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Discrete curvature and rigidity of Fuchsian manifolds

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Abstract

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Discrete curvature and rigidity of Fuchsian manifolds

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This thesis is devoted to some applications of cone-manifolds and discrete curvature to problems in 3-dimensional hyperbolic geometry.

First, we prove a realization and rigidity result for a specific family of hyperbolic cone-3-manifolds. This allows us to give a new variational proof of the existence and uniqueness of a hyperbolic cone-metric on S_g with prescribed curvature in a given discrete conformal class. Here S_g is a closed orientable surface of genus $g > 1$. This also provides a new proof of the fact that every hyperbolic cusp-metric on S_g can be uniquely realized as a convex surface in a Fucsian manifold.

A *Fuchsian manifold* is a hyperbolic manifold homeomorphic to $S_g \times [0; +\infty)$ with geodesic boundary $S_g \times \{0\}$. They are known as toy cases for studying geometry of non-compact hyperbolic 3-manifolds and hyperbolic 3-manifolds with boundary.

Second, we consider *compact Fuchsian manifolds with boundary*, i.e., hyperbolic manifolds homeomorphic to $S_g \times [0; 1]$ with geodesic boundary $S_g \times \{0\}$. We use cone-manifolds to prove that a compact Fuchsian manifold with convex boundary is uniquely determined by the induced metric on $S_g \times \{1\}$. It is distinguishing that except convexity we do not put any other condition on the boundary, so it may be neither smooth nor polyhedral.

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To my family

Chapter 1

Introduction

1.1 Realization and rigidity problems in Euclidean 3-space

The first source of our inspiration is

Problem 1.1.1 (The Weyl problem)**.** Let g be a Riemannian metric on S^2 with positive Gaussian curvature. Does there exist a convex body $G \subset \mathbb{E}^3$ with smooth boundary isometric to (S^2, g)? If G^1 and G^2 are two such bodies, does it follow that they are ambiently isometric?

By a *convex body* in a 3-dimensional space we mean a compact convex set with non-empty interior. A smooth convex body naturally has the induced Riemannian metric on its boundary. By *ambiently isometric* we mean that there exists an isometry of the ambient space (here $\mathbb{E}^3$) mapping G^1 to G^2.

The Weyl problem is a particular case of a general framework for questions in geometry: how can one reconstruct a geometric shape based on some its characteristics. In the case of the Weyl problem the characteristic is the induced boundary metric. The study of surfaces originated from studying surfaces embedded in space. Since an intrinsic definition of a Riemannian manifold was introduced, it appeared very natural to ask, which intrinsically defined metric structures can be isometrically embedded into the space. Problems of this type are also called *realization problems*. A subsequent question to ask is if the induced metric determines the shape completely. Such problems are also called *rigidity problems*.

The Weyl problem asks specifically about realizations of metrics in the class of convex surfaces, i.e., the boundaries of convex bodies. A smooth convex surface has a non-negative Gaussian curvature everywhere. Convexity is a nice regularity assumption and not much is known if one discards it. Global rigidity does not hold without convexity. However, an important open question is the *flexibility problem* asking if there exists a closed smooth surface in $\mathbb{E}^3$ that can be continuously isometrically deformed in the class of smooth surfaces. Such a surface does not exist among the convex ones, but the general question is still widely open.

A program to resolve the Weyl problem with the help of non-linear PDEs was initiated by Weyl himself [114]. In [68] Lewy solved the existence part for analytic metrics. In [79] Nirenberg proposed a solution for C^4-smooth class. It was extended to C^3-smooth class by Heinz [49]. The uniqueness part was resolved by Cohn-Vossen [28] in the analytic class and the solution was simplified by Zhitomirsky [115]. It was improved to C^3-smooth class by Herglotz [50] and to C^2-smooth by Sacksteder [96]. Guang–Li [47] and Hong–Zuily [55] independently extended the Weyl problem to non-negative curvature.

One can point out that geometry is not limited to smooth shapes. It is natural to look for an analogue of the Weyl problem for non-smooth bodies. A Riemannian metric is not exactly a metric, but a metric tensor g. It determines an actual metric d, which is a *path metric*, i.e., the distance between two points is equal to the infimum of lengths of all curves connecting them (if d is complete, then the Arzela-Ascoli theorem says that there exists a shortest path between any two points). We define in this fashion the induced path metric on general surfaces in $\mathbb{E}^3$. One can quickly discover that there are some pathological examples of surfaces (e.g., similar to the Koch snowflake) where the induced path metric does not behave well. However, the induced path metric still behaves nicely if we continues to restrict ourselves to convex surfaces. Thereby, the following natural generalization arises:

Problem 1.1.2 (The generalized Weyl problem)**.** Let d be a metric on the 2-sphere S^2. What are the conditions on d such that (S^2, d) is isometric to the boundary of a convex body $G \subset \mathbb{E}^3$ endowed with the induced path metric? If G^1 and G^2 are two such bodies, does it follow that they are ambiently isometric?

First obvious condition is that d is *intrinsic*, i.e., it coincides with the path metric induced by d. Next, one needs to come up with an analogue of the curvature bound. This was done by Alexandrov. First, he stated and resolved the polyhedral case of the problem [5], [6]. To describe it we need

Definition 1.1.3. A *Euclidean cone-metric* d on a surface S is locally isometric to the Euclidean plane $\mathbb{E}^2$ except finitely many points called *conical points*. At a conical point v the metric d is locally isometric to a cone with angle $\lambda_v(d) \neq 2\pi$. The number $\nu_v(d) := 2\pi - \lambda_v(d)$ is called the *curvature* of v. We denote the set of conical points of d by $V(d)$. A cone-metric is called *convex* if for every $v \in V(d)$, we have $\nu_v(d) > 0$. It is called *concave* if for every $v \in V(d)$, we have $\nu_v(d) < 0$.

Theorem 1.1.4. For every convex Euclidean cone-metric d on S^2 there exists a convex polyhedron $P \subset \mathbb{E}^3$ (possibly degenerate) such that (S^2, d) is isometric to the boundary of P. Moreover, P is unique up to ambient isometry.

To prove it Alexandrov introduced a beautiful continuity method. First, prove the uniqueness part, which follows from a refinement of Cauchy's global rigidity theorem for compact convex polyhedra. Next consider the set $\mathfrak{D}_n$ of Euclidean cone-metrics on S^2 with n conical points (up to some equivalence that we omit here). Let $\mathfrak{P}_n$ be the set of compact convex polyhedra P in $\mathbb{E}^3$ with n vertices. The induced metric on the boundary of P is in $\mathfrak{D}_n$. This defines the realization map $\mathfrak{I} : \mathfrak{P}_n \to \mathfrak{D}_n$, which is a local homeomorphism with respect to the natural topologies on $\mathfrak{P}_n$, $\mathfrak{D}_n$ due to the rigidity statement (we omit some details needed to make this rigorous). One can show that this map is proper and $\mathfrak{D}_n$ is connected and simply connected. Thus, an elementary topological argument shows that $\mathfrak{I}$ is a homeomorphism. This proves Theorem 1.1.4.

Despite deep beauty and simplicity of this argument it has one important disadvantage: it is purely non-constructive. A Euclidean cone-metric can be described via a gluing pattern of triangles. Let $\mathcal{T}$ be a geodesic triangulation of (S^2, d) with vertices $V(\mathcal{T}) = V(d)$. It allows to define d in a combinatorial way: we have $2n - 4$ oriented Euclidean triangles with edges identified in pairs. If d is convex, then (S^2, d) has a polyhedral realization P. But generally $\mathcal{T}$ has nothing to do with the face decomposition of P. There can be plenty of geodesic triangulations of (S^2, d) and generically only one of them is preferred as the face triangulation of P. It is a deep question how to find this triangulation and construct subsequently the polyhedral realization starting from an arbitrary geodesic triangulation. The Alexandrov approach leaves this problem totally uncovered. A first attempt to bridge this gap was undertaken by Volkov, a student of Alexandrov, in his thesis using a variational approach. It was published in [111]. It seems that Volkov's thesis was the first ever place where a discrete analogue of the curvature of cone-manifolds was considered, which plays an important role in our work. Another variational proof was given in [14] by Bobenko and Izmestiev with a computer realization performed by Sechelmann [102]. Izmestiev [57] applied this method to the case of a *convex cap*, which provided a nice simple introduction to the approach. When a face triangulation of P is given, one may also study its realization with the help of polynomial invariants. We refer to [31] for this viewpoint.

As a next steep, Alexandrov [5], [7] found a necessary and sufficient condition for the induced path metric on the boundary of a 3-dimensional convex body. There are plenty of equivalent formulations and we use the one via angles. Let S be a connected surface with a complete intrinsic metric d and ψ, χ be two shortest paths emanating from a point $p \in (S, d)$. Further, let $q \in \psi$ be a point at distance x from p and $r \in \chi$ be the point at distance y from p. Consider the Euclidean triangle with the sides x, y, $d(q, r)$. Let $\lambda(x, y)$ be the angle of this triangle opposite to the side of length $d(q, r)$.

Definition 1.1.5. We say that (S, d) has *curvature bounded from below by 0 in the sense of Alexandrov* (or (S, d) is a *CBB(0) metric space* for short), if d is complete, intrinsic and for each $p \in S$ there exists a neighbourhood $U \ni p$ such that the function $\lambda(x, y)$ is a nonincreasing function of x and y for every ψ, χ emanating from p, in the range $x \in [0; x_0]$, $y \in [0; y_0]$ where the respective points q, r belong to U.

Theorem 1.1.6. For every CBB(0) metric d on S^2 there exists a convex body $G \subset \mathbb{E}^3$ such that (S^2, d) is isometric to the boundary of G.

The idea of the proof is to approximate (S^2, d) by Euclidean convex cone-metrics, use Theorem 1.1.4 and investigate what happens in the limit. Finally, in a series of works (see, e.g., [87]) culminated in the book [88] Pogorelov proved the rigidity theorem for general convex bodies:

Theorem 1.1.7. Let G^1 and G^2 be convex bodies in $\mathbb{E}^3$ and $f : \partial G^1 \to \partial G^2$ be an isometry. Then f extends to an ambient isometry.

This resolved completely the generalized Weyl problem in $\mathbb{E}^3$. Moreover, Pogorelov proved [88] also *the regularity theorem* stating that the realization of a C^k-smooth Riemannian metric is C^{k-1}-smooth. This allows to obtain a resolution of the Weyl problem as a particular case.

Theorem 1.1.7 if of the greatest interest to us because a large part of this thesis is devoted to its generalization, which will be described further. Contrary to the smooth case, where the realization problem appears to be quite harder than the rigidity, known proofs of Theorem 1.1.7 are based on much more complicated ideas rather than its existence companion Theorem 1.1.6. Pogorelov himself proposed two approaches to Theorem 1.1.7, both are quite lengthy and intricate. In both of them the starting point is that the global rigidity can be reduced to the local one: if there exist two non-congruent convex bodies G^1 and G^2 with isometric boundaries, then for every $\varepsilon > 0$ there exists a convex body G^ε that is ε-close in the Hausdorff sense to G^1, is not congruent to G^1, but has isometric boundary to ∂G^1. In the first approach Pogorelov replaces G^2 with G^ε and constructs on their boundaries a pair of f-isometric simple closed curves bounding regions that need to be intrinsically isometric, but have non-equal generalized Gaussian curvature. This can not happen because Alexandrov proved that the generalized Gaussian curvature is intrinsic (this is a generalization of the Gauss-Bonnet theorem to non-smooth surfaces). Another solution to Theorem 1.1.7 can be concluded from the infinitesimal rigidity of convex surfaces, which was proved in the non-regular case also by Pogorelov. Here the family of bodies G^ε can be used to construct a vector field on G^1 providing a non-trivial infinitesimal deformation of G^1 (one should be especially careful with flat parts of ∂G^1 where an infinitesimal deformation could be non-trivial).

It is worth to mention that both Alexandrov and Pogorelov provided plenty of generalizations of their results to convex surfaces with boundaries as well as to non-compact ones.

It is natural to investigate the quantitative stability of Theorem 1.1.7. In other words, one can ask: if the boundaries of G^1, G^2 are sufficiently close in the intrinsic sense, how close are G^1, G^2 themselves?

Definition 1.1.8. Let $f : (M^1, d^1) \to (M^2, d^2)$ be a homeomorphism between metric spaces. It is called an ε-isometry if for any $p, q \in (M^1, d^1)$

$$|d^1(p, q) - d^2(f(p), f(q))| \leq \varepsilon.$$

Problem 1.1.9 (The Cohn-Vossen problem)**.** Let G^1, G^2 be convex bodies in $\mathbb{E}^3$, $\varepsilon > 0$ and $f : \partial G^1 \to \partial G^2$ be an ε-isometry. Do there exist a monotonously increasing function $s : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ and a constant C that depends only on global geometry of ∂G^1 (e.g., its diameter) such that f extends to a $Cs(\varepsilon)$-isometry between G^1 and G^2?

Volkov claimed a solution to this problem in [110]. The proof is completely unrelated to both Pogorelov's approaches, so, in particular, it gives a third way to prove Theorem 1.1.7. The paper [110] is highly innovative without any doubts. However, we have some concerns on it. There are some gaps that can be bridged after some non-trivial work. But more important, some essential arguments are written in a very brief and cryptic way making them hard to interpret. A large part of this thesis can be considered as a revival of paper [110] with an application to a rigidity problem for a family of hyperbolic 3-manifolds. We use many important ideas from [110] and its influence on this thesis can not be diminished. But in some steps of the proof we have to choose a different road due to our inability to understand some parts of [110]. We think that a proof of Theorem 1.1.7 can be obtained by the means of the present manuscript, but they are insufficient to resolve Problem 1.1.9.

In higher dimensions the situation is in some way more surprising. The rigidity problem in the smooth case is easier: all smooth surfaces that are "curved enough" are rigid regardless any convexity assumption. This was proven by Killing. We refer to [104, Chapter 12] for an exposition. A proof for polytopes is due to Alexandrov [6]. However, it is interesting if one can establish a higher dimensional analogue of Theorem 1.1.7. Speaking about realization problems, one can show that by dimensional reasons not all metrics on the n-sphere with positive sectional curvature can be realized even locally on the boundaries of convex bodies. There is no known criterion to distinguish the induced metrics of convex surfaces neither in smooth nor in general case.

We should mention that plenty of work was done on higher dimensional CBB(0)-spaces (and, more generally, CBB(k)-spaces) due to their relation to limits of smooth manifolds.

There are some other ways to determine a convex body in $\mathbb{E}^n$ by its boundary characteristics. For instance, there are *area measures* and *curvature measures* normally defined with the help of the well-known Steiner formula for the volume of a parallel body to a convex body. The determination by the $(n-1)$-area measure is called the *Minkowsi problem* and is a classical geometry problem. It is in some sense dual to the zero curvature measure, prescribing which is called the *Alexandrov problem*. The problem of prescribing the first area measure is called the *Christoffel problem* and is also widely studied. Other measures are significantly less developed. A good reference on this topic is Schneider's book [101].

1.2 Realization and rigidity problems in hyperbolic 3-space

It is natural to consider the problems from the previous section also for convex bodies in 3-dimensional space forms. Except $\mathbb{E}^3$ they can be reduced to the spherical space $\mathbb{S}^3$ and the hyperbolic space $\mathbb{H}^3$ after performing a rescaling.

A *spherical cone-metric* or a *hyperbolic cone-metric* can be defined in the same way as in Definition 1.1.3 by replacing $\mathbb{E}^2$ with the standard sphere $\mathbb{S}^2$ or the hyperbolic plane $\mathbb{H}^2$. Similarly one can define CBB(1) and CBB(-1) metrics if one replaces a Euclidean triangle in the definition of $\lambda(x, y)$ (see the paragraph before Definition 1.1.5) by a spherical or a hyperbolic one. In this way one can also define a CBB(k) metric for any $k \in \mathbb{R}$.

Here we restrict our exposition to the hyperbolic case. A solution to the generalized Weyl problem in $\mathbb{H}^3$ was sketched already by Alexandrov and Pogorelov themselves [7], [88].

Theorem 1.2.1. For every convex hyperbolic cone-metric d on S^2 there exists a convex polyhedron $P \subset \mathbb{H}^3$ such that (S^2, d) is isometric to the boundary of P. Moreover, P is unique up to an ambient isometry.

Theorem 1.2.2. For every CBB(-1) metric d on S^2 there exists a convex body $G \subset \mathbb{H}^3$ such that (S^2, d) is isometric to the boundary of G.

Theorem 1.2.3. Let G^1 and G^2 be convex bodies in $\mathbb{H}^3$ and $f : \partial G^1 \to \partial G^2$ be an isometry. Then f extends to an ambient isometry.

For a modern approach to the smooth case one can see [62].

The proofs of Theorem 1.2.1 and Theorem 1.2.2 can be obtained in an identical way to their Euclidean counterparts. However, a proof of Theorem 1.2.3 appears to be quite different from the proof of Theorem 1.1.7 given by Pogorelov, because the latter uses the linear-algebraic structure of $\mathbb{E}^3$. In order to prove Theorem 1.2.3, Pogorelov developed a groundbreaking concept of the Pogorelov map $\mathbb{H}^3 \times \mathbb{H}^3 \to \mathbb{E}^3 \times \mathbb{E}^3$. It produces a correspondence between pairs of bodies with isometric boundaries (and pairs of congruent bodies) in both spaces. However, one needs to be careful with convexity. Pogorelov developed all necessary tools and completed the proof of rigidity in the spherical space, but the details of the proof in the hyperbolic space were furnished in further works of other people [42, 74].

There are some substantial distinctive features of the hyperbolic space that allow to consider additional generalizations of aforementioned results. First, there are much more possible topological types for the boundary of a non-compact convex body (by a non-compact convex body we simply mean a non-compact closed convex set with non-empty interior and non-empty boundary). Its boundary can be homeomorphic to any domain on sphere. Indeed, the boundary at infinity $\partial_\infty \mathbb{H}^3$ is homeomorphic to S^2. Consider a domain $R \subset \partial_\infty \mathbb{H}^3$ distinct from $\partial_\infty \mathbb{H}^3$ such that its complement $\overline{R}$ is not contained in the boundary at infinity of a plane (arbitrary R is homeomorphic to R' satisfying this condition). The convex hull of $\overline{R}$ is a non-compact convex body with boundary homeomorphic to R.

In particular, one can consider polyhedra with some vertices belonging to $\partial_\infty H^3$ (called *ideal* vertices) and even *hyperideal* vertices outside of it (defined, e.g., using the projective model). Of particular interest and beauty are results for *ideal polyhedra*, i.e., polyhedra with all vertices on $\partial_\infty \mathbb{H}^3$. The induced metric is a *hyperbolic cusp-metric*.

Definition 1.2.4. A *hyperbolic cusp-metric* on a surface S is a complete hyperbolic metric of finite area on $S \backslash V(d)$, where $V(d)$ is a finite set. Points of $V(d)$ are called *cusps*.

Although a hyperbolic cusp-metric is not really a metric on S, but on S with punctures, we abuse the notation by saying that it is a metric on S.

In [94] Rivin proved an analogue of Theorem 1.2.1 in this case:

Theorem 1.2.5. For every hyperbolic cusp-metric d on S^2 there exists a convex ideal polyhedron $P \subset \mathbb{H}^3$ such that (S^2, d) is isometric to the boundary of P. Moreover, P is unique up to an ambient isometry.

The proof follows the lines of the Alexandrov continuity method used to prove Theorems 1.1.4 and 1.2.1. Springborn [105] provided a variational proof of Theorem 1.2.5 based on ideas from [14] and discrete conformal geometry.

In [95, 93] Rivin also proved a "dual result" to Theorem 1.2.5 providing a characterization of ideal polyhedra in terms of dihedral angles:

Theorem 1.2.6. Let P^* be the graph of a 3-polytope together with a weight function $w : E(P^*) \to (0; \pi)$ such that

(1) For each face R of P^* we have $\sum_{e \in E(R)} w(e) = 2\pi$;

(2) For each cycle C in the 1-skeleton of P^* that is not the boundary of a face we have $\sum_{e \in C} w(e) > 2\pi$.

Then there exists a convex ideal polyhedron $P \subset \mathbb{H}^3$ such that P^* is the dual graph of P and w is equal to the exterior dihedral angles of P. Moreover, it is unique up to ambient isometry.

In particular, this allowed to answer an old question of Steiner to characterize the combinatorial types of polyhedra that admit a realization inscribed in a sphere. Indeed, Theorem 1.2.6 provides a polynomial-time algorithm to decide whether or not a given combinatorial type admits such a realization. A classification of ideal polyhedra with non-obtuse dihedral angles was done previously by Andreev [8]. A generalization of Theorems 1.2.5 and 1.2.6 to hyperideal polyhedra was given by Schlenker [99] (for Theorem 1.2.6 this was also done independently by Bao and Bonahon in [9]).

It is important for us to clarify in which sense Theorem 1.2.6 is dual to Theorem 1.2.5. Theorem 1.2.6 is a deep consequence of the hyperbolic – de Sitter duality, wich is a substantial tool allowing to enhance greatly the understanding of convex hyperbolic geometry.

The outward normal to an oriented plane in $\mathbb{H}^3$ naturally belongs to the *de Sitter space* $d\mathbb{S}^3$, which is a Lorenzian manifold of constant curvature 1. Thus, $d\mathbb{S}^3$ can be seen as the space of oriented planes of $\mathbb{H}^3$. It can be easily described with the help of the hyperboloid model, for the details we refer to Sections 2.2.1 and 2.2.2. If $G \subset \mathbb{H}^3$ is a convex body, then the set of outwards normals to all supporting planes defines the polar convex body $G^* \subset d\mathbb{S}^3$. It has a space-like boundary. We also get the *Gauss map* $\mathcal{G} : \partial G \rightrightarrows \partial G^*$, which is a multivalued map sending a boundary point to the set of its outwards normals. In the Euclidean case the image of a convex body under the Gauss map is always the 2-sphere $\mathbb{S}^2$ with the standard metric. Contrary to this, the boundary metric of the dual body G^* in the hyperbolic space varies and defines G uniquely. For polytopes this was proven by Rivin–Hodgson [54] with a complete metric description of the dual boundary.

Definition 1.2.7. A metric d on a surface is called *large* if the lengths of all closed geodesics are strictly greater than 2π.

Theorem 1.2.8. For every large concave spherical cone-metric d on S^2 there is a convex polytope $P \subset \mathbb{H}^3$ such that (S^2, d) is isometric to the boundary of its dual P^*. Moreover, P is unique up to an ambient isometry.

The proof also is based on the Alexandrov continuity method, however, it is slightly more intricate than the previously mentioned instances. The existence in Theorem 1.2.6 is proven in [95] as a carefully examined limit case of Theorem 1.2.8. However, as usual with proofs by approximation, this does not allow to prove the uniqueness, which is proven in [93] by a totally different approach. This approach happens to be very important to the present thesis and we will make a further glimpse on it in subsequent sections. Schlenker [97] also proved a smooth strictly convex analogue of Theorem 1.2.8:

Theorem 1.2.9. For every large smooth metric d on S^2 with sectional curvature $K < 1$ there is a strictly convex body $G \subset \mathbb{H}^3$ with smooth boundary such that (S^2, d) is isometric to the boundary of its dual G^*. Moreover, G is unique up to an ambient isometry.

No generalizations of Theorem 1.2.9 to non-smooth and non-polyhedral convex bodies seem to be known.

1.3 Realization and rigidity problems in hyperbolic 3-manifolds

By a hyperbolic manifold we will always mean a complete hyperbolic manifold.

Except the generalized Weyl problem, the second main source of our inspiration is theory of hyperbolic 3-manifolds originated in breakthrough works of Thurston from 70s, who was the first to understand their distinguished importance to the whole 3-dimensional topology. Without going into much details, we recall that Thurston proposed *the geometrization program* to understand the topology of 3-manifolds: each 3-manifold can be canonically decomposed into pieces and each peace can be endowed with one of eight canonical geometries (see, e.g. [108, 109]). Hence, to get a picture of 3-dimensional topology, it is enough to investigate the topology of geometric manifolds of each type. Thurston himself proved a large piece of this puzzle, namely, the hyperbolization theorem for Haken manifolds. The geometrization program was deeply developed by plenty of researchers with its notable culmination in the works of Perelman [83, 84, 85].

Manifolds admitting each of canonical geometries can be described in purely topological terms. Among all canonical geometries hyperbolic manifolds constitute the largest and the most mysterious class. In particular, it was shown that in some sense a random 3-manifold is hyperbolic [70]. Hence, much of the efforts of researchers were directed toward developing a deeper understanding of hyperbolic 3-manifolds.

One of the most motivational for us properties of hyperbolic manifolds (closed or of finite volume) in dimensions starting from 3 is their *rigidity*, which can be quite unexpected on the first sight especially if we compare it with dimension 2, where the moduli space of hyperbolic metrics on a surface is homeomorphic to a multidimensional ball. The rigidity of hyperbolic manifolds was first proved by Mostow [78] for closed manifolds and extended to the case of finite volume by Prasad [89]. Notable alternative proofs are due to Thurston and Gromov [108].

Theorem 1.3.1. Let M^1, M^2 be hyperbolic n-dimensional manifolds of finite volume (without boundary) and $n \geq 3$. Then every isomorphism $\pi_1(M^1) \xrightarrow{\sim} \pi_1(M^2)$ is induced by a unique isometry $M^1 \xrightarrow{\sim} M^2$.

In particular this implies that if M^1 is homotopically equivalent to M^2, then they are isometric. An important viewpoint (which was actually the initial viewpoint of Mostow contrary to the geometric vision of Gromov) is that Mostow's rigidity is a result on lattices in specific Lie groups. Since Mostow's paper appeared, this paved the way for a vast amount of results in this direction.

Theory of hyperbolic 3-manifolds is intimately related to theory of Kleinian groups. Each complete hyperbolic n-manifold M (without boundary) is isometric to the quotient $\mathbb{H}^n/\Gamma$, where Γ is a discrete subgroup of isometries of $\mathbb{H}^n$ without elliptic elements. The group Γ is defined up to conjugacy and is isomorphic to $\pi_1(M)$. The condition not to contain elliptic isometries is equivalent that the action of Γ on $\mathbb{H}^3$ is free, or alternatively, that Γ does not have torsion. The condition that Γ is discrete is equivalent that the action is properly discontinuous. A *Kleinian group* is a discrete subgroup of $\mathrm{Iso}_+(\mathbb{H}^3)$, i.e., of the group of orientation-preserving isometries of $\mathbb{H}^3$. Thus, Kleinian groups without torsion are fundamental groups of oriented hyperbolic 3-manifolds. We refer the reader, e.g., to [71] for more details.

We use the representation of M as $\mathbb{H}^n/\Gamma$ to define the boundary at infinity $\partial_\infty M$ as the quotient $\partial_\infty \mathbb{H}^n/\Gamma$ with the natural topology on $M \cup \partial_\infty M$, although $\partial_\infty M$ can be defined purely intrinsically with the help of equivalence classes of geodesic rays.

We are mostly interested in results that unify the (generalized) Weyl problem and the Mostow rigidity. On the side of the Mostow rigidity they can be seen as the exploration of additional (boundary) invariants that together with topology determine compact hyperbolic 3-manifolds with boundary up to isometry.

We say that a hyperbolic manifold has *convex boundary* if each boundary point has a neighbourhood isometric to a convex set in $\mathbb{H}^3$. We say that it has *polyhedral boundary* if each boundary point has a neighbourhood isometric to a polyhedral set in $\mathbb{H}^3$.

Theory of hyperbolic 3-manifolds with convex boundary is significantly connected with theory of open hyperbolic 3-manifolds. An open manifold is a connected non-compact manifold without boundary. Every hyperbolic 3-manifold with convex boundary can be embedded as a convex subset in a canonical open hyperbolic 3-manifold. However, we need to define convex subsets in some strong sense.

Definition 1.3.2. A subset F of a hyperbolic manifold M is called *totally convex* if F contains **every** geodesic segment between any two points of F. The *convex hull* of a set $V \subset M$ is the inclusion minimal closed totally convex set containing V.

Take a hyperbolic 3-manifold F with convex boundary. Its universal cover can be developed as a convex subset of $\mathbb{H}^3$. One can show that the image of $\pi_1(F)$ under the corresponding holonomy representation is a Kleinian group Γ without torsion. The open manifold $M(F) := \mathbb{H}^3/\Gamma$ is called *the extension of* F. We obtain the natural isometric embedding $F \hookrightarrow M(F)$, which induces an isomorphism of the fundamental groups. The image of F under this embedding is a totally convex subset of M. We refer, e.g., to [27, Section I.2.4] for more details of this construction.

One can prove that a compact hyperbolic 3-manifold F with convex polyhedral boundary can be represented as the convex hull of finitely many points in $M(F)$. In a similar way, a hyperbolic 3-manifold F with polyhedral boundary is called *ideal* if it is isometric to the convex hull of finitely many points in $\partial_\infty M$ of an open hyperbolic 3-manifold M. An ideal manifold can also be defined in an intrinsic way, but the definition is harder and does not provide a good intuition.

Now we review the state of art of Weyl-type problems for hyperbolic 3-manifolds. Our main attention is restricted to the following family of examples:

Definition 1.3.3. A *Fuchsian group* is a Kleinian group that is isomorphic to $\pi_1(S_g)$, where S_g is a closed oriented surface of genus g, and leaves invariant a plane in $\mathbb{H}^3$. A *Fuchsian manifold* $\overline{F}$ is the quotient of a **closed invariant half-space** of $\mathbb{H}^3$ under a Fuchsian group of isometries. It has geodesic boundary (homeomorphic to S_g), which we denote by $\partial_\downarrow \overline{F}$.

We highlight that our definition is not traditional because normally Fuchsian manifolds are defined as quotients of the whole $\mathbb{H}^3$, not just of a half-space. However, we think that the modified definition is natural due to the evident symmetry of Fuchsian manifolds defined in the traditional sense. Our modification allows us to simplify some statements.

A Fuchsian manifold is homeomorphic to $S_g \times [0; +\infty)$. Fuchsian manifolds can be considered as toy cases for realization and rigidity problems in hyperbolic 3-manifolds. Their study with respect to such questions takes the origin in classical works of Pogorelov [88, Section VI.12] and Gromov [43, Section 3.2.4].

Definition 1.3.4. A *compact Fuchsian manifold F with convex boundary* is a manifold isometric to a compact totally convex subset with non-empty interior of a Fuchsian manifold $\overline{F}$. It has two boundary components: the geodesic boundary $\partial_\downarrow F$ coinciding with $\partial_\downarrow \overline{F}$ and the non-geodesic component $\partial^\uparrow F$. This first is called the *lower boundary* and the second is called the *upper boundary*.

Intrinsically, a compact Fuchsian manifold F with convex boundary can be described as a hyperbolic 3-manifold homeomorphic to $S_g \times [0; 1]$ with one boundary component totally geodesic and the second one locally convex. The extension construction described above shows the equivalence of both descriptions.

Definition 1.3.5. An *ideal Fuchsian manifold F with convex polyhedral boundary* is the convex hull of finitely many points in $\partial_\infty \overline{F}$ of a Fuchsian manifold F. The boundary components of F are defined similarly.

An analogue of Theorem 1.2.1 in the Fuchsian case was proven by Fillastre [32]:

Theorem 1.3.6. For every convex hyperbolic cone-metric d on S_g there exists a compact Fuchsian manifold F with convex polyhedral boundary such that (S_g, d) is isometric to $\partial^\uparrow F$. Moreover, F is unique up to isometry.

Schlenker [98] and Fillastre [33] also proved an analogue of Theorem 1.2.5:

Theorem 1.3.7. For every hyperbolic cusp-metric d on S_g there exists an ideal Fuchsian manifold F with convex polyhedral boundary such that (S_g, d) is isometric to $\partial^\uparrow F$. Moreover, F is unique up to isometry.

In [33] Fillastre generalized it further to metrics that have simultaneously conical points, cusps and so-called *complete ends of infinite area*, which we do not define.

Assume that we have a compact hyperbolic 3-manifold F with convex boundary. Let G be its universal cover developed to $\mathbb{H}^3$. It is a convex set. For each boundary component of ∂G its Gauss dual is a space-like convex surface in the de Sitter space $d\mathbb{S}^3$. The action of $\pi_1(F)$ on $\mathbb{H}^3$ by the holonomy representation can be extended to the action on $d\mathbb{S}^3$. The dual surface ∂G^* is invariant under this action. After taking the quotient we obtain a metric space, which we denote ∂F^*. In this framework Schlenker [98] and Fillastre [34] also proved an analogue of Theorem 1.2.8:

Theorem 1.3.8. For every concave large spherical cone-metric d on S_g there exists a compact Fuchsian manifold F with convex polyhedral boundary such that (S_g, d) is isometric to the dual $\partial^\uparrow F^*$ of its upper boundary. Moreover, F is unique up to isometry.

Theorem 1.3.6 is a generalization of Theorem 1.2.1 to surfaces of genus greater than one. In order to cover the case of genus one we need to introduce another family of examples.

Definition 1.3.9. An *open hyperbolic (rank-2) 3-cusp* M is a quotient of $\mathbb{H}^3$ by a parabolic subgroup of orientation-preserving isometries isomorphic to $\mathbb{Z}^2$. We will omit the words "rank-2" and "3-". A *hyperbolic cusp with compact convex boundary* is a hyperbolic 3-manifold F isometric to a totally convex subset with compact boundary of an open hyperbolic cusp M.

An open hyperbolic cusp M is homeomorphic to $T^2 \times \mathbb{R}$, where T^2 is the topological 2-torus. Its boundary at infinity $\partial_\infty M$ consists of a component homeomorphic to T^2 at one side and one point at another. Every hyperbolic cusp with compact convex boundary is homeomorphic to $T^2 \times [0; +\infty)$.

In [35], [36] Fillastre and Izmestiev proved analogues of Theorems 1.1.4 and 1.2.8 for this family.

Theorem 1.3.10. For every convex hyperbolic cone-metric d on T^2 there exists a hyperbolic cusp F with compact convex polyhedral boundary such that (T^2, d) is isometric to ∂F. Moreover, F is unique up to isometry.

Theorem 1.3.11. For every concave large spherical cone-metric d on T^2 there exists a hyperbolic cusp F with compact convex polyhedral boundary such that (T^2, d) is isometric to the boundary of its dual ∂F^*. Moreover, F is unique up to isometry.

It is remarkable that last two theorems are proven not by the traditional Alexandrov continuity method like the aforementioned results, but by a variational approach similar to one in [14]. Hence, their proof can be called constructive.

To proceed further we need to introduce an important tool to study open hyperbolic 3-manifolds:

Definition 1.3.12. The *convex core* of a hyperbolc 3-manifold M is the intersection of all closed totally convex subsets of M.

In the case of open hyperbolic cusps, the convex core is empty unless we consider $M \cup \partial_\infty M$. Their convex core is the component $\partial_\infty M$ that is a point. Except this case, the convex core of an open hyperbolic manifold M is non-empty and is homotopically equivalent to M. It contains all closed geodesics of M. The convex core of a closed hyperbolic 3-manifold is the manifold itself. The convex core of a Fuchsian manifold $\overline{F}$ is its lower boundary $\partial_\downarrow \overline{F}$.

Definition 1.3.13. An open hyperbolic 3-manifold is called *co-compact* if it has compact convex core.

Labourie [63] (existence) and Schlenker [100] (uniqueness and an alternative proof of existence) proved an analogue of the Weyl problem in the co-compact case:

Theorem 1.3.14. Let F be a compact 3-manifold with boundary such that its interior admits a co-compact hyperbolic metric. Then for each Riemannian metric g on ∂F of sectional curvature $K > -1$ there exists a unique hyperbolic metric on F such that ∂F is smooth, convex and the induced metric on ∂F is isometric to g.

Theorem 1.3.15. Let F be a compact 3-manifold with boundary such that its interior admits a co-compact hyperbolic metric. Then for each large Riemannian metric g on ∂F of sectional curvature $K < 1$ there exists a unique hyperbolic metric on F such that ∂F is smooth, convex and the induced metric on ∂F^* is isometric to g.

The uniqueness here is up to isometry isotopic to identity.

In another preprint [98] of Schlenker an analogue of Theorem 1.2.6 was proven, but the preprint was never published.

Theorem 1.3.16. Let F be a compact 3-manifold with incompressible boundary and P^* be a finite graph on ∂F, such that all faces are homeomorphic to disks, together with a weight function $w : E(P^*) \to (0;\pi)$ such that

(1) For each face R of P^* we have $\sum_{e \in E(R)} w(e) = 2\pi$;

(2) For each cycle C in the 1-skeleton of P^* that is not the boundary of a face we have $\sum_{e \in C} w(e) > 2\pi$.

Then there exists a unique hyperbolic metric d on F such that (F,d) is an ideal hyperbolic 3-manifold with convex polyhedral boundary, P^* is the dual graph of the edges of (F,d) and w is equal to the exterior dihedral angles.

Towards general metrics Fillastre, Izmestiev and Veronelli [37] proved the following analogue of Theorem 1.2.2:

Theorem 1.3.17. For every CBB(-1) metric d on T^2 there exists a hyperbolic cusp with compact convex boundary F such that (T^2,d) is isometric to ∂F.

Finally, Slutskiy [103] proved a similar statement for quasi-Fuchsian manifolds.

Definition 1.3.18. A *quasi-Fuchsian group* is a Kleinian group that is isomorphic to $\pi_1(S_g)$ and its limit set in $\partial_\infty \mathbb{H}^3$ is a Jordan curve. A *quasi-Fuchsian manifold M* is the quotient of $\mathbb{H}^3$ under a quasi-Fuchsian group of isometries. A *compact quasi-Fuchsian manifold with convex boundary* is a manifold F isometric to a compact totally convex subset with non-empty interior of a quasi-Fuchsian manifold M.

Theorem 1.3.19. For every pair of CBB(-1) metrics d_1, d_2 on S^2 there exists a compact quasi-Fuchsian manifold with convex boundary F such that (S_g,d_1), (S_g,d_2) are isometric to the boundary components of F.

Quasi-Fuchsian manifolds serve as an intermediate case between Fuchsian and much wider class of co-compact manifolds. While remaining topologically still simple, quasi-Fuchsian manifolds contain already all geometric difficulties that appear when we try to proceed from the Fuchsian case to the co-compact one while solving realization and rigidity problems. The main feature of the geometry of Fuchsian manifolds is that it is essentially a product geometry, hence, we can consider them as a kind of (2+1)-dimensional case while quasi-Fuchsian manifolds show in the full power the 3-dimensional nature of problems under consideration. Thereby, it is natural to consider any geometric problem that can be formulated for co-compact manifolds, first, in the Fuchsian case and then try to extend a solution to the quasi-Fuchsian family. It happens that there are not many obstructions to extend a solution further from the quasi-Fuchsian to the co-compact case.

As we see, there is no analogue of Theorem 1.1.7 for hyperbolic 3-manifolds with boundary except the ball (Theorem 1.2.3). Probably, this is because Pogorelov's proof of Theorem 1.1.7 is quite specific to the Euclidean space and the Pogorelov map used in the proof of Theorem 1.2.3 does not match well to the equivariance with respect to a group action. One of the results of this thesis is a proof of such an analogue for the Fuchsian case, but we postpone the statement and the discussion until Section 1.6.

Note that Fuchsian and quasi-Fuchsian manifolds can be defined naturally also in the Lorenzian setting. There is an active research area investigating realization problems in the Lorentzian case, which, except the de Sitter space, also include the Minkowski space and the Anti de Sitter space. For the details we refer to [64, 34, 39, 25, 61, 107].

An interesting work direction is to describe in a similar way the deformation spaces of open hyperbolic 3-manifolds. An existing approach was developed in the time when Kleinian groups attracted more attention than hyperbolic 3-manifolds, but it applies perfectly to the latter.

Definition 1.3.20. An open hyperbolic 3-manifold is called *geometrically finite* if its convex core has finite volume.

Theorem 1.3.21. *Let M be an open geometrically finite hyperbolic 3-manifold. Then every conformal structure on $\partial_\infty M$ is induced by a unique choice of a geometrically finite hyperbolic metric on M.*

This result is obtained in the works of several researchers including Ahlfors, Bers, Kra, Marden and Maskit. We refer to [12] for an expository account. The boundary of the deformation space consists of both geometrically finite and geometrically infinite structures. An important problem posed by Thurston was to describe the boundary in the geometrically infinite case with the help of specific invariants called *ending laminations*. This was done by Minsky, Brock and Canary [75], [20].

However, Thurston himself proposed also an alternative approach to describe the interior of the deformation space. Note that the induced metric on boundary of the convex core $C(M)$ is hyperbolic. If follows from the work of Labourie [63] that any such metric on the convex core of an open co-compact hyperbolic 3-manifold M is induced by a co-compact metric on M. The question, whether this metric on M is unique, is open and is called *the induced metric conjecture*.

This question is just a particular case of the generalized Weyl problem applied to compact hyperbolic 3-manifolds with convex boundary. Indeed, one can show that if the boundary metric is hyperbolic, then such a manifold is isometric to the convex core of an open manifold. The boundary of a convex core is generally non-smooth despite it is smooth intrinsically: it can be bent along a geodesic lamination.

This lamination is another important piece of data. It carries a *pleating measure*, which is a transverse measure to the lamination generalizing exterior dihedral angles. The *pleating lamination*, i.e., the lamination with the pleating measure, is a substitute for the dual metric, which is degenerated in this case. In [18] Bonahon and Otal described the admissible pleating laminations in the case of incompressible boundary at infinity. It was extended to the compressible case by Lecuire [66]. The question whether a pleating lamination determines uniquely the metric of an open geometrically finite hyperbolic 3-manifold is called *the pleating lamination conjecture*. It was proven by Bonahon [17] for quasi-Fuchsian manifolds that are sufficiently close to Fuchsian, but the general case remains open.

1.4 Hyperbolic cone-3-manifolds

Cone-manifolds are natural generalizations of cone-metrics on surfaces to higher dimensions. We restrict ourselves only to dimension 3. Besides, we restrict our discussion only to orientable hyperbolic manifolds.

Definition 1.4.1. A *hyperbolic cone-metric* d on a 3-manifold M is locally isometric to a hyperbolic cone over the 2-sphere with a spherical cone-metric. Points that do not have a neighbourhood isometric to a hyperbolic ball constitute the *singular locus* of (M, d).

Let M be a 3-manifold with a stratification into cells combinatorially equivalent to polyhedra. One can replace each cell with a hyperbolic polyhedron so that the gluing maps are isometries. This defines a cone-metric on M. If one is interested in finding a complete hyperbolic metric on M, then one can start from a cone-metric and then try to deform it in order to resolve cone-singularities. This approach is successful in some cases. In this section we give a quick review of the most interesting results related to hyperbolic cone-3-manifolds with a special attention on rigidity problems.

Basically, hyperbolic cone-3-manifolds were introduced by Thurston in his proof of the celebrated hyperbolic Dehn filling theorem. We need to extend the notion of hyperbolic cusp-metrics to 3-dimensional case:

Definition 1.4.2. A *hyperbolic cusp-3-manifold* is an open complete hyperbolic manifold of finite volume.

A hyperbolic cusp-3-manifold M is homeomorphic to the interior of a manifold $\widetilde{M}$ with finitely many boundary components homeomorphic to a torus. One can take a solid torus for each boundary component of $\widetilde{M}$ and glue them with $\widetilde{M}$ along the boundary. The homeomorphism class of the resulting manifold depends only on the isotopy class of the curve on the boundary of $\widetilde{M}$ glued with the meridian of the respective solid torus. The *hyperbolic Dehn filling theorem* states that for all except finitely many fillings the resulting manifold is hyperbolic. A proof was done with the help of cone-manifolds. An important problem is to bound the number of exceptional Dehn fillings.

A significant result on cone-3-manifolds is

Theorem 1.4.3. Let M be a closed hyperbolic cone-3-manifold with cone-angles at most 2π. Then it is locally rigid with respect to the cone-angles.

It means that the set of cone-structures with a fixed singular set can be locally parametrized by cone-angles. It is also proven that there are no infinitesimal deformations of a cone-metric preserving the angles. Theorem 1.4.3 was first proven by Hodgson and Kerckhoff [51] in the case when the singular locus of M is a link (however, they proved it not only in the closed case, but in the more general case of finite volume). It was extended to the present form by Weiss [113] and Montcouquiol [76].

Theorem 1.4.3 has several nice applications. With its finite volume version Hodgson and Kerckhoff [52] proved a universal bound 60 for the number of exceptional surgeries of a one-cusped manifold and 114 in the case of multiple cusps. One should note that this is pretty far from the optimal bound 10 (for a single cusp) due to Lackenby and Meyerhoff [65]. However, the approach through cone-manifolds has some advantages. First, it does not use such heavy machinery as the geometrization theorem. Second, it shows that all resulting hyperbolic manifolds are continuous deformations of the initial cusp-manifold through cone-structures.

Another application is a proof of the infinitesimal hyperbolic Stoker conjecture stating that every first order deformation of a convex hyperbolic polytope that preserves the dihedral angles is trivial. This is due to Mazzeo and Montcouquiol [72]. It can be extended to local rigidity. There is also a Euclidean result with the additional condition that face angles are preserved.

One more application is the previously mentioned characterization of admissible pleating laminations on the boundaries of convex cores [18] by Bonahon and Otal.

Bromberg [22] proved a local rigidity result for geometrically finite manifolds, which is a common generalization Theorem 1.4.3 and Theorem 1.3.21:

Theorem 1.4.4. Let M be an open geometrically finite hyperbolic cone-3-manifold without rank-1 cusps, with singular locus at a link and with all cone angles at most 2π. Then it is locally rigid with respect to the cone-angles and the conformal structure at $\partial_\infty M$.

In [53] Hodgson and Kerckhoff dropped the bound on cone-angles in favour of the lower bound on the radius of tubes around singularities (in the case of finite volume manifolds and link-type singularities). This allowed them to obtain new quantitative results on the shape of the hyperbolic Dehn surgery space. A similar extension of Theorem 1.4.4 was done by Bromberg in [21]. This allowed a significant progress towards *the density conjecture* claiming that every complete hyperbolic 3-manifold with finitely generated fundamental group is a limit of geometrically finite 3-manifolds [23], [19]. It was also used in several quantitative estimates in [21] on how the geometry of a geometrically finite manifold changes under the drilling operation. This is an important operation on the way to understand deformation spaces of open hyperbolic 3-manifolds.

Moroianu and Schlenker [77] proved a version of local rigidity in the co-compact case allowing the singular locus to be non-compact:

Theorem 1.4.5. Let M be an open co-compact hyperbolic cone-3-manifold with all cone angles at most 2π. Then it is locally rigid with respect to the cone-angles and the conformal structure at $\partial_\infty M$.

The manifolds of this type are sometimes called *manifolds with particles* due to the physical interpretation of cone-singularities along time-like geodesics in Lorentzian 3-manifolds as trajectories of moving massive particles. For more background in this setting we refer, e.g., to [10, 11].

Weiss [112] proved a global rigidity result (preceded by Kojima [60] when the singular locus is a link):

Theorem 1.4.6. Let $f : M^1 \to M^2$ be a homeomorphism between two compact hyperbolic cone-3-manifolds that maps the singular locus of M^1 onto the singular locus of M^2, preserves the cone-angles, and let the cone-angles be at most π. Then f is isotopic to an isometry.

Weiss also proved a version for hyperbolic cone-3-manifolds of finite volume, but we omit its exact statement.

Fillastre and Izmestiev [35] proved a global rigidity statement for convex cusps with boundary:

Theorem 1.4.7. Let $f : M^1 \to M^2$ be a homeomorphism between two hyperbolic 3-cusps with compact convex boundaries and cone-singularities along geodesics passing from the (compact) boundary to $\partial_\infty M$ that maps the singular locus of M^1 onto the singular locus of M^2, preserves the cone-angles and is an isometry at the boundary. Then f is isotopic to an isometry.

They also obtained [36] a local rigidity result in the dual case: Lorentzian 3-cusps with cone singularities along time-like geodesics and convex boundaries are locally rigid.

In this thesis we will prove a rigidity result for hyperbolic cone-3-manifolds of a specific type. See Section 1.6 for the formulation.

Another notable application of hyperbolic cone-3-manifolds is a proof of the orbifold geometrization theorem [16, 29].

1.5 Discrete conformality and angle structures

The last topic that influenced our work significantly is the discrete conformal geometry. Its main motivation comes from the intention to find a good discrete analogue of the notion of *conformal equivalence* for the case of cone-metrics on surfaces. There were several (non-equivalent) attempts to give such a definition. The most prominent in the recent years became the following:

Definition 1.5.1. We say that a geodesic triangulation $\mathcal{T}$ of a cone-surface (S_g, d) with $V(\mathcal{T}) \supseteq V(d)$ is *Delaunay* if when we develop any two adjacent triangles to the model plane ($\mathbb{E}^2$, $\mathbb{H}^2$ or $\mathbb{S}^2$), the circumscribed disc of each triangle does not contain the opposite vertex of the other triangle in the interior.

Definition 1.5.2. Two cone-metrics d' and d'' on a surface S with finite marked point-set $V \supseteq (V(d_1) \cup V(d_2))$ are called *discretely conformally equivalent* if there exists a sequence of pairs $\{(d_k, \mathcal{T}_k)\}_{k=1}^m$, where d_k is a cone-metric on S, $\mathcal{T}_k$ is a Delaunay triangulation of (S, d_k) with $V(\mathcal{T}_k) = V$, $d_1 = d'$, $d_m = d''$ and for every k either

(i) $d_k = d_{k+1}$ in the sense that (S, d_k) is isometric to (S, d_{k+1}) by an isometry isotopic to identity with respect to V, or

(ii) $\mathcal{T}_k = \mathcal{T}_{k+1}$ and there exists a function $u : V \to \mathbb{R}$ such that for every edge e of $\mathcal{T}_k$ with vertices v_1 and v_2 we have

in the Euclidean case

$$\mathrm{len}_e(d_k) = \exp(u(v_1) + u(v_2))\mathrm{len}_e(d_{k+1}),$$

in the hyperbolic case

$$\sinh\left(\frac{\mathrm{len}_e(d_k)}{2}\right) = \exp(u(v_1) + u(v_2))\sinh\left(\frac{\mathrm{len}_e(d_{k+1})}{2}\right),$$

in the spherical case

$$\sin\left(\frac{\mathrm{len}_e(d_k)}{2}\right) = \exp(u(v_1) + u(v_2))\sin\left(\frac{\mathrm{len}_e(d_{k+1})}{2}\right),$$

where $\mathrm{len}_e(d_k)$ is the length of e in d_k.

Of course, all cone-metrics in these definitions are assumed to be of the same type, i.e., either Euclidean or hyperbolic or spherical. The scaling in the hyperbolic and spherical cases comes from the *Delaunay condition* for inscribed quadrilateral, which we do not discuss.

In this setting Gu, Guo, Luo, Sun, Wu [46, 45] succeeded to prove fundamental results that are reminiscent to the work of Kazdan and Warner [59] in the smooth setting:

Theorem 1.5.3. Let d be a Euclidean cone-metric on a closed surface S with a marked set $V \supseteq V(d)$ and $\kappa' : V \to (-\infty; 2\pi)$ be a function satisfying

$$\sum_{v \in V} \kappa'(v) = 2\pi(2 - 2g). \tag{1.5.1}$$

Then there exists a unique up to scaling Euclidean cone-metric d' discretely conformally equivalent to d such that $\kappa_v(d') = \kappa'(v)$ for all $v \in V$.

Theorem 1.5.4. Let d be a hyperbolic cone-metric on a closed surface S with a marked set $V \supseteq V(d)$ and $\kappa' : V \to (-\infty; 2\pi)$ be a function satisfying

$$\sum_{v \in V} \kappa'(v) > 2\pi(2 - 2g). \tag{1.5.2}$$

Then there exists a unique hyperbolic cone-metric d' discretely conformally equivalent to d such that $\kappa_v(d') = \kappa'(v)$ for all $v \in V$.

Of the main importance are the following corollaries that are discrete analogues of the well-known uniformization theorem:

Corollary 1.5.5. Every Euclidean cone-metric on T^2 is discretely conformally equivalent to a unique up to scaling Euclidean metric.

Corollary 1.5.6. Every hyperbolic cone-metric on S_g is discretely conformally equivalent to a unique hyperbolic metric.

Uniqueness everywhere is up to isometry isotopic to identity with respect to the marked set V (in addition to scaling in the Euclidean case).

Proofs of Theorem 1.5.4 and 1.5.3 are done in the spirit of the Alexandrov continuity method by establishing that the realization map sending a cone-metric in a given discrete conformal class to its curvatures is a homeomorphism. The authors also note that the metrics can be found with the help of a variational approach without stating the functional explicitly. One of the main results of this thesis is an alternative variational proof of Theorem 1.5.4 with the help of an explicit variational approach on a space of cone-manifolds with convex ideal boundary.

The viewpoint that discrete uniformization problems can be interpreted in terms of geometry of ideal polyhedra was significantly elaborated in the paper [15]. It was shown there that the discrete uniformization problem of a given cone-metric d is equivalent to a realization problem for some cusp-metric associated to d. Such problems can be solved with the help of the volume functional on the space of admissible angle structures. As mentioned above, Springborn [105] gave a variational proof of Theorem 1.2.5 using these interpretations from discrete conformal geometry.

Theorems 1.5.3 and 1.5.4 were also used in [44] in order to approximate the uniformization maps in the smooth setting with the help of the discrete uniformization. It was shown there that discrete conformal structures converge in some sense to the smooth ones.

The magical variational properties of the volume functional mentioned above were noted and highlighted by Rivin in [93] in an investigation of Euclidean angle structures assigned to a triangulated surface. Let S be an oriented surface with a topological triangulation $\mathcal{T}$. Consider the space $\mathfrak{A}(\mathcal{T}) \subset (0; \pi)^{3|F(\mathcal{T})|}$, where $|F(\mathcal{T})|$ is the number of triangles, of angle assignments to triangles of $\mathcal{T}$ such that the sum of angles in each triangle is equal to π. These are called *Euclidean angle structures*. One can introduce the *dihedral angle* of an edge of $\mathcal{T}$, which is defined as the sum of angles opposite to e

in all (one or two) triangles adjacent to e. Consider $\Delta \in (0; 2\pi)^{|E(\mathcal{T})|}$ and by $\mathfrak{A}(\mathcal{T}, \Delta)$ denote the set of angle structures with dihedral angles given by Δ. Rivin proved

Theorem 1.5.7. If $\Delta \in (0; \pi)^{|E(\mathcal{T})|}$ and $\mathfrak{A}(\mathcal{T}, \Delta)$ is non-empty, then there exists a unique up to similarity Euclidean cone-metric d on S realizing $\mathcal{T}$ such that Δ is the dihedral angle function of $\mathcal{T}$ in (S, d).

The condition $\Delta \in (0; \pi)^{|E(\mathcal{T})|}$ (instead of $\Delta \in (0; 2\pi)^{|E(\mathcal{T})|}$) is some kind of convexity condition. For an interior edge e its dihedral angle can be interpreted as the intersection angle between circumscribed circles of triangles adjacent to e. Therefore, $\Delta \in (0; \pi)^{|E(\mathcal{T})|}$ means that the triangulation is Delaunay and Theorem 1.5.7 means that a Euclidean cone-metric on a closed surface is defined up to similarity by the intersection angles of circumscribed circles of its Delaunay triangulation. An easy geometric construction allows to derive from (a slight modification of Theorem 1.5.7) the uniqueness part of Theorem 1.2.8.

This result is notable for us because of its proof idea. We briefly describe it here. An ideal tetrahedron P in $\mathbb{H}^3$ is defined uniquely by its dihedral angles. In particular the volume of P is the function of the dihedral angles. An horosection of any vertex of P is a Euclidean triangle defined up to similarity with angles equal to dihedral angles of P. This allows us to assign the *hyperbolic volume* to each angle structure from $\mathfrak{A}(\mathcal{T})$. If $\mathfrak{A}(\mathcal{T})$ is non-empty, then it is a relatively open convex polyhedron. It is magical that critical points of the volume on $\mathfrak{A}(\mathcal{T})$ correspond to angle structures coming from actual metrics. The convexity condition implies that the minimal point of the volume over $\mathfrak{A}(\mathcal{T})$ exists.

Leibon [67] extended this result to hyperbolic cone-metrics on surfaces S_g of higher genus.

This approach, especially in Leibon's version, can be seen dual to ours. Instead of volume and angle structures we consider the discrete curvature functional over the space of manifolds with cone-singularities. It is important to note that in all papers [93], [67] and [15] the problems were considered only for a fixed triangulation. In our approach to Theorem 1.5.4 we consider a space of cone-metrics with varying (Delaunay) triangulations. It seems to us that when one needs to vary combinatorics, it is more productive to work with cone-structures in (2+1) dimensions and discrete curvature rather than with angle structures and the volume functional.

The magical properties of the volume functional were noted as the possible application towards the hyperbolization problem of hyperbolic cusp-3-manifolds. This is called the Casson-Rivin approach. As we mentioned before, a hyperbolic cusp-3-manifold M is homeomorphic to the interior of a manifold $\widetilde{M}$ with finitely many boundary components consisting of tori. An important problem was to determine, when a manifold of the latter type admits a hyperbolic cusp-metric. It was resolved by Thurston as a part of his hyperbolization theorem for Haken manifolds [109]. It states that a necessary and sufficient condition is the absence of essential embedded spheres, tori, disks and annuli. We skip here the definition of an essential surface. The cusp-metric is unique due to Theorem 1.3.1.

Consider a topological ideal triangulation of $\widetilde{M}$, i.e., a decomposition into cells homeomorphic to truncated tetrahedra. Try to replace each topological tetrahedron with a hyperbolic ideal one. They can be parametrized by dihedral angles. Therefore, we can start from assigning dihedral angles to edges in all tetrahedra.

An assignment is called an *angle structure* if

(1) angles are in $(0; \pi)$;
(2) opposite angles in a tetrahedron are equal;
(3) for each vertex of a tetrahedron the sum of adjacent angles is equal to π;

(4) around each edge the sum of adjacent angles is 2π.

The gluing of ideal tetrahedra corresponding to an angle structure produces a hyperbolic metric on M, which is, however, incomplete in general. One is interested to find an angle structure coming from a complete metric. The volume functional can be defined over the set of all angle structures. It is strictly convex and its critical points correspond to complete metrics. One can try to investigate the critical points. The main difficulty is that a random topological ideal triangulation might be non-realizable geometrically in the actual cusp-metric. This corresponds to the case when the critical point belongs to the boundary of the space of angle structures. This requires us to change a triangulation, but it has not been understood yet how to perform this change. We refer to [41] as a perfect introduction in the Casson-Rivin approach towards the geometrization. It was successfully realized in some particular cases: see [48, 40].

1.6 Statements of the results

Now we are going to formulate the main results of this thesis. The first result concerns *ideal Fuchsian cone-manifolds with convex polyhedral boundary*. Intuitively they are like ideal Fuchsian manifolds with convex polyhedral boundary, but with cone-type singularities along geodesics that emanate from ideal points of the upper boundary and are orthogonal to the lower boundary. A rigorous definition is somehow cumbersome and we refer for it to Chapter 3. Similarly to the non-singular case they have two boundary components: the upper $\partial^{\uparrow}P$ and the lower $\partial_{\downarrow}P$. We will prove the following realization and rigidity result:

Theorem A. Let d be a hyperbolic cusp-metric on S_g with $V = V(d)$ and let $\kappa' : V \to (-\infty; 2\pi)$ be a function such that

$$\sum_{v \in V} \kappa'(v) > 2\pi(2-2g).$$

Then there exists a unique up to isometry ideal Fuchsian cone-manifold P with convex polyhedral boundary such that (S_g, d) is isometric to $\partial^{\uparrow}P$ and $\kappa_v(P) = \kappa'(v)$, where $\kappa_v(P)$ is 2π minus the angle of the cone-singularity of the ideal point in $\partial^{\uparrow}P$ corresponding to v.

The main point of this thesis is the investigation of how cone-manifolds can be applied to other geometric problems. The original motivation of our work on Theorem A was to give a variational proof of Theorem 1.3.7 (which is a particular case of Theorem A). After this was done we became aware about the work [45] on discrete conformality proving Theorem 1.5.4. We thank Boris Springborn who pointed out that our approach provides a new variational proof of Theorem 1.5.4. Indeed, we will prove that Theorem A is equivalent to Theorem 1.5.4. Hence, one can see Theorem A as a way to interpret discrete conformality in terms of cone-manifolds.

The second main result of our thesis is the global rigidity of compact Fuchsian manifolds with convex boundaries:

Theorem B. Let F^1 and F^2 be two compact Fuchsian manifolds with convex boundaries and $f : \partial^{\uparrow}F^1 \to \partial^{\uparrow}F^2$ be an isometry between the upper boundaries. Then f extends to an isometry between F^1 and F^2.

This is an analogue of Pogorelov's global rigidity of convex bodies stated in Theorem 1.1.7 and Theorem 1.2.3. As we wrote in the previous sections, this is the first known result of this type for general metrics on manifolds more complicated than a ball. As mentioned in Section 1.1, our main intention was to revive the approach of Volkov [110]. We followed his general framework, however, we had to interpret many steps on our own. We hope that it should be possible to write a proof of Theorem 1.1.7 and Theorem 1.2.3 with the help of our interpretations. The proof of Theorem B proceeds by a polyhedral approximation. When we have a sequence of polyhedral manifolds converging to a manifold with general boundary metric, it is not enough to prove just the global rigidity of polyhedral manifolds. One needs also some sort of stability for them. Volkov's brilliant idea was to use cone-manifolds and the discrete curvature functional to control it. In the setting of Theorem B those are compact Fuchsian cone-manifolds with convex polyhedral boundary. They are very similar to the objects of Theorem A. The only difference are vertices: in one case they are "ordinary" points, in the other case they are ideal. This similar nature of the objects that we use in Theorem A and Theorem B allows us to consider them in parallel. This seems to us quite interesting on its own: we develop simultaneously the theory of cone-manifolds with compact boundary and with ideal boundary and compare all the differences that appear.

Some previous papers that relied heavily on Volkov's ideas are [14, 57, 35, 36].

The content of this thesis is as follows. In Chapter 2 we go through all preliminaries from metric geometry that we will use. In Chapter 3 we define Fuchsian cone-manifolds with convex boundary and establish their basic properties. In Chapter 4 we introduce our main tool, namely, the discrete curvature functional, use it to enhance our understanding of Fuchsian cone-manifolds and prove Theorem A. In Chapter 5 we investigate compact Fuchsian manifolds with general convex boundary and prove Theorem B modulo several tough lemmas concerning the stability of Fuchsian manifolds with polyhedral boundary. Chapter 6 is devoted to the proofs of these lemmas.

Theorem A is the subject of paper [90]. Theorem B is the subject of manuscript [91].

1.7 Further work directions

In the previous sections of the introduction we already mentioned some important open problems in the areas of our research. We recollect some of them here and give few more remarks.

Problem 1.7.1. Write down a proof of the Cohn-Vossen problem (Problem 1.1.9).

As we mentioned in Chapter 1.1, the Cohn-Vossen problem was presumably solved by Volkov in [110], but we can not reconstruct some important steps of his proof. We are also unaware of any researchers that claim to understand this proof despite we are aware of some attempts made.

In addition, one can state an analogue of the Cohn-Vossen problem for compact Fuchsian manifolds with boundary, which is a natural extension of our Theorem B. As we mentioned, for a proof of Theorem B we have to establish some kind of stability for polyhedral manifolds. One can expect that it should be promoted to a global stability. However, in our proof we use crucially that this is a local stability result for polyhedral manifolds close to a manifold with general boundary. Thereby, our proof in its current form can not be extended to a global stability result.

Problem 1.7.2. Prove an analogue of the Cohn-Vossen problem for compact Fuchsian manifolds with convex boundary.

Next, it is very tempting to extend Theorem B to compact hyperbolic manifolds with boundary.

Problem 1.7.3. Let F^1 and F^2 be compact hyperbolic manifolds with convex boundaries and $f : F^1 \to F^2$ be a homeomorphism that is isotopic to an isometry on the boundaries of F^1, F^2. Is it true that f is isotopic to an isometry?

Important particular cases are manifolds with polyhedral boundaries and manifolds with entirely hyperbolic boundaries. The latter are the convex cores of co-compact hyperbolic manifolds. This would imply the induced metric conjecture formulated in Section 1.3. A dual version also has a significant importance. The study of degenerations in the dual version could lead to a proof of the pleating lamination conjecture from Section 1.3. We hope that the means of the present thesis can be used to a further research in this direction.

As another problem, one can prove a common generalization of Theorem 1.3.10 and Theorem B:

Problem 1.7.4. Let F^1 and F^2 be two cusps with compact convex boundaries and $f : \partial F^1 \to \partial F^2$ be an isometry between their boundaries. Is it true that f extends to an isometry between F^1 and F^2?

This should be possible to prove using exactly the same techniques as in our proof of Theorem B. However, we note that there are unexpected topological difficulties to prove in this setting Main Lemma I formulated in Chapter 5.

There is one more promising direction of the research that we would like to point out. In Section 1.5 we described the unrealized Casson-Rivin approach to the geometrization of hyperbolic cusp-3-manifolds. There is a dual approach based on cone-manifolds. Instead of assigning angle structures to topological tetrahedra we may assign lengths and interpret each tetrahedron as a truncated ideal tetrahedron in $\mathbb{H}^3$. By gluing them together we obtain a complete cone-cusp-metric on M with cone-angles along the edges of the triangulation. If there is a cusp-metric without cone-singularities realizing the given triangulation, then it can be found as a critical point of the discrete curvature functional defined in Chapter 4. This functional is strictly concave over the space of admissible lengths. Hence, we are in a dual setting to the angle structures and the volume functional. However, there are some differences between these viewpoints. As we mentioned before, in the (2+1)-dimensional setting that we explore in this thesis the approach via cone-manifolds tends to be more compatible with changing combinatorics.

Problem 1.7.5. Investigate the geometrization of hyperbolic cusp-3-manifolds through cone-manifolds and discrete curvature.

Last, the discrete curvature functional described in Chapter 4 can be defined for arbitrary cone-3-manifolds in essentially the same way. Moreover, questions of infinitesimal rigidity (and, therefore, of local rigidity) can be transformed to questions on the signature of its Hessian in a similar way like we do in Chapter 4. In Section 1.4 we explained why these rigidity questions are important to 3-dimensional topology. Therefore, it is interesting if the discrete curvature might lead to their further understanding.

Problem 1.7.6. Investigate the rigidity of hyperbolic cone-3-manifolds with the help of the discrete curvature functional.

Chapter 2

Preliminaries

2.1 CBB(-1) metrics

First, we briefly sketch the facts that we need about metrics with curvature bounded from below by -1 in the sense of Alexandrov (CBB(-1) for short). For a detailed exposition of CBB(k) metrics we refer to [4, 7, 26, 88].

Let S be a connected oriented surface and d be a complete intrinsic metric on S. Then there exists a shortest path between any two points. This is a corollary of the Arzela–Ascoli theorem, see [26, Theorem 2.5.23].

Definition 2.1.1. A *geodesic* is a rectifiable curve (possibly closed), which is locally distance minimizing.

In Section 1.1 we defined a CBB(0) metric space. Here we reproduce the definition in the CBB(-1) case.

Let ψ, χ be two shortest paths in (S, d) sharing an endpoint p. Let $q \in \psi$ be the point at distance x from p and $r \in \chi$ be the point at distance y from p. Consider the hyperbolic triangle with side lengths x, y and $d(q, r)$ and let $\lambda(x, y)$ be the angle opposite to the side of length $d(q, r)$.

Definition 2.1.2. We say that (S, d) is *CBB(-1) metric space* if d is complete, intrinsic and for each $p \in S$ there exists a neighbourhood $U \ni p$ such that the function $\lambda(x, y)$ is a nonincreasing function of x and y for every ψ, χ emanating from p, in the range $x \in [0; x_0]$, $y \in [0; y_0]$ where the respective points q, r belong to U.

For other definitions of CBB(k)-spaces we refer to [26, Chapter 4] and [7, Chapter VII].

Definition 2.1.2 implies that the angle between ψ and χ, which we define as $\lim_{x,y\to 0} \lambda(x, y)$, exists. Denote it by $\mathrm{ang}(\psi, \chi, d)$. In this way the angle could be defined between any two rectifiable curves sharing an endpoint, which are possibly not geodesics, but the angle might not exist. We say that an (oriented) curve ψ emanating from a point p *has a direction* if the angle $\mathrm{ang}(\psi, \psi, d)$ exists (it is equal to zero if it exists). If two curves with the same endpoint have a direction, then there exists the angle between them.

It is important to note that the locality in Definition 2.1.2 can be dropped. Namely, for any three points $p, q, r \in (S, d)$ the angle between shortest paths from p to q, r is at least the respective angle in the *comparison triangle* for p, q, r, i.e., in the hyperbolic triangle with the side lengths $d(p, q)$, $d(q, r)$ and $d(p, q)$. This is called *the Toponogov globalization theorem*, which was proved in the general case by Perelman. We refer to [26, Theorem 10.3.1].

For geodesics in CBB(-1) spaces the *non-overlapping property* holds: if two geodesics have a segment in common, they can be covered by a larger geodesic.

Sometimes we need a slightly stronger version: if two shortest paths have two points in common, then either these are their endpoints or they have a segment in common.

The shortest paths ψ and χ emanating from a point p divide a sufficiently small neighbourhood of p into two *sectors* U and U'. Let $\psi_1, \dots, \psi_k$ be shortest paths emanating from p belonging to U enumerated in the order from ψ to χ. The angle $\mathrm{ang}(U, d)$ is defined as the supremum of the sums $\mathrm{ang}(\psi, \psi_1, d) + \ldots + \mathrm{ang}(\psi_k, \chi, d)$ over all finite collections of shortest paths from p in U. The smallest of $\mathrm{ang}(U, d)$, $\mathrm{ang}(U', d)$ is equal to $\mathrm{ang}(\psi, \chi, d)$. The total angle $\lambda_p(d)$ of p is equal to $\mathrm{ang}(U, d) + \mathrm{ang}(U', d)$.

A *geodesic polygon* is a submanifold of (S, d) with piecewise geodesic boundary. It is called *convex* if there is a shortest path between any two of its points that belongs to the polygon. It might be worth to note [2, Chapter III, Theorem 3]:

Lemma 2.1.3. Assume that (S_g, d) is a compact CBB(-1) space. Then it admits a decomposition into finitely many arbitrarily small convex geodesic triangles.

The area of a Borel set in (S, d) is defined intrinsically as its Hausdorff measure. See [26, Chapters 1.7, 2.6].

We sketch the concept of *intrinsic curvature* ν_I of a Borel set in (S, d).

For a point $p \in (S, d)$ we define $\nu_I(p, d) := 2\pi - \lambda_p(d)$.

For a relatively open geodesic segment ψ we define $\nu_I(\psi) := 0$.

For an open geodesic triangle T we define its curvature $\nu_I(T, d) := \alpha + \beta + \gamma - \pi$, where α, β, γ are the angles of T.

These three types of sets are called *primitive sets.* An *elementary set* is a set that can be represented as a finite disjoint union of primitive sets. Then its curvature is the sum of the curvatures of these primitive sets. It does not depend on a representation. Then for a closed set in (S, d) its curvature is defined as the infimum of the curvatures of its elementary supersets. For an open set its curvature is defined as the supremum of the curvatures of its closed subsets. This defines a Borel measure on (S, d): for the details we refer to [2, Chapter V].

However, we will mostly use the *extrinsic curvature* of a Borel set B, namely,

$$\nu(B, d) := \nu_I(B, d) + \mathrm{area}(B, d).$$

In what follows we will omit the word extrinsic.

2.2 Hyperbolic geometry

2.2.1 Hyperboloid model

Let $\mathbb{R}^{1,3}$ be the 4-dimensional Minkowski space, i.e., a real vector space equipped with the scalar product

$$\langle x, y \rangle := -x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3.$$

We identify

$$\mathbb{H}^3 = \{x \in \mathbb{R}^{1,3} : \langle x, x \rangle = -1,\ x_0 > 0\}.$$

By $\overline{\mathbb{H}}^3$ we denote the union of $\mathbb{H}^3$ with its boundary at infinity. We identify $\mathbb{R}^{1,2}$ with the plane $\{x : x_3 = 0\}$ and $\mathbb{H}^2$ with $\mathbb{H}^3 \cap \mathbb{R}^{1,2}$.

Define the future cone

$$\mathbb{L} = \{x \in \mathbb{R}^{1,3} : \langle x, x \rangle = 0,\ x_0 > 0\},$$

and the three-dimensional de Sitter space

$$d\mathbb{S}^3 = \{x \in \mathbb{R}^{1,3} : \langle x, x\rangle = 1\}.$$

There is a natural correspondence between ideal points of $\overline{\mathbb{H}}^3$ and generatrices of $\mathbb{L}$. An horosphere is the intersection of $\mathbb{H}^3$ and an affine plane K having the light-like normal vectors. Define its polar dual $k \in \mathbb{L}$ by the equation

$$\langle k, x\rangle = -1,$$

for all $x \in K$. Abusing the notation, we will sometimes use the same letter both for the horosphere and for the defining plane.

It is natural to interpret $d\mathbb{S}^3$ the space of oriented planes in $\mathbb{H}^3$. An oriented hyperbolic plane M in $\mathbb{H}^3$ is the intersection of $\mathbb{H}^3$ with an oriented linear two-dimensional time-like subspace of $\mathbb{R}^{1,3}$ with space-like unit normal m. Again, in our notation we will not distinguish these planes in $\mathbb{R}^{1,3}$ from the corresponding planes in $\mathbb{H}^3$. (The same also holds for hyperbolic lines in $\mathbb{H}^2$.)

We denote a point $A \in \mathbb{H}^3$ by capital letter when we think regardless the model, but its defining vector in the hyperboloid model is denoted by a.

We will need the following interpretation of scalar products between vectors of $\mathbb{R}^{1,3}$ in terms of distances in $\mathbb{H}^3$ (see [92] or [108]):

Lemma 2.2.1. 1. If $A_1, A_2 \in \mathbb{H}^3$, then

$$\langle a_1, a_2\rangle = -\cosh \operatorname{dist}(A_1, A_2).$$

2. If $A \in \mathbb{H}^3$ and M is an oriented plane in $\mathbb{H}^3$, then

$$\langle a, m\rangle = \sinh \operatorname{dist}(A, M),$$

where the distance is signed and depends on the half-space with respect to M containing A.

3. If $A \in \mathbb{H}^3$ and K is an horosphere, then

$$\langle a, k\rangle = -e^{\operatorname{dist}(A,K)}$$

where the distance is signed: it is positive if A is outside of the horoball bounded by K and negative otherwise.

4. If M_1, M_2 are oriented planes in $\mathbb{H}^3$, then

$$\langle m_1, m_2\rangle = \begin{cases} \cos \angle M_1, M_2 & \text{if } M_1, M_2 \text{ intersect in } \mathbb{H}^3 \\ \pm 1 & \text{if } M_1, M_2 \text{ are asymptotically parallel} \\ \pm\cosh \operatorname{dist}(M_1, M_2) & \text{if } M_1, M_2 \text{ are ultraparallel} \end{cases}$$

Here the sign depends on the mutual orientations of M_1, M_2.

5. If K_1, K_2 are horospheres, then

$$\langle k_1, k_2\rangle = -2e^{\operatorname{dist}(K_1,K_2)}$$

where the distance between two horospheres is the length of the common perpendicular taken with the minus sign if these horospheres intersect.

6. If M is an oriented plane in $\mathbb{H}^3$ and K is a horosphere, then

$$\langle m, k\rangle = \pm e^{\mathrm{dist}(M,K)}$$

where the distance between a plane and a horosphere is the length of the common perpendicular taken with the minus sign if the plane intersects the horosphere. The sign on the right hand side depends on at which halfspace with respect to M the center of K lies.

2.2.2 Hyperbolic convex bodies and duality

Let G be a closed convex set in $\mathbb{H}^3$ with non-empty interior distinct from $\mathbb{H}^3$. Then its boundary ∂G is homeomorphic to an open subset of the sphere S^2, see Section 1.2. First, we recall a fundamental result from convex geometry [73]:

Lemma 2.2.2 (The hyperbolic Busemann-Feller lemma)**.** Let $p, q \in \mathbb{H}^3\backslash G$ and p', q' be their nearest points on the boundary of G. Then the path distance between p' and q' on ∂G is at most the hyperbolic distance between p and q.

Corollary 2.2.3. Let $\psi \subset \mathbb{H}^3\backslash G$ be a rectifiable curve and ψ' be its nearest point projection to the boundary of G. Then ψ' is rectifiable and its length is at most the length of ψ.

Another well-known result is [7, Chapter XII]

Lemma 2.2.4. The boundary ∂G equipped with the induced path metric is CBB(-1).

In particular, the curvature measure ν is defined for ∂G as in Section 2.1.

In Section 1.2 we mentioned a well-known duality between convex sets is $\mathbb{H}^3$ and in $d\mathbb{S}^3$. Define the dual convex set $G^* \subset d\mathbb{S}^3$ as the set of all planes that do not intersect $\mathrm{int}(G)$ and are oriented outwards. We refer to [13, 38] for more details.

For each Borel set $U \subset \partial G$ define its dual $U^* \subset \partial G^*$ as the set of all planes tangent to G and passing through points of U. It is folklore that

$$\nu(U) = \mathrm{area}(U^*).$$

However, we are unaware of any sources that prove this in general case, hence, we are going to prove this now.

We restrict ourselves to the case, when G is a convex body in $\mathbb{H}^3$, i.e. compact, convex and, as before, with non-empty interior. We follow the framework from [13].

We work in the hyperboloid model. Define

$$d\mathbb{S}^3_+ := \{x \in d\mathbb{S}^3 : x_0 > 0\}.$$

Assume without loss of generality that the point $o = (1, 0, 0, 0)$ is in the interior of G. Consider the cone $\mathcal{C}(G)$ in $\mathbb{R}^{1,3}$ from the origin over G. Define its dual cone

$$\mathcal{C}(G)^* := \{x \in \mathbb{R}^{1,3} : \forall y \in \mathcal{C}(G), \quad \langle x, y\rangle \le 0\},$$

which is a convex cone containing the future cone of $\mathbb{R}^{1,3}$ in its interior. Then $G^* = \mathcal{C}(G)^* \cap d\mathbb{S}^3_+$ (see [13]).

Define the Gauss map $\mathcal{G} : \partial G \rightrightarrows \partial G^*$ as the multivalued map sending a point on ∂G to the set of its outward unit normals. The set ∂G^* is space-like and comes with a well-defined area measure $\sigma_{\partial G^*}$. This is proven in [13, Lemma 2.1].

Consider S^2 as the unit sphere in $T_o\mathbb{H}^3$, let $\mathcal{P} : S^2 \to \partial G$ be the radial projection, i.e., a map that sends a point from the sphere to the endpoint on ∂G of the geodesic in the respective direction, and let $r : S^2 \to \mathbb{R}_{>0}$ be the function measuring the length of the this geodesic. We pull back $\sigma_{\partial G^*}$ to S^2 via $\mathcal{G} \circ \mathcal{P}$ and denote the obtained measure by μ. We also pull back the area measure $\sigma_{\partial G}$ to S^2 via $\mathcal{P}$ and denote it by σ (note that it is not the same σ as in [13]).

We pull back our curvature measure ν to S^2 via $\mathcal{P}$ and, abusing the notation, continue to denote it by ν.

Lemma 2.2.5 ([13], Proposition A.3)**.** There exists a sequence G_k such that
(1) G_k are smooth and strictly convex;
(2) r_k converges to r uniformly;
(3) $\{G_k\}$ converges to G in the Hausdorff sense;
(4) the curvature measures μ_k converge weakly to μ.

Now we are ready to prove

Lemma 2.2.6. The measures ν and μ coincide. Thus, for each Borel set $U \subset \partial G$

$$\nu(U) = \text{area}(U^*).$$

Proof. First, we claim that $\nu_k = \mu_k$. Indeed, for $p \in S^2$ let $K_I(p)$ and $K(p)$ be its intrinsic and extrinsic (Gaussian) curvatures at $\mathcal{P}(p) \in \partial G$. The Gauss equation says that $K_I(p) + 1 = K(p)$. Proposition 2.2.1 from [13] states that $\mu_k = K_k d\sigma_k$. We claim that also $\nu_k = K_k d\sigma_k$. Due to the Gauss equation, it is enough to show that $\nu_{I,k} = K_{I,k} d\sigma_k$. Take an open geodesic triangle T. The Gauss-Bonnet theorem shows that $\nu_{I,k}(T) = \int_T K_{I,k} d\sigma_k$. The measures of other elementary sets (geodesic arcs and points) are zero for smooth metrics. The proof of $\nu_k = \mu_k$ for any Borel set follows from the definition of ν_I and elementary properties of the Hausdorff measure $d\sigma$. As $\nu_k = \nu_{I,k} + \sigma_k$, we get $\nu_k = K_k d\sigma_k = \mu_k$.

Theorem 7 from [2, Chapter 7] says that $\nu_{I,k}$ converge weakly to the intrinsic curvature measure ν_I of ∂G. Theorem 9 from [2, Chapter 8] says that the area measures σ_k converge weakly to σ. Thus, ν_k converge weakly to ν and, therefore, $\nu = \mu$. □

2.2.3 Ideal triangles

From now on, unless something different is stated, we consider ideal points always equipped with horospheres (or horocycles). Under this agreement we use the word *distance* between two points even in the cases when one of them or both are ideal. In the latter case, by distance we mean the signed distance between the corresponding horospheres: we write it with the minus sign if the horospheres intersect. In the former case, the distance means the signed distance from the non-ideal point to the horosphere at the ideal point. Similarly, we speak about the *length* of a segment even if one or two of its endpoints are ideal. Moreover, if $A \in \overline{\mathbb{H}}^3$ is an ideal point, then by a we denote the polar vector in $\mathbb{R}^{1,3}$ to the respective horosphere at A.

Lemma 2.2.7. Let $A_1A_2A_3$ be an ideal hyperbolic triangle with side lengths l_1, l_2 and l_3 respectively, and let α_1 be the length of the part of the horosphere at A_1 inside the triangle. Then

$$\alpha_1^2 = e^{l_1 - l_2 - l_3}.$$

A proof can be found in [81, Proposition 2.8]. In what follows we will need a semi-ideal version of this lemma:

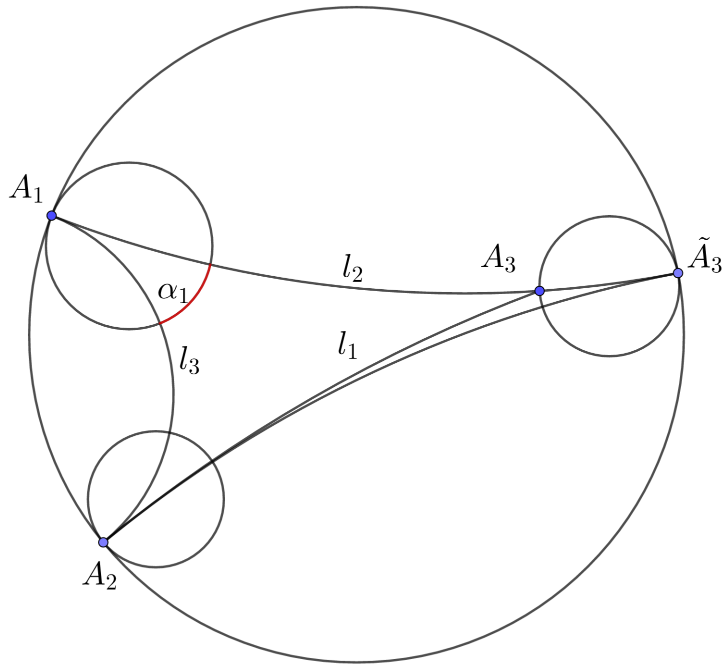

FIGURE 2.1: To the proof of Lemma 2.2.8.

Lemma 2.2.8. Let $A_1A_2A_3$ be a hyperbolic triangle with ideal vertices A_1 and A_2, non-ideal vertex A_3, side lengths l_1, l_2 and l_3 respectively, and let α_1 be the length of the part of the horosphere at A_1 inside the triangle. Then

$$\alpha_1^2 = e^{l_1 - l_2 - l_3} - e^{-2l_2}.$$

Proof. Let $\widetilde{A}_3$ be the intersection of the ray A_1A_3 with boundary at infinity and put the horocycle at $\widetilde{A}_3$ passing through A_3 (see Figure 2.1). Denote the side lengths of this new ideal decorated triangle by $\widetilde{l}_1$, $\widetilde{l}_2 = l_2$ and $\widetilde{l}_3 = l_3$. From Lemma 2.2.7 it follows that $\alpha_1^2 = e^{\widetilde{l}_1 - \widetilde{l}_2 - \widetilde{l}_3} = e^{\widetilde{l}_1 - l_2 - l_3}$. Hence, we need to calculate $\widetilde{l}_1$.

In the hyperboloid model we have

$$\widetilde{a}_3 = \lambda a_3 + \mu a_1,$$

$$\langle \widetilde{a}_3, a_1 \rangle = -\lambda e^{l_2} = -2e^{l_2}.$$

Hence, we obtain that $\lambda = 2$. Now calculate

$$\langle \widetilde{a}_3, a_3 \rangle = -\lambda - \mu e^{l_2} = -1.$$

We obtain $\mu = -e^{-l_2}$. It remains to evaluate

$$\langle \widetilde{a}_3, a_2 \rangle = -2e^{\widetilde{l}_1} = -2e^{l_1} + 2e^{l_2 - l_2}.$$

We get $e^{\widetilde{l}_1} = e^{l_1} - e^{l_3 - l_2}$. Finally, $\alpha_1^2 = e^{l_1 - l_2 - l_3} - e^{-2l_2}$. □

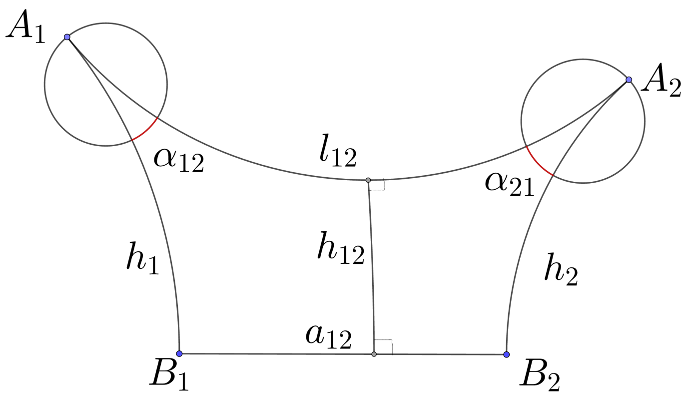

FIGURE 2.2: An ideal trapezoid. Ideal vertices are equipped with horocycles.

2.3 Trapezoids and prisms

Trapezoids and prisms are objects of the greatest importance in our investigation of Fuchsian cone-manifolds.

Definition 2.3.1. A *trapezoid* is the convex hull of a segment $A_1A_2 \subset \overline{\mathbb{H}}^2$ and its orthogonal projection to a line such that the segment A_1A_2 does not intersect this line. It is called *ultraparallel* if the line A_1A_2 is ultraparallel to the second line. It is called *ideal* if both A_1 and A_2 are ideal. It is called *compact* if both A_1 and A_2 are non-ideal. If some vertices are ideal, then we consider them equipped with horocycles, which we call *canonical*.

By B_i denote the image of A_i under the projection, $i = 1, 2$. We refer to A_1A_2 as to the *upper edge*, to B_1B_2 as to the *lower edge* and to A_iB_i as to the *lateral edges*. The vertices A_i are also called *upper* and B_i are called *lower*. We denote by l_{12} the length of A_1A_2, by a_{12} the length of B_1B_2, by h_i the length of A_iB_i, by α_{12} and α_{21} the angles at vertices A_1 and A_2 (or the lengths of horocycles if the vertices are ideal) and by h_{12} the distance from the line A_1A_2 to the line B_1B_2 in the case of an ultraparallel trapezoid. We note that every ideal trapezoid is ultraparallel.

Definition 2.3.2. A *prism* is the convex hull of a triangle $A_1A_2A_3 \subset \overline{\mathbb{H}}^3$ and its orthogonal projection to a plane such that the triangle $A_1A_2A_3$ does not intersect this plane. It is called *ultraparallel* if the plane $A_1A_2A_3$ is ultraparallel to the second plane. It is called *ideal* if all A_i are ideal. It is called *compact* if all A_i are non-ideal. If some vertices are ideal, then we consider them equipped with horospheres, which we call *canonical*.

Similarly to trapezoids, by B_i we denote the image of A_i under the projection, $i = 1, 2, 3$, and we distinguish edges and faces of a prism into *upper*, *lower* and *lateral*. The lateral faces of a prism are trapezoids. The dihedral angles of edges B_iB_j are equal $\pi/2$. The dihedral angles of edges A_1A_2, A_2A_3 and A_3A_1 are denoted by ϕ_3, ϕ_1 and ϕ_2 respectively. The dihedral angle of an edge A_iB_i is denoted by ω_i.

We do not allow degenerations of the upper triangle, but we consider degenerate prisms with collinear B_1, B_2 and B_3, when the upper plane is orthogonal to the lower one. However, soon we will restrict ourselves only to ultraparallel ones, which do not degenerate.

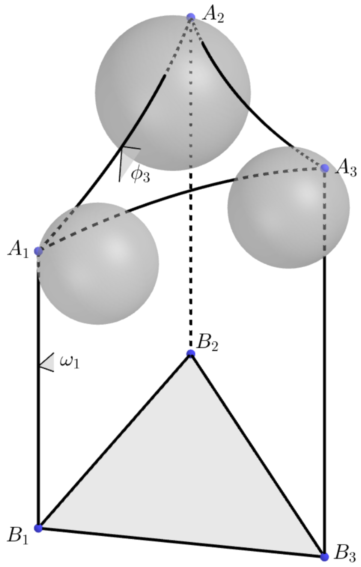

FIGURE 2.3: An ideal prism. Ideal vertices are equipped with horospheres.

It is natural to see a trapezoid (or a prism) as a triangle (respectively, a tetrahedron) with one hyperideal vertex dual to the lower edge (lower face).

The proof of next two lemmas is straightforward: see Thurston's book [108, Chapter 2.6] for a general approach allowing to obtain these formulas.

Lemma 2.3.3 (The cosine laws)**.** For a compact trapezoid we have

$$\cos\alpha_{12} = \frac{\cosh l_{12}\sinh h_1 - \sinh h_2}{\sinh l_{12}\cosh h_1},$$

$$\cosh a_{12} = \frac{\sinh h_1 \sinh h_2 + \cosh l_{12}}{\cosh h_1 \cosh h_2}.$$

Lemma 2.3.4 (The sine law)**.** For a compact trapezoid we have

$$\frac{\sinh a_{12}}{\sinh l_{12}} = \frac{\sin\alpha_{12}}{\cosh h_2} = \frac{\sin\alpha_{21}}{\cosh h_1}.$$

In particular, for right-angled trapezoids we will use the following identities:

Corollary 2.3.5. Assume that in a compact trapezoid $\alpha_{21} = \pi/2$. Then

$$\sinh h_1 = \cosh l_{12}\sinh h_2, \quad \tanh h_1 = \cosh a_{12}\tanh h_2,$$

$$\sin\alpha_{12} = \frac{\cosh h_2}{\cosh h_1}, \quad \cot\alpha_{12} = \sinh l_{12}\tanh h_2.$$

Now we investigate similar identities for ideal trapezoids. In this case we can not follow the Thurston approach and start with a lemma for a semi-ideal trapezoid:

Lemma 2.3.6. Let $A_1A_2B_2B_1$ be an ultraparallel trapezoid with $A_1 \in \partial_\infty\mathbb{H}^2$, $A_2 \in \mathbb{H}^2$ and $\alpha_{21} = \pi/2$. Then

$$e^{h_1} = e^{l_{12}}\sinh h_2,$$

$$\tanh a_{12} = \frac{1}{\cosh h_2}.$$

Proof. Consider the point $A \in A_1B_1$ inside the horodisk at A_1. Let l be the length AA_2 and h be the length AB_1. Then from Corollary 2.3.5 we have

$$\sinh h_2 = \frac{\sinh h}{\cosh l}.$$

Now let h' be the modulo of the length AA_1, hence $h = h_1 + h'$. Extend (if necessarily) AA_2 to the intersection point A' with the horocycle at A_1, let l' be the modulo of the length AA', l'' be the length A_1A' taken with the minus sign if A_1 is inside the horodisk, then $l = l'' + l'$. Move the point A to A_1 and consider the limit of the expression:

$$\sinh h_2 = \lim_{A \to A_1} \frac{\sinh h_1 \cosh h' + \cosh h_1 \sinh h'}{\cosh l' \cosh l'' + \sinh l' \sinh l''} =$$

$$= \lim_{A \to A_1} \frac{\sinh h_1 + \cosh h_1 e^{h'}}{\cosh l'' + \sinh l'' e^{l'}} = e^{h_1 - l_{12}}.$$

This is because $h' - l'$ tends to zero and l'' tends to l_{12}.

The second formula is obtained similarly. □

From this we obtain analogues of cosine and sine laws for ideal trapezoids:

Corollary 2.3.7. For an ideal trapezoid we have

$$\cosh a_{12} = 1 + 2e^{l_{12} - h_1 - h_2}.$$

Corollary 2.3.8. For an ideal ultraparallel trapezoid we have

$$\alpha_{12}^2 = e^{h_2 - h_1 - l_{12}} + e^{-2h_1}.$$

A compact or an ideal trapezoid or a prism is clearly determined uniquely by the lengths of the lower and lateral edges. It follows from the cosine laws that they are also uniquely defined the lengths of the upper and the lateral edges. We will actually need this also in the mixed ultraparallel case when some vertices are ideal and some are not. The requirement of ultraparallelism is not necessary, but otherwise we need to consider too many cases. For ultraparallel ones this follows easily from Lemma 2.3.6 by cutting (or extending) respective trapezoids. We omit the details in the mixed case and state

Corollary 2.3.9. An ultraparallel trapezoid or a prism is determined up to isometry by the lengths of the upper and the lateral edges.

We end the section with two purely computational corollaries of cosine/sine laws that will be used further.

Corollary 2.3.10. For an ideal trapezoid we have

$$\cosh a_{12} = 1 + \frac{2}{\sinh^2 h_{12}}.$$

Corollary 2.3.11. For a compact trapezoid we have

$$\frac{\cot \alpha_{12}}{\cosh h_1} = \frac{\cosh a_{12} \tanh h_1 - \tanh h_2}{\sinh a_{12}}.$$

2.4 Cone-metrics and cusp-metrics

2.4.1 Cone-metrics and triangulations

Let S_g be a closed oriented connected surface of genus g.

Definition 2.4.1. A *topological triangulation* of S_g with vertex set V is a collection of simple disjoint paths with endpoints in V that cut S_g into triangles with vertices in V. Two topological triangulations with the same vertex set V are *equivalent* if they are isotopic with respect to V (so the edges are not allowed to pass through points of V during the isotopy). A *triangulation* $\mathcal{T}$ of S_g with vertex set V is an equivalence class of topological triangulations.

This definition allows loops and multiple edges between two vertices, thereby a triangulation from our definition might be not simplicial. By $V(\mathcal{T})$ we denote the set of vertices of $\mathcal{T}$ and by $E(\mathcal{T})$ we denote the set of edges of $\mathcal{T}$ considered as isotopy classes of the respective paths.

We give the definition of a hyperbolic cone-metric similar to its Euclidean analogue from Section 1.1:

Definition 2.4.2. A *hyperbolic cone-metric* d on S_g is locally isometric to the metric of hyperbolic plane except finitely many points called *conical points*. At a conical point v the metric d is locally isometric to the metric of a hyperbolic cone with angle $\lambda_v(d) \neq 2\pi$. The number $\nu_v(d) := 2\pi - \lambda_v(d)$ is called the *curvature* of v. We denote the set of conical points of d by $V(d)$. A hyperbolic cone-metric is called *convex* if for every $v \in V(d)$ we have $\lambda_v(d) < 2\pi$. Metrics are considered up to isometry isotopic to identity with respect to $V(d)$.

Lemma 2.4.3 ([7], Chapter XII)**.** A convex hyperbolic cone-metric is CBB(-1).

From now on we restrict ourselves almost only to the hyperbolic case and omit the word “hyperbolic” saying just cone-metrics (except some special cases).

We note that on a cone-metric there can be multiple shortest paths between two points.

Definition 2.4.4. A *geodesic triangulation* is a topological triangulation such that all edges are geodesics.

Definition 2.4.5. Let $\mathcal{T}$ be a triangulation of S_g and d be an intrinsic metric. We say that $\mathcal{T}$ is *realized* by d if there is a geodesic triangulation of (S_g, d) in the class $\mathcal{T}$.

Note that if d is a cone-metric, then we do not require in this definition neither $V(\mathcal{T}) \subseteq V(d)$ nor the converse. We highlight that degenerated triangles are not allowed because the edges are defined up to isotopy with respect to $V(\mathcal{T})$. Sometimes when we write $e \in E(\mathcal{T})$ and the realization of $\mathcal{T}$ is evident, then we mean the respective realization of e. Similarly, when we consider a triangle T of $\mathcal{T}$ we frequently mean its realization in a metric.

We recall that a geodesic on a convex cone-metric can not pass through conical points. We will frequently need the following result that follows from [58, Proposition 4]:

Lemma 2.4.6. If d is a cone-metric and $V \supseteq V(d)$, then there exists a triangulation $\mathcal{T}$ realized by d with $V(\mathcal{T}) = V$. Moreover, any set of disjoint geodesic paths with vertices in V can be extended to a geodesic triangulation.

In this case the metric d is uniquely determined by $\mathcal{T}$ and the edge lengths. Fix $V \subset S_g$. By $\mathfrak{D}(V)$ we denote the space of cone-metrics d on S_g with $V(d) \subseteq V$ up to isometry isotopic to identity with respect to V. By $\mathfrak{D}_c(V) \subset \mathfrak{D}(V)$ we denote the subspace of convex metrics. By $\mathfrak{D}_{sc}(V) \subset \mathfrak{D}_c(V)$ we denote the subspace of strictly convex metrics on V, i.e., $d \in \mathfrak{D}_{sc}(V)$ if and only if $V(d) = V$ and d is convex.

Let $\mathcal{T}$ be a triangulation of S_g with $V(\mathcal{T}) = V$. By $\mathfrak{D}(V, \mathcal{T}) \subset \mathfrak{D}(V)$ we denote the set of cone-metrics realizing $\mathcal{T}$. Similarly we denote its subsets $\mathfrak{D}_c(V, \mathcal{T})$ and $\mathfrak{D}_{sc}(V, \mathcal{T})$. Recall that if $n := |V|$, then any triangulation of S_g with vertices at V has $2(n+2g-2)$ triangles and $N = 3(n+2g-2)$ edges. Then the edge lengths map $l : \mathfrak{D}(V, \mathcal{T}) \to \mathbb{R}^N$ is an embedding. Abusing the notation, we frequently identify $\mathfrak{D}(V, \mathcal{T})$ with its image under this map. Let us study some basic properties.

The set $\mathfrak{D}(V, \mathcal{T}) \subset \mathbb{R}^N$ is an open polyhedron defined by the strict triangles inequalities for all triangles of $\mathcal{T}$. For every $v \in V$ the total angle of v defines an analytic function $\lambda_v : \mathfrak{D}(V, \mathcal{T}) \to (0; \infty)$. Then $\mathfrak{D}_c(V, \mathcal{T})$ is the subset of $\mathfrak{D}(V, \mathcal{T})$ satisfying inequalities $\lambda_v(d) \leq 2\pi$. It is a semi-analytic set.

If $d \in \mathfrak{D}(V)$ realizes two triangulations, then the transition maps are smooth. This endows $\mathfrak{D}(V)$ with a smooth manifold structure. The set $\mathfrak{D}_{sc}(V)$ is its open subset.

Lemma 2.4.7. Let $d \in \mathfrak{D}(V, \mathcal{T})$. Define a 1-parameter family of cone-metrics $d_t \in \mathfrak{D}(V, \mathcal{T})$ by $l_e(d_t) = t \cdot l_e(d)$ for each $e \in E(\mathcal{T})$, where $t \in [1; +\infty)$. Then $\lambda_v(d_t)$ is strictly monotonously decreasing for every $v \in V$.

Proof. It is clear that all strict triangle inequalities are satisfied after multiplying by t, therefore equations $l_e(d_t) = t \cdot l_e(d)$ indeed define a metric $d_t \in \mathfrak{D}(V, \mathcal{T})$. It remains to prove that if all edge lengths of a hyperbolic triangle ABC are multiplied with the same factor $t > 1$, then its angles become strictly smaller.

Let $A'B'C'$ be a triangle with the increased side lengths. Consider $B'' \in A'B'$ and $C'' \in A'C'$ such that $A'B'' = AB$ and $A'C'' = AC$. It suffices to show that $B''C'' < BC$.

Now let $A'_0B'_0C'_0$ be a Euclidean comparison triangle for $A'B'C'$ and $B''_0 \in A'_0B'_0$, $C''_0 \in A'_0C'_0$ be points such that $A'_0B''_0 = AB$ and $A'_0C''_0 = AC$. Because of scaling we get $B''_0C''_0 = BC$. But $B''_0C''_0 > B''C''$ because of elementary properties of comparison geometry (as hyperbolic plane has curvature -1). $\square$

In what follows we will need the following corollary. Let $d \in \mathfrak{D}_c(V, \mathcal{T})$, $\sigma > 0$ and $\mathfrak{B}(d, \sigma)$ be the open ball centered at d of radius σ in $(\mathbb{R}^N, l_\infty)$. Define $\mathfrak{B}_c(d, \sigma) = \mathfrak{D}_c(V, \mathcal{T}) \cap \mathfrak{B}(d, \sigma)$ and $\mathfrak{B}_{sc}(d, \sigma) = \mathfrak{D}_{sc}(V, \mathcal{T}) \cap \mathfrak{B}(d, \sigma)$.

Corollary 2.4.8. For sufficiently small σ the set $\mathfrak{B}_{sc}(d, \sigma)$ is connected.

Proof. If $d \in \mathfrak{D}_{sc}(V, \mathcal{T})$, then the statement is clear as $\mathfrak{D}_{sc}(V, \mathcal{T})$ is an open set.

Assume that $d \in \mathfrak{D}_c(V, \mathcal{T})$, but not in $\mathfrak{D}_{sc}(V, \mathcal{T})$. As $\mathfrak{D}_c(V, \mathcal{T})$ is semi-analytic, it is locally connected, i.e., for sufficiently small σ the set $\mathfrak{B}_c(d, \sigma)$ is connected. Now consider two points in $\mathfrak{B}_{sc}(d, \sigma)$. They can be joined by a path d_t in $\mathfrak{B}_c(d, \sigma)$. As $\mathfrak{B}(d, \sigma)$ is open, then for $t_0 > 1$ sufficiently close to 1, if we multiply all side lengths of d_t by t_0, then the new path d'_t still belongs to $\mathfrak{B}(d, \sigma)$. On the other hand, d'_t belongs to $\mathfrak{D}_{sc}(V, \mathcal{T})$ by Lemma 2.4.7. It remains to connect the endpoints of the old path with the endpoints of the new one. $\square$

2.4.2 Cusp-metrics and decorations

Recall the definition of a cusp-metric from Section 1.2:

Definition 2.4.9. A *hyperbolic cusp-metric* d on S_g is a complete hyperbolic metric of finite area on $S_g \backslash V(d)$, where $V(d)$ is a finite set. Points of $V(d)$ are called *cusps*.

Similarly to cone-metrics, we will say just a cusp-metric dropping the word "hyperbolic". As every complete hyperbolic manifold, (S_g, d) can be represented as $\mathbb{H}^2/\Gamma$, where Γ is a discrete subgroup of $\mathrm{Iso}_+(\mathbb{H}^2)$ isomorphic to $\pi_1(S_g)$ unique up to conjugation. The fundamental domain of the action of Γ in the case of a cusp-metric is an ideal polygon.

Definition 2.4.10. A *decoration* of a cusp-metric space (S_g, d) is a choice of a horocycle at each cusp.

After we chose a decoration, we can speak about the distance between two cusps as we agreed in Section 2.2.3: we consider the signed distance between the respective horocycles taken with minus sign if they intersect.

Despite d is actually a metric on a surface with punctures $S_g \backslash V(d)$ rather than on S_g, we can still speak about realizations of triangulations with some vertices at points of $V(d)$. Then the edges are infinite geodesics approaching the cusps like ideal points of $\overline{\mathbb{H}}^2$. It is well-known that

Lemma 2.4.11 ([71], Proposition 7.4.6)**.** For each cusp-metric d on S_g each triangulation $\mathcal{T}$ with $V(\mathcal{T}) = V(d)$ is realized by d.

In a similar way as with cone-metrics, for a finite set $V \subset S_g$ we define the space $\mathfrak{C}(V)$ of cusp-metrics d on S_g with $V(d) = V$ up to isometry isotopic to identity with respect to V. By $\widetilde{\mathfrak{C}}(V)$ we denote the space of decorated cusp-metrics. These spaces are studied much better rather than the respective spaces for cone-metrics: see [81, 82] as the main references. After we fix a triangulation $\mathcal{T}$ with $V(\mathcal{T}) = V$ we have again the edge-length map $l : \widetilde{\mathfrak{C}}(V) \to \mathbb{R}^N$ with $N = 3(n+2g-2)$. It is classical that

Lemma 2.4.12 ([81], Theorem 3.1)**.** The map l is a bijection.

After we chose a decoration of (S_g, d), every other decoration can be decoded by $h \in \mathbb{R}^n$: each coordinate represents the signed distance from a new horocycle to the initial one. This defines a fibration $\widetilde{\mathfrak{C}}(V) \to \mathfrak{C}(V)$ with the fibre $\mathbb{R}^n$.

Each decorated cusp-metric has a canonical decomposition into ideal polygons, which is of great importance for their study. We devote the next subsection to it.

2.4.3 Epstein–Penner decompositions

Let d be a cusp-metric on S_g with a decoration. Consider the hyperboloid model of $\mathbb{H}^2$ and represent (S_g, d) as $\mathbb{H}^2/\Gamma$. By $K_i^1, K_i^2, \ldots$ denote horocycles in the orbit of the horocycle at i-th cusp under the action of Γ. By $\mathcal{K}$ denote the union of their polar vectors k_i^j.

Let C be the convex hull of the set $\{k_i^j\}$ in $\mathbb{R}^{1,2}$. Its boundary ∂C is divided into two parts $\partial_l C \sqcup \partial_t C$ consisting of light-like points and time-like points. Below we describe well-known properties of this construction. For proofs we refer to [71, Chapter 5.1.7], [30] and [81].

Lemma 2.4.13.
- The convex hull C is 3-dimensional.
- The set $\partial_l C = C \cap \mathbb{L}$ is the set of points λk_i^j for $\lambda \geq 1$.
- Every time-like ray intersects $\partial_t C$ exactly once.

- The boundary $\partial_t C$ is decomposed into countably many polygons. The supporting plane containing each polygon is space-like. This decomposition is Γ-invariant and projects to a decomposition of (S_g, d) into finitely many ideal polygons.

Definition 2.4.14. This decomposition is called *the Epstein–Penner decomposition* of (S_g, d) with respect to the given decoration.

Definition 2.4.15. *An Epstein–Penner triangulation* of (S_g, d) is a geodesic triangulation with vertices at cusps that refines the Epstein–Penner decomposition for some decoration.

Chapter 3

Fuchsian cone-manifolds

3.1 Basic definitions

Let d be either a cone-metric or a decorated cusp-metric on S_g. In the latter case we will refer to the horodisks of the chosen decoration as to the *canonical horodisks* and to their boundary as to the *canonical horocycles*.

Definition 3.1.1. A *representable triple* is a triple $(d, \mathcal{T}, h)$, where $\mathcal{T}$ is a triangulation of S_g with $V(\mathcal{T}) = V$, $d \in \mathfrak{D}_c(V, \mathcal{T})$ or $d \in \widetilde{\mathfrak{C}}(V)$ and $h : V \to \mathbb{R}_{>0}$ is a function on V such that for every triangle T of $\mathcal{T}$ there exists a prism with the lengths of the upper edges defined by the side lengths of T in d and the lengths of the lateral edges determined by h. We write $h(v)$ as h_v.

Definition 3.1.2. Let $(d, \mathcal{T}, h)$ be a representable triple. Take all prisms determined by h, $\mathcal{T}$ and d and glue them isometrically according to $\mathcal{T}$. In the ideal case we choose the gluing isometries such that the canonical horocycles match together. The resulting intrinsic metric space P together with the canonical isometry from (S_g, d) to the upper boundary $\partial^\uparrow P$ provided by our construction, is called a *marked Fuchsian cone-manifold with polyhedral boundary*. In what follows we will mostly omit the words "marked" and "with polyhedral boundary" saying only a *Fuchsian cone-manifold* for short. If d is a decorated cusp-metric, then we call P *decorated ideal*. We will omit the word "decorated" implying that this is always the case. If d is a cone-metric, then we call P *compact*.

We note that we will never mix ideal and compact Fuchsian cone-manifolds together. However, one can definitely explore metrics having both cone singularities and cusps, like it is done in [33], and build cone-manifolds with their help. We consider this unnecessary for our needs as this provides some technical difficulties, which we would like to avoid. It is more interesting for us to build both theories in parallel and compare one with another. Thereby, all statements that we make are restricted to one of these classes, although frequently they are similar.

We say that a representable triple $(d, \mathcal{T}, h)$ is a *representation* of P and write $P = P(d, \mathcal{T}, h)$. The upper and lower boundaries of P are defined naturally. When a representation of P is given, we will not attach significance whether $p \in (S_g, d)$ or $p \in \partial^\uparrow P$ using the canonical isometry as an identification. We also say that the triangulation $\mathcal{T}$ is *compatible with* P. The function h is called the *height function* of P.

We say that an isometry $f : P^1 \to P^2$ between two Fuchsian cone-manifolds, where $P^1 = P(d^1, \mathcal{T}^1, h^1)$ and $P^2 = P(d^2, \mathcal{T}^2, h^2)$ with $V(\mathcal{T}^1) = V(\mathcal{T}^2) = V$, is a *marked isometry* if its composition with the canonical isometries of $\partial^\uparrow P^1$, $\partial^\uparrow P^2$ induces an isometry between (S_g, d^1) and (S_g, d^2) isotopic to identity with respect to V. We will consider Fuchsian cone-manifolds up to marked isometry.

For a Fuchsian cone-manifold $P = P(d, \mathcal{T}, h)$ and $v \in V(\mathcal{T})$ we denote by $\omega_v(P)$ the sum of dihedral angles of the respective lateral edges in all prisms incident to v in P and define the *particle curvature* of v in P as $\kappa_v(P) := 2\pi - \omega_v(P)$. If all $\kappa_v(P) = 0$, then P is a Fuchsian manifold with polyhedral boundary. We will also call it a *polyhedral Fuchsian manifold* for short. Let $\mathcal{T}$ be a triangulation compatible with P. For $e \in E(\mathcal{T})$ denote by $l_e(d)$ its length in the metric d, by $\phi_e(P)$ its dihedral angle in P and by $\theta_e(P) := \pi - \phi_e(P)$ the *curvature* of e. A Fuchsian cone-manifold P is called *convex* if the dihedral angles $\phi_e(P)$ of all edges of P are not greater than π. Convex Fuchsian cone-manifolds are our main objects and we consider non-convex ones only sometimes in intermediate steps of proofs.

The upper boundary $\partial^\uparrow P$ admits a canonical stratification into vertices, edges and faces. We start from compact cone-manifolds. First, for a point $p \in \partial^\uparrow P$ we define naturally its *spherical link* as the gluing of the spherical links of p in all prisms containing it. If this link is a hemisphere, then p is called *regular*. If the link is a spherical lune, then p is called a *ridge point*. Otherwise, it is a *vertex* of P. It is standard to define *faces* as connected components of regular points and edges as connected components of ridge points. If $\kappa_v(P) \neq 0$, then the spherical link of v is a spherical polygon with a conical singularity in the interior.

However, we want to change this notation slightly. First, every compact cone-manifold P under our consideration carries a marked point set $V \subset \partial^\uparrow P$ containing the vertices of P. We will refer to points of V as to vertices of P despite some of them might not be actual vertices of P. Next, we call *edge* any geodesic segment in $\partial^\uparrow P$ between two points of V (that may coincide) that is geodesic in P. If it is an actual edge of P, then we say that it is a *strict edge*. Otherwise, we call it a *flat edge*. For faces we continue to keep the same definition.

The case of ideal cone-manifolds is simpler because the set of vertices V is always the set of cusps of (S_g, d). For edges and faces of ideal cone-manifolds we follow the same convention as in the compact case. We need to make a remark on links of the vertices of ideal cone-manifolds. Canonical horodisks of (S_g, d) in each prism of $P(d, \mathcal{T}, h)$ give rise to sectors of horoballs inside the prism, which we also call canonical. The boundary of such a sector is a Euclidean triangle. Gluing these triangles around the vertex produces a Euclidean polygon with a conical singularity in the interior (in case of $\kappa_v \neq 0$). We will refer to it as to the *horospherical link* of v.

We denote the set of the faces P by $\mathcal{R}(P)$ and call it the *face decomposition* of P. A triangulation $\mathcal{T}$ is compatible with P if and only if it refines the face decomposition $\mathcal{R}(P)$.

In the compact case it might happen that even an actual vertex of P is isolated in the sense that there are no strict edges emanating from it. Then its spherical link is a hemisphere with a conical singularity in its center. In case of polyhedral cusps with particles an example is given in [35]. It is easy to adapt it to higher genus and we do not provide it here. We provide another interesting example: we show that faces can be homotopically non-trivial.

Example 3.1.3. Namely, we construct an example of a compact convex Fuchsian cone-manifold containing a homotopically nontrivial curve in the interior of a face.

Consider two ultraparallel planes $M^\uparrow$ and $M_\downarrow$ in $\mathbb{H}^3$. Let AB be their common perpendicular, $A \in M^\uparrow$, $B \in M_\downarrow$.

Let $R_A = A_1A_2A_3A_4$ and $R_C = C_1C_2C_3C_4$ be two regular 4-gons in $M^\uparrow$ with center A such that points A, A_i and C_i are on the same line for every i and R_A is in the interior of R_C. Let B_i, D_i be the orthogonal projections of A_i, C_i to $M_\downarrow$ respectively. Consider the annulus $R_1 = \overline{R_C \backslash R_A}$ in $M^\uparrow$. Join each point of R_1 with its

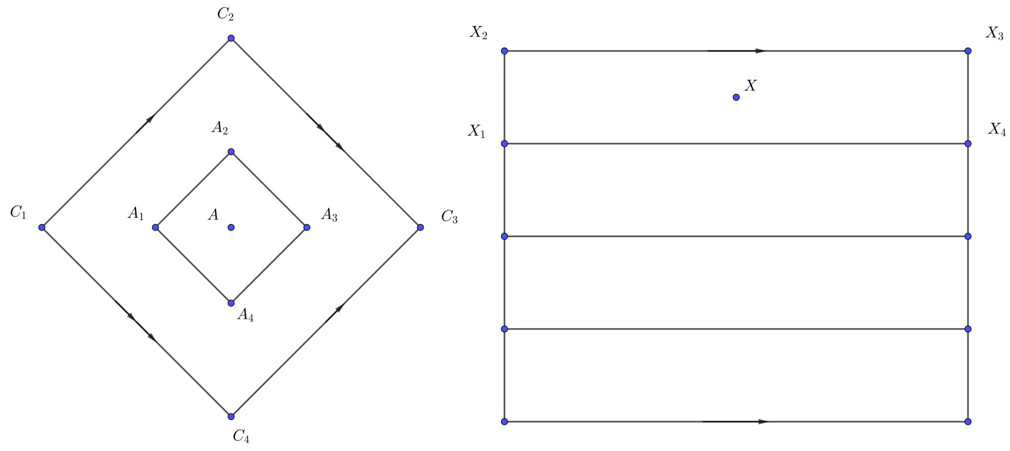

FIGURE 3.1: Gluing schemes for the upper boundaries of Block I' and Block II.

orthogonal projection to $M_\downarrow$. We call the obtained polyhedron Block I (see Figure 3.1 left).

Consider two other ultraparallel planes $N^\uparrow$ and $N_\downarrow$ in $\mathbb{H}^3$, let XY be their common perpendicular, $X \in N^\uparrow$, $Y \in N_\downarrow$.

Now we consider a 4-gon $R_X = X_1X_2X_3X_4 \subset N^\uparrow$ symmetric with respect to X such that $X_1X_2 = X_3X_4 = A_1A_2$ and $X_2X_3 = X_1X_4 = t$. We assume t be very large. Let Y_i be the orthogonal projections of X_i to $N_\downarrow$. Join each point of R_X with its orthogonal projection to $N_\downarrow$. We choose the distance XY so that the lateral edges X_iY_i are equal to A_iB_i. Note that if we choose t sufficiently large, then the dihedral angles of the obtained polyhedron at X_1X_2, X_3X_4 are small as well as the angles of X_i in the upper face R_X.

Take four copies of this polyhedron and glue the lateral faces of the sides corresponding to X_2X_3, X_1X_4 subsequently. We obtain a polyhedral tube. We call it Block II (see Figure 3.1 right). Its upper boundary is a cylinder with two isometric boundary components. By construction, the lateral faces of Block II are isometric to lateral faces of Block I coming from points A_i (recall that a trapezoid is determined by the upper and the lateral edges).

Take a copy of Block I. Identify its lateral face $C_1C_2D_1D_2$ with $C_3C_4D_3D_4$ and $C_2C_3D_2D_3$ with $C_1C_4D_1D_4$ as in Figure 3.1 left. Thereby, the upper boundary becomes a torus with a hole coming from R_A. We call it Block I'. The remaining lateral boundary of Block I' is isometric to a connected component of the lateral boundary of Block II. Take two copies of Block I'. We can glue a copy of Block II identifying each lateral boundary component with the remaining part of the lateral boundary of Blocks I'.

In the upper boundary it looks like we connected two tori with holes by a tube. Thus, the upper boundary becomes a surface of genus two endowed with a cone-metric. It is convex if t is sufficiently large. The obtained cone-manifold is a compact Fuchsian cone-manifold. It is also convex if t is sufficiently large. It has two homotopically non-trivial faces coming from Blocks I'.

Manipulating more with Blocks I and Blocks II we can obtain such examples for any genus ≥ 2.

It is interesting that in the ideal case this can not happen: see Lemma 3.2.3.

We rephrase from Section 1.2 the following results of Fillastre and Schlenker:

Theorem 3.1.4. [32] For each convex cone-metric d on S_g there exists a unique up to isometry compact convex polyhedral Fuchsian manifold P with $\partial^{\uparrow}P$ isometric to (S_g, d).

Theorem 3.1.5. [33, 98] For each cusp-metric d on S_g there exists a unique up to isometry ideal convex polyhedral Fuchsian manifold P with $\partial^{\uparrow}P$ isometric to (S_g, d).

Remark 3.1.6. We note that the proofs of the uniqueness in Theorems 3.1.4, 3.1.5 can be strengthened to the uniqueness up to marked isometry. This is equivalent to the following: if the upper boundary metric d of a convex Fuchsian manifold P admits a self-isometry, then it extends to an isometry of P. We briefly sketch the argument in the compact case to convince the reader.

Let $\mathfrak{P}(V)$ be the set of marked convex polyhedral Fuchsian manifolds with vertices at V up to isometry isotopic to identity with respect to V. For $P \in \mathfrak{P}(V)$ the induced metric of $\partial^{\uparrow}P$ is in $\mathfrak{D}_{sc}(V)$. This defines a map $\mathfrak{I} : \mathfrak{P}(V) \to \mathfrak{D}_{sc}(V)$. After setting a natural topology on $\mathfrak{P}(V)$, it is proven in [32] that $\mathfrak{I}$ is a homeomorphism. Noting that $\mathfrak{I}$ is equivariant with respect to the action of the mapping class group of (S_g, V) we obtain Theorem 3.1.4.

One of the most important tools in our study of cone-manifolds is the following.

Definition 3.1.7. Let $P(d, \mathcal{T}, h)$ be a Fuchsian cone-manifold. The function $\widetilde{h} : \partial^{\uparrow}P \to \mathbb{R}_{>0}$ assigning to a point $p \in \partial^{\uparrow}P$ its distance to $\partial_{\downarrow}P$ is called the *extended height function* of P.

In the compact case it coincides with h at the vertices of $\mathcal{T}$.

Lemma 3.1.8. Take $p, q \in \partial^{\uparrow}P(d, \mathcal{T}, h)$. Then $\widetilde{h}(p) \leq \widetilde{h}(q) + d(p, q)$.

The proof is straightforward.

Now we examine some important properties of Fuchsian cone-manifolds.

3.2 Ultraparallelism

Lemma 3.2.1. Let $P = P(d, \mathcal{T}, h)$ be an ideal convex Fuchsian cone-manifold. Then each prism of P is ultraparallel.

Proof. Suppose the contrary. We construct an infinite sequence of pairwise distinct prisms of P.

Assume that there exists a prism Π of P embedded to $\mathbb{H}^3$ as $A_1A_2A_3B_1B_2B_3$ such that its upper boundary plane $M^{\uparrow}$ intersects the lower boundary plane $M_{\downarrow}$ in line L. If Π is degenerated, then we set $M_{\downarrow}$ as the plane containing the lower boundary and orthogonal to $M^{\uparrow}$.

The intersection of $M^{\uparrow}$ with $\partial_{\infty}\mathbb{H}^3$ is a circle. The line L divides it into two arcs. All points A_1, A_2 and A_3 belong to the same arc and one of them lies between the two others. Assume that this point is A_1. Then we call the edge A_2A_3 *heavy* and two other edges *light*.

Let ϕ be the (smallest) dihedral angle between $M^{\uparrow}$ and $M_{\downarrow}$. For every $p \in M^{\uparrow}$ we have

$$\sinh \widetilde{h}(p) = \sinh \operatorname{dist}(p, M_{\downarrow}) = \sinh(\operatorname{dist}(p, L)) \sin \phi, \tag{3.2.1}$$

by the sine law in a right-angled hyperbolic triangle.

It follows that the distances from the light edges to $M_\downarrow$ are both strictly bigger than the distance from the heavy edge. For the dihedral angles of the upper edges we have $\phi_1 > \pi/2$ and ϕ_2, $\phi_3 < \pi/2$.

The edge corresponding to A_2A_3 can not be glued in $\mathcal{T}$ neither with the edge corresponding to A_1A_2 nor to A_1A_3 because these edges have bigger distances to $M_\downarrow$ than A_2A_3. Therefore, the second prism Π' of P adjacent to this edge is distinct from Π. Embed the it as $A_2A_3A_4B_2B_3B_4$ in $\mathbb{H}^3$ so that it is glued with Π over the face $A_2A_3B_2B_3$ via an orientation-reversing isometry. Then $B_4 \in M_\downarrow$.

The total dihedral angle at A_2A_3 is less or equal than π. Hence, the plane $A_2A_3A_4$ also intersects $M_\downarrow$. The light edges and the heavy edge are defined for Π' in the same way. Moreover, it is clear that in Π' the edge A_2A_3 is light. Hence, we see that the distance from the new heavy edge to $M_\downarrow$ is strictly less than the distance from A_2A_3. Now for this edge we choose the next prism adjacent to it and continue this process. The distances from the heavy edges to $M_\downarrow$ are strictly decreasing in the obtained sequence of prisms. Thereby, the prisms are all distinct. But the number of prisms in P is finite. We get a contradiction.

If there exists only a prism with $M^\uparrow$ intersecting $M_\downarrow$ at infinity, then the arguments are the same, but instead of the intersection line L we fix an horosphere at the ideal intersection point and consider the distances to it. □

The proof above uses sufficiently the geometry of ideal points. Thus the proof in the compact case is slightly more elaborate, however, quite similar.

Lemma 3.2.2. Let $P = P(d, \mathcal{T}, h)$ be a compact convex Fuchsian cone-manifold. Then each prism of P is ultraparallel.

Proof. Similarly, we suppose the contrary and construct an infinite sequence of pairwise distinct prisms of P.

We start also the same: take a prism Π of P developed to $\mathbb{H}^3$ as $A_1A_2A_3B_1B_2B_3$ such that its upper boundary plane $M^\uparrow$ intersects the lower boundary plane $M_\downarrow$ in line L under dihedral angle ϕ.

We call the distance function w to L over the triangle $A_1A_2A_3$ the *weight function* and its minimum the *weight* $w(\Pi)$. It is easy to see that the point p, where $w(\Pi)$ is attained, is unique and belongs to the boundary of $A_1A_2A_3$. We have

$$\sinh \widetilde{h}(p) = \sinh w(p) \sin \phi.$$

Suppose that p is an interior point of an edge (say, A_2A_3). Then the line A_2A_3 is ultraparallel to L. We observe that that the dihedral angle of Π at A_2A_3 is greater than $\pi/2$. Take next the prism Π' adjacent to Π at the edge corresponding to A_2A_3. Embed it to $\mathbb{H}^3$ adjacent to $A_1A_2A_3B_1B_2B_3$. Then its upper boundary plane also intersects the lower boundary plane $M_\downarrow$ in line L'. Let ϕ' be the respective dihedral angle. Due to the convexity condition, the angle of Π' at A_2A_3 is smaller than $\pi/2$ and $\phi \le \phi' \le \pi/2$. To see this one considers the orthogonal plane section to lines L and A_2A_3, which exists as they are ultraparallel. We get $w'(p) \le w(p)$, where w' is the weight function of Π' defined as the distance to L', and $w'(p) = w(p)$ if and only if $\phi = \phi'$.

If the weight of Π' is attained at p, then its dihedral angle at the edge containing p is also greater than $\pi/2$. This can not happen as the total angle is at most π. Therefore, it can not be attained at p and $w(\Pi') < w(\Pi)$.

Now suppose that $w(\Pi)$ is attained at a vertex of Π (say, A_3). One can see that among the edges incident to A_3 there exists at least one (say, A_2A_3) with the dihedral angle greater than $\pi/2$. Take the adjacent prism Π' and embed it to $\mathbb{H}^3$ adjacent to $A_1A_2A_3B_1B_2B_3$. Similarly to the case before, convexity implies that the upper plane of Π' intersects $M_\downarrow$ in line L' with the dihedral angle ϕ' satisfying $\phi \le \phi' \le \pi/2$. If $\phi' > \phi$, then $w'(A_3) < w(A_3)$ and, therefore, $w(\Pi') < w(\Pi)$. Assume that $w(\Pi') = w(\Pi)$, then $w(\Pi')$ is also attained at A_3 and $\phi' = \phi$, so the upper plane of Π' coincides with $M^\uparrow$.

By L_0 denote the line in $M^\uparrow$ through A_3 orthogonal to L oriented outwards to L. Assume that A_3A_2 lies to the left of L_0 and γ be the angle between L_0 and A_3A_2. The condition that $w(\Pi)$ is attained at A_3 implies that $\gamma \le \pi/2$.

Let A_3A_4 be the edge of Π' incident to A_3. As $w(\Pi) = w(\Pi')$, then A_3A_4 also lies to the left of L_0 and for its angle γ' with L_0 we have $\gamma < \gamma' \le \pi/2$.

In this way we construct an infinite sequence of prisms of P. Two prisms in this sequence are distinguished either by their weight, which decreases monotonously, or, if the weights are equal, by the angle γ, which strictly increases as long as the weight remains constant. Clearly, both $w(\Pi)$ and γ are independent of the embedding. This shows a contradiction.

The asymptotically parallel case does not show any difference. $\square$

Let $P(d, \mathcal{T}, h)$ be convex and $\widetilde{h}$ be the extended height function. Take $T \in \mathcal{T}$ and embed the prism Π containing T to $\mathbb{H}^3$. Lemmas 3.2.1 and 3.2.2 say that this prism is ultraparallel. Let A and B be the closest points on the upper and lower boundary planes respectively. Due to Corollary 2.3.5, $\widetilde{h}$ satisfies for every $p \in T$

$$\sinh \widetilde{h}(p) = \sinh AB \cosh(\mathrm{dist}(p, A)). \tag{3.2.2}$$

In particular, its restriction to an edge of P has the form

$$\mathrm{arsinh}(b \cosh(x - a)). \tag{3.2.3}$$

We highlight that it frequently happens that A and B do not belong to Π. However, neither $\mathrm{dist}(x, A)$ nor the distance AB do depend on the choice of representation or embedding.

With the help of this expression we can prove

Lemma 3.2.3. Let R be a face of an ideal convex Fuchsian cone-manifold P. Then R is simply connected.

Proof. First, we prove that if R is not simply connected, then there is a closed geodesic in R.

Let $\mathcal{T}$ be a triangulation compatible with P. Choose a simple homotopically nontrivial closed curve ψ in R transversal to edges of $\mathcal{T}$. Develop all triangles that intersect ψ to $\mathbb{H}^2$ (each triangle is developed once). We obtain an ideal polygon R'. The triangulation $\mathcal{T}$ is lifted to a triangulation of R'. All inner edges of R' are lifts of flat edges of P.

Let $\rho : R' \to R$ be the inverse to the developing map. It is injective in the interior, but glue at least two boundary edges of R' to a flat edge of R. Denote them by AB and CD so that $\rho(AB) = \rho(CD)$, $\rho(A) = \rho(C)$ and $\rho(B) = \rho(D)$ (note that A may coincide with C and B may coincide with D). For a point $X \in AB$ there is a unique point $Y \in CD$ such that $\rho(Y) = \rho(X)$. Hyperbolic segment XY projects to a closed curve in R, which is a geodesic except possibly one point $\rho(X) = \rho(Y)$. Clearly, $\rho(XY)$ is a closed geodesic if and only if $\angle BXY + \angle XYD = \pi$. It is clear that as X tends to B, the point Y tends to D and this sum tends to 2π. Similarly, as X tends

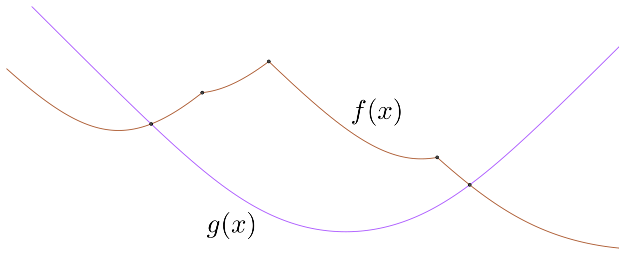

FIGURE 3.2: An $\mathcal{F}(-1)$-concave function.

to C, this sum tends to 0. Therefore, there exists X such that this sum is equal to π. In this case $\rho(XY)$ is a closed geodesic $\psi' \subset R$.

Consider the distance function $\widetilde{h}$ of P. Its restriction to ψ' must be periodic, because ψ' is a closed geodesic. On the other hand ψ' intersects no strict edges. Therefore, the restriction of ρ to ψ' has the form (3.2.3), which is not periodic. We obtain a contradiction. □

This implies an important

Corollary 3.2.4. There are finitely many triangulations compatible with an ideal convex Fuchsian cone-manifold.

3.3 Heights define a convex cone-manifold

We need the notion of a $\mathcal{F}(-1)$-concave function due to Alexander and Bishop [3].

Definition 3.3.1. By $\mathcal{F}(-1)$ we denote the set of twice continuously differentiable functions $g : \mathbb{R} \to \mathbb{R}$ satisfying $g'' = g$. A continuous function $f : I \to \mathbb{R}$ is $\mathcal{F}(-1)$-*concave* if for each $[x_1; x_2] \subseteq I$ we have $f \geq g$, where $g \in \mathcal{F}(-1)$, $g(x_1) = f(x_1)$ and $g(x_2) = f(x_2)$ (see Figure 3.2).

A continuous function $f : (S_g, d) \to \mathbb{R}$ is $\mathcal{F}(-1)$-*concave* if it becomes $\mathcal{F}(-1)$-concave when restricted to every unit speed geodesic.

An element of $\mathcal{F}(-1)$ is a linear combination of $\sinh x$ and $\cosh x$ and is defined uniquely by values at two points.

Consider a convex Fuchsian cone-manifold $P(d, \mathcal{T}, h)$ (it does not matter ideal or compact) and a unit speed geodesic $\psi : [0; \tau] \to (S_g, d)$. Let $x_1, \ldots, x_k \in (0; \tau)$ be its intersection points with strict edges of P. We set $x_0 := 0$ and $x_{k+1} := \tau$. Due to (3.2.3), on each segment $[x_i; x_{i+1}]$ the restriction of $\widetilde{h}$ has the form

$$\widetilde{h}(x) = \operatorname{arsinh}(a_i \cosh(x - b_i))$$

for some positive a_i. Moreover, the convexity implies that for each kink point x_i we have the left derivative greater than the right derivative (see $f(x)$ on Figure 3.2).

Lemma 3.3.2. The function $\sinh \widetilde{h}$ is $\mathcal{F}(-1)$-concave.

Proof. Let $g : [x^0; x^1] \to \mathbb{R}$ be in $\mathcal{F}(-1)$ and $f : [x^0; x^1] \to \mathbb{R}$ be a function such that

(1) $g(x^0) = f(x^0)$, $g(x^1) = f(x^1)$;

(2) there is a subdivision

$$[x^0; x^1] = [x_0; x_1] \cup \ldots \cup [x_k; x_{k+1}]$$

($x^0 = x_0$ and $x_{k+1} = x^1$) such that the restriction of f to each $[x_i; x_{i+1}]$ is $\mathcal{F}(-1)$;

(3) at every point x_i the left derivative of $\widetilde{h}(x)$ is greater than the right derivative.

We claim that $f(x) \geq g(x)$ for all $x \in [x^0; x^1]$. Due to the discussion above, this will imply that $\sinh \widetilde{h}$ is $\mathcal{F}(-1)$-concave.

Claim 1. Two distinct $\mathcal{F}(-1)$-functions on $\mathbb{R}$ can not coincide at more than one point.

This follows from the definition of the $\mathcal{F}(-1)$-class.

Claim 2. Let $f_i(x) : \mathbb{R} \to \mathbb{R}$ be the extension to $\mathbb{R}$ of the restriction $f|_{[x_i;x_{i+1}]}$. For all $x \notin [x_i; x_{i+1}]$ we have $f_i(x) > f(x)$.

Indeed, consider $j > i$ and prove by induction over $j - i$ that for $x \in [x_j; x_{j+1}]$ we have $f_i(x) > f_j(x)$. The base case $j = i + 1$ follows from Claim 1 and the fact that in the kink point x_{i+1} the derivative of f_i is greater or equal than the derivative of f_{i+1}. The inductive step is obvious. The case $j < i$ is the same.

Claim 3. For all $x \in (x_0; x_1]$ we have $f(x) \geq g(x)$.

Indeed, Claim 1 and the assumption $f(x^0) = g(x^0)$ imply that the sign of the difference $f(x) - g(x)$ is constant on $(x_0; x_1]$. Assume that it is negative, i.e., for every $x > x^0$ we have $f_0(x) < g(x)$. Also assume $k \neq 0$. Substituting $x = x^1$ and using Claim 2 we obtain that $f(x^1) = f_k(x^1) < f_0(x^1) < g(x^1)$, which contradicts the statement.

Now we may assume that the sign of the difference $f(x) - g(x)$ is positive on $(x_0; x_1]$: if it is zero, then we just cut off $[x_0; x_1]$ and proceed in the same way. Under this assumption, suppose that for some i and $x' \in (x_i; x_{i+1}]$, $x' \neq x^1$, we have $f(x') = g(x')$. Due to Claim 1 and the assumption, we have $f_i(x) < g(x)$ for all $x > x'$. Thus, due to Claim 2 we have $f(x) < g(x)$ for all $x > x'$. We obtain a contradiction.

It implies that $f(x) \geq g(x)$ over the interval $[x^0; x^1]$. □

We note a simple property of $\mathcal{F}(-1)$-concave functions similar to a property of ordinary concave functions. Let $f : I \to \mathbb{R}$ be a $\mathcal{F}(-1)$-concave function and $g \in \mathcal{F}(-1)$ coincides with f at points $x_1, x_2 \in I$. Then for all $x \notin [x_1; x_2]$ we get $f(x) \leq g(x)$. If for some $x \in (x_1; x_2)$ we have $f(x) > g(x)$, then $f(x) > g(x)$ for all $x \in (x_1; x_2)$ and for all $x \notin [x_1; x_2]$ we get $f(x) < g(x)$.

Now we are going to prove that a convex Fuchsian cone-manifold is uniquely defined by d and h. If we know d and h, then it remains only to restore its face decomposition. Contrary to ultraparallelism, this is easier to do for compact cone-manifolds.

Lemma 3.3.3. Let $\mathcal{T}^1$ and $\mathcal{T}^2$ be two triangulations with

$$V(\mathcal{T}^1) = V(\mathcal{T}^2) = V,$$

$P^1 = P(d, \mathcal{T}^1, h)$ and $P^2 = P(d, \mathcal{T}^2, h)$ be two compact convex Fuchsian cone-manifolds. Then P^1 is marked isometric to P^2 with respect to V.

Proof. Take the intersection point v' of two edges $e^1 \in E(\mathcal{T}^1)$ and $e^2 \in E(\mathcal{T}^2)$. The restriction of $\sinh \widetilde{h}^1$ to e^1 is $\mathcal{F}(-1)$. Due to Lemma 3.3.2, the restriction of $\sinh \widetilde{h}^2$ to e^1 is $\mathcal{F}(-1)$-concave. As they coincide at the endpoints of e^1, we get $\sinh \widetilde{h}^2(v') \geq \sinh \widetilde{h}^1(v')$. Applying the same reasoning to e^2, we get $\sinh \widetilde{h}^2(v') \leq \sinh \widetilde{h}^1(v')$. Hence, $\widetilde{h}^2(v') = \widetilde{h}^1(v')$.

Let V' be V together with all the intersection points between the edges of $\mathcal{T}^1$ and $\mathcal{T}^2$. We obtained $\widetilde{h}^1|_{V'} = \widetilde{h}^2|_{V'}$. The union of geodesic triangulations $\mathcal{T}_1$ and $\mathcal{T}_2$ decomposes the metric space (S_g, d) into geodesic polygons with vertices in V'. Let $\mathcal{T}'$

be a geodesic triangulation refining this decomposition. The convex Fuchsian cone-manifold $P(d, \mathcal{T}', \widetilde{h}^1|_{V'})$ is marked isometric to both P^1, P^2 due to Corollary 2.3.9. This finishes the proof. $\square$

Now we prove a similar statement in the ideal case.

Lemma 3.3.4. Let $\mathcal{T}^1$ and $\mathcal{T}^2$ be two triangulations with

$$V(\mathcal{T}^1) = V(\mathcal{T}^2) = V,$$

$P^1 = P(d, \mathcal{T}^1, h)$ and $P^2 = P(d, \mathcal{T}^2, h)$ be two ideal convex Fuchsian cone-manifolds. Then P^1 is marked isometric to P^2 with respect to V.

Proof. The idea is just the same: to prove that the distance functions coincide at the intersection point v' of any two edges $e^1 \in E(\mathcal{T}^1)$ and $e^2 \in E(\mathcal{T}^2)$ and use Corollary 2.3.9 to construct a marked isometry. Let us only prove $\widetilde{h}^2(v') \geq \widetilde{h}^1(v')$, the converse inequality is obtained similarly. To this purpose we need to investigate the behaviour of distance functions near cusps. But first, we recall Claim 1 from the proof of Lemma 3.3.2:

Claim 1. Two $\mathcal{F}(-1)$-functions on $\mathbb{R}$ can not coincide at more than one point.

Another easy statement is

Claim 2. Let

$$g_1(x) = b_1 \cosh(x - a_1),$$
$$\rho_2(x) = b_2 \cosh(x - a_2)$$

and for $x_0 \in \mathbb{R}$ we have $g_1(x_0) = g_2(x_0)$ and $\dot{g}_1(x_0) > \dot{g}_2(x_0)$. Then $a_2 > a_1$.

Indeed, we have

$$\dot{g}_1(x_0) = b_1 \sinh(x_0 - a_1) > b_2 \sinh(x_0 - a_2) = \dot{g}_2(x_0).$$

Using $g_1(x_0) = g_2(x_0)$ and the fact that b_1, b_2 are positive we obtain that this is equivalent to

$$\tanh(x_0 - a_1) = \frac{\sinh(x_0 - a_1)}{\cosh(x_0 - a_1)} > \frac{\sinh(x_0 - a_2)}{\cosh(x_0 - a_1)} = \tanh(x_0 - a_2).$$

The function tanh is strictly increasing. This shows the desired statement.

Claim 3. Let ψ_1 and ψ_2 be two distinct geodesic lines in $\mathbb{H}^2$ meeting at a point $A \in \partial_\infty \mathbb{H}^2$ and ultraparallel to a line ψ_0. Let A be decorated by an horocycle and ψ_1, ψ_2 be parametrized by the (signed) distance to this horocycle. Denote the distance functions from ψ_1 and ψ_2 to ψ_0 by

$$g_1(x) = \operatorname{arsinh}(b_1 \cosh(x - a_1)),$$

$$g_2(x) = \operatorname{arsinh}(b_2 \cosh(x - a_2))$$

respectively. Then $g_1(x) - g_2(x)$ has a constant nonzero sign. Besides, if $g_1(x) > g_2(x)$, then $a_2 > a_1$.

Indeed, the claim on constant sign is straightforward. For the second claim, let $A_i \in \psi_i$ be the closest point from ψ_i to ψ_0 for $i = 1, 2$. Recall from (3.2.2) that b_i is the hyperbolic sine of the distance from ψ_i to ψ_0 and a_i is the distance from A_i to the horocycle. Observe that the sign of $g_1(x) - g_2(x)$ is the sign of $b_1 - b_2$. Also note that $a_1 = a_2$ if and only if $b_1 = b_2$ (if and only if lines ψ_1 and ψ_2 coincide) and a_i decreases as b_i grows.

Now return to the problem. Let $g : \mathbb{R} \to \mathbb{R}$ be an $\mathcal{F}(-1)$-function obtained by restriction of $\sinh \widetilde{h}^1$ to e^1 and $f : \mathbb{R} \to \mathbb{R}$ is obtained from $\sinh \widetilde{h}^2$. Then we can decompose $\mathbb{R} = (-\infty; x_0] \cup [x_0; x_1] \cup \ldots \cup [x_k; +\infty)$ with respect to the intersections of e^1 with strict edges of P^2. For convenience, we set $x_{-1} = -\infty$ and $x_{k+1} = +\infty$. For every $i = 0, \ldots, k+1$, the restriction of f to $(x_{i-1}; x_i)$ has the form: $b_i \cosh(x - a_i)$. We denote the extension to $\mathbb{R}$ of this restriction by f_i. The subdivision is induced by intersections with only strict edges of P, therefore, each pair (a_i, b_i) is distinct from the pair (a_{i+1}, b_{i+1}) and at every kink point x_i the left derivative of f is strictly greater than the right derivative. As in Claim 2 from the proof of Lemma 3.3.2, we have $f_i(x) > f(x)$ for each $x \notin [x_i; x_{i+1}]$.

By Claim 3 the sign of $g(x) - f_0(x)$ is constant. Suppose that $f_0(x) < g(x)$. Then by Claim 3 we have $a < a_0$. For all $x \in \mathbb{R}$ we see that $f(x) \leq f_0(x) < g(x)$. By Claim 2 and induction we get $a_0 < a_1 < \ldots < a_{k+1}$. Therefore, $a < a_{k+1}$. On the other hand, consider the parametrization of e^1 by distance to the horosphere at another endpoint. Then the new distance functions are $g(l_{e^1} - x)$ and $f(l_{e^1} - x)$, where l_{e^1} is the length of edge e^1. We apply Claim 3 one more time and obtain $l_{e^1} - a < l_{e^1} - a_{k+1}$. This is equivalent to $a_{k+1} < a$ and gives a contradiction. We also obtain the same contradiction if we suppose that $f_k(x) < g(x)$.

Now suppose that for $1 \leq i \leq k$ and a point $x' \in [x_{i-1}; x_i]$ we have $f(x') < g(x')$. We know that f_i is strictly bigger than f outside of $[x_{i-1}; x_i]$. Also by Claim 1 we see that either for all $x \in (-\infty; x']$ or for all $x \in [x'; +\infty)$ we have $g(x) > f_i(x)$. Altogether this gives us that either $f_0(x) < g(x)$ or $f_k(x) < g(x)$. However, we already showed that this is impossible. Thereby, $f(x) \geq g(x)$ everywhere and, in particular, $\widetilde{h}^2(v') \geq \widetilde{h}^1(v')$ as desired. □

Hence, while we restrict ourselves to convex Fuchsian cone-manifolds, we can drop the triangulation from its definition. However, we need the base set V, i.e., we represent the cone-manifold simply as $P = P(d, V, h)$, where $V(d) \subseteq V$ in the compact case and $V = V(d)$ in the ideal case.

Definition 3.3.5. We call a function h on V to be *admissible* for (d, V) if there exists a convex Fuchsian cone-manifold $P(d, V, h)$. By $H(d, V) \subset \mathbb{R}^n$ we denote the set of all admissible h for the pair (d, V), where $n := |V|$ and functions on V are associated with points in $\mathbb{R}^n$.

Let $V(d) \subseteq V \subset V'$, $P(d, V, h)$ be a compact convex Fuchsian cone-manifold and $\widetilde{h} : \partial^{\uparrow} P \to V$ is the extension of h. The extension gives us a height function $h' \in H(d, V')$. This defines the canonical embedding $H(d, V) \hookrightarrow H(d, V')$. In what follows, when we need such an extension, we will abuse the notation and write just $P = P(d, V', h)$ in the sense that we extend h to V' using $\widetilde{h}$.

In the opposite direction, assume that $P(d, V', h)$ is a compact convex Fuchsian cone-manifold, $V \subset V'$ and for any $v \in V' \backslash V$ we have $\nu_v(d) = 0$ and $\kappa_v(d) = 0$. Let $h|_V$ be the restriction to the set V of h. Then P can be represented as $P(d, V, h|_V)$. In such case we will also simply write $P(d, V, h)$ instead.

For a triangulation $\mathcal{T}$ denote by $H(d, V, \mathcal{T}) \subset H(d, V)$ the set of all admissible $h \in \mathbb{R}^n$ such that $\mathcal{T}$ is compatible with $P(d, V, h)$. This defines a subdivision of $H(d, V, \mathcal{T})$ into cells corresponding to different triangulations. It is evident that the boundary of $H(d, V, \mathcal{T})$ is piecewise analytic.

3.4 The space of admissible heights in the ideal case

In this section we will describe $H(d,V)$ for a cusp-metric d. Actually, we are going to prove that $H(d,V)=\mathbb{R}^n$. This is very different from the compact case.

Let (S_g,d) be a decorated cusp-surface. Recall from Section 2.4.2 that every other decoration can be decoded by $h\in\mathbb{R}^n$: each coordinate represents a distance from a new horocycle to the initial one. We also recall that Epstein–Penner triangulations were discussed in Section 2.4.3.

Lemma 3.4.1. Let $\mathcal{T}$ be a triangulation with $V=V(\mathcal{T})$. The triple $(d,\mathcal{T},h)$ is representable and the respective ideal Fuchsian cone-manifold $P(d,V,h)$ is convex if and only if $\mathcal{T}$ is an Epstein–Penner triangulation of (S_g,d) for the decoration defined by h.

Clearly, this implies $H(d,V)=\mathbb{R}^n$. Moreover, the face decomposition $\mathcal{R}(P)$ of $P=P(d,V,h)$ is exactly the Epstein–Penner decomposition of (S_g,d) with the decoration defined by h.

Proof. Let $h\in\mathbb{R}^n$ and $\mathcal{T}$ be one of its Epstein–Penner triangulations. Represent S_g as $\mathbb{H}^2/\Gamma$ and lift $\mathcal{T}$ to a triangulation $\widetilde{\mathcal{T}}$ of $\mathbb{H}^2$. As in Section 2.4.3, we denote the polar vector to a horosphere K by k, the Epstein–Penner convex hull by C and the set of vertices of C by $\mathcal{K}$. Let $T=A_1A_2A_3$ be a triangle of $\widetilde{\mathcal{T}}$, K_1, K_2 and K_3 be the horocycles at A_1, A_2 and A_3 defined by h, i.e. at distances equal to h_1, h_2, h_3 from the canonical ones. The affine plane $M^{\uparrow}=M^{\uparrow}(T)\subset\mathbb{R}^{1,2}$ spanned by the points k_1, k_2 and k_3 is a supporting plane of C. By Lemma 2.4.13, $M^{\uparrow}$ is space-like, which means that its normal $m^{\uparrow}$ (in the direction of C) is time-like. Let $\mathcal{K}_M=M^{\uparrow}\cap\mathcal{K}$. As $M^{\uparrow}$ is a supporting plane to C, for $k\in\mathcal{K}$ we have

$$\langle m^{\uparrow},k\rangle=\begin{cases}-1 & \text{if } k\in\mathcal{K}_M,\\ <-1 & \text{otherwise.}\end{cases}$$

Now assume that $\mathbb{R}^{1,2}\hookrightarrow\mathbb{R}^{1,3}$ as $\{x\in\mathbb{R}^{1,3}:x_3=0\}$ and $\mathbb{H}^2$ is embedded in $\mathbb{H}^3$ respectively. Extend each horocycle to an horosphere. We continue to denote them by K_i. Let $m_{\downarrow}$ be the intersection point of $d\mathbb{S}^3$ with the ray $m^{\uparrow}+\lambda e_3$, $\lambda>0$.

By construction we have

$$\langle m_{\downarrow},k\rangle=\begin{cases}-1 & \text{if } k\in\mathcal{K}_M,\\ <-1 & \text{otherwise.}\end{cases}$$

Let $M_{\downarrow}=M_{\downarrow}(T)\subset\mathbb{H}^3$ be the plane obtained from the time-like linear plane in $\mathbb{R}^{1,3}$ orthogonal to $m_{\downarrow}$. From Lemma 2.2.1 we see that each horosphere K (with $k\in\mathcal{K}$) lies in the closed halfspace $M_{\downarrow -}$ and $M_{\downarrow}$ is tangent to K if and only if $k\in\mathcal{K}_M$ (otherwise $M_{\downarrow}$ does not intersect K). We summarize our discussion in the following description (we proved only in one direction, but the converse is clear):

Claim 1. A triangle $A_1A_2A_3$ is contained in a face of the Epstein–Penner decomposition if and only if all canonical horospheres are on one side from the common tangent plane to the horospheres K_1, K_2 and K_3.

By B_1, B_2 and B_3 denote the tangent points of $M_{\downarrow}$ with K_1, K_2 and K_3 respectively. We see that the prism $A_1A_2A_3B_1B_2B_3$ is an ideal prism with lateral edges h_1, h_2 and h_3. It follows that the triple $(d,\mathcal{T},h)$ is representable. Construct the complex $P=P(d,\mathcal{T},h)$.

It remains to check its convexity. Take two adjacent triangles $T' = A_1A_2A_3$, $T'' = A_2A_3A_4$ of $\widetilde{\mathcal{T}}$ and the corresponding ideal prisms. In the construction above, points A_1, A_2, A_3 and A_4 lie in the same plane and the lateral faces of the prisms are not glued. To glue them, we should bend these prisms around the edge A_2A_3. The question is in which direction do we bend.

Clearly, $k_4 \in \mathcal{K}_{M(T')}$ if and only if the plane $M'_\downarrow = M_\downarrow(T')$ coincides with the plane $M''_\downarrow = M_\downarrow(T'')$, which is equivalent to the condition $\phi_1 + \phi_4 = \pi$ (edge A_2A_3 is flat). From now on assume that $M'_\downarrow$ and $M''_\downarrow$ are distinct planes.

Let Y be the intersection point of A_1A_4 and A_2A_3. Parametrize the geodesic line A_1A_4 by length and let y be the coordinate of Y. By f denote the distance function $\widetilde{h}$ of P restricted to A_1A_4. It has a kink point at y and we need to check that it is concave. Let $f_1(x)$ and $f_2(x)$ be the distance functions from A_1A_4 to the planes $M'_\downarrow$ and $M''_\downarrow$ respectively. Thus, f_1 coincides with f over $(-\infty; y]$ and f_2 coincides over $[y; +\infty)$. We have not proved yet that $M'_\downarrow$ and $M''_\downarrow$ are ultraparallel to $\mathbb{H}^2$. However, both A_1 and A_4 are in the same halfspaces $M'_{\downarrow-}$ and $M''_{\downarrow-}$. Therefore, the whole line A_1A_4 belongs to these halfspaces and f_1, f_2 have the form (3.2.3). By Claim 1, the function $f_1(x) - f_2(x)$ has constant sign over the segments $(-\infty, y)$ and $(y, +\infty)$. If it is positive for $x > y$, then f is concave at y and A_2A_3 corresponds to a strict edge of P.

Consider x approaching $+\infty$. Take a sphere centered at the corresponding point $X \in A_1A_4$ (i.e. the set of points of $\mathbb{H}^3$ equidistant to X) tangent to $M''_\downarrow$. This sphere tends to the horosphere K_4 at A_4 as x approaches $+\infty$. This horosphere belongs to the interior of $M'_{\downarrow-}$, hence for some sufficiently large x, the sphere at X does not intersect $M'_\downarrow$. It implies that $f_2(x) < f_1(x)$ and A_2A_3 is strict.

We proved that if $\mathcal{T}$ is Epstein–Penner for h, then (d, V, h) is representable and the respective cone-manifold P is compatible with $\mathcal{T}$. Suppose that $\mathcal{T}'$ is another triangulation compatible with P. Triangulations $\mathcal{T}$ and $\mathcal{T}'$ can differ only in flat edges. By Lemma 3.2.3, faces of P are ideal polygons, hence $\mathcal{T}$ and $\mathcal{T}'$ can be connected by a sequence of flips of flat edges. Let $\mathcal{T}_i$ be an Epstein–Penner triangulation for h and $\mathcal{T}_{i+1}$ be obtained from $\mathcal{T}_i$ by flipping an edge A_2A_3 to A_1A_4. We saw before that an edge A_2A_3 between triangles $A_1A_2A_3$ and $A_2A_3A_4$ is flat if and only if all k_1, k_2, k_3, k_4 are in the same face of C. This means that then $\mathcal{T}_{k+1}$ also is Epstein–Penner for h. $\square$

3.5 Ideal cone-manifolds and discrete conformality

Here we show that Theorem A is equivalent to Theorem 1.5.4 as we claimed this in Section 1.6. We need two facts.

Lemma 3.5.1. Let d be a cusp-metric on S_g and $C \subset H(d, V) = \mathbb{R}^n$ be a compact. Then there are finitely many $\mathcal{T}$ such that $C \cap H(d, V, \mathcal{T}) \neq \emptyset$.

We note that this is just a particular case of a theorem of Akiyoshi [1]:

Theorem 3.5.2. For each hyperbolic cusp-metric d on S_g there are finitely many Epstein–Penner triangulations of (S_g, d).

However, the proof of our local version is much simpler, hence we provide it here.

Proof of Lemma 3.5.1. By compactness, there exists $m = m(C)$ such that for each $h \in C$ the distance function $\widetilde{h}$ of $P = P(d, V, h)$ is bounded from below by m. From Corollary 2.3.10 the lengths of lower edges of P are uniformly bounded from above.

From Corollary 2.3.7 the lengths of upper edges of P are uniformly bounded from above. But the lengths of geodesics between cusps on a decorated surface (S_g, d) form a discrete set (see e.g. [81, Lemma 4.1]). □

Let d' be a cone-metric on S_g realizing a geodesic triangulation $\mathcal{T}$ with $V(d) \subseteq V(\mathcal{T}) = V$. Take a triangle T of $\mathcal{T}$ and develop it to $\mathbb{H}^3$ as $B_1B_2B_3$. There is a unique up to isometry (non-decorated) ideal prism that has $B_1B_2B_3$ as its lower face. Its lateral edges are rays from each point B_i orthogonal to the plane $B_1B_2B_3$. Glue all these prisms together and obtain a (non-decorated) ideal Fuchsian cone-manifold, which we denote $P_\downarrow(d', \mathcal{T})$ with the lower boundary isometric to (S_g, d'). Gluing isometries are uniquely defined if we fix the horosphere at each upper vertex passing through the respective lower vertex and match them together: one can see that this is the only way of gluing to obtain a complete metric space.

Lemma 3.5.3. The complex $P_\downarrow(d', \mathcal{T})$ is convex if and only if $\mathcal{T}$ is a Delaunay triangulation of (S_g, d'). Besides, any two ideal cone-manifolds with isometric lower boundaries are isometric.

Proof. In [67, Section 3] Leibon provides a geometric observation showing that the intersection angle between circumscribed circles of two adjacent triangles $B_1B_2B_3$ and $B_2B_3B_4$ is equal to the dihedral angle of the respective upper edge A_2A_3. Clearly, a triangulation is Delaunay if and only if all these intersection angles are at most π. This proves the first claim. Besides, if a diagonal switch transforms a Delaunay triangulation to Delaunay, then it is done in an inscribed quadrilateral and the Leibon observation shows that it flips a flat edge in the upper boundary and, thereby, does not change the cone-manifold. The fact that two Delaunay triangulations of (S_g, d') can be connected by a sequence of diagonal flips through Delaunay triangulations is proved in [45, Proposition 16]. This settles the second claim.

□

Thus, we denote by $P_\downarrow(d')$ the unique (non-decorated) ideal convex Fuchsian cone-manifold that has (S_g, d') as its lower boundary.

Lemma 3.5.4. Theorem 1.5.4 is equivalent to Theorem A.

Proof. It is enough to show that d' is discretely conformally equivalent to d'' if and only if the upper boundaries of $P_\downarrow(d')$ and $P_\downarrow(d'')$ are isometric.

Assume that the upper boundaries of $P_\downarrow(d')$, $P_\downarrow(d'')$ are both isometric to (S_g, d) for a cusp-metric d. Choose a decoration on (S_g, d) and consider $H(d, V)$ identified with $\mathbb{R}^n$ with the help of chosen decoration. First, assume that $P_\downarrow(d'), P_\downarrow(d'') \in H(d, V, \mathcal{T})$ for some triangulation $\mathcal{T}$. By Lemma 3.5.3, $\mathcal{T}$ is Delaunay for both d' and d''. Take $e \in E(\mathcal{T})$ and recall that its lengths in d' and d'' are denoted by $l_e(d')$ and $l_e(d'')$ respectively. By h'_{v_1} and h'_{v_2} denote the heights of the trapezoid containing e in $P_\downarrow(d')$, by h''_{v_1} and h''_{v_1} in $P_\downarrow(d'')$. Then from Corollary 2.3.7 we see that

$$\sinh\left(\frac{l_e(d')}{2}\right) = \sinh\left(\frac{l_e(d'')}{2}\right) \exp\left(\frac{h''_{v_1} - h'_{v_1}}{2} + \frac{h''_{v_1} - h'_{v_1}}{2}\right).$$

Thus, d' is discretely conformally equivalent to d''.

Assume that d' and d'' are in the different cells $H(d, V, \mathcal{T}')$ and $H(d, V, \mathcal{T}'')$. The decomposition $H(d, V) = \bigcup H(d, V, \mathcal{T})$ is locally finite due to Lemma 3.5.1 and the boundaries of cells $\mathcal{H}(d, V, \mathcal{T})$ are piecewise analytic. Then points corresponding to $P_\downarrow(d')$ and $P_\downarrow(d'')$ can be connected by a path in $H(d, V)$ transversal to the boundaries of all cells and intersecting them m times. Each intersection point defines a cone-manifold with the same upper boundary metric. Denote their lower boundaries metrics by $d_1, \ldots, d_m$. Define also $d_0 = d'$, $d_{m+1} = d''$. A segment between d_i and d_{i+1} of the path belongs to $H(d, V, \mathcal{T}_i)$ for some triangulation $\mathcal{T}_i$. By Lemma 3.5.3, $\mathcal{T}_i$ is Delaunay for both d_i and d_{i+1}. By the previous argument, they are discretely conformally equivalent. Then so are d_0 and d_{m+1}.

In the opposite direction, assume that d' and d'' are discretely conformally equivalent and have a common Delaunay triangulation $\mathcal{T}$. Then there exists a function $u : V \to \mathbb{R}$ such that for each edge e of $\mathcal{T}$ with endpoints v_1 and v_2 we have

$$\sinh\left(\frac{l_e(d')}{2}\right) = \exp(u(v_1) + u(v_2)) \sinh\left(\frac{l_e(d'')}{2}\right).$$

Consider $P_\downarrow(d')$ and $P_\downarrow(d'')$, then $\mathcal{T}$ is compatible with both these cone-manifolds due to Lemma 3.5.3. Metric spaces (S_g, d') and (S_g, d'') come with a homeomorphism between them isotopic to identity on S_g with respect to V. This allows us to identify the upper boundary metrics of $P_\downarrow(d')$ and $P_\downarrow(d'')$ with elements of $\mathfrak{C}(V)$. Choose an horosphere at each vertex of the upper boundaries in both $P_\downarrow(d')$, $P_\downarrow(d'')$. Let h'_v and h''_v be the signed distances from the horospheres at $v \in V$ to the lower boundary in $P_\downarrow(d')$ and $P_\downarrow(d'')$ respectively. We can choose the horospheres such that for every v, $\frac{h''_v - h'_v}{2} = u(v)$. Then Corollary 2.3.7 shows that for each $e \in E(\mathcal{T})$ its length in the upper boundary of $P_\downarrow(d')$ is the same as in the upper boundary of $P_\downarrow(d'')$ (with respect to the chosen horosections). Therefore, the upper boundary metrics of $P_\downarrow(d')$ and $P_\downarrow(d'')$ together with the chosen decorations coincide due to Lemma 2.4.12.

The case, when d' and d'' are discretely conformally equivalent and do not have a common Delaunay triangulation T, is inductively reduced to the last case. $\square$

3.6 The space of admissible heights in the compact case

If d is a convex cone-metric, then we do not have a complete description of $H(d, V)$. In this section we collect the facts about $H(d, V)$ that will be used in the proof of Theorem B.

Lemma 3.6.1. Let $d \in \mathfrak{D}_c(V)$ and $h \in H(d, V)$. Define h_t by $\sinh h_{t,v} = e^t \sinh h_v$ for all $v \in V$. Then $h_t \in H(d, V)$ for all $t \in \mathbb{R}$, and all Fuchsian cone-manifolds $P_t = P(d, V, h_t)$ have the same face decomposition.

Proof. Consider two adjacent prisms such that the dihedral angle of the adjacent upper edge is π. Develop them to $\mathbb{H}^3$ in the half-space model, let $M_\downarrow$ be the plane containing the lower boundaries and $M^\uparrow$ — upper. By $A \in M^\uparrow$ and $B \in M_\downarrow$ denote the closest points. Assume that the line AB intersects the ideal boundary at point C and B is between C and A. The homothety ρ with center C is a hyperbolic isometry. Consider the images of all upper vertices of the prisms and their projections to $M_\downarrow$. We obtain two new prisms with upper boundaries isometric to those of former prisms and the dihedral angle of the common upper edge remains equal to π.

Assume that $\cosh(\rho(A)B) = e^t \cosh AB$. Let h_ρ be the distance function of the new vertices. Due to formula (3.2.2), $\sinh h_{\rho,v} = e^t \sinh h_v$ for each vertex v of the prisms.

Let $\mathcal{T}$ be a triangulation compatible with P, so $P = P(d, \mathcal{T}, h)$. We saw that all flat edges remain flat in all $P(d, \mathcal{T}, h_t)$. But for sufficiently small t all strict edges remain to be strict. Thus if t is sufficiently small, then $P(d, \mathcal{T}, h_t)$ is convex and $h_t \in H(d, V)$. However, if a strict edge becomes flat, then it is flat in all P_t. Thus, $h_t \in H(d, V)$ for all $t \in \mathbb{R}$ and all P_t have the same face decomposition. $\square$

Lemma 3.6.2. For each $V \subset S_g$ and $d \in \mathfrak{D}_c(V)$ there are finitely many $\mathcal{T}$ with $V(\mathcal{T}) = V$ for which there is $h \in H(d, V)$ such that $\mathcal{T}$ is compatible with $P(d, V, h)$.

Proof. Due to Lemma 3.6.1, it is enough to prove that there are finitely many $\mathcal{T}$ compatible with convex cone-manifolds $P = P(d, V, h)$ with $h_v \geq m$ for some $m > 0$ and all $v \in V$.

Let e be an edge of P. Develop the trapezoid containing e as $A_1A_2B_1B_2 \subset \mathbb{H}^2$. Let A and B be the closest points on the lines containing its upper and lower boundaries. Assume that A_1 lies between A_2 and A. Denote the distance AB by h, AA_1 by l_1 and AA_2 by l_2. Then from Corollary 2.3.5 we get

$$\sinh h_1 = \cosh l_1 \sinh h, \quad \sinh h_2 = \cosh l_2 \sinh h.$$

Then we have

$$\frac{\sinh h_2}{\sinh h_1} = \cosh(h_2 - h_1) + \sinh(h_2 - h_1)\coth h_1 = \frac{\cosh l_2}{\cosh l_1} \geq \cosh l_{12}.$$

Due to Lemma 3.1.8, we have $h_2 - h_1 \leq \mathrm{diam}(S_g, d)$. Then we get

$$\cosh l_{12} \leq \cosh\big(\mathrm{diam}(S_g, d)\big)(1 + \coth m).$$

If A lies between A_1 and A_2, then one of l_1, l_2 is at least $l_{12}/2$. Assume that this is l_1. Note that $h_1 - h \leq \mathrm{diam}(S_g, d)$. Considering the trapezoid AA_1BB_1, we get similarly

$$\cosh\frac{l_{12}}{2} \leq \cosh\big(\mathrm{diam}(S_g, d)\big)(1 + \coth m).$$

Hence there exists a constant $M = M(d, m)$ such that the lengths of edges of P are at most M. But the lengths of geodesics between points of V in (S_g, d) form a discrete set by an argument from [56, Proposition 1]. Thus there are finitely many of them that can appear as edges of some P. Therefore, there are finitely many of realizable triangulations. □

Define

$$H(d, V, m, M) = \{h \in H(d, V) : \min_{v \in V} h_v \geq m, \max_{v \in V} h_v \leq M\}.$$

Lemma 3.6.3. For every $d \in \mathfrak{D}_c(V)$ and $0 < m \leq M$ the set $H(d, V, m, M)$ is compact.

Proof. From Lemma 3.6.2 there are finitely many triangulations compatible with $P(d, V, h)$ with $h \in H(d, V)$. For every triangulation $\mathcal{T}$ the set of heights h bounded between m and M such that $(d, \mathcal{T}, h)$ is a realizable triple, is compact. The set of admissible heights h, such that $\mathcal{T}$ is compatible with the convex cone-manifold $P(d, V, h)$, is a closed subset of this set, hence, also compact. Finally, $H(d, V, m, M)$ is compact as a finite union of compact sets. □

Now we are going to establish two easy, but fundamental properties used in several parts of our proof: even if h is on the boundary of $H(d, V)$, the height of a point with non-positive particle curvature can always be decreased and the heights of all points with positive particle curvatures can always be simultaneously increased. Together with this we show how non-positive/positive particle curvature impacts on the upper boundary structure.

Lemma 3.6.4. Let $P(d, V, h)$ be a compact convex Fuchsian cone-manifold, $v \in V$ and either $\kappa_v(P) < 0$ or $\kappa_v(P) = 0$ and $\nu_v(d) > 0$. Then all angles at v of faces incident to v are strictly less than π.

Proof. This is due to Volkov [110]. A proof can also be found in [57, Lemma 5.3]. □

Lemma 3.6.5. Let $A_1A_2A_3B_1B_2B_3$ and $A_2A_3A_4B_2B_3B_4$ be two incident compact prisms in $\mathbb{H}^3$ such that the total dihedral angle ϕ of the common edge A_2A_3 is π. By λ denote the total angle of A_2 in the upper faces. Then

$$\operatorname{sgn} \frac{\partial \phi}{\partial h_3} = \operatorname{sgn}(\lambda - \pi).$$

Proof. For the angle α_{23} at A_2 in the trapezoid $A_2A_3B_2B_3$ from Lemma 2.3.3 we have

$$\frac{\partial \alpha_{23}}{\partial h_3} > 0.$$

Consider the spherical link of A_2 in the union of both prisms. It consists of two spherical triangles glued altogether. When h_3 changes, only the common edge α_{23} changes with positive derivative. Let ϕ^+, ϕ^- be the dihedral angles of A_2A_3 in two prisms and λ^+, λ^- be the angles at A_2 of their upper boundaries. Then

$$\frac{\partial \phi^+}{\partial \alpha_{23}} = -\frac{\cot \lambda^+}{\sin \alpha_{23}}, \quad \frac{\partial \phi^-}{\partial \alpha_{23}} = -\frac{\cot \lambda^-}{\sin \alpha_{23}}.$$

So we get

$$\frac{\partial \phi}{\partial h_3} = -\frac{\cot \lambda^+ + \cot \lambda^-}{\sin \alpha_{23}} \cdot \frac{\partial \alpha_{23}}{\partial h_3}.$$

This implies the desired. □

Lemma 3.6.6. Let $P = P(d, V, h)$ be a compact convex Fuchsian cone-manifold and $v \in V$ be a vertex that has angle less than π in all faces. For $\xi > 0$ define h' by $h'_v := h_v - \xi$ and $h'_u := h_u$ for all $u \in V$, $u \neq v$. Then $h' \in H(d, V)$ provided ξ is sufficiently small.

Proof. The proof ideas are essentially due to Volkov [110]. It was also reproduced for cusps with particles in [35]. We sketch it here because the ideas from the proof are used further.

Let v be a vertex such that it has angle less than π in all faces of P. We claim that each face can be triangulated such that all new flat edges become convex after we decrease the height of v. Let R be a face incident to v and u_1, u_2 be two vertices incident to v. Consider the shortest path in R connecting u_1 and u_2 and homotopic to the polygonal curve u_1vu_2 (see Figure 3.3). This path is a polygonal curve that cuts off R a polygon R' such that it has angles greater than π at all vertices except v, u_1, u_2. Triangulate it by diagonals from v. Lemma 3.6.5 shows that this triangulation is a desired one. □

Lemma 3.6.7. If v is an isolated vertex of a compact Fuchsian cone-manifold $P = P(d, V, h)$, then $\kappa_v(P) = \nu_v(P) \geq 0$.

Proof. Consider a triangulation $\mathcal{T}$ compatible with P and start developing all prisms containing v to $\mathbb{H}^3$ one by one preserving the incidence. We get a chain of prisms with all lower boundaries belonging to the same plane $M_\downarrow$ and all upper boundaries belonging to the same plane $M^\uparrow$ as all upper edges are flat. It is easy to see that

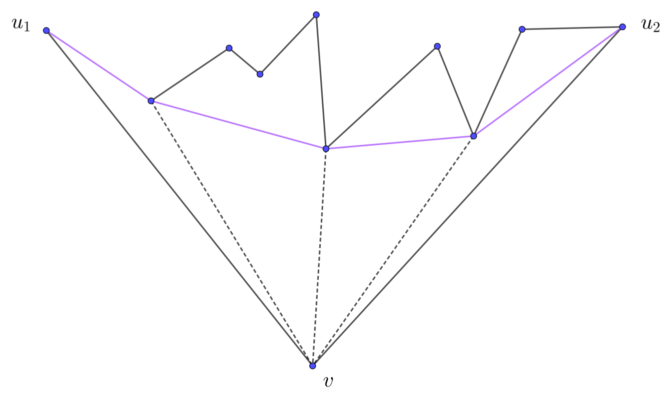

FIGURE 3.3: To the proof of Lemma 3.6.6.

as $\nu_v(d) \geq 0$, then $\kappa_v(P) \geq 0$ with the equality in one case implying the equality in the second case. Thereby, our chain of prisms is not self-intersecting. Assume that $\nu_v(d) > 0$. The first and the last upper edge of the development are images of the same edge of P, therefore the distance function restricted to these edges is the same. Due to condition that the sum of the dihedral angles at this edge is π, we see that v is developed to the closest point from $M^\uparrow$ to $M_\downarrow$. This gives the desired equality. □

Lemma 3.6.8. Let $P = P(d, V, h)$ be a compact convex Fuchsian cone-manifold and $V = V(d)$. For $\xi > 0$ define h' by $\sinh h'_v := e^\xi \sinh h_v$ if $\kappa_v(P) > 0$ and $h'_u := h_u$ otherwise. Then $h' \in H(d, V)$ provided ξ is sufficiently small.

Proof. Lemma 3.6.2 shows that there are finitely many triangulations compatible with P. Consider ξ sufficiently small such that for any $\mathcal{T}$ compatible with P all strict edges of P remain strict in $P(d, \mathcal{T}, h')$. We start from an arbitrary triangulation $\mathcal{T}$ and use the *flip algorithm* to find a triangulation $\mathcal{T}'$ such that $P(d, \mathcal{T}', h')$ is convex. The flip algorithm will also be used in some further parts of the proof.

We simply take a concave edge e of $P(d, \mathcal{T}, h')$ and flip it, i.e., replace by the other diagonal in the quadrangle formed by two triangles incident to e. There are two questions: why the algorithm can not run infinitely and why a concave edge can be flipped.

By induction it is easy to see that each triangulation that appears is compatible with P. Indeed, if this is true for the current triangulation, then by the choice of ξ all strict edges of P are strict in $P(d, \mathcal{T}, h')$. Therefore, they can not be flipped and the new triangulation is also compatible with P. The extended height function $\widetilde{h}$ is pointwise non-decreasing after each flip. Moreover, it strictly increases at all the interior points of the quadrangle defining the flip. This means that no triangulation appears twice during the algorithm. As there are finitely many of them, the algorithm can not run infinitely.

There are two cases, when an edge can not be flipped. First, it can be an edge of a triangle with two sides glued altogether, so there is no quadrangle at all. Second, it can be a diagonal in a concave quadrangle. We illustrate this by Figure 3.4.

We prove that under our conditions this can not happen. Suppose that for some $P(d, \mathcal{T}, h')$ an edge e is incident only to one triangle T, v is the vertex incident only to e and u is the other vertex of e. Then $\nu_v(d) > \pi$. The edge e is flat in P, thus, v is isolated in P. Lemma 3.6.7 implies that $\kappa_v(P) > \pi$, so $\sinh h'_v = e^\xi \sinh h_v$. If also $\sinh h'_u = e^\xi \sinh h_u$, then e remains flat in $P(d, \mathcal{T}, h')$ (see the proof of Lemma 3.6.1).

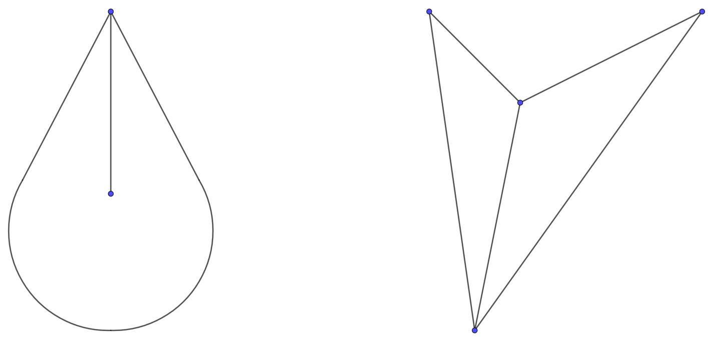

FIGURE 3.4: Two cases when an edge of a triangulation can not be flipped.

Consider the case $h'_u = h_u$. Note that as T is isosceles, then both angles of T at u are smaller than $\pi/2$. Therefore, e becomes convex in $P(d, \mathcal{T}, h')$ due to Lemma 3.6.5.

Suppose that for some $P(d, \mathcal{T}, h')$ an edge e is a diagonal of a concave quadrilateral and v is the vertex of this quadrilateral with total angle at least π. The edge e is flat in P. Lemma 3.6.4 shows that $\kappa_v(P) > 0$ and, therefore, $\sinh h'_v = e^{\xi} \sinh h_v$. If this is true for all vertices of the quadrilateral, then e remains flat in $P(d, \mathcal{T}, h')$. We prove that in all other cases e becomes convex.

Indeed, denote three other vertices by u_1, u_2, u_3 such that e connects v and u_3. If only v increased its height, then the claim follows from Lemma 3.6.5. Otherwise, we first increase the heights of all u_1, u_2, u_3 by $\sinh h'_u = e^{\xi} \sinh h_u$. The edge e stays flat. If we decrease now the height of u_3, then e becomes convex due to Lemma 3.6.5. If we decrease the heights of u_1 or u_2, then it is clear that the dihedral angle of e decreases. This also applies if we need to decrease heights of several of them. Thus, all cases are considered. □

Lemma 3.6.9. Let $P = P(d, V, h)$ be a compact convex Fuchsian cone-manifold such that all faces are strictly convex hyperbolic polygons. Then h is in the interior of $H(d, V)$.

Proof. The proof is similar to [35, Proposition 3.15(2)]. Lemma 3.6.2 shows that there are finitely many triangulations compatible with P. We can choose a small enough neighbourhood N of h in $\mathbb{R}^n$ such that for any such triangulation $\mathcal{T}$ and any $h' \in N$ all strict edges of P remain strict in $P(d, \mathcal{T}, h')$. Then the flip algorithm from the proof of Lemma 3.6.8 performs inside each face of P separately. Therefore, it finds in finitely many steps a triangulation $\mathcal{T}'$ such that $P(d, \mathcal{T}', h')$ is convex. □

Lemma 3.6.10. If a face R contains a curve, which is not homotopic to a point in $\partial^{\uparrow} P$, then it contains angle greater than π.

Proof. It is clear that R has at least two boundary components. We cut it along geodesic segments until it becomes simply connected (the existence of a cut follows, e.g., from Lemma 2.4.6). Let ψ_1 and ψ_2 be two boundary components before the last cut, which is made along the geodesic segment ψ producing a polygon R'. Each cut does not increase the values of the angles, hence it is enough to prove that there is angle at least π before the last cut.

If there are no angles greater than π in R', then R' is convex. Develop the prisms having upper boundaries in R' to $\mathbb{H}^3$. Let χ_1, χ_2 be geodesic segments, which are images of ψ, $M_\downarrow$ be the plane containing the lower boundaries and $M^\uparrow$ — upper. Denote by $A \in M^\uparrow$ and $B \in M_\downarrow$ their closest point. The distance function from χ_1 and χ_2 to $M_\downarrow$ coincides at respective points. Therefore, the trapezoids containing χ_1 and χ_2 are isometric. There exists a unique orientation preserving isometry of $\mathbb{H}^3$ mapping one trapezoid to the other. Clearly, this isometry preserves $M_\downarrow$, $M^\uparrow$. This means that this is a rotation along the line AB at the angle smaller than 2π, which maps χ_1 to χ_2. This shows that R can be developed to a convex hyperbolic cone. Note that the apex does not belong to R. Assume that ψ_2 is developed closer to the apex than ψ_1. Write the formula for the area of the cone-polygon bounded by ψ in terms of the angles. We see that ψ_2 contains in R an angle greater than π. □

From Lemmas 3.6.4, 3.6.7, 3.6.9 and 3.6.10 we conclude

Corollary 3.6.11. Let $P = P(d, V, h)$ be a compact convex Fuchsian cone-manifold with $V = V(d)$ such that for all $v \in V$ we have $\kappa_v(P) \leq 0$. Then the graph of the strict edges of P is connected and all faces are strictly convex hyperbolic polygons. In particular, $h \in \operatorname{int}(H(d, V))$.

Remark 3.6.12. A crucial difficulty for some of our arguments is given in the fact that $H(d, V)$ is not convex for a cone-metric d. However, it becomes convex if we do the coordinate change $h_v \to \sinh h_v$. But in these coordinates the discrete curvature functional, defined in the next chapter, does not have the magical properties that we need.

Fix $V \subset S_g$. For a subset $\mathfrak{U} \subseteq \mathfrak{D}_c(V)$ define

$$\mathfrak{H}(\mathfrak{U}) = \{(d, h) : d \in \mathfrak{U}, h \in H(d, V)\}.$$

This set is endowed with the topology induced from the product topology. Any point of $\mathfrak{H}(\mathfrak{U})$ can be considered as a Fuchsian cone-manifold with the vertex set V.

Chapter 4

Discrete curvature

4.1 First variation formulas

Let $P = P(d, \mathcal{T}, h)$ be a Fuchsian cone-manifold. Define its *discrete curvature* as

$$S(P) = S(d, \mathcal{T}, h) = -2\mathrm{vol}(P) + \sum_{v \in V(\mathcal{T})} \kappa_v(P) h_v + \sum_{e \in E(\mathcal{T})} \theta_e(P) l_e(d).$$

Note that S is independent from the choice of $\mathcal{T}$ compatible with P. This defines a continuous functional S on $H(d, V)$ in the coordinates h_v called *the discrete curvature functional* or *the discrete Hilbert–Einstein functional*.

If d is a cone-metric, then we also consider variations of S as d varies in $\mathfrak{D}_c(V)$. For a triangulation $\mathcal{T}$ with the vertex set $V(\mathcal{T}) = V$ define the set $\mathfrak{H}(\mathfrak{D}_c(V), \mathcal{T})$ as the set of pairs (d, h), where $d \in \mathfrak{D}_c(V, \mathcal{T})$, $h \in H(d, V)$ and $\mathcal{T}$ is compatible with the cone-manifold $P(d, V, h)$. The boundary of the set $\mathfrak{H}(\mathfrak{D}_c(V), \mathcal{T})$ is piecewise analytic. In the chart $\mathfrak{H}(\mathfrak{D}_c(V), \mathcal{T})$ on $\mathfrak{H}(\mathfrak{D}_c(V))$ the functional S is a continuous function of heights h_v and lengths of the upper edges $l_e(d)$.

Lemma 4.1.1. S is continuously differentiable over $\mathfrak{H}(\mathfrak{D}_c(V))$ and

$$\frac{\partial S}{\partial h_v} = \kappa_v, \quad \frac{\partial S}{\partial l_e} = \theta_e.$$

Proof. The set $\mathfrak{H}(\mathfrak{D}_c(V))$ is locally modelled in $\mathbb{R}^{n+N}$, where $n = |V|$ and $N = 3(n + 2g - 2)$.

Assume first that (d, h) belongs to the interior of some $\mathfrak{H}(\mathfrak{D}_c(V), \mathcal{T})$. Then the combinatorics of Fuchsian cone-manifolds in a neighbourhood of (d, h) does not change and all the dihedral angles can be written as sums of dihedral angles in the same prisms. For a single prism $\Pi = A_1A_2A_3B_1B_2B_3$ each dihedral angle is a smooth function of its lengths and by the Schlaffli formula we get

$$-2d\mathrm{vol}(\Pi) = h_1 d\omega_1 + h_2 d\omega_2 + h_3 d\omega_3 + l_{12} d\phi_{12} + l_{13} d\phi_{13} + l_{23} d\phi_{23}.$$

Summing this for all prisms we obtain

$$-2d\mathrm{vol}(P) = -\sum_{v \in V} h_v d\kappa_v - \sum_{e \in E(\mathcal{T}} l_e d\theta_e.$$

Therefore,

$$dS(P) = \sum_{v \in V} \kappa_v dh_v + \sum_{e \in E(\mathcal{T})} \theta_e dl_e.$$

This shows the first derivative formula. As the angles are smooth functions of the lengths, we see that S is smooth in this case.

Now assume that (d, h) is not in the interior of any cell. This means that either some vertices of V have the total angle 2π or some edges of $\mathcal{T}$ are flat in P. In any case, all boundary conditions are piecewise analytic. Consider a chart over (d, h) in $\mathbb{R}^{n+N}$. Let $\bar{x} \in \mathbb{R}^{n+N}$ be the point corresponding to (d, h) and $\bar{e}$ be a vector such that $\bar{x} + \xi\bar{e} \in \mathfrak{H}(\mathfrak{D}_c(V))$ for all sufficiently small $\xi > 0$. Then, as all boundary conditions are piecewise analytic, there exists $\mathcal{T}$ such that $\bar{x} + \xi\bar{e} \in \mathfrak{H}(\mathfrak{D}_c(V), \mathcal{T})$ for all sufficiently small ξ. From the previous argument, the partial derivatives depend continuously on $\bar{e}$. Then S is continuously differentiable and its first derivatives are the respective curvatures. □

In the ideal case we need to vary S only over $H(d, V)$:

Lemma 4.1.2. Let d be a cusp-metric. Then S is continuously differentiable over $H(d, V)$ and

$$\frac{\partial S}{\partial h_v} = \kappa_v.$$

We omit the proof because it is exactly the same: we only need to use the Schläffli formula for partially ideal polyhedra. It is given in [93, Theorem 14.5].

For the proof of Theorem A we also need to introduce a *modified* discrete curvature functional. Let $\kappa' : V \to \mathbb{R}$ be a function. We write κ'_v instead of $\kappa'(v)$. Define

$$S_{\kappa'}(P) := S(P) - \sum_{v \in V} h_v \kappa'_v. \tag{4.1.1}$$

We have an immediate

Corollary 4.1.3. $S_{\kappa'}$ is continuously differentiable over $H(d, V)$ and

$$\frac{\partial S}{\partial h_v} = \kappa_v - \kappa'_v.$$

Corollary 4.1.3 implies that if h is a critical point of $S_{\kappa'}$ over $H(d, V)$, then for all $v \in V$ we have $\kappa_v = \kappa'_v$.

4.2 Second variation formulas in the compact case

Now we are going to investigate the second derivatives of S. To this purpose we need a new notation. By $\vec{E}(\mathcal{T})$ we denote the set of *oriented* edges of $\mathcal{T}$ in the sense that each edge $e \in E(\mathcal{T})$ gives rise to two oriented edges with respect to two different possible orientations. By $\vec{E}_v(\mathcal{T})$ we denote the set of oriented edges starting at v and by $\vec{E}_{vu}(\mathcal{T})$ we denote the set of oriented edges starting at v and ending in u. In particular, $\vec{E}_{vv}(\mathcal{T})$ is the set of loops from v to itself and each loop is counted twice. For P compatible with $\mathcal{T}$ and $\vec{e} \in \vec{E}(\mathcal{T})$ by $\phi^+_{\vec{e}}$ and $\phi^-_{\vec{e}}$ we denote the dihedral angles of $\vec{e}$ in the right and the left prisms incident to $\vec{e}$ respectively. By $\alpha_{\vec{e}}$ we denote the angle of the trapezoid containing $\vec{e}$ at the vertex, where $\vec{e}$ starts. By $l_{\vec{e}}$ and $a_{\vec{e}}$ we continue to denote the lengths of $\vec{e}$ in the upper and the lower boundaries of P respectively.

Lemma 4.2.1. Let $d \in \mathfrak{D}_c(V)$. Then S is twice continuously differentiable over $H(d, V)$ and

$$v \neq u: \quad \frac{\partial^2 S}{\partial h_v \partial h_u} = \frac{\partial \kappa_u}{\partial h_v} = \frac{\partial \kappa_v}{\partial h_u} = \sum_{\vec{e} \in \vec{E}_{vu}(\mathcal{T})} \frac{\cot \phi^+_{\vec{e}} + \cot \phi^-_{\vec{e}}}{\sin \alpha_{\vec{e}}} \cdot \frac{1}{\cosh h_v \sinh a_{\vec{e}}} \geq 0;$$

$$v = u: \quad \frac{\partial^2 S}{\partial h_v^2} = \frac{\partial \kappa_v}{\partial h_v} = -\sum_{\vec{e}\in \vec{E}_v(\mathcal{T})} \frac{\cot \phi_{\vec{e}}^+ + \cot \phi_{\vec{e}}^-}{\sin \alpha_{\vec{e}}} \cdot \frac{\coth a_{\vec{e}}}{\cosh h_v} +$$

$$+ \sum_{\vec{e}\in \vec{E}_{vv}(\mathcal{T})} \frac{\cot \phi_{\vec{e}}^+ + \cot \phi_{\vec{e}}^-}{\sin \alpha_{\vec{e}}} \cdot \frac{1}{\cosh h_v \sinh a_{\vec{e}}} =$$

$$= -\sum_{u\neq v} \frac{\partial \kappa_u}{\partial h_v} \sum_{\vec{e}\in \vec{E}_{vu}(\mathcal{T})} \cosh a_{\vec{e}} - \sum_{\vec{e}\in \vec{E}_{vv}(\mathcal{T})} \frac{\cot \phi_{\vec{e}}^+ + \cot \phi_{\vec{e}}^-}{\sin \alpha_{\vec{e}}} \cdot \frac{\cosh a_{\vec{e}} - 1}{\cosh h_v \sinh a_{\vec{e}}} \leq 0.$$

Proof. Similarly to the proof of Lemma 4.1.1, it is enough to prove the claim for h in the interior of a set $H(d, V, \mathcal{T})$ of heights such that $\mathcal{T}$ is compatible with the respective cone-manifold. This is reduced to computations for a single prism $\Pi = A_1A_2A_3B_1B_2B_3$.

Fix a spherical section at the vertex A. From the cosine rule we get

$$\cos \omega_1 = \frac{\cos \lambda_1 - \cos \alpha_{12} \cos \alpha_{13}}{\sin \alpha_{12} \sin \alpha_{13}}.$$

Differentiating the spherical cosine rule, we obtain

$$\frac{\partial \omega_1}{\partial \alpha_{12}} = -\frac{\cot \phi_{12}}{\sin \alpha_{12}}, \quad \frac{\partial \omega_1}{\partial \alpha_{13}} = -\frac{\cot \phi_{13}}{\sin \alpha_{13}}.$$

Differentiating the cosine rule for trapezoids from Lemma 2.3.3, we see

$$\frac{\partial \alpha_{12}}{\partial h_1} = -\frac{\coth a_{12}}{\cosh h_1}, \quad \frac{\partial \alpha_{13}}{\partial h_1} = -\frac{\coth a_{13}}{\cosh h_1}, \quad \frac{\partial \alpha_{12}}{\partial h_2} = \frac{1}{\cosh h_1 \sinh a_{12}}.$$

So we have

$$\frac{\partial \omega_1}{\partial h_1} = \frac{\partial \omega_1}{\partial \alpha_{12}} \cdot \frac{\partial \alpha_{12}}{\partial h_1} + \frac{\partial \omega_1}{\partial \alpha_{13}} \cdot \frac{\partial \alpha_{13}}{\partial h_1} = \frac{\cot \phi_{12} \coth a_{12}}{\sin \alpha_{12} \cosh h_1} + \frac{\cot \phi_{13} \coth a_{13}}{\sin \alpha_{13} \cosh h_1}, \tag{4.2.1}$$

$$\frac{\partial \omega_1}{\partial h_2} = \frac{\partial \omega_1}{\partial \alpha_{12}} \cdot \frac{\partial \alpha_{12}}{\partial h_2} = -\frac{\cot \phi_{12}}{\sin \alpha_{12} \cosh h_1 \sinh a_{12}}. \tag{4.2.2}$$

Consider $v \neq u$ and sum up a formula of type (4.2.2) for each prism incident to both v and u. Then we get the desired formulas for $\frac{\partial \kappa_v}{\partial h_u} = -\frac{\partial \omega_v}{\partial h_u}$. For $v = u$ we sum up the formulas of type (4.2.1) in all prisms incident to v plus a formula of type (4.2.2) for each oriented loop and get $\frac{\partial \kappa_v}{\partial h_v}$. This finishes the proof. □

Lemma 4.2.2. Fix $d \in \mathfrak{D}_c(V)$ and identify $H(d, V)$ with a subset of $\mathbb{R}^n$. The Hessian of S over $H(d, V)$ is non-positively defined. Moreover, its kernel is spanned by vectors $\bar{x}^u = (x_v^u)_{v\in V}$ defined by $x_v^u = 1$ if $v = u$ and 0 otherwise, where u is an isolated vertex of P.

Proof. We have

$$\bar{x}^T \mathrm{Hess}(S) \bar{x} = \sum_{u,v\in V} \frac{\partial \kappa_u}{\partial h_v} x_u x_v = -\sum_{\substack{u,v\in V\\ u\neq v}} \frac{\partial \kappa_u}{\partial h_v} (x_u - x_v)^2 + \sum_{v\in V} x_v^2 \sum_{u\in V} \frac{\partial \kappa_u}{\partial h_v}$$

From Lemma 4.2.1 we see

$$\sum_{u\in V} \frac{\partial \kappa_u}{\partial h_v} = \frac{\partial \kappa_v}{\partial h_v} + \sum_{\substack{u\in V\\ u\neq v}} \frac{\partial \kappa_u}{\partial h_v} = -\sum_{u\neq v} \frac{\partial \kappa_u}{\partial h_v} \sum_{\vec{e}\in \vec{E}_{vu}(\mathcal{T})} (\cosh a_{\vec{e}} - 1) -$$

$$- \sum_{\vec{e} \in \vec{E}_{vv}(\mathcal{T})} \frac{\cot \phi_{\vec{e}}^{+} + \cot \phi_{\vec{e}}^{-}}{\sin \alpha_{\vec{e}}} \cdot \frac{\cosh a_{\vec{e}} - 1}{\cosh h_v \sinh a_{\vec{e}}} \leq 0,$$

because $\frac{\partial \kappa_u}{\partial h_v} \geq 0$ if $u \neq v$.

Hence, the Hessian is defined non-positively. Also $\frac{\partial \kappa_u}{\partial h_v} = 0$ if and only if there are no strict edges in P between u and v. Thus, it is easy to see that $\bar{x}^T \mathrm{Hess}(S) \bar{x} = 0$ if and only if $\bar{x}$ has non-zero coordinates only on isolated vertices of P. □

Thus, S is concave over $H(d, V)$.

Remark 4.2.3. Although the Hessian of S can be degenerate and, moreover, critical points might be non-isolated, one can show that it can not be degenerate along any segment in $H(d, V)$. Thus, S is strictly concave over $H(d, V)$. In what follows we will not need it, so we do not prove this. This would lead to a proof of local rigidity of compact convex Fuchsian cone-manifolds.

Together with Lemma 3.6.7 we get

Corollary 4.2.4. Let $P = P(d, V, h)$ be a convex Fuchsian cone-manifold with $V = V(d)$ such that $\kappa_v(P) \leq 0$ for all $v \in V$. Then $\mathrm{Hess}(S)$ is non-degenerate at P.

4.3 Behaviour of S in the compact case

The set $H(d, V)$ is non-compact and we need to understand the behaviour of S outside a compact set.

Lemma 4.3.1. Let $d \in \mathfrak{D}_c(V)$. For each $K \in \mathbb{R}$ there exists $M = M(K, d, V) > 0$ such that if $h \in H(d, V)$ and for some vertex $v \in V$ one has $h_v \geq M$, then we get $S(d, V, h) \leq K$.

Proof.

Claim 1. For each $\varepsilon > 0$ there exists $\delta > 0$ such that if the side lengths of a hyperbolic triangle are at most δ, then the sum of its angles is at least $\pi - \varepsilon$.

This is because the area of a hyperbolic triangle is equal to the sum of its angles minus π. By computing the Euler characteristic we get

Claim 2. For each $\varepsilon > 0$ there exists $\delta > 0$ such that if for a Fuchsian cone-manifold $P = P(d, \mathcal{T}, h)$ for each $e \in E(\mathcal{T})$ the length a_e of the projection of e to $\partial_{\downarrow} P$ is at most δ, then

$$\sum_{v \in V} \kappa_v(P) \leq 2\pi(2 - 2g) + \varepsilon.$$

The number of triangulations compatible with some $P(d, V, h)$ is finite for fixed d, V due to Lemma 3.6.2. Due to Lemma 2.3.3 if a trapezoid has fixed upper edge and growing heights, then its lower edge becomes smaller. Thus, we conclude

Claim 3. For each $\delta > 0$ there exists $M_0 = M_0(\delta, d, V) > 0$ such that if $h \in H(d, V)$ and for each $v \in V$ we have $h_v \geq M_0$, then for each edge e of $P = P(d, V, h)$ we have $a_e \leq \delta$.

From the finiteness of the set of compatible triangulations we also get

Claim 4. There exists $C = C(d, V) > 0$ such that for any convex Fuchsian cone-manifold $P = P(d, V, h)$ compatible with a triangulation $\mathcal{T}$ we have

$$\Big| \sum_{e \in E(\mathcal{T})} \theta_e(P) l_e(d) \Big| \leq C.$$

Finally, the triangle inequality for trapezoids (Lemma 3.1.8) gives

Claim 5. For every $M_0 > 0$ there exists $M = M(M_0, d) > 0$ such that if $h \in H(d, V)$ and for some $v \in V$ we have $h_v \geq M$, then for each $u \in V$ we get $h_u \geq M_0$.

Combining all these claims together we see that for every $\varepsilon > 0$ and $M_0 > 0$ we can choose $M > 0$ such that if $h \in H(d, V)$ and $h_v \geq M$ for some $v \in V$, then we get

$$S(d, V, h) \leq -\mathrm{vol}(P) + M_0(2\pi(2 - 2g) + \varepsilon) + C.$$

As $2 - 2g < 0$ and M_0 is arbitrary, this finishes the proof. □

Define

$$H_S(d, V, m, K) = \{h \in H(d, V) : \min_{v \in V} h_v \geq m, S(d, V, h) \geq K\}.$$

Lemma 3.6.3 and Lemma 4.3.1 together give

Corollary 4.3.2. For every $d \in \mathfrak{D}_c(V)$, $m \in \mathbb{R}_{>0}$ and $K \in \mathbb{R}$ the set $H_S(d, V, m, K)$ is compact.

Now we turn to cone-manifolds with small heights.

Lemma 4.3.3. If $h^n \in H(d, V)$ is a sequence such that for $w \in V$ we have $h^n_w \to 0$, then for all $v \in V$ we get $h^n_v \to 0$ and $\kappa^n_v \to \nu_v(d) \geq 0$, where κ^n_v are particle curvatures in $P^n = P(d, V, h^n)$. Also for every edge e of all P^n we have $\phi^n_e \to \pi$, where ϕ^n is the dihedral angle of e in P^n.

Proof. Lemma 3.6.2 implies that there are finitely many triangulations $\mathcal{T}$ compatible with some P^n. Hence, up to taking a subsequence we may suppose that the same triangulation $\mathcal{T}$ is compatible with all P^n.

Take v connected with w by an edge e of $\mathcal{T}$. If for some $m > 0$ we have $h^n_v \geq m$, then the trapezoid corresponding to e becomes not ultraparallel for large enough n. This contradicts Lemma 3.2.2. Hence, $h^n_v \to 0$. By the connectivity of the edge graph of $\mathcal{T}$, this extends to all vertices.

Take an oriented edge $\vec{e} \in \vec{E}_v(\mathcal{T})$ emanating from $v \in V$. Develop the trapezoid containing $\vec{e}$ from P^n to $\mathbb{H}^2$ as $A_1A_2B_1B_2$. Let A and B be the closest points on the lines containing the upper and the lower boundary respectively. We have $A_1B_1 = h^n_v$. Denote AB by $h^n_{\vec{e}}$. Clearly, $h^n_{\vec{e}} \leq h^n_v$, hence $h^n_{\vec{e}} \to 0$. From Corollary 2.3.5 we get

$$\sin \alpha_{\vec{e}} = \frac{\cosh h^n_{\vec{e}}}{\cosh h^n_v} \to 1.$$

Hence, $\alpha_{\vec{e}} \to \pi/2$. Now we take a triangle T of $\mathcal{T}$ and the prism from P^n containing T. Let T be incident to $v \in V$, $\lambda^n_{v,T}$ be the angle of T at v, $\omega^n_{v,T}$ be the dihedral angle of the respective lateral edge and $\phi^n_{e,T}$ be the dihedral angle of e in the prism. Consider the spherical section at v. As all $\alpha_{\vec{e}} \to \pi/2$, we obtain $\omega^n_{v,T} \to \lambda^n_{v,T}$ and $\phi^n_{e,T} \to \pi/2$. Thereby, $\omega_v(P^n) \to \lambda_v(d)$ and $\kappa_v(P^n) \to \nu_v(d) \geq 0$. □

Thus, when some heights tend to zero, the upper boundary falls down to the lower boundary as we may expect. Lemma 4.3.3 implies

Corollary 4.3.4. Let $o \in \mathbb{R}^n$ be the origin. The functional S can be C^1-smoothly extended to o over $H(d, V)$ by putting $S(o) = 0$.

Now we are ready to prove the maximum principle.

Lemma 4.3.5. Let $d \in \mathfrak{D}_{sc}(V)$, $P^1 = P(d, V, h^1)$ be the convex polyhedral Fuchsian manifold realizing d and $P^2 = P(d, V, h^2)$ be a convex Fuchsian cone-manifold distinct from P^1. Then $S(P^1) > S(P^2)$.

Proof. We want to show that there exists at least one global maximum of S over $H(d, V)$ and there are $0 < m < M$ such that all global maxima are contained in $H^o(d, V, m, M)$, where

$$H^o(d, V, m, M) := \{h \in H(d, V) : m < h_v < M \text{ for all } v \in V\}.$$

Indeed, an upper bound follows from Lemma 4.3.1. To obtain a lower bound take P^1. We can contract it to o with the help of Lemma 3.6.1. Lemma 4.3.3 implies that all κ_v become positive at some point. Thus, S strictly decreases starting from some point of the deformation. This means that we can find $0 < m < M$ such that $H^o(d, V, m, M)$ contains all global maxima and at least one global maximum exists.

Let $P = P(d, V, h)$ be a convex Fuchsian cone-manifold that is a local maximum (not necessarily strict) of S over $H^o(d, V, m, M)$. We claim that it is actually a convex polyhedral Fuchsian manifold. Indeed, if there is $v \in V$ such that $\kappa_v(P) < 0$, then due to Lemma 3.6.6 we can decrease h_v while staying in $H^o(d, V, m, M)$. Due to Lemma 4.1.1 this increases S. If there is a vertex $v \in V$ such that $\kappa_v(P) > 0$, then Lemma 3.6.8 shows that we can increase simultaneously the heights h_v of all such vertices and stay in $H^o(d, V, m, M)$. Again, this increases S.

Hence, P is a convex polyhedral Fuchsian manifold. Theorem 3.1.4 and Remark 3.1.6 show that $P = P^1$ is unique. □

Here we appealed to Theorem 3.1.4 because the set $H^o(d, V, m, M)$ may be non-convex. However, one can see that it is contractible. With the help of elementary Morse theory one can obtain an alternative proof of uniqueness and, thus, reprove Theorem 3.1.4.

4.4 Second variation formulas in the ideal case

We use the definitions of $\vec{E}$, $\vec{E}_v$ and $\vec{E}_{vu}$ from the previous section.

Lemma 4.4.1. Fix $d \in \widetilde{\mathfrak{C}}(V)$. Then S is twice continuously differentiable over $H(d, V)$ and

$$v \neq u: \quad \frac{\partial^2 S}{\partial h_v \partial h_u} = \frac{\partial \kappa_u}{\partial h_v} = \frac{\partial \kappa_v}{\partial h_u} = \sum_{\vec{e} \in \vec{E}_{vu}(\mathcal{T})} \frac{\cot \phi_{\vec{e}}^+ + \cot \phi_{\vec{e}}^-}{2\alpha_{\vec{e}}^2} \cdot \left(-e^{-2h_v} + \alpha_{\vec{e}}^2\right) \geq 0;$$

$$v = u: \quad \frac{\partial^2 S}{\partial h_v^2} = \frac{\partial \kappa_v}{\partial h_v} = -\sum_{\vec{e} \in \vec{E}_v(\mathcal{T})} \frac{\cot \phi_{\vec{e}}^+ + \cot \phi_{\vec{e}}^-}{2\alpha_{\vec{e}}^2} \cdot \left(e^{-2h_v} + \alpha_{\vec{e}}^2\right) +$$

$$+ \sum_{\vec{e} \in \vec{E}_{vv}(\mathcal{T})} \frac{\cot \phi_{\vec{e}}^+ + \cot \phi_{\vec{e}}^-}{2\alpha_{\vec{e}}^2} \cdot \left(-e^{-2h_v} + \alpha_{\vec{e}}^2\right) =$$

$$= -\sum_{u \neq v} \frac{\partial \kappa_u}{\partial h_v} - \sum_{\vec{e} \in \vec{E}_v(\mathcal{T})} \frac{\cot \phi_{\vec{e}}^+ + \cot \phi_{\vec{e}}^-}{\alpha_{\vec{e}}^2} \cdot e^{-2h_v} < 0.$$

Proof. The proof is similar to the compact case. Let $A_1A_2A_3B_3B_2B_1$ be an ideal prism. The solid angle at the vertex A cuts a Euclidean triangle out of the canonical horosphere at A with side lengths equal to α_{12}, α_{13} and λ_1; its respective angles are ϕ_{12}, ϕ_{13} and ω_1. By the cosine law we have

$$\cos\omega_1 = \frac{\alpha_{12}^2+\alpha_{13}^2-\lambda_1^2}{2\alpha_{12}\alpha_{13}}.$$

We calculate the derivatives of ω_1:

$$\frac{\partial\omega_1}{\partial\alpha_{12}} = -\frac{\cot\phi_{12}}{\alpha_{12}}, \qquad \frac{\partial\omega_1}{\partial\alpha_{13}} = -\frac{\cot\phi_{13}}{\alpha_{13}}.$$

Calculate the derivatives of α_{12} from Corollary 2.3.8:

$$\frac{\partial\alpha_{12}}{\partial h_1} = \frac{-\alpha_{12}^2-e^{-2h_1}}{2\alpha_{12}}, \qquad \frac{\partial\alpha_{12}}{\partial h_2} = \frac{\alpha_{12}^2-e^{-2h_1}}{2\alpha_{12}}.$$

Then

$$\frac{\partial\omega_1}{\partial h_1} = \frac{\partial\omega_1}{\partial\alpha_{12}}\frac{\partial\alpha_{12}}{\partial h_1} + \frac{\partial\omega_1}{\partial\alpha_{13}}\frac{\partial\alpha_{13}}{\partial h_1} = \tag{4.4.1}$$

$$= \frac{\cot\phi_{12}}{2\alpha_{12}^2}\left(\alpha_{12}^2+e^{-2h_1}\right) + \frac{\cot\phi_{13}}{2\alpha_{13}^2}\left(\alpha_{13}^2+e^{-2h_1}\right),$$

$$\frac{\partial\omega_1}{\partial h_2} = \frac{\partial\omega_1}{\partial\alpha_{12}}\frac{\partial\alpha_{12}}{\partial h_2} = \frac{\cot\phi_{12}}{2\alpha_{12}^2}\left(-\alpha_{12}^2+e^{-2h_1}\right), \tag{4.4.2}$$

$$\frac{\partial\omega_1}{\partial h_3} = \frac{\partial\omega_1}{\partial\alpha_{13}}\frac{\partial\alpha_{13}}{\partial h_3} = \frac{\cot\phi_{13}}{2\alpha_{13}^2}\left(-\alpha_{13}^2+e^{-2h_1}\right).$$

The rest is identical to the proof of Lemma 4.2.1. However, the principal difference is that

$$\frac{\partial^2 S}{\partial h_v^2} = \frac{\partial\kappa_v}{\partial h_v} \neq 0.$$

Indeed, the equality to zero would mean that for each oriented edge $\vec{e}\in\vec{E}_v(\mathcal{T})$ we have

$$\cot\phi_{\vec{e}}^+ + \cot\phi_{\vec{e}}^- = 0,$$

i.e., all edges emanating from v are flat. But this implies that there exists a non-simply connected face of P, which is impossible by Lemma 3.2.3. □

Corollary 4.4.2. The functions S and $S_{\kappa'}$ are strictly concave over $H(d,V)$ for $d\in\widetilde{\mathfrak{C}}(V)$.

Proof. Indeed, here we have

$$\bar{x}^T\mathrm{Hess}(S)\bar{x} = \sum_{u,v\in V}\frac{\partial\kappa_u}{\partial h_v}x_ux_v = -\sum_{\substack{u,v\in V\\ u\neq v}}\frac{\partial\kappa_u}{\partial h_v}(x_u-x_v)^2 + \sum_{v\in V}x_v^2\sum_{u\in V}\frac{\partial\kappa_u}{\partial h_v}.$$

For each $v\in V$ we have

$$\sum_{u\in V}\frac{\partial\kappa_u}{\partial h_v} = -\sum_{\vec{e}\in\vec{E}_v(\mathcal{T})}\frac{\cot\phi_{\vec{e}}^+ + \cot\phi_{\vec{e}}^-}{\alpha_{\vec{e}}^2}\cdot e^{-2h_v} < 0.$$

Thus $\mathrm{Hess}(S)=\mathrm{Hess}(S_{\kappa'})$ are negatively defined. □

4.5 Behaviour of S in the ideal case

In the ideal case $H(d,V)=\mathbb{R}^n$ is convex and $S_{\kappa'}$ is strictly concave. Thus, $S_{\kappa'}$ has at most one maximum point in $H(d,V)$. This shows uniqueness in Theorem A.

Now we want to give a new proof of the existence. To this purpose we need to study the behaviour of S outside of a compact set in a similar vein as in the compact case. The tricky part is that (S_g,d) is non-compact.

In what follows we fix V and $d\in\widetilde{\mathfrak{C}}(V)$. We start from the case when all heights are sufficiently negative.

Lemma 4.5.1. For every $\varepsilon>0$ there exists $C_1>0$ such that if for some $v\in V$ and $h\in\mathbb{R}^n$ we have $h_v<-C_1$, then $\omega_v(P)<\varepsilon$ in $P=P(d,V,h)$.

Proof. Fix $\varepsilon>0$ and some ideal triangulation $\mathcal{T}$ of (S_g,d). Take P compatible with $\mathcal{T}$ satisfying the assumptions of the lemma. Recall that for $\vec{e}\in\vec{E}_{vu}(\mathcal{T})$ Lemma 2.3.8 gives

$$\alpha_{\vec{e}}^2=e^{h_u-h_v-l_e}+e^{-2h_v}.$$

Hence,

$$\alpha_e^2\geq e^{-2h_v}.$$

Consider two consecutive edges $\vec{e}_1$ and $\vec{e}_2\in\vec{E}_v(\mathcal{T})$. They cut a Euclidean triangle out of the canonical horosphere at v in P with the side length $\alpha_{\vec{e}_1}$, $\alpha_{\vec{e}_2}$ and λ. If $h_v<-C_1$, then both lengths $\alpha_{\vec{e}_1}$ and $\alpha_{\vec{e}_2}$ are at least e^{C_1} and λ is bounded from above by the total length of the canonical horocycle at v on (S_g,d). The angle ω between sides of lengths $\alpha_{\vec{e}_1}$ and $\alpha_{\vec{e}_2}$ in this triangle (which is the dihedral angle of the lateral edge from v in the prism containing $\vec{e}_1$ and $\vec{e}_2$) decreases as C_1 grows. We choose large enough $C_1>0$ such that if $h_v<-C_1$, then the angle at v in every such triangle is less than $\varepsilon/(6(n+2g-2))$.

Note that the number of triangles incident to one cusp is bounded from above by three times the total number of triangles of $\mathcal{T}$, which can be calculated from the Euler characteristic and is equal to $2(n+2g-2)$. Therefore, the total dihedral angle $\omega_v<\varepsilon$. □

Now we deal with the case when there is at least one sufficiently large positive coordinate. The key lemma is

Lemma 4.5.2. For every $\varepsilon>0$ and $C_1>0$ there exists $C_2>0$ such that if for some $v\in V$ and $h\in\mathbb{R}^n$ we have $h_v\geq C_2$ and for every u we have $h_u\geq -C_1$, then at every point $p\in S_g$ the value of the extended distance function $\widetilde{h}_P(p)\geq\varepsilon$, where $P=P(d,V,h)$.

Proof. Let J_v be the canonical horodisk at v on (S_g,d) and G_v be its boundary. Below $sd(p,G)$ means the signed distance from a point to an horocycle on (S_g,d). Our proof is based on the following simple

Claim 1. Let $t\in\mathbb{R}$ and $D(u,t):=\{p\in S_g \mid sd(p,G_u)\leq t\}$. If $p\in D(u,t)$, then $\widetilde{h}_P(p)\geq h_u-t$.

Indeed, the set $D(u,t)$ is a horodisk centered at $u\in V$. Let $\mathcal{T}$ be a triangulation compatible with P. First, we consider the case when t is small enough, so $D(u,t)$ is contained in the union of triangles of $\mathcal{T}$ incident to u. If $p\in D(u,t)$, then in this case there is a triangle vv_1v_2 containing p. Develop the prism with this triangle to $\mathbb{H}^3$ and let $M_\downarrow$ be the plane passing through the lower boundary. The horodisk $D(u,t)$ is extended to the horoball E in the development and h_u-t is the signed distance from E to M. We have $p\in E$, therefore $\widetilde{h}_P(p)=\mathrm{dist}_{\mathbb{H}^3}(p,M_\downarrow)\geq h_u-t$.

If $D(u,t)$ does not meet this condition, then consider sufficiently small $t_0 \leq t$ such that $D(u,t_0)$ does. For each $p \in D(u,t)$ there exists $q \in D(u,t_0)$ such that $d(p,q) \leq t - t_0$ and $\widetilde{h}_P(q) \geq h_u - t_0$. Then the desired bound follows from Lemma 3.1.8.

Define $t := -C_1 - \varepsilon$ and $D := \cup_{u \neq v} D(u,t)$, where $D(u,t)$ is defined in Claim 1. Then Claim 1 implies that if $p \in D$, then $\widetilde{h}_P(p) \geq \varepsilon$.

Define $z := \sup\{d(p, J_u) : p \in S_g \backslash D\}$. Note that $0 \leq z < \infty$. Indeed, if $p \in J_v$, then $d(p, J_v) = 0$. But the closure of $S_g \backslash (D \cup J_v)$ is compact (possibly empty).

Take $C_2 = \varepsilon + z \geq \varepsilon > 0$. If $p \in J_v$, then $\widetilde{h}_P(p) \geq h_v \geq \varepsilon$ due to Claim 1. If $p \notin D$, then there exists $q \in J_v$ such that $d(p,q) \leq z$ and $\widetilde{h}_P(q) \geq h_v$. Then by Lemma 3.1.8 we obtain $\widetilde{h}_P(p) \geq \varepsilon$. This finishes the proof. □

From Corollary 2.3.10 we see that

Lemma 4.5.3. For every $\varepsilon > 0$ there exists $C > 0$ such that if the distance h_e from an edge e in the upper boundary of a complex P to the lower boundary is greater than C, then $a_e < \varepsilon$, where a_e is the length of the corresponding lower edge.

Combining two last Lemmas with the fact that the sum of angles of a sufficiently small hyperbolic triangle is close to π we obtain

Lemma 4.5.4. For every $\varepsilon > 0$ and $C_1 > 0$ there exists $C_2 > 0$ such that if for some $v \in V$ and $h \in \mathbb{R}^n$ we have $h_v \geq C_2$ and for every u we have $h_u \geq -C_1$, then for $P = P(d, V, h)$ we have

$$\sum_{v \in V} \omega_v(P) \geq 2\pi(n + 2g - 2) - \varepsilon.$$

Now we are able to prove that $S_{\kappa'}$ has a maximal point at $\mathbb{R}^n$.

Lemma 4.5.5. Consider a cube Q in $\mathbb{R}^n$: $Q := \{h \in \mathbb{R}^n : \max_{v \in V} |h_v| \leq q\}$. If

$$\sum_{v \in V} \kappa'_v > 2\pi(2 - 2g), \tag{4.5.1}$$

then for sufficiently large q, the maximum of $S_{\kappa'}(d, V, h)$ over Q is attained in the interior of Q.

Proof. Let $\mu_v := 2\pi - \kappa'_v$ and $\mu := \min(\mu_v : v \in V)$. The condition (4.5.1) can be rewritten as

$$2\pi(n + 2g - 2) > \sum_{v \in V} \mu_v.$$

Take C_1 from Lemma 4.5.1 for $\varepsilon = \mu$ and C_2 from Lemma 4.5.4 for C_1 and $\varepsilon = \varepsilon_0$ where

$$0 < \varepsilon_0 < 2\pi(n + 2g - 2) - \sum_{v \in V} \mu_v.$$

Let $q > \max\{C_1, C_2\}$. The cube Q is convex and compact, S is concave, therefore S reaches its maximal value over Q at some point $h^0 \in Q$. Suppose that $h^0 \in \partial Q$. Then there are two possibilities: either there is $v \in V$ such that $h^0_v < -C_1 < 0$ or for every $v \in V$ we have $h^0_v \geq -C_1$. Consider $P^0 = P(d, V, h^0)$.

In the first case by Lemma 4.5.1 we have $\omega_v(P^0) < \mu \leq \mu_v$. Therefore,

$$\mu_v - \omega_v(P^0) = \kappa_v(P^0) - \kappa'_v = \left.\frac{\partial S_{\kappa'}}{\partial h_v}\right|_{h^0} > 0.$$

Let e_v be the coordinate vector corresponding to v. For small enough $\nu > 0$ we can see that $S_{\kappa'}(d, V, h^0 + \nu e_v) > S_{\kappa'}(d, V, h^0)$ and $h^0 + \nu e_v \in Q$, which is a contradiction.

In the second case consider v such that $|h_v^0| = q$. Then $h_v^0 = q > C_2$ (because if $h_v^0 = -q$, then the first case holds). Therefore, by Corollary 4.5.4 we have

$$\sum_{v \in V} \omega_v(P^0) \geq 2\pi(n + 2g - 2) - \varepsilon_0 > \sum_{v \in V} \mu_v.$$

Consider two sets $I = \{v \in V : h_v^0 = q\}$ and $J = V \backslash I$. Clearly, if $v \in J$, then $\frac{\partial S_{\kappa'}}{\partial h_v}\Big|_{h^0} = 0$. Therefore, $\omega_v(P^0) = \mu_v$. Then we have

$$\sum_{v \in I} \omega_v(P^0) > \sum_{v \in I} \mu_v.$$

Hence, for some $v \in I$ we obtain $\omega_v(P^0) > \mu_v$ and so $\frac{\partial S_{\kappa'}}{\partial h_v}\Big|_{h^0} < 0$. Therefore, for small enough $\nu > 0$ we get $S_{\kappa'}(d, V, h^0 - \nu e_v) > S_{\kappa'}(d, V, h^0)$ and $h^0 - \nu e_v \in Q$, which is a contradiction. $\square$

This finishes the proof of Theorem A.

Chapter 5

Rigidity of compact Fuchsian manifolds with convex boundary

Since now we consider only **compact** convex Fuchsian cone-manifolds. For brevity we will omit the word compact until the end of this manuscript. In this chapter we prove Theorem B modulo the main lemmas formulated in Section 5.4.

5.1 Height functions

Let F be a convex Fuchsian manifold with boundary. We say that the function $h : \partial^{\uparrow} F \to \mathbb{R}_{>0}$ assigning to a point of the upper boundary its distance to the lower boundary is the *height function* of F. We remark that if F is a polyhedral Fuchsian manifold, then according to the notation from Subsection 3.1 we should denote this function by $\widetilde{h}$ and by h we denote only its restriction to the vertices. However, in this section we consider the height functions always defined over the whole upper boundary and denote them by h slightly abusing the notation.

Consider the universal cover G of F developed to the Klein model of $\mathbb{H}^3$. We consider the Klein model endowed simultaneously with the hyperbolic metric and with the metric of the Euclidean unit ball. We note that orthogonality and convexity are the same in both metrics. Let $Ox_1x_2x_3$ be the Euclidean coordinates. We assume that the geodesic plane $\partial_{\downarrow} G$ is developed to the (open) unit disk in the Ox_1x_2 plane. By $\rho : \partial^{\uparrow} G \to \partial_{\downarrow} G$ we denote the orthogonal projection map, which is a homeomorphism due to convexity, and define $h_{\downarrow} := h \circ \rho^{-1} : \partial_{\downarrow} G \to \mathbb{R}_{>0}$, where h is the height function extended to $\partial^{\uparrow} G$. By $h_{\mathbb{E}} : \partial_{\downarrow} G \to \mathbb{R}_{>0}$ we denote the Euclidean distance between $x \in \partial_{\downarrow} G$ and $\rho^{-1}(x)$. The convex surface $\partial^{\uparrow} G$ is the graph of $h_{\mathbb{E}}$. Thereby, $h_{\mathbb{E}}$ is a Lipschitz function over every compact subset of $\partial_{\downarrow} G$ and is differentiable almost everywhere.

Comparing the Euclidean and hyperbolic metrics in the Klein model we compute

$$h_{\mathbb{E}}(x) = h_{\mathbb{E}}(x_1, x_2) = \sqrt{1 - x_1^2 - x_2^2} \tanh h_{\downarrow}. \tag{5.1.1}$$

This implies that $h_{\downarrow}$ is also differentiable almost everywhere and is Lipschitz over every compact subset of $\partial_{\downarrow} G$. In particular, if $\psi_{\downarrow} \subset \partial_{\downarrow} G$ is a rectifiable curve, then $\psi = \rho^{-1}(\psi_{\downarrow})$ is also rectifiable. Note that this is similar to the treatment of *horoconvex functions* done in [37, Subsection 2.2], [61].

By Corollary 2.2.3 if $\psi \subset \partial^{\uparrow} G$ is Lipschitz, then its orthogonal projection to $\partial_{\downarrow} G$ is also Lipschitz.

Denote $x_h = \operatorname{artanh}(x_3/\sqrt{1 - x_1^2 - x_2^2})$. Let $g_{\downarrow}$ be the metric tensor of $\partial_{\downarrow} G$ in the coordinates x_1, x_2 and g_h be the metric tensor of $\mathbb{H}^3$ in the coordinates x_1, x_2, x_h. Using the expression of the metric tensor in the Klein model we get

$$g_h = \cosh^2 x_h g_\downarrow + dx_h^2. \tag{5.1.2}$$

Now we are ready to prove the main result of this subsection:

Lemma 5.1.1. Let F^1 and F^2 be two Fuchsian manifolds with convex boundaries, $f : \partial^\uparrow F^1 \to \partial^\uparrow F^2$ be an isometry and h^1, h^2 be the height functions. Assume that $h^1 = h^2 \circ f$. Then f extends to an isometry of F^1 and F^2.

Proof. We develop both universal covers G^1, G^2 to the Klein model such that both lower boundaries coincide with the plane $M = Ox_1x_2$. Let ρ^1, ρ^2 be the projection maps from the upper boundaries $\partial^\uparrow G^1$, $\partial^\uparrow G^2$ to M and $h^1_\downarrow = h^1 \circ (\rho^1)^{-1}$, $h^2_\downarrow = h^2 \circ (\rho^2)^{-1}$. We will lift f to an isometry of $\partial^\uparrow G^1$ and $\partial^\uparrow G^2$. Define

$$f_\downarrow := \rho^2 \circ f \circ (\rho^1)^{-1} : M \to M.$$

It is enough to prove that $f_\downarrow$ is an isometry. Indeed, if $f_\downarrow$ is a an isometry and $h^1_\downarrow = h^2_\downarrow$, then the natural extension of $f_\downarrow$ to G^1 with the help of the orthogonal projection is an equivariant isometry of G^1 to G^2.

Let $\psi^1_\downarrow : [0;\tau] \to M$ be a rectifiable curve parametrized by length and $\psi^2_\downarrow := f_\downarrow \circ \psi^1_\downarrow$. As the hyperbolic metric on M is intrinsic, it suffices to prove that the length $l(\psi^1_\downarrow) = l(\psi^2_\downarrow)$. Suppose that this is not true. Then there exists a subset $I \subset [0;\tau]$ of strictly positive Lebesgue measure that either $g_\downarrow(\dot\psi^2_\downarrow, \dot\psi^2_\downarrow) > 1$ or $g_\downarrow(\dot\psi^2_\downarrow, \dot\psi^2_\downarrow) < 1$ almost everywhere on I.

Assume that the first case hold, the second is done similarly. There exists an open interval $(\tau_1, \tau_2) \subset I$. Let $\psi^1 := (\rho^1)^{-1} \circ \psi^1_\downarrow$ and $\psi^2 := (\rho^2)^{-1} \circ \psi^2_\downarrow$. As f is an isometry between $\partial^\uparrow G^1$ and $\partial^\uparrow G^2$, we have $l(\psi^1|_{(\tau_1,\tau_2)}) = l(\psi^2|_{(\tau_1,\tau_2)})$. Due to (5.1.2) one gets

$$l(\psi^1|_{(\tau_1,\tau_2)}) = \int_{\tau_1}^{\tau_2} \left(\cosh^2 h^1_\downarrow(\psi^1_\downarrow) g_\downarrow(\dot\psi^1_\downarrow, \dot\psi^1_\downarrow) + (\dot h^1_\downarrow(\psi^1_\downarrow))^2\right)^{1/2} dt,$$

$$l(\psi^2|_{(\tau_1,\tau_2)}) = \int_{\tau_1}^{\tau_2} \left(\cosh^2 h^2_\downarrow(\psi^2_\downarrow) g_\downarrow(\dot\psi^2_\downarrow, \dot\psi^2_\downarrow) + (\dot h^2_\downarrow(\psi^2_\downarrow))^2\right)^{1/2} dt.$$

For all $\tau_1 < t < \tau_2$ (almost all for the third inequality) we have

$$h^1_\downarrow(\psi^1_\downarrow(t)) = h^2_\downarrow(\psi^2_\downarrow(t)),$$

$$g_\downarrow(\dot\psi^1_\downarrow(t), \dot\psi^1_\downarrow(t)) = 1,$$

$$g_\downarrow(\dot\psi^2_\downarrow(t), \dot\psi^2_\downarrow(t)) > 1.$$

Thus, $l(\psi^1|_{(\tau_1,\tau_2)}) \neq l(\psi^2|_{(\tau_1,\tau_2)})$ and we get a contradiction. □

Choose an arbitrary homeomorphism $f^1 : S_g \to \partial^\uparrow F^1$ and let d be the pull-back by f^1 of the intrinsic metric of $\partial^\uparrow F^1$. Then (S_g, d) is a CBB(-1) metric space and f^1 is an isometry. The pair (F^1, f^1), where f^1 is defined up to isotopy, is called a *marked Fuchsian manifold with boundary.* Define $f^2 := f \circ f^1$, which is also an isometry. The height functions h^1 and h^2 can be pulled back to the functions on S_g, which we continue to denote by h^1 and h^2. Lemma 5.1.1 shows that if h^1 and h^2 coincide as functions on S_g, then Theorem B follows. We denote the marked manifolds (F^1, f^1) and (F^2, f^2) as $F(d, h^1)$ and $F(d, h^2)$ respectively.

5.2 Flat points

We call a point $p \in (S_g, d)$ *flat* if there exists an open neighborhood $U \ni p$ such that $\nu(U) = 0$. Otherwise, we say that p is *non-flat*. In particular, if $\nu_p(d) \neq 0$, then p is non-flat. The main result of this subsection is

Lemma 5.2.1. Assume that for each non-flat $p \in (S_g, d)$ we have $h^1(p) = h^2(p)$. Then this is true for any p.

It is reminiscent of the fact from Euclidean convex geometry that the support of m-th curvature measure is the closure of the set of m-extreme points: see [101, Theorem 4.5.1].

First we need to establish some auxiliary facts (note that they are used only for the proof of Lemma 5.2.1).

Lemma 5.2.2. Let $G \subset \mathbb{H}^3$ be a convex body and M be a supporting plane such that $R = M \cap G$ has non-empty interior relative to M. Define $\overline{R} = \partial G \backslash R$. Then $\nu(\overline{R}) > 0$.

Proof. We prove it with the help of duality from Section 2.2.2 and Lemma 2.2.6. In particular, we use the notation from there.

Define the equator

$$(S^2)^* := \{x \in d\mathbb{S}^3 : x_0 = 0\}$$

and the projection map $\mathcal{Q} : \partial G^* \to (S^2)^*$ sending a point $p \in \partial G^*$ to the endpoint of a geodesic from p orthogonal to $(S^2)^*$. It is easy to see that it is a homeomorphism: for the details see [13, Section 1]. Due to Lemma 2.1 from [13] and the duality $\nu = \mu$ from Lemma 2.2.6, to prove $\nu(\overline{R}) > 0$ it is enough to show that $\mathcal{Q}((\overline{R})^*)$ contains an open set. Let C be the convex cone in $\mathbb{H}^3$ with the apex o over ∂R. Consider a plane K through o that is tangent to C, but not to ψ, oriented outwards. The cone C is strictly convex, therefore the set of dual points K^* for all such planes K form an open subset of $(S^2)^*$. Consider a geodesic segment in $d\mathbb{S}^3$ connecting K^* and $\mathcal{Q}^{-1}(K^*) \in \partial G^*$. It corresponds to a path of mutually ultraparallel planes all orthogonal to the same ray from o directed towards $\overline{R}$. Thus, $\mathcal{Q}^{-1}(K^*) \in (\overline{R})^*$. This finishes the proof. □

We need the last Lemma for a proof of the following fact:

Lemma 5.2.3. Let $F = F(d, h)$ be a Fuchsian manifold with convex boundary. Then all flat points of (S_g, d) belong to the convex hull in F of non-flat points.

Proof. Consider the universal cover G of F in $\mathbb{H}^3$. We say that a segment in $\partial^\uparrow G$ is *extrinsically geodesic* if it is a geodesic segment in $\mathbb{H}^3$. Note that $\partial^\uparrow G$ does not contain an extrinsically geodesic ray. Indeed, if such a ray intersects with the boundary at infinity of $\partial_\downarrow G$, then h goes to zero along this ray. Otherwise, h goes to infinity. Both conclusions contradict to the compactness of F.

Recall that a point $p \in \partial^\uparrow G$ is *extreme* if it does not belong to the relative interior of an extrinsically geodesic segment in $\partial^\uparrow G$. A point p is called *exposed* if there exists a plane M in $\mathbb{H}^3$ such that $M \cap G = \{p\}$. As $\partial^\uparrow G$ does not contain extrinsically geodesic rays, it is straightforward that $\partial^\uparrow G$ is contained in the convex hull of its extreme points. Clearly, being extreme or exposed depends only on the preimage of p from $\partial^\uparrow F$.

Consider $\mathbb{H}^3$ in the Klein model. Note that the notions of extreme and exposed points are purely affine, hence, if we consider G as a Euclidean convex set, then extreme and exposed points remain the same. For Euclidean closed convex sets the Straszewicz theorem [106] says that the the extreme points belong to the closure of the set of exposed points. One can see, e.g., [101, Theorem 1.4.7] for a modern proof under additional assumption of compactness. To make G closed we only need to add the boundary at infinity of $\partial_\downarrow G$, which does not change the result because $\partial^\uparrow G$ does not contain extrinsically geodesic rays.

Our plan is to prove that an exposed point is non-flat. The set of non-flat points is closed by definition, hence, extreme points are also non-flat and this finishes the proof due to the discussion above. We use some ideas from Olovyanishnikov [80].

Suppose that $p \in \partial^\uparrow G$ is a flat exposed point. Let U be a neighbourhood of p such that $\nu(U) = 0$ and M be a supporting plane to G such that $M \cap G = \{p\}$. We push M slightly inside by a hyperbolic isometry with the axis passing through p orthogonally to M and obtain the plane M' with $M' \cap G = \chi$ where χ is a closed curve bounding a compact set $U' \subset U$. Lemma 5.2.2 implies that $\nu(U') > 0$. This is a contradiction with $\nu(U) = 0$. □

Now we recall Definition 3.3.1 of an $\mathcal{F}(-1)$-concave function and present the last tool we need:

Lemma 5.2.4. Let $F = F(d, h)$ be a Fuchsian manifold with convex boundary. Then the function $\sinh h$ is $\mathcal{F}(-1)$-concave.

Proof. Let $f : I \to \mathbb{R}$ be a function. We will use two facts that can be proven in the same way as in the theory of concave functions.

Claim 1. Assume that for every $x \in I$ there exists a neighbourhood of x in I for which f is $\mathcal{F}(-1)$-concave. Then f is $\mathcal{F}(-1)$-concave over I.

Claim 2. Assume that f has the left and the right derivatives at every point. Then f is $\mathcal{F}(-1)$-concave if and only if for every sufficiently close t and t' with $t < t'$ (resp. $t > t'$) we have $f(t') \leq g(t')$, where g is a unique $\mathcal{F}(-1)$ function such that $f(t) = g(t)$ and $\dot{g}(t)$ is equal to the right (resp. the left) derivative of f at t.

The proof of the next Claim is also straightforward.

Claim 3 ([3], Lemma 2.1). Let ψ be a line in $\mathbb{H}^2$ and g be the hyperbolic sine of the distance to ψ. Then the restriction of g to every unit speed geodesic is in $\mathcal{F}(-1)$.

We adapt Liberman's method [69], [7, Lemma 1 in Chapter IV.6]. Let $\widetilde{\psi} : [0; \tau] \to (S_g, d)$ be a unit speed geodesic and $\widetilde{L}$ be the union of all segments connecting points of the image of $\widetilde{\psi}$ with its orthogonal projections to $\partial_\downarrow F$. By abuse of notation, we denote $h \circ \widetilde{\psi}(t)$ by $h(t)$. The surface $\widetilde{L}$ is ruled, hence it can be developed isometrically to a subset L of $\mathbb{H}^2$. We denote its boundary components by ψ and $\psi_\downarrow$.

The curve $\psi_\downarrow$ is a geodesic segment. The set L is convex. Indeed, otherwise there are two points $p, q \in \psi$ such that the geodesic segment connecting them lies above ψ. Its length is smaller than the length of ψ between p and q. On the other hand, this segment corresponds to a curve that lies above $\partial^\uparrow F$ in the ambient Fuchsian manifold $\overline{F}$ and connects the preimages $\widetilde{p}$ and $\widetilde{q}$ of p and q under the developing map. Due to Corollary 2.2.3, the length of this curve is at least the length of its orthogonal projection to $\partial^\uparrow F$. Hence, it is at least the length of $\widetilde{\psi}$ between $\widetilde{p}$ and $\widetilde{q}$. This is a contradiction.

The number $h(t)$ is equal to the distance from $\psi(t)$ to $\psi_\downarrow$. Let $p = \psi(0)$ and $p_\downarrow$ be the base of the perpendicular from p to $\psi_\downarrow$. Due to convexity of L, there exists the right half-tangent χ to ψ at p. This is well-known in the Euclidean case and extends to the hyperbolic case with the help of the Klein model. Parametrize χ with the unit speed and denote the distance from $\chi(t)$ to $\psi_\downarrow$ by $h^+(t)$. By α denote the angle between χ and the segment $pp_\downarrow$.

Claim 4. The right derivatives of h and h^+ exist at zero and both are equal to $\cos\alpha$.

Indeed, define $l(t) := d_{\mathbb{H}^2}(\psi(t), \psi(0))$. The convexity of L implies

$$\lim_{t\to 0+} \frac{l(t)}{t} = 1.$$

In the Euclidean case this is Lemma 2 from [7, Chapter IV.6]. If we consider the Klein model and put $\psi(0)$ in the origin, then we see that this is true in the hyperbolic case also. By $\alpha(t)$ we denote the angle between segments $p\psi(t)$ and $pp_\downarrow$. With the help of Lemma 2.3.3 we get

$$\lim_{t\to 0+} \frac{h(t)-h(0)}{t} = \lim_{t\to 0+} \frac{\sinh h(t) - \sinh h(0)}{t\cosh h(0)} =$$

$$= \lim_{t\to 0+} \frac{\sinh h(t) - \sinh h(0)}{l(t)\cosh h(0)} = \lim_{t\to 0+} \frac{\cosh l(t)\sinh h(t) - \sinh h(0)}{\sinh l(t)\cosh h(0)} =$$

$$= \lim_{t\to 0+} \cos\alpha(t) = \cos\alpha.$$

For $h^+(t)$ the same computations hold.

Thereby, the functions h and h^+ have the same right derivatives at 0, $h(0) = h^+(0)$ and $\sinh h^+ \in \mathcal{F}(-1)$ due to Claim 3. Let us now prove that for sufficiently small t we have $h(t) \leq h^+(t)$. One can prove similar statement for any sufficiently close points p and q on ψ. Then it follows from Claim 1 and Claim 2 that $\sinh h$ is $\mathcal{F}(-1)$-concave.

Let $q = \psi(t)$ be a point such that ψ is a shortest path between p and q. By $q_\downarrow$ denote the base of perpendicular from q and by α denote the angle between ψ and the segment $pp_\downarrow$. Assume that $\alpha \geq \pi/2$. Define $t' = d_{\mathbb{H}^2}(p,q) \leq t$. We have $h^+(t') \geq h(t) = d_{\mathbb{H}^2}(q, q_\downarrow)$. Indeed, to get $\chi(t')$ we need to rotate the segment pq from the line $\psi_\downarrow$. As $t' \leq t$ and $\alpha \geq \pi/2$, we get $h^+(t) \geq h^+(t') \geq h(t)$.

Now we assume that $\alpha < \pi/2$. Extend the segment $qq_\downarrow$ to the intersection with χ and denote the intersection point by $q' = \chi(t')$. We have $h^+(t') \geq h(t)$. Define $t'' = t' + d_{\mathbb{H}^2}(\chi(t'), q)$. Due to convexity, $t'' \geq t$. Consider the point $q'' = \chi(t'')$. As the triangle $q''q'q$ is isosceles, we get $\angle q_\downarrow q q'' > \pi/2$. Therefore, $h^+(t'') \geq h(t)$.

We need to get $h^+(t) \geq h^+(t'')$. As $\alpha < \pi/2$, if χ is not ultraparallel to $\psi_\downarrow$, then this is true for any $t \leq t''$. Otherwise, we assume that t is sufficiently small, so q'' lies on the same side with q' from the closest point of χ to $\psi_\downarrow$. Thus, $h^+(t) \geq h^+(t'') \geq h(t)$ and the proof is finished. □

Now we have all the ingredients that we need.

Proof of Lemma 5.2.1. Let $p \in (S_g, d)$ be a flat point. Due to Lemma 5.2.3, p belongs to the convex hull of non-flat points. The Caratheodory theorem (normally it is stated for Euclidean convex bodies, but this does not matter because of the Klein model) implies that it belongs to the convex hull of at most four non-flat points. If this number can not be reduced to at most three, then p must be an interior point of F. Thereby, p belongs either to the relative interior of a segment ψ with non-flat endpoints q_1 and q_2 or to a triangle T with non-flat vertices q_1, q_2 and q_3 that is extrinsically geodesic in F^1. Consider the first case. Parametrize ψ by length. Note that the distance function restricted to an extrinsically geodesic segment is $\mathcal{F}(-1)$: this is Claim 3 from the previous proof. As h^1 coincides with h^2 at the endpoints of

ψ, by Lemma 5.2.4 we have $h^2|_\psi \geq h^1|_\psi$. In particular $h^2(p) \geq h^1(p)$. In the second case, let ψ be the edge of T opposite to q_1. Similarly, we have $h^2|_\psi \geq h^1|_\psi$. Connect q_1 with a point $q \in \psi$ by a geodesic χ passing through p. This is possible as T is extrinsically geodesic in F^1, so it is isometric to a hyperbolic triangle. Applying the same arguments to χ we still get that $h^2(p) \geq h^1(p)$. In the same way one can show that $h^1(p) \geq h^2(p)$. Thus, $h^1(p) = h^2(p)$. □

We showed that to prove Theorem B it is enough to show that h^1 coincides with h^2 at all non-flat points.

5.3 Polyhedral approximation

Let (S_g, d) be a CBB(-1) metric space.

Theorem 5.3.1. There exists a convex Fuchsian manifold with boundary F such that $\partial^\uparrow F$ with the induced path metric is isometric to (S_g, d).

The proof of Theorem 5.3.1 is not written, but it is a straightforward simplification of the proof of Theorem 1.3.19 proving a similar statement for two metrics and the quasi-Fuchsian case, see [103]. An explanation of the relation between Theorem 1.3.19 and Theorem 5.3.1 is given in the introduction of [37]. Briefly, the proof of Theorem 1.3.19 is based on a smooth approximation and the smooth realization theorem. The smooth Fuchsian realization theorem is available in [43]. Other details remain unchanged.

Note that F can be embedded into a unique up to isometry complete Fuchsian manifold $\overline{F}$.

We need to define several classes of triangulations:

Definition 5.3.2. A geodesic triangulation $\mathcal{T}$ of (S_g, d) is called *short* if all edges are shortest paths and all angles are strictly smaller than π. It is called *strictly short* if additionally all edges are unique shortest paths between their endpoints.

Definition 5.3.3. A geodesic triangulation $\mathcal{T}$ of (S_g, d) is called *δ-fine* if for every triangle T we have $\mathrm{diam}(T, d) < \delta$ and $\nu(T, d) < \delta$.

Lemma 5.3.4. For every $\delta > 0$ there exists a strictly short δ-fine triangulation $\mathcal{T}$ of (S_g, d). Moreover, each triangulation $\mathcal{T}'$ can be refined to a short δ-fine triangulation.

Proof. The proof is the same as in CBB(0) case [88, Lemma in Chapter III.1, p. 134]. However, Pogorelov uses the intrinsic curvature ν_I instead of ν. This can be easily fixed as the area of sufficiently small triangles uniformly diminishes. □

Let F be a convex Fuchsian manifold with boundary from Theorem 5.3.1 with $\partial^\uparrow F$ isometric to (S_g, d). This isometry makes F marked. We will consider various convex surfaces in F. We mark all of them with the help of the vertical projection map (along the perpendiculars to $\partial_\downarrow F$). Their intrinsic metrics and height functions are transferred to S_g with the help of marking maps. In particular, this induces a marking on $\partial_\downarrow F$.

Consider a sequence of positive numbers $\mu_m \to 0$ and a sequence of finite μ_m-dense sets $V_m \subset (S_g, d)$. Take the convex hull of V_m in F and obtain a convex Fuchsian manifold F_m with polyhedral boundary. By d_m we denote its boundary metric.

Lemma 5.3.5. The metrics d_m converge uniformly to d.

Proof. We follow the ideas from [37, Lemma 3.7] and [26, Lemma 10.2.7]. Unfortunately, some details are less straightforward in our setting.

Let h, h_m be the height functions of $\partial^\uparrow F$, $\partial^\uparrow F_m$. The sequence $\partial^\uparrow F_m$ converges to $\partial^\uparrow F$ in the Hausdorff sense. This implies that h_m converges to h uniformly, i.e., $\zeta_m = \sup_{p \in S_g}(h(p) - h_m(p)) \to 0$ (note that $h_m \leq h$).

Claim 1. Let $\partial^\uparrow F^1$, $\partial^\uparrow F^2$ be two convex surfaces in a marked Fuchsian manifold $\overline{F}$, h^1 and h^2 be their height functions and d^1, d^2 be their intrinsic metrics. Assume that $h^1 \leq h^2$ and

$$\zeta := 2 \sup_{p \in S_g} h^2(p) - h^1(p).$$

Then $d^1 \leq d^2 + \zeta$.

The proof is the same as in [37, Lemma 2.12]. Claim 1 and $h_m \leq h$ implies $d_m \leq d + \zeta_m$ with $\zeta_m \to 0$.

Due to (5.1.1), for each t the function $\widetilde{h}_t$ defined by the equation

$$\tanh \widetilde{h}_t = e^{-t} \tanh h$$

is the height function of another convex surface in F. We can choose a sequence $t_m \to 0$ such that $\widetilde{h}_{t_m} \leq h_m$ and $\widetilde{h}_{t_m}$ converges uniformly to h. Denote $\widetilde{h}_{t_m}$ by $\widetilde{h}_m$, the respective surface by $\partial^\uparrow \widetilde{F}_m$ and let $\widetilde{d}_m$ be the intrinsic metric of $\partial^\uparrow \widetilde{F}_m$. From Claim 1 we obtain $\widetilde{d}_m \leq d_m + \widetilde{\zeta}_m$ with $\widetilde{\zeta}_m \to 0$.

It remains to relate d with $\widetilde{d}_m$. First, we check how the distances in F change. Fix m, let $p, q \in \partial^\uparrow F$ and $\widetilde{p}, \widetilde{q} \in \partial^\uparrow \widetilde{F}_m$ be the corresponding points. Define $l := d_F(p, q)$, $\widetilde{l} := d_F(\widetilde{p}, \widetilde{q})$. We want to show that there exist numbers $\xi_m \to 0$ independent of p, q such that $l \leq \widetilde{l}(1 + \xi_m)$. We need

Claim 2. Let $A_1A_2B_1B_2$ and $\widetilde{A}_1\widetilde{A}_2B_1B_2$ be two (equally oriented) ultraparallel trapezoids with

$$\tanh \widetilde{h}_1 = e^{-t} \tanh h_1,$$

$$\tanh \widetilde{h}_2 = e^{-t} \tanh h_2$$

(here we use the notation from Subsection 2.3 and mark the parameters of the trapezoid $\widetilde{A}_1\widetilde{A}_2B_1B_2$ with tilde). By h_0 we denote the distance between lines A_1A_2 and B_1B_2. For $0 < m < M$ by $\mathrm{Trap}(t, m, M)$ we denote the space of such pairs of trapezoids with $h_0 \geq m$ and $h_1, h_2 \leq M$. Define the function $\mathrm{rat} : \mathrm{Trap}(t, m, M) \to \mathbb{R}_{>0}$ as $l_{12}/\widetilde{l}_{12}$. Fix m, M and let $t \to 0$. Then

$$\sup_{\mathrm{Trap}(t,m,M)} \mathrm{rat} \to 1.$$

First, we finish the proof of the Lemma and then give a proof of Claim 2. As F is compact, the function h is bounded from below and from above by positive constants. Let $p', q' \in \partial^\uparrow G \subset \mathbb{H}^3$ be the lifts of p, q to the universal cover such that $d_{\mathbb{H}^3}(p', q') = d_F(p, q)$. The line passing through p', q' is ultraparallel to the lower plane $\partial_\downarrow G$. Let r be the closest point from this line to $\partial_\downarrow G$. If r lies outside the segment $p'q'$, then it lies above $\partial^\uparrow G$ and $d_{\mathbb{H}^3}(r, \partial_\downarrow G)$ is at least the infimum of h. If r lies between p' and q', then due to Corollary 2.3.5 we have

$$\sinh h(p') = \sinh d_{\mathbb{H}^3}(r, \partial_\downarrow G) \cosh d_{\mathbb{H}^3}(p', r).$$

As $d_{\mathbb{H}^3}(p', r) \leq d_F(p, q) \leq \mathrm{diam}(F)$, we get that there exists $m = m(F) > 0$ such that $d_{\mathbb{H}^3}(r, \partial_\downarrow G) \geq m$. Thus, we can apply Claim 2 and get $l \leq \widetilde{l}(1 + \xi_m)$ with $\xi_m \to 0$.

Now connect $\widetilde{p}$ and $\widetilde{q}$ by a shortest path $\widetilde{\psi} \subset \partial^\uparrow \widetilde{F}_m$. Let $\psi \subset \partial^\uparrow F$ be its image under the vertical projection. It is rectifiable. Consider a polygonal approximation of $\widetilde{\psi}$ with sufficiently small segments. It corresponds to a polygonal approximation of ψ of total length multiplied by at most $1+\xi_m$. As the lengths of ψ and $\widetilde{\psi}$ are the suprema of the lengths of their polygonal approximations, we get $d \leq \widetilde{d}_m(1+\xi_m)$.

In total, we obtain $|d - d_m| \to 0$ uniformly.

□

Proof of Claim 2. The space Trap of all ultraparallel trapezoids up to isometry can be parametrized by h_0, h_1 and a_{12} with $h_0 > 0$, $h_1 \geq h_0$ and $a_{12} \in \mathbb{R}\backslash\{0\}$: first we choose two ultraparallel lines at distance h_0. Let A be the closest point on the upper line. We choose A_1 to the right from A such that its distance to the lower line is h_1. Then we choose A_2 to the right from A_1 if a_{12} is positive and to the left if a_{12} is negative. One can see that two trapezoids are isometric if and only if the parameters are the same.

The trapezoid $\widetilde{A}_1\widetilde{A}_2B_1B_2$ is restored uniquely by $A_1A_2B_1B_2$, hence for each t we can consider $\mathrm{Trap}(t,m,M)$ as a subset of Trap. The closure $\overline{\mathrm{Trap}}(t,m,M)$ is compact (note that $m \leq h_0 \leq h_1 \leq M$) and is obtained by adding the degenerated trapezoids with $a_{12} = 0$. We want to show that the function rat extends there continuously. Indeed, by Lemma 2.3.4 we get

$$\lim_{a_{12}\to 0} \mathrm{rat}(h_0,h_1,a_{12}) = \lim_{a_{12}\to 0} \frac{l_{12}}{\widetilde{l}_{12}} = \lim_{a_{12}\to 0} \frac{\sinh l_{12}}{\sinh \widetilde{l}_{12}} = \frac{\cosh h_1 \sin \widetilde{\alpha}_{12}}{\cosh \widetilde{h}_1 \sin \alpha_{12}}.$$

Applying Lemma 2.3.4 the second time we eliminate the angles:

$$\lim_{a_{12}\to 0} \mathrm{rat}(h_0,h_1,a_{12}) = \frac{\cosh^2 h_1 \cosh \widetilde{h}_0}{\cosh^2 \widetilde{h}_1 \cosh h_0} > 0.$$

Now for arbitrary $\tau > 0$ we consider the set

$$\mathrm{TRAP}(\tau,m,M) = \{\cup \mathrm{Trap}(t,m,M) : 0 \leq t \leq \tau\}$$

and its closure $\overline{\mathrm{TRAP}}(\tau,m,M)$. The function rat is a continuous function over this compact and is equal to 1 as $t = 0$. The proposition follows. □

Corollary 5.3.6. $\mathrm{diam}(S_g,d_m) \to \mathrm{diam}(S_g,d)$, $\quad \mathrm{area}(S_g,d_m) \to \mathrm{area}(S_g,d)$.

Proof. The first statement is trivial. For the second we use Theorem 9 of [2, Chapter 8].

□

Let T be a geodesic triangle in (S_g,d). By $T(d) = (l_1,l_2,l_3,\lambda_1,\lambda_2,\lambda_3)(d)$ denote the 6-tuple of its side lengths and angles in the metric d. We will always consider these 6-tuples as points of $\mathbb{R}^6$ endowed with l_∞ metric.

Lemma 5.3.7. Let $\mathcal{T}$ be a strictly short triangulation of (S_g,d). Consider a sequence of finite μ_m-dense sets $V_m \subset (S_g,d)$ with $\mu_m \to 0$ such that all of them contain $V(\mathcal{T})$ and none of them contains a point in the interior of an edge of $\mathcal{T}$. By d_m denote the pull back of the upper boundary metric of the convex hull of V_m in F defined as above.

Then $\mathcal{T}$ is realized by infinitely many d_m. Moreover, the realizations can be chosen such that they are short and for every triangle T we have (after taking a subsequence)
(1) $T(d_m) \to T(d)$;
(2) $\mathrm{diam}(T,d_m) \to \mathrm{diam}(T,d)$;
(3) $\mathrm{area}(T,d_m) \to \mathrm{area}(T,d)$.

Proof. Consider $u, v \in V(\mathcal{T})$ connected by an edge $e \in E(\mathcal{T})$. Connect u and v by a shortest path $\psi_m(e)$ in $\partial^\uparrow F_m$. By $\psi(e)$ we denote the realization of e in $\partial^\uparrow F$. By [7, Chapter II.1, Theorems 4 and 5] applied to the intrinsic metric space F, for every $e \in E(\mathcal{T})$ the sequence $\{\psi_m(e)\}$ converges uniformly (up to taking a subsequence) in F as parametrised curves to a rectifiable path $\psi'(e) \subset \partial^\uparrow F$ of length at most $l_e(d)$. As $\psi(e)$ is the unique shortest path in $\partial^\uparrow F$ between its endpoints, we have $\psi(e) = \psi'(e)$.

Choose $\xi_1 > 0$ such that ξ_1-neighborhoods of vertices of $\mathcal{T}$ in F do not intersect. Then we choose $\xi_2 > 0$ such that for every pair of edges e' and e'', ξ_2-neighborhoods of $\psi(e'), \psi(e'')$ intersect only if e' and e'' share an endpoint v and if they do, then the intersection lies in the ξ_1-neighborhood of v. For sufficiently large m, $\psi_m(e)$ belongs to ξ_2-neighborhood of $\psi(e)$. This means that $\psi_m(e')$ and $\psi_m(e'')$ can intersect only if e' and e'' share an endpoint. But in this case they can not intersect except at this endpoint by the non-overlapping property: if two shortest paths have two points in common, then either these are their endpoints or they have a segment in common, see [7, Chapter II.3, Theorem 1].

It is clear now that the union of $\psi_m(e)$ gives a realization of $\mathcal{T}$ in d_m. We will continue to work with this realization. The convergence of side lengths of each triangle is already shown. Now we proceed to diameters and areas.

Let $\rho : F \to \partial^\uparrow F$ be the vertical projection map. We claim that $\rho(\psi_m(e))$ converge uniformly to $\psi(e)$ in the metric d. To see this, first project everything to $\partial_\downarrow F$. Denote the images by $\psi_{m\downarrow}(e)$ and $\psi_\downarrow(e)$. The curves $\psi_m(e)$ converge uniformly to $\psi(e)$ in F and the projection to $\partial_\downarrow F$ contract F-distances. Therefore, $\psi_{m\downarrow}(e)$ converge uniformly to $\psi_\downarrow(e)$ in $\partial_\downarrow F$. It remains to recall that $\partial^\uparrow F$ is the graph of a Lipschitz function h, hence $d \leq C d_\downarrow$ for some constant C, where $d_\downarrow$ is the intrinsic metric of $\partial_\downarrow F$.

The convergence of the diameters follows easily from this and the uniform convergence of the metrics. For the convergence of areas we apply Theorem 8 of [2, Chapter 8].

Let us prove the convergence of angles. Consider $v \in V(\mathcal{T})$ and let $U_1, \dots, U_k$ be the sectors between the edges of $\mathcal{T}$ emanating from v in the cyclic order. The claim that

$$\operatorname{ang}(U_i, d) \leq \liminf_{m\to\infty} \operatorname{ang}(U_i, d_m)$$

follows line by line the proof in the Euclidean case, see [7, Chapter IV.4, Theorem 2].

Assume that for $\xi > 0$ and some i we have

$$\operatorname{ang}(U_i, d) < \liminf_{m\to\infty} \operatorname{ang}(U_i, d_m) - \xi.$$

Then for some m we have $\lambda_v(d) < \lambda_v(d_m) - \xi$. This contradicts to

Claim 1. For each $v \in V(\mathcal{T})$ and each m, $\lambda_v(d_m) \leq \lambda_v(d)$.

□

Proof of Claim 1. Consider the universal covers $\partial^\uparrow G, \partial^\uparrow G_m \subset \mathbb{H}^3$ of $\partial^\uparrow F$ and $\partial^\uparrow F_m$. We use the Klein model of $\mathbb{H}^3$ embedded into $\mathbb{E}^3$ as the interior of the unit Euclidean ball $B(1)$. By $d_{\mathbb{E}}$ and $d_{\mathbb{H}}$ denote the Euclidean and the hyperbolic metrics on $B(1)$ respectively. We assume that v is developed to the origin. Both $\partial^\uparrow G$ and $\partial^\uparrow G_m$ are also convex surfaces in the Euclidean metric. By d' we denote the intrinsic metric on $\partial^\uparrow G$ induced by $d_{\mathbb{E}}$. We want to show that $\lambda_v(d) = \lambda_v(d')$.

Let $B(r)$ be the Euclidean ball with the raidus $r < 1$ and center v. Comparing the Euclidean and the hyperbolic metric one can see that the identity map between $(B(r), d_{\mathbb{E}})$ and $(B(r), d_{\mathbb{H}})$ is bi-Lipschitz with constant $C(r)$ such that $C(r) \to 1$ as

$r \to 0$, where $d_{\mathbb{E}}$ and $d_{\mathbb{H}}$ are the Euclidean and the hyperbolic metrics respectively on $B(r)$.

Let ψ and χ be two geodesics on $\partial^{\uparrow}G$ in the metric d emanating from v. Due to the bi-Lipscitz equivalence of small neighborhoods of v it is easy to see that the angle $\mathrm{ang}(\psi,\chi,d')$ is also defined and is equal to $\mathrm{ang}(\psi,\chi,d)$. Let $p\in\psi$, $q\in\chi$ be two points, connect them by shortest paths ψ', χ' with v in the metric d. By definition, $\mathrm{ang}(\psi',\chi',d')\to\mathrm{ang}(\psi,\chi,d')$. Using the definition of the total angle as the supremum of sums of angles between consecutive geodesics, we get $\lambda_v(d)\leq\lambda_v(d')$. The converse inequality could be obtained in the same way.

Consider the tangent cone to $\partial^{\uparrow}G$ at v in the Euclidean metric (see [7, Chapter IV.5]). Its total angle is equal to $\lambda_v(d')=\lambda_v(d)$ [7, Chapter IV.6, Theorem 3]. Similarly there exists the tangent cone to $\partial^{\uparrow}G_m$ at v with the total angle $\lambda_v(d_m)$. The latter tangent cone is inscribed in the former. We note that for the Euclidean convex cones the total angle is equal to the area of the Gaussian image. This shows the desired inequality. □

5.4 Statements of the stability lemmas

Our proof of Theorem B is based on several lemmas, but even their statements are somehow cumbersome. Hence, we are going to discuss each one before we formulate it.

From Lemma 4.3.5 we know that if $P^1=P(d,V,h^1)$ is a convex polyhedral Fuchsian manifold (so it does not have cone-singularities) and $P^2=P(d,V,h^2)$ is a convex (polyhedral) Fuchsian cone-manifold distinct from P^1 with isometric boundary to P^1, then $S(P^1)>S(P^2)$. We want to give a better quantitative lower bound on $S(P^1)-S(P^2)$. Naturally, such bound should depend on $\max_{v\in V}|h^2_v-h^1_v|$ and on global geometry of P^1. However, to our purposes it is enough to treat only the case when there exists $v\in V$ such that $h^2_v>h^1_v$.

An important difficulty arises on our way: the bound we get depends on the curvature ν_v of a point v with $h^2_v\neq h^1_v$. This is unsatisfactory for our purposes: when we try to apply such a bound in the proof of Theorem B we realize that we can not guarantee a lower bound on the curvature of any particular point v with $h^1_v\neq h^2_v$. However, results of Subsection 5.2 allow us to guarantee a curvature bound in a small neighbourhood of such a point. This means that we should be able to handle the case when we have a lower bound on the curvature of a set of points where h^1 is distinct from h^2. We conjecture that the presence of ν in our Main Lemma I is excessive and it should be possible to give a bound without it (this should be helpful for a resolution of the Cohn-Vossen problem for Fuchsian manifolds).

Lemma 5.4.1 (Main Lemma I)**.** Let d be a convex cone-metric on S_g, $V=V(d)$, $P^1=P(d,V,h^1)$ be a convex polyhedral Fuchsian manifold and $P^2=P(d,V,h^2)$ be a convex (polyhedral) Fuchsian cone-manifold. Let $W\subseteq V$ be a subset of vertices. Define

$$\nu:=\sum_{w\in W}\nu_w(d),\quad m:=\min_{w\in W}\tanh h^1_w,\quad \tau:=\min_{w\in W}\ln\left(\frac{\sinh h^2_w}{\sinh h^1_w}\right),$$

$$M:=\max_{w\in W}\cosh^2\Big(\operatorname{arsinh}\left(e^{\tau}\sinh h^1_w\right)\Big).$$

Assume that for all $w \in W$ we have $h^2_w > h^1_w$, thereby, $\tau > 0$. Then

$$S(P^1) - S(P^2) \geq \nu m \left(M e^{\tau/M} - \tau \right).$$

Note that if $W = \{w\}$, then $M = \cosh^2 h^2_w$.

Remark 5.4.2. The function $\nu m \left(M e^{\tau/M} - \tau \right)$ is increasing in ν, m, M and τ in the range $\nu > 0$, $m > 0$ and $M > \tau > 0$. The definition of M implies that $M > \tau$.

A natural way to approximate a metric d by cone-metrics (when we do not have a convex isometric embedding) is to take a sufficiently fine geodesic triangulation $\mathcal{T}$ and replace each triangle by a triangle from a model space (e.g. $\mathbb{H}^2$) with the same side lengths. For such approximations the curvature converges weakly as a measure. However, if we pick a particular point or a triangle, then its curvature can be quite distorted. In our proof we need to overcome this. To this purpose we propose to replace each triangle of $\mathcal{T}$ by a triangle with the same side lengths, angles and one conical point in the interior. This is the subject of Main Lemma II.

Recall from Subsection 5.3 that $T(d)$ means the 6-tuple of side lengths and angles of a triangle T in metric d. We consider these 6-tuples as elements of $\mathbb{R}^6$ endowed with l_∞-norm. We also recall definitions of short (Definition 5.3.2) and δ-fine (Definition 5.3.3) triangulations.

Definition 5.4.3. Let d be a cone-metric on S_g and $\mathcal{T}$ be a triangulation of S_g. We call d *swept with respect to* $\mathcal{T}$ if d realizes $\mathcal{T}$ and each triangle T of $\mathcal{T}$ in (S_g, d) has at most one conical point in the interior.

Definition 5.4.4. Let (S_g, d) be a CBB(-1) metric space and $\mathcal{T}$ be its geodesic triangulation. A convex cone-metric $\widehat{d}$ on S_g is called a *sweep-in* of d with respect to $\mathcal{T}$ if

(1) $\widehat{d}$ is swept with respect to $\mathcal{T}$;
(2) for each triangle T of $\mathcal{T}$ we have $T(d) = T(\widehat{d})$.

Lemma 5.4.5 (Main Lemma II)**.** There exists an absolute constant $\delta > 0$ such that if (S_g, d) is a CBB(-1) metric space and $\mathcal{T}$ is its short δ-fine triangulation, then there exists a unique sweep-in $\widehat{d}$ of d with respect to $\mathcal{T}$.

Next we turn to Main Lemma III, which appears to be decomposed into three parts. It is the core of our proof strategy. Assume that we have two cone-metrics d^1 and d^2 that are very close in some way and there is a convex polyhedral Fuchsian manifold $P^1 = P(d^1, V(d^1), h^1)$. We claim that there is a convex Fuchsian cone-manifold $P^{21} = P(d^2, V(d^2), h^{21})$ such that $\widetilde{h}^{21}(p) \geq \widetilde{h}^1(p)$ for some "essential" points p and $S(P^{21})$ can not be significantly smaller than $S(P^1)$. The idea behind this is that we can transform d^1 to d^2 with the help of some elementary operations and modify P^1 along this transformation so that (1) we control the change of heights of some points; (2) we control the decrement of S.

Lemma 5.4.6 (Main Lemma IIIA)**.** There exists an absolute constant $\delta > 0$ with the following properties. Let d be a convex cone-metric on S_g, $\mathcal{T}$ be a short δ-fine triangulation of (S_g, d), $\widehat{d}$ be a sweep-in of d with respect to $\mathcal{T}$ and $P = P(d, V(d), h)$ be a convex Fuchsian cone-manifold. There exists $\widehat{h} \in H(\widehat{d}, V(\widehat{d}))$ such that

(1) $\widehat{h}_v \geq h_v$ for each $v \in V(\mathcal{T}) \cap V(d)$;
(2) $S(\widehat{P}) = S(\widehat{d}, V(\widehat{d}), \widehat{h}) \geq S(P)$.

Lemma 5.4.7 (Main Lemma IIIB)**.** For any numbers $\varepsilon, A, D > 0$ there exists $\delta = \delta(\varepsilon, A, D)$ with the following properties. Let d be a convex cone-metric on S_g with

$$\mathrm{diam}(S_g, d) < D, \quad \mathrm{area}(S_g, d) < A;$$

$\mathcal{T}$ be a δ-fine triangulation of (S_g, d), $\widehat{d}$ be a sweep-in of d with respect to $\mathcal{T}$ and $\widehat{P} = P(\widehat{d}, V(\widehat{d}), \widehat{h})$ be a convex Fuchsian cone-manifold. Then there exists $h \in H(d, V(d))$ such that

(1) $h_v \geq \widehat{h}_v$ for each $v \in V(\mathcal{T}) \cap V(d)$;
(2) $S(P) = S(d, V(d), h) \geq S(\widehat{P}) - \varepsilon$.

Lemma 5.4.8 (Main Lemma IIIC)**.** Let $\widehat{d}$ be a convex cone-metric on S_g swept with respect to a triangulation $\mathcal{T}$. For each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, \widehat{d}, \mathcal{T}) > 0$ with the following properties. Let d^1 and d^2 be convex cone-metrics on S_g swept with respect to $\mathcal{T}$ and for each triangle T of $\mathcal{T}$ we have

$$||T(d^1) - T(\widehat{d})||_\infty < \delta, \quad ||T(d^2) - T(\widehat{d})||_\infty < \delta.$$

Let $P^1 = P(d^1, V(d^1), h^1)$ be a convex Fuchsian cone-manifold. Then there exists $h^{21} \in H(d^2, V(d^2))$ such that

(1) $h^{21}_v \geq h^1_v$ for each $v \in V(\widehat{d})$;
(2) $S(P^{21}) = S(d^2, V(d^2), h^{21}) \geq S(P^1) - \varepsilon$.

5.5 Proof of Theorem B

Suppose the contrary. Because the heights of non-flat points determine uniquely a compact Fuchsian manifold with convex boundary (Lemma 5.1.1 and Lemma 5.2.1), there exists a non-flat point $u \in (S_g, d)$ such that $h^1(u) \neq h^2(u)$. Without loss of generality assume that $h^2(u) - h^1(u) > 0$. Define

$$H := \frac{\min\{h^1(u), h^2(u) - h^1(u)\}}{3}.$$

Let U be an (open) geodesic triangle with $\mathrm{diam}(U, d) < H$ containing u in the interior. Define

$$D := 2\mathrm{diam}(S_g, d), \quad A := 2\mathrm{area}(S_g, d), \quad \nu := \nu(U, d)/2 > 0$$

$$m := \tanh(h^1(u) - H), \quad \tau := \ln\left(\frac{\sinh h^2(u) - H}{\sinh h^1(u) + H}\right) > 0,$$

$$M := \cosh^2\Big(\mathrm{arsinh}\left(e^\tau \sinh(h^1(u) - H)\right)\Big),$$

$$\varepsilon = \frac{1}{5}\nu m\left(Me^{\tau/M} - \tau\right) > 0.$$

Take δ_1 as the minimum of δ from Main Lemma II, IIIA and from IIIB for these ε, A, D. Construct a short δ_1-fine geodesic triangulation $\mathcal{T}$ of (S_g, d) with the help of Lemma 5.3.4. We can assume that

(1) $u \in V(\mathcal{T})$,
(2) $\mathcal{T}$ subdivides U in the sense that there is a subset $\mathcal{W}$ of triangles of $\mathcal{T}$ such that $U = \cup_{T \in \mathcal{W}} T$.

Indeed, if this is not the case, we can refine $\mathcal{T}$ to a triangulation $\mathcal{T}'$ for which this is true. It might happen that the new triangulation is not strictly short. But due to

Lemma 5.3.4, we can refine it further, finally obtaining a triangulation fulfilling all our demands, which we continue to denote by $\mathcal{T}$.

So we have $\sum_{T\in\mathcal{W}} \nu(T,d) = 2\nu$.

Due to Main Lemma II, there exists a sweep-in $\widehat{d}$ of d with respect to $\mathcal{T}$. Let δ_2 be δ from Main Lemma IIIC for ε, $\widehat{d}$ and $\mathcal{T}$.

We have two convex Fuchsian manifolds with boundary $F^1 = F(d,h^1)$ and $F^2 = F(d,h^2)$. Choose a sufficiently dense set $V \subset (S_g,d)$ containing $V(\mathcal{T})$. Define $P^1 = P(d^1, V(d^1), h^1)$ and $P^2 = P(d^2, V(d^2), h^2)$ to be convex polyhedral Fuchsian manifolds obtained by taking the convex hull of V in F^1 and F^2 respectively. Here d^1 and d^2 are induced metrics on the boundaries pulled back to S_g with the help of the vertical projections, and, abusing the notation, we denote restrictions of h^1, h^2 to $V(d^1)$, $V(d^2)$ still by h^1 and h^2.

Because u is non-flat in (S_g,d), we have $u \in (V(d^1)\cap V(d^2))$.

Due to the convergence of inscribed polyhedral manifolds (Lemma 5.3.7 and Corollary 5.3.6) we can choose V such that

(1) both d^1 and d^2 realize $\mathcal{T}$;

(2) $\operatorname{diam}(S_g,d^1) < D$, $\operatorname{diam}(S_g,d^2) < D$, $\operatorname{area}(S_g,d^1) < A$, $\operatorname{area}(S_g,d^2) < A$;

(3) $\mathcal{T}$ is δ_1-fine on d^1, d^2;

and for each triangle T of $\mathcal{T}$ we have

(4) $||T(d^1) - T(d)||_\infty < \delta_2$, $||T(d^2) - T(d)||_\infty < \delta_2$.

Because $T(d) = T(\widehat{d})$ we can rewrite (4) as

(4) $||T(d^1) - T(\widehat{d})||_\infty < \delta_2$, $||T(d^2) - T(\widehat{d})||_\infty < \delta_2$.

By W we denote all the points of V in the triangles of $\mathcal{W}$, i.e., $W = V \cap U$. As all the angles and the areas of the triangles of $\mathcal{T}$ in d^1 can be chosen to be arbitrarily close to those in d, we can assume also that

$$\nu' := \sum_{T\in\mathcal{W}} \nu(T,d^1) = \sum_{w\in W} \nu_w(d^1) > \nu.$$

Let $\widehat{d}^1$, $\widehat{d}^2$ be sweep-ins of d^1, d^2 with respect to $\mathcal{T}$. They exist due to Main Lemma II as d^1, d^2 are δ_1-fine.

First we apply Main Lemma IIIA to P^2 and $\widehat{d}^2$. We get $\widehat{h}^2 \in H(\widehat{d}^2, V(\widehat{d}^2))$ such that

(1) $\widehat{h}^2_u \geq h^2_u$;

(2) $S(\widehat{P}^2) = S(\widehat{d}^2, V(\widehat{d}^2), \widehat{h}^2) \geq S(P^2)$.

Now we apply Main Lemma IIIC to $\widehat{P}^2$ and $\widehat{d}^1$. We get $\widehat{h}^{12} \in H(\widehat{d}^1, V(\widehat{d}^1))$ such that

(1) $\widehat{h}^{12}_u \geq \widehat{h}^2_u$;

(2) $S(\widehat{P}^{12}) = S(\widehat{d}^1, V(\widehat{d}^1), \widehat{h}^{12}) \geq S(\widehat{P}^2) - \varepsilon$.

Finally, we take $\widehat{P}^{12}$, d^1 and apply Main Lemma IIIB. We get $h^{12} \in H(d^1, V(d^1))$ such that

(1) $h^{12}_u \geq \widehat{h}^{12}_u$;

(2) $S(P^{12}) = S(d^2, V(d^1), h^{12}) \geq S(\widehat{P}^{12}) - \varepsilon$.

Summarizing all last three steps together we see:

(1) $h^{12}_u \geq h^2_u$;

(2) $S(P^{12}) = S(d^2, V(d^1), h^{12}) \geq S(P^2) - 2\varepsilon$.

We get $h^{12}_u \geq h^2_u \geq h^1_u + 3H$.

Now we do the same but starting from P^1 and tranforming d^1 to $\widehat{d}^1$ to $\widehat{d}^2$ to d^2. We obtain $h^{21} \in H(d^2, V(d^2))$ such that (here we need only properties (2))

$$S(P^{21}) = S(d^2, V, h^{21}) \geq S(P^1) - 2\varepsilon.$$

Let

$$m' := \min_{w \in W} \tanh(h^1_w), \quad \tau' := \min_{w \in W} \ln \left(\frac{\sinh h^{12}_w}{\sinh h^1_w} \right),$$

$$M' := \max_{w \in W} \cosh^2 \left(\operatorname{arsinh} \left(e^{\tau} \sinh(h^1_w) \right) \right)$$

and ν' is defined above. Then Main Lemma I implies that

$$S(P^1) - S(P^{12}) \geq \nu' m' \left(M' e^{\tau'/M'} - \tau' \right).$$

Recall that we chose U such that $\operatorname{diam}(U, d) < H$. Thus, for each $w \in W$ we have $d(u, w) < H$. We know $d^1 \leq d$, $d^2 \leq d$ due to Corollary 2.2.3. Thus, for each $w \in W$ we have $d^1(u, w) < H$, $d^2(u, w) < H$. From this we conclude the bounds on heights with the help of Lemma 3.1.8: for every $w \in W$ we obtain

$$h^{12}_w \geq h^{12}_u - H \geq h^2_u - H,$$

$$h^1_u - H \leq h^1_w \leq h^1_u + H.$$

Due to Remark 5.4.2 we get

$$\nu' m' \left(M' e^{\tau'/M'} - \tau' \right) \geq \nu m \left(M e^{\tau/M} - \tau \right) = 5\varepsilon.$$

By Lemma 4.3.5 we have

$$S(P^1) - S(P^{12}) \geq 0.$$

Summing this up we have

$$4\varepsilon \geq S(P^2) - S(P^{12}) + S(P^1) - S(P^{21}) \geq 5\varepsilon.$$

Thus, $\varepsilon \leq 0$ and we obtain a contradiction.

Chapter 6

Stability lemmas

6.1 Proof of Main Lemma I

6.1.1 Dual area

Let $P = P(d, V, h)$ be a (compact) convex Fuchsian cone-manifold. In Section 3.1 we defined the spherical link of a vertex $v \in V$. Intrinsically it is a disk with a spherical cone-metric that has (at most) one conical point in the interior, piecewise geodesic boundary and the angles of all kink points of the boundary are less than π. Geometry of spherical links is highly connected with the behaviour of discrete curvature. In order to get the quantitative bound of Main Lemma I, first we prove an inequality concerning spherical links, which will be used further in estimating better the derivatives of the discrete curvature.

Lemma 6.1.1. Let R be a disk with a spherical cone-metric that has a single conical point Z in the interior of total angle ω and curvature $\kappa := 2\pi - \omega$, and piecewise geodesic boundary with kink points $A_1, \ldots, A_m$ denoted in the cyclic order. By α_i denote the length ZA_i, by ϕ_i^+ and ϕ_i^- denote the angles ZA_iA_{i+1} and ZA_iA_{i-1} respectively. Assume that $\kappa \leq 0$, $m \geq 1$, for each $i = 1 \ldots m$ we have $\phi_i^- + \phi_i^+ < \pi$ and the perimeter λ of R is (strictly) less than 2π. Define $\nu := 2\pi - \lambda$. Then

$$\sum_{i=1}^{m} \frac{\cot \phi_i^+ + \cot \phi_i^-}{\sin \alpha_i} \cdot \cot \alpha_i \geq \nu - \kappa. \tag{6.1.1}$$

Proof. Note that under our assumptions both sides of (6.1.1) are greater than 0.

We include the case $\kappa = 0$ under our consideration. Then there is no conical point, but an arbitrary marked point Z in the interior of R. Actually, we start from treating this case, so suppose that $\kappa = 0$, which is equivalent to $\omega = 2\pi$.

In this case we develop the polygon R to the unit sphere such that Z coincides with the south pole. Then R determines a solid convex cone with the apex at the origin. Consider its polar cone. It defines the polar spherical polygon $\widetilde{R}$ that belongs to the open hemisphere centered at the north pole $\widetilde{Z}$. Project it from the center of sphere to the tangent plane at $\widetilde{Z}$. We get a Euclidean polygon $\bar{R}$. If all $\alpha_i < \pi$, then $\widetilde{Z}$ is in the interior of both $\widetilde{R}$, $\overline{R}$. If there are some $\alpha_i > \pi$, then $\widetilde{Z}$ is outside from both of them.

Regardless the location of $\widetilde{Z}$, one can compute that the left-hand side of (6.1.1) is exactly the area of $\bar{R}$ and the right-hand side is the area of $\widetilde{R}$ (in particular this will follow from the general discussion below). Hence, (6.1.1) is true because the central projection increases areas.

In the general case $\kappa \neq 0$, one can still consider similar polar polygons (which will contain cone-singularities) as long as all $\alpha_i < \pi$. Otherwise the polar interpretation breaks down and $\bar{R}$, $\widetilde{R}$ can be defined only in some virtual sense.

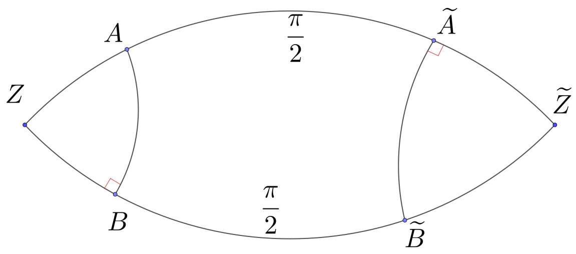

FIGURE 6.1: An orthosceme and its dual.

Now we proceed to more details. We call a spherical triangle $O = ZAB$ a *(spherical) orthoscheme* if $\angle B = \pi/2$, the sides ZA and ZB are either both at most $\pi/2$ or both at least $\pi/2$ and O is equipped with plus or minus sign. If ZA is equal to $\pi/2$, then the orthoscheme is called *singular*. If $ZB = \pi/2$, then also automatically $ZA = \pi/2$ and we call O *double-singular*. From simple trigonometry we conclude that if O is non-singular, then

$$\operatorname{sgn}(\pi/2 - \angle A) = \operatorname{sgn}(\pi/2 - ZA) = \operatorname{sgn}(\pi/2 - ZB) \neq 0,$$

$$AB < \pi/2, \quad \angle Z < \pi/2.$$

Define its *dual* orthoscheme $\widetilde{O} = \widetilde{Z}\widetilde{A}\widetilde{B}$ as a spherical triangle with $\angle\widetilde{A} = \pi/2$, $\angle\widetilde{Z} = \angle Z$ and $\widetilde{Z}\widetilde{A} = |\pi/2 - ZA|$. One can conclude

$$\widetilde{Z}\widetilde{B} = |\pi/2 - ZB|, \quad \widetilde{A}\widetilde{B} = |\pi/2 - \angle A|, \quad \angle\widetilde{B} = \pi/2 - AB.$$

See Figure 6.1. For a non-singular orthoscheme O the sign of $\widetilde{O}$ is defined as the sign of O if $ZA, ZB < \pi/2$ or as the converse sign otherwise. For a singular orthoscheme, the dual orthoscheme is degenerate and we assign zero as its sign.

This duality is not entirely reflexive. By definition, only orthoschemes with both $ZA, ZB < \pi/2$ can be obtained as duals and each such non-degenerate orthoscheme is obtained as the dual of exactly two orthoschemes.

Let O be an orthoscheme. Consider the unit sphere and place the dual orthoscheme $\widetilde{O}$ such that $\widetilde{Z}$ becomes the north pole. Then $\widetilde{O}$ belongs to the open northern hemisphere. Project it from the center of the sphere to the tangent plane at $\widetilde{Z}$. The image is called *the Euclidean dual orthoscheme* to O and is denoted by $\bar{O} = \bar{Z}\bar{A}\bar{B}$ (where $\widetilde{Z} = \bar{Z}$). One can compute the signed areas, i.e., the areas taken with the sign of the dual orthoshemes:

$$\operatorname{sarea} \widetilde{O} = \angle Z - AB, \quad \operatorname{sarea} \bar{O} = \frac{\cot \angle B}{\sin ZA} \cot ZA.$$

Let ZA_1A_2 be a spherical triangle with both ZA_1, ZA_2 at least $\pi/2$ or both at most $\pi/2$, but at least one (say, ZA_1) not equal to $\pi/2$. Put it on the unit sphere and consider the geodesic great circle containing segment A_1A_2. Then there are two perpendiculars from Z to it. Let ZB be the one such that $\operatorname{sgn}(\pi/2 - ZB) = \operatorname{sgn}(\pi/2 - ZA_1)$. The characteristic function of triangle ZA_1A_2 is the signed sum of the characteristic function of triangles ZBA_1, ZBA_2. Consider the latter triangles as orthoschemes endowed with signs coming from this sum. We say that ZA_1A_2 is *canonically decomposed* into the orthoschemes ZBA_1, ZBA_2. In the case of $ZA_1 = ZA_2 = \pi/2$ the canonical decomposition of ZA_1A_2 into orthoschemes is ZA_1A_2 itself taken with the plus sign (then it is a double-singular orthoscheme).

Now we turn to the spherical cone-polygon R. Since now we will call the kink points *vertices* of R. Cut it into triangles ZA_iA_{i+1}. If there are i such that $ZA_i < \pi/2$ and $ZA_{i+1} > \pi/2$ (or the converse), then there exists a unique $A \in A_iA_{i+1}$ such that $ZA = \pi/2$ and we add A to the set of vertices. Hence, we may assume that for all i either both ZA_i, ZA_{i+1} are at least $\pi/2$ or both are at most $\pi/2$. We decompose each triangle canonically into orthoschemes and call it *the canonical decomposition* of R. By $\mathcal{O}$ denote the set of orthoschemes of the canonical decomposition. The total angle ω is the signed sum of angles at Z in all orthoschemes, similarly the perimeter λ. Thereby,

$$\sum_{O\in\mathcal{O}} \operatorname{sarea} \widetilde{O} = \omega - \lambda = \nu - \kappa$$

and the inequality (6.1.1) is equivalent to

$$\sum_{O\in\mathcal{O}} \operatorname{sarea} \bar{O} \geq \sum_{O\in\mathcal{O}} \operatorname{sarea} \widetilde{O}.$$

Take A_i and consider two orthoschemes incident to ZA_i. Denote them by O_i^+ and O_i^- respectively. Define $\phi_i := \phi_i^+ + \phi_i^-$. We have all $\phi_i < \pi$, except possibly some i with $\alpha_i = \pi/2$ (which correspond to additionally added vertices). One can see that if $\alpha_i < \pi/2$, then the sum of the dual orthoschemes $\widetilde{O}_i^+ + \widetilde{O}_i^-$ is a positively oriented spherical triangle $\widetilde{Z}\widetilde{B}_{i-1}\widetilde{B}_i$ with $\widetilde{B}_{i-1}\widetilde{B}_i = \pi - \phi_i$. Here B_i is the base of perpendicular from Z to the geodesic A_iA_{i+1} in the canonical decomposition and the sum can be defined rigorously as the polygon determined by the signed sum of the characteristic functions of $\widetilde{O}_i^+$, $\widetilde{O}_i^-$. We call it a positive pair. We obtain that if $\alpha_i < \pi/2$, then

$$\operatorname{sarea} \bar{O}_i^+ + \operatorname{sarea} \bar{O}_i^- = \operatorname{sarea} \left(\bar{O}_i^+ + \bar{O}_i^-\right) \geq$$

$$\geq \operatorname{sarea} \left(\widetilde{O}_i^+ + \widetilde{O}_i^-\right) = \operatorname{sarea} \widetilde{O}_i^+ + \operatorname{sarea} \widetilde{O}_i^-$$

because the central projection increases areas.

Similarly, if $\alpha_i = \pi/2$, then

$$\operatorname{sarea} \bar{O}_i^+ + \operatorname{sarea} \bar{O}_i^- = \operatorname{sarea} \widetilde{O}_i^+ + \operatorname{sarea} \widetilde{O}_i^- = 0.$$

Therefore, if all $\alpha_i \leq \pi/2$, then the proof is finished.

It remains to consider the case, when some $\alpha_i > \pi/2$, which is much more subtle.

First, we investigate the boundary structure. Let S^1 be an oriented circle of length ω. It naturally parametrizes the set of geodesics emanating from Z. We say that $0 \in S^1$ corresponds to the segment ZA_1. If ZA is the segment corresponding to $t \in S^1$, where A belongs to the boundary of R, then t is the angle of the sector (in the sense of Alexandrov: see Section 2.1) between ZA_1, ZA in the positive direction. We remark that if the angles of both sectors between ZA_1, ZA are at least π, then the angle between geodesics ZA_1, ZA in the sense of Alexandrov is equal to π, hence, it might be not equal to the both sector angles: this is a feature of negative curvature. Let $\alpha(t)$ be the length of ZA. It defines a continuous function α on S^1. We now consider the case when for some $t \in S^1$ we have $\alpha(t) > \pi/2$.

Lemma 3.6.4 implies that no geodesic segment in the boundary of R has length at least π. This clearly means that no geodesic segment in R has length at least π. Otherwise we can extend this segment until it intersects the boundary and cut off the part of the polygon that does not contain Z. Then we obtain the cone-polygon with non-positive curvature, small perimeter and geodesic segment in the boundary of length at least π, which can not happen.

In particular this means that for each $t \in S^1$ we get $\alpha(t) + \alpha(t+\pi) < \pi$. Then there also exists $t \in S^1$ such that $\alpha(t) < \pi/2$.

Claim 1. There exist exactly two $t_1, t_2 \in S^1$ such that $\alpha(t_1) = \alpha(t_2) = \pi/2$.

With the preliminary discussion this means that S^1 is subdivided by t_1, t_2 into two complementary open intervals I, J such that $\alpha|_I > \pi/2$ and $\alpha|_J < \pi/2$.

Indeed, consider t such that $\alpha(t) > \pi/2$. Let I be the maximal interval containing t for which $\alpha|_I > \pi/2$. We saw that $I \neq S^1$. Let t_1, t_2 be the left and the right endpoints of I, so $\alpha(t_1) = \alpha(t_2) = \pi/2$. Let A_{i_2} be the vertex of R corresponding to t_2. Then $\phi^-_{i_2} > \pi/2$. Consider the interval $I^+ = (t_2; t_2 + \pi]$. Develop the part of R corresponding to I^+ to the unit sphere. From convexity, $\phi^-_{i_2} > \pi/2$ and $\alpha(t_2) = \pi/2$ one can see that $\alpha|_{I^+} < \pi/2$.

Assume that there are more points such that $\alpha(t) = \pi/2$. Take the leftmost of them to t_2 and denote it by t'_1. We showed that $t'_1 - t_2 > \pi$. Together with $\alpha(t_2) = \alpha(t'_1) = \pi/2$ this implies that the part of perimeter of R between the segments from Z corresponding to t_2 and t'_1 is greater than π. Similarly, we consider t'_2 that is the rightmost to t_1 such that $\alpha(t'_2) = \pi/2$. We get that the segments corresponding to t'_2 and t_1 also cut off the part of perimeter of length greater than π. The intervals $[t_2; t'_1)$ and $(t'_2; t_1]$ do not intersect. Hence, the perimeter of R is greater than 2π, which is a contradiction. Claim 1 is proven.

By A_{i_1}, A_{i_2} we denote the points at the boundary of R corresponding to t_1, t_2. Due to our previous agreement, they are vertices even if their angles are π. By $\mathcal{A}_I, \mathcal{A}_J$ we denote the sets of vertices corresponding to the segments $I, J \subset S^1$.

We are going to discuss now what happens with dual orthoschemes. We focus on Euclidean duals, but we note that exactly the same happens with spherical duals. We think that our discussion is easier to imagine on the Euclidean plane.

Take $A_i \in \mathcal{A}_I$, we get $\alpha_i > \pi/2$. For two orthoschemes incident to ZA_i the sum of their Euclidean duals $\bar{O}^+_i + \bar{O}^-_i$ is a negatively oriented triangle $\bar{Z}\bar{B}_{i-1}\bar{B}_i$. We call it a negative pair.

Consider all $A_i \in \mathcal{A}_I$ and start developing all dual triangles $\bar{Z}\bar{B}_{i-1}\bar{B}_i$ one by one to the Euclidean plane. The sides $\bar{B}_{i-1}\bar{B}_i$ constitute a polygonal curve $\bar{B}_{i_1}\bar{B}_{i_1+1}\ldots\bar{B}_{i_2-1}$, which we denote by $\bar{B}_\smile$. See Figure 6.2. Define

$$\omega_\smile := \sum_{i_1}^{i_2-2} \angle \bar{B}_i\bar{Z}\bar{B}_{i+1}.$$

The angle $\bar{B}_{i-1}\bar{B}_i\bar{B}_{i+1}$ as seen from $\bar{Z}$ is

$$\angle \bar{B}_{i-1}\bar{B}_i\bar{Z} + \angle \bar{Z}\bar{B}_i\bar{B}_{i+1} = \pi + \lambda_i > \pi,$$

where $\lambda_i = A_iA_{i+1} < \pi$. This means that the polygonal curve $\bar{B}_\smile$ is convex as seen from $\bar{Z}$. From this one can show that $\omega_\smile < \pi$ and $\bar{B}_\smile$ is not self-intersecting.

We remark that $\omega_\smile \neq \angle A_{i_1}ZA_{i_2}$ or $\angle A_{i_1+1}ZA_{i_2-1}$. The angle $\omega_\smile$ can be seen as follows. Take all orthoschemes for triangles $ZA_{i_1}A_{i_1+1}, \ldots ZA_{i_2-1}A_{i_2}$ except the first and the last ones, which are singular. The (signed) sum of their angles at Z is negative and is equal to $-\omega_\smile$.

Our plan is to cover the (negatively oriented) polygon $\bar{Z}\bar{B}_{i_1}\ldots\bar{B}_{i_2-1}$ by (positively oriented) triangles constructed from remaining orthoschemes. This will finish the proof: we obtain (6.1.1) for the sum of all negative pairs with the positive pairs,

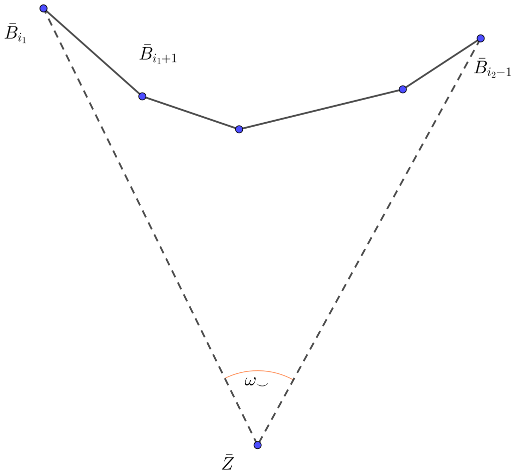

FIGURE 6.2: Polygonal curve $\bar{B}_\smile$.

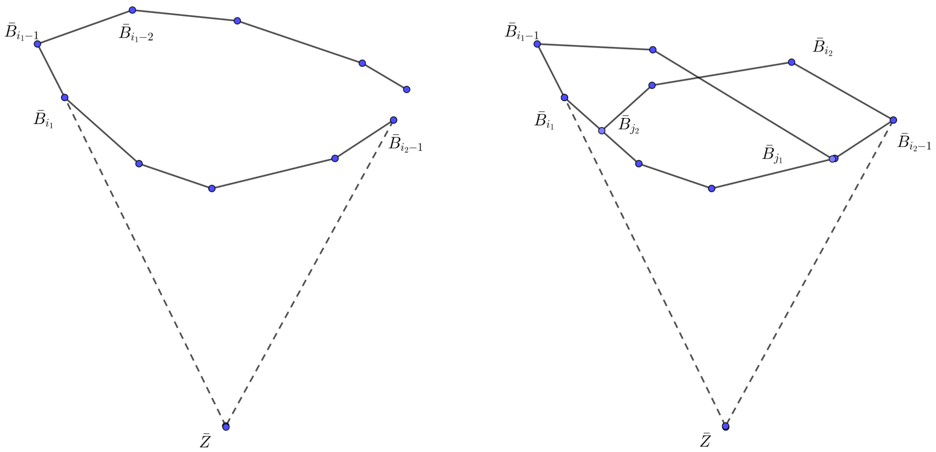

FIGURE 6.3: Left: curve $\bar{B}_{\frown 1}$ is above $\bar{B}_\smile$.
Right: Curves $\bar{B}_{\frown 1}$ and $\bar{B}_{\frown 2}$ intersect.

which we used to cover. All remained orthoschemes are split into positive pairs, for which the inequality also holds. If this is not possible, then we will prove $\lambda > 2\pi$, which is a contradiction.

If we do a similar construction with all $A_i \in \mathcal{A}_J$, then we obtain a concave polygonal curve $\bar{B}_\frown = \bar{B}_{i_2} \dots \bar{B}_{i_1-1}$ seen from $\bar{Z}$ under total angle $\omega_\frown$. Then $\omega_\frown \geq \omega_\smile$. Indeed, this follows from

$$\pi - \omega_\smile + \pi + \omega_\frown = \omega \geq 2\pi$$

that we obtain from the decomposition of R into orthoschemes.

Now we return to $\bar{B}_\smile$ and start to develop positively oriented triangles $\bar{Z}\bar{B}_{i_1-1}\bar{B}_{i_1-2}, \dots$ to the right from the ray $\bar{Z}\bar{B}_{i_1}$. We note that if $\phi_{i_1} = \pi$, then $\bar{B}_{i_1-1}$ coincides with $\bar{B}_{i_1}$, otherwise, $\bar{B}_{i_1}$ lies in the interior of the segment $\bar{Z}\bar{B}_{i_1-1}$. The new polygonal curve is concave (it is a part of polygonal curve $\bar{B}_\frown$). As $\omega_\frown \geq \omega_\smile$, there are two possibilities: either the new polygonal curve is entirely above $\bar{B}_\smile$ (except possibly $\bar{B}_{i_2-1}$) or it intersects $\bar{B}_\smile$ transversely.

In the first case the proof is finished. In the second case if the intersection point is not a vertex of the upper polygonal curve, then we declare it a vertex. Denote it by B_{j_1}. Then we start to develop the triangles $\bar{Z}\bar{B}_{i_2}\bar{B}_{i_2+1},\dots$ to the left from the ray $\bar{Z}\bar{B}_{i_2-1}$. Similarly, either the new polygonal curve is above $\bar{B}_{\smile}$ or it intersects it. Then declare the intersection point to be a vertex and denote it by $\bar{B}_{j_2}$.

We need to consider the case, when both upper polygonal curves intersect $\bar{B}_{\smile}$. If $\bar{B}_{j_2}$ is to the left from $\bar{B}_{j_1}$ (or they coincide), then the upper polygonal curves intersect (possibly non-transversely). Let $\bar{B}$ be an intersection point. The polygon $\bar{Z}\bar{B}_{i_1}\dots\bar{B}_{i_2-1}$ is covered by the union of polygons $\bar{Z}\bar{B}_{i_1-1}\dots\bar{B}$ and $\bar{Z}\bar{B}\dots\bar{B}_{i_2}$. They are made from distinct dual orthoschemes because $\angle\bar{B}_{i_1-1}\bar{Z}\bar{B}+\angle\bar{B}\bar{Z}\bar{B}_{i_2}=\omega_{\smile}\leq\omega_{\frown}$.

The last remaining case is when $\bar{B}_{j_1}$ lies to the left from $\bar{B}_{j_2}$. We denote polygonal curve $\bar{B}_{i_1-1}\dots\bar{B}_{j_1}$ by $\bar{B}_{\frown 1}$ and polygonal curve $\bar{B}_{j_2}\dots\bar{B}_{i_2}$ by $B_{\frown 2}$. We will prove that $\lambda>2\pi$. Our plan is to find points A_j, A_k on the boundary of R such that

(1) the angles of both sectors between ZA_j and ZA_k are at least π;

(2) $ZA_j+ZA_k>\pi$.

Indeed, this means that both parts of the boundary between A_j and A_k have lengths greater than π. This will finish the proof.

Assume that $\bar{B}_{j_1}$ lies on the segment $\bar{B}_{l_1-1}\bar{B}_{l_1}$ of $\bar{B}_{\smile}$ (possibly $\bar{B}_{l_1}$ coincides with $\bar{B}_{j-1}$ as a point, however we distinguish then $\bar{B}_{j-1}$ as a vertex of $\bar{B}_{\frown 1}$ and $\bar{B}_{l_1}$ as a vertex of $\bar{B}_{\smile}$). We claim that the angle subtended by $A_{j_1+1}A_{j_1+2}\dots A_{l_1}$ is greater than π.

Indeed, we consider the polygon formed by $\bar{B}_{\smile}$, $\bar{B}_{\frown 1}$ in the tangent plane to the unit sphere at the north pole. Consider the solid cone from the origin over this polygon and take its polar cone. The polar cone intersects the sphere in a polygon consisting of two parts: one is $ZA_{j_1+1}A_{j_1+2}\dots A_{l_1}$ from R and the second is a geodesic spherical triangle formed by connecting ZA_{l_1} with ZA_{j_1+1} via a geodesic segment. The length of this segment is smaller than π, hence it is seen from Z with angle smaller than π. This shows that the angle subtended by $A_{j_1+1}A_{j_1+2}\dots A_{l_1}$ is greater than π.

Similarly, let $\bar{B}_{j_2}$ belongs to segment $\bar{B}_{l_2-1}\bar{B}_{l_2}$ on $\bar{B}_{\smile}$. Then the angle subtended by $A_{l_2}A_{l_2+1}\dots A_{j_2}$ is at least π.

Now take an arbitrary point of $\bar{B}_{\smile}$ between $\bar{B}_{j_1}$ and $\bar{B}_{j_2}$. Let it belong to a segment $\bar{B}_{k-1}\bar{B}_k$ on $\bar{B}_{\smile}$. We see that the angles of both sectors between ZA_{j_1+1} and ZA_k are at least π; similarly for ZA_{j_2} and ZA_k. It remains to show that either $ZA_{j_1+1}+ZA_k>\pi$ or $ZA_{j_2}+ZA_k>\pi$. We look at three lines: L_{j_1+1} through $\bar{B}_{j_1},\bar{B}_{j_1+1}$; L_{j_2} through $\bar{B}_{j_2},\bar{B}_{j_2-1}$ and L_k through $\bar{B}_{k-1},\bar{B}_k$. Due to their location with respect to $\bar{Z}$, either the distance from $\bar{Z}$ to $\bar{L}_{j_1}$ or the distance from $\bar{Z}$ to $\bar{L}_{j_2}$ is smaller than the distance from $\bar{Z}$ to L_k. Without loss of generality, assume that this is to L_{j_2}. Going to the spherical duals recall that for every i the distance from $\widetilde{Z}$ to the geodesic $\widetilde{B}_i\widetilde{B}_{i-1}$ is $|\pi/2-ZA_i|$. Then our conclusion on the lines is equivalent to

$$ZA_k-\pi/2>\pi/2-ZA_{j_2},$$

which means $ZA_{j_2}+ZA_k>\pi$ as desired. □

6.1.2 Comparing discrete curvature

Now we investigate how maxima of the discrete curvature behave under some constraints. Let $d\in\mathfrak{D}_{sc}(V)$ and $P^0=P(d,V,h^0)$ be the convex polyhedral Fuchsian manifold. By Lemma 4.3.5 P^0 maximizes S over $H(d,V)$. Consider $W\subseteq V$. For each $t\geq 0$ define the set

$$H(t)=\{h\in H(d,V):\sinh h_w\geq e^t\sinh h^0_w \text{ for every } w\in W\},$$

which is non-empty due to Lemma 3.6.1.

The core of our proof of Main Lemma I is the following

Lemma 6.1.2. The maximum of S over $H(t)$ is attained at a unique point $h(t)$. Moreover, the map $t \to h(t)$ is C^1 and satisfies

(1) for each $v \in V \backslash W$ we have $\kappa_v(t) = 0$ and $0 < \dot{h}_v(t) < \tanh h_v(t)$;

(2) for each $w \in W$ we have $\sinh h_w(t) = e^t \sinh h_w^0$ and $\dot{h}_w(t) = \tanh h_w(t)$.

Here $\kappa_w(t)$ are curvatures $\kappa(P(t))$ in the cone-manifold $P(t) = P(d, V, h(t))$. The property $\dot{h}_w(t) = \tanh h_w(t)$ follows directly from $\sinh h_w(t) = e^t \sinh h_w^0$.

Proof. For reader's convenience we advise to assume during the first read that W is a single point w.

Let $\check{h}(t)$ be any of maximum points of S over $H(t)$. By $\check{\kappa}_v(t)$ we denote the particle curvatures in the cone-manifold $\check{P}(t) := P(d, V, \check{h}(t))$.

Claim 1. For every $v \in V \backslash W$ we have $\check{\kappa}_v(t) \geq 0$.

Suppose the contrary. Because of Lemma 3.6.4 and Lemma 3.6.6 the height of v can be decreased. This deformation stays in $H(t)$ and increases S due to Lemma 4.1.1, which contradicts to the choice of $\check{h}(t)$.

Claim 2. For every $v \in V$ we have $\check{\kappa}_v(t) \leq 0$.

The proof is the same as of the previous Claim, but we use Lemma 3.6.8.

We obtain that for each $\check{h}(t)$ and each $v \in V \backslash W$, we have $\check{\kappa}_v(t) = 0$. For $w \in W$ we have $\check{\kappa}_w(t) \leq 0$. Also if $t > 0$, then there exists $w \in W$ such that $\check{\kappa}_w(P) < 0$ (we use the fact that the critical point of S is unique due to Theorem 3.1.4 and Remark 3.1.6). Lemma 3.6.6 implies that if for some $w \in W$ we have $\check{\kappa}_w(P) < 0$, then $\sinh \check{h}_w(t) = e^t \sinh h_w^0$; otherwise we can increase S while staying in $H(t)$.

For all $t \geq 0$ Corollary 3.6.11 implies that $\check{h}(t) \in \mathrm{int}(H(d, V))$ and Corollary 4.2.4 implies that the Hessian of S is non-degenerate at $\check{h}(t)$.

Take $t_0 > 0$ and some $\check{h}(t_0)$. Let us show that there is a unique piecewise C^1-curve $h(t) : (-\infty; t_0] \to H(d, V)$ such that:

(0) $h(t_0) = \check{h}(t_0)$;

(1) for any $v \in V \backslash W$, we have $\kappa_v(t) = 0$ (here we write $\kappa_v(t)$ for particle curvatures of the cone-manifold $P(t) = P(d, V, h(t))$);

(2) for $w \in W$ if for some $t' > t$ we have $\kappa_w(t') = 0$, then $\kappa_w(t) = 0$; otherwise

$$\sinh h_w(t) = e^{t - t_0} \sinh \check{h}_w(t_0) = e^t \sinh h_w^0.$$

Due to this definition, if for some t we have $h(t) = h^0$, then $h(t)$ stabilizes, i.e., for each $t' \leq t$ we have $h(t') = h^0$.

First, we see that if $h(t)$ is defined for some $t \leq t_0$ and $h(t) \neq h^0$, then there exists $\Delta t > 0$ such that h is defined uniquely for all $t' \in (t - \Delta t; t]$. Indeed, due to Corollary 4.2.4 and Corollary 3.6.11, the Hessian of S is non-degenerate at $h(t)$ and $h(t) \in \mathrm{int}(H(d, V))$ (note that the κ_w can not be positive for $w \in W$ as they are non-negative for $t = t_0$ and once they become 0, they stay equal to 0). Then our claim follows from the Implicit Function Theorem.

Assume that $h(t)$ is defined uniquely by conditions (0)-(2) over the interval $(t'; t_0]$ for some $t' < t_0$, and $h(t) \neq h^0$ for all $t \in (t'; t_0]$. The second assumption implies that there exists $w \in W$ such that $\kappa_w(t) < 0$ for all $t \in (t'; t_0]$. Thus,

$$\lim_{t \to t'} \sinh h_w(t) = e^{t'} \sinh h_w^0.$$

By Lemma 4.3.3 there exists $m > 0$ such that for all $t \in (t'; t_0]$ and all $v \in V$ we have $h_v(t) \geq m$. By Claim 5 from the proof of Lemma 4.3.1 there exists $M > 0$ such that for all $t \in (t'; t_0]$ and all $v \in V$ we get $h_v(t) \leq M$. Then for all $t \in (t'; t_0]$ we have $h(t) \in H(d, V, m, M)$, which is compact by Lemma 3.6.3. Then there exists a unique point $h(t') \in H(d, V)$ that is a continuous extension of $h(t)$. Moreover, by continuity for all $v \in V$ we have $\kappa_v(t') \leq 0$. Then the Hessian of S is non-degenerate at $h(t')$ and $h(t') \in \mathrm{int}(H(d, V))$. Due to the Implicit Function Theorem, $h(t)$ is C^1 in some right neighbourhood of t'.

Thereby, $h(t)$ is uniquely defined for all $t \leq t_0$. There might be kink points corresponding to those t, when κ_w becomes 0 for a vertex $w \in W$.

It follows that either there exists $\check{t}_0 < t_0$ such that $h(\check{t}_0) = h^1$ or there is $w \in W$ such that $\kappa_w(t) < 0$ for all $t \in (-\infty; t_0]$. In the second case $h_w(t) \to 0$ as $t \to -\infty$. Then Lemma 4.3.3 excludes this case as it shows that κ_w becomes positive for t close to $-\infty$.

Thus, there exists $\check{t}_0 < t_0$ such that $h(\check{t}_0) = h^0$. There is $w \in W$ such that for all $t > \check{t}_0$ we have $\kappa_w(t) < 0$. This means that

$$\sinh h_w(\check{t}_0) = e^{\check{t}_0} \sinh h_w^0.$$

As it is equal to $\sinh h_w^0$, then $\check{t}_0 = 0$.

Our current aim is

Claim 3. For all $w \in W$ and all $0 < t \leq t_0$ we have $\kappa_w(t) < 0$.

This requires some preparation. Fix t that is not a kink point of $h(t)$ and define

$$W(t) = \{w \in W : \kappa_w(t) < 0\}.$$

If $w \in W(t)$, then $\dot{h}_w(t) = \tanh h_w(t)$.

Claim 4. For each $v \in V$ we have $\dot{h}_v(t) > 0$.

Indeed, we already saw this for $v \in W(t)$. If $v \notin W(t)$, then $\kappa_v(t)$ is identically 0. Lemma 4.2.1 gives

$$-\dot{\kappa}_v = \frac{1}{\cosh h_v} \sum_{\vec{e} \in \vec{E}_v} \frac{\cot \phi_{\vec{e}}^+ + \cot \phi_{\vec{e}}^-}{\sin \alpha_{\vec{e}}} \cdot \frac{\cosh a_{\vec{e}} \dot{h}_v - \dot{h}_u}{\sinh a_{\vec{e}}} = 0.$$

Let v be a vertex, where the minimum of $\dot{h}_v(t)$ is attained over the vertices not from $W(t)$. Because of Lemma 3.6.7 there exists at least one strict edge emanating from v, so the sum is non-empty. If $\dot{h}_v < 0$, then is easy to see that each summand is negative, which is a contradiction. If $\dot{h}_v = 0$, then looking at the sum we see that for every u connected with v we also have $\dot{h}_u = 0$. Repeating this and noting that the graph of strict edges is connected due to Corollary 3.6.11, we obtain a contradiction. This proves Claim 4.

Claim 5. For every $v \in V \backslash W(t)$, we have $\dot{h}_v < \tanh h_v$

Define $x_v(t) := \tanh h_v(t)$. For every $v \in V$ we get

$$-\frac{\partial \kappa_v}{\partial h_v} x_v - \sum_{u \neq v} \frac{\partial \kappa_v}{\partial h_u} x_u =$$

$$= \frac{1}{\cosh h_v} \sum_{\vec{e} = \overrightarrow{vu} \in \vec{E}_v} \frac{\cot \phi_{\vec{e}}^+ + \cot \phi_{\vec{e}}^-}{\sin \alpha_{\vec{e}}} \cdot \frac{\cosh a_{\vec{e}} \tanh h_v - \tanh h_u}{\sinh a_{\vec{e}}}.$$

In this sum, $\vec{e} = \overrightarrow{vu}$ is an oriented edge from v to u and the sum is over all edges starting at v. With the help of Lemma 2.3.11 we obtain

$$-\frac{\partial \kappa_v}{\partial h_v} x_v - \sum_{u \neq v} \frac{\partial \kappa_v}{\partial h_u} x_u = \frac{1}{\cosh^2 h_v} \sum_{\vec{e} \in \vec{E}_v} \frac{\cot \phi^+_{\vec{e}} + \cot \phi^-_{\vec{e}}}{\sin \alpha_{\vec{e}}} \cdot \cot \alpha_{\vec{e}}.$$

Recall that $V = V(d)$, thus we have $\nu_v(d) > 0$. From Lemma 6.1.1 we see

$$\frac{\partial \kappa_v}{\partial h_v} x_v - \sum_{u \neq v} \frac{\partial \kappa_v}{\partial h_u} x_u > 0. \tag{6.1.2}$$

If $v \notin W(t)$, then we also have

$$-\frac{\partial \kappa_v}{\partial h_v} \dot{h}_v - \sum_{u \neq v} \frac{\partial \kappa_v}{\partial h_u} \dot{h}_u = 0. \tag{6.1.3}$$

Multiplying (6.1.2) with $\frac{\dot{h}_v}{x_v} > 0$ and subtracting (6.1.3) from it we write

$$\sum_{u \neq v} \frac{\partial \kappa_v}{\partial h_u} \left(\dot{h}_u - x_u \frac{\dot{h}_v}{x_v} \right) > 0.$$

The coefficients $\frac{\partial \kappa_v}{\partial h_u}$ are non-negative. Thus, there exists u connected with v such that

$$\dot{h}_u - x_u \frac{\dot{h}_v}{x_v} > 0,$$

or equivalently

$$\frac{\dot{h}_u}{x_u} > \frac{\dot{h}_v}{x_v}.$$

Consider the maximum of the expression $\frac{\dot{h}_v}{x_v}$ over V. We obtain that it is attained at points of $W(t)$, where it is equal 1. Therefore, for every $v \notin W(t)$ we get $\dot{h}_v < \tanh h_v$ and Claim 5 is proven.

Let t is a kink point of $h(t)$ such that κ_w becomes 0 the first time for some $w \in W$. Then the right derivative of h_w at t is $\tanh h_w(t)$, but the left derivative is smaller. For other v the derivatives of h_v remain continuous.

Suppose that there exist $t \in (0; t_0]$ such that for some $w \in W$ we have $\kappa_w(t) = 0$. Take t to be the largest with this property. We have $\sinh h_w(t) \geq e^t \sinh h^1_w$. However, for all $t' \in [0; t)$, we get $\dot{h}_w < \tanh h_w$. This gives $\sinh h_w(0) > \sinh h^1_w$, which is a contradiction. This proves Claim 3.

From all the above discussion we can see that the maximum point $\check{h}(t)$ over $H(t)$ is unique for all $t \geq 0$ and these points form a C^1-curve $h(t) \in H(d, V)$ that is determined by the conditions:

(0) $h(0) = h^1$;

(1) for any $v \in V$, $v \notin W$, we have $\kappa_v(t) = 0$;

(2) for any $w \in W$ we have

$$\sinh h_w(t) = e^t \sinh h^1_w.$$

Moreover, for any $w \in W$ we obtained $\dot{h}_w(t) = \tanh h_w(t)$ and for any $v \notin W$ we got $0 < \dot{h}_v < \tanh h_v$.

This finishes the proof. □

Main Lemma I follows from Lemma 6.1.2 quite easily.

Proof of Main Lemma I. Let $h(t)$, $H(t)$ and $P(t) = P(d, V, h(t))$ be taken from Lemma 6.1.2 (note that P^1 from the notation of Main Lemma I is $P(0) = P^0$ in the notation of Lemma 6.1.2). Define $S(t) = S(P(t))$. We have $h^2 \in H(\tau)$. As $h(t)$ is the maximal point of S over $H(t)$, we see that $S(P^2) \leq S(\tau)$.

From Lemma 4.1.1 we obtain

$$-\dot{S}(t) = -\sum_{w \in W} \kappa_w \tanh h_w(t).$$

We have

$$S(P^1) - S(P^2) \geq S(0) - S(\tau) = -\int_0^\tau dS =$$

$$= -\int_0^\tau \sum_{w \in W} \kappa_w \tanh h_w dt \geq -m \cdot \int_0^\tau \sum_{w \in W} \kappa_w dt.$$

From Lemma 6.1.2 for any $w \in W$ we get

$$-\dot{\kappa}_w \geq \frac{1}{\cosh h_w} \sum_{\vec{e} = \overrightarrow{wu} \in \vec{E}_w} \frac{\cot \phi_{\vec{e}}^+ + \cot \phi_{\vec{e}}^-}{\sin \alpha_{\vec{e}}} \cdot \frac{\cosh a_{\vec{e}} \tanh h_w - \tanh h_u}{\sinh a_{\vec{e}}}$$

Lemma 2.3.11 transforms the right hand side as

$$\frac{1}{\cosh h_w} \sum_{\vec{e} = \overrightarrow{wu} \in \vec{E}_w} \frac{\cot \phi_{\vec{e}}^+ + \cot \phi_{\vec{e}}^-}{\sin \alpha_{\vec{e}}} \cdot \frac{\cosh a_{\vec{e}} \tanh h_w - \tanh h_u}{\sinh a_{\vec{e}}} =$$

$$= \frac{1}{\cosh^2 h_w} \sum_{\vec{e} \in \vec{E}_w} \frac{\cot \phi_{\vec{e}}^+ + \cot \phi_{\vec{e}}^-}{\sin \alpha_{\vec{e}}} \cdot \cot \alpha_{\vec{e}}$$

Combining this with Lemma 6.1.1 we get

$$-\dot{\kappa}_w \geq \frac{\nu_w - \kappa_w}{M}.$$

Define $\kappa := \sum_{w \in W} \kappa_w \leq 0$. We obtain $-\dot{\kappa} \geq \frac{\nu - \kappa}{M}$. Rewrite it as

$$\frac{d(\nu - \kappa(t))}{\nu - \kappa(t)} \geq \frac{dt}{M}.$$

Integrating this we see

$$-\kappa(t) \geq \nu \left(\exp\left(\frac{t}{M}\right) - 1 \right).$$

Finally, we get

$$S(P^1) - S(P^2) \geq -m \cdot \int_0^\tau \kappa dt \geq$$

$$\geq \nu m \cdot \int_0^\tau \left(\exp\left(\frac{t}{M}\right) - 1 \right) dt =$$

$$= \nu m \left(M \exp\left(\frac{\tau}{M}\right) - \tau \right).$$

This finishes the proof. $\square$

6.2 Proof of Main Lemma II

Main Lemma II states that if we have a sufficiently fine short triangulation of (S_g, d), then we can replace each triangle by a triangle with a swept cone-metric and the same sides and angles. To this purpose it is enough to treat each triangle separately.

Definition 6.2.1. By a (topological) triangle T we mean a topological disc with three marked points at the boundary called vertices. By a metric on T we always mean a metric such that the boundary is geodesic except (possibly) the vertices. Cone-metrics or CBB(-1) metrics on T are defined naturally. We note that we call a cone metric d on T *convex* if in addition to the condition $\nu_p(d) \leq 2\pi$ for the interior points of T, we also require that the angles of the vertices of T are at most π. One can see that this condition also holds for CBB(-1) metrics due to their definition. A metric on T is called *short* if the sides of T are shortest paths and the angles of vertices are defined and strictly smaller than π. A cone-metric d on T is called *swept* if $|V(d)| \leq 1$. Here we note that by $V(d)$ we mean the set of interior singularities of d, i.e., we do not include the vertices of T in $V(d)$. Metric $\widehat{d}$ on T is a *sweep-in* of metric d if $\widehat{d}$ is swept and T has the same side lengths and angles in d and $\widehat{d}$.

Main Lemma II follows from

Lemma 6.2.2. There exists an absolute constant $\delta > 0$ such that if d is a short CBB(-1) metric on a triangle T with $\operatorname{diam}(T, d) < \delta$ and $\nu(T, d) < \delta$, then there exists a unique sweep-in $\widehat{d}$ of d on T.

The plan is first to prove the existence for convex cone-metrics in Subsection 6.2.1 via the *curvature merging process*. We also give some quantitative estimates on $\widehat{d}$ that will be crucial further in the proof of Main Lemmas IIIA-B. Next, we prove the uniqueness in Subsection 6.2.2. The proof is very easy and relies on elementary geometry. Last, in Subsection 6.2.3 we prove Lemma 6.2.2 in the CBB(-1) case with the help of approximation by cone-metrics. Note that the condition on d to be short in the statement of Lemma 6.2.2 seems to be non-essential. It is not used in the proof for cone-metrics. In the proof for CBB(-1) metrics it is used to avoid some annoying degenerations that we want to exclude from the present manuscript to simplify the exposition.

6.2.1 Curvature merging

The main proof idea of Lemma 6.2.2 in the cone-metric case is that we can take two cone-singularities, cut the metric along a geodesic connecting them and glue in there a piece of a hyperbolic cone so that the number of cone-singularities is reduced. The main difficulty is that we can not do this in all circumstances. In order to perform this operation, the length of the geodesic and curvatures of the singularities should satisfy some mutual conditions. Hence, we should control carefully what happens with the metric on each step in order to be able to perform the next one.

We start from two simple lemmas on hyperbolic triangles.

Lemma 6.2.3. Let $\Delta > 0$ and define

$$\eta = \eta(\Delta) = 2 \arcsin \frac{1}{\sqrt{2}\cosh \frac{\Delta}{2}} \in (0; \pi/2).$$

Then for any numbers $a \in (0; \Delta)$ and $\beta, \gamma > 0$ such that $\beta + \gamma < \eta$ there exists a hyperbolic triangle ABC with $BC = a$, $\angle B = \beta$ and $\angle C = \gamma$. Moreover, $\angle A > \pi/2$.

Proof. Suppose that the triangle ABC exists. Write the dual cosine law for $\angle A = \alpha$:

$$\cos\alpha = -\cos\beta\cos\gamma + \sin\beta\sin\gamma\cosh a. \tag{6.2.1}$$

One can prove that ABC exists if and only if the right hand side is in the range of cosine of a non-zero angle, i.e., it belongs to $(-1;1)$. Due to elementary trigonometry, the lower bound is always true. We are interested in $\angle A > \pi/2$, i.e., when the right hand side of (6.2.1) is less than 0.

Let us fix a and fix the sum of β and γ. Then if one considers the right hand side of (6.2.1) as the function of β, by taking the derivative it follows that α is minimized when $\beta = \gamma$.

Now let us check what are the values of β and γ in the extreme example when $\alpha = \pi/2$ and $\beta = \gamma$. It is easy to compute that

$$\beta = \gamma = \arcsin\frac{1}{\sqrt{2}\cosh\frac{a}{2}} = \frac{\eta(a)}{2}.$$

Moreover, if $\beta = \gamma$ and they are smaller than $\eta(a)/2$, then $\alpha > \pi/2$. This discussion shows that for any $\beta, \gamma > 0$ such that $\beta + \gamma < \eta(a)$ the triangle ABC exists and $\angle A > \pi/2$. It remains only to note that $\eta(a)$ decreases as a increases. □

Lemma 6.2.4. Let ABC by a hyperbolic triangle with sides a, b, c and angles α, β, γ respectively such that $\alpha > \pi/2$. Then

$$\pi - \alpha < \beta + \gamma\cosh a.$$

Proof. Define

$$\alpha' = \pi - \beta - \gamma.$$

We have

$$\cos\alpha' = \sin\beta\sin\gamma - \cos\beta\cos\gamma,$$

$$\cos\alpha = \sin\beta\sin\gamma\cosh a - \cos\beta\cos\gamma,$$

$$\cos\alpha - \cos\alpha' = \sin\beta\sin\gamma(\cosh a - 1).$$

We have $\pi > \alpha' > \alpha > \pi/2$. From the concativity of cosine over $[\pi/2;\pi]$ we obtain

$$\frac{\alpha' - \alpha}{\cos\alpha - \cos\alpha'} < \frac{1}{\sin\alpha'}.$$

From this we get

$$\alpha' - \alpha < \frac{\cos\alpha - \cos\alpha'}{\sin\alpha'} = \frac{\sin\beta\sin\gamma(\cosh a - 1)}{\sin(\beta+\gamma)}.$$

Note that $\beta < \beta + \gamma < \pi/2$ and $\sin\gamma < \gamma$. Therefore,

$$\alpha' - \alpha < \gamma(\cosh a - 1).$$

This is equivalent to

$$\pi - \alpha = \beta + \gamma + \gamma(\cosh a - 1) < \beta + \gamma\cosh a.$$

□

Fix $\Delta > 0$ and choose $\eta = \eta(\Delta)$ from Lemma 6.2.3. Then choose $\delta_\nu > 0$ such that

$$\delta_\nu < \frac{2\eta}{\cosh\Delta},$$
$$\delta_\nu < \frac{\Delta}{\sinh\Delta}. \tag{6.2.2}$$

After that we choose $\delta_D > 0$ such that

$$\delta_D < \Delta - \delta_\nu \sinh\Delta. \tag{6.2.3}$$

The part of Lemma 6.2.2 for cone-metrics is

Lemma 6.2.5. Let d be a convex cone-metric on T with $\operatorname{diam}(T,d) < \delta_D$ and $\nu(T,d) < \delta_\nu$. Then there exists a sweep-in $\widehat{d}$ of d on T such that

$$\operatorname{diam}(T,d) \le \operatorname{diam}(T,\widehat{d}) < \Delta, \quad \nu(T,d) \le \nu(T,\widehat{d}) < 2\eta.$$

Proof. Put $d_0 = d$. At first step take two points $v_0, v_1 \in V(d_0)$ and connect them by a shortest path χ_1.

Consider the hyperbolic triangle ABC such that

$$BC = \operatorname{len}(\chi_1, d_0) = d_0(v_0, v_1), \quad \angle B = \nu_{v_0}(d_0)/2, \quad \angle C = \nu_{v_1}(d_0)/2.$$

We have

$$BC \le \operatorname{diam}(T,d_0) < \delta_D < \Delta,$$
$$\angle B + \angle C = \frac{\nu_{v_0}(d_0) + \nu_{v_1}(d_0)}{2} \le \frac{\nu(T,d)}{2} < \frac{\delta_\nu}{2} < \eta.$$

Thus, Lemma 6.2.3 implies that ABC exists and BC is its greatest side.

We take two copies of ABC and glue isometrically the sides corresponding to AB and AC. We call the obtained figure a *bigon* and denote it by X. It has the piecewise geodesic boundary consisting of two segments equal to BC, two kink points at the boundary, which we continue to denote by B and C abusing the notation, and a conical point in the interior, which we continue to denote by A.

Cut (T,d_0) along χ_1. Glue isometrically the boundary of X with the boundary of the cut: the vertex B is glued with v_0, C is glued with v_1. Denote the resulting metric by d_1. We have a natural map

$$g_1 : (T\backslash\chi_1, d_0) \to (T\backslash X, d_1),$$

which is a local isometry. Note that when we write $(T\backslash A, d)$, where A is a subset of the metric space (T,d), we mean the induced (extrinsic) metric on $(T\backslash A, d)$, in contrast to all previous text, when we considered induced intrinsic metrics.

We identify $V(d_0)$ with its g_1-image. The point A will be also denoted by $w_1 \in (T,d_1)$. Then

$$\nu_{v_0}(d_1) = \nu_{v_1}(d_1) = 0,$$
$$\nu_{w_1}(d_1) > \nu_{v_0}(d_0) + \nu_{v_1}(d_1),$$
$$V(d_1) = (V(d_0) \cup \{w_1\})\backslash\{v_0, v_1\}.$$

Claim 1. The map $g_1 : (T\backslash\chi_1, d_0) \to (T\backslash X, d_1)$ does not decrease the distances.

Indeed, let $p', p'' \in (T\backslash X, d_1)$. Connect them by a shortest path in (T,d_1). If it does not intersect X, then the statement is trivial. If it intersects X, then it intersects X in a segment. This is because of the strong non-overlapping condition

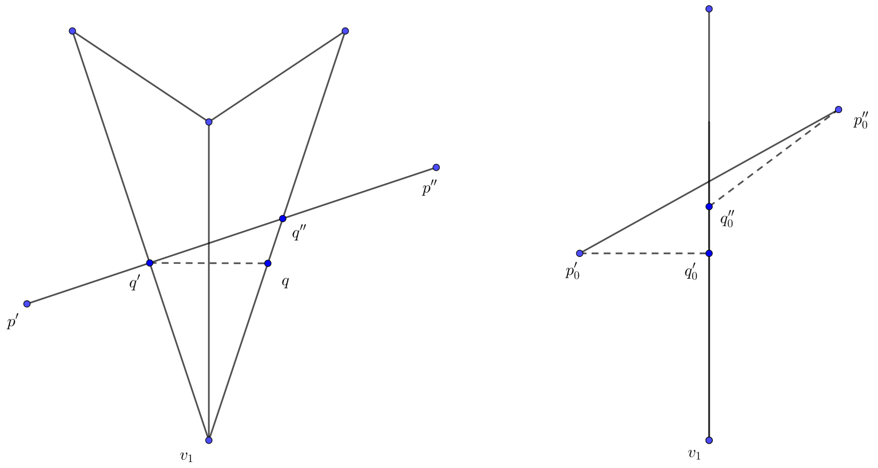

FIGURE 6.4: To the proof of Claim 1.

(see Section 2.1). Let the shortest path intersect the boundary of X in points $q', q'' \in (T, d_1)$ such that q' is closer to p' and q'' is closer to p''. See Figure 6.4. Choose points $q_0', q_0'' \in \chi_1 \subset (T, d_0)$ such that

$$d_0(v_1, q_0') = d_1(v_1, q'),$$

$$d_0(v_1, q_0'') = d_1(v_1, q'').$$

Without loss of generality we assume that q_0' lies between v_1 and q_0''. Then choose a point $q \in (T, d_1)$ on the boundary of X between v_1 and q'' such that $d_1(v_1, q) = d_1(v_1, q')$. It is clear that $d_1(q'', q) \leq d_1(q', q'')$. Define $p_0' = g_1^{-1}(p')$, $p_0'' = g_1^{-1}(p'')$. Then

$$\begin{aligned} d_0(p_0', p_0'') \leq d_0(p_0', q_0') + d_0(q_0', q_0'') + d_0(q_0'', p_0'') \leq \\ \leq d_1(p', q') + d_1(q, q'') + d_1(q'', p'') \leq \\ \leq d_1(p', q') + d_1(q', q'') + d_1(q'', p'') = d_1(p', p''). \end{aligned}$$

Claim 1 is shown. In particular, we get

Claim 2. $\operatorname{diam}(T, d_0) \leq \operatorname{diam}(T, d_1)$.

Claim 3. $\operatorname{diam}(T, d_1) < \operatorname{diam}(T, d_0) + \nu_{v_1}(d_0) \sinh(\operatorname{diam}(T, d_0))$.

Indeed, consider in $\mathbb{H}^2$ two triangles $AB'C$ and $AB''C$ sharing the edge AC, each is isometric to the triangle ABC. As $\angle AB'C = \angle AB''C > \pi/2$, the point A belongs to the interior of the triangle $B'B''C$.

Let ψ be the (unique) shortest path connecting w_1 and v_0 in $X \subset (T, d_1)$. Cut X along ψ and consider the developing map

$$\rho : X \backslash \psi \to CB'AB''.$$

One can see that it does not decrease the distances. Then

$$\operatorname{diam}(X) \leq \operatorname{diam}(B'B'') = \max(B'B'', B'C).$$

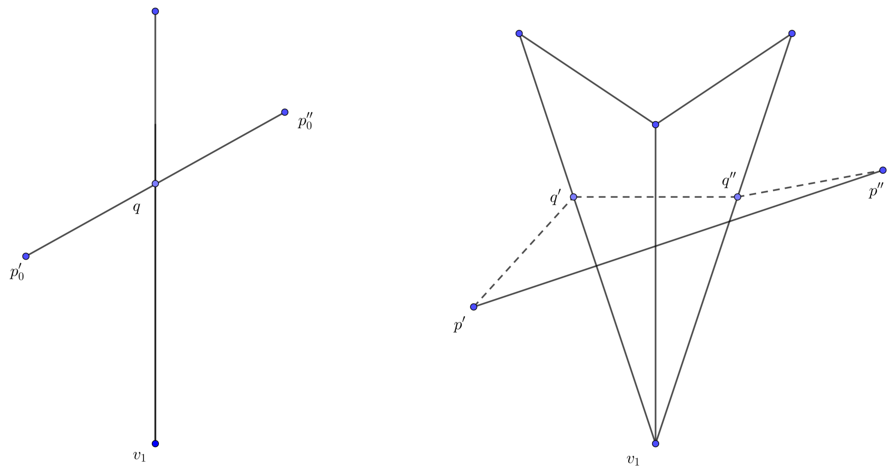

FIGURE 6.5: To the proof of Claim 3.

Note that $B'C \leq \operatorname{diam}(T, d_0)$ and

$$B'B'' < 2\sinh(B'B''/2) = 2\sin(\nu_{v_1}(d_0)/2)\sinh B'C <$$

$$< \nu_{v_1}(d_0)\sinh(\operatorname{diam}(T, d_0)).$$

Consider $p' \in (T\backslash X, d_1)$ and $p'' \in X \subset (T, d_1)$. Connect them by a shortest path. Let it intersect the boundary of X in a point with ρ-image in CB'. In the triangle $B'B''C$ draw the segment through $\rho(p'')$, which cuts off equal segments from the segments CB' and CB''. Consider its intersection point with CB' and let $q \in (T, d_1)$ be its preimage under ρ. Let $q_0 \in \chi_1 \subset (T, d_0)$ be a point such that $d_0(v_0, q_0) = d_1(v_0, q)$ and $p_0' = g^{-1}(p') \in (T\backslash\chi_1, d_0)$. Then

$$d_1(p', p'') \leq d_1(p', q) + d_1(p'', q) \leq d_0(p_0', q_0) + B'B'' <$$

$$< \operatorname{diam}(T, d_0) + \nu_{v_1}(d_0)\sinh(\operatorname{diam}(T, d_0)).$$

Now we consider $p', p'' \in (T\backslash X, d_1)$. Let $p_0' = g_1^{-1}(p') \in (T, d_0)$ and $p_0'' = g_1^{-1}(p'') \in (T, d_0)$. If $d_0(p_0', p_0'') \neq d_1(p', p'')$, then all shortest paths connecting p_0' and p_0'' in d_0 intersect χ_0. Consider an arbitrary shortest path between p_0' and p_0''. Let it intersect χ_0 at the point q. We consider the points $q', q'' \in (T, d_1)$ at the borders of X such that $d_1(q', v_1) = d_1(q'', v_1) = d_0(q, v_1)$. See Figure 6.5. Then

$$d_1(p', p'') \leq d_1(p', q') + d_1(q', q'') + d_1(q'', p'') \leq$$

$$\leq d_0(p_0', p_0'') + B'B'' < \operatorname{diam}(T, d_0) + \nu_{v_1}(d_0)\sinh(\operatorname{diam}(T, d_0)).$$

This finishes the proof of Claim 3.

Claim 4. $\nu_{v_0}(d_0) + \nu_{v_1}(d_0) < \nu_{w_1}(d_1) < \nu_{v_0}(d_0) + \nu_{v_1}(d_0)\cosh(\operatorname{diam}(T, d_0))$.

Indeed, the lower bound is trivial, the upper is just Lemma 6.2.4 applied to triangle ABC.

We continue the process of merging the curvature by gluing new bigons. At every step we take the point that we obtained after the last step, take another conical point in T and cut the metric along a shortest path connecting both points. Each time we should check that there exists a bigon to glue in the cut.

In particular, let $v_2 \in V(d_1)$ be distinct from w_1. Connect them by a shortest path χ_2. Note that $\nu_{v_2}(d_1) = \nu_{v_2}(d_0)$. From Claim 3 and bounds (6.2.2), (6.2.3) we check the length of χ_2

$$\mathrm{len}(\chi_2, d_1) = d_1(w_1, v_2) \leq \mathrm{diam}(T, d_1) <$$

$$< \mathrm{diam}(T, d_0) + \nu_{v_1}(d_0)\sinh(\mathrm{diam}(T, d_0)) < \delta_D + \delta_\nu \sinh(\Delta) < \Delta.$$

Next we check the curvature of w_1 from Claim 4

$$\begin{aligned}\nu_{w_1}(d_1) &< \nu_{v_0}(d_0) + \nu_{v_1}(d_0)\cosh(\mathrm{diam}(T, d_0)) < \\ &< (\nu_{v_0}(d_0) + \nu_{v_1}(d_0))\cosh\Delta < \\ &< (\nu_{v_0}(d_0) + \nu_{v_1}(d_0) + \nu_{v_2}(d_1))\cosh\Delta - \nu_{v_2}(d_1) \leq \\ &\leq \nu(T, d_0)\cosh\Delta - \nu_{v_2}(d_1) < \\ &< \delta_\nu \cosh\Delta - \nu_{v_2}(d_1) < 2\eta - \nu_{v_2}(d_1).\end{aligned}$$

Then Lemma 6.2.3 shows that the bigon exists to glue with the cut along χ_2. Its boundary segments are greater than the distances from boundary vertices to the interior vertex. We glue the bigon and obtain a new metric d_2 and a local isometry g_2. We identify $V(d_1)$ in (T, d_1) with its g_2-image in (T, d_2). Let w_2 be a new vertex. Then

$$V(d_2) = (V(d_1) \cup \{w_2\}) \backslash \{w_1, v_2\}.$$

Claims 3 and 4 with (6.2.2) and (6.2.3) show that

$$\begin{aligned}\mathrm{diam}(T, d_2) &< \mathrm{diam}(T, d_1) + \nu_{v_2}\sinh\mathrm{diam}(T, d_1) < \\ &< \mathrm{diam}(T, d_0) + \nu_{v_1}(d_0)\sinh(\mathrm{diam}(T, d_0)) + \nu_{v_2}(d_0)\sinh(\mathrm{diam}(T, d_1)) < \\ &< \mathrm{diam}(T, d_0) + \nu(T, d_0)\sinh\Delta < \Delta,\end{aligned}$$

$$\begin{aligned}\nu_{w_2}(d_2) &< \nu_{w_1}(d_1) + \nu_{v_2}(d_1)\cosh(\mathrm{diam}(T, d_1)) < \\ &< \nu_{v_0}(d_0) + \nu_{v_1}(d_0)\cosh(\mathrm{diam}(T, d_0)) + \nu_{v_2}(d_0)\cosh\Delta < \\ &< (\nu_{v_0}(d_0) + \nu_{v_1}(d_0) + \nu_{v_2}(d_0))\cosh\Delta.\end{aligned}$$

If there is another point $v_3 \in V(d_2)$, then we repeat the process. The same arguments as before show that the new bigon exists. We continue until all curvature inside T is merged into one point. Note that the side lengths of T and its angles do not change. Thus, we obtain a sweep-in $\widehat{d}$ of d on T. By induction, $\mathrm{diam}(T, \widehat{d}) < \Delta$ and $\nu(T, \widehat{d}) < 2\eta$. □

Now we make a brief sidestep from metrics on T to metrics on S_g. We claim that if we perform the sweeping process described in this section for each triangle of T, then the diameter of the resulting metric can be estimating in exactly the same way as for a single triangle. In other words, we state

Lemma 6.2.6. Let Δ be a positive number and δ_ν, δ_D be positive numbers satisfying (6.2.2), (6.2.3). If (S_g, d) is a cone-metric space and $\mathcal{T}$ is its short δ-fine triangulation, where $\delta := \min\{\delta_\nu, \delta_D\}$, then there exists a unique sweep-in $\widehat{d}$ of d with respect to $\mathcal{T}$. Moreover, $\mathrm{diam}(S_g, \widehat{d})$ satisfies

$$\mathrm{diam}(S_g, d) \leq \mathrm{diam}(S_g, \widehat{d}) < \mathrm{diam}(S_g, d) + \nu(S_g, d)\sinh\Delta.$$

The existence here follows from Lemma 6.2.5 and the uniqueness will be proven in the next section. Thereby, what we remark now is the diameter estimate. The proof is exactly the same as the proof of similar bounds in Lemma 6.2.5. We only need to note that proofs of Claim 1, Claim 3 from the proof of Lemma 6.2.5, which are responsible for these bounds, work without changes if instead of points in T we consider points in S_g.

6.2.2 Swept triangles

Definition 6.2.7. Since now we call a metric space $(T, \widehat{d})$, where T is a triangle and $\widehat{d}$ is a swept metric, a *swept triangle*. It is called *convex* if $\widehat{d}$ is convex. It is called *strict* if $|V(\widehat{d})| = 1$ and the curvature of the conical point is positive. Note that a strict swept triangle might be non-convex: this happens if some boundary angles are greater than π.

Let $(T, \widehat{d})$ be a swept triangle. We also denote it as $OA_1A_2A_3$, where A_1, A_2, A_3 are vertices of T and O is the interior cone point of $\widehat{d}$. In case $V(\widehat{d}) = 0$ we assume that O is some marked point in the interior of T, i.e., a swept triangle is always a triangle with a marked point inside. By l_i denote the length of $A_{i-1}A_{i+1}$ (we assume that A_i are enumerated modulo three) and by λ_i denote the angle $A_{i-1}A_iA_{i+1}$. We naturally call l_i the *side lengths* of $OA_1A_2A_3$ and λ_i its *angles*. There are unique geodesics from O to each vertex. By x_i denote the length of the geodesic OA_i and by β_i denote the sector angle $A_{i-1}OA_{i+1}$. One can see that $0 < \beta_i < \pi$.

We note that a swept triangle is uniquely determined by x_i, β_i (up to isometry). This gives us a parametrization of the set of strict triangles up to isometry by the following subset of $\mathbb{R}^6$:

$$SCT = \{(x_1, x_2, x_3, \beta_1, \beta_2, \beta_3) : x_i > 0,\ 0 < \beta_i < \pi,\ \beta_1 + \beta_2 + \beta_3 < 2\pi\}.$$

First we prove the uniqueness of a sweep-in in Lemma 6.2.2. It is easy to see that if a swept cone triangle has the side lengths and angles coinciding to the side lengths and angles of a hyperbolic triangle, then it is isometric to this hyperbolic triangle. The rest of the proof follows from

Lemma 6.2.8. A strict swept triangle is uniquely determined by the side lengths and angles (up to isometry)

Proof. We show that the distances x_i in a strict swept triangle $OA_1A_2A_3$ are uniquely determined by side lengths and angles. This also shows that β_i are uniquely determined, which implies the claim.

Cut $OA_1A_2A_3$ along the geodesic OA_1. Develop it in $\mathbb{H}^2$. Let $A_1'A_2A_3A_1''O$ be the resulting polygon. One can show that it is not self-intersecting even if some boundary angles are greater than π.

The side lengths l_1, l_2, l_3 and the angles λ_2 and λ_3 determine the polygonal line A_1BCA_2 in $\mathbb{H}^2$ up to isometry. The point O belongs to the perpendicular bisector to segment $A_1'A_1''$. We can see that as O moves along this bisector, the sum of angles $OA_1'A_2$ and $OA_1''A_3$ changes monotonously. Then there exists no more than one position where this sum is equal to the angle λ_1. This finishes the proof. □

Consider the map $\Theta : SCT \to \mathbb{R}^6$, which associates to a strict swept triangle the 6-tuple of its side lengths and angles. Lemma 6.2.8 states that this map is injective. Clearly it is continuous, thereby by the Brouwer Invariance of Domain Theorem we get

Corollary 6.2.9. The image $\Theta(SCT)$ is open and Θ is a homeomorphism onto its image.

The map Θ will be used prominently in the next subsection.

6.2.3 Sweeping-in CBB(−1)-metrics

As we wrote before, the proof of Lemma 6.2.2 for CBB(−1) metrics is done via an approximation by cone-metrics. Assume that (T, d) be a triangle with a CBB(−1)-metric. Let (T, d^n) be a sequence of convex cone-metrics converging to (T, d) in some sense. There is a sweep-in $\widehat{d^n}$ for each metric d^n. We need to investigate the convergence of swept triangles $(T, \widehat{d^n})$. It might happen that such a sequence converges not to a swept triangle. For instance, it might happen that some $x_i = 0$ in the limit. The tough part of the proof is to show that such degeneracies can not happen when d^n converges to a CBB(−1)-metric.

The approximation can be done as in Subsection 5.3. To this purpose first we need to realize (T, d) on a convex surface. This can be done with the help of Alexandrov's gluing theorem:

Theorem 6.2.10 (The gluing theorem)**.** Let (M^1, d^1) and (M^2, d^2) be two surfaces with boundaries and CBB(−1)-metrics. Let $f : \partial M^1 \to \partial M^2$ be an isometry (it is possible to consider only some boundary components of S^1). Then the gluing of (S^1, d^1) and (S^2, d^2) along f is a CBB(−1)-metric space.

A proof is identical to the CBB(0)-case proven by Alexandrov in [7, Section VIII.1]. Another proof in the CBB(k)-case for arbitrary dimension can be found in [86].

Let (T, d) be a triangle with a CBB(−1)-metric. Take two its copies and glue along the boundary. We obtain a metric $\widetilde{d}$ on the 2-sphere S^2. Theorem 6.2.10 implies that it is CBB(−1). We fix the inclusion map $\rho : (T, d) \hookrightarrow (S^2, \widetilde{d})$. It maps the geodesic segments of ∂T to geodesics. If d is short, then the ρ-image of T is a short triangle in $(S^2, \widetilde{d})$.

By Alexandrov's realization theorem (Theorem 1.2.2), there exists a convex body $\widetilde{G} \subset \mathbb{H}^3$ with the boundary isometric to $(S^2, \widetilde{d})$. There exists an isometric involution of $\partial\widetilde{G}$. By Theorem 1.2.3 it extends to an isometry of $\mathbb{H}^3$. Due to the classification of isometries, it is a reflection σ with respect to a plane M. Pick a point $o \in (\mathrm{int} G \cap M)$.

Similarly to Subsection 5.3, we may consider a sequence of μ_m-dense sets $V_m \subset (T, d)$ with $\mu_m \to 0$ such that the vertices of T are in each V_m, but there are no other points from ∂T in each V_m. Consider the ρ-image of V_m in ∂G and its σ-symmetrization. Let G_m be the convex hull of the obtained set. The metrics of ∂G_m can be pulled back to cone-metrics on S^2 with the help of central projection from o. Due to σ-symmetry, they can be transferred to cone-metrics d_m on T with geodesic boundary outside the vertices. We may prove an analogue of Lemma 5.3.5 and Lemma 5.3.7:

Lemma 6.2.11. Let $\mathcal{T}$ be a strictly short triangulation of (T, d). Consider a sequence of finite μ_m-dense sets $V_m \subset (T, d)$ with $\mu_m \to 0$ as above and requiring additionally that all of them contain $V(\mathcal{T})$ and none of them contains a point in the interior of an edge of $\mathcal{T}$. By d_m denote the metric on T defined as above.

Then d_m converge to d uniformly and the triangulation $\mathcal{T}$ is realized by infinitely many d_m. Moreover, the realizations can be chosen such that they are short and for every triangle T' of $\mathcal{T}$ we have (after taking a subsequence)

(1) $T'(d_m) \to T'(d)$;

(2) $\operatorname{diam}(T', d_m) \to \operatorname{diam}(T', d)$;
(3) $\operatorname{area}(T', d_m) \to \operatorname{area}(T', d)$.

Here a strictly short triangulation means that all edges of $\mathcal{T}$ are unique shortest paths between their endpoints except the edges of T, for which it is enough just to be shortest paths. In particular, if d is a short metric on T, then we may consider a triangulation consisting of only T itself. However, to rule out degenerations in the proof below, we will need to consider some triangulations refining T and use Lemma 6.2.11 in its full power.

A proof of Lemma 6.2.11 can be done exactly in the same way as our proof of Lemma 5.3.7. Instead of distance functions to the lower boundary one should use the distance function to the point o. We do not provide here a proof of Lemma 6.2.11. We note that if we want to prove only Main Lemma II, then it is enough to prove the existence of the sweep-in only for metrics coming from short convex triangles on the upper boundary of a convex compact Fuchsian manifold. In this case we may just use Lemma 5.3.7 instead of Lemma 6.2.11 in the proof below. However, in a more general Lemma 6.2.2 we do not know if the triangle comes this way and we use Lemma 6.2.11 to overcome this.

Then there exists a unique sweep-in $\widehat{d}$ of d.

Proof of Lemma 6.2.2. We already proved the uniqueness and proved the existence in the case of cone-metrics. Now we consider the existence in the general case.

Let Δ be an arbitrary positive number and $\eta = \eta(\Delta)$ be from Lemma 6.2.3. Take δ_D, δ_ν satisfying (6.2.2) and (6.2.3). Define δ as the minimum of δ_D, δ_ν. Let d be a short CBB(-1)-metric on T with $\operatorname{diam}(T, d) < \delta$, $\nu(T, d) < \delta$ as in the statement of Lemma 6.2.2. Lemma 6.2.11 shows that there exists a convex cone-metric d' on T such that $\operatorname{diam}(T, d') < \delta_D$, $\nu(T, d') < \delta_\nu$ and $||T(d') - T(d)||_\infty < \varepsilon$ for arbitrary $\varepsilon > 0$.

By Lemma 6.2.5 we construct a convex cone-metric $\widehat{d'}$ on T such that $(T, \widehat{d'})$ is a convex swept triangle and $T(\widehat{d'}) = T(d')$. Thus, we have a swept triangle with side lengths and angles arbitrarily close to those of (T, d). We need to understand what happens in the limit.

Recall the definitions of $SCT \subset \mathbb{R}^6$ and the map Θ from Subsection 6.2.2. We have $T(d) \in \overline{\Theta(SCT)}$. Assume that $T(d)$ does not correspond to the 6-tuple of side lengths and angles of a hyperbolic triangle. Then $T(d')$ also does not, provided that ε is sufficiently small. It follows that $T(d') = T(\widehat{d'}) \in \Theta(SCT)$ and $T(d) \in \overline{\Theta(SCT)}$. Define $Z' := \Theta^{-1}(T(d'))$. By x_i' and β_i' we denote the respective components of Z'.

For positive numbers μ and Δ we define the following subset of SCT:

$$SCT(\mu, \Delta) = \{(x_1, x_2, x_3, \beta_1, \beta_2, \beta_3) : 0 < x_i < \Delta,\ \mu < \beta_i < \pi,\ \beta_1 + \beta_2 + \beta_3 < 2\pi\}.$$

We want to show $Z' \in SCT(\mu, \Delta)$. Lemma 6.2.5 says that $\operatorname{diam}(T, \widehat{d'}) < \Delta$. Then $x_i' < \Delta$ for all i. We prove that it is possible to choose $\mu = \mu(d)$ such that if ε is sufficiently small, then $\beta_i' > \mu$ for all i.

Indeed, denote (T, d') as $OA_1A_2A_3$ like in Subsection 6.2.2. For every $\xi > 0$ there exists $\mu > 0$ such that if $\angle A_1OA_2 \leq \mu$, then either $A_1A_2 < \xi$ or $\angle OA_1A_2 > \pi - \xi$ or $\angle OA_2A_1 > \pi - \xi$ (note that OA_1, OA_2 are smaller than Δ). Take ξ smaller than the length of any edge of (T, d) and than the difference of π and any angle of (T, d). Choose μ for this ξ. Then if ε is sufficiently small, we see that $\beta_i' > \mu$ and $Z' \in SCT(\mu, \Delta)$.

The image $\Theta(SCT(\mu, \Delta))$ is a bounded subset of $\mathbb{R}^6$. Note that the map Θ can be extended continuously to the boundary of $SCT(\mu, \Delta)$. This is clear for those boundary points that $\beta_1 + \beta_2 + \beta_3 = 2\pi$ or/and some $\beta_i = 2\pi$, but all $x_i > 0$. Now assume that for some Z in the boundary of $SCT(\mu, \Delta)$ we have $x_1 = 0$, but $x_2, x_3 > 0$. Define $l_2 := x_3$, $l_3 := x_2$ and $\lambda_1 := 2\pi - \beta_2 - \beta_3$. Determine l_1, λ_2, λ_3 as the third side length and two adjacent angle in the hyperbolic triangle with two sides of lengths x_2, x_3 and the angle β_1 between them. One can see that this defines a continuous extension of Θ to Z. This construction naturally generalizes to boundary points with several $x_i = 0$ simultaneously. The map Θ is not injective on the subsets of the boundary corresponding to the conditions $\beta_1 + \beta_2 + \beta_3 = 2\pi$ and $x_i = 0$.

As $\varepsilon > 0$ can be chosen arbitrarily small, $T(d)$ belongs to the closure $\overline{\Theta(SCT(\mu, \Delta))}$. But $\overline{SCT(\mu, \Delta)}$ is compact, therefore

$$\overline{\Theta(SCT(\mu, \Delta))} = \Theta(\overline{SCT(\mu, \Delta)}).$$

That means that there exists $Z \in \overline{SCT(\mu, \Delta)}$ such that $T(d) = \Theta(Z)$. In particular, $\beta_i(Z) > 0$ for all i. Here β_i and x_i are the components of Z. If $Z \in SCT$, then the corresponding swept triangle is convex and the proof is finished. If $Z \notin SCT$, then we have the following cases:

(i) $x_i = 0$ for some i;

(ii) $\beta_1 + \beta_2 + \beta_3 = 2\pi$ and $x_i > 0$ for all i;

(iii) $\beta_i = \pi$ for some i, but $\beta_1 + \beta_2 + \beta_3 < 2\pi$ and $x_i > 0$ for all i.

In the second case $T(d)$ corresponds to a hyperbolic triangle. In the third case assume that $\beta_1 = \pi$. Consider the respective angle λ_1 of (T, d). One can see that it is smaller than the respective angle of the comparison hyperbolic triangle $l_1 = x_2 + x_3$, l_2 and l_3. This contradicts to the definition of CBB(-1)-metric (Definition 2.1.2) and the condition that sides of (T, d) are shortest paths.

To finish our proof it remains to show that (i) can not happen. This appears to be uneasy.

If $\nu(T, d) = 0$, then $T(d)$ corresponds to a hyperbolic triangle. This can be seen from approximating by cone-metrics as above and noting that $2\pi - \beta'_1 - \beta'_2 - \beta'_3$ converges to $\nu(T, d)$. Assume that $\nu(T, d) > 0$. The idea is the following. If $x_1 = 0$ and a cone-metric d' is sufficiently close to d, then x'_1 is sufficiently small. We are going to construct arbitrarily close convex cone-metrics to d, for which x'_1 is bounded away from zero. This is done by paying the attention on how we do the approximation. Subdivide (T, d) into several triangles so that one of them contains almost all curvature and is not adjacent to the vertex A_1 of T. Then we find a cone-metric d' close to d realizing this new triangulation. When we merge the curvature inside T in d', the resulting swept triangle does not depend on the process of merging. Hence, we can first merge all curvature inside “the biggest part” of T and then all other curvature. We can show that the resulting conical point after the final merging is not close to A_1.

Now we proceed to the details. Consider (T, d) with $x_1 = 0$ where x_1 is defined as above. Denote the vertices of T by v_1, v_2, v_3 and the respective edges by e_1, e_2, e_3. Fix $t > 0$ and choose points $p_2 \in e_3$ and $p_3 \in e_2$ at distance t from v_1. Connect p_2 with v_2 and p_3 with v_3 by shortest paths. If there are several of them, then consider those that are the closest to v_1. Let them intersect in the point v. By ψ_2 and ψ_3 we denote the segments from v to v_2 and v_3 respectively. Note that geodesics in a CBB(-1)-metric do not pass through points p with $\nu_p(d) \neq 0$. Therefore, $\nu_v(d) = 0$. Let $U(t)$ be the (open) region bounded by e_2, e_3, ψ_2, ψ_3 and $T'(t)$ be the (open) triangle bounded by e_1, ψ_2 and ψ_3. Define

$$\nu'(t) := \nu(T'(t), d), \quad \nu''(t) := \nu(U(t), d).$$

We have $\nu'(t) + \nu''(t) = \nu(T, d)$. If $t_1 < t_2$, then $U(t_1) \subset U(t_2)$. The intersection of all $U(t)$ is empty. Thus, one can see that $\nu''(s)$ tends to zero as t tends to zero. We choose $t_0 > 0$ such that

$$\frac{2\sinh\Delta}{\Delta} \cdot \frac{\tan(\nu''(t_0)\cosh\Delta)}{\nu'(t_0)} < 1 - \xi$$

for some arbitrary number $\xi \in (0; 1)$.

From now on we denote $T'(t_0)$, $U(t_0)$, $\nu'(t_0)$ and $\nu''(t_0)$ by T', U, ν' and ν'' . Also we define

$$y := \inf_{p \in T'} d(v_1, p).$$

All sides of the triangle T' are shortest paths. Therefore, with the help of Lemma 5.3.4 we can construct a short triangulation $\mathcal{T}$ of T containing T'. Due to Lemma 6.2.11 there exists a sequence of convex cone-metrics $\{d_n\}$ on T converging to d uniformly and realizing $\mathcal{T}$. Define

$$\nu'_n := \nu(T', d_n), \quad \nu''_n := \nu(U, d_n), \quad y_n := \inf_{p \in T'} d_n(v_1, p).$$

Here by U, T' we mean their realizations in d_n. From Lemma 6.2.11 we can choose $\{d_n\}$ so that $\nu'_n \to \nu'$, $\nu''_n \to \nu''$ and $T(d_n) \to T(d)$ as n goes to infinity. From the uniform convergence one can get $y_n \to y$.

Now choose n such that

$$\begin{gathered}
\operatorname{diam}(T, d_n) < \delta_D, \\
\nu(T, d_n) < \delta_\nu, \\
y_n > \frac{y}{2}, \\
\frac{2\sinh\Delta}{\Delta} \cdot \frac{\tan(\nu''_n\cosh\Delta)}{\nu'_n} < 1 - \xi, \\
\lambda_{1,n} < \frac{\lambda_1 + \pi}{2} < \pi, \\
x_{1,n} < \min\left\{\frac{\xi y}{2}, \operatorname{artanh}\left(\cos\frac{\lambda_1 + \pi}{2} \cdot \tanh\frac{y}{2}\right)\right\}.
\end{gathered} \tag{6.2.4}$$

Here $\lambda_{1,n}$, λ_1 are angles at the vertex v_1 in (T, d_n), (T, d) respectively and $x_{1,n}$ is the distance from v_1 to the cone point in the sweep-in of (T, d_n). Since now we denote d_n, y_n, ν'_n and ν''_n by $\widetilde{d}$, $\widetilde{y}$, $\widetilde{\nu}'$ and $\widetilde{\nu}''$.

By Lemma 6.2.8 we see that if we merge all curvature inside $(T, \widetilde{d})$ in one point (by gluing bigons), then the resulting swept triangle does not depend on the choice of a merging algorithm. Lemma 6.2.5 says that its diameter is smaller than Δ. Claim 2 of Lemma 6.2.5 shows that after each merging the diameter of T is not decreasing, therefore Δ is an upper bound on the diameter of T during the whole process.

First, we merge all curvature inside T' to point w'. By Claim 1 of Lemma 6.2.5 the distance from w' to v_1 is at least y. Now we merge all curvature in U to a point w''. We note that U is not convex, hence we do not expect that w'' stays in U. Let d_* be the resulting cone-metric on T. We would like to glue a bigon along the shortest path ψ connecting w' and w''. From Claim 4 of Lemma 6.2.5 we see that $\widetilde{\nu}' \leq \nu_{w'}(d_*) < \widetilde{\nu}'\cosh\Delta$ and $\widetilde{\nu}'' \leq \nu_{w''}(d_*) < \widetilde{\nu}''\cosh\Delta$. Then from inequalities (6.2.2) we get $\nu_{w'}(d_*) + \nu_{w''}(d_*) < 2\eta$. Together with $\operatorname{len}(\psi, d_*) < \Delta$, it follows from Lemma 6.2.3 that the last bigon exists.

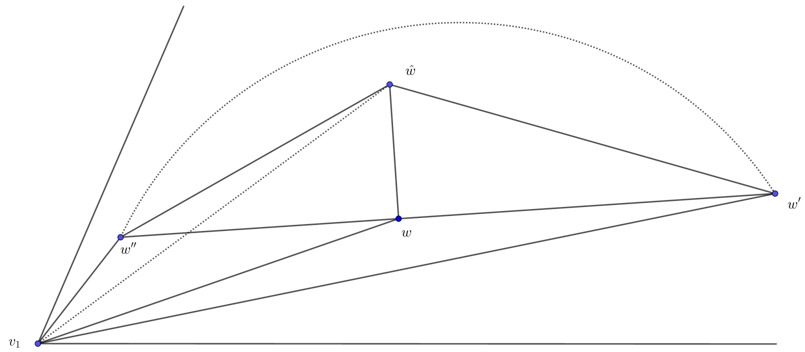

Figure 6.6: Positions of points.

Let $\widehat{w}$ be the resulting curvature point, $\widehat{d}_*$ be the resulting metric and w be the base of perpendicular from $\widehat{w}$ to the boundary segment of the glued bigon, which is intersected by a shortest path from $\widehat{w}$ to v_1. See Figure 6.6. Then $\widehat{d}_*(v_1, \widehat{w}) \geq d_*(v_1, w)$.

Note that $\widehat{d}_*(v_1, \widehat{w}) = x_{1,n}$. By the choice of n there is the upper bound (6.2.4) on $x_{1,n}$. We are going to obtain a contradiction with this bound. We consider two cases. First, suppose that $d_*(v_1, w'') \geq y/2$. Then we claim that

$$\tanh d_*(v_1, w) \geq \left(\cos \frac{\lambda_1 + \pi}{2} \cdot \tanh \frac{y}{2}\right).$$

Indeed, elementary trigonometry shows that for each hyperbolic triangle ABC with $\angle A \leq \frac{\lambda_1+\pi}{2} < \pi$ and AB, $BC \geq \frac{y}{2}$ we have the hyperbolic tangent of the distance from A to the segment BC is at least $\cos \frac{\lambda_1+\pi}{2} \cdot \tanh \frac{y}{2}$. We apply this to the triangle $v_1 w' w''$. This contradicts to (6.2.4).

In the other case, we suppose $d_*(v_1, w'') \leq y/2 \leq d_*(v_1, w')$. Write

$$d_*(v_1, w) \geq d_*(v_1, w') - d_*(w', w).$$

Now we bound $d_*(w', w)$. Recall that $\nu_{w'}(d_*) \geq \widetilde{\nu}'$.

$$d_*(w', w) \leq \sinh(d_*(w', w)) = \sinh(d_*(w, w''))\frac{\tan(\nu_{w''}(d_*)/2)}{\tan(\nu_{w'}(d_*)/2)} \leq$$

$$\leq \sinh(d_*(w', w''))\frac{\tan(\nu_{w''}(d_*)/2)}{\tan(\widetilde{\nu}'/2)} \leq d_*(w', w'') \cdot \frac{\sinh \Delta}{\Delta} \cdot \frac{\tan(\nu_{w''}(d_*))}{\widetilde{\nu}'}.$$

Recall that $\nu_{w''}(d_*) < \widetilde{\nu}'' \cosh \Delta$.
Also $d_*(w', w'') \leq d_*(v_1, w') + d_*(v_1, w'') \leq 2d_*(v_1, w')$ due to our assumption.
In total we get

$$d_*(v_1, w) \geq d_*(v_1, w') - d_*(w', w) \geq$$

$$\geq d_*(v_1, w')\left(1 - \frac{2\sinh \Delta}{\Delta} \cdot \frac{\tan(\widetilde{\nu}'' \cosh \Delta)}{\widetilde{\nu}'}\right) > \frac{\xi y}{2} > x_{1,n}.$$

This gives the last contradiction. It follows that $x_1 > 0$ and the proof is over. □

6.3 Proof of Main Lemmas III

The rough idea of our proofs of Main Lemmas IIIA-C is as follows. Assume that we have a path of metrics d_t parametrized by $t \in [0;\tau]$ and a convex Fuchsian cone-manifold $P_0 = P(d_0, V, h_0)$. We want to transform this cone-manifold continuously along the path d_t so that (1) heights of the vertices can move only in one direction (up); (2) we control the change of discrete curvature S (for our purposes it is enough to control it also only in one direction). There are two aspects impacting the discrete curvature: the change of heights and the change of the upper boundary metric. A great observation is that we can move up only vertices with positive curvature. This always increases S and it remains to bound the change of S under the metric change. Another idea is that we may consider separately two types of small deformations of P_t: (1) transform heights with fixed boundary metric (Section 6.3.1); (2) transform boundary metric with fixed heights. Our transformation is decomposed virtually into a discrete process. A "continuous induction" argument (Lemma 6.3.13) allows us to combine this into a continuous one. We note that for technical reasons we can control efficiently only heights of vertices with $\nu_v(d) \neq 0$. Thereby, we pay plenty of attention to the change of the singularity set $V(d_t)$.

Last, we make remarks on metric paths d_t. For proofs of Main Lemmas IIIA-B we turn the operations of gluing/cutting a bigon from the proof of Lemma 6.2.5 into continuous processes. This is done in Subsection 6.3.2. For our proof of Main Lemma IIIC we introduce a process of dissolving/creating cone-points of small curvature. This is described in Subsection 6.3.7. The rest of the proof of Main Lemma IIIC follows from an indirect transformation based on a study of local properties of the space $\mathfrak{H}(\mathfrak{D}_c(V))$.

6.3.1 Changing heights with fixed boundary

Recall that $\mathfrak{D}_{sc}(V)$ and $\mathfrak{D}_c(V)$ denote the sets of convex cone-metrics on S_g with $V(d) = V$ and with $V(d) \subseteq V$ respectively.

Lemma 6.3.1. Let $P = P(d, V, h)$ be a convex Fuchsian cone-manifold with $d \in \mathfrak{D}_{sc}(V)$. Define $\mathfrak{N}(d, \zeta) \subset \mathfrak{D}_c(V)$ as the set of convex cone-metrics d' such that for every $u, v \in V$,

$$|d'(u, v) - d(u, v)| < \zeta.$$

Then there exist $\zeta > 0$ and $h' \in H(d, V)$ such that:

(1) $h'_v \geq h_v$ for every $v \in V$;
(2) $S(d, V, h') \geq S(d, V, h)$;
(3) $h' \in H(d', V)$ for every metric $d' \in \mathfrak{N}(d, \zeta)$.

Proof. Consider the face decomposition $\mathcal{R} = \mathcal{R}(P)$. By a triangle of $\mathcal{R}$ we mean a triangle of a triangulation $\mathcal{T}$ compatible with P such that $V = V(\mathcal{T})$. Recall that due to Lemma 3.6.2, there are finitely many triangulations compatible with P and, therefore, finitely many triangles of $\mathcal{R}$.

First we see that in some cases we can simply take $h' = h$. Consider a triangulation $\mathcal{T}$ compatible with P and the Fuchsian cone-manifold $P' = P(d', \mathcal{T}, h)$. It might happen that P' is not convex. Let us use the flip algorithm described in Lemma 3.6.8: take an arbitrary concave edge and flip it. Provided ζ is sufficiently small, strict edges of P remain strict in all intermediate cone-manifolds during the algorithm and, therefore, can not be flipped. We refer to the proof of Lemma 3.6.8 for details. Thus, all triangulations that appear are compatible with P. We also recall that the extension $\widetilde{h}$ of the height function is pointwise non-decreasing during the process and is increased at least in one point. Hence, the algorithm can not run infinitely.

If every concave edge can always be flipped, then the algorithm terminates with a triangulation $\mathcal{T}'$ such that the Fuchsian cone-manifold $P'' = P(d', \mathcal{T}', h)$ is convex. Recall from the proof of Lemma 3.6.8 that there are two types of edges that can not be flipped. With respect to this we define the notion of *obstructive* vertices.

(a) A vertex $v \in V$ is called *obstructive of type I* if v is isolated and there exists a triangle T of $\mathcal{R}$ with two edges adjacent to v glued together. We say that T is *associated* to v.

(b) A vertex $v \in V$ is called *obstructive of type II* if there exist two triangles T_1 and T_2 of $\mathcal{R}$ sharing a flat edge incident to v and the total angle of v at this edge is at least π (it might happen that this edge is incident to v twice, then we count only one incidence). We say that T_1 and T_2 are *associated* to v.

Otherwise a vertex is called *non-obstructive*. Suppose that all vertices are non-obstructive in P. It is possible to choose ζ small enough so in any $P' = P(d', \mathcal{T}, h)$ all vertices remain non-obstructive. The discussion above shows that in this case any concave edge can be flipped, the flip algorithm terminates and we can take $h' = h$.

Now we assume that there are obstructive vertices. Our aim is to find $h' \in H(d, V)$ such that properties (1) and (2) from the statement of Lemma 6.3.1 are satisfied and there are no obstructive vertices in $P' = P(d, V, h')$.

There might be plenty of triangles associated to an obstructive vertex. We remark that being obstructive of type II is a stronger property than having an angle at least π in a face of $\mathcal{R}$. Indeed, take a hyperbolic polygon $A_1A_2B_2B_1$ with $A_1B_1 = A_2B_2$, $\angle A_1 = \angle A_2 > \pi/2$ and $\angle B_1 = \angle B_2 < \pi/2$. Glue the sides A_1B_1 and A_2B_2. The angle of vertex A obtained from A_1 and A_2 is greater than π. However, one can see that it is not obstructive of type II, i.e., for any triangulation of the respective surface, any two adjacent triangles have total angle smaller than π at v.

Because an obstructive vertex v of type I is isolated and $V = V(d)$, Lemma 3.6.7 implies that $\kappa_v(P) > 0$ (one can see even more that $\kappa_v(P) > \pi$ in this case). Lemma 3.6.4 shows that $\kappa_v(P) > 0$ also for an obstructive vertex v of type II.

Claim 1. Let v be obstructive of type I, T be the associated triangle and u be the other vertex of T. Then $h_u > h_v$.

Indeed, develop the prism containing T as the prism $A_1A_2A_3B_1B_2B_3$ in $\mathbb{H}^3$, where A_1 corresponds to v, A_2 and A_3 correspond to u.

The lengths A_1A_2 and A_1A_3 are equal. The distance to the lower plane is the same at the corresponding points of A_1A_2 and A_1A_3. It implies that the prism has a plane symmetry with respect to the plane orthogonal to $B_1B_2B_3$, passing through A_1 and the midpoint of A_2A_3. Then the dihedral angles of A_1A_2 and A_1A_3 are equal to $\pi/2$. Hence, A_1 is the closest point from the upper plane to the lower. From this we see that $h_u = A_2B_2 > A_1B_1 = h_v$.

Claim 2. Let v be obstructive of type II and T_1, T_2 be the respective associated triangles. Then there exists a vertex u of T_1 or T_2 such that $h_u > h_v$.

Indeed, develop the prisms containing T_1 and T_2 as a polyhedron in $\mathbb{H}^3$. Let A be the vertex of this polyhedron corresponding to v. It has angle at the upper face greater than π, therefore it lies in the interior of the convex hull of other upper boundary vertices. By the convexity of the distance function, one boundary vertex has height strictly greater than h_v.

Together these claims imply

Claim 3. There exists a triangle associated to an obstructive vertex and containing a non-obstructive vertex.

Indeed, consider the set of pairs (v, u), where v is obstructive and u belongs to a triangle associated with v. Take a pair (v, u) among them such that u has the maximum height. Then u is non-obstructive.

Now we describe a change of heights satisfying the conditions (1) and (2) of the Lemma such that all strict edges remain strict and at least one new strict edge appears. This new edge is a flat edge of P. There are finitely many of them, therefore, the process terminates with a Fuchsian cone-manifold without obstructive vertices.

The deformation is the same as described in Lemma 3.6.8, but it is applied now to the set of obstructive vertices, which is a subset of vertices with positive curvature. Namely, for $\xi > 0$ define h' by $\sinh h'_v := e^{\xi} \sinh h_v$ if v is obstructive and $h'_v := h_v$ otherwise. One can see that the argument from the proof of Lemma 3.6.8 works without modifications to show that for small enough ξ we get $h' \in H(d, V)$. Since $\kappa_v(P) > 0$ for every obstructive v, this deformation increases S.

Consider $P' = P(d, V, h')$. It is clear that for sufficiently small ξ, strict edges of P remain strict in P' and non-obstructive vertices remain non-obstructive. By Claim 3 there is a triangle T associated to an obstructive vertex that contains a non-obstructive vertex. In the end of the proof of Lemma 3.6.8 it is shown that a flat edge of T necessarily becomes strict. □

The assumption $d \in \mathfrak{D}_{sc}(V)$ was essential in our proof of Lemma 6.3.1, but we need also to deal with $d \in \mathfrak{D}_c(V)$. In this case we can guarantee the following weaker version:

Lemma 6.3.2. Let $P = P(d, V, h)$ be a convex Fuchsian cone-manifold. Define $\mathfrak{N}(d, \zeta) \subset \mathfrak{D}_c(V)$ as the set of convex cone-metrics d' such that for every $u, v \in V$,

$$|d'(u, v) - d(u, v)| < \zeta.$$

Then for every $\mu > 0$ there exist $\zeta > 0$ and $h' \in H(d, V)$ such that:

(1) $h'_v \geq h_v$ for every $v \in V$;
(2) $S(d, V, h') \geq S(d, V, h) - \mu$;
(3) for every metric $d' \in \mathfrak{N}(d, \zeta)$, we have $h' \in H(d', V)$.

Proof. The proof is the same to Lemma 6.3.1. There is one thing that goes wrong: there can be obstructive vertices of type II with $\kappa_v(P) = \nu_v(d) = 0$. This tiny aspect appears to be important: when we move such a vertex up, κ_v becomes negative due to Lemma 6.3.17 below. Thus, moving up such vertices may produce an unpredictable negative impact on S. Nevertheless, we can control it. At each step of moving obstructive points higher (see the end of the proof of Lemma 6.3.1) we can choose ξ small enough such that if S decreases, then it decreases by arbitrarily small amount. We have an upper bound for the number of steps (all edges that appear are flat edges of $\mathcal{R}$ and there are finitely many of them), therefore the decrement of S can be kept under μ. □

Now we consider deformations of the boundary metric.

6.3.2 Gluing a bigon continuously

We return to the operation described in the proof of Lemma 6.2.5 of modifying a cone-metric by gluing a bigon. We turn it into a continuous process. After this we will learn how to transform a Fuchsian cone-manifold along with gluing a bigon to its upper boundary.

Let $d_0 \in \mathfrak{D}_{sc}(V)$, $u, v \in V$, χ be a shortest path between u and v and

$$\begin{aligned} \operatorname{len}(\chi, d_0) = d_0(u, v) < \Delta, \\ \nu_u(d_0) + \nu_v(d_0) < 2\eta. \end{aligned} \tag{6.3.1}$$

Here $\Delta > 0$ is arbitrary and $\eta = \eta(\Delta)$ is taken from Lemma 6.2.3. On χ choose a point w such that

$$\sinh(d_0(u, w)) \tan(\nu_u(d_0)/2) = \sinh(d_0(w, v)) \tan(\nu_v(d_0)/2).$$

Define $V' = V \cup \{w\}$. Now we consider d_0 as an element of $\mathfrak{D}_c(V')$.

As in the proof of Lemma 6.2.5, we consider a hyperbolic triangle ABC such that

$$BC = d_0(u, v), \quad \angle B = \nu_u(d)/2, \quad \angle C = \nu_v(d_0)/2.$$

Lemma 6.2.3 implies that ABC exists and BC is its greatest side. Let AH be the height of ABC from point A, so $H \in BC$. Then

$$\sinh BH \tan \angle B = \sinh CH \tan \angle C.$$

Define $\tau = AH$. Note that

$$\tau \leq \sinh \tau \leq \sin(\nu_v(d_0)/2) \sinh(d_0(u, v)) < \nu_v(d_0) \sinh \Delta. \tag{6.3.2}$$

For every $t \in [0; \tau]$ let $A_t \in AH$ be the point such that $HA_t = t$.

Cut (S_g, d_0) along χ. Glue there a bigon X_t as in Section 6.2.1 consisting of two copies of A_tBC. We obtain a metric d_t on S_g. The set V can be naturally defined in all metric spaces (S_g, d_t). We associate the remaining point $w \in V'$ with the vertex A_t of $X_t \subset (S_g, d_t)$. Thus we get a continuous path of convex cone-metrics $d_t \subset \mathfrak{D}_c(V')$ parametrized by $t \in [0; \tau]$. Moreover, for each $t \in (0; \tau)$ we have $d_t \in \mathfrak{D}_{sc}(V')$. Also note that $V(d_\tau) = V' \backslash \{u, v\}$ and $\nu_u(d_0) + \nu_v(d_0) < \nu_w(d_\tau)$.

By $\psi_1, \psi_2 \subset X_t$ we denote the unique shortest paths from w to u and v in each metric d_t respectively. Remark that for any $0 \leq t' < t'' \leq \tau$ the metric $d_{t''}$ can be obtained from $d_{t'}$ by cutting $d_{t'}$ along ψ_1 and ψ_2 and gluing there the conical 4-gon $X_{t',t''}$ obtained from two copies of the triangle $A_{t'}A_{t''}B$ and two copies of $A_{t'}A_{t''}C$. Treating this operation similar as in the proof of Claim 1 of Lemma 6.2.5 one can show that the diameter of d_t is non-decreasing with respect to t.

Assume that we have a convex Fuchsian cone-manifold $P_0 = P(d_0, V, h_0)$. We want to transform it into a convex Fuchsian cone-manifold $P_\tau = P(d_\tau, V', h_\tau)$ and estimate the change of S.

6.3.3 Changing boundary with fixed heights under gluing a bigon

All the notation is taken from the previous subsection. Here we are going to estimate the change of S under a small deformation of the boundary metric with the fixed heights. We do it modulo several quantitative results that we prove separately in Subsection 6.3.4. After this in Subsection 6.3.5 we combine it with results of Subsection 6.3.1 to get the full deformation.

For a metric d on S_g define $\alpha(d) := \operatorname{arccot}(2\cosh(\operatorname{diam}(S_g, d)))$.

Lemma 6.3.3. Let $0 \le t' < t'' \le \tau$. Assume that there is

$$h \in \bigcap_{t \in [t';t'']} H(d_t, V')$$

and a triangulation $\mathcal{T}_h$ with $V(\mathcal{T}_h) = V'$ such that every convex Fuchsian cone-manifold $P_t = P(d_t, V', h)$ is compatible with $\mathcal{T}_h$. Then

$$\begin{aligned} 0 \le S(d_{t''}, V', h) - S(d_{t'}, V', h) \le \\ \le \left(\frac{(\pi + \Delta) \sinh \Delta}{\Delta} + 1 \right) \frac{2\pi(t'' - t')}{\sin^2 \alpha(d_\tau)}. \end{aligned} \tag{6.3.3}$$

Proof. By $S(t)$ denote $S(P_t)$. As all P_t are compatible with $\mathcal{T}_h$, $S(t)$ is a function of the upper lengths $l_e(d_t) = l_e(t)$ for $e \in E(\mathcal{T}_h)$. From Lemma 4.1.1

$$\frac{\partial S}{\partial l_e} = \theta_e = \pi - \phi_e.$$

We have

$$S(t'') - S(t') = \int_{t'}^{t''} \langle \text{grad } S,\ \dot{l} \rangle dt = \int_{t'}^{t''} \sum_{e \in E(\mathcal{T}_h)} (\pi - \phi_e) \dot{l}_e dt.$$

First we suppose that the edges of $E(\mathcal{T}_h)$ changing their lengths over $[t';t'']$ are split into three groups:

(1) edges that intersect ψ_1 once;
(2) edges that intersect ψ_2 once;
(3) edges with w as one (and only one) of the endpoints.

In the general case it might happen that an edge belongs to several types, or intersects ψ_1 or ψ_2 more than once, or has w as both endpoints. First we give a proof in the simple case when neither of this happens, and then show how to reduce the general case to the simple one.

We denote the set of edges of each type by E_1, E_2 and E_3 respectively.

Fix $t \in (0;\tau]$ and consider (S_g, d_t). Triangulate the bigon $X_t \subset (S_g, d_t)$ in the natural way and extend it to a triangulation $\mathcal{T}_0$ in any way. This is possible because of Lemma 2.4.6, which allows to extend any partial triangulation to a full triangulation. Note that for all $t \in (0;\tau]$ the metric d_t realizes $\mathcal{T}_0$. In the metric d_0 our triangulation degenerates, but this provides no additional difficulties to the discussion below. The deformation d_t can be described easily in the chart $\mathfrak{D}(V', \mathcal{T}_0)$ as only two edges of $\mathcal{T}_0$ (ψ_1 and ψ_2) change their lengths in d_t.

Take $e \in E_1$. Develop the path of triangles of $\mathcal{T}_0$ intersecting e in the moment $t \in [t';t'']$ to $\mathbb{H}^2$. The development results in a (possibly non-convex) triangulated polygon R_0 in $\mathbb{H}^2$. See Figure 6.7. One can see that R_0 is not self-intersecting. Denote the image of the endpoints of e in the development by D_1 and D_2. By construction, $D_1 D_2$ is in the interior of R_0 and intersects each interior edge of its triangulation. Note that a triangle of $\mathcal{T}_0$ may be repeated several times in the triangulation of R_0.

As $e \in E_1$ and the situation is simple, there exists exactly one interior edge of R_0 that is the image of ψ_1 under the developing map. Denote its endpoints by A_0 and B_0 so that A_0 corresponds to w and B_0 to u, and by C_1, C_2 denote the points corresponding to v. Thereby, the triangles $A_0 B_0 C_1$ and $A_0 B_0 C_2$ are isometric to the triangle $A_t BC$ from the bigon X_t, the distances $B_0 D_1$ and $B_0 D_2$ does not change during the deformation and $l_e = D_1 D_2$.

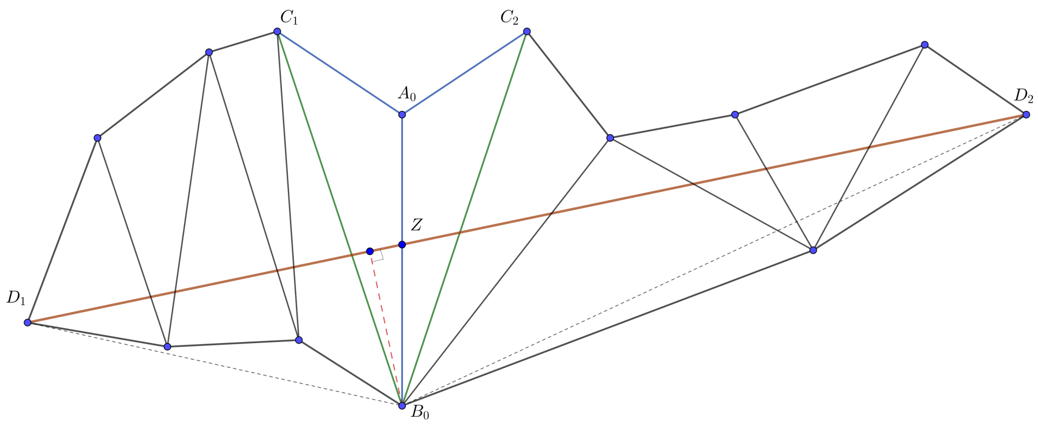

FIGURE 6.7: Polygon R_0 for an edge of type (1).

Let y_e be the hyperbolic sine of the length of perpendicular from the point B_0 to D_1D_2, ρ be the angle $C_1B_0C_2$ and ρ_e be the angle $D_1B_0D_2$. We claim

$$\frac{\partial l_e}{\partial \rho} = y_e. \tag{6.3.4}$$

Indeed, differentiating the cosine rule for the triangle $D_1B_0D_2$ we obtain

$$\frac{\partial l_e}{\partial \rho_e} = y_e.$$

However, $\frac{\partial \rho_e}{\partial \rho} = 1$, therefore we get (6.3.4). It is easy to see that $\dot{\rho} \geq 0$, therefore $\dot{l}_e \geq 0$.

Let D_1D_2 and A_0B_0 intersect in a point Z. By x_e denote $\sinh B_0Z$. By γ_e denote the smallest angle at which D_1D_2 intersects A_0B_0. We have

$$y_e = x_e \sin \gamma_e. \tag{6.3.5}$$

In P_t consider the geodesic ray from the point corresponding to Z orthogonal to $\partial_\downarrow P_t$. Let this ray make the angles β_e^- and β_e^+ with the links of ψ_1. Define $\beta_e := \beta_e^- + \beta_e^+$. Corollary 6.3.5 shows that

$$\alpha(d_t) \leq \beta_e^-, \beta_e^+ \leq \pi - \alpha(d_t).$$

Together with Corollary 6.3.6 and Lemma 6.3.9 this gives

$$\pi - \phi_e \leq \frac{\pi - \beta_e}{\sin^2 \alpha(d_t) \sin \gamma_e}.$$

Combined this with (6.3.4) and (6.3.5) we get

$$(\pi - \phi_e)\frac{\partial l_e}{\partial \rho} \leq \frac{\pi - \beta_e}{\sin^2 \alpha(d_t) \sin \gamma_e} x_e \sin \gamma_e \leq \frac{\pi - \beta_e}{\sin^2 \alpha(d_\tau)} x_e. \tag{6.3.6}$$

Consider the union of all segments connecting points of ψ_1 with their orthogonal projections to $\partial_\downarrow P_t$. This set is isometric to a convex hyperbolic polygon, which is a union of trapezoids with the angles at the upper boundary equal to β_e. The total length of the upper boundary is equal to A_tB. Apply Corollary 6.3.11 and get

$$\sum_{e\in E_1}(\pi-\beta_e)\leq \pi+A_tB<\pi+\Delta. \tag{6.3.7}$$

Here we used $A_tB<\Delta$ for all t.

Define $c(t)=\sinh A_tB$. We have

$$x_e\leq \sinh A_0B_0=\sinh A_tB\leq \sinh A_{t''}B=c(t'').$$

From this, (6.3.6) and (6.3.7) we obtain

$$\sum_{e\in E_1}(\pi-\phi_e)\frac{\partial l_e}{\partial\rho}\leq\frac{1}{\sin^2\alpha(d_\tau)}\sum_{e\in E_1}(\pi-\beta_e)x_e<\frac{(\pi+\Delta)c(t'')}{\sin^2\alpha(d_\tau)}.$$

The map $t\to\rho$ is monotone. Substitute ρ in the integral

$$0\leq\int_{t'}^{t''}\sum_{e\in E_1}(\pi-\phi_e)\dot{l}dt=\int_{\rho(t')}^{\rho(t'')}\sum_{e\in E_1}(\pi-\phi_e)\frac{\partial l_e}{\partial\rho}d\rho<$$

$$<\frac{\pi+\Delta}{\sin^2\alpha(d_\tau)}c(t'')(\rho(t'')-\rho(t')).$$

Now we consider the triangles $A_{t'}BC$, $A_{t''}BC$ embedded in the triangle ABC. Note that $\angle A_{t'}BA_{t''}=\frac{1}{2}(\rho(t'')-\rho(t'))<\pi/2$ and $\sinh A_{t''}B=c(t'')$. Therefore,

$$c(t'')(\rho(t'')-\rho(t'))=2\sinh A_{t''}B\cdot\angle A_{t'}BA_{t''}\leq$$

$$\leq\pi\sinh A_{t''}B\cdot\sin\angle A_{t'}BA_{t''}\leq\pi\sinh A_{t'}A_{t''}\leq$$

$$\leq\frac{\pi\sinh\Delta}{\Delta}A_{t'}A_{t''}=\frac{\pi\sinh\Delta}{\Delta}(t''-t').$$

Here we used the concativity of sin and the convexity of sinh.

In total we get

$$0\leq\int_{t'}^{t''}\sum_{e\in E_1}(\pi-\phi_e)\dot{l}dt\leq\frac{\pi(\pi+\Delta)\sinh\Delta}{\Delta\sin^2\alpha(d_\tau)}(t''-t').$$

The same bound holds for the edges of the second type:

$$0\leq\int_{t'}^{t''}\sum_{e\in E_2}(\pi-\phi_e)\dot{l}dt\leq\frac{\pi(\pi+\Delta)\sinh\Delta}{\Delta\sin^2\alpha(d_\tau)}(t''-t')$$

Now consider edges of the third type. By Lemma 6.3.7 and Lemma 6.3.8 we have

$$\sum_{e\in E_3}(\pi-\phi_e)\leq\omega_w(P_t)\leq\frac{2\pi}{\sin\alpha(d_t)}\leq\frac{2\pi}{\sin^2\alpha(d_\tau)}.$$

Recall here that $\omega_w(P_t)$ is the total cone angle of w in the cone-manifold P_t.

Let $e\in E_3$. Develop the path of triangles of $\mathcal{T}_0$ intersecting e in the moment t to $\mathbb{H}^2$. Let A_0 be the image of w under the developing map, D be the image of the second endpoint of e, B_0C_0 be the image of the boundary segment of X_t intersected by e.

Consider the moment $t + \Delta t$ and develop this path of triangles so that the image of each triangle coincides with its image under the previous development, except the first one, which is developed to $A_1B_0C_0$. Then $l_e(t) = A_0D$, $l_e(t + \Delta t) = A_1D$. Note that the angle $\angle DA_0A_1$ is not acute, therefore $l_e(t+\Delta t) \geq l_e(t)$, which implies $\dot{l}_e \geq 0$.

From the triangle inequality we see

$$l_e(t + \Delta t) - l_e(t) = DA_1 - DA_0 \leq A_0A_1 = \Delta t$$

Hence, $\dot{l}_e \leq 1$. Then

$$0 \leq \int_{t'}^{t''} \sum_{e \in E_3} (\pi - \phi_e)\dot{l}_e dt \leq \int_{t'}^{t''} \sum_{e \in E_3} (\pi - \phi_e) dt \leq$$

$$\leq \frac{2\pi}{\sin^2 \alpha(d_\tau)}(t'' - t').$$

Summing up all three estimates we get exactly the estimate (6.3.3). This finishes the proof in the simple case.

Now it remains to consider the general case, when an edge e may intersect each of ψ_1 and ψ_2 several times and in the same time may have endpoints at w. As before, we develop the path of triangles intersecting e in the moment t to $\mathbb{H}^2$ as a triangulated polygon R_0. Orient e and enumerate all the triangles of R_0 with respect to the orientation. By k denote the number of edges in R_0 (including the interior edges of the triangulation of R_0) and by $z_1(t), \ldots, z_k(t)$ denote their lengths in the moment t. The lengths $z_1(t), \ldots, z_k(t)$ and the combinatorics determine R_0 up to isometry. Thus, the length l_e can be considered as the composite function $l_e(t) = l_e(z_1(t), \ldots, z_k(t))$. We consider the derivative $\dot{l}_e$ as the composite derivative

$$\dot{l}_e = \sum \frac{\partial l_e}{\partial z_i} \frac{\partial z_i}{\partial t}$$

and decompose this sum into *elementary deformations*, which have types (1), (2) and (3) similar to the edge types in the simple case.

We note that for all t the angle BA_tC is obtuse: this is because we use Lemma 6.2.3 to construct the bigons. This means that if e enters X_t in the metric d_t, then it intersects either ψ_1 or ψ_2 and then leaves X_t (or e ends in w). This allows us to simplify the exposition, however a proof can be conducted without this restriction.

(1) Assume that T_{i-1} and T_i are two subsequent triangles of R_0 that are the images of X_t under the developing map, and they share an edge that is the image of ψ_1 intersected by e. Let z_{i_1} be the length of this edge and z_{i_2}, z_{i_3} be the lengths of the images of ψ_2 (as e leaves X_t immediately, the edges of T_{i-1}, T_i at the boundary of R_0 are the images of ψ_2). Then we say that the sum of coordinate derivatives

$$L(e, i_1, i_2, i_3) = \frac{\partial l_e}{\partial z_{i_1}} \frac{\partial z_{i_1}}{\partial t} + \frac{\partial l_e}{\partial z_{i_2}} \frac{\partial z_{i_2}}{\partial t} + \frac{\partial l_e}{\partial z_{i_3}} \frac{\partial z_{i_3}}{\partial t}$$

is an elementary deformation of the first type.

An elementary deformation of the first type is analyzed in the same way as the deformation of an edge of the first type in the simple case. Let $y(e, i_1, i_2, i_3)$ be the hyperbolic sine of the length of perpendicular from the image of u in $T_{i-1} \cup T_i$ to the image of e and consider z_{i_j} as the function of ρ, which is the angle at vertex u in the bigon X_t. Then

$$L(e, i_1, i_2, i_3) = y(e, i_1, i_2, i_3)\frac{\partial \rho}{\partial t}.$$

(2) An elementary deformation of the second type is defined similarly, but swapping ψ_1 and ψ_2.

(3) Assume that w is the starting vertex of e and z_{i_1}, z_{i_2} are the lengths of the edges of T_1 that are the images of ψ_1 and ψ_2. Then we say that the sum of coordinate derivatives

$$L(e, i_1, i_2) = \frac{\partial l_e}{\partial z_{i_1}} \frac{\partial z_{i_1}}{\partial t} + \frac{\partial l_e}{\partial z_{i_2}} \frac{\partial z_{i_2}}{\partial t}.$$

is an elementary deformation of the third type. Similarly, if w is the ending vertex of e, then the sum of coordinate derivatives of the lengths of the last triangle T_k also form an elementary deformation of the third type.

As in the case of an edge of the third type, we have

$$0 \le L(e, i_1, i_2) \le 1.$$

We see that every coordinate derivative

$$\frac{\partial l_e}{\partial z_i} \frac{\partial z_i}{\partial t}$$

either belongs to exactly one elementary deformation or is equal to zero. Therefore, the total derivative $\dot{l}_e$ can be decomposed into the sum of elementary deformations.

We do this for every edge e of $\mathcal{T}_h$ and group elementary deformations of each type together for all edges. Then we decompose the sum

$$\sum_{e \in E(\mathcal{T}_h)} (\pi - \phi_e) \dot{l}_e$$

into three summands corresponding to elementary deformations of each type. The integral of each summand is estimated in the same way as in the simple case. □

6.3.4 Slopes and angle estimates

All results of this subsection are auxiliary for the proof of Lemma 6.3.3 and a similar Lemma 6.3.22 below.

Let $P = P(d, V, h)$ be a convex Fuchsian cone-manifold, $p \in (S_g, d)$ be a point and T be a face triangle containing p. Embed the prism containing T to $\mathbb{H}^3$. Lemma 3.2.2 says that this prism is ultraparallel. By A_p we denote the image of p, by B_p its projection to the lower plane, by AB denote the common perpendicular between the upper and the lower boundary planes. Define

$$\mathrm{slope}(p, P, T) := \angle B_p A_p A.$$

It is clear that it does not depend on the choice of an embedding and if T is contained in a face R of P, then $\mathrm{slope}(p, P, T)$ does not depend on T, so we can denote it as $\mathrm{slope}(p, P, R)$. We have $0 < \mathrm{slope}(p, P, R) \le \pi/2$.

Recall that $\alpha(d) := \operatorname{arccot}(2\cosh(\operatorname{diam}(S_g, d)))$.

Lemma 6.3.4. For each $p \in (S_g, d)$ and a face $R \in \mathcal{R}(P)$ with $p \in \overline{R}$ we have

$$\alpha(d) \le \mathrm{slope}(p, P, R).$$

Proof. In the notation above we get (using Corollary 2.3.5)

$$\cot(\mathrm{slope}(p, P, R)) = \sinh AA_p \tanh AB \le \cosh AA_p \tanh AB =$$

$$= \frac{\sinh A_pB_p}{\cosh AB} = \frac{\sinh(A_pB_p - AB + AB)}{\cosh AB} =$$
$$= \sinh(A_pB_p - AB) + \cosh(A_pB_p - AB)\tanh AB \leq$$
$$\leq 2\cosh(A_pB_p - AB).$$

We claim that there exists a point $q \in S_g$ such that $\widetilde{h}(q) \leq AB$. Indeed, we can assume to this purpose that $p \in R$, i.e, p is not a conical point and does not belong to a strict edge, because this does not change AB. Consider the unit speed geodesic $\psi \subset (S_g, d)$ starting from p in the direction $-\mathrm{grad}_\mathrm{p}\widetilde{h}$. Suppose that ψ extends to the distance AA_p from p (it might not happen in the only case when ψ ends in a conical point before). Let $q \in \psi$ be the point at the distance AA_p. Consider the function f on ψ defined by

$$\sinh f(x) = \cosh(x - AA_p)\sinh AB.$$

We recall the definition of $\mathcal{F}(-1)$-function and $\mathcal{F}(-1)$-concave function from Section 3.3. By definition, $\sinh f$ is $\mathcal{F}(-1)$; $\sinh \widetilde{h}$ is $\mathcal{F}(-1)$-concave due to Lemma 3.3.2 and $f(x)$ coincides with the restriction of $\widetilde{h}$ to ψ in a neighbourhood of p. Then $\sinh \widetilde{h}(q) \leq f(q) = \sinh AB$.

In the case that ψ ends in a conical point before moving at distance AA_p we can perturb slightly the direction of ψ to avoid this. By the argument above, we obtain q_ε such that $\widetilde{h}(q_\varepsilon) \leq AB + \varepsilon$ for some $\varepsilon > 0$. We can make ε arbitrarily small and consider a limit point q of the set $\{q_\varepsilon\}$. By continuity of $\widetilde{h}$, we have $\widetilde{h}(q) \leq AB$.

We get

$$\cot(\mathrm{slope}(p, P, R)) \leq 2\cosh(\widetilde{h}(v) - \widetilde{h}(q)) \leq 2\cosh(\mathrm{diam}(S_g, d))$$

due to Lemma 3.1.8. This proves the desired inequality. □

The following corollary easily follow after a local development.

Corollary 6.3.5. Let $p \in (S_g, d)$ and α be the angle between a geodesic ray from p in $\partial^\uparrow P$ and the geodesic ray from p orthogonal to $\partial_\downarrow P$. Then

$$\alpha(d) \leq \alpha \leq \pi - \alpha(d).$$

The next one is just slightly more involved.

Corollary 6.3.6. Let e be an edge of P, ϕ_e^-, ϕ_e^+ be the dihedral angles of e in the prisms containing e. Then

$$\alpha(d) \leq \phi_e^-, \phi_e^+ \leq \pi - \alpha(d).$$

Proof. Let Π be the prism containing e with dihedral angle ϕ_e^- and T be the upper boundary triangle of Π. Embed it to $\mathbb{H}^3$ and let A be as above. By A_e denote the closest point to A on the line containing e and by B_e denote its projection to the lower boundary plane. Then $\angle AA_eB_e$ is either ϕ_e^- or $\pi - \phi_e^-$ and $\angle AA_eB_e \leq \pi/2$. The only issue is that A_e may lie outside of e. Then take any $p \in e$ and note that $\mathrm{slope}(p, P, T) \leq \angle AA_eB_e$. Apply Lemma 6.3.4 and finish the proof. □

Lemma 6.3.7. Let $P = P(d, V, h)$ be a convex Fuchsian cone-manifold. Then for every $v \in V$ we have

$$\omega_v(P) \leq \frac{2\pi}{\sin\alpha(d)}.$$

Proof. Let $M^\uparrow$ and $M_\downarrow$ be ultraparallel hyperbolic planes in $\mathbb{H}^3$, $A \in M^\uparrow$ and $B \in M_\downarrow$ be the closest points, $A_1 \in M^\uparrow$ be a point, B_1 be its projection to $M_\downarrow$ and $\alpha = \angle B_1 A_1 A$.

Consider an angle of value λ in $M^\uparrow$ with the vertex A_1. Let ω be the value its orthogonal projection to $M_\downarrow$. We claim that

$$\omega \leq \frac{\lambda}{\sin\alpha}. \tag{6.3.8}$$

First, assume that A belongs to one of the two rays bounding λ and $\lambda \leq \pi/2$. We call it an angle of the first type. In this case we claim

$$\tan\omega = \frac{\tan\lambda}{\sin\alpha}. \tag{6.3.9}$$

Indeed, let A_2 belongs to the second ray of the angle and $\angle AA_2A_1 = \pi/2$. It may coincide with A_1, but this produces no difficulties. Let B_2 be its orthogonal projection to $M_\downarrow$. We have three orthogonal trapezoids. Applying Corollary 2.3.5 several times we get

$$\tanh A_1A_2 = \tanh B_1B_2 \cosh A_2B_2,$$

$$\sinh AA_2 = \sinh BB_2 \cosh A_2B_2,$$

$$\sin\alpha = \frac{\cosh AB}{\cosh A_1B_1}, \quad \tan\lambda = \frac{\tanh AA_2}{\sinh A_1A_2}, \quad \tan\omega = \frac{\tanh BB_2}{\sinh B_1B_2}.$$

The equation (6.3.9) follows from this.

Let $f(x) = \arctan\left(\frac{\tan x}{\sin\alpha}\right)$. We have $f(0) = 0$ and

$$f'(x) = \frac{1}{\sin\alpha + (1-\sin\alpha)\sin^2 x} \leq \frac{1}{\sin\alpha}.$$

This proves (6.3.8) for angles of the first type. We can also see that $f(x)$ is monotonously increasing and concave for $0 \leq x \leq \pi/2$.

If we have an angle of value λ that is the union of two angles of the first type, then (6.3.8) holds by additivity. We call it an angle of the second type.

Let an angle of value λ be the difference of two angles of the first type of values λ_1 and λ_2. We call it an angle of the third type. Then by concativity of f for the projections of these angles we have

$$\omega = \omega_1 - \omega_2 \leq (\lambda_1 - \lambda_2) f'(\lambda_2) \leq \frac{\lambda}{\sin\alpha}.$$

Now consider an arbitrary angle at A_1. Let s be a line in $M^\uparrow$ through A_1 orthogonal to AA_1. Take the part of our angle that lies on the other side from A with respect to s and replace this part with its centrally symmetric image in $M^\uparrow$ with respect to A_1. This does not change neither the total value of the angle nor the total value of the projection. Now we have at most two angles and each of them belong to one of three types described above. Then they satisfy (6.3.8) and so does the initial angle.

Consider a convex Fuchsian cone-manifold $P = P(d, V, h)$. Subdivide its face decomposition $\mathcal{R}(P)$ into triangles. For a vertex $v \in V$ incident to a triangle T let $\lambda_{v,T}$, $\omega_{v,T}$ be the angles of the prism containing T in the upper and lower faces corresponding to v. We showed that

$$\omega_{v,T} \leq \frac{\lambda_{v,T}}{\sin(\mathrm{slope}(v, P, T))}.$$

Thus,

$$\omega_v(P) = \sum_{T \text{ is incident } v} \omega_{v,T} \leq \left(\sum_{T \text{ is incident } v} \frac{\lambda_{v,T}}{\sin(\operatorname{slope}(v,P,T))} \right) \leq$$

$$\leq \frac{2\pi}{\sin \alpha(d)}.$$

□

Lemma 6.3.8. Let $P = P(d, V, h)$ be a convex Fuchsian cone-manifold. Then for every $v \in V$ we have

$$\sum_{e \text{ is incident } v} (\pi - \phi_e(P)) < \omega_v(P).$$

Proof. Consider the spherical link of P at v. It is a spherical polygon with a conical point of total angle $\omega_v(P)$ and boundary angles $\phi_e(P)$. Then its area is

$$\omega_v(P) + \sum_{e \text{ is incident } v} (\phi_e(P) - \pi) > 0.$$

□

Lemma 6.3.9. Consider two half-planes M^-, M^+ in $\mathbb{H}^3$ sharing a line χ (see Figure 6.8). Let $p \in \chi$ and $\psi^- \subset M^-$, $\psi^+ \subset M^+$ be geodesic rays from p making an intrinsic geodesic in the union of M^- and M^+. By $\gamma > 0$ denote the (smallest) angle of this geodesic and χ.

Let ψ be a geodesic ray from p in the (convex) dihedral angle spanned by M^- and M^+ and M be the half plane spanned by χ and ψ. By ϕ^-, ϕ^+ denote the dihedral angles between M and M^-, M^+ respectively. By β^-, β^+ denote the angles between ψ and ψ^-, ψ^+ respectively. Define $\phi = \phi^- + \phi^+$, $\beta := \beta^- + \beta^+$. By α denote the (smallest) angle between ψ and χ.

Assume that for some α_0 with $0 < \alpha_0 < \pi/2$ we have

$$\alpha_0 \leq \alpha, \phi^-, \phi^+, \beta^-, \beta^+ \leq \pi - \alpha_0.$$

Then

$$\pi - \phi \leq \frac{\pi - \beta}{\sin^2 \alpha_0 \sin \gamma}.$$

Proof. We consider the function $\frac{\pi-\beta}{(\pi-\phi)\sin\gamma}$ as the function of ϕ^-, ϕ^+, γ and α and find its minimal value under the given restrictions.

Note that $\beta \leq \pi$. As $0 < \gamma, \alpha < \pi$, then $\phi = \pi$ if and only if $\beta = \pi$. Assume that $\phi, \beta < \pi$.

Consider a spherical section at p, so geodesic rays become points and half-planes become half-circles. Assume also that β^+ belongs to the spherical triangle with the other sides γ and α (otherwise, switch M^- and M^+). From the cosine law we get

$$\cos \beta^+ = \cos\gamma\cos\alpha + \sin\gamma\sin\alpha\cos\phi^+,$$

$$\cos \beta^- = -\cos\gamma\cos\alpha + \sin\gamma\sin\alpha\cos\phi^-,$$

$$\cos\beta^+ + \cos\beta^- = \sin\gamma\sin\alpha(\cos\phi^+ + \cos\phi^-).$$

As $\beta \leq \pi$, then from elementary trigonometry

$$\pi - \beta \geq \cos\beta^- + \cos\beta^+ \geq 0$$

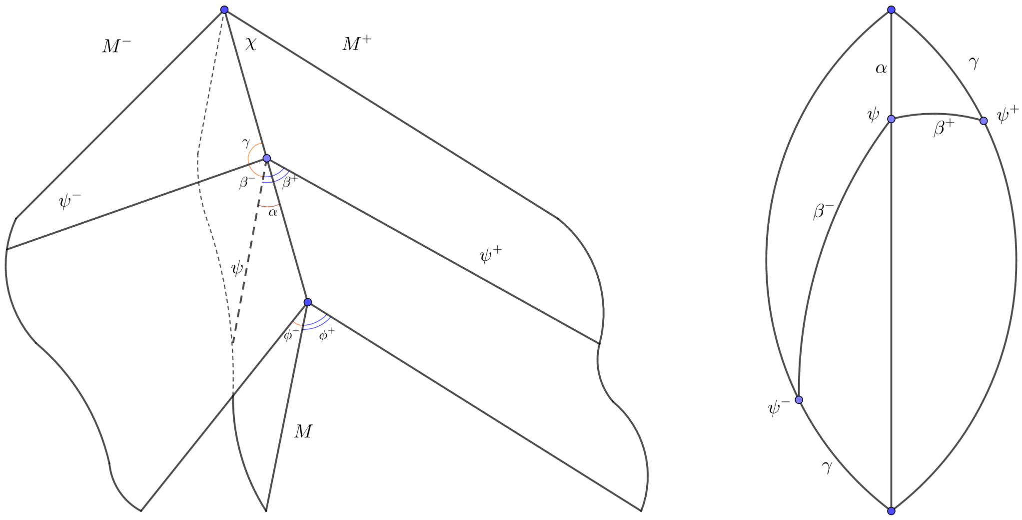

FIGURE 6.8: To the statement of Lemma 6.3.9.
Left: In space. Right: in the spherical link of p.

with equality if and only if $\beta = \pi$. Thus,

$$\pi - \beta \geq \sin\gamma \sin\alpha(\cos\phi^+ + \cos\phi^-)$$

and

$$\frac{\pi - \beta}{(\pi - \phi)\sin\gamma} \geq \frac{\sin\alpha(\cos\phi^- + \cos\phi^+)}{\pi - \phi}.$$

Assume that $\phi^- \leq \phi^+$. Fix $\phi^- \leq \pi/2$ and consider the function

$$f(\phi^+) := \frac{\cos\phi^- + \cos\phi^+}{\pi - \phi}$$

as a function of $\phi^+ \in [\phi^-; \pi - \phi^-)$. It suffices to prove $f(\phi^+) \geq \sin\alpha_0$. Using L'Hopital's rule we compute the limit of f as $\phi^+ \to \pi - \phi^-$:

$$\lim_{\phi^+ \to \pi - \phi^-} f(\phi^+) = \sin(\phi^-) \geq \sin\alpha_0.$$

Now we compute its derivative

$$\frac{\partial f}{\partial \phi^+} = \frac{1}{(\pi - \phi)^2}\left(-\sin\phi^+(\pi - \phi) + \cos\phi^- + \cos\phi^+\right).$$

We see that if ϕ^+ is a critical point, then f is equal to $\sin\phi^+ \geq \sin\alpha_0$.

It remains to check what happens as $\phi^+ = \phi^-$. From elementary trigonometry we have

$$f(\phi^-) = \frac{2\cos\phi^-}{\pi - 2\phi^-} \geq \sin\phi^- \geq \sin\alpha_0.$$

In total we get

$$\frac{\pi - \beta}{(\pi - \phi)\sin\gamma} \geq \sin^2\alpha_0,$$

which is equivalent to the desired inequality. □

Lemma 6.3.10. Let $A_1A_2B_2B_1$ be a trapezoid in $\mathbb{H}^2$. Then

$$\text{area}(A_1A_2B_2B_1) \leq A_1A_2. \tag{6.3.10}$$

Proof. We claim that is enough to prove (6.3.10) for a right-angled trapezoid. Indeed, if both $\angle A_1$, $\angle A_2$ are less than $\pi/2$, then the trapezoid is ultraparallel and, moreover, can be cut into two right-angled ones. If we know (6.3.10) for both of them, then it holds also for the initial one. Suppose that $\angle A_1 > \pi/2$. Then we can rotate A_1A_2 around A_2 until $\angle A_1$ becomes $\pi/2$ so that the area only grows.

Thereby, it is enough to prove (6.3.10) for $\angle A_1 = \pi/2$. We use the notation from Subsection 2.3. The area of the trapezoid is $\pi/2 - \alpha_{21}$. From Corollary 2.3.5 we have

$$\tan(\pi/2 - \alpha_{21}) = \cot \alpha_{21} = \tanh h_1 \sinh l_2 < \sinh l_2.$$

Therefore, the area of the trapezoid is less than $\arctan \sinh A_1A_2$. Consider the function $f(x) = \arctan(\sinh x)$. We have

$$f'(x) = \frac{1}{\cosh x} \leq 1.$$

As $f(0) = 0$, it follows that $f(x) \leq x$ for $x > 0$. This finishes the proof. □

Corollary 6.3.11. Let $B_0A_0 \dots A_{m+1}B_{m+1}$ be a convex polygon in $\mathbb{H}^2$ with $\angle B_0 = \angle B_{m+1} = \pi/2$, α_k be the angle $A_{k-1}A_kA_{k+1}$ and L be the sum of lengths $A_0A_1 + \dots + A_mA_{m+1}$. Then

$$\sum_{k=1}^{m} (\pi - \alpha_k) \leq \pi + L.$$

Proof. Decompose the polygon into trapezoids $A_{k-1}A_kB_{k-1}B_k$. Denote the angle $B_kA_kA_{k+1}$ by α_k^+, the angle $B_kA_kA_{k-1}$ by α_k^- and the length A_kA_{k+1} by l_k. Lemma 6.3.10 shows $\pi - \alpha_k^+ - \alpha_{k+1}^- \leq l_k$.

Then

$$\sum_{k=1}^{m} (\pi - \alpha_k) = \pi/2 - \alpha_1^+ + \sum_{k=1}^{m-1} (\pi - \alpha_k^- - \alpha_{k+1}^+) + \pi/2 - \alpha_m^- \leq$$

$$\leq \pi + \sum_{k=1}^{m-1} l_k + \leq \pi + L.$$

□

6.3.5 Changing heights and boundary simultaneously

Lemma 6.3.3 gives us a Lipschitz bound on the change of S when we change the boundary metric with fixed heights as we glue a bigon. Below we will obtain some other bounds of this type. Lemma 6.3.13 combines such bounds with the results of Subsection 6.3.1 to get a full deformation.

We need to prepare slightly. Recall that

$$H_S(d, V, m, K) = \{h \in H(d, V) : \min_{v \in V} h_v \geq m, S(d, V, h) \geq K\}.$$

For $m \in \mathbb{R}_{>0}$ and $K \in \mathbb{R}$ define the sets

$$\mathfrak{H}_S(\mathfrak{U}, m, K) = \{(d, h) : d \in \mathfrak{U}, h \in H_S(d, V, m, K)\},$$

$$H_S(\mathfrak{U}) = \bigcup_{d\in\mathfrak{U}} H_S(d, V, m, K).$$

We claim

Lemma 6.3.12. If $\mathfrak{U}$ is compact, then so are $\mathfrak{H}_S(\mathfrak{U}, m, K)$, $H_S(\mathfrak{U}, m, K)$.

Proof. By Corollary 4.3.2, $H_S(d, V, m, K)$ is compact. The projection map $(d, h) \to d$ mapping $\mathfrak{H}(\mathfrak{D}_c(V))$ onto $\mathfrak{D}_c(V)$ is continuous, closed and have compact fibers. Therefore, it is proper. Then if $\mathfrak{U}$ is compact, $\mathfrak{H}_S(\mathfrak{U}, m, K)$ also is compact. The set $H_S(\mathfrak{U}, m, K)$ is compact as the image of $\mathfrak{H}_S(\mathfrak{U}, m, K)$ under the projection map $(d, h) \to h$. □

Note that except the proof of Lemma 6.3.13 just below, Lemma 6.3.12 will be also prominent in our proof of Main Lemma IIIC. Now we are ready to prove

Lemma 6.3.13. Let $d_t \subset \mathfrak{D}_c(V)$, $t \in [0;\tau]$, be a continuous path of metrics such that $d_t \in \mathfrak{D}_{sc}(V)$ for $t \in (0;\tau)$, and $P_0 = P(d_0, V, h_0)$ be a convex Fuchsian cone-manifold.

Assume that there exists a constant $C \geq 0$ with the following property. Let $[t';t''] \subseteq [0;\tau]$ and let there be $h \in H(d_t, V)$ and a triangulation $\mathcal{T}_h$ compatible with Fuchsian cone-manifolds $P_t = P(d_t, V, h)$ for all $t \in [t', t'']$. Then

$$S(d_{t''}, V, h) \geq S(d_{t'}, V, h) - C(t'' - t').$$

Let $I \subseteq [0;\tau]$ be the set of t such that there exists $h_t \in H(d_t, V)$ with the properties:

(1) $h_{t,v} \geq h_{0,v}$ for every $v \in V$;

(2) $S(d_t, V, h_t) \geq S(d_0, V, h_0) - Ct$.

We claim that $I = [0;\tau]$.

Proof. Note that $0 \in I$. First suppose that $d_0 \in \mathfrak{D}_{sc}(V)$. Take $t \in I$ such that $t \neq \tau$. We prove that there is $\xi > 0$ such that $[t;t+\xi) \subset I$. Apply Lemma 6.3.1 to $P_t = P(d_t, V, h_t)$. We obtain $\xi > 0$ and h' such that:

(1) $h'_v \geq h_{t,v} \geq h_{0,v}$ for every $v \in V$;

(2) $S(d_t, V, h') \geq S(d_t, V, h_t)$;

(3) $h' \in H(d_{t'}, V)$ for every $t' \in [t;t+\xi)$.

We may also assume that Fuchsian cone-manifolds $P_{t'} = P(d_{t'}, V, h')$ for every $t' \in [t;t+\xi)$ have a common compatible triangulation provided ξ is sufficiently small. From the assumption of the lemma we get

$$S(d_{t'}, V, h') \geq S(d_t, V, h') - C(t' - t) \geq$$

$$\geq S(d_t, V, h_t) - C(t' - t) \geq S(d_0, V, h_0) - Ct'.$$

Therefore, $t' \in I$.

Suppose that $I \neq [0;\tau]$. Let $t \in (0;\tau]$ be the smallest value not in I. Consider a monotonous sequence $\{t_n\}$ converging to t from the left. Since $t_n \in I$, there exists a sequence $\{h_n \in H(d_{t_n}, V)\}$ such that

(1) $h_{n,v} \geq h_{0,v}$ for every $v \in V$;

(2) $S(d_{t_n}, V, h_n) \geq S(d_0, V, h_0) - Ct_n$.

Define $\mathfrak{U} = \{d_t : t \in [0;\tau]\} \subset \mathfrak{D}_c(V)$. Due to property (1) there also exists a lower bound $m > 0$ for all $h_{n,v}$. Also define $K := S(d_0, V, h_0) - C\tau$. We have all $h_n \in H_S(\mathfrak{U}, m, K)$. By Lemma 6.3.12 the set $H_S(\mathfrak{U}, m, K)$ is compact. Then there is a limit point $h_t \in H(d_t, V)$ for $\{h_n\}$, and for every $v \in V$ one has $h_{t,v} \geq h_{0,v}$. The discrete curvature S is continuous over $H_S(\mathfrak{U}, m, K)$, therefore $S(d_t, V, h_t) \geq S(d_0, V, h_0) - Ct$.

Hence, $t \in I$ and we obtained a contradiction.

Now consider the case $d_0 \in \mathfrak{D}_c(V)$, but not in $\mathfrak{D}_{sc}(V)$. Apply Lemma 6.3.2 to d_0 (with arbitrary $\mu > 0$). For $t \in (0;\tau)$ we can still apply Lemma 6.3.1 to d_t. In this way for any $\mu > 0$ and $t \in [0;\tau]$ we obtain $h_{t,\mu} \in H(d_t, V)$ such that

(1) $h_{t,\mu,v} \geq h_{0,v}$ for every $v \in V$;

(2) $S(d_t, V, h_{t,\mu}) \geq S(d_0, V, h_0) - Ct - \mu$.

Fix t and consider a sequence $\mu_n \to 0$. The heights h_{t,μ_n} are uniformly bounded from below. The discrete curvature $S(d_t, V, h_{t,\mu_n})$ is also uniformly bounded from below. Hence, by Corollary 4.3.2 they belong to a compact set and there exists the limit point $h_t \in H(d_t, V)$ such that

(1) $h_{t,v} \geq h_{0,v}$ for every $v \in V$;

(2) $S(d_t, V, h_t) \geq S(d_0, V, h_0) - Ct$. □

Note that Lemma 6.3.13 does not hold for paths in $\mathfrak{D}_c$ due to remarks in the proof of Lemma 6.3.2.

Now we return to the path of metrics d_t described in Subsection 6.3.2. Take a convex Fuchsian cone-manifold $P_0 = P(d_0, V, h_0)$. Applying Lemma 6.3.13 together with the lower bound of Lemma 6.3.3 we get (here $C = 0$):

Corollary 6.3.14. There exists $h_\tau \in H(d_\tau, V')$ such that

(1) $h_{\tau,v} \geq h_{0,v}$ for every $v \in V'$;

(2) $S(d_\tau, V', h_\tau) \geq S(d_0, V, h_0)$.

Consider this path of metrics in the inverse direction, i.e., define $d'_t = d_{\tau-t}$ for $t \in [0;\tau]$. Let $P'_0 = P(d'_0, V', h_0)$ be a convex Fuchsian cone-manifold. Applying the upper bound of Lemma 6.3.3 we have the following:

Corollary 6.3.15. There exist $h_\tau \in H(d'_\tau, V')$ such that

(1) $h_{\tau,v} \geq h_{0,v}$ for every $v \in V'$;

(2) $S(d'_\tau, V', h_\tau) \geq$
$\geq S(d'_0, V, h_0) - \left(\frac{(\pi+\Delta)\sinh\Delta}{\Delta} + 1\right)\frac{2\pi\nu_v(d_0)\sinh\Delta}{\sin^2\alpha(d_\tau)}$.

For a proof of Main Lemma IIIC we will also need the following variation of Lemma 6.3.13:

Lemma 6.3.16. Let $d_t \subset \mathfrak{D}_c(V)$, $t \in [0;\tau]$, be a continuous path of metrics such that $d_t \in \mathfrak{D}_{sc}(V)$ for $t \in (0;\tau)$, and $P_0 = P(d_0, V, h_0)$ be a convex Fuchsian cone-manifold.

Assume that there exists a constant $C \geq 0$ with the following property. Let $[t';t''] \subseteq [0;\tau]$ and let there be $h \in H(d_t, V)$ for all $t \in [t', t'']$ such that

(1*) $h_v \geq h_{0,v}$ for every $v \in V$;

(2*) $S(d_{t'}, V, h) \geq S(d_0, V, h_0) - Ct'$.

Let there be a triangulation $\mathcal{T}_h$ compatible with Fuchsian cone-manifolds $P_t = P(d_t, V, h)$ for all $t \in [t', t'']$. Then

$$S(d_{t''}, V, h) \geq S(d_{t'}, V, h) - C(t'' - t').$$

Let $I \subseteq [0;\tau]$ be the set of t such that there exists $h_t \in H(d_t, V)$ with the properties:

(1) $h_{t,v} \geq h_{0,v}$ for every $v \in V$;

(2) $S(d_t, V, h_t) \geq S(d_0, V, h_0) - Ct$.

We claim that $I = [0;\tau]$.

This Lemma with its monstrous statement is just a more restrictive version of Lemma 6.3.13 and its proof is exactly the same: in the proof of Lemma 6.3.13 we actually work only with heights h satisfying (1*) and (2*). We need this version because in the proof of Main Lemma IIIC we can prove a Lipschitz bound that holds only on heights satisfying (1*) and (2*), but not globally.

6.3.6 Proof of Main Lemmas IIIA-IIIB

Lemma 6.3.17. Let $P = P(d, V, h)$ be a convex Fuchsian cone-manifold and $v \in V$ be a vertex with $\nu_v(d) = 0$. Then $\kappa_v(P) \leq 0$.

Proof. Assume that $\kappa_v(P) > 0$. Consider a spherical section of P at v and the solid cone X (with particle singularity) defined by it. Let $x \in \partial X$ be a point such that the ray vx determines the maximal angle with the axis (axis comes from the perpendicular from v to $\partial_\downarrow P$). Consider the 2-dimensional cone spanned by the rays vx and the axis. It splits the dihedral angle at the ray vx into two angles at most $\pi/2$ as x formed the maximal angle with the axis.

Cut X along this cone and glue there an orthogonal prism with dihedral angle $\kappa_v(P)$ at the axis. The resulting cone is convex. Its particle curvature will be zero, hence it can be embedded in $\mathbb{H}^3$. But its surface angle is greater than 2π, which is a contradiction. $\square$

Proof of Main Lemma IIIA. Choose any $\Delta > 0$ and δ_ν, δ_D with the help of (6.2.2) and (6.2.3). Put $\delta := \min\{\delta_\nu, \delta_D\}$. Then we can merge all the curvature inside each triangle of $\mathcal{T}$ as described in the proof of Lemma 6.2.5. We are going to use the previous results of this section to deform the cone-manifold P.

Let v_0 and v_1 be the first two vertices that we merge together, w_1 be the new vertex, d_1 be the obtained metric, $V_1' = V \cup \{w_1\}$, $V_1 = V(d_1) = V_1' \backslash \{v_0, v_1\}$, $V' = V(\mathcal{T}) \cap V(d)$. Note that $V' \subset V_1 \subset V_1'$. We transform the metric d to d_1 continuously through $\mathfrak{D}_{sc}(V_1')$ as described in Section 6.3.2 and transform respectively the cone-manifold $P = P(d, V, h)$. Using Corollary 6.3.14 we obtain $h_1' \in H(d_1, V_1')$ such that

(1) $h_{1,v}' \geq h_v$ for every $v \in V'$;

(2) $S(d_1, V_1', h_1') \geq S(d, V, h)$.

After this we have $\nu_{v_0}(d_1) = 0$. By Lemma 6.3.17 it has curvature $\kappa_{v_0}(P_1') \leq 0$ in $P_1' = P(d_1, V_1', h_1')$. If it is less than zero, then by Lemma 3.6.6 we can decrease its height. The functional S is increased under this deformation. Consider the minimal value of h_{v_0} such that it can not be decreased (with all other heights fixed). It is greater than zero. Indeed, the number of triangulations compatible with cone-manifolds defined by $H(d, V)$ is finite. In every such triangulation there exists a prism containing v_0 and other vertex of v (possibly, they are connected with flat edges). But for every prism, if we decrease sufficiently the height of one upper vertex with the fixed height of another vertex, then the prism will not be ultraparallel anymore. This contradicts to Lemma 3.2.2. Then for the minimal value of h_{v_0} we get a convex cone-manifold with the particle curvature of v_0 equal to 0.

Do the same with the vertex v_1. Let h_1 be the modified height function and $P_1 = P(d_1, V_1', h_1)$. It has no particle curvatures at points of $V_1' \backslash V_1$, therefore it can be rewritten as $P_1 = P(d_1, V_1, h_1)$. We have

$$S(P_1) \geq S(P_1').$$

By repeating this procedure we obtain $\widehat{h} \in H(\widehat{d}, V(\widehat{d}))$ such that

(1) $\widehat{h}_v \geq h_v$ for every $v \in V'$;

(2) $S(\widehat{d}, V(\widehat{d}), \widehat{h}) \geq S(d, V, h)$.

The property (1) holds because $V(\mathcal{T}) \cap V(d) = V' \subset V(d_i)$ for all intermediate metrics d_i. This finishes the proof. $\square$

Proof of Main Lemma IIIB. The proof of Main Lemma IIIB is just a bit more elaborate as now we have to control better the decrement of S. We start again by choosing any $\Delta > 0$, δ_ν, δ_D from (6.2.2) and (6.2.3) and put $\delta := \min\{\delta_\nu, \delta_D\}$. Lemma 6.2.5 shows that $\widehat{d}$ can be obtained from d via gluings bigons. Let m be the number of steps to obtain $\widehat{d}$. Now we do this process in the reverse order, i.e., we cut bigons from $\widehat{d}$ to obtain d.

Let the metric $\widehat{d}$ be obtained from the metric d_m by merging w_m with v_m into w_{m+1}. Note that $\nu_{v_m}(d_m) = \nu_{v_m}(d)$. Define $V'_m = V(\widehat{d}) \backslash \{w_{m+1}\}$, $V_m = V(d_m) = V'_m \cup \{w_m, v_m\}$, $V' = V(\mathcal{T}) \cap V(d)$. Then $V' \subset V_m \subset V'_m$.

We have to make a remark on the diameters. Define

$$\widehat{D} = \widehat{D}(\Delta, D, A) := D + (A + 2\pi(2-2g)) \sinh \Delta.$$

Recall that Lemma 6.2.6 implies that $\widehat{d}$ has the largest diameter among all metrics on our way and it satisfies

$$\operatorname{diam}(S_g, \widehat{d}) \leq \operatorname{diam}(S_g, d) + \nu(S_g, d) \sinh \Delta \leq \widehat{D}.$$

Set

$$M = M(\Delta, D, A) := \frac{1}{\sin^2(\operatorname{arccot}(2\cosh \widehat{D}))}.$$

Recall that for a metric d on S_g we defined $\alpha(d) = \operatorname{arccot}(2\cosh(\operatorname{diam}(S_g, d))$. Hence, M serves as an upper bound for $\frac{1}{\sin^2 \alpha(d_i)}$ for all intermediate d_i. Also $V' \subset V(d_i)$ for all i.

We apply Corollary 6.3.15 and get $h'_m \in H(d_m, V'_m)$ such that

(1) $h'_{m,v} \geq \widehat{h}_{m,v}$ for each $v \in V'$;

(2) $S(d_m, V'_m, P'_m) \geq S(\widehat{P}) - \left(\frac{(\pi+\Delta)\sinh\Delta}{\Delta} + 1\right) 2\pi M \nu_{v_m}(d) \sinh \Delta$.

Now we have $\nu_{w_{m+1}}(d_m) = 0$. Thus, by Lemma 6.3.17 we get $\kappa_{w_{m+1}}(P'_m) \leq 0$. Similarly to the proof of Main Lemma IIIA, we decrease the height of w_{m+1} until we get $P_m = P(d_m, V'_m, h_m)$ with $\kappa_{w_{m+1}}(P_m) = 0$, so $P_m = P(d_m, V_m, h_m)$ and $S(P_m) \geq S(P'_m)$.

After repeating these operations we obtain $h \in H(d, V)$ with

(1) $h_v \geq \widehat{h}_v$ for each $v \in V'$;

(2) $S(d, V, h) \geq S(\widehat{P}) - \left(\frac{(\pi+\Delta)\sinh\Delta}{\Delta} + 1\right) 2\pi M \nu(S_g, d) \sinh \Delta$.

We have

$$\nu(S_g, d) = \operatorname{area}(S_g, d) + 2\pi(2-2g) \leq A + 2\pi(2-2g).$$

We see that the decrement is bounded by $o(\Delta)$ as $\Delta \to 0$ tends to 0 and we can choose Δ arbitrarily (observe that M is bounded for fixed D, A and $\Delta \to 0$). This finishes the proof. $\square$

6.3.7 Dissolving curvature of swept triangles

In order to prove Main Lemma IIIC we should describe how to transform strict swept triangles with small curvature to hyperbolic ones. We apply the results from previous sections to obtain a simultaneous transformation of a Fuchsian cone-manifold and bound the change of the discrete curvature.

Definition 6.3.18. A swept triangle is called *tetrahedral* if it is strict and is isometric to the surface of a tetrahedron (possibly degenerated) with a face excluded. A swept triangle is called *short* if it is strict, convex and all sides are shortest paths.

A tetrahedral swept triangle might be non-short if it has some angles at least π. Denote a swept triangle by $OA_1A_2A_3$ as in Subsection 6.2.2. Recall that x_i is the length OA_i and β_i is $\angle OA_{i-1}A_{i+1}$. A strict swept triangle is tetrahedral if and only if β_i satisfy the triangle inequalities. Indeed, if they do, then there exists a solid trihedral cone in $\mathbb{H}^3$ (possibly degenerated) with angles β_i and we take the points at distances x_i from the apex at the corresponding rays. The converse direction is obvious. It is clear that the tetrahedron is unique up to isometry.

Lemma 6.3.19. A short swept triangle $OA_1A_2A_3$ is tetrahedral.

Proof. Assume that $\beta_3 > \beta_1 + \beta_2$. Develop the triangles OBC and OAC to $\mathbb{H}^2$ as the convex polygon $OA_1'A_2A_3$. Due to convexity, the segment $A_1'A_2$ belongs to the union of the triangles. Then develop the triangle OA_1A_2 as the triangle OA_2A_1'' such that A_1'' lies on the same side from OA_2 as A_1' (recall that $\beta_3 < \pi$). As $\beta_3 > \beta + \beta_2$, it is easy to see that $A_1''A_2 > A_1'A_2$, which contradicts that A_1A_2 is a shortest path. $\square$

Let l_1, l_2, l_3 be three numbers satisfying the strict triangle inequalities and λ_1^0, λ_2^0, λ_3^0 be the angles of the corresponding hyperbolic triangle. Define $TCT(l_1, l_2, l_3) \subset \mathbb{R}^3$ as the set of triples $Z = (x_1, x_2, x_3)$ such that there exists a tetrahedral swept triangle $OA_1A_2A_3$, where the side lengths of $A_1A_2A_3$ are equal to l_1, l_2 and l_3 and the lengths OA_1, OA_2 and OA_3 are equal to x_1, x_2 and x_3. The boundary of $TCT(l_1, l_2, l_3)$ corresponds to the degenerations of the triangle inequalities for the angles β_i. The closure of $TCT(l_1, l_2, l_3)$ is formed by adding points with $\beta_1 + \beta_2 + \beta_3$, i.e., that correspond to hyperbolic triangles with O belonging to $A_1A_2A_3$. We consider λ_j as continuous functions over $TCT(l_1, l_2, l_3)$. Note that $\lambda_j(Z) \geq \lambda_j^0$ for each Z.

Consider $OA_1A_2A_3$ as a tetrahedron. Let $\widetilde{\tau}$ be the distance from point O to the triangle $A_1A_2A_3$. It defines another continuous function over $TCT(l_1, l_2, l_3)$. Moreover, it can be continuously extended to the closure of $TCT(l_1, l_2, l_3)$ in $\mathbb{R}^3$. It is equal to zero at all points from the closure that are not in $TCT(l_1, l_2, l_3)$.

Lemma 6.3.20. For every $\xi > 0$ there exists $\delta > 0$ such that if for all $j = 1, 2, 3$ we have $\lambda_j(Z) < \lambda_j^0 + \delta$, then $\widetilde{\tau}(Z) < \xi$.

Proof. For any $Z \in TCT(l_1, l_2, l_3)$ we have $\lambda_j(Z) > \lambda_j^0$ for some j. Moreover, one can show that for every $\xi > 0$ there exists $C > 0$ such that if $x_i > C$ for some i, then $\lambda_j(Z) - \lambda_j^0 > \xi$ for some j.

Now suppose the converse to the statement of the lemma. Then there exists $\xi > 0$ and a sequence Z_n such that $\tau(Z_n) \geq \xi$ for all n, but $\lambda_j(Z_n) \to \lambda_j^0$ for all j. Then $x_{n,i} \leq C$ for all i, all sufficiently large n and a constant C defined above. Then Z_n belongs to a compact in the closure of $TCT(l_1, l_2, l_3)$ and has a limit point Z. We have $\lambda_j(Z) = \lambda_j^0$ for all j. This means that Z does not belong to $TCT(l_1, l_2, l_3)$. However, this implies $\widetilde{\tau}(Z) = 0$. $\square$

Consider $Z_0 \in TCT(l_1, l_2, l_3)$, the corresponding tetrahedron $OA_1A_2A_3$ and a point O' in the interior of $A_1A_2A_3$. Let $\tau = OO'$ and $O_t \in OO'$ be the point at distance t from O. This gives us a path of swept triangles $Z_t \in TCT(l_1, l_2, l_3)$ parametrized by $t \in [0; \tau]$. Note that τ can be chosen arbitrarily close to $\widetilde{\tau}(Z_0)$.

Lemma 6.3.21. We have $\angle O_tA_1A_2 + \angle O_tA_1A_3 < \angle OA_1A_2 + \angle OA_1A_3$.

Proof. Consider a spherical section of the tetrahedron $OA_1A_2A_3$ at point A. The inequality easily follows from the fact that the resulting spherical triangle for $O_tA_1A_2A_3$ is strictly contained in the spherical triangle for $OA_1A_2A_3$ and the fact that the perimeter of convex figures decreases under inclusion. □

In other words, the angles of swept triangles determined by Z_t are strictly decreasing.

Consider now a Fuchsian cone-manifold $P = P(d_0, V, h)$, $d_0 \in \mathfrak{D}_{sc}(V)$. Assume that Z is a short swept triangle in d_0 with the conical point w and vertices from V. We perform the deformation of Z transforming it to a hyperbolic triangle as described above and obtain a path of metrics $d_t \in \mathfrak{D}(V)$, $t \in [0;\tau]$. Because the angles of Z are strictly decreasing and all other angles remain the same, we have that $d_t \in \mathfrak{D}_{sc}(V)$ except the last point $d_\tau \in \mathfrak{D}_c(V)$ because $\nu_w(d_\tau) = 0$. We would like to transform simultaneously the Fuchsian cone-manifold with the help of Lemma 6.3.13 and control the change of S. To this purpose we need an analogue of Lemma 6.3.3.

It is easy to see that the diameter of metrics d_t does not increase as well as the diameter of T. Let Δ be an upper bound for the diameter of Z. We denote the three vertices of T by v_1, v_2, v_3, the shortest paths from w to them by ψ_1, ψ_2 and ψ_3, the edges of Z by e_1, e_2 and e_3 and the angles of Z by λ_1, λ_2, λ_3 respectively. Define $\Lambda := \lambda_1 + \lambda_2 + \lambda_3$.

Lemma 6.3.22. Let $0 \leq t' < t'' \leq \tau$. Assume that there is

$$h \in \bigcap_{t \in [t';t'']} H(d_t, V)$$

and a triangulation $\mathcal{T}_h$ with $V(\mathcal{T}_h) = V$ such that every convex Fuchsian cone-manifold $P_t = P(d_t, V, h)$ is compatible with $\mathcal{T}_h$. Then

$$\frac{-1}{\sin^2 \alpha(d_0)} \left((\pi + \Delta)\Delta \left(\Lambda(t'') - \Lambda(t')\right) + 2\pi(t'' - t')\right) \leq$$

$$\leq S(d_{t''}, V, h) - S(d_{t'}, V, h) \leq \frac{2\pi(t'' - t')}{\sin \alpha(d_0)}.$$

Proof. The proof is similar to the proof of Lemma 6.3.3. By $S(t)$ denote $S(d_t, V, h)$. We have

$$S(t'') - S(t') = \int_{t'}^{t''} \sum_{e \in E(\mathcal{T}_h)} (\pi - \theta_e)\dot{l}_e dt.$$

First, for simplicity we assume that each edge of $\mathcal{T}_h$ that changes its the length during the deformation belongs to only one of the following four types:

(1) edges that intersect ψ_1 once;
(2) edges that intersect ψ_2 once;
(3) edges that intersect ψ_3 once;
(4) edges that have w as one (and only one) of the endpoints.

Denote these types by E_1, E_2, E_3 and E_4. Triangulate Z naturally into 3 triangles and extend this to a triangulation $\mathcal{T}_0$ of (S_g, d_0) with vertex set V. Then $d_t \in \mathcal{D}_{sc}(V, \mathcal{T}_0)$ for all $t \in [0;\tau]$, and only three edges of $\mathcal{T}_0$ change their lengths: ψ_1, ψ_2 and ψ_3.

Similarly to Lemma 6.3.3 for $e \in E_1$ we consider the path of triangles of $\mathcal{T}_0$ along e, develop it to $\mathbb{H}^2$ and denote by $y_e(t)$ the hyperbolic sine of the length of the perpendicular from the image of v_1 under the developing map to the line containing the image of e. Then we obtain

$$\frac{\partial l_e}{\partial \lambda_1} = y_e, \quad \frac{\partial l_e}{\partial \lambda_2} + \frac{\partial l_e}{\partial \lambda_3} = 0.$$

By Lemma 6.3.21, λ_1 is strictly decreasing, therefore $\dot{l}_e(t) \leq 0$. Then as in the proof of Lemma 6.3.3 using the bound $\operatorname{len}(\psi_1, d_t) < \Delta$ we get the estimate

$$-\frac{(\pi+\Delta)\Delta}{\sin^2 \lambda(d_0)}(\lambda_1(t'') - \lambda_1(t')) \leq \int_{t'}^{t''} \sum_{e \in E_1} (\pi - \theta_e)\dot{l} dt \leq 0.$$

Similar estimates hold for edges of the second and the third type.

We proceed with the fourth type. By Lemma 6.3.8 and Lemma 6.3.7 we have

$$\sum_{e \in E_4} (\pi - \theta_e) \leq \omega_w \leq \frac{2\pi}{\sin \alpha(d_0)}.$$

We claim that for $e \in E_4$ we have

$$-1 \leq \dot{l}_e(t) \leq 1.$$

Assume that e leaves Z through e_3. Let v_4 be the second vertex of e. Develop to $\mathbb{H}^2$ the path of triangles of $\mathcal{T}_0$ intersecting e in the moment t. Let A_1 and A_2 be the images of v_1 and v_2, B_1 and C be the images of w and v_4 respectively. For some Δt develop the path of triangles intersecting e in the moment $t + \Delta t$ in such a way that the image of each triangle coincides with its image under the previous developments, except the first one, which is developed to a triangle $A_1A_2B_2$. We have $l_e(t) = B_1C$, $l_e(t + \Delta t) = B_2C$, so

$$-B_1B_2 \leq l_e(t + \Delta t) - l_e(t) \leq B_1B_2.$$

One can see that $B_1B_2 \leq \Delta t$. Indeed, there is a hyperbolic tetrahedron $A_1A_2O_1O_2$ with $A_1A_2O_1 = A_1A_2B_1$, $A_1A_2O_2 = A_1A_2B_2$ and $O_1O_2 = \Delta t$ (this tetrahedron is a subset of the tetrahedron $OA_1A_2A_3$ defining our deformation). Clearly if we rotate the triangle $A_1A_2O_2$ around the edge A_1A_2, the distance O_1O_2 is minimized, when O_2 is in the plane $A_1A_2O_1$. But in this moment $O_1O_2 = B_1B_2$. Therefore, $B_1B_2 \leq \Delta t$ and we obtain the estimate of $\dot{l}_e(t)$. Thus, we have

$$\frac{-2\pi}{\sin \lambda(d_0)} \leq \int_{t'}^{t''} \sum_{e \in E_4} (\pi - \theta_e)\dot{l}_e dt \leq \frac{2\pi}{\sin \lambda(d_0)}.$$

The proof of the general case is done via elementary deformations of the four types as in the proof of Lemma 6.3.3. However, for some reasons the treatment here must be more intricate.

Consider the general case and an edge $e \in E(\mathcal{T}_h)$ with an orientation. As before, develop the path of triangles intersecting e at moment t as a triangulated polygon R_0 in $\mathbb{H}^2$. Enumerate triangles of R_0 with respect to the orientation. By k denote the number of edges in R_0 and by $z_1(t), \dots, z_k(t)$ denote their lengths. So the length $l_e(t)$ can be considered as a function $l_e(t) = l_e(z_1(t), \dots, z_k(t))$.

All z_i are constant except those that are the images of some ψ_j under the developing map. The length of each ψ_j is completely determined by l_j and λ_j due to Lemma 6.2.8. Assume that l_j are fixed, but λ_j vary. Then z_i may be considered as functions of λ_j. In this way we view each $z_i(t)$ as a function $z_i(\lambda_1(t), \lambda_2(t), \lambda_3(t))$. We are going to consider triple coordinate derivatives

$$\frac{\partial l_e}{\partial z_i}\frac{\partial z_i}{\partial \lambda_j}\frac{\partial \lambda_j}{\partial t}(t).$$

How can e pass through Z? It might happen that w is an endpoint of e. Then e immediately leaves Z. Consider an intermediate event when e passes through Z on its way. Suppose that $\nu_w(d_t) < \pi$. Then up to an index permutation, there can be the following two types of events:

(.1) e enters Z at e_2, intersect ψ_1 and leaves Z through e_3;

(.2) e enters Z at e_2, then intersects ψ_3, then ψ_2 and leaves through e_3.

If one of the vertices of Z is an endpoint of e and e enters/finishes at Z then we consider this as a limiting case of (.1) or (.2).

Consider an event of type (.1). Let T_{i-1} and T_i be the respective triangles of R_0. Denote their vertices by O', A'_j with respect to their preimages under the developing map. Let z_{i_1}, z_{i_2} and z_{i_3} be the lengths of edges corresponding to ψ_1, ψ_2 and ψ_3 respectively. We claim that

$$\sum_{s=1}^{3}\frac{\partial l_e}{\partial z_{i_s}}\frac{\partial z_{i_s}}{\partial \lambda_2}+\sum_{s=1}^{3}\frac{\partial l_e}{\partial z_{i_s}}\frac{\partial z_{i_s}}{\partial \lambda_3}=0.$$

Indeed, this corresponds to a deformation of R_0 when the point O' moves while $\angle A'_2A'_1A'_3$ stays constant as well as all edges of R_0 except incident to O'. Then the image of e under the development does not change its length.

Define

$$L(e,i_1,i_2,i_3,\lambda_j)(t):=\sum_{s=1}^{3}\frac{\partial l_e}{\partial z_{i_s}}\frac{\partial z_{i_s}}{\partial \lambda_j}\frac{\partial \lambda_j}{\partial t}(t).$$

Then $L(e,i_1,i_2,i_3,\lambda_2)+L(e,i_1,i_2,i_3,\lambda_3)=0$. We say that $L(e,i_1,i_2,i_3,\lambda_1)$ is an elementary deformation of type (1.1). It is equal to the hyperbolic sine of the length of the perpendicular to the image of e. Similarly we define elementary deformations of types (2.1) and (3.1).

In an event of type (.2) there are four edges of R_0 that change their length (ψ_1 has two images). Let their lengths be z_{i_1}, z_{i_2}, z_{i_3} and z_{i_4}. Define

$$L(e,i_1,i_2,i_3,i_4,\lambda_j)(t):=\sum_{s=1}^{4}\frac{\partial l_e}{\partial z_{i_s}}\frac{\partial z_{i_s}}{\partial \lambda_j}\frac{\partial \lambda_j}{\partial t}(t).$$

We have $L(e,i_1,i_2,i_3,i_4,\lambda_1)(t)=0$. Two other expressions are called elementary deformations of types (2.2) and (3.2). They are equal to the hyperbolic sines of the lengths of the respective perpendiculars.

Similarly we define elementary deformations in the events obtained from (.1) and (.2) by a permutation of indices.

An elementary deformation of the type (4) is defined in the same way as an elementary deformation of type (3) in the proof of Lemma 6.3.3.

We have

$$\dot{l}_e(t)=\sum_{i=1}^{k}\sum_{j=1}^{3}\frac{\partial l_e}{\partial z_i}\frac{\partial z_i}{\partial \lambda_j}\frac{\partial \lambda_j}{\partial t}(t).$$

Some of these triple derivatives form elementary deformations. Due to the discussion above, the sum of all others is zero.

Consider elementary deformations of every edge and decompose

$$\sum_{e\in E(\mathcal{T}_h)} (\pi - \theta_e(t))\dot{l}_e(t)$$

into sums of elementary deformations of each type. Then we sum together all elementary deformations of types (1.1) and (1.2) and give the same estimate for its integral as for type (1) edges in the simple case. We do similarly with other types. This gives the same estimate for S as in the simple case.

The assumption $\nu_w(d_t) < \pi$ is not essential. In the other case, e can spiral several times in Z around w. We say that it is an event of type $(.n)$ if it crosses n internal edges. Some λ_j may be reproduced several times in the triangles of R_0 corresponding to this event. Taking this into account, one can see that this produces no difficulties to the proof. □

Now we can transform a convex Fuchsian cone-manifold $P_0 = P(d_0, V, h_0)$ along the path of metrics d_t, $t \in [0;\tau]$, described above. With the help of Lemma 6.3.13 and the lower bound Lemma 6.3.22 we get

Corollary 6.3.23. Assume that for some $\xi > 0$ and for all $j = 1, 2, 3$ we have $\lambda_j(0) - \lambda_j(\tau) < \xi$ and $\tau < \xi$.

Then there exists $h_\tau \in H(d_\tau, V)$ such that

(1) $h_\tau(v) \geq h_0(v)$ for every $v \in V$;

(2) $S(d_\tau, V, h_\tau) \geq S(d_0, V, h_0) - \frac{(3(\pi+\Delta)\Delta+2\pi)\xi}{\sin^2\alpha(d_0)}$.

Now consider the same path of metrics in the inverse direction, i.e., $d'(t) := d(\tau - t)$ for $t \in [0;\tau]$. Let $P'_0 = P(d'_0, V, h_0)$ be a convex Fuchsian cone-manifold. Then from Lemma 6.3.13 and the upper bound Lemma 6.3.22 we get

Corollary 6.3.24. There exists $h_\tau \in H(d'_\tau, V)$ such that

(1) $h_\tau(v) \geq h_0(v)$ for every $v \in V$;

(2) $S(d'_\tau, V, h_\tau) \geq S(d'_0, V, h_0) - \frac{2\pi\tau}{\sin\alpha(d'_\tau)}$.

6.3.8 Proof of Main Lemma IIIC

Let $\mathfrak{S}(\mathcal{T})$ be the set of convex cone-metrics swept with respect to $\mathcal{T}$ (defined up to isometry isotopic to identity with respect to $V(\mathcal{T})$). Hence, $\widehat{d} \in \mathfrak{S}(\mathcal{T})$. By $\mathfrak{S}(\mathcal{T}, \widehat{d}, \delta) \subset \mathfrak{S}(\mathcal{T})$ denote the set of metrics $d \in \mathfrak{S}(\mathcal{T})$ such that for each triangle T of $\mathcal{T}$ we have $||T(d) - T(\widehat{d})||_\infty < \delta$. By the assumptions of Main Lemma IIIC we have d^1, $d^2 \in \mathfrak{S}(\mathcal{T}, \widehat{d}, \delta)$.

Define

$$\widehat{V} := V(\widehat{d}) \cup V(\mathcal{T}), \quad D := 2\mathrm{diam}(S_g, \widehat{d}),$$
$$M := \sin^{-2}(\mathrm{arccot}\, D), \quad \Delta := 2\max_{T\in\mathcal{T}} \mathrm{diam}(T, \widehat{d}).$$

By $\widehat{\mathcal{T}}$ we denote the triangulation obtained by refining each triangle of $\mathcal{T}$ that has a conical point in $\widehat{d}$ into three. Naturally $\widehat{d} \in \mathfrak{D}_c(\widehat{V}, \widehat{\mathcal{T}})$. Recall from Subsection 2.4.1 that with the help of the edge-length map, $\mathfrak{D}_c(\widehat{V}, \widehat{\mathcal{T}})$ is considered as a subset of $\mathbb{R}^N$, where $N = |E(\widehat{\mathcal{T}})|$, endowed with l_∞-metric.

Our strategy is as follows. If a triangle T of $\mathcal{T}$ is non-strict in $\widehat{d}$, then its curvature in d^1, d^2 is small provided that δ is small. Then we use the results of Subsection 6.3.7 to dissolve this curvature. Doing this for each triangle of this kind we obtain two metrics in $\mathfrak{D}_c(\widehat{V}, \widehat{\mathcal{T}})$ that are very close with respect to l_∞-metric. By an indirect argument we connect them with a short path and transform the Fuchsian cone-manifold along the path.

There exists δ_0 such that for each $d \in \mathfrak{S}(\mathcal{T}, \widehat{d}, \delta_0)$ the following hold
(1) $V(\widehat{d}) \subseteq V(d)$;
(2) $\mathrm{diam}(S_g, d) < D$;
(3) $\max_{T \in \mathcal{T}} \mathrm{diam}(T, d) < \Delta$.

Let k be the number of triangles of $\mathcal{T}$ that are non-strict in $\widehat{d}$. Choose $\xi > 0$ small enough so that

$$M\left(3(\pi + \Delta)\Delta + 2\pi\right)\xi < \frac{\varepsilon}{3k}.$$

Then also $2\pi M \xi < \varepsilon/3k$.

Take a triangle T of $\mathcal{T}$ that is non-strict in $\widehat{d}$. By Lemma 6.3.20 there exists $\delta_1(T)$ such that if $d \in \mathfrak{S}(\mathcal{T}, \widehat{d}, \delta_1(T))$, then $\tau < \xi$, where τ is the distance from the conical point O to an interior point of the triangle $A_1A_2A_3$ for a tetrahedral realization of the swept triangle (T, d). We also assume that $\delta_1(T) < \xi$. Hence, if $d \in \mathfrak{S}(\mathcal{T}, \widehat{d}, \delta_1(T))$, if $\lambda(d)$ is an angle of (T, d) and if $\lambda(\widehat{d})$ is the same angle of $(T, \widehat{d})$, then $\lambda(d) < \lambda(\widehat{d}) + \xi$.

Take δ_1 as the minimum of $\delta_1(T)$ over all such triangles T.

For $\sigma_0 > 0$ define $\mathfrak{B}_c(\widehat{d}, \sigma_0)$, $\mathfrak{B}_{sc}(\widehat{d}, \sigma_0)$ as in Section 2.4.1 to be the intersections of the open ball in $(\mathbb{R}^N, l_\infty)$ of radius σ_0 with center at $\widehat{d}$ with $\mathfrak{D}_c(\widehat{V}, \widehat{\mathcal{T}})$, $\mathfrak{D}_{sc}(\widehat{V}, \widehat{\mathcal{T}})$ respectively. By Corollary 2.4.8 the set $\mathfrak{B}_{sc}(\widehat{d}, \sigma_0)$ is connected for sufficiently small σ_0. Its boundary is locally piecewise analytic, thus $\mathfrak{B}_{sc}(\widehat{d}, \sigma_0)$ is an open, connected and bounded subset of $\mathbb{R}^N$ with Lipschitz boundary. From [24, Chapters 2.5.1-2.5.2] it is quasiconvex: there exists a constant $C_1' = C_1'(\widehat{d}, \mathcal{T})$ such that every two points of $\mathfrak{B}_{sc}(\widehat{d}, \sigma_0)$ at the distance σ can be connected through $\mathfrak{B}_{sc}(\widehat{d}, \sigma_0)$ by a path of length at most $C_1'\sigma$. By Lemma 2.4.7, the set $\mathfrak{B}_c(\widehat{d}, \sigma_0)$ belongs to the closure of $\mathfrak{B}_{sc}(\widehat{d}, \sigma_0)$ Then for any $C_1 > C_1'$ the set $\mathfrak{B}_c(\widehat{d}, \sigma_0)$ is strictly C_1-quasiconvex, i.e., every two points of $\mathfrak{B}_c(\widehat{d}, \sigma_0)$ at the distance σ can be connected through $\mathfrak{B}_{sc}(\widehat{d}, \sigma_0)$ by a path of length at most $C_1\sigma$. Take any such C_1. We also assume that the closure $\overline{\mathfrak{B}}_c(\widehat{d}, \sigma_0)$ is in $\mathfrak{D}_c(\widehat{V}, \widehat{\mathcal{T}})$ (therefore, no triangles become degenerate in the closure).

Recall that

$$H_S(d, V, m, K) = \{h \in H(d, V) : \min_{v \in V} h_v \geq m, S(d, V, h) \geq K\},$$

$$\mathfrak{H}_S(\overline{\mathfrak{B}}_c(\widehat{d}, \sigma_0), m, K) = \{(d, h) : d \in \overline{\mathfrak{B}}_c(\widehat{d}, \sigma_0), h \in H_S(d, V, m, K)\} \subset \mathfrak{H}(\mathfrak{D}_c(\widehat{V}, \widehat{\mathcal{T}})).$$

By Lemma 4.1.1, S is continuously differentiable over $\mathfrak{H}(\mathfrak{D}_c(\widehat{V}, \widehat{\mathcal{T}}))$. Put

$$K := S(d^1, V(d^1), h^1) - \varepsilon.$$

As $\overline{\mathfrak{B}}_c(\widehat{d}, \sigma_0)$ is compact, it follows from Lemma 6.3.12 that so is $\mathfrak{H}_S(\overline{\mathfrak{B}}_c(\widehat{d}, \sigma_0), m, K)$. Hence, there exists a constant $C_2 = C_2(\widehat{d}, \mathcal{T})$ such that S is C_2-Lipschitz over $\mathfrak{H}_S(\overline{\mathfrak{B}}_c(\widehat{d}, \sigma_0), m, K)$.

Choose $\sigma > 0$ such that $C_1C_2\sigma < \varepsilon/3$ and $\sigma < \sigma_0$. By Corollary 6.2.9 we can choose δ_2 sufficiently small such that if $d \in \mathfrak{S}(\mathcal{T}, \widehat{d}, \delta_2)$, then $d \in \mathfrak{B}_c(\widehat{d}, \sigma/2)$.

Now put $\delta := \min\{\delta_0, \delta_1, \delta_2\}$ and take $d^1, d^2 \in \mathfrak{S}(\mathcal{T}, \widehat{d}, \delta)$. We have

$$V(\widehat{d}) \subseteq (V(d^1) \cap V(d^2)).$$

Let T be a triangle of $\mathcal{T}$ such that T is strict in d^1, but T is non-strict in $\widehat{d}$. Denote the conical point of T in d_1 by w. Define $d_0 := d^1$, $V_0 := V(d^1)$. Consider a deformation d_t through $\mathfrak{D}_{sc}(V_0)$, $t \in [0; \tau)$, described in Section 6.3.7 that dissolves the curvature of w. It follows from the discussion above that $\tau < \xi$ and the variations of angles of T are smaller than ξ. Denote the resulting metric by d_1. By Corollary 6.3.23 we obtain a convex Fuchsian cone-manifold $P_1' = P(d_1, V_0, h_1')$ such that

(1) $h'_{1,v} \geq h^1_v$ for every $v \in V(\widehat{d})$;

(2) $S(d_1, V_0, h_1') \geq S(d^1, V^1, h^1) - \frac{(3(\pi+\Delta)\Delta+2\pi)\xi}{\sin^2 \alpha(d_0)} \geq S(d^1, V^1, h^1) - \frac{\varepsilon}{3k}$.

Then we reduce the height of w until its particle curvature disappears as in the proofs of Main Lemmas IIIA-B. This increases S and produces $h_1 \in H(d_1, V_1)$, where $V_1 = V_0 \backslash \{w\}$ such that

$$S(d_1, V_0, h_1) \geq S(d_1, V_0, h_1').$$

We do this for every such triangle and obtain a metric $\widehat{d}^1 \in \mathfrak{D}_c(\widehat{V}, \widehat{\mathcal{T}})$. We note that the diameters of all intermediate metrics are smaller than D. We get $\widehat{h}^1 \in H(\widehat{d}^1, \widehat{V})$ such that

(1) $\widehat{h}^1_v \geq h^1_v$ for every $v \in V(\widehat{d})$;

(2) $S(\widehat{d}^1, \widehat{V}, \widehat{h}^1) \geq S(d^1, V^1, h^1) - \varepsilon/3$.

Now we take d^2 and replace each strict swept triangle of $\mathcal{T}$ that is hyperbolic in $\widehat{d}$ by the hyperbolic triangle with the same side lengths (of d^2). Denote the obtained metric by $\widehat{d}^2$. Note that $\widehat{d}^1, \widehat{d}^2 \in \mathfrak{S}(\mathcal{T}, \widehat{d}, \delta)$. Thus, $\widehat{d}^1, \widehat{d}^2 \in \mathfrak{B}_c(\widehat{d}, \sigma/2)$. Hence, they can be connected through $\mathfrak{B}_{sc}(\widehat{d}, \sigma_0)$ via a curve d_t of length at most $C_1\sigma$.

We take the Fuchsian cone-manifold $\widehat{P}^1 = P(\widehat{d}^1, \widehat{V}, \widehat{h}^1)$ that we obtained before. We want to show that there exist a height function $\widehat{h}^{21} \in H(\widehat{d}^2, \widehat{V})$ such that

(1) $\widehat{h}^{21}_v \geq \widehat{h}^1_v$ for every $v \in V(\widehat{d})$;

(2) $S(\widehat{d}^2, \widehat{V}, \widehat{h}^{21}) \geq S(\widehat{d}^1, \widehat{V}, \widehat{h}^1) - C_1C_2\sigma \geq S(\widehat{d}^1, \widehat{V}, \widehat{h}^1) - \varepsilon/3$.

This follows from Lemma 6.3.16 together with the fact that S is C_2-Lipschitzian over $\mathfrak{H}_S(\overline{\mathfrak{B}}_c(\widehat{d}, \sigma_0), m, K)$.

It remains to reverse the process of transforming $\widehat{d}^2$ to d^2 and transform simultaneously the Fuchsian cone-manifold $P^{21} = P(\widehat{d}^2, \widehat{V}, \widehat{h}^{21})$ with the help of Corollary 6.3.24. We obtain $h^{21} \in H(d^2, V^2)$ such that

(1) $h^{21}_v \geq \widehat{h}^{21}_v$ for every $v \in V(\widehat{d})$;

(2) $S(d^2, V^2, h^{21}) \geq S(\widehat{d}^2, \widehat{V}, \widehat{h}^{21}) - k2\pi M\xi \geq S(\widehat{d}^2, \widehat{V}, \widehat{h}^{21}) - \varepsilon/3$.

In total we get

(1) $h^{21}_v \geq h^1_v$ for each $v \in V(\widehat{d})$;

(2) $S(d^2, V^2, h^{21}) \geq S(d^1, V^1, h^1) - \varepsilon$.

This finishes the proof.

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