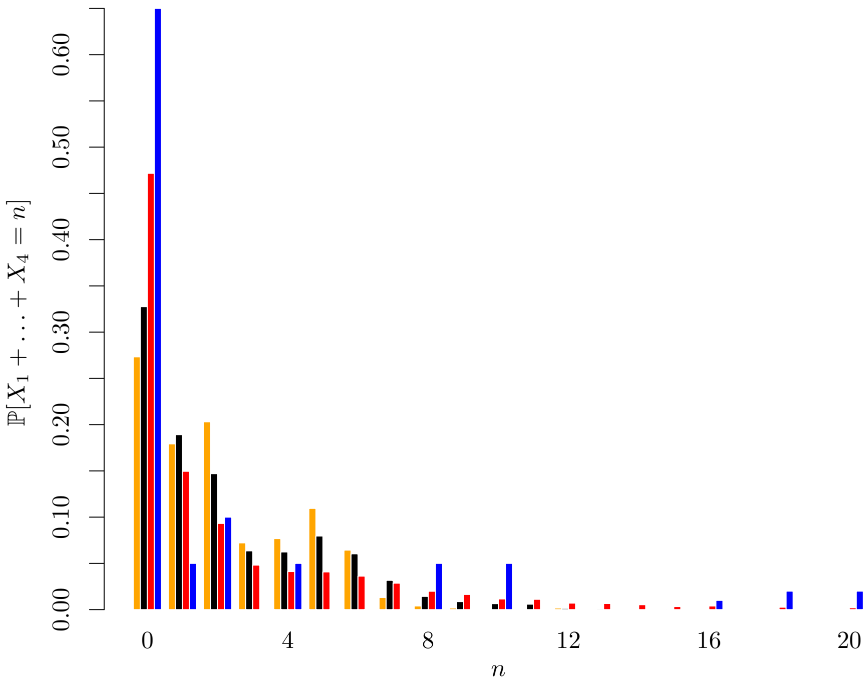

Figure 6.36: Probability mass function of S for X_i as defined above, $i = 1, \ldots, 4$, coupled by a Gaussian copula. Orange: $C^{\mathrm{Ga}}_{-\frac{1}{3}}$, Black: C^{Ga}_{0}, Red: $C^{\mathrm{Ga}}_{0.5}$, Blue: C^{Ga}_{1}

6.2 Risk Measures for S

The bounds on Value-at-Risk and Expected Shortfall are calculated according to the methodologies described in Chapter 5 using $N = 10^5$ and tolerance $\epsilon = 10^{-8}$.

Example 6.21. Let $d = 3$ and consider Poisson-distributed random variables $X_1 \sim \mathrm{Poi}(3)$, $X_2 \sim \mathrm{Poi}(5)$ and $X_3 \sim \mathrm{Poi}(8)$. For fundamental copulas follows:

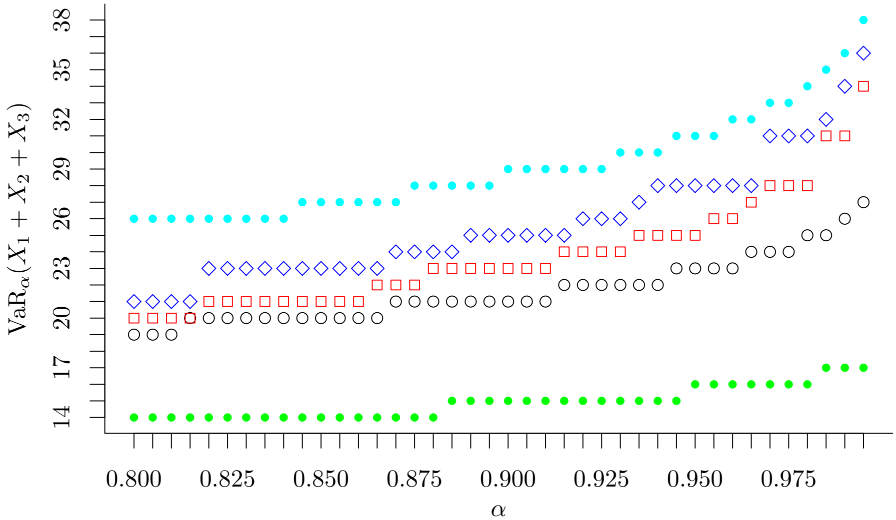

Figure 6.37: $\mathrm{VaR}_\alpha(S)$ for $X_1 \sim \mathrm{Poi}(3)$, $X_2 \sim \mathrm{Poi}(5)$ and $X_3 \sim \mathrm{Poi}(8)$ coupled by fundamental copulas. Green: lower bound, Black: Π, Red: $0.5\Pi + 0.5M$, Blue: M, Cyan: upper bound

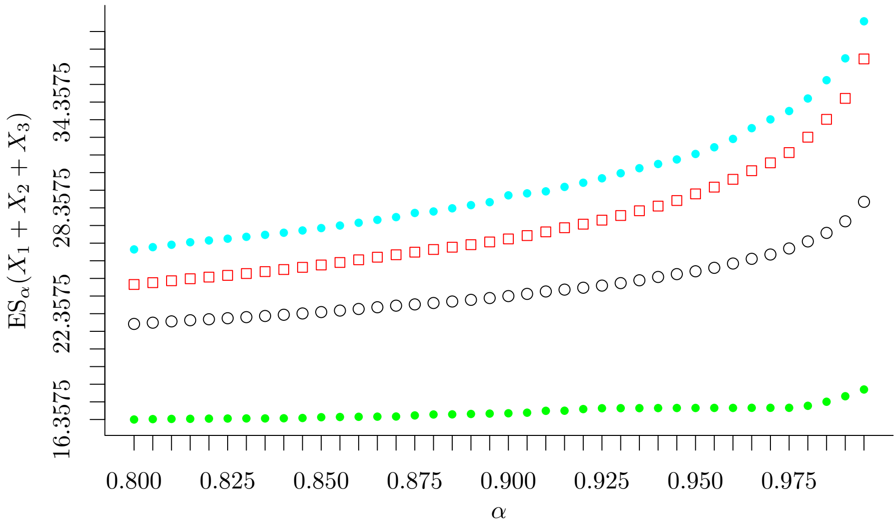

Figure 6.38: $\mathrm{ES}_\alpha(S)$ for $X_1 \sim \mathrm{Poi}(3)$, $X_2 \sim \mathrm{Poi}(5)$ and $X_3 \sim \mathrm{Poi}(8)$ coupled by fundamental copulas. Green: lower bound, Black: Π, Red: $0.5\Pi + 0.5M$, Cyan: upper bound

For the subsequent illustrations based on the Gaussian copula we used the same correlation matrices as in Example 6.10.

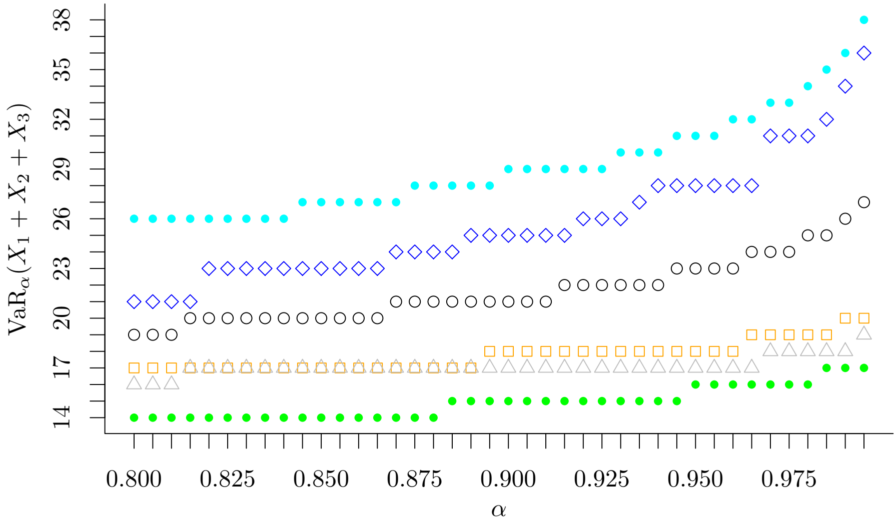

Figure 6.39: $\mathrm{VaR}_\alpha(S)$ for $X_1 \sim \mathrm{Poi}(3)$, $X_2 \sim \mathrm{Poi}(5)$ and $X_3 \sim \mathrm{Poi}(8)$ coupled by a Gaussian copula. Green: lower bound, Grey: C_V^{Ga}, Orange: $C_{-0.5}^{\mathrm{Ga}}$, Black: C_0^{Ga}, Blue: C_1^{Ga}, Cyan: upper bound

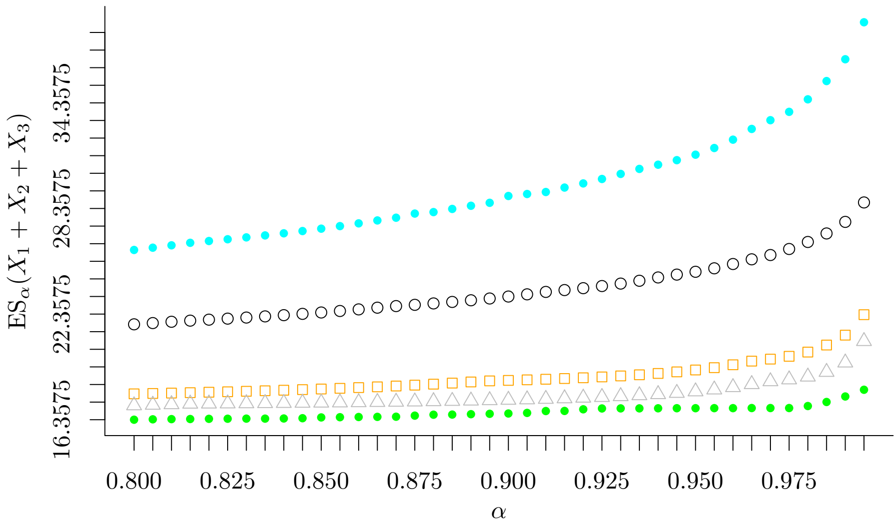

Figure 6.40: $\mathrm{ES}_\alpha(S)$ for $X_1 \sim \mathrm{Poi}(3)$, $X_2 \sim \mathrm{Poi}(5)$ and $X_3 \sim \mathrm{Poi}(8)$ coupled by a Gaussian copula. Green: lower bound, Grey: C_V^{Ga}, Orange: $C_{-0.5}^{\mathrm{Ga}}$, Black: C_0^{Ga}, Cyan: upper bound

For the subsequent illustrations based on the t-copula with $\nu = 1$ we used the same correlation matrices as in Example 6.10.

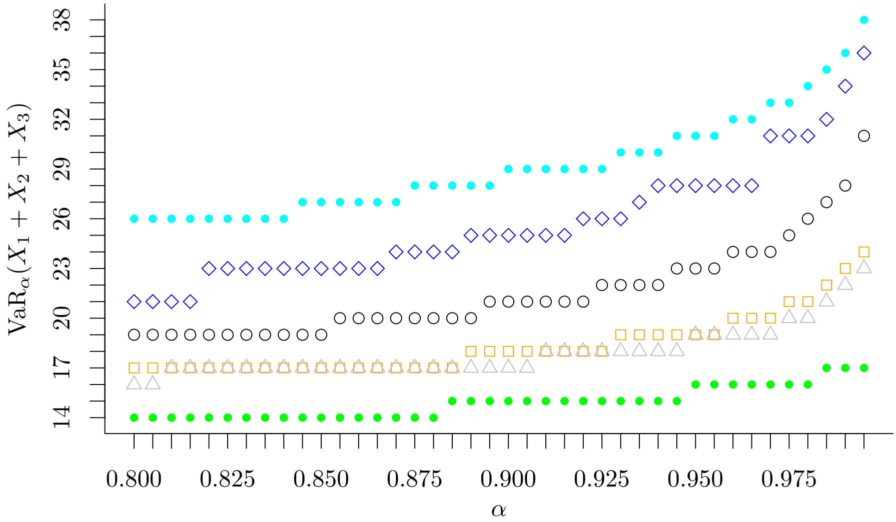

Figure 6.41: $\mathrm{VaR}_\alpha(S)$ for $X_1 \sim \mathrm{Poi}(3)$, $X_2 \sim \mathrm{Poi}(5)$ and $X_3 \sim \mathrm{Poi}(8)$ coupled by a t-copula with $\nu = 1$. Green: lower bound, Grey: $C^{\mathrm{t}}_{1,V}$, Orange: $C^{\mathrm{t}}_{1,-0.5}$, Black: $C^{\mathrm{t}}_{1,0}$, Blue: $C^{\mathrm{t}}_{1,1}$, Cyan: upper bound

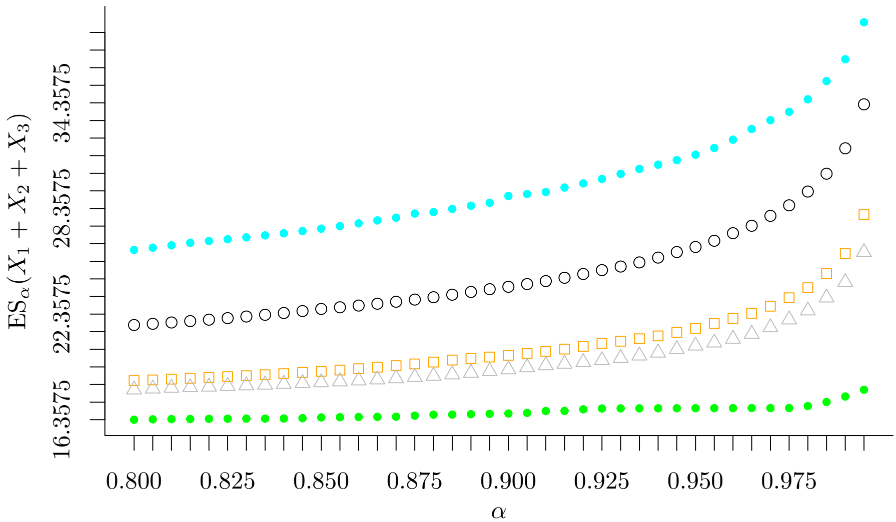

Figure 6.42: $\mathrm{ES}_\alpha(S)$ for $X_1 \sim \mathrm{Poi}(3)$, $X_2 \sim \mathrm{Poi}(5)$ and $X_3 \sim \mathrm{Poi}(8)$ coupled by a t-copula with $\nu = 1$. Green: lower bound, Grey: $C^{\mathrm{t}}_{1,V}$, Orange: $C^{\mathrm{t}}_{1,-0.5}$, Black: $C^{\mathrm{t}}_{1,0}$, Cyan: upper bound

The following illustrations are based on a Gumbel copula with various parameters:

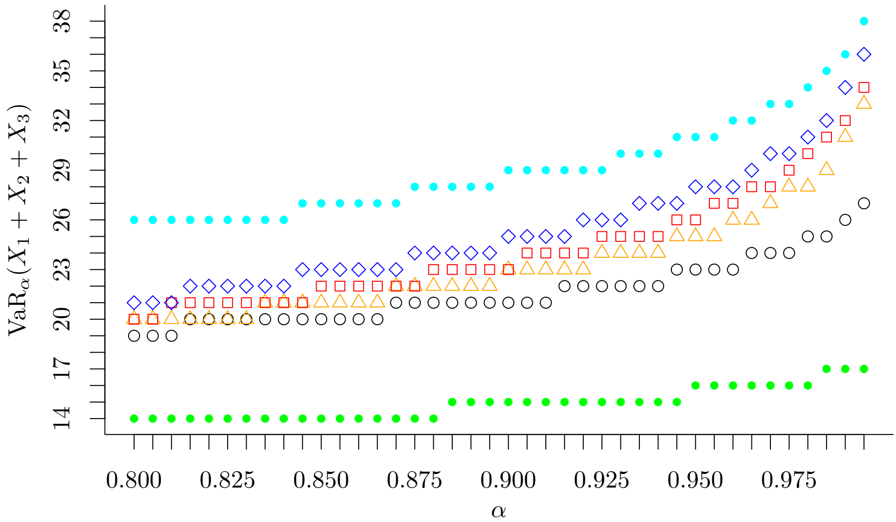

Figure 6.43: $\mathrm{VaR}_\alpha(S)$ for $X_1 \sim \mathrm{Poi}(3)$, $X_2 \sim \mathrm{Poi}(5)$ and $X_3 \sim \mathrm{Poi}(8)$ coupled by a Gumbel copula. Green: lower bound, Black: C_1^{Gu}, Orange: $C_{1.25}^{\mathrm{Gu}}$, Red: $C_{1.5}^{\mathrm{Gu}}$, Blue: C_{10}^{Gu}, Cyan: upper bound

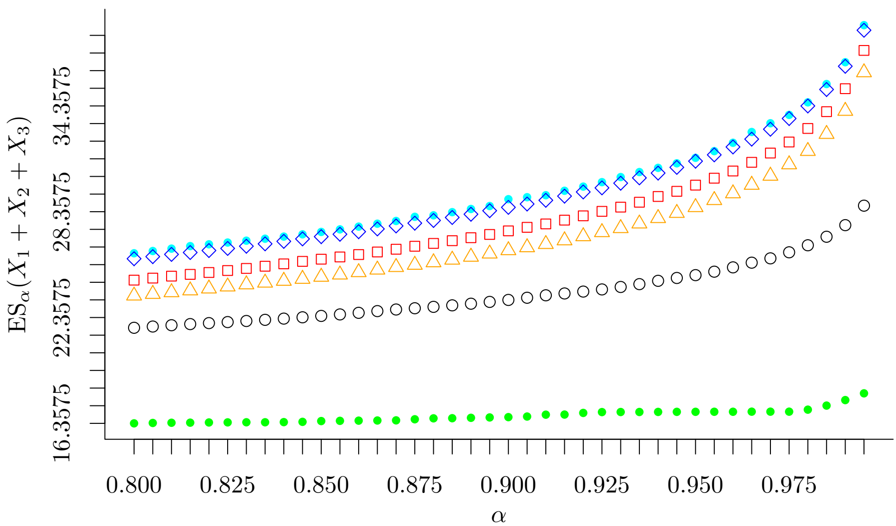

Figure 6.44: $\mathrm{ES}_\alpha(S)$ for $X_1 \sim \mathrm{Poi}(3)$, $X_2 \sim \mathrm{Poi}(5)$ and $X_3 \sim \mathrm{Poi}(8)$ coupled by a Gumbel copula. Green: lower bound, Black: C_1^{Gu}, Orange: $C_{1.25}^{\mathrm{Gu}}$, Red: $C_{1.5}^{\mathrm{Gu}}$, Blue: C_{10}^{Gu}, Cyan: upper bound

The illustrations below are based on a Frank copula with various parameters:

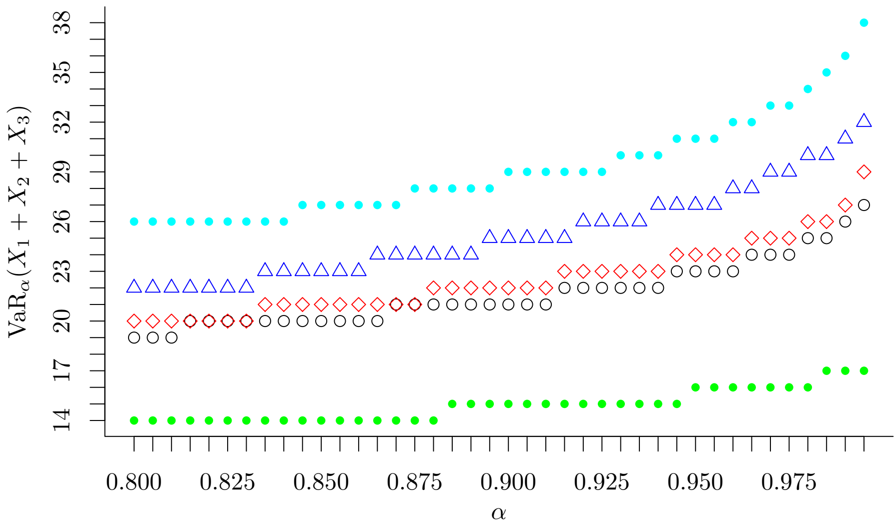

Figure 6.45: VaR$_\alpha(S)$ for $X_1 \sim$ Poi(3), $X_2 \sim$ Poi(5) and $X_3 \sim$ Poi(8) coupled by a Frank copula. Green: lower bound, Black: $C^{\mathrm{Fr}}_{0.01}$, Red: C^{Fr}_{1}, Blue: C^{Fr}_{10}, Cyan: upper bound

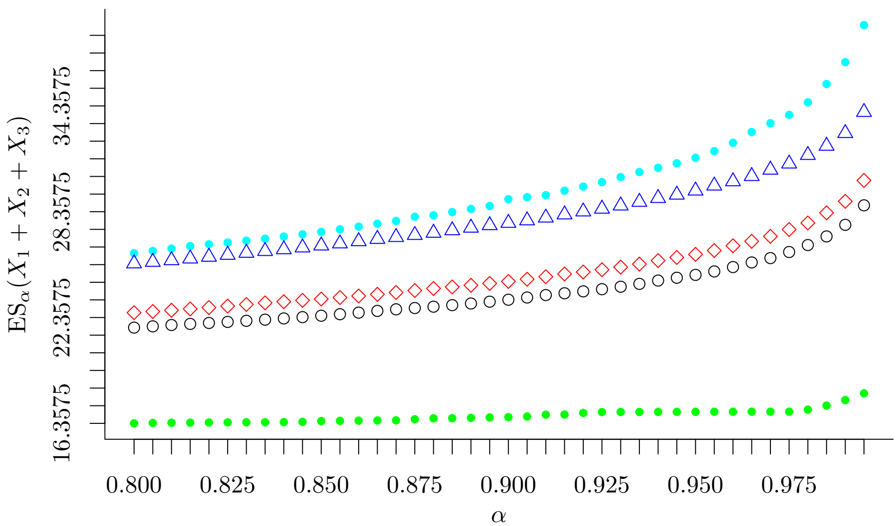

Figure 6.46: ES$_\alpha(S)$ for $X_1 \sim$ Poi(3), $X_2 \sim$ Poi(5) and $X_3 \sim$ Poi(8) coupled by a Frank copula. Green: lower bound, Black: $C^{\mathrm{Fr}}_{0.01}$, Red: C^{Fr}_{1}, Blue: C^{Fr}_{10}, Cyan: upper bound

Example 6.22. Let $d = 3$ and consider negative binomial-distributed random variables $X_1 \sim \mathrm{NB}(5, 0.9)$, $X_2 \sim \mathrm{NB}(5, 0.7)$ and $X_3 \sim \mathrm{NB}(5, 0.3)$. For fundamental copulas follows:

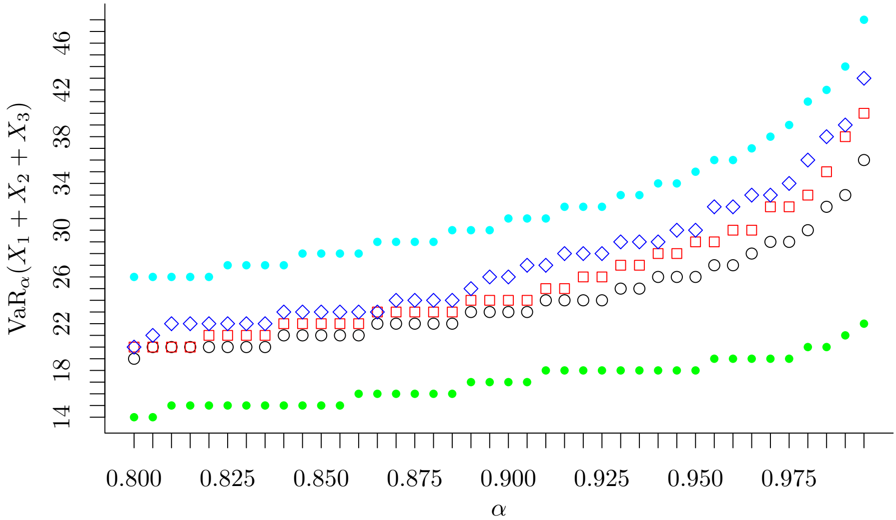

Figure 6.47: $\mathrm{VaR}_\alpha(S)$ for $X_1 \sim \mathrm{NB}(5, 0.9)$, $X_2 \sim \mathrm{NB}(5, 0.7)$ and $X_3 \sim \mathrm{NB}(5, 0.3)$ coupled by fundamental copulas. Green: lower bound, Black: Π, Red: $0.5\Pi + 0.5M$, Blue: M, Cyan: upper bound

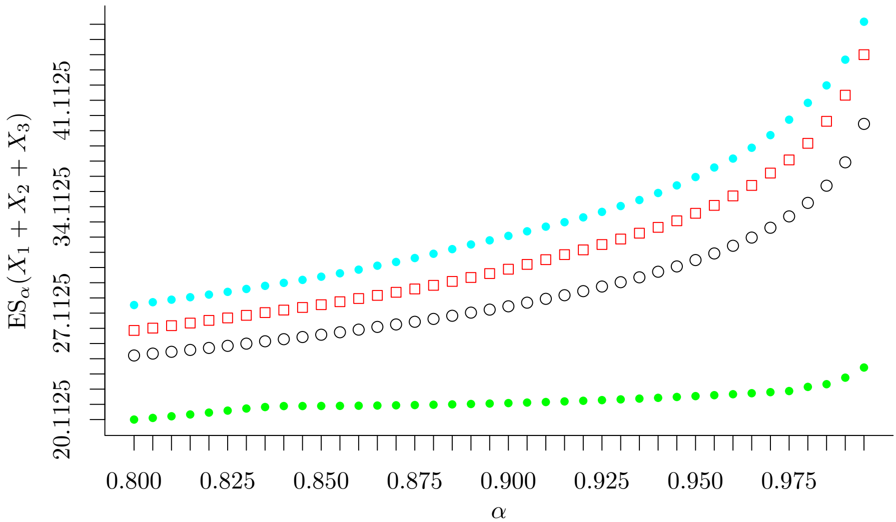

Figure 6.48: $\mathrm{ES}_\alpha(S)$ for $X_1 \sim \mathrm{NB}(5, 0.9)$, $X_2 \sim \mathrm{NB}(5, 0.7)$ and $X_3 \sim \mathrm{NB}(5, 0.3)$ coupled by fundamental copulas. Green: lower bound, Black: Π, Red: $0.5\Pi + 0.5M$, Cyan: upper bound

For the subsequent illustrations based on the Gaussian copula we used the same correlation matrices as in Example 6.17.

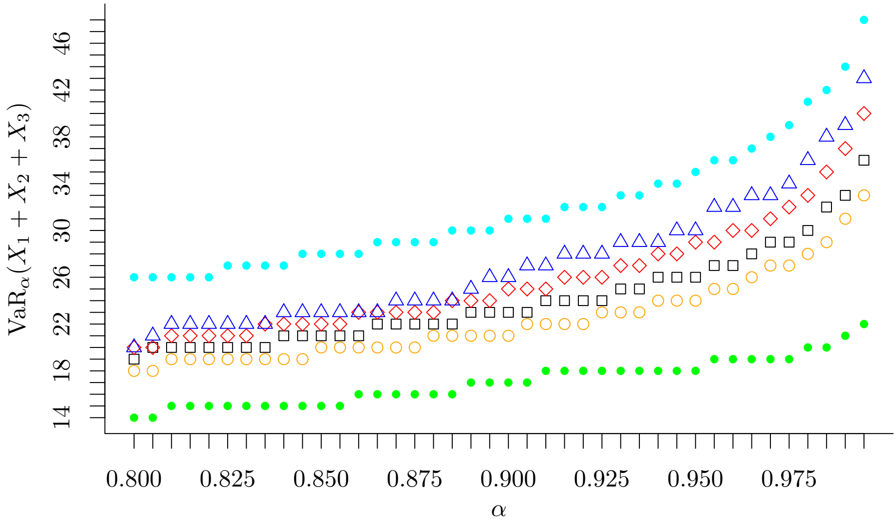

Figure 6.49: $\mathrm{VaR}_\alpha(S)$ for $X_1 \sim \mathrm{NB}(5, 0.9)$, $X_2 \sim \mathrm{NB}(5, 0.7)$ and $X_3 \sim \mathrm{NB}(5, 0.3)$ coupled by a Gaussian copula. Green: lower bound, Orange: $C^{\mathrm{Ga}}_{-0.5}$, Black: C^{Ga}_{0}, Red: $C^{\mathrm{Ga}}_{0.5}$, Blue: C^{Ga}_{1}, Cyan: upper bound

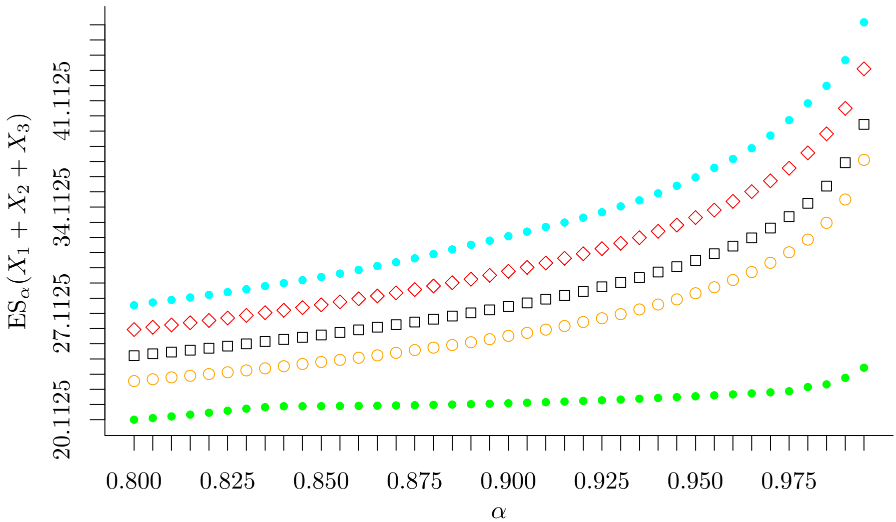

Figure 6.50: $\mathrm{ES}_\alpha(S)$ for $X_1 \sim \mathrm{NB}(5, 0.9)$, $X_2 \sim \mathrm{NB}(5, 0.7)$ and $X_3 \sim \mathrm{NB}(5, 0.3)$ coupled by a Gaussian copula. Green: lower bound, Orange: $C^{\mathrm{Ga}}_{-0.5}$, Black: C^{Ga}_{0}, Red: $C^{\mathrm{Ga}}_{0.5}$, Cyan: upper bound

For the subsequent illustrations based on the t-copula with $\nu = 1$ we used the same correlation matrices as in Example 6.17.

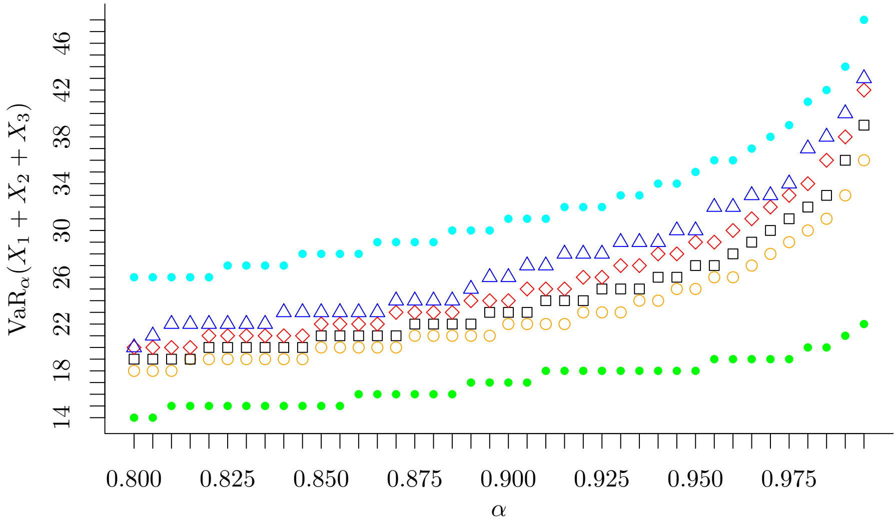

Figure 6.51: $\mathrm{VaR}_\alpha(S)$ for $X_1 \sim \mathrm{NB}(5, 0.9)$, $X_2 \sim \mathrm{NB}(5, 0.7)$ and $X_3 \sim \mathrm{NB}(5, 0.3)$ coupled by a t-copula with $\nu = 1$. Green: lower bound, Orange: $C^{\mathrm{t}}_{1,-0.5}$, Black: $C^{\mathrm{t}}_{1,0}$, Red: $C^{\mathrm{t}}_{1,0.5}$, Blue: $C^{\mathrm{t}}_{1,1}$, Cyan: upper bound

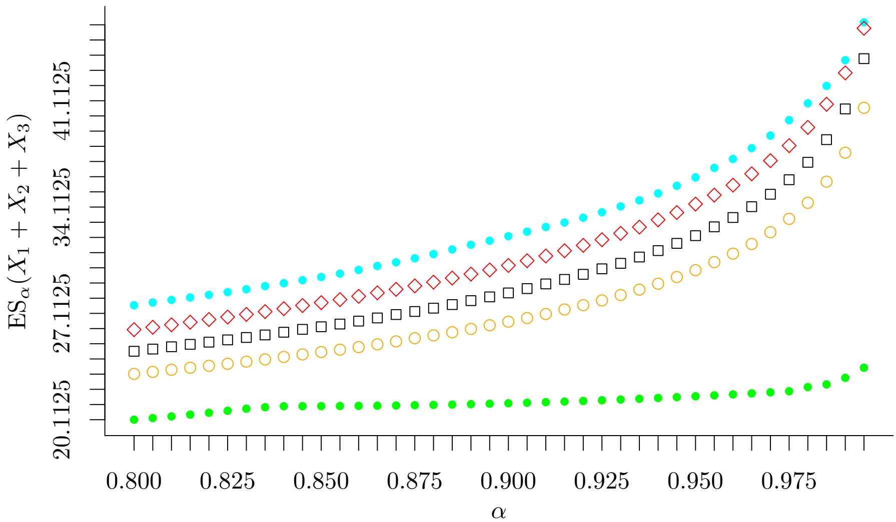

Figure 6.52: $\mathrm{ES}_\alpha(S)$ for $X_1 \sim \mathrm{NB}(5, 0.9)$, $X_2 \sim \mathrm{NB}(5, 0.7)$ and $X_3 \sim \mathrm{NB}(5, 0.3)$ coupled by a t-copula with $\nu = 1$. Green: lower bound, Orange: $C^{\mathrm{t}}_{1,-0.5}$, Black: $C^{\mathrm{t}}_{1,0}$, Red: $C^{\mathrm{t}}_{1,0.5}$, Cyan: upper bound

The following illustrations are based on a Gumbel copula with various parameters:

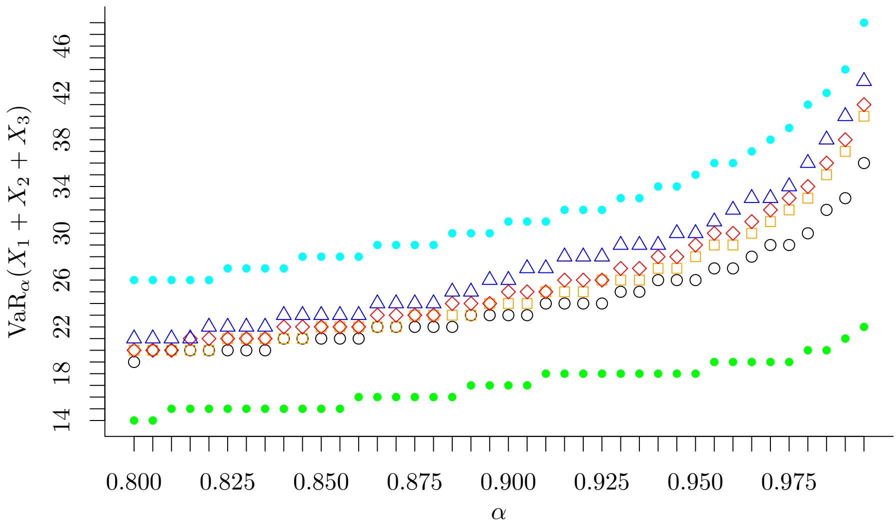

Figure 6.53: $\mathrm{VaR}_\alpha(S)$ for $X_1 \sim \mathrm{NB}(5, 0.9)$, $X_2 \sim \mathrm{NB}(5, 0.7)$ and $X_3 \sim \mathrm{NB}(5, 0.3)$ coupled by a Gumbel copula. Green: lower bound, Black: C_1^{Gu}, Orange: $C_{1.25}^{\mathrm{Gu}}$, Red: $C_{1.5}^{\mathrm{Gu}}$, Blue: C_{10}^{Gu}, Cyan: upper bound

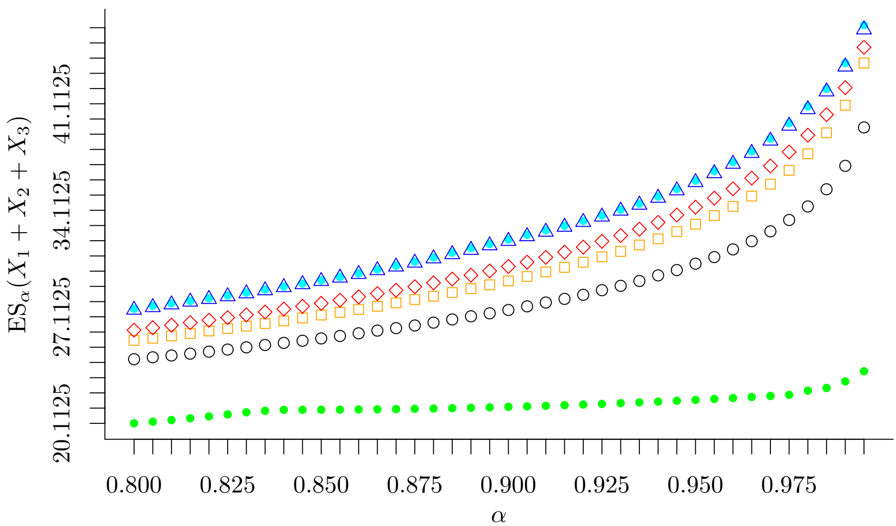

Figure 6.54: $\mathrm{ES}_\alpha(S)$ for $X_1 \sim \mathrm{NB}(5, 0.9)$, $X_2 \sim \mathrm{NB}(5, 0.7)$ and $X_3 \sim \mathrm{NB}(5, 0.3)$ coupled by a Gumbel copula. Green: lower bound, Black: C_1^{Gu}, Orange: $C_{1.25}^{\mathrm{Gu}}$, Red: $C_{1.5}^{\mathrm{Gu}}$, Blue: C_{10}^{Gu}, Cyan: upper bound

The subsequent illustrations are based on a Frank copula with various parameters:

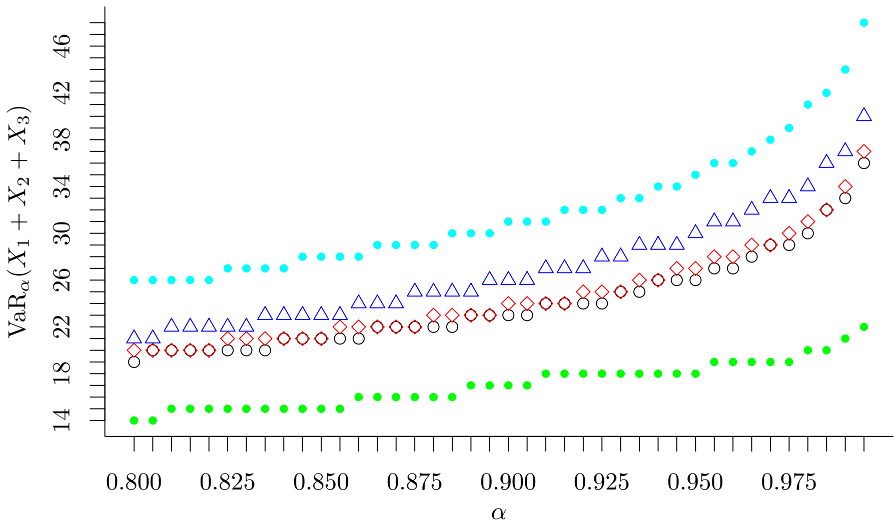

Figure 6.55: $\mathrm{VaR}_\alpha(S)$ for $X_1 \sim \mathrm{NB}(5, 0.9)$, $X_2 \sim \mathrm{NB}(5, 0.7)$ and $X_3 \sim \mathrm{NB}(5, 0.3)$ coupled by a Frank copula. Green: lower bound, Black: $C^{\mathrm{Fr}}_{0.01}$, Red: C^{Fr}_{1}, Blue: C^{Fr}_{10}, Cyan: upper bound

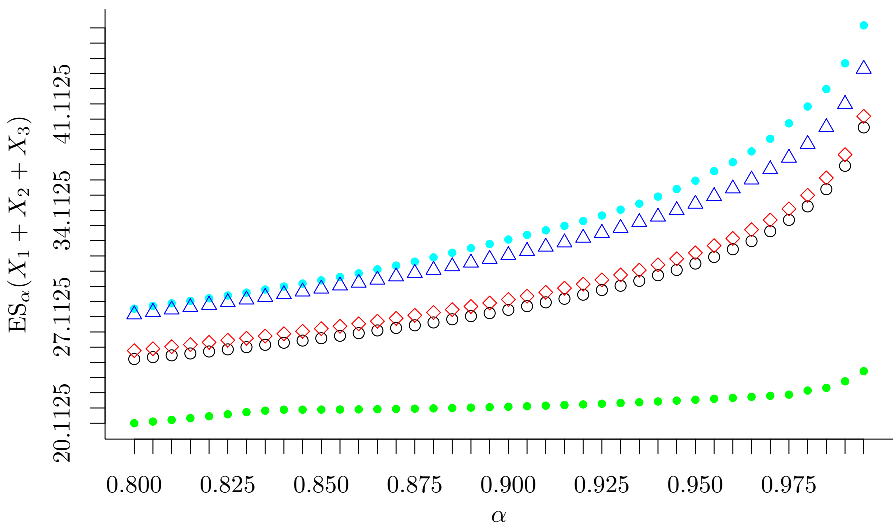

Figure 6.56: $\mathrm{ES}_\alpha(S)$ for $X_1 \sim \mathrm{NB}(5, 0.9)$, $X_2 \sim \mathrm{NB}(5, 0.7)$ and $X_3 \sim \mathrm{NB}(5, 0.3)$ coupled by a Frank copula. Green: lower bound, Black: $C^{\mathrm{Fr}}_{0.01}$, Red: C^{Fr}_{1}, Blue: C^{Fr}_{10}, Cyan: upper bound

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Appendix

Lemma A1. *The function F defined by*

$$\begin{aligned} F : \mathbb{R}^2 &\to [0,1] \\ (x,y) &\mapsto \frac{1}{1+\mathrm{e}^{-x}+\mathrm{e}^{-y}} \end{aligned} \tag{6.1}$$

is a 2-dimensional distribution function.

Proof. To verify that F is a bivariate distribution function, we will validate properties (DF1)–(DF5). At first it is obvious that F is a monotonously increasing function. Furthermore, F as a composition of continuous functions is continuous and therefore right-continuous. It is also generally known that F fulfils properties (DF3) and (DF4). This leaves property (DF5) to prove. For this purpose we observe that F as defined in (6.1) has a bivariate density function

$$f(x,y) = \frac{2\mathrm{e}^{-x}\mathrm{e}^{-y}}{(1+\mathrm{e}^{-x}+\mathrm{e}^{-y})^3}, \quad x,y \in \mathbb{R}.$$

As one easily verifies, $f \geq 0$ holds for all $x,y \in \mathbb{R}$, such that property (DF5) applies. ❑

Lemma A2. *The function W as defined in (2.9) is a copula in dimension $d = 2$.*

Proof. To confirm that W is a copula for $d = 2$ we will verify properties (C1)–(C4). Obviously, W is increasing and for $u_1, u_2 \in [0,1]$ it holds that $W(u_1, 0) = W(0, u_2) = 0$. Additionally, $W(u_1, 1) = u_1$ and $W(1, u_2) = u_2$. For characteristic (C4) take $a = (a_1, a_2), b = (b_1, b_2) \in [0,1]^2$ with $a < b$. We have to show that[101]

$$\begin{aligned} V_W((a,b]) &= W(a_1,a_2) - W(a_1,b_2) - W(b_1,a_2) + W(b_1,b_2) \\ &= (a_1+a_2-1)^+ - \underbrace{(a_1+b_2-1)^+}_{:=A} - \underbrace{(a_2+b_1-1)^+}_{:=B} + (b_1+b_2-1)^+ \\ &\geq 0. \end{aligned} \tag{6.2}$$

[101] $(\cdot)^+ := \max\{\cdot, 0\}$

Evidently, $V_W((a,b]) \geq 0$ if $A = 0$ and $B = 0$. For the other possibilities of A and B we make a case distinction:

- Case 1: $A > 0$ and $B = 0$.
  As $a_i < b_i, i = 1, 2$, formula (6.2) reduces to
  $$(a_1 + a_2 - 1)^+ - (a_1 + b_2 - 1) + (b_1 + b_2 - 1)^+ > (a_1 + a_2 - 1)^+ \geq 0.$$
- Case 2: $A = 0$ and $B > 0$.
  Analogous to case 1.
- Case 3: $A > 0$ and $B > 0$.
  Substituting the given conditions in (6.2) shortens the formula to:
  $$(a_1 + a_2 - 1)^+ - (a_1 + a_2 - 1) = \begin{cases} 0 & \text{for } a_1 + a_2 \geq 1, \\ 1 - (a_1 + a_2) & \text{for } a_1 + a_2 < 1. \end{cases}$$

Thus, we have proved property (C4) and can conclude that W is a copula for $d = 2$. ❑

Below we present an alternative proof for Proposition 4.3:

Alternative proof for Proposition 4.3. Let d be arbitrary in $\mathbb{N}$ such that $d \geq 2$. We will prove that the formula given in (4.2) coincides with (4.8) by mathematical induction on $n \in \mathbb{N}_0$:

Base case: The statement holds for $n = 0$.

It is obvious that both (4.2) and (4.8) reduce to

$$c_0 = C(F_1(0), \ldots, F_d(0))$$

and thus the base induction assertion is proven.

Induction hypothesis (IH): The statement holds for general $n \in \mathbb{N}_0$.

Inductive step: Given that the statement holds for general $n \in \mathbb{N}_0$ it holds for $n + 1$.

We have to show that

$$\sum_{j \in \mathcal{J}^d_{n+1}} \sum_{i \in \mathcal{I}^d} \operatorname{sign}(i)\, C(F_1(j_1 - i_1), \ldots, F_d(j_d - i_d))$$
$$= \sum_{k=0}^{\min\{d-1, n+1\}} (-1)^k \binom{d-1}{k} c_{n+1-k} \tag{6.3}$$

holds. To achieve this, we use the fact that

$$\mathcal{J}_{n+1}^d = \mathcal{J}_n^d \cup \overline{\mathcal{J}}_{n+1}^d$$

and rewrite the left-hand side of the above equation as follows:

$$\begin{aligned}
&\sum_{j \in \mathcal{J}_{n+1}^d} \sum_{i \in \mathcal{I}^d} \operatorname{sign}(i)\, C(F_1(j_1 - i_1), \dots, F_d(j_d - i_d)) \\
&= \sum_{j \in \mathcal{J}_n^d} \sum_{i \in \mathcal{I}^d} \operatorname{sign}(i)\, C(F_1(j_1 - i_1), \dots, F_d(j_d - i_d)) \\
&\quad + \sum_{j \in \overline{\mathcal{J}}_{n+1}^d} \sum_{i \in \mathcal{I}^d} \operatorname{sign}(i)\, C(F_1(j_1 - i_1), \dots, F_d(j_d - i_d)) \\
&\overset{\text{(IH)}}{=} \sum_{k=0}^{\min\{d-1,n\}} (-1)^k \binom{d-1}{k} c_{n-k} \\
&\quad + \sum_{j \in \overline{\mathcal{J}}_{n+1}^d} \sum_{i \in \mathcal{I}^d} \operatorname{sign}(i)\, C(F_1(j_1 - i_1), \dots, F_d(j_d - i_d)).
\end{aligned}$$

Consequently, we can rewrite (6.3) as

$$\begin{aligned}
&\sum_{j \in \overline{\mathcal{J}}_{n+1}^d} \sum_{i \in \mathcal{I}^d} \operatorname{sign}(i)\, C(F_1(j_1 - i_1), \dots, F_d(j_d - i_d)) \\
&= \sum_{k=0}^{\min\{d-1,n+1\}} (-1)^k \binom{d-1}{k} c_{n+1-k} - \sum_{k=0}^{\min\{d-1,n\}} (-1)^k \binom{d-1}{k} c_{n-k}.
\end{aligned}$$

The right-hand side of the above equation can now be rewritten as follows:[102]

$$\begin{aligned}
&\sum_{k=0}^{\min\{d-1,n+1\}} (-1)^k \binom{d-1}{k} c_{n+1-k} - \sum_{k=0}^{\min\{d-1,n\}} (-1)^k \binom{d-1}{k} c_{n-k} \\
&= \sum_{k=0}^{\min\{d-1,n+1\}} (-1)^k \binom{d-1}{k} c_{n+1-k} - \sum_{k=1}^{\min\{d-1,n\}+1} (-1)^{k-1} \binom{d-1}{k-1} c_{n+1-k} \\
&= \sum_{k=0}^{\min\{d-1,n+1\}} (-1)^k \binom{d-1}{k} c_{n+1-k} - \sum_{k=0}^{\min\{d-1,n\}+1} (-1)^{k-1} \binom{d-1}{k-1} c_{n+1-k} \\
&= \sum_{k=0}^{\min\{d-1,n+1\}} (-1)^k \binom{d-1}{k} c_{n+1-k} - \sum_{k=0}^{\min\{d-1,n\}} (-1)^{k-1} \binom{d-1}{k-1} c_{n+1-k} \\
&\quad - (-1)^{\min\{d-1,n\}} \binom{d-1}{\min\{d-1,n\}} c_{n-\min\{d-1,n\}}.
\end{aligned} \tag{6.4}$$

[102] Note that for $n, k \in \mathbb{N}_0$ such that $n < k$ we have that $\binom{n}{k} = 0$.

Without loss of generality we can assume that $d \leq n+1$, as in the opposite case we can still set the upper limits of the outer sums to $d-1$, as the sums are simply extended by 0-terms. As a result, we can transform (6.4) as follows:

$$\begin{aligned}
&= \sum_{k=0}^{\min\{d-1,n+1\}} (-1)^k \binom{d-1}{k} c_{n+1-k} - \sum_{k=0}^{\min\{d-1,n\}} (-1)^{k-1} \binom{d-1}{k-1} c_{n+1-k} \\
&\quad - (-1)^{\min\{d-1,n\}} \binom{d-1}{\min\{d-1,n\}} c_{n-\min\{d-1,n\}} \\
&= \sum_{k=0}^{d-1} (-1)^k \left[\binom{d-1}{k} + \binom{d-1}{k-1} \right] c_{n+1-k} + (-1)^d c_{n+1-d} \\
&= \sum_{k=0}^{d} (-1)^k \binom{d}{k} c_{n+1-k}.
\end{aligned}$$

It remains to be shown that

$$\begin{aligned}
&\sum_{j \in \bar{\mathcal{J}}_{n+1}^d} \sum_{i \in \mathcal{I}^d} \operatorname{sign}(i)\, C(F_1(j_1 - i_1), \dots, F_d(j_d - i_d)) \\
&= \sum_{k=0}^{d} (-1)^k \binom{d}{k} c_{n+1-k}
\end{aligned}$$

holds. For this purpose we define

$$\bar{\mathcal{I}}_l^d = \{ i = (i_1, \dots, i_d) \in \mathcal{I}^d : i_1 + \dots + i_d = l \}, \quad l \in \mathbb{N}_0,\ l \leq d.$$

Then we have that

$$\begin{aligned}
&\sum_{j \in \bar{\mathcal{J}}_{n+1}^d} \sum_{i \in \mathcal{I}^d} \operatorname{sign}(i)\, C(F_1(j_1 - i_1), \dots, F_d(j_d - i_d)) \\
&= \sum_{k=0}^{d} (-1)^k \sum_{i \in \bar{\mathcal{I}}_k^d} \sum_{j \in \bar{\mathcal{J}}_{n+1}^d} C(F_1(j_1 - i_1), \dots, F_d(j_d - i_d)) \\
&= \sum_{k=0}^{d} (-1)^k \binom{d}{k} c_{n+1-k},
\end{aligned}$$

which proves the induction claim. ❑

Below we present an alternative proof for Theorem 4.8:

Alternative proof for Theorem 4.8. Assuming that (4.11) holds, the following applies for $l \in \mathbb{N}_0,\ l \leq n$:

$$c_{n-l} = p_{n-l} + \sum_{k=1}^{n-l} \binom{k+d-1}{d-1} p_{n-l-k}$$

$$
\begin{aligned}
&= \sum_{k=0}^{n-l} \binom{k+d-1}{d-1} p_{n-l-k} \\
&= \sum_{k=l}^{n} \binom{k-l+d-1}{d-1} p_{n-k} \\
&= \sum_{k=0}^{n} \binom{k-l+d-1}{d-1} p_{n-k},
\end{aligned} \tag{6.5}
$$

where the last step can be justified by the fact that the sum is extended by 0, since the binomial coefficient is 0 for $k < l$. This allows us to observe further:

$$
\begin{aligned}
c_{n-l} - c_{n-1-l} &= \sum_{k=0}^{n} \binom{k-l+d-1}{d-1} p_{n-k} - \sum_{k=0}^{n-1} \binom{k-l+d-1}{d-1} p_{n-1-k} \\
&= \sum_{k=0}^{n} \binom{k-l+d-1}{d-1} p_{n-k} - \sum_{k=1}^{n} \binom{k-l+d-2}{d-1} p_{n-k} \\
&= \sum_{k=0}^{n} \binom{k-l+d-1}{d-1} p_{n-k} - \sum_{k=0}^{n} \binom{k-l+d-2}{d-1} p_{n-k} \\
&= \sum_{k=0}^{n} \binom{k-l+d-2}{d-2} p_{n-k}.
\end{aligned} \tag{6.6}
$$

Given that (4.11) holds, with the results from (6.5) and (6.6) we get the following:

$$
\begin{aligned}
p_n &= c_n - \sum_{k=1}^{n} \binom{k+d-1}{d-1} p_{n-k} \\
&= c_n - \sum_{k=1}^{n} \binom{k+d-1}{d-1} p_{n-k} - \underbrace{\left(c_{n-1} - p_{n-1} - \sum_{k=1}^{n-1} \binom{k+d-1}{d-1} p_{n-1-k} \right)}_{=0} \\
&= c_n - c_{n-1} - \sum_{k=1}^{n} \binom{k+d-1}{d-1} p_{n-k} + p_{n-1} + \sum_{k=2}^{n} \binom{k+d-2}{d-1} p_{n-k} \\
&= c_n - c_{n-1} - \sum_{k=1}^{n} \binom{k+d-1}{d-1} p_{n-k} + \sum_{k=1}^{n} \binom{k+d-2}{d-1} p_{n-k} \\
&= c_n - c_{n-1} - \sum_{k=1}^{n} \binom{k+d-2}{d-2} p_{n-k} \\
&= c_n - c_{n-1} - \sum_{k=1}^{n} \binom{k+d-2}{d-2} p_{n-k} \\
&\quad + \sum_{l=1}^{d-2} (-1)^l \binom{d-2}{l} \underbrace{\left(c_{n-l} - c_{n-1-l} - \sum_{k=1}^{n} \binom{k-l+d-2}{d-2} p_{n-k} \right)}_{=0}
\end{aligned}
$$

$$
\begin{aligned}
&= c_n - c_{n-1} + \sum_{l=1}^{d-2}(-1)^l \binom{d-2}{l}(c_{n-l} - c_{n-1-l}) \\
&\quad - \sum_{k=1}^{n}\binom{k+d-2}{d-2}p_{n-k} - \sum_{l=1}^{d-2}(-1)^l\binom{d-2}{l}\sum_{k=1}^{n}\binom{k-l+d-2}{d-2}p_{n-k} \\
&= \sum_{l=0}^{d-2}(-1)^l\binom{d-2}{l}(c_{n-l} - c_{n-1-l}) \\
&\quad - \sum_{k=1}^{n}\left[\sum_{l=0}^{d-2}(-1)^l\binom{d-2}{l}\binom{k-l+d-2}{d-2}\right]p_{n-k}.
\end{aligned}
$$

We will now prove that for any $k, d \in \mathbb{N},\ d \geq 2$, the following applies:

$$
\sum_{l=0}^{d-2}(-1)^l\binom{d-2}{l}\binom{k-l+d-2}{d-2} = 1.
$$

We have that[103]

$$
\begin{aligned}
&\sum_{l=0}^{d-2}(-1)^l\binom{d-2}{l}\binom{k-l+d-2}{d-2} \\
&= \sum_{l=0}^{d-2}(-1)^l\binom{d-2}{l}[x^{d-2}](1+x)^{k-l+d-2} \\
&= [x^{d-2}](1+x)^k\sum_{l=0}^{d-2}(-1)^l\binom{d-2}{l}(1+x)^{d-2-l} \\
&= [x^{d-2}](1+x)^k x^{d-2} \\
&= 1.
\end{aligned}
$$

Therefore it holds that:

$$
\begin{aligned}
&= \sum_{l=0}^{d-2}(-1)^l\binom{d-2}{l}(c_{n-l} - c_{n-1-l}) \\
&\quad - \sum_{k=1}^{n}\left[\sum_{l=0}^{d-2}(-1)^l\binom{d-2}{l}\binom{k-l+d-2}{d-2}\right]p_{n-k} \\
&= \sum_{l=0}^{d-2}(-1)^l\binom{d-2}{l}c_{n-l} - \sum_{l=0}^{d-2}(-1)^l\binom{d-2}{l}c_{n-1-l} - \sum_{k=1}^{n}p_{n-k} \\
&= \sum_{l=0}^{d-2}(-1)^l\binom{d-2}{l}c_{n-l} - \sum_{l=1}^{d-1}(-1)^l\binom{d-2}{l-1}c_{n-l} - \sum_{k=1}^{n}p_{n-k} \\
&= \sum_{l=0}^{d-2}(-1)^l\left[\binom{d-2}{l} + \binom{d-2}{l-1}\right]c_{n-l} + (-1)^{d-1}c_{n-d+1} - \sum_{k=1}^{n}p_{n-k}
\end{aligned}
$$

[103] Here, $[x^n]$ denotes the reading of the coefficient of x^n.

$$= \sum_{l=0}^{d-1} (-1)^l \binom{d-1}{l} c_{n-l} - \sum_{k=1}^{n} p_{n-k}.$$

To summarize, we have that

$$\begin{aligned} p_n &= \sum_{l=0}^{d-1} (-1)^l \binom{d-1}{l} c_{n-l} - \sum_{k=1}^{n} p_{n-k} \\ \iff \sum_{k=0}^{n} p_{n-k} &= \sum_{l=0}^{d-1} (-1)^l \binom{d-1}{l} c_{n-l} \\ &= \sum_{l=0}^{\min\{d-1,n\}} (-1)^l \binom{d-1}{l} c_{n-l}, \end{aligned}$$

which by Proposition 4.3 is a true statement and thus completes the proof. ❑

R: Calculation of the Distribution of S

```
##     Calculation of the distribution function of S as in (4.8)
##     with given copula C and univariate margins F_1,...,F_d.
##
##     Required input:
##          o) Dimension d ∈ ℕ, d ≥ 2
##          o) n ∈ ℕ_0
##          o) Copula C
##          o) List of univariate marginal distributions
##             F_1,...,F_d
##
##     Output: ℙ[S ≤ n]
##     ------------------------------------------------------------------

helper.evalCopula=function(j,copula,margins) {
     valMargins=numeric(length(j))
     for (i in 1:length(valMargins)) {
          valMargins[i]=margins[[i]](j[i])
     }
     return(copula(valMargins))
}

helper.setNextCombination=function(j,target) {
     for (i in 1:length(j)) {
          j[i]=j[i]+1
          if (j[i]<=target&&sum(j)<=target) {
               break
          } else {
               j[i]=0
          }
     }
     return(j)
}

S.distribution=function(d,n,copula,margins) {
     if (n<0) {
          return(0)
     }
     n=floor(n)
     j=numeric(d)
     result=0
     for (k in 0:min(d-1,n)) {
          j=numeric(d)
          if (sum(j)==n-k) {
               result=result+((-1)^k)*choose(d-1,k)*helper.
                    evalCopula(j,copula,margins)
          }
          repeat {
               j=helper.setNextCombination(j,n-k)
               if (sum(j)==0) {
                    break
               } else if (sum(j)==n-k) {
                    result=result+((-1)^k)*choose(d-1,k)*helper.
                         evalCopula(j,copula,margins)
               }
          }
     }
     return(result)
}
```

Program R1: Calculation of the distribution of S as in (4.8)

R: Calculation of the Probability Mass Function of S as in (4.12)

```
##     Calculation of the probability mass function of S as
##     in (4.12) with given copula C and univariate margins
##     F_1,...,F_d.
##
##     Required input:
##          o) Dimension d ∈ ℕ, d ≥ 2
##          o) n ∈ ℕ_0
##          o) Copula C
##          o) List of univariate marginal distributions
##             F_1,...,F_d
##
##     Output: ℙ[S = n]
##     ----------------------------------------------------------------

helper.evalCopula=function(j,copula,margins) {
     valMargins=numeric(length(j))
     for (i in 1:length(valMargins)) {
          valMargins[i]=margins[[i]](j[i])
     }
     return(copula(valMargins))
}

helper.setNextCombination=function(j,target) {
     for (i in 1:length(j)) {
          j[i]=j[i]+1
          if (j[i]<=target&&sum(j)<=target) {
               break
          } else {
               j[i]=0
          }
     }
     return(j)
}

S.probabilityMassFunction=function(d,n,copula,margins) {
     if (n<0) {return(0)}
     j=numeric(d)
     result=0
     for (k in 0:min(d,n)) {
          j=numeric(d)
          if (sum(j)==n-k) {
               result=result+((-1)^k)*choose(d,k)*helper.evalCopula(j,copula,margins)
          }
          repeat {
               j=helper.setNextCombination(j,n-k)
               if (sum(j)==0) {
                    break
               } else if (sum(j)==n-k) {
                    result=result+((-1)^k)*choose(d,k)*helper.evalCopula(j,copula,margins)
               }
          }
     }
     return(result)
}
```

Program R2: Calculation of the probability mass function of S as in (4.12)

R: Recursion for the Probability Mass Function of S

```
##     Recursion for the calculation of the probability mass
##     function of S with given copula C and univariate
##     discrete marginal distributions F_1,...,F_d.
##
##     Required input:
##          o) Dimension d ∈ ℕ, d ≥ 2
##          o) n ∈ ℕ_0
##          o) Copula C
##          o) List of univariate marginal distributions
##             F_1,...,F_d
##          o) Support of the univariate marginals
##
##     Output: ℙ[S = n]
##     -----------------------------------------------------------------

helper.setNextIndex=function(ind,maxValues) {
    for (i in 1:length(ind)) {
        ind[i]=ind[i]+1
        if (ind[i]<=maxValues[i]) {
            break
        } else {
            ind[i]=1
        }
    }
    return(ind)
}

helper.calcCombinations=function(d,n,support) {
    j=numeric(d)
    ind=rep(1,d)
    number=1
    maxValues=numeric(d)
    combinations=list()
    for (i in 1:d) {
        maxValues[i]=length(support[[i]])
    }
    repeat {
        for (i in 1:d) {
            j[i]=support[[i]][ind[i]]
        }
        if (sum(j)<=n) {
            combinations[[number]]=c(sum(j),j)
            number=number+1
        }
        ind=helper.setNextIndex(ind,maxValues)
        if (sum(ind)==d) {
            break
        }
    }
    return(combinations)
}

S.probabilityMassFunctionRecursion=function(d,n,copula,margins,
    support,combinations=list(),firstRun=TRUE) {
    result=NULL
    valMargins=numeric(d)
    combinationsN = list()
    ind=1
```

```
    if(firstRun==TRUE) {
        combinations=helper.calcCombinations(d,n,support)
    }
    uniqueComb=NULL
    for (i in 1:length(combinations)) {
        uniqueComb[i]=combinations[[i]][1]
    }
    if (sum(uniqueComb==n)==0) {
        return(0)
    }
    if (n==0) {
        for (i in 1:d) {
            valMargins[i]=margins[[i]](0)
        }
        result=copula(valMargins)
        return(result)
    } else {
        for (i in 1:length(combinations)) {
            if(combinations[[i]][1]==n) {
                combinationsN[[ind]]=combinations[[i]][2:(d+1)]
                ind=ind+1
            }
        }
        if (length(combinationsN)==0) {
            return(0)
        } else {
            tmpCopula=0
            tmpSum=0
            for (i in 1:length(combinationsN)) {
                for (k in 1:d) {
                    valMargins[k]=margins[[k]](combinationsN[[i
                        ]][k])
                }
                tmpCopula=tmpCopula+copula(valMargins)
            }
            for (k in 1:n) {
                tmpSum=tmpSum+choose(k+d-1,d-1)*S.
                    probabilityMassFunctionRecursion(d,n-k,
                    copula,margins,support,combinations,FALSE)
            }
            result=tmpCopula-tmpSum
            return(result)
        }
    }
}
```

Program R3: Recursion for the probability mass function of S as in (4.11)

R: Calculation of the Probability Mass Function of S by Integration over Copula Densities

```
##     Calculation of the probability mass function of $S$ by
##     integration of the copula density as described in
##     Section 4.5.
##
##     Required input:
##          o) Dimension $d \in \mathbb{N}$, $d \geq 2$
##          o) $n \in \mathbb{N}_0$
##          o) Copula density $c$
##          o) List of univariate marginal distributions
##             $F_1, \ldots, F_d$
##          o) Support of the univariate marginals
##
##     Output: $\mathbb{P}[S = n]$
##     -----------------------------------------------------------------

##     ▷ "cubature" is a R-package for multidimensional
##     integration over hypercubes, c.f. [2].

library("cubature")

helper.setNextIndex=function(ind,maxValues) {
    for (i in 1:length(ind)) {
        ind[i]=ind[i]+1
        if (ind[i]<=maxValues[i]) {
            break
        } else {
            ind[i]=1
        }
    }
    return(ind)
}

helper.calcCombinations=function(d,n,support) {
    j=numeric(d)
    ind=rep(1,d)
    number=1
    maxValues=numeric(d)
    combinations=list()
    for (i in 1:d) {
        maxValues[i]=length(support[[i]])
    }
    repeat {
        for (i in 1:d) {
            j[i]=support[[i]][ind[i]]
        }
        if (sum(j)==n) {
            combinations[[number]]=j
            number=number+1
        }
        ind=helper.setNextIndex(ind,maxValues)
        if (sum(ind)==d) {
            break
        }
    }
    return(combinations)
}

```

```
S.probabilityMassFunctionIntegration=function(d,n,copulaDensity,
    margins,support) {
    J=helper.calcCombinations(d,n,support)
    result=0
    for (j in J) {
        limUpper=NULL
        limLower=NULL
        for (i in 1:length(j)) {
            limUpper[i]=margins[[i]](j[i])
            limLower[i]=margins[[i]](j[i]-1)
        }
        int=adaptIntegrate(copulaDensity,lowerLimit=limLower,
            upperLimit=limUpper)
        result=result+int$integral
    }
    return(result)
}
```

Program R4: Calculation of the probability mass function of S by integration over copula densities as described in Section 4.5

R: Rearrangement Algorithm for the Calculation of Sharp Bounds on the Distribution of S

```
##     Rearrangement Algorithm for the calculation of pointwise
##     sharp bounds on the distribution function of S as
##     described in Section 4.3.
##
##     ▷ For an R-Package and additional information about the
##     Rearrangement Algorithm in R we refer interested readers
##     to [14].
##
##     Required input:
##          o) Dimension d ∈ ℕ, d ≥ 2
##          o) n ∈ ℕ_0
##          o) N ∈ ℕ
##          o) Tolerance ϵ > 0
##          o) List of univariate quantile functions
##             F_1^←,...,F_d^←
##
##     Output: lower and upper bound on ℙ[S ≤ n]
##     ------------------------------------------------------------------

helper.rearrangeMatrix=function(d,M,tolerance,func) {
    M=apply(M,2,sample)
    absDiff=Inf
    result=Inf
    while(absDiff>tolerance) {
        tmp=result
        for (j in 1:d) {
            rankBy=rowSums(M[,(1:d)[-j]])
            M[,j]=sort(M[,j],decreasing=TRUE)[rank(rankBy)]
        }
        result=func(rowSums(M))
        absDiff=abs(result-tmp)
    }
    return(result)
}

S.upperBound=function(d,n,quantiles,tolerance,N) {
    left=0
    right=1
    repeat {
        result=(left+right)/2
        X=matrix(ncol=d,nrow=N)
        for (i in 1:N) {
            for (j in 1:d) {
                X[i,j]=quantiles[[j]](result*(i-1)/N)
            }
        }
        nApprox=helper.rearrangeMatrix(d,X,tolerance,max)
        if (abs(n+1-nApprox)<tolerance||abs(left-right)<
            tolerance) {
            break
        }
        if (nApprox>n+1) {
            left=result*2-right
            right=result
        } else {
            right=result*2-left
            left=result
        }
```

```
    }
    return(result)
}

S.lowerBound=function(d,n,quantiles,tolerance,N) {
    left=0
    right=1
    repeat {
        result=(left+right)/2
        X = matrix(ncol=d,nrow=N)
        for (i in 1:N) {
            for (j in 1:d) {
                X[i,j]=quantiles[[j]](result+(1-result)*(i-1)/N)
            }
        }
        nApprox=helper.rearrangeMatrix(d,X,tolerance,min)
        if (abs(n+1-nApprox)<tolerance||abs(left-right)<
            tolerance) {
            break
        }
        if (nApprox<n+1) {
            right=result*2-left
            left=result
        } else {
            left=result*2-right
            right=result
        }
    }
    return(result)
}
```

Program R5: Rearrangement Algorithm for the calculation of sharp bounds on the distribution of S as described in Section 4.3

R: Calculation of Sharp Bounds on Value-at-Risk and Expected Shortfall

```
##      Rearrangement Algorithm for the calculation of sharp
##      bounds on risk measures $\mathrm{VaR}_\alpha(S)$ and $\mathrm{ES}_\alpha(S)$.
##
##      ▷ For an R-Package and additional information about the
##      Rearrangement Algorithm in R we refer interested readers
##      to [14].
##
##      Required input:
##           o) Dimension $d \in \mathbb{N},\ d \geq 2$
##           o) $\alpha \in (0,1)$
##           o) $N \in \mathbb{N}$
##           o) Tolerance $\epsilon > 0$
##           o) List of univariate quantile functions
##              $F_1^{\leftarrow},\ldots,F_d^{\leftarrow}$
##
##      Output: lower and upper bound on $\mathrm{VaR}_\alpha(S)$ and $\mathrm{ES}_\alpha(S)$
##      ----------------------------------------------------------------

VaR.lowerBound=function(quantiles,tolerance,alpha,N,d) {
     iterations=1e2
     X=matrix(0,nrow=N,ncol=d)
     for (j in 1:d) {
          for (i in 1:N) {
               X[i,j]=quantiles[[j]]((alpha*(i-1))/N)
          }
     }
     X=apply(X,2,sample)
     VaR_L=-Inf
     for (i in 1:iterations) {
          for (j in 1:d) {
               rankBy=rowSums(X[,(1:d)[-j]])
               X[,j]=sort(X[,j],decreasing=TRUE)[rank(rankBy)]
          }
          tmp=max(rowSums(X))
          if (tmp>VaR_L) {
               VaR_L=tmp
          }
     }
     return(VaR_L)
}

VaR.upperBound=function(quantiles,tolerance,alpha,N,d) {
     iterations=1e2
     X=matrix(0,nrow=N,ncol=d)
     for (j in 1:d) {
          for (i in 1:N) {
               X[i,j]=quantiles[[j]]((alpha)+((1-alpha)*i)/N)
          }
     }
     X=apply(X,2,sample)
     VaR_U=Inf
     for (i in 1:iterations) {
          for (j in 1:d) {
               rankBy=rowSums(X[,(1:d)[-j]])
               X[,j]=sort(X[,j],decreasing=TRUE)[rank(rankBy)]
          }
          tmp=min(rowSums(X))
          if (tmp<VaR_U) {
```

```
            VaR_U=tmp
        }
    }
    return(VaR_U)
}

ES.lowerBound=function(quantiles,tolerance,alpha,N,d) {
    iterations=1e2
    X=matrix(0,nrow=N,ncol=d)
    for (j in 1:d) {
        for (i in 1:N) {
            X[i,j]=quantiles[[j]]((i-1)/N)
        }
    }
    X=apply(X,2,sample)
    ES_L=-Inf
    for (i in 1:iterations) {
        for (j in 1:d) {
            rankBy=rowSums(X[,(1:d)[-j]])
            X[,j]=sort(X[,j],decreasing=TRUE)[rank(rankBy)]
        }
        Y=sort(rowSums(X))
        tmp=sum(Y[(floor(N*alpha)+1):N])/(N*(1-alpha))
        if (tmp>ES_L) {
            ES_L=tmp
        }
    }
    return(ES_L)
}

ES.upperBound=function(quantiles,alpha,N,d) {
    helper.univariateES=function(qDist,N,alpha) {
        X=matrix(0,nrow=N)
        for (i in 1:N) {
            X[i]=qDist(alpha+((1-alpha)*(i-1))/N)
        }
        return(sum(X)/N)
    }
    tmp=numeric(d)
    for (j in 1:d) {
        tmp[j]=helper.univariateES(quantiles[[j]],N,alpha)
    }
    ES_U=sum(tmp)
    return(ES_U)
}
```

Program R6: Adapted Rearrangement Algorithm for the calculation of sharp bounds on risk measures $\mathrm{VaR}_\alpha(S)$ and $\mathrm{ES}_\alpha(S)$ as described in Chapter 5

Declaration of Authorship

I, Martin Schmidt, hereby declare that I have authored the present master thesis independently and did not use any sources other than those specified. I have not yet submitted the work to any other examining authority in the same or comparable form. It has not been published yet.

Vienna, 27.02.2019

Martin Schmidt